Principal series and generalized principal series Whittaker functions with peripheral K-types on the real symplectic group of rank 2
(実二次シンプレクティック群上の主系列表現及び一般主系列表現の周辺的 K-type を持つ Whittaker 関数)

YASUKO HASEGAWA

Typeset by $\mathcal{AMS}$-TEX

CONTENTS

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INTRODUCTION

We are interested in Whittaker functions belonging to the standard representations of the real symplectic group of $G = \mathrm{Sp}(2, \mathbb{R})$. At the present time, the explicit expressions of Whittaker functions with minimal K-types of $G = \mathrm{Sp}(2, \mathbb{R})$ are obtained by Ishii [I] for the principal series with respect to minimal parabolic subgroup (the P_{min}-series) ; Miyazaki-Oda [MO1] for the generalized principal series representations induced from the Jacobi maximal parabolic subgroup (the P_{J}-series); and Oda [O1] for the discrete series representations. Among standard series of representations, only the case of the generalized principal series representations induced from the Siegel maximal parabolic subgroup (the P_{S}-series) have not yet handled.

Our motivation here is to have explicit formulas of Whittaker functions for P_{S}-series with peripheral K-types. Our strategy is, one way to have explicit formulas of the system of partial differential equations for the radial part of Whittaker functions belonging to the P_{S}-series, and the other way to have more formulas for the principal series Whittaker functions. The reason of this strategy is because we want to derive the solution, i.e., the Whittaker functions belonging to the P_{S}-series from the Whittaker functions belonging to the principal series, utilizing the embedding of the P_{S}-series to P_{min}-series.

We deduce explicit formulas for Whittaker functions with "non-minimal" small K-types in the P_{min}-series from the fundamental formulas by Ishii in [I]. This is the first main result of this paper (Chapter I, Theorem 8.5 and Theorem 8.6).

To give explicit formulas of Whittaker functions for all the scalar K-types, we use shift operators which move K-type of Whittaker functions from Ishii's formula to general scalar K-types. The shift operators, the so called Maass shift operators, were firstly introduced by Maass in the context of global theory of automorphic forms in [M1]. Also in a different context, G. J. Heckman, E. M. Opdam ([HO], [OP]) and T. Koornwinder ([K]) discussed similar operator. In our situation, they are iterated composites of the gradient operators and the injections or projections in the Clebsh-Gordan decomposition (i.e., composites of the Dirac-Schmid operators).

The second main result of this paper is to find explicit formulas of the holonomic system for the radial part of the vector-valued Whittaker function of the P_{S}-principal series with peripheral K-types (Chapter II, Theorem 3.1). Here the peripheral K-types of a P_{S}-principal series Π are the K-type whose dimensions are smallest. We expect the solution space of the obtained holonomic system has dimension eight, i.e., the order of the Weyl group.

In Chapter III, for a special P_{S}-series induce from the discrete series D_2 of $\mathrm{SL}^{\pm}(2,\mathbb{R})$, we give explicit formulas of power series Whittaker functions with $\tau_{(1,-1)}$ K-type on G (i.e., the secondary Whittaker functions). Eight linearly independ solutions are obtained utilizing various embedding of the P_{S}-series into the P_{min}-series.

Let us explain the contents of this paper. In Chapter I, section 1 we recall the structure of $G = \mathrm{Sp}(2,\mathbb{R})$, its standard subgroups and their Lie algebras. In section 3, we define the principal series representation. In section 5, we recall the definition of Whittaker functions satisfying the system of two differential equations which is expressed in section 7.

In section 8, applying the shift operators introduced here, we show power series expansions of eight Whittaker functions in Theorems 8.5 and 8.6.

In Chapter II, we show the holonomic system for P_{S}-principal series Whittaker function (Theorem 3.1). In section 1, we recall the basic facts on the P_{S}-principal series, and in section 2, we investigate the $(\mathfrak{g}, K)$-module structure of this representation around the peripheral K-types. Passing to the Whittaker model, we get a holonomic system for the radial part of the Whittaker functions with peripheral K-types (Theorem 3.1).

In Chapter III, we consider one example. Here the P_{S}-series is induced from the discrete series D_2 of $\mathrm{SL}^{\pm}(2,\mathbb{R})$. We consider eight different embedding $\Pi \hookrightarrow \pi$ of the P_{S}-series into P_{min}-series (section 1).

In section 2, starting from a Whittaker function with sccalar K-type $\tau_{(l,l)}$ belonging to π, we shift the K-type from $\tau_{(l,l)}$ to a constituent in Π, by application of the Whittaker realization of some elements in $S(\mathfrak{p}_{\pm}) \hookrightarrow \mathrm{U}(\mathfrak{g}_{\mathbb{C}})$. Then we have four solutions either in Theorem 2.1 or in Theorem 2.5, which satisfy the holonomic system of Theorem 3.1 of Chapter II.

The important object for application to automorphic forms is the Whittaker function with moderate growth, which should be a linear combination of (eight power series) Whittaker functions discussed here. If this part is developed in the near future, then we can hope to have applications for automorphic forms, say, for Eisenstein cohomology classes in the mixed Hodge structures discussed by Oda and Schwermer [OS].

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CHAPTER I

PRINCIPAL SERIES WHITTAKER FUNCTIONS ON THE REAL SYMPLECTIC GROUP OF RANK 2

I-1. Lie groups and Lie algebras

Let G be the real symplectic group of degree two

$$G = \mathrm{Sp}(2, \mathbb{R}) = \left\{ g \in \mathrm{SL}(4, \mathbb{R}) \,\middle|\, {}^t g J_2 g = J_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}. \tag{1.1}$$

For a Cartan involution $\theta(g) = {}^t g^{-1}$, $g \in G$ of G, its fixed part

$$K = \{ g \in G \mid \theta(g) = g \} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \right\} \tag{1.2}$$

is a maximal compact subgroup of G.

Let $\mathfrak{g}$ be the Lie algebra of G which is given by

$$\mathfrak{g} = \mathfrak{sp}(2, \mathbb{R}) = \{ X \in \mathrm{M}_4(\mathbb{R}) \mid JX + {}^t X J = 0 \} \tag{1.3}$$

If we denote the differential of θ again by θ, then we have $\theta(X) = -{}^t\overline{X}$ for $X \in \mathfrak{g}$. Hence the subspaces

$$\mathfrak{p} = \{ X \in \mathfrak{g} \mid \theta(X) = X \} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \,\middle|\, {}^t A = A, {}^t B = B; A, B \in \mathrm{M}_2(\mathbb{R}) \right\} \tag{2.4}$$

and

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \,\middle|\, A, B \in \mathrm{M}_2(\mathbb{R}); {}^t A = -A, {}^t B = B \right\} \tag{1.5}$$

give the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The linear map κ is

$$\mathfrak{u}(2) \ni A + \sqrt{-1} B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{k} \tag{1.6}$$

which defines an isomorphism of Lie algebras from $\mathfrak{k}$ to the unitary Lie algebra

$$\mathfrak{u}(2) = \{C \in \mathrm{M}_2(\mathbb{C}) \mid {}^t\overline{C} + C = 0\}. \tag{1.7}$$

We recall a basis of $\mathfrak{u}(2)_{\mathbb{C}}$:

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.8}$$

For $1 \le i, j \le 4$, let $E_{i,j}$ be the matrix unit with its (i,j)-entry 1 and remaining entries 0. Moreover we put $H_i = E_{i,i} - E_{2+i,2+i}$. We take $\mathfrak{a} = \mathbb{R}H_1 \oplus \mathbb{R}H_2$. Then $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$.

Let

(1.9)
$$N = \left\{ n(n_0, n_1, n_2, n_3) = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \cdot \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array} \right) \;\middle|\; n_i \in \mathbb{R} \right\}$$

be the maximal unipotent radical of G. Fix an unitary character η of N, for $n(n_0, n_1, n_2, n_3) \in N$,

$$\eta(n(n_0, n_1, n_2, n_3)) = \exp(2\pi\sqrt{-1}(c_0 n_0 + c_3 n_3)) \tag{1.10}$$

with some c_0, $c_3 \in \mathbb{R}$. In this paper we assume that η is non-degenerate, i.e., $c_0 c_3 \neq 0$. Then, hereafter, we may assume that $c_0 = c_3 = 1$ without loss of generality. Let

$$A = \left\{ a(a_1, a_2) = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i \in \mathbb{R}_{>0} \right\} \tag{1.11}$$

be a maximal split torus of G. We have the Iwasawa decomposition $G = NAK$ of G, where A and N are the analytic subgroup with Lie algebra $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Moreover, we have an Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ of $\mathfrak{g}$.

Let $M = Z_K(A)$ be the centralizer of A in K.

$$M = \{\mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}\}. \tag{1.12}$$

Then the minimal parabolic subgroup $P = NAM$ of G has the Langlands decomposition.

I-2. Root system of $(\mathfrak{g}, \mathfrak{a})$

Let $\{e_1 = (1,0), e_2 = (0,1)\}$ be a standard basis of the 2-dimensional Euclidean plane $\mathbb{R}^2$. The root system of $(\mathfrak{g}, \mathfrak{a})$ is given by $\Psi = \{\pm 2e_1, \pm 2e_2, \pm e_1, \pm e_2\}$. We denote a positive root by $\Psi_+ = \{2e_1, 2e_2, e_1 + e_2, e_1 - e_2\}$ and generators E_α of $\mathfrak{g}_\alpha$:

$$\begin{aligned} E_{2e_1} &= \begin{pmatrix} 0_2 & e_{11} \\ 0_2 & 0_2 \end{pmatrix}, \quad E_{e_1+e_2} = \begin{pmatrix} 0_2 & e_{12}+e_{21} \\ 0_2 & 0_2 \end{pmatrix}, \\ E_{2e_2} &= \begin{pmatrix} 0_2 & e_{22} \\ 0_2 & 0_2 \end{pmatrix}, \quad E_{e_1-e_2} = \begin{pmatrix} e_{12} & 0_2 \\ 0_2 & -e_{12} \end{pmatrix}. \end{aligned} \tag{2.1}$$

Here $e_{i,j}$ $(i, j \geq 2)$ are the matrix units in $\mathrm{M}_2(\mathbb{C})$.

I-3. Principal series representations

We put σ to be an irreducible unitary representation of M which is determined by $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$ with $\gamma_1 = \mathrm{diag}(-1,1,-1,1)$ and $\gamma_2 = \mathrm{diag}(1,-1,1,-1)$. We define $\sigma_i \in \widehat{M}$ $(i = 0,\ 1,\ 2,\ 3)$ such that

$$\begin{aligned} &\sigma_0(\gamma_1) = \sigma_0(\gamma_2) = 1, && \sigma_1(\gamma_1) = \sigma_1(\gamma_2) = -1, \\ &\sigma_2(\gamma_1) = 1, \sigma_2(\gamma_2) = -1, && \sigma_3(\gamma_1) = -1, \sigma_3(\gamma_2) = 1. \end{aligned} \tag{3.1}$$

For $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, let e^ν be a quasi-character of A, which obtained from a linear form $\nu \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a},\ \mathbb{C})$. We suppose the condition that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers.

The representation $1_N \otimes e^{\nu+\rho} \otimes \sigma$ of P is defined as $(1_N \otimes e^{\nu+\rho} \otimes \sigma)(l) = 1_N(n)e^{\nu+\rho}(a)\sigma(m)$, for $l = n \cdot a \cdot m$, $n \in N$, $a \in A$ and $m \in M$. Here $\rho = (2,1)$ is the half-sum of the positive roots.

DEFINITION 3.1. *We call an induced representation*

$$\pi = \mathrm{Ind}_P^G(1_N \otimes e^{\nu+\rho} \otimes \sigma) \tag{3.2}$$

of G from the minimal parabolic subgroup $P = NAM$ the principal series representation of G. i.e. π is the right regular representation of G on the space $H_{\nu,\sigma}$ which is the completion of

$$\left\{ f : G \to \mathbb{C} \text{ smooth} \;\middle|\; \begin{array}{c} f(namg) = e^{\nu+\rho}(a)\sigma(m)f(g), \\ m \in M, a \in A, g \in G \end{array} \right\} \tag{3.3}$$

with respect to the norm

$$\|f\|^2 = \int_K |f(k)|^2 dk. \tag{3.4}$$

If $\sigma = \sigma_1$ or $\sigma = \sigma_2$, then we call π an even principal series representation of G. If $\sigma = \sigma_3$ or $\sigma = \sigma_4$, then we call π an odd principal series representation of G. $\square$

The representation space of this parabolic induction is identified with a closed subspace of $L^2(K)$

$$L^2_{(M\cap K, \sigma|_{M\cap K})} = \{f \in L^2(K) \mid f(mk) = \sigma(m)f(k), m \in M, k \in K\}. \tag{3.5}$$

I-4. The peripheral K-types of the principal series representation

Let (τ, V_τ) be a K-type of π. According to the theory of highest weight, the irreducible representations of K are parameterized by the set of dominant weights $L^+ = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid \lambda_1 \geq \lambda_2\}$. We denote by $\tau_\lambda = \mathrm{S}^{\lambda_1 - \lambda_2} \otimes \det^{\lambda_2}$ the representation corresponding to $\lambda = (\lambda_1, \lambda_2)$. The dimension of the representation space V_λ associated to λ is $d = \lambda_1 - \lambda_2 + 1$.

LEMMA 4.1 ([MO1], P.9, PROPOSITION 3.2). *(a)(i) If $\sigma = \sigma_0$, then $\tau_{l,l}$ (l is even) occurs in π with multiplicity one and the minimal K-type of π is $\tau_{(0,0)}$.*
(ii) If $\sigma = \sigma_1$, then $\tau_{(l,l)}$ (l is odd) occurs in π with multiplicity one and the minimal K-type of π are $\tau_{(1,1)}$ and $\tau_{(-1,-1)}$.
(b) If $\sigma = \sigma_2$ or $\sigma = \sigma_3$, then $\tau_{(l+1,l)}$ occurs in π with multiplicity one and the minimal K-type of π are $\tau_{(1,0)}$ and $\tau_{(0,-1)}$. $\square$

I-5. Whittaker functions

For the pair (N, η) defined above. Let $C^\infty\text{-}\mathrm{Ind}_N^G(\eta)$ be the representation of G induced from η as C^∞-function. Then the representation space of $C^\infty\text{-}\mathrm{Ind}_N^G(\eta)$ is the space

$$C^\infty_\eta(N\backslash G) = \{f \in C^\infty(G) \mid f(rg) = \eta(r)f(g), (r,g) \in N \times G\} \tag{5.1}$$

and

$$C_\eta^{\infty,\mathrm{mod}}(N\backslash G) = \{f \in C_\eta^\infty(N\backslash G) \mid f \text{ is moderate growth}\}. \tag{5.2}$$

By the right translation, $C_\eta^\infty(N\backslash G)$ is a smooth G-module and we denote by the same symbol its underlying $(\mathfrak{g}_\mathbb{C}, K)$-module ($\mathfrak{g}_\mathbb{C}$ is the complexification of $\mathfrak{g}$). For any finite-dimensional K-module (τ, V_τ), we put

$$\begin{aligned} &C_{\eta,\tau}^\infty(N\backslash G/K) \\ &\quad = \{\varphi : G \longrightarrow V_\tau, C^\infty \mid \varphi(rgk) = \eta(\tau)\tau(k^{-1})\varphi(g), (r,g,k) \in N \times G \times K\}. \end{aligned} \tag{5.3}$$

Then the function $\varphi \in C_{\eta,\tau}^\infty(N\backslash G/K)$ is determined by its restriction $\varphi|_A$ to A, because of the Iwasawa decomposition $G = NAK$ of G.

If we denote by (τ^*, V_{τ^*}) the contragradient representation of (τ, V_τ) and $\langle \cdot,\cdot \rangle$ the canonical paring on $V_{\tau^*} \times V_\tau$. Then the relation

$$\iota(v^*)(g) = \langle v^*, \varphi_\iota(g)\rangle, \quad v \in V_{\tau^*}, g \in G, \tag{5.4}$$

defines an isomorphism between $\iota \in \mathrm{Hom}_K(\tau^*, C^\infty\text{-}\mathrm{Ind}_N^G(\eta))$ and $\varphi_\iota \in C_{\eta,\tau}^\infty(N\backslash G$ $/K)$.

For an irreducible admissible representation (π, H_π) of G, we choose a K homomorphism $i \in \mathrm{Hom}_K(\tau^*, \pi|_K)$. Let

$$\mathcal{I}_{\eta,\pi} = \mathrm{Hom}_{(\mathfrak{g}_\mathbb{C},K)}(\pi, C^\infty\text{-}\mathrm{Ind}_N^G(\eta)) \tag{5.5}$$

be the intertwining space between $(\mathfrak{g}_\mathbb{C}, K)$-modules π and $C^\infty\text{-}\mathrm{Ind}_N^G(\eta)$ consisting of all K-finite vectors. For each $T \in \mathcal{I}_{\eta,\pi}$, we define an element $T_i \in C_{\eta,\tau}^\infty(N\backslash G/K)$ by

$$T(i(v^*))(g) = \langle v^*, T_i(g)\rangle, \quad v^* \in V_{\tau^*}, g \in G. \tag{5.6}$$

Definition 5.1. *We call the subspace*

$$\mathrm{Wh}(\pi,\eta,\tau) = \bigcup_{i\in Hom_K(\tau^*,\pi|_K)} \{T_i \in C^\infty_{\eta,\tau}(N\backslash G/K) \mid T \in \mathcal{I}_{\iota,\pi}\} \tag{5.7}$$

of $C^\infty_{\eta,\tau}(N\backslash G/K)$ the space of Whittaker functions with respect to (π,η,τ). Moreover we denote by $\mathcal{I}^0_{\iota,\pi}$ the space of $\mathcal{I}_{\iota,\tau}$ consisting of the intertwining operators whose images in $C^\infty_\eta(N\backslash G)$ are moderate growth functions and define the subspace

$$\mathrm{Wh}(\pi,\eta,\tau)^{\mathrm{mod}} = \bigcup_{i\in Hom_K(\tau^*,\pi|_K)} \{T_i \in \mathrm{Wh}(\pi,\eta,\tau) \mid T \in \mathcal{I}^0_{\iota,\pi}\}. \tag{5.8}$$

of $Wh(\pi,\eta,\tau)$. An element in $Wh(\pi,\eta,\tau)^{\mathrm{mod}}$ is called Whittaker function of moderate growth. □

Because G has a Iwasawa decomposition $G = NAK$, $\varphi_\iota \in \mathrm{Wh}(\pi,\eta,\tau)$ is determined by its restriction $\varphi_\iota|_A$ to A. We call $\varphi_\iota the|_A$ A-radial part of $\varphi_\iota|_A$.

I-6. The standard basis

Each irreducible representation, or simple module τ_λ of $\mathfrak{k}_\mathbb{C} = \mathfrak{gl}(2,\mathbb{C})$ has a monomial basis parameterized by Gelfand-Tsetlin patterns $M = \begin{pmatrix} \lambda_1 & & \lambda_2 \\ & \alpha & \end{pmatrix} \in G(\lambda)$, $\lambda = (\lambda_1,\lambda_2)$, consisting of 3 integer satisfying the inequalities $\lambda_2 \le \alpha \le \lambda_1$. Then the following is well-known ([O1], p. 267).

Proposition 6.1. *There exists a basis $\{f(M)\}_{M\in G(\lambda)}$ in τ_λ of $\mathfrak{k}_\mathbb{C}$ such that*

$$\begin{aligned} &e_{11}f(M) = \alpha f(M), && e_{22}f(M) = (\lambda_1+\lambda_2-\alpha)f(M),\\ &e_{12}f(M) = (\lambda_1-\alpha)f(M_{+1}), && e_{21}f(M) = (\alpha-\lambda_2)f(M_{-1}). \end{aligned} \tag{6.1}$$

Here $M_{+1} = \begin{pmatrix} \lambda_1 & & \lambda_2 \\ & \alpha+1 & \end{pmatrix}$ and $M_{-1} = \begin{pmatrix} \lambda_1 & & \lambda_2 \\ & \alpha-1 & \end{pmatrix}$ for $M = \begin{pmatrix} \lambda_1 & & \lambda_2 \\ & \alpha & \end{pmatrix} \in G(\lambda)$. □

For all $M \in G^*(\lambda) = \{M \in G(\lambda) \mid f(M) \in V^*_\lambda[\sigma]\}$, where $V^*_\lambda[\sigma]$ is the σ-isotypic component of V^*_λ, we define a K-homomorphism $S_\lambda(M)$ from V_λ to the representation space H_π of G such that

$$V_\lambda \cong \mathbb{C}f(M)^* \otimes_\mathbb{C} V_\lambda \to \widehat{\bigoplus}_\lambda (V^*_\lambda[\sigma]) \otimes_\mathbb{C} V_\lambda \cong H_\pi. \tag{6.2}$$

Then $\{S_\lambda(M)\}_{M\in G^*(\lambda)}$ become a basis of the space $\mathrm{Hom}_K(V_\lambda, H_\pi)$. We call the basis the induced basis from the monomial basis.

$L^2(K)$ models of the principal series

Now we define natural sets of orthogonal basis in the irreducible constituents in the K-finite vectors on $H_{\nu,\sigma} \hookrightarrow L^2(K)$. The point here is that we can specify canonical basis in irreducible K-modules in each K-isotypic component in $L^2(K)$. Firstly recall the canonical isomorphism

$$H_{\nu,\sigma} \cong L^2_{(M,\sigma)}(K) \qquad f \mapsto f|_K$$

with

$$L^2_{(M,\sigma)}(K) := \left\{ s \in L^2(K) \mid s(mk) = \sigma(m)s(k)\text{for a.e. } m \in M, k \in K \right\}.$$

We define the tautological representation

$$S : x = K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto S(x) = \begin{pmatrix} s_{11}(x) & s_{12}(x) \\ s_{21}(x) & s_{22}(x) \end{pmatrix} = A + \sqrt{-1}B \in \mathrm{U}(2).$$

Let $\mathrm{S}^k(S(x))$ be a square matrix of degree $k+1$ associated with $S(x)$ which is defined as follows. For two independ variables U and V, define two linear forms

$$U' = s_{11}U + s_{22}V \quad \text{and} \quad V' = s_{21}U + s_{22}V,$$

or equivalently by $(U', V') = (U, V) \cdot S(x)$. Then by using homogeneous forms $\{(U')^{k-i}(V')^i\}_{0\leq i\leq k}$ of degree k, we define a $(k+1)\times(k+1)$ matrix $\mathrm{S}^k(S(x))$ by

$$((U')^k, \cdots, (U')^{k-i}(V'), \cdots, (V')^k) = (U^k, \cdots, U^{k-i}, V^i, \cdots, V^k) \cdot \mathrm{S}^k(S(x)).$$

The $k+1$ entries of each row vector of $\mathrm{S}^k(S(x))$ make a canonical basis of a simple K-module $\tau_{(k,0)}$ in $L^2(K)$ with highest weight $(k, 0)$.

elementary functions

Then we define a $k+1$ column vectors $\{\mathrm{s}_i^{(k)}\}_{0\leq i\leq k}$ of $k+1$ elementary functions by

$$(6.3) \qquad (\mathrm{s}_0^{(k)}, \mathrm{s}_1^{(k)}, \cdots, \mathrm{s}_k^{(k)}) = {}^t\mathrm{Sym}^k(S(x)).$$

And also we put $\Delta = \det S(x)$. We set for even k,

$$(6.4)\qquad \mathrm{S}^{(k)}_{[0,2,\cdots,k]} = (\mathbf{s}^{(k)}_0, \mathbf{s}^{(k)}_2, \cdots \mathbf{s}^{(k)}_k) \quad \text{and} \quad \mathrm{S}^{(k)}_{[1,3,\cdots,k-1]} = (\mathbf{s}^{(k)}_1, \mathbf{s}^{(k)}_3, \cdots \mathbf{s}^{(k)}_{k-1}).$$

Moreover $\Delta(x)$ denotes the determinant $\det S(x)$ of the matrix $S(x)$ as a function on $L^2(K)$. Then for each pair of integers m, n, the entries of the vector $\Delta(x)^m \mathbf{s}^{(n)}_i(x)$ form a set of canonical basis of $\tau_{(m+n,m)}$ for each i $(0 \le i \le n)$.

Let $\langle \Delta(x)^m \mathbf{s}^n_i(x)\rangle$ be the irreducible K-module in $L^2(K)$ generated by the entries of $\Delta(x)^m \mathbf{s}^{(n)}_i(x)$. Then for $\sigma = \sigma_0$, the $\tau_{(m+n,m)}$-isotypic component in H_π is given by

$$\begin{cases} \bigoplus_{i=0}^{n/2} \langle \Delta(x)^m \mathbf{s}^{(n)}_{2i}(x)\rangle, & \text{if } m+n \text{ and } m \text{ are even,} \\ \bigoplus_{i=1}^{(n-1)/2} \langle \Delta(x)^m \mathbf{s}^{(n)}_{2i-1}(x)\rangle & \text{if } m+n \text{ and } m \text{ are odd.} \end{cases}$$

For $\sigma = \sigma_1$, the $\tau_{(m+n)}$-isotypic component in H_π.

$$\begin{cases} \bigoplus_{i=1}^{(n-1)/2} \langle \Delta(x)^m \mathbf{s}^{(n)}_{2i-1}(x)\rangle, & \text{if } m+n \text{ and } m \text{ are even,} \\ \bigoplus_{i=0}^{n/2} \langle \Delta(x)^m \mathbf{s}^{(n)}_{2i}(x)\rangle & \text{if } m+n \text{ and } m \text{ are odd.} \end{cases}$$

I-7. System of partial differential equations for Whittaker functions

From now on, we consider only those principal series with $\sigma \in \widehat{M}$ such that $\sigma = \sigma_0$ or $\sigma = \sigma_1$. Let us fix a Whittaker functional T in $\mathcal{I}_{\eta,\pi}$. Let $\mathrm{f}^{(l,l)} \in H_\pi$ be an K-finite element, which corresponds to Δ^l in $L^2(K)$, via the model $H_\pi \cong L^2_{(M,\sigma)}(K)$ with even l for $\sigma = \sigma_0$, and odd l for $\sigma = \sigma_1$.

Then we define the function φ on A by

$$\Phi^{(l,l)}(a) = a^\rho(y) \cdot \varphi^{(l,l)}(a) = T(\mathrm{f}^{(l,l)})|_A.$$

Here $T \in I_{\eta,\pi}$ be a Whittaker functional. Moreover we use the coordinate $y = (y_1, y_2) = (a_1/a_2, a_2^2)$, and the symbol $\partial_i = y_i(\partial/\partial y_i)$ $i = 1, 2$ and $a^\rho(y) = y_1^2 y_2^{3/2} = a_1^2 a_2$. We give two partial differential equations characterizing the Whittaker functions with one or two dimensional K-type, respectively, which were obtained by Miyazaki and Oda [MO1], p.28, Theorem 10.1.

PROPOSITION 7.1 ([MO1], P.28, THEOREM 10.1). *Let $\Phi^{(l,l)}(y)$ be an A-radial part of the even principal series Whittaker function with 1-dimensional K-type. If* *we put* $\Phi^{(l,l)}(y) = a^{\rho}(y)\varphi^{(l,l)}(y)$, *then* $\varphi^{(l,l)}(y)$ *satisfy*

$$
\begin{aligned}
(7.1) \qquad & \Big\{2\partial_1^2 + 4\partial_2^2 - 4\partial_1\partial_2 - 2(2\pi y_1)^2 - 4(2\pi y_2)^2 - 4l(2\pi y_2)\Big\}\varphi^{(l,l)}(y) \\
& = (\nu_1^2 + \nu_2^2)\varphi^{(l,l)}(y)
\end{aligned}
$$

and

$$
\begin{aligned}
(7.2) \qquad & \Big[(\partial_1 + l + 1)(\partial_1 - l - 1)(\partial_1 - 2\partial_2 + l + 1)(\partial_1 - 2\partial_2 - l - 1) \\
& +(2\pi y_1)^4 - 2(2\pi y_1)^2\{(\partial_1 + 1)(\partial_1 - 2\partial_2 + 1) - l(l+2)\} \\
& +4l(2\pi y_1)^2(2\pi y_2) - 4(2\pi y_2)(2\pi y_2 + l)(\partial_1 + l + 1)(\partial_1 - l - 1)\Big]\varphi^{(l,l)}(y) \\
& = \{\nu_1^2 - (l+1)^2\}\{\nu_2^2 - (l+1)^2\}\varphi^{(l,l)}(y). \quad \square
\end{aligned}
$$

We remark that the Proposition 7.1 is essentially equivalent to what is obtained by computing the action of the elements of the center $Z(\mathfrak{g})$ of the universal embeloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, which are the generators of degree 2 and 4. . This system is a holonomic system of rank 8.

I-8. Formal power series solutions and the shift operator

In this section we consider a formal power series solution

$$
(8.1) \qquad \varphi^{(l,l)}_{(\tau_1,\tau_2)}(y) = \sum_{m,n\geq 0} a^{(l,l)}_{m,n}(2\pi y_1)^{2m+\tau_1}(2\pi y_2)^{n+\tau_2}
$$

around $(y_1, y_2) = (0,0)$. The holonomic system of partial differential equations has regular singularities along two divisors $y_1 = 0$ and $y_2 = 0$ which are of simple normal crossing at $(y_1, y_2) = (0,0)$.

LEMMA 8.1. *The characteristic indices* (τ_1, τ_2) *are elements of the set*

$$
\begin{aligned}
(8.2) \qquad \Lambda &= \{w(\nu_1, (\nu_1+\nu_2)/2) \mid w \in W \cong \mathfrak{S}_2 \ltimes (\mathbb{Z}/2Z)^2\} \\
&= \{(\varepsilon_1\nu_1, (\varepsilon_1\nu_1 + \varepsilon_2\nu_2)/2), (\varepsilon_2\nu_2, (\varepsilon_1\nu_1 + \varepsilon_2\nu_2)/2) \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}\},
\end{aligned}
$$

where W *is the Weyl group.* $\square$

PROOF. This is given by Ishii ([I], p. 9) $\square$

We have the following recurrence relation which the coefficients $a_{m,n} = a_{m,n}^{(l,l)}$ satisfy

(8.3)

$$2\{2m^2-2mn+n^2+2(\tau_1-\tau_2)m+(2\tau_2-\tau_1)n\}a_{m,n}-2a_{m,n-2}-2la_{m,n-1}-a_{m-1,n}=0,$$

and

(8.4)

$$\begin{aligned}&\big[(2m+\tau_1+l+1)(2m+\tau_1-l-1)(2m-2n-2\tau_2-\tau_1+l+1)\\ &\times(2m-2n-2\tau_2-\tau_1-l-1)-\{\tau_1^2-(l+1)^2\}\{(2\tau_2-\tau_1)^2-(l+1)^2\}\big]a_{m,n}\\ &+a_{m-2,n}-2\{(2m+\tau_1+1)(2m-2n-(2\tau_2-\tau_1)+1)-l(l+2))\}a_{m-1,n}\\ &+4la_{m-1,n-1}-4(2m+\tau_1+l+1)(2m-\tau_1-l-1)\big(a_{m,n-2}+la_{m,n-1}\big)=0.\end{aligned}$$

The solutions of the above recurrence relations of $a_{m,n}$ are obtained as follows.

Iwasawa decomposition

For $x = {}^t x \in \mathrm{M}_n(\mathbb{C})$, we put

$$p_\pm(x) = \begin{pmatrix} x & \pm\sqrt{-1}x \\ \pm\sqrt{-1}x & -x \end{pmatrix}. \tag{8.5}$$

We define $\mathfrak{p}_\pm$ the image of $p_\pm$ respectively. Both of $\mathfrak{p}_\pm$ is stable under the adjoint action of K, and the action $\mathrm{Ad}_{\mathfrak{p}_\pm}$ of the element $\big((1+\sqrt{-1})/\sqrt{2}\big)\cdot 1_2$ in the center of K defines a complex structure.

Put $X_{\pm ii} = p_\pm(e_{ii})$ for $i = 1,\ 2$ and $X_{\pm 12} = p_\pm((e_{12}+e_{21})/2)$, then we have $\mathfrak{p}_+ = \mathbb{C}X_{+11}\oplus\mathbb{C}X_{+12}\oplus\mathbb{C}X_{+22}$, and $\mathfrak{p}_- = \mathbb{C}X_{-11}\oplus\mathbb{C}X_{-12}\oplus\mathbb{C}X_{-22}$. Then Iwasawa decomposition tells that

$$X_{\pm ii} = \pm 2\sqrt{-1}E_{2e_i} + H_{i,i+2} \pm \kappa(e_{ii}), \tag{8.6}$$

for $i = 1,\ 2$, and

(8.7)

$$X_{+,12} = E_{e_1-e_2} + \sqrt{-1}E_{e_1+e_2} + \kappa(e_{21}), \quad X_{-,12} = E_{e_1-e_2} - \sqrt{-1}E_{e_1+e_2} - \kappa(e_{12}).$$

shift operators

If we use two matrices of shift operators

$$\mathcal{C}_{\pm} = \begin{pmatrix} X_{\pm 11} & X_{\pm 12} \\ X_{\pm 12} & X_{\pm 22} \end{pmatrix} \tag{8.8}$$

then for $\det(\mathcal{C}_{\pm}) \in Sym^2(\mathfrak{p}_{\pm})$, we have a lemma.

LEMMA 8.2.

$$\mathrm{Ad}_{\mathfrak{p}_{\pm}}(k)(\det \mathcal{C}_{\pm}) = \det(S(k))^{\pm 2} \cdot \det \mathcal{C}_{\pm}.$$

□

This implies that $(\det \mathcal{C}_{\pm})\mathrm{f}^{(l,l)}$ is a scalar multiple of $\mathrm{f}^{(l+2,l+2)}$.

$$(\det \mathcal{C}_{\pm})\mathrm{f}^{(l,l)} = (\nu_1 + l + 1)(\nu_2 + l + 1)\mathrm{f}^{(l+2,l+2)}.$$

PROOF. This is shown substantially in the proof of theorem 10.1 of [MO1], utilizing Harish-Chandra hypergeometric functions. The other proof is to compute the value of $\det(\mathcal{C}_{\pm})\mathrm{f}^{(l,l)}$ at the identity $e \in K$ utilizing the Iwasawa decomposition (see Hayata, [H], Proposition 3.15 or [O2], Theorems 5.1 and 5.2). □

Passing to the Whittaker realization via $T \in \mathcal{I}_{\eta,\pi}$, we have the following proposition.

PROPOSITION 8.3. *The operator* $\det \mathcal{C}_{+}$ *acts by*

$$a^{-\rho}(y)\rho_A(T(\det \mathcal{C}_{+})) \cdot a^{\rho}(y)\varphi^{(l,l)}_{(\tau_1,\tau_2)}(a) = (\nu_1 + l + 1)(\nu_2 + l + 1)\varphi^{(l+2,l+2)}_{(\tau_1,\tau_2)}(a),$$

where ρ_A *is the* A*-radial part.* □

Now we have to compute the Whittaker realization $T(\det \mathcal{C}_{+})$ of

$$\begin{aligned} \det \mathcal{C}_{+} = \big\{ & 2\sqrt{-1}E_{2e_2}H_{1,3} + H_{1,3}H_{2,4} + H_{1,3}\kappa(e_{22}) + 2\sqrt{-1}E_{2e_2}\kappa(e_{11}) \\ & + H_{2,4}\kappa(e_{11}) + \kappa(e_{11})\kappa(e_{22}) - E^2_{e_1-e_2} - H_{2,4} - 2\sqrt{-1}E_{2e_2} - \kappa(e_{22}) \big\}, \end{aligned}$$

which is already rewritten in the normal order $\mathrm{U}(\mathfrak{g}) = \mathrm{U}(\mathfrak{n})\mathrm{U}(\mathfrak{a})\mathrm{U}(\mathfrak{k})$ with respect to the Iwasawa decomposition.

PROPOSITION 8.4. *The action of $a^{-\rho}(y)T(\det C_+)a^{\rho}(y)$ on $\varphi = \varphi^{(l,l)}_{(\tau_1,\tau_2)}$ is given by*

$$a^{-\rho}(y)\rho_A(T(\det C_+))a^{\rho}(y)\varphi \tag{8.10}$$

$$= \{(\partial_1 + l + 1)(2\partial_2 - \partial_1 + l + 1) + 2\pi c_3 y_2(\partial_1 + l + 1) + 4\pi^2 c_0^2 y_1^2\}\varphi. \quad \square$$

PROOF. The same calculations were given by [H], p.21, Lemma 6.2. The actions of $H_{1,3}$, $H_{2,4} \in \mathfrak{a}_{\mathbb{C}}$ on $\varphi = \varphi^{(l,l)}_{(\tau_1,\tau_2)}(y)$ are given by

$$H_{1,3}\varphi = \partial_1\varphi, \qquad H_{2,4}\varphi = (2\partial_2 - \partial_1)\varphi.$$

The actions of the elements $E_{e_1+e_2}$, $E_{e_1-e_2}$, E_{2e_1} and E_{2e_2} in $\mathfrak{n}_{\mathbb{C}}$ are the following:

$$E_{e_1+e_2}\varphi = 0, \quad E_{e_1-e_2}\varphi = 2\pi\sqrt{-1}c_0 y_1\varphi,$$

$$E_{2e_1}\varphi = 0, \qquad E_{2e_2}\varphi = 2\pi\sqrt{-1}c_3 y_2\varphi.$$

The actions of elements $\kappa(e_{11})$ and $\kappa(e_{22})$ in $\mathfrak{k}_{\mathbb{C}}$ are

$$\kappa(e_{11})\varphi = l\varphi, \quad \text{and} \quad \kappa(e_{22})\varphi = l\varphi.$$

Finally we have to consider the $a^{\rho}(y)$ twist in the unknown functions : $\Phi = a^{\rho}(y)\varphi$. $\square$

THEOREM 8.5. *Suppose that ν_1, ν_2 and $\nu_1 \pm \nu_2$ are not integers. For $l \in \mathbb{Z}$, let $\varphi^{(2l,2l)}_{(\tau_1,\tau_2)}(y)$ be the Whittaker function with K-type $\tau_{(2l,2l)}$, and $a^{(2l,2l)}_{0,0} = 1$. Then the coefficients $a^{(2l,2l)}_{m,n}$ are given as follows.*

$$a^{(2l,2l)}_{m,n} = \sum_{\substack{(i,j) \\ 0 \le i+j \le |l| \\ n-j:even}} 2^{2l-2i-j} a^{(0,0)}_{m-i,n-j} \binom{|l|}{i}\binom{|l|-i}{j} \tag{8.9}$$

$$\times \left(\frac{2m+\tau_1+1}{2}\right)_{|l|-i} \left(\frac{-2m-\tau_1+2n+2\tau_2+1}{2}+i\right)_{|l|-i-j},$$

with

$$a^{(0,0)}_{m,n} = {}_3F_2\left(\left.\begin{matrix} -n, m+\frac{\tau_1}{2}+1, -m-\frac{\tau_1}{2} \\ \frac{\tau_1}{2}+1, \tau_2-\frac{\tau_1}{2}+1 \end{matrix}\right| 1\right)$$

$$\times \frac{1}{m!n!(\tau_1-\tau_2+1)_m(\tau_2+1)_n 2^{2m+\tau_1+2n+2\tau_2}}.$$

Here the generalized hypergeometric series are denoted by

$$
{}_3F_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}\,\middle|\, z\right) = \sum_{i\geq 0} \frac{(a)_i(b)_i(c)_i}{(d)_i(e)_i}\cdot\frac{z^i}{i!},
$$

with Pochhammer symbol $(a)_i = \Gamma(a+i)/\Gamma(a)$. $\square$

PROOF. Suppose that $\sigma=\sigma_0$ or $\sigma=\sigma_1$, and ν_1, ν_2 and $\nu_1\pm\nu_2$ are not integers. For $l\in\mathbb{Z}$, we consider the formal power series solutions

$$
\varphi^{(l,l)}_{(\tau_1,\tau_2)}(y) = \frac{1}{(\nu_1+l+1)(\nu_2+l+1)}\sum_{m,n\geq 0} a^{(l,l)}_{m,n}(2\pi y_1)^{2m+\tau_1}(4\pi y_2)^{n+\tau_2},
$$

We put for $k\in\mathbb{Z}$,

$$
\begin{aligned}
\varphi^{(l+k,l+k)}_{(\tau_1,\tau_2)}(y) &= a^{-\rho}(y)\cdot\rho_A(T(\det C_+))^k a^{\rho}(y)\varphi^{(l,l)}_{(\tau_1,\tau_2)}(y)\\
&= \frac{1}{(\nu_1+l+1)^k(\nu_2+l+1)^k}\sum_{m,n\geq 0} a^{(l+k,l+k)}_{m,n}(2\pi y_1)^{2m+\tau_1}(4\pi y_2)^{n+\tau_2}.
\end{aligned}
$$

The explicit formulas of $\varphi^{(2l,2l)}_{(\tau_1,\tau_2)}(y)$ and $\varphi^{(2l+1,2l+1)}_{(\tau_1,\tau_2)}(y)$ for $l\in\mathbb{Z}$, can be given recursively.

In fact, by using the Proposition 8.4, we have the corresponding recurrence relations among $a^{(l,l)}_{m,n}$.

$$
\begin{aligned}
(8.11)\qquad a^{(l+2,l+2)}_{m,n} &= (2m+\tau_1+l+1)(-2m+2n+2\tau_2-\tau_1+l+1)a^{(l,l)}_{m,n}\\
&+a^{(l,l)}_{m-1,n}+2(2m+\tau_1+l+1)a^{(l,l)}_{m,n-1}.
\end{aligned}
$$

Now we use induction. If $l=0$, then the formula of Theorem 8.5 for the coefficients of $\varphi^{(0,0)}_{(\tau_1,\tau_2)}(y)$ is due to Ishii ([I], p.11, Theorem 2.1). The inductive step $l\mapsto l+2$

is confirmed, l is true for recurrence if we check the developed form of (8.11) :

$$\sum_{(i,j)\in\mathbb{Z}^2} a^{(0,0)}_{m-i,n-j}\binom{l+1}{i}\binom{l+1-i}{j}\left(\frac{2m+\tau_1+1}{2}\right)_{l+1-i}$$
$$\times\left(\frac{2n-2m+2\tau_2-\tau_1+1}{2}+i\right)_{l+1-i-j}$$
$$=(2m+\tau_1+2l+1)(2n-2m+2\tau_2-\tau_1+2l+1)\sum_{(p,q)\in\mathbb{Z}^2} a^{(0,0)}_{m-p,n-q}$$
$$\times\binom{l}{p}\binom{l-p}{q}+\sum_{(r,s)\in\mathbb{Z}^2} a^{(0,0)}_{m-1-rn-s}\binom{l}{r}\binom{l-r}{s}\left(\frac{2m+\tau_1-1}{2}\right)_{l-r}$$
$$\times\left(\frac{2n-2m+2\tau_2-\tau_1+3}{2}+r\right)_{l-r-s}+(2m+\tau_1+2l+1)\sum_{t,u\in\mathbb{Z}^2} a^{(0,0)}_{m-t,n-1-u}$$
$$\times\binom{l}{t}\binom{l-t}{u}\left(\frac{2m+\tau_1+1}{2}\right)_{l-t}\left(\frac{2n-2m+2\tau_2-\tau_1-1}{2}+t\right)_{l-t-u}.$$

This can be checked by comparing both side of the coefficients of $a^{(0,0)}_{m-i,n-j}$. We can check that the solutions are truly the unique solutions (8.3). □

REMARK. As is well-known, we do not need to check (8.4), because the "Casimir recurrence" (8.3) determine the possible solution uniquely. But we did check.

THEOREM 8.6. *Suppose that $\sigma=\sigma_1$, ν_1, ν_2 and $\nu_1\pm\nu_2$ are not integers. For $l\in\mathbb{Z}$, let $\varphi^{(2l+1,2l+1)}_{(\tau_1,\tau_2)}(y)$ be the Whittaker function with K-type $\tau_{(2l+1,2l+1)}$ and $a^{(2l+1,2l+1)}_{(0,0)}=1$. Then the coefficients $a^{(2l+1,2l+1)}_{m,n}$ are given as follows.*

$$a^{(2l+1,2l+1)}_{m,n}=\sum_{\substack{(i,j)\\0\le i+j\le l\\m-i,n-j:even}} 2^{2l-2i-j}a^{(1,1)}_{m-i,n-j}\binom{l}{i}\binom{l-i}{j}\tag{8.12}$$
$$\times\left(\frac{2m+\tau_1+1}{2}\right)_{l-i}\left(\frac{-2m-\tau_1+2n+2\tau_2+1}{2}+i\right)_{l-i-j}$$

and for $l\le 0$,

$$a^{(2l+1,2l+1)}_{m,n}=\sum_{\substack{(i,j)\\0\le i+j\le |l|\\m-i,n-j:even}} 2^{2|l|-2i-j}a^{(-1,-1)}_{m-i,n-j}\binom{|l|}{i}\binom{|l|-i}{j}$$
$$\times\left(\frac{2m+\tau_1+1}{2}\right)_{|l|-i}\left(\frac{-2m-\tau_1+2n+2\tau_2+1}{2}+i\right)_{|l|-i-j},$$

where

$$a^{(1,1)}_{m,2t} = a^{(-1,-1)}_{m,2t} = {}_3F_2\left(\begin{matrix} -t, m+\frac{\tau_1}{2}+\frac{1}{2}, -m-\frac{\tau_1}{2}-\frac{1}{2} \\ \frac{\tau_1}{2}+\frac{1}{2}, \tau_2-\frac{\tau_1}{2}+\frac{1}{2} \end{matrix}\middle| 1\right)$$
$$\times \frac{1}{m!t!(\tau_1-\tau_2+1)_m(\tau_2+1)_t 2^{2m+\tau_1+n+\tau_2}},$$

for $n = 2t$, *if* $n = 2t+1$, *then we have*

$$a^{(1,1)}_{m,2t+1} = -a^{(1,1)}_{m,2t+1} = {}_3F_2\left(\begin{matrix} -t, m+\frac{\tau_1}{2}+\frac{3}{2}, -m-\frac{\tau_1}{2}+\frac{1}{2} \\ \frac{\tau_1}{2}+\frac{3}{2}, \tau_2-\frac{\tau_1}{2}+\frac{3}{2} \end{matrix}\middle| 1\right)$$
$$\times \frac{2(2m+\tau_1+1)}{(\tau_1+1)(2\tau_2-\tau_1+1)m!t!(\tau_1-\tau_2+1)_m(\tau_2+1)_t 2^{2m+\tau_1+n+\tau_2}}. \quad \square$$

PROOF. This proof is completely the same as that of Theorem 8.5. If $k = 0$, then the results the above theorem are due to Ishii, ([I], p. 11, Theorem 2.2). We omit details. $\square$

CHAPTER II

THE HOLONOMIC SYSTEM FOR WHITTAKER FUNCTION OF P_{S}-PRINCIPAL SERIES REPRESENTATIONS

In this chapter, we discuss the P_{S}-principal series representation. We calculate the holonomic system on A, which is satisfied by the radial part of vector-valued Whittaker functions belonging to the P_{S}-principal series with peripheral K-types, i.e. K-types with smallest dimension.

II-1. The generalized principal series representation with respect to the Siegel parabolic subgroup

The Siegel maximal parabolic subgroup P_{S} of G has the Langlands decomposition, where $P_{\mathrm{S}} = N_{\mathrm{S}} A_{\mathrm{S}} M_{\mathrm{S}}$. Namely,

$$N_{\mathrm{S}} = \left\{ \begin{pmatrix} 1_2 & T \\ 0 & 1_2 \end{pmatrix} \,\middle|\, T = {}^tT = \begin{pmatrix} n_1 & n_2 \\ n_2 & n_3 \end{pmatrix} \in \mathrm{M}_2(\mathbb{R}) \right\},$$
$$A_{\mathrm{S}} = \left\{ \mathrm{diag}(a, a, a^{-1}, a^{-1}) \in G \mid a \in \mathbb{R}_{>0} \right\},$$

and

$$M_{\mathrm{S}} = \left\{ \begin{pmatrix} A_{\mathrm{S}} & 0_2 \\ 0_2 & {}^tA_{\mathrm{S}}^{-1} \end{pmatrix} \in G \,\middle|\, A_{\mathrm{S}} \in \mathrm{GL}(2, \mathbb{R}), \det A_{\mathrm{S}} = \pm 1_2 \right\}.$$

Before the definition of the P_{S}-principal series, we recall some basic facts on the representations of $M_{\mathrm{S}} \cong \mathrm{SL}^{\pm}(2, \mathbb{R})$ and its identity component $M_{\mathrm{S}}^0 \cong \mathrm{SL}(2, \mathbb{R}) = G_0$. We write

$$G_0 = \mathrm{SL}(2, \mathbb{R}), \quad K_0 = \mathrm{SO}(2), \quad N_0 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$
$$A_0 = \{ a_0 = diag(r, r^{-1}) \mid r \in \mathbb{R}_{>0} \}, \quad M_0 = \{ diag(\varepsilon, \varepsilon) \mid \varepsilon \in \{\pm 1\} \}.$$

For a character σ' in $\widehat{M_0}$, a linear form $\nu_0 \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_0, \mathbb{C})$ ($\mathfrak{a}_0 = \mathrm{Lie}(A_0)$), the half sum of the positive root ρ_0 and the minimal parabolic subgroup $P_0 = N_0 M_0 A_0$,

$$\pi_0 = \mathrm{Ind}_{P_0}^{G_0}(1_{N_0} \otimes \sigma' \otimes e^{\nu_0 + \rho_0})$$

is a principal series representation of $\mathrm{SL}(2,\mathbb{R})$. The Hilbert space of the principal series representation π_0 is defined as

$$H_0 = \left\{ f : G_0 \to \mathbb{C} \;\middle|\; \begin{array}{c} f(n_0 m_0 a_0 x) = \sigma'(m_0) e^{(\nu_0+\rho_0)\log(a_0)} f(x), \\ n_0 \in N_0, m_0 \in M_0, a_0 \in A_0, x \in G_0 \text{ and } f|_{K_0} \in L^2(K_0) \end{array} \right\},$$

with the usual right quasi-regular action of G_0.

Let D_k^+ (resp D_k^-) be the holomorphic (resp antiholomorphic) discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ with Blattner parameter k (resp $-k$). If we put $\nu_0(\log(a_0)) = (k-1)\log r$ and $\mathrm{sgn}(\sigma') = (-1)^k$, then these are injective M_{S}^0-homomorphisms from $D_k^{\pm}$ to π_0, and the quotient $\pi_0/(D_k^+ \oplus D_k^-)$ is a finite dimensional representation of dim $k-1$.

Now we define the P_{S}-principal series. Then D_k denotes the discrete series representation $\mathrm{Ind}_{\mathrm{SL}(2,\mathbb{R})}^{\mathrm{SL}^{\pm}(2,\mathbb{R})}(D_k^+)$ of $\mathrm{SL}(2,\mathbb{R}) = M_{\mathrm{S}}$, where representations space H_{D_k} is the direct sum $H_{D_k^+} \oplus H_{D_k^-}$ of the representations spaces of $D_k^{\pm}$ of $\mathrm{SL}(2,\mathbb{R})$. Let $e^{\nu_{\mathrm{S}}}$ be the quasi-character of A_{S}, which is obtained from a linear form $\nu_{\mathrm{S}} \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$. The representation $1_{N_{\mathrm{S}}} \otimes D_k \otimes e^{\nu_{\mathrm{S}}+\rho}$ of P_{S} is defined as $(D_k \otimes e^{\nu_{\mathrm{S}}+\rho})(l) = e^{\nu_{\mathrm{S}}+\rho}(a) D_k(m)$, for $l = n \cdot a \cdot m$, $n \in N_{\mathrm{S}}, a \in A_{\mathrm{S}}$, $m \in M_{\mathrm{S}}$. Here ρ is the half-sum of the positive restricted roots.

Definition 1.1. *We call an induced representation*

$$\Pi = \mathrm{Ind}_{P_S}^{G}(1_{N_{\mathrm{S}}} \otimes e^{\nu_{\mathrm{S}}+\rho} \otimes D_k)$$

of G from P_{S} the P_{S}-principal series representation of G. □

The representation space $H = H_\Pi$ of Π is

$$\left\{ f \to H_{D_k}, \text{measurable on } G \;\middle|\; \begin{array}{c} f(namx) = e^{\nu_{\mathrm{S}}+\rho}(a) D_k(m) f(x), \\ (a \in A_{\mathrm{S}}, m \in M_{\mathrm{S}}, x \in G), \int_K \| f \|_{H_{D_k}} dk < \infty \end{array} \right\}.$$

By restriction of f to K, this representation space is naturally identified with a closed subspace of $L^2(K; H_{D_k})$

$$\{ f : K \to H_{D_k}, \text{square-integrable} \mid f(mx) = D_k(m) f(x), (m \in M_{\mathrm{S}} \cap K, x \in K) \}.$$

This space is the L^2-induction of the unitary $(M_{\mathrm{S}} \cap K)$-module $(D_k|_{M_{\mathrm{S}}\cap K}, H_{D_k})$ to the larger compact group K. Via the natural isomorphism $M_{\mathrm{S}} \cap K \cong \mathrm{O}(2)$ as a $(M_{\mathrm{S}}\cap K)$-module H_{D_k} is a Hilbert space direct sum $\bigoplus_{a=0}^{\infty} \mathbb{C}v_{k+2a} \oplus \bigoplus_{a=0}^{\infty} \mathbb{C}v_{-(k+2a)}$. Here $v_m \in H_{D_k}$ $(m \in \mathbb{Z})$ is the element in H_{D_k} corresponding to $\chi_m \in L^2(K_0)$:

$$\chi_m \left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right) = e^{\sqrt{-1}m\theta} \qquad (\theta \in \mathbb{R})$$

by the restriction $v_m \mapsto v_m|_{K_0} \in L^2(K_0)$. Therefore $L^2_{(M_{\mathrm{S}}\cap K, D_k|_{M_{\mathrm{S}}\cap K})}(K)$ is the completion of the direct sum

$$\bigoplus_{a=0}^{\infty} L^2_{(M_{\mathrm{S}}\cap K, \chi_{k+2a})}(K) \boxtimes \mathbb{C}v_{k+2a} \oplus \bigoplus_{a=0}^{\infty} L^2_{(M_{\mathrm{S}}\cap K, \chi_{-k-2a})}(K) \boxtimes \mathbb{C}v_{-k-2a}.$$

By the Frobenius reciprocity law, the multiplicities of the K-type of Π are fully described as follows.

PROPOSITION 1.2. *The multiplicity of $\tau_{(l_1,l_2)}$ in the P_{S}-principal series representation Π is given by*

$$[\Pi : \tau_{(l_1,l_2)}] = \begin{cases} 0, & \text{if} \quad l_1 - l_2 \not\equiv k \mod 2 \quad \text{or} \quad l_1 - l_2 < k, \\ \left[\frac{l_1 - l_2 - k}{2}\right] + 1, & \text{if} \quad l_1 - l_2 \equiv k \mod 2 \quad \text{and} \quad l_1 - l_2 \geq k. \end{cases} \qquad \square$$

Therefore we will pay attention to Whittaker functions with multiplicity one K-type, i.e., $l_1 = l + k$, $l_2 = l$ for $l \in \mathbb{Z}$. These K-type $\tau_{(l+k,l)}$ $(l \in \mathbb{Z})$ are called the *peripheral* K-types of Π.

REMARK 1.3. *The $D_k \boxtimes L^2(K)$ model of the P_{S}-principal series.*

Now we want have an expression of the elements in the peripheral K-type $\tau_{(l+k,l)}$ in Π. In general for $\mathrm{f} \in H_\pi$, $(x \in K)$, $\mathrm{f}|_K$ has an expression

$$\mathrm{f}|_K(x) = \sum_{a=0}^{\infty} f_{k+2a}(x) \boxtimes v_{k+2a} + \sum_{a=0}^{\infty} f_{-k-2a}(x) \boxtimes v_{-k-2a}$$

with $f_{k+2a}(x) \in L^2_{(M_{\mathrm{S}}\cap K), \chi_{k+2a}}(K)$ and $f_{-k-2a} \in L^2_{(M_{\mathrm{S}}\cap K, \chi_{-k-2a})}(K)$. The intertwining property $\mathrm{f}|_K(m_{\mathrm{S}}x) = D_k(m_{\mathrm{S}}x)\mathrm{f}|_K(x)$ $(m_{\mathrm{S}} \in M_{\mathrm{S}} \cap K)$ implies that $f_m(m_{\mathrm{S}}^0 x) = \chi_m(m_{\mathrm{S}}^0) f_m(x)$ for $m_{\mathrm{S}}^0 \in M_{\mathrm{S}}^0$. Moreover when $\mathrm{f}|_K$ belongs to the peripheral K-type, $f_{k+2a}(x)$ and $f_{-k-2a}(x)$ vanishes for $a > 0$. And we have $f_k(x) = t^{(k)}_{+,0}(x)\Delta^l(x)\ c_+$, $f_{-k}(x) = t_{-,0}(x)\Delta^l(x)c_-$ with some functions c_+ and c_- on $M^0_{\mathrm{S}}\backslash M_{\mathrm{S}}$. Here the functions $t^{(k)}_{\pm,a}(x)$ are defined as follows.

Firstly we recall the definition the vectors of function $s^{(k)}_i$ in the subsection I-6. For $k \in \mathbb{N}$, we introduce $2(k+1)$ vectors of length $k+1$ by

$$\left(\mathrm{t}^{(k)}_{+,0}, \cdots, \mathrm{t}^{(k)}_{+,k}\right) = \left(\mathrm{t}^{(k)}_{-,k}, \cdots, \mathrm{t}^{(k)}_{-,0}\right) = \left(s^{(k)}_0, \cdots, s^{(k)}_k\right) {}^t\mathrm{Sym}\left(\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\right),$$

this means

$$\begin{aligned} \mathrm{t}^{(k)}_{\pm,0} &= s^{(k)}_0 \mp is^{(k)}_1 - s^{(k)}_2 + \cdots + (-i)^a s^{(k)}_a + \cdots + (-1)^{k/2} s^{(k)}_k; \\ \mathrm{t}^{(k)}_{\pm,-2} &= ks^{(k)}_0 \mp i(k-2)s^{(k)}_1 - (k-4)s^{(k)}_2 + \cdots + (\pm i)(k-2a)s^{(k)}_a \\ &\quad + \cdots + (-1)^{k/2}(-k)s^{(k)}_k; \end{aligned}$$

etc.

Note here that the same vectors have the double names $\mathrm{t}^{(k)}_{+,a} = \mathrm{t}^{(k)}_{-,k-a}$ $(0 \le a \le k)$. We note that the vectors $\mathrm{t}^{(k)}_{\pm,a}$ of functions are diagonalized by the action of M^0_{S}.

Then these functions $\mathrm{t}^{(k)}_{\pm,0}$ in $L^2(K)$ are utilized to describe K-finite functions in the representation space of the P_{S}-principal series, because they exhaust those functions which are right $\tau_{(k,0)}$-isotypic under K and left $\chi_{\pm k}$-isotypic under $M^0_{\mathrm{S}} \cong \mathrm{SO}(2)$ up to scalar multiples in $L^2(K)$. Thus we see that the canonical basis of the peripheral K-type $\tau_{(l+k,l)}$ is expressed as

$$c_+ \cdot t^{(k)}_{+,0}\Delta^l \boxtimes v_k + c_- \cdot t^{(k)}_{-,0}\Delta^l \boxtimes v_{-k},$$

invariant under the action of M_{S} (or more precisely under $M^0_{\mathrm{S}}\backslash M_{\mathrm{S}}$).

When the P_{S}-principal series Π (with even k) is embedded in a principal series π of Chapter I, these functions $c_\pm$ are mapped to some elements in the finite dimensional $\mathbb{C}$-vector space $\mathrm{Hom}_K(\tau_{(l+k,l)}, \pi\ )$.

II-2. The $(\mathfrak{g}, K)$-module structure for peripheral K-types

Since K-types $\tau_{(l+k,l)}$ occurs in the P_{S}-principal series occurs with multiplicity one, there is the unique injective K-homomorphism $\iota_{(l+k,l)} : W_{(l+k,l)} \longrightarrow H_\Pi$, where $W_{(l+k,l)}$ be a representation space of K with respect to highest weight $(l+k,l)$. Let $\left\{ f \begin{pmatrix} l+k,l \\ \alpha \end{pmatrix} \right\}$ be the canonical basis defined in Chapter I, Proposition 6.1. Then we write

$$\iota_{(l+k,l)} \left(f \begin{pmatrix} l+k,l \\ l+k-i \end{pmatrix} \right) = f_i, \quad 0 \le i \le k.$$

Then we have various annihilator among $\{f_i\}_{0\le i \le k}$. First we introduce the Casimir element defined as follows.

DEFINITION 2.1. *Let $C(X,Y) = \mathrm{Tr}(XY)$ be the real part of the trace form. Choose a basis $X_1, \cdots, X_n$ of $\mathfrak{g}$, and let $(g_{i,j}) = (C(X_i,X_j))^{-1}$. Then we define Casimir element by*

$$\frac{1}{2}C = \sum_{i,j} g_{i,j} X_i X_j. \quad \square$$

Explicitly, it can be written as

$$C = H_1^2 + H_2^2 - 4H_1 - 2H_2 + 2E_{e_1-e_2}E_{-e_1+e_2} + 4E_{2e_1}E_{-2e_1}$$
$$+ 2E_{e_1+e_2}E_{-e_1-e_2} + 4E_{2e_2}E_{-2e_2},$$

which is equal to

$$H_1^2 + H_2^2 - 4H_1 - 2H_2 + 2E_{e_1-e_2}^2 + 4E_{2e_2}^2 + 2E_{e_1+e_2}^2 + 4E_{2e_2}^2$$
$$- 4E_{2e_1}(E_{2e_1} - E_{-2e_1}) - 2E_{e_1+e_2}(E_{e_1+e_2} - E_{-e_1-e_2}) - 4E_{2e_2}(E_{2e_2} - E_{-2e_2}).$$

PROPOSITION 2.2. *Let C be the Casimir element. Then we have*

$$C \cdot f_i = \chi_\Pi(C) f_i, \quad \text{for each} \quad 0 \le i \le k, \tag{2.1}$$

with $\chi_\Pi(C) = \nu_{\mathrm{S}}/2 + (k-1)^2/2 - 5$. Here $\chi_\Pi : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ be the infinitesimal character of the quasi-simple representation Π with $Z(\mathfrak{g}_{\mathbb{C}})$ the center of the universal embeloping algebra $U(\mathfrak{g})$. Annihilator equations are

$$X_{+22} f_{i-1} - 2X_{+12} f_i + X_{+11} f_{i+1} = 0 \tag{2.2}$$

and

$$X_{-11}f_{i-1} + 2X_{-12}f_i + X_{-22}f_{i+1} = 0. \quad \square \tag{2.3}$$

PROOF. (2.1) is the Casimir equation. We determine the value $\chi_\Pi(C)$. We have to compute $\chi_\Pi(C) = (\nu_S + \rho - 3)^2/2 + (k-1)^2/2 - 5$. We refer to the book of Knapp, [KV], p. 665, Proposition 11.43 to determine the infinitesimal character of Π. Put $P_S = N_S A_S M_S$, where $M_S = \mathrm{SL}^{\pm}(2,\mathbb{R}) \cdot \{\pm 1\}$. $\mathrm{Lie} A_S = \mathfrak{a}_S$ is equal to

$$\exp\left\{\mathbb{R}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} e^{t_S} & & & \\ & e^{t_S} & & \\ & & e^{-t_S} & \\ & & & e^{-t_S} \end{pmatrix} \middle| \; t_S \in \mathbb{R}\right\}.$$

On the other hand, the set of M_S is

$$\exp\left\{\mathbb{R}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}\right\} = \left\{\begin{pmatrix} e^{t_O} & & & \\ & e^{-t_O} & & \\ & & e^{-t_O} & \\ & & & e^{t_O} \end{pmatrix} \middle| \; t_O \in \mathbb{R}\right\}.$$

Since A is included in $A_S M_S$, we compare the following equation in $\mathfrak{a}$ to get a representative of the infinitesimal character.

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & -t_1 & \\ & & & -t_2 \end{pmatrix} = \begin{pmatrix} t_O + t_S & & & \\ & -t_O + t_S & & \\ & & -t_O - t_S & \\ & & & t_O - t_S \end{pmatrix}.$$

We have $(t_1 + t_2)/2 = t_S$ and $(t_1 - t_2)/2 = t_O$. And also we get

$$(\nu_S + \rho_S)(t_S) = \frac{1}{2}\{(\nu_S + \rho_S)(t_1) + (\nu_S + \rho_S)(t_2)\} \quad \text{and} \quad \nu_O(t_O) = \frac{1}{2}\{\nu_O(t_1) - \nu_O(t_2)\}.$$

Then we add both equations.

$$\nu_O(t_O) + (\nu_S + \rho_S)(t_S) = \frac{1}{2}(\nu_S + \nu_O + \rho_S)(t_1) + \frac{1}{2}(\nu_S - \nu_O + \rho_S)(t_2).$$

Since the first term of the right hand side of the above equation is equal to $(\nu + \rho)(H_2)$ and the second term of it is $(\nu + \rho)(H_2)$, then we have

$$(\nu + \rho)(H_1) = \frac{1}{2}(\nu_S + \rho_S + k) \quad \text{and} \quad (\nu + \rho)(H_2) = \frac{1}{2}(\nu_S + \rho_S - k)$$

with the Blattner parameter k. This means $((\nu_{\rm S}+k-1)/2, (\nu_{\rm S}-k+1)/2)$ is the infinitesimal character of Π. Therefore we have $\chi_\Pi(C) = (\nu_{\rm S}+(k-1))^2/4+(\nu_{\rm S}-(k-1))^2/4-5 = \nu_{\rm S}^2/2+(k-1)^2/2-5 = (\nu_{\rm S}+\rho-3)^2/2+(k-1)^2/2-5$.

We consider the canonical surjection

$$\begin{array}{ccccc} \mathfrak{p}_\pm \otimes W_{(l+k,l)} & \longrightarrow & \mathfrak{p}_\pm W_{(l+k,l)} \hookrightarrow H_\Pi & & \\ (X \otimes v) & \longmapsto & Xv, & & (X \in \mathfrak{p}_\pm, \nu \in W_{(l+k,l)}). \end{array}$$

which is a K-homomorphism. Since $\mathfrak{p}_+ \cong W_{(2,0)}$ and $\mathfrak{p}_- \cong W_{(0,-2)}$, we have the Clebsh-Gordan decomposition

$$\mathfrak{p}_+ \otimes W_{(l+k,l)} \cong W_{(l+k,l)} \oplus W_{(l+k,l+1)} \oplus W_{(l+k+2,l)}$$

and

$$\mathfrak{p}_- \otimes W_{(l+k,l)} \cong W_{(l+k-2,l)} \oplus W_{(l+k-1,l-1)} \oplus W_{(l+k,l-2)}.$$

Here we note that $\{X_{+11}, X_{+12}, X_{+22}\}$ in $\mathfrak{p}_+$ and $\{X_{-22}, -X_{-12}, X_{-11}\}$ in $\mathfrak{p}_-$ are canonical basis, respectively. Recall here Lemma 3.1 of [O1]. Corresponding to this decomposition, the elements $\{X_{+22}f_i - 2X_{+12}\ f_{i+1} + X_{+11}f_{i+2} \mid 0 \le i \le k-2\}$ form a canonical basis of the K-types $\tau_{(l+k,l+2)}$ in H_Π. But this K-type does not occur in H_Π. Hence we have (2.2). The relation (2.3) is obtained similarly, since the set $\{X_{-11}f_{i-1} + 2X_{-12}f_i + X_{-22}f_{i+1}\}$ $(1 \le i \le k)$ also forms a canonical basis of the K-type $\tau_{(l+k-2,l)}$. □

One might suspect that $\{f_i\}$ would have more relations under the action of $\mathrm{U}(\mathfrak{g})$. We explain below the reason that we believe these relation given in the above proposition should be enough.

REMARK 2.3. We have

$$\begin{aligned} (2.4) \qquad -\frac{1}{4}\{\nu_{\rm S}^2-(2l+k+1)\}f_i &= -\frac{1}{4}(X_{-11}X_{+11}+X_{-22}X_{+22})f_i \\ &+\frac{1}{4}X_{-11}X_{+22}f_{i-2}+\frac{1}{4}X_{-22}X_{+11}f_{i+2}. \end{aligned}$$

However this relation is equivalent to the Casimir equation.

OUTLINE OF PROOF. We explain the heuristic reason why these equations given above are enough. We take a canonical basis $\{f_i\}_{0\le i\le k}$ of $\tau_{(l+k+1,l+1)}$. By the Clebsh-Gordan decomposition ([O1], Lemma 3.2)

$$\left\{-\frac{i}{k}X_{+22}f_{i-1}-\frac{k-2i}{k}X_{+12}f_i+\frac{k-i}{k}X_{+11}f_{i+1}\,\middle|\,0\le i\le k\right\}$$

is another canonical basis. Here we set $f_{-1}=f_{k+1}=0$. Then there exist a scalar γ_+ independent of i such that

$$\gamma_+f_i=-\frac{i}{k}X_{+22}f_{i-1}-\frac{k-2i}{k}X_{+12}f_i+\frac{k-i}{k}X_{+11}f_{i+1}. \tag{2.5}$$

Moreover, if $1\le i\le k-1$, adding $-\frac{k-2i}{k}$ times of the relation (2.2), we have

$$\gamma_+f_i=-\frac{1}{2}X_{+22}f_{i-1}+\frac{1}{2}X_{+11}f_{i+1}. \tag{2.6}$$

Changing the role of $\{f_i\}$ and $\{f_i\}$, we have

$$\gamma_-f_i=-\frac{i}{k}X_{-11}f_{i-1}+\frac{k-2i}{k}X_{-12}f_i+\frac{k-i}{k}X_{-22}f_{i+1} \tag{2.7}$$
$$=-\frac{1}{2}X_{-11}f_{i-1}+\frac{1}{2}X_{-22}f_{i+1}\quad(1\le i\le k-1), \tag{2.8}$$

with a constant γ_-. The replacing f_i in (2.8) by (2.6), we have

$$\begin{aligned}(\gamma_+\gamma_-)f_i&=\frac{1}{4}X_{-11}(X_{+22}f_{i-2}-X_{+11}f_i)+\frac{1}{4}X_{-22}(-X_{+22}f_i+X_{+11}f_{i+2})\\&=-\frac{1}{4}(X_{-11}X_{+11}+X_{-22}X_{+22})f_i+\frac{1}{4}X_{-11}X_{+22}f_{i-2}+\frac{1}{4}X_{-22}X_{+11}f_{i+2}\\&=\frac{1}{4}(X_{-11}X_{+11}+X_{-22}X_{+22})f_i+\frac{1}{4}X_{+22}X_{-11}f_{i-2}+\frac{1}{4}X_{+11}X_{-22}f_{i+2}\\&=-\frac{1}{4}(X_{-11}X_{+11}+X_{-22}X_{+22}+X_{+22}X_{-22}+\frac{1}{4}X_{+11}X_{-11})f_i\\&\quad-\frac{1}{2}X_{+22}X_{-12}f_{i-1}-\frac{1}{2}X_{+11}X_{-12}f_{i+1}.\end{aligned}$$

Here we apply the relation (2.3) twice to remove f_{i-2} and f_{i+2}. Furthermore the second line of the last right hand side is rewritten as

$$\begin{aligned}&-\frac{1}{2}[X_{+22},X_{-12}]f_{i-1}-\frac{1}{2}[X_{+11},X_{-12}]f_{i+1}-\frac{1}{2}X_{-12}X_{+22}f_{i-1}-\frac{1}{2}X_{-12}X_{+11}f_{i+1}\\&=-\frac{1}{2}X_{-12}X_{+12}f_i-\frac{1}{2}X_{-12}X_{+12}f_i+\frac{1}{2}X_{-12}X_{+11}f_{i+1}+\frac{1}{2}X_{-12}X_{+22}f_{i-1}.\end{aligned}$$

Here we apply (2.2) twice.

Finally, we have to decide $\gamma_\pm$. Since the computation takes some space more, we omit this. We have to compare the last equation with the Casimir equation.

Meanwhile we can rewrite the Casimir equation as

$$\frac{1}{4}\Big\{X_{+11}X_{-11}+X_{-11}X_{+11}+X_{+22}X_{-22}+X_{-22}X_{+22}\Big\}f_i$$
$$+\frac{1}{2}\Big\{X_{+12}X_{-12}+X_{-12}X_{+12}\Big\}f_i+C_{\mathfrak{k}}f_i=\Big\{\frac{(\nu_{\mathrm{S}}-\rho+3)^2}{2}+\frac{(k-1)^2}{2}\Big\}f_i,$$

where

$$C_{\mathfrak{k}}=\frac{1}{2}\Big\{\kappa(e_{11})^2+\kappa(e_{22})^2+\kappa(e_{12})\kappa(e_{21})+\kappa(e_{21})\kappa(e_{12})\Big\}$$

is the Casimir operator of the K-module $W_{(l+k,l)}$, hence it is equal to the scalar. This is the outline of the proof. $\square$

REMARK 2.4. The center $Z(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$ has two generators, one is the Casimir operator, the other is an operator C_4 of degree 4. We should have a system of equations

$$C_4\cdot f_i=\chi_\Pi(C_4)f_i \qquad (0\le i\le k)$$

The equation which is essentially equivalent to this is obtained by the composition $\det C_-\cdot\det C_+$ or $\det C_+\cdot\det C_-$. However $\det C_+$ is the composition of the operators (2.5) twice : $\tau_{(l+k,l)}\longrightarrow\tau_{(l+k+1,l+1)}\longrightarrow\tau_{(l+k+2,l+2)}$. Therefore by the former remark (2.3) implies that $\det C_-\cdot\det C_+$ is essentially the "square" of the Casimir operator modulo the annihilation relation (2.2) and (2.3).

II-3. System of partial differential equations for Whittaker functions

Let $\Phi\in\mathrm{Hom}(\Pi,C_\eta^\infty(N\backslash G))$ be an intertwining operator, i.e., an algebraic Whittaker functional on Π. Let $\Phi(f_i)|_A=\varphi_i^{(l+k,l)}(y)$ be the A-radial part of the Whittaker function of the canonical basis $\{f_i\}_{0\le i\le k}$ of $\tau_{(l+k,l)}$ in Π. We will use these notation below.

We introduce the coordinate associated with positive simple roots $\{e_1-e_2,2e_2\}$, $y=(y_1,y_2)=(a_1/a_2,a_2^2)$ and Euler operator $\partial_i=y_i(\partial/\partial y_i)$, $i=1$, 2.

THEOREM 3.1. *Let $\Phi^{(l+k,l)}(y)$ be the radial part of the even P_{S}-principal series Whittaker function with $(k+1)$-dimensional K-type. Then $\varphi^{(l+k,l)}(y) = {}^t(\varphi_0^{(l+k,l)}(y), \varphi_1^{(l+k,l)}(y), \cdots, \varphi_k^{(l+k,l)}(y))$ satisfy*

(3.1)
$$\begin{aligned}
&\{2\partial_1^2 + 4\partial_2^2 - 4\partial_1\partial_2 - 8\pi^2 c_0^2 y_1^2 - 16\pi^2 c_3^2 y_2^2 \\
&+8\pi(l+i)c_3 y_2 - (\nu_{\mathrm{S}} + \rho - 3)^2/2 - (k-1)^2/2\}\varphi_i^{(l+k,l)}(y) \\
&-4\pi\sqrt{-1}ic_0 y_1 \varphi_{i-1}^{(l+k,l)}(y) + 4\pi\sqrt{-1}(k-i)c_0 y_1 \varphi_{i+1}^{(l+k,l)}(y) = 0, \qquad 0 \le i \le k,
\end{aligned}$$

(3.2)
$$\begin{aligned}
&(\partial_1 + l - k + i)\varphi_i^{(l+k,l)}(y) - 4\pi\sqrt{-1}c_0 y_1 \varphi_{i-1}^{(l+k,l)}(y) \\
&+(2\partial_2 - \partial_1 - 4\pi c_3 y_2 + l + i - 1)\varphi_{i-2}^{(l+k,l)}(y) = 0, \qquad 2 \le i \le k,
\end{aligned}$$

(3.3)
$$\begin{aligned}
&(\partial_1 - l - k - i + 2)\varphi_{i-2}^{(l+k,l)}(y) + 4\pi\sqrt{-1}c_0 y_1 \varphi_{i-1}^{(l+k,l)}(y) \\
&+(2\partial_2 - \partial_1 + 4\pi c_3 y_2 - l - i + 1)\varphi_i^{(l+k,l)}(y) = 0, \qquad 2 \le i \le k. \quad \square
\end{aligned}$$

REMARK 3.2. We note that a part of equations (3.2) is equivalent to (cf. Remark 2.3)

(3.4)
$$\begin{aligned}
&\{(\partial_1 - l - k + i)(\partial_1 + l + k - i + 2) \\
&+(2\partial_2 - \partial_1 + 4\pi y_2 - l - i - 1)(2\partial_2 - \partial_1 - 4\pi y_2 + l + i + 1) \\
&-(\nu_{\mathrm{S}} + \rho - 2l - k - 4)(\nu_{\mathrm{S}} + \rho + 2l + k - 2)\}\varphi_i^{(l+k,l)}(y) \\
&-(\partial_1 - l - k + i)(2\partial_2 - \partial_1 - 4\pi y_2 + l + i - 1)\varphi_{i-2}^{(l+k,l)}(y) \\
&-(\partial_1 + l + k - i)(2\partial_2 - \partial_1 + 4\pi y_2 - l - i - 1)\varphi_{i+2}^{(l+k,l)}(y) = 0, \\
&\hspace{20em} 2 \le i \le k-2.
\end{aligned}$$

Logically speaking, we do not need this. But they might be useful for the investigation in the future.

PROOF OF THEOREM 3.1. First we note that (3.1) is the Whittaker realization of (2.1). We have to take care of the change caused by the multiplicity $a^\rho(y) =$

$y_1^2 y_2^{3/2} = a^\rho(y)$. Then the action on the A-radial part $\varphi_i^{(l+k,l)}(y)$ of the K-invariant element in the Whittaker model is given by the operator

$$\begin{aligned}&\{2\partial_1^2 + 4\partial_2^2 - 4\partial_1\partial_2 - 8\pi^2 y_1^2 - 16\pi^2 y_2^2 + 8\pi(l+i)y_2\}\varphi_i^{(l+k,l)}(y)\\ &- 4\pi\sqrt{-1} i y_1 \varphi_{i-1}^{(l+k,l)}(y) + 4\pi\sqrt{-1}(k-i) y_1 \varphi_{i+1}^{(l+k,l)}(y)\end{aligned}$$

which is given after the $a^\rho(y)$-shift.

Next we will prove (3.2) and (3.3). There are the Whittaker realizations of (2.2) and (2.3). We compute them by using the Iwasawa decomposition in Chapter I, (8.6) and (8.7). Then its action on the A-radial part $\varphi_i^{(l+k,l)}(y)$ of the K-invariant element in the Whittaker model is given by the operators

$$(\partial_2 - 4\pi y_2 + i + l - 1)\varphi_{i-1}^{(l+k,l)}(y) - 4\pi\sqrt{-1} y_1 \varphi_i^{(l+k,l)} + (\partial_1 + l + i + k - 1)\varphi_{i+1}^{(l+k,l)}(y),$$

and

$$(\partial_1 - k - l + i + 2)\varphi_{i-1}^{(l+k,l)}(y) + 4\pi\sqrt{-1} y_1 \varphi_i^{(l+k,l)}(y) + (\partial_2 + 4\pi y_2 - l - i)\varphi_{i+1}^{(l+k,l)}(y).$$

We can checked by the target K-types. The left hand side of them are not occur in the K-types decomposition of P_S-principal series. Hence their eigenvalues are 0. □

CONJECTURE. (3.1), (3.2) *and* (3.3) *are holonomic system of rank 8.* □

The conjecture means that the solution space of (3.1), (3.2) and (3.3) are naturally isomorphic to the space of algebraic Whittaker functions $\mathrm{Hom}_{(\mathfrak{g},K)}(\Pi, C^\infty$ -$\mathrm{Ind}_N^G(\eta))$, which has the dimension 8, the order of the Weyl group (see [M2], p. 238, theorem 6.2.1).

REMARK 3.3. Heuristically speaking, it is easy to see that the rank of solution space is dominated by $8 \cdot (k+1)$. But to determine the exact value of the rank seems to be difficult just starting from the equations.

II-4. The characteristic indices

Around $(y_1, y_2) = (0,0)$. Similarly as the case of other groups and other representations, we may effect the point $(y_1, y_2) = (0,0)$ is a regular singularity. We want to determine the characteristic indices at $(0,0)$. To have the characteristic equations, it is enough to erase the terms divisible either y_1 or y_2.

Therefore we consider a formal power series solution

$$\varphi_i^{(l+k,l)}(y) = \sum_{m,n\geq 0} c_{m,n}^{(i)} \left(2\pi\sqrt{-1}y_1\right)^{m+\mu_1} \left(2\pi y_2\right)^{n+\mu_2} \tag{4.1}$$

LEMMA 4.1. *The characteristic indices of* (4.1) *are as follows*

$$\begin{aligned} &(i) \quad \mu_1 = \pm\frac{1}{2}\nu_{\mathrm{S}} + \frac{1}{2}(k-1), && \mu_2 = \pm\frac{1}{2}\nu_S, \\ &(ii) \quad \mu_1 = \pm\frac{1}{2}\nu_{\mathrm{S}} + \frac{1}{2}(k-1), && \mu_2 = \frac{1}{2}(k-1). \quad \square \end{aligned}$$

We remark that (i) (we call this pair case 1) is $\mu_1 - \mu_2 = (k-1)/2$, (ii) (we call this pair case 2) is $\mu_2 = (k-1)/2$.

PROOF. Similarly as the case of other groups and other representations, we may expect the point $(y_1, y_2) = (0,0)$ is a regular singularity. We want to determine the characteristic indices at $(0,0)$.

Firstly we set $c_0 = c_3 = 0$ in (3.2) and (3.3) of Theorem 3.1 :

$$\begin{cases} (\partial_1 + l - k + i)\varphi_i^{(l+k,l)}(y) + (2\partial_2 - \partial_1 + l + i - 1)\varphi_{i-2}^{(l+k,l)}(y) = 0, \\ (\partial_1 - l - k - i + 2)\varphi_{i-2}^{(l+k,l)}(y) + (2\partial_2 - \partial_1 - l - i + 1)\varphi_i^{(l+k,l)}(y) = 0. \end{cases}$$

If we may set $\varphi_i = c_{0,0}^{(i)} y_1^{\mu_1} y_2^{\mu_2}$, then

$$\begin{cases} (\mu_1 + l - k + i)c_{0,0}^{(i)} + (2\mu_2 - \mu_1 + l + i - 1)c_{0,0}^{(i-2)} = 0, \\ (\mu_1 - l - k + i + 2)c_{0,0}^{(i-2)}(2\mu_2 - \mu_1 - l - i + 1)c_{0,0}^{(i)} = 0. \end{cases}$$

To have non zero solution for $c_{0,0}^{(i-2)}, c_{0,0}^{(i)}$, we need

$$(\mu_1 + l - k + i)(\mu_1 - l - k + i + 2) - (2\mu_2 - \mu_1 + l + i - 1)(2\mu_2 - \mu_1 - l - i + 1) = 0.$$

This last equation is simplified as $(\mu_1 - k + 1)^2 = (2\mu_2 - \mu_1)^2$. Here we have either $\mu_1 - \mu_2 = (k-1)/2$ or $\mu_2 = (k-1)/2$. In the case 1, $c_{0,0}^{(i)} = -c_{0,0}^{(i-2)}$, and in the case 2, $(\mu_1 - k + l + i)c_{0,0}^{(i)} = (\mu_1 - k - l - i + 2)c_{0,0}^{(i-2)}$.

To determine μ_1 completely, we use the equation (4.1) by putting $c_0 = c_3 = 0$ and setting the term $8\pi(l+i)y_2 = 0$.

$$\Big\{2\partial_1^2 + 4\partial_2^2 - 4\partial_1\partial_2 - \frac{(\nu_S+\rho-3)^2}{3} - \frac{(k-1)^2}{2}\Big\}\varphi_i^{(l+k,l)} = 0.$$

Therefore

$$2\mu_1^2 + 4\mu_2^2 - 4\mu_1\mu_2 = \frac{(\nu_S+\rho-3)^2}{2} + \frac{(k-1)^2}{2}.$$

In either case

$$2\mu_1 + 4\mu_2^2 - 4\mu_1\mu_2 = \mu_1^2 + (2\mu_2-\mu_1)^2 = \mu_1^2 + (\mu_1 - k + 1)^2.$$

Then

$$2\mu_1^2 - 2(k-1)\mu_1 = \mu_1\{\mu_1 - (k-1)\} = \frac{(\nu_S+\rho-3)^2}{4} - \frac{(k-1)^2}{4}.$$

If $\mu_1 - \mu_2 = (k-1)/2$, then $\{\mu_1 - (k-1)/2\}^2 = (\nu_S+\rho-3)^2/4$. If $\mu_2 = (k-1)/2$, then $\mu_1 = \pm(\nu_S+\rho-3)/2 + (k-1)/2$. Thus we have our lemma. □

We remark that the characteristic indices (i) , (ii) are same properties. Namely, (i) are $\mu_1 - \mu_2 = (k-1)/2$, (ii)) are $\mu_2 = (k-1)/2$.

CHAPTER III

SOLUTIONS FOR THE FORMAL POWER SERIES WHEN $k = 2$ AND $l = -1$

III-1. The holonomic system of rank 8

We discuss here a very special case, hoping that this result would give some insight for the general case. Here we consider only those P_{S}-principal with minimal K-types of dimension three. This means that we take $D_k = D_2$ in the definition of the P_{S}-principal series.

If $k = 2$, our holonomic system is reduced to the following.

PROPOSITION 1.1. *Let $\varphi^{(1,-1)}(y)$ be the radial part of the even P_{S}-principal series Whittaker function with 3-dimensional K-type. Then $\varphi^{(1,-1)}(y) = {}^t(\varphi_0^{(1,-1)}(y), \varphi_1^{(1,-1)}(y), \varphi_2^{(1,-1)}(y))$ satisfy*

(1.1)

$$\begin{aligned}&\{2\partial_1^2 + 4\partial_2^2 - 4\partial_1\partial_2 - 8\pi^2 c_0^2 y_1^2 - 16\pi^2 c_3^2 y_2^2 + 8\pi(i-1)c_3 y_2 \\ &- \nu_{\mathrm{S}}^2/2 - 1/2\}\varphi_i^{(1,-1)}(y) - 4\pi\sqrt{-1}ic_0 y_1 \varphi_{i-1}^{(1,-1)} + 4\pi\sqrt{-1}(2-i)c_0 y_1 \varphi_{i+1}^{(1,-1)} = 0,\end{aligned}$$

$$\text{for} \quad 0 \le i \le 2,$$

(1.2)

$$(\partial_1 - 1)\varphi_2^{(1,-1)}(y) - 4\pi\sqrt{-1}c_0 y_1 \varphi_1^{(1,-1)}(y) + (2\partial_2 - \partial_1 - 4\pi c_3 y_2)\varphi_0^{(1,-1)}(y) = 0,$$

and

(1.3)

$$(\partial_1 - 1)\varphi_0^{(1,-1)}(y) + 4\pi\sqrt{-1}c_0 y_1 \varphi_1^{(1,-1)}(y) + (2\partial_2 - \partial_1 + 4\pi c_3 y_2)\varphi_2^{(1,-1)}(y) = 0.$$

□

We know that the system (1.1), (1.2) and (1.3) are a holonomic system of rank at most 24 ($= 8 \times 3$). And we expect that the rank of the above holonomic system is 8.

a heuristic argument for rank 8

Here is a heuristic enumeration of the rank of our holonomic system. We have an equation

$$(\det \mathcal{C}_-)(\det \mathcal{C}_+) \begin{pmatrix} f_0^{(1,-1)} \\ f_1^{(1,-1)} \\ f_2^{(1,-1)} \end{pmatrix} = (\nu_1 - 4)^2(\nu_2 - 4)^2 \cdot \begin{pmatrix} f_0^{(1,-1)} \\ f_1^{(1,-1)} \\ f_2^{(1,-1)} \end{pmatrix}$$

as we remarked in Chapter II, Lemma 2.4, this is derived from those relations which we already discussed. The Whittaker realization of this equation is of the form

$$\{\partial_1^2(2\partial_2 - \partial_1)^2 + \text{lower terms}\} \begin{pmatrix} \varphi_0^{(1,-1)} \\ \varphi_1^{(1,-1)} \\ \varphi_2^{(1,-1)} \end{pmatrix} = (\nu_1 - 4)^2(\nu_2 - 4)^2 \cdot \begin{pmatrix} \varphi_0^{(1,-1)} \\ \varphi_1^{(1,-1)} \\ \varphi_2^{(1,-1)} \end{pmatrix}.$$

These equation together with the Casimir equation (1.1) implies ${}^t(\varphi_0^{(1,-1)} \ \varphi_1^{(1,-1)} \ \varphi_2^{(1,-1)})$ are solutions of a holonomic system of rank 24 (Compare with the system in [MO], Theorem 10.1, which has rank 8). But we have the relations (1.2) and (1.3) compatible with these equations. Their derivations $\partial_1^a(2\partial_2 - \partial_1)^b \cdot (1.2) = 0$, $\partial_1^a(2\partial_2 - \partial_1)^b \cdot (1.3) = 0$ should give $2 \times 8 = 16$ other relations among 24 independents

$$\left\{\partial_1^a(2\partial_2 - \partial_1)^b \cdot \varphi_i^{(1,-1)} \,\middle|\, i = 0, 1, 2, \quad a, b \in \mathbb{N} \times \mathbb{N}\right\}.$$

Thus remaining rank is $24 - 16 = 8$. To have a rigorous proof, we have to fined, say, an integrable affine connection on a rank 8 vector bandle over the space $\{(y_1, y_2) \in \mathbb{C}^2\}$ equivalent to our holonomic system. Probably we need a more "dirty" computation which we cannot yet find.

The idea of the construction of solutions

Before the statement of the result, we explain how to get our power series solutions for the K-type $\tau_{(1,-1)}$ in the P_{S}-principal series from those of the $P_{\min}$-principal series.

Recall the definition the principal series representations π_0 of $M_{\mathrm{S}}^0 = G_0 = \mathrm{SL}(2, \mathbb{R})$ given in Chapter II, p. 25.

$$\pi_0 = \mathrm{Ind}_{P_0}^{G_0}(1_{N_0} \otimes e^{\nu_0 + \rho_0} \otimes \sigma)$$

is principal series representation of $\mathrm{SL}(2,\mathbb{R})$. Set $\nu_0 = 1$. Then we have an exact sequence of G_0-module

$$0 \longrightarrow D_2^+ \oplus D_2^- \longrightarrow \pi_0 \longrightarrow \mathbb{C} \longrightarrow 0$$

of the representations of $\mathrm{SL}(2,\mathbb{R})$. The irreducible decomposition the induction $\mathrm{Ind}_{\mathrm{SL}(2,\mathbb{R})}^{\mathrm{SL}^{\pm}(2,\mathbb{R})}$ gives two exact sequences of $\mathrm{SL}^{\pm}(2,\mathbb{R})$.

$$0 \longrightarrow D_2^+ \oplus D_2^- \longrightarrow \pi_0 \longrightarrow \mathbb{C}(\pm) \longrightarrow 0$$

Here $\mathbb{C}(+)$ is the trivial representation of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ and $\mathbb{C}(-)$ the determinant representation of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ ([V], pp. 68–77, Proposition 1.4.8). Then we have four embeddings $\Pi \hookrightarrow \pi$ of the P_{S}-series Π into a principal series representation π with either $\sigma \in \widehat{M}_{\min}$ is $\sigma = \sigma_0$ or $\sigma = \sigma_1$ and with $\nu_1 = (\nu_{\mathrm{S}}^* + k - 1)/2$, $\nu_2 = (\nu_{\mathrm{S}}^* - k + 1)/2$, or $\nu_1 = (\nu_{\mathrm{S}}^* + k - 1)/2$, $\nu_2 = (\mp\nu_{\mathrm{S}} - k + 1)/2$.

III-2. Construction of 8 independent power series solutions

To have a Whittaker function with K-type $\tau_{(1,-1)}$ belonging to Π, we start from the Whittaker function with K-type $\tau_{(1,1)}$ or $\tau_{(-1,-1)}$ (or the Whittaker function with K-type $\tau_{(0,0)}$) belonging to π. Applying some adequate elements in $S(\mathfrak{p}_{\pm})$, we shift those K-types of π to the K-type $\tau_{(1,-1)}$ in Π.

THEOREM 2.1. *The case of the embedding* $\sigma = \sigma_1$, $(\nu_1, \nu_2) = ((\nu_{\mathrm{S}} + k - 1)/2, (\nu_{\mathrm{S}} - k + 1)/2)$.

We define the functions $\varphi_{(\mu_1,\mu_2)}^{(1,-1)}(y)$ *by Chapter II,* (3.1). *A power series solutions* $\Phi^{(1,-1)}(y) = a^{\rho}(y)\varphi_{(\mu_1,\mu_2)}^{(1,-1)}(y) = {}^t\big(\Phi_0^{(1,-1)}(y), \Phi_1^{(1,-1)}(y), \Phi_2^{(1,-1)}(y)\big)$ *with the characteristic indices (i) in Lemma 5.2 is given by*

$$\Phi_0^{(1,-1)}(y) = a^{\rho}(y) \sum_{m,n\geq 0} \Big[\Big\{-2\Big(2(m+n) + (\mu_1 - \mu_2) + 1\Big)P_{m,n}(\mu_1,\mu_2)$$
$$+ 4c_3 Q_{m,n-1}(\mu_1,\mu_2)\Big\} + 4c_3\Big\{Q_{m,n}(\mu_1,\mu_2) + P_{m,n}(\mu_1,\mu_2)\Big\}(\pi y_2)\Big]$$
$$\times \frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n},$$
$$\Phi_1^{(1,-1)}(y) = -a^{\rho}(y)\cdot(4\pi c_0 y_1) \sum_{m,n\geq 0} P_{m,n}(\mu_1,\mu_2)\cdot \frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}$$

and

$$\Phi_2^{(1,-1)}(y) = a^\rho(y) \sum_{m,n\geq 0} \Big[\Big\{2\Big(2(m-n)+(\mu_1-\mu_2)+1\Big)P_{m,n}(\mu_1,\mu_2) - 4c_3Q_{m,n-1}\Big\}$$
$$+\Big\{2(2n+\mu_2)Q_{m,n}(\mu_1,\mu_2)+4c_3P_{m,n}(\mu_1,\mu_2)\Big\}(\pi y_2)\cdot \frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}.$$

Here $(\mu_1,\mu_2) = (\nu_{\mathrm{S}}^*/2+1/2, \nu_{\mathrm{S}}^*/2)$ *are the characteristic indices (i) of Lemma 5.2 with* $\nu_{\mathrm{S}}^* = \nu_{\mathrm{S}}$ *or* $\nu_{\mathrm{S}}^* = -\nu_{\mathrm{S}}$.

The case of the embedding $(\nu_1,\nu_2) = ((\nu_{\mathrm{S}}+k-1)/2, -(\nu_{\mathrm{S}}+k-1)/2)$.

The explicit formulas of $\Phi^{(1,-1)}(y)$ *with characteristic indices (ii) in Lemma 5.2 is a power series solution*

$$\Phi_0^{(1,-1)}(y) = a^\rho(y) \sum_{m,n\geq 0} \Big[4\Big\{nP_{m,n}(\mu_1,\mu_2)+c_3Q_{m,n-1}(\mu_1,\mu_2)\Big\}$$
$$+\Big\{2\Big(2(n-m)+(\mu_2-\mu_1)-1\Big)Q_{m,n}(\mu_1,\mu_2)+4c_3P_{m,n}(\mu_1,\mu_2)\Big\}(\pi y_2)\Big]$$
$$\times\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n},$$

$$\Phi_1^{(1,-1)}(y) = -a^\rho(y)\cdot(2\pi c_0 y_1)\sum_{m,n\geq 0} Q_{m,n}(\mu_1,\mu_2)\cdot\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}$$

and

$$\Phi_2^{(1,-1)}(y) = a^\rho(y)\sum_{m,n\geq 0}\Big[\Big\{2(\mu_2+n)P_{m,n}(\mu_1,\mu_2)+4c_3Q_{m,n-1}(\mu_1,\mu_2)\Big\}$$
$$+\Big\{2\Big(2(m-n)+(\mu_1-\mu_2)+1\Big)Q_{m,n}(\mu_1,\mu_2)-4c_3P_{m,n}(\mu_1,\mu_2)\Big\}\Big]$$
$$\times\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}.$$

Here $(\mu_1,\mu_2) = (\nu_{\mathrm{S}}^*/2+1/2, 1/2)$ *are the characteristic indices (ii) of Lemma 5.2. We set*

$$P_{m,n}(\mu_1,\mu_2) = {}_3F_2\left(\begin{matrix}-n,\ m+\frac{\mu_1}{2}+\frac{1}{2},\ -m-\frac{\mu_1}{2}-\frac{1}{2}\\ \frac{\mu_1}{2}+\frac{1}{2},\ \mu_2-\frac{\mu_1}{2}+\frac{1}{2}\end{matrix}\,\Big|\,1\right),$$
$$Q_{m,n}(\mu_1,\mu_2) = \frac{2(2m+\mu_1+1)}{(\mu_1+1)(2\mu_2-\mu_1+1)}{}_3F_2\left(\begin{matrix}-n,\ m+\frac{\mu_1}{2}+\frac{3}{2},\ -m-\frac{\mu_1}{2}+\frac{1}{2}\\ \frac{\mu_1}{2}+\frac{3}{2},\ \mu_2-\frac{\mu_1}{2}+\frac{3}{2}\end{matrix}\,\Big|\,1\right).\ \square$$

Before the proof of this theorem, we prepare two lemmas on the $(\mathfrak{g},K)$-module of the principal series. Recall the notation of I-6, (6.3) and (6.4):

$$S_{[0]}^{(2)} = {}^t(\,s_{11}^2\quad s_{11}s_{12}\quad s_{12}^2\,),\quad S_{[2]}^{(2)} = {}^t(\,s_{21}^2\quad s_{21}s_{22}\quad s_{22}^2\,)$$

for $S(x) = \begin{pmatrix} s_{11} & s_{12}\\ s_{21} & s_{22}\end{pmatrix} \in \mathrm{U}(2)$ $(x\in K)$.

LEMMA 2.2. *Let* ${}^t(\, f_0 \quad f_1 \quad f_2\,)$ *be a vector of elements in* H_π *corresponding to the vector* $\{S^{(2)}_{[0]} - S^{(2)}_{[2]}\}\Delta^{-1}$ *of elements in* $L^2(K)$, *which is the canonical basis of* $\tau_{(1,-1)}$.

If $\nu_1 = (\nu_{\mathrm{S}} + k - 1)/2$ *and* $\nu_2 = (\nu_{\mathrm{S}} - k + 1)/2$, *we have*

(i)
$$\mathcal{C}_{+;(+2)}\Delta^{-1} - \mathcal{C}_{-;(+2)}\Delta = \frac{1}{3}\cdot\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}.$$

with $\mathcal{C}_{+;(+2)} = \frac{1}{6}\cdot\begin{pmatrix} X_{+11} \\ X_{+12} \\ X_{+22}\end{pmatrix}$ *and* $\mathcal{C}_{-;(+2)} = \frac{1}{6}\cdot\begin{pmatrix} X_{-22} \\ -X_{-12} \\ X_{-11}\end{pmatrix}$.

Moreover we have

(ii)
$$\mathcal{C}_{+;(-2)}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = 0 \quad \textit{and} \quad \mathcal{C}_{-;(-2)}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = 0,$$

for $\mathcal{C}_{+;(-2)} = (\, X_{+22} \quad -2X_{+12} \quad X_{+11}\,)$ *and* $\mathcal{C}_{-;(-2)} = (\, X_{-11} \quad 2X_{-12} \quad X_{-22}\,)$. □

PROOF. First we quote some special cases of the formulas in Theorem 5.1 of [O2].

PROPOSITION A (A PART OF THEOREM 5.1). *Suppose* ν_1 *and* ν_2 *are generic and* $\sigma = \sigma_1$. *We have*

(2.1)
$$\mathcal{C}_{+;(-2)}\left(\mathrm{S}^{(2)}_{[0]}\Delta^{-1} \quad \mathrm{S}^{(2)}_{[2]}\Delta^{-1}\right) = \Delta\,(\,\nu_2+\rho_2-1 \quad \nu_1+\rho_1-3\,),$$

(2.2)
$$\mathcal{C}_{-;(-2)}\left(\mathrm{S}^{(2)}_{[0]}\Delta^{-1} \quad \mathrm{S}^{(2)}_{[2]}\Delta^{-1}\right) = \Delta^{-1}\,(\,\nu_2+\rho_2-1 \quad \nu_1+\rho_1-3\,),$$

(2.3)
$$\mathcal{C}_{+;(+2)}\Delta^{-1} = \left(S^{(2)}_{[0]} \quad S^{(2)}_{[2]}\right)\frac{1}{6}\cdot\begin{pmatrix}\nu_1+\rho_1-1 \\ \nu_2+\rho_2-1\end{pmatrix},$$

(2.4)
$$\mathcal{C}_{-;(+2)}\Delta = \left(\mathrm{S}^{(2)}_{[0]} \quad \mathrm{S}^{(2)}_{[2]}\right)\frac{1}{6}\cdot\begin{pmatrix}\nu_2+\rho_2-1 \\ \nu_1+\rho_1-1\end{pmatrix}. \quad □$$

In order to prove the first formula of the statement (ii), we multiply the vector $\begin{pmatrix} 1 \\ -1\end{pmatrix}$ from the right for the both sides of (2.1) in the above Proposition A, to get $\mathcal{C}_{+;(-2)}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = \Delta\{(\nu_2+\rho_2-1)-(\nu_1+\rho_1-3)\}$. Set here $k=2$, $\nu_1 = (\nu_{\mathrm{S}}+k-1)/2$

and $\nu_2 = (\nu_{\mathrm{S}} - k + 1)/2$, then we find that the left side is 0. The second formula of (ii) is shown similarly.

In order to show (i) of our lemma, subtract (2.4) from (2.3) to get

$$\begin{aligned} \mathcal{C}_{+;(+2)}\Delta^{-1} - \mathcal{C}_{-;(+2)}\Delta &= \begin{pmatrix} \mathrm{S}^{(2)}_{[0]} & \mathrm{S}^{(2)}_{[2]} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{6} \cdot \Big\{ (\nu_1 + \rho_1) - (\nu_2 + \rho_2 - 1) \Big\} \\ &= {}^t(f_0 \quad f_1 \quad f_2) \frac{1}{6} \cdot \Big\{ (\nu_1 + \rho_1) - (\nu_2 + \rho_2 - 1) \Big\}. \end{aligned}$$

Rewrite the parameters ν_1 and ν_2 by ν_{S}, then we have $(\nu_1 + \rho_1) - (\nu_2 + \rho_2) = 2$ hence we get (i) in the lemma. □

REMARK. The embedding consider here is corresponding to the map

$$t^{(k)}_{+,0} c_+ \otimes v_+ + t_{-,0} c_- \otimes v_- \mapsto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathrm{Hom}_K(\tau_{(1,-1)}, \pi_0) \cong \mathbb{C}^2$$

(cf. Chapter II, Remark 1.3).

LEMMA 2.3. *Define $\bar{f}_0$, $\bar{f}_1$ and $\bar{f}_2$ as follows :*

$${}^t(\bar{f}_0 \quad \bar{f}_1 \quad \bar{f}_2) = \Big\{ \mathrm{S}^{(2)}_{[0]} + \mathrm{S}^{(2)}_{[2]} \Big\} \Delta^{-1}.$$

Assume that $k = 2$. If $\nu_1 = (\nu_{\mathrm{S}} + 1)/2$ and $\nu_2 = (-\nu_{\mathrm{S}} + 1)/2$, then we have

(i)
$$\mathcal{C}_{+;(+2)}\Delta^{-1} + \mathcal{C}_{-;(+2)}\Delta = \frac{1}{3} \cdot \begin{pmatrix} \bar{f}_0 \\ \bar{f}_1 \\ \bar{f}_2 \end{pmatrix},$$

(ii)
$$\mathcal{C}_{+;(+2)} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = 0 \quad \textit{and} \quad \mathcal{C}_{-;(+2)} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} = 0.$$

PROOF. The proof is similar to that of Lemma 2.2. When we assume $k = 2$, $\nu_1 = (\nu_{\mathrm{S}} + k - 1)/2$ and $-\nu_2 = (\nu_{\mathrm{S}} - k + 1)/2$, we have $(\nu_1 + \rho_1 - 3) + (\nu_2 + \rho_2 - 1) = 0$.

Multiply $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to the right on the both sides of (2.1) and (2.2) in Proposition A, to get the annihilator relations (ii).

To get (i), we add (2.3) and (2.4). □

PROOF OF THEOREM 2.1. The explicit formulas of $\Phi^{(1,1)}$ and $\Phi^{(-1,-1)}$ are given by Ishii([I], p.11 Theorem 2.2). These are the Whittaker realizations of $\Delta^{\pm 1}$. If we calculate the Whittaker realization of the left hand sides of (i) of Lemma 2.2 and Lemma 2.4, then we obtain the power series in Theorem 2.1. Since we have the relation (ii) of either Lemma 2.2 or Lemma 2.4, these power series satisfies the equations (1.2) and (1.3).

Lastly, since the Casimir operator is in the center $Z(\mathfrak{g})$, the annihilation $(C-\chi_\pi(C))Yf^{(l,l)}=0$ holds if we apply some elements Y in $S(\mathfrak{p}_\pm)$. Moreover $\chi_\pi(C)=\chi_\Pi(C))$, if we have an embedding $\Pi\hookrightarrow\pi$. Thus our power series satisfies the Casimir equations (1.1). □

REMARK. The embedding considered here is corresponding to the map

$$t^{(k)}_{+,0}c_+\otimes v_+ + t_{-,0}c_-\otimes v_- \mapsto \begin{pmatrix}1\\1\end{pmatrix}\in \mathrm{Hom}_K(\tau_{(1,-1)},\pi_0)\cong\mathbb{C}^2$$

(cf. Chapter II, Remark 1.3).

THEOREM 2.4. *The case of the embedding* $\sigma=\sigma_0$, $(\nu_1,\nu_2)=((\nu_S^*+1)/2,(\nu_S^*-1)/2)$ *or* $((\nu_S^*+1)/2,-(\nu_S^*+1)/2)$.

Another 4 solutions with characteristic indices (i) and (ii) are as follows :

$$\Phi_0^{(1,-1)}=\gamma a^\rho(y)\cdot(2\pi\sqrt{-1}c_0y_1)\sum_{m,n\geq0}\Big(4n+2\mu_2+4\pi c_3y_2\Big)$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\ m+\frac{\mu_1}{2}+1,\ -m-\frac{\mu_1}{2}\\ \frac{\mu_1}{2}+1,\ \mu_2-\frac{\mu_1}{2}+1\end{matrix}\,\middle|\,1\right)\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n},$$
$$\Phi_1^{(1,-1)}=\gamma a^\rho(y)\sum_{m,n\geq0}\Big(2\mu_2(\mu_1-\mu_2)+16\pi^2c_3^2y_2^2\Big)$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\ m+\frac{\mu_1}{2}+1,\ -m-\frac{\mu_1}{2}\\ \frac{\mu_1}{2}+1,\ \mu_2-\frac{\mu_1}{2}+1\end{matrix}\,\middle|\,1\right)\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}$$

and

$$\Phi_2^{(1,-1)}=\gamma a^\rho(y)\cdot(2\pi\sqrt{-1}c_0y_1)\sum_{m,n\geq0}\Big(4n+2\mu_2-4\pi c_3y_2\Big)$$
$$\times\,{}_3F_2\left(\begin{matrix}-n,\ m+\frac{\mu_1}{2}+1,\ -m-\frac{\mu_1}{2}\\ \frac{\mu_1}{2}+1,\ \mu_2-\frac{\mu_1}{2}+1\end{matrix}\,\middle|\,1\right)\frac{(\pi y_1)^{2m+\mu_1}(\pi y_2)^{2n+\mu_2}}{m!n!(\mu_1+\mu_2+1)_m(\mu_2+1)_n}.$$

Here (μ_1,μ_2) *is either* $(\nu_S^*/2+1/2,\nu_S^*/2)$ *or* $(\nu_S^*/2+1/2,1/2)$, *and* $\gamma=(\nu_S^*+5)(\nu_S^*+1)/4$.

PROOF. We again quote special cases of Theorem 5.1 of [O2].

PROPOSITION B. *Assume that* ν_1, ν_2 *are generic and* $\sigma = \sigma_0$. *Let* Δ^0 *be the canonical generator of* $\tau_{(0,0)}$, $S^{(2)}_{[0]}$, $S^{(2)}_{[2]}$ *be two canonical basis of* $\tau_{(2,0)}$ *inside* H_π, $S^{(2)}_{[0]}\Delta^{-2}$, $S^{(2)}_{[2]}\Delta^{-2}$ *be the canonical basis of* $\tau_{(0,-2)}$ *inside* H_π. *Moreover let* $S^{(2)}_{[1]}\Delta^{-1}$ *be the canonical basis of* $\tau_{(1,-1)}$ *inside* H_π.

Let $\mathcal{C}_{+;(+2)}$ *and* $\mathcal{C}_{-;(+2)}$ *be the same symbols as in Lemma 2.2, and*

$$\mathcal{C}_{+;(0)} = \begin{pmatrix} -X_{+12} & X_{+11} & 0 \\ -\frac{1}{2}X_{+22} & 0 & \frac{1}{2}X_{+11} \\ 0 & -X_{+22} & X_{+12} \end{pmatrix}$$

and

$$\mathcal{C}_{-;(0)} = \begin{pmatrix} X_{-12} & X_{-22} & 0 \\ -\frac{1}{2}X_{-11} & 0 & \frac{1}{2}X_{-22} \\ 0 & -X_{-11} & -X_{-12} \end{pmatrix},$$

respectively. Then we have

(i)
$$\mathcal{C}_{-;(+2)}\Delta^0 = \left(S^{(2)}_{[0]}\Delta^{-2} \quad S^{(2)}_{[2]}\Delta^{-2} \right) \frac{1}{6} \cdot \begin{pmatrix} \nu_2+\rho_2 \\ \nu_1+\rho_1 \end{pmatrix},$$
$$\mathcal{C}_{+;(+2)}\Delta^0 = \left(S^{(2)}_{[0]}\Delta^{0} \quad S^{(2)}_{[2]}\Delta^{0} \right) \frac{1}{6} \cdot \begin{pmatrix} \nu_1+\rho_1 \\ \nu_2+\rho_2 \end{pmatrix},$$

and

(ii)
$$\mathcal{C}_{-;(0)} \left(S^{(2)}_{[0]}\Delta^{0} \quad S^{(2)}_{[2]}\Delta^{0} \right) = \frac{1}{2} \left(-(\nu_2-1) \quad \nu_1 \right) S^{(2)}_{[1]}\Delta^{-1},$$
$$\mathcal{C}_{+;(0)} \left(S^{(2)}_{[0]}\Delta^{-2} \quad S^{(2)}_{[2]}\Delta^{-2} \right) = \frac{1}{2} \left(-\nu_1 \quad (\nu_2-1) \right) S^{(2)}_{[1]}\Delta^{-1}.$$

with $(\rho_1, \rho_2) = (2,1)$. $\square$

By composing the operators in the above Proposition B, we have the formulas :

(2.4)
$$\mathcal{C}_{+;(0)} \circ \mathcal{C}_{-;(+2)}\Delta^0 = \begin{pmatrix} -(X_{+12}X_{-22} + X_{+11}X_{-12}) \\ \frac{1}{2}(X_{+11}X_{-11} - X_{+22}X_{-22}) \\ (X_{+22}X_{-12} + X_{+12}X_{-11}) \end{pmatrix} \Delta^0 = \gamma_- S^{(2)}_{[1]}\Delta^{-1},$$

and

(2.5)
$$\mathcal{C}_{-;(0)} \circ \mathcal{C}_{+;(+2)}\Delta^0 = \begin{pmatrix} X_{-12}X_{+11} + X_{-22}X_{+12} \\ -\frac{1}{2}X_{-11}X_{+11} + \frac{1}{2}X_{-22}X_{+22} \\ -(X_{-11}X_{+12} + X_{+12}X_{-11}) \end{pmatrix} \Delta^0 = \gamma_+ S^{(2)}_{[1]}\Delta^{-1},$$

with

$$\gamma_- = \frac{1}{2}(\nu_2^2 - 1) + \frac{1}{2}\nu_1(\nu_1+2) \text{ and } \gamma_+ = -\frac{1}{2}\nu_1(\nu_1+2) + \frac{1}{2}(\nu_2^2-1).$$

Here we specialize the parameter (ν_1, ν_2) so that either $(\nu_1, \nu_2) = ((\nu_S^* + 1)/2, (\nu_S^* - 1)/2)$ or $((\nu_S^* + 1)/2, -(\nu_S^* + 1)/2)$. Then $\gamma_- = \gamma_+ = \frac{1}{4}(\nu_S^* + 5)(\nu_S^* + 1) \neq 0$.

Now using the Iwasawa decomposition for (2.4), we have

$$\begin{pmatrix} E_{e_1-e_2}(H_{2,4} + H_{1,3}) - 2\sqrt{-1}E_{e_1-e_2}E_{2e_2} \\ H_{1,3}^2 - H_{2,4}^2 - 4E_{2e_2}^2 \\ E_{e_1-e_2}(H_{2,4} + H_{1,3}) + 2\sqrt{-1}E_{e_1-e_2}E_{2e_2} \end{pmatrix} \Delta^0 = \frac{1}{4}(\nu_S^* + 5)(\nu_S^* + 1)S_{[1]}^{(2)}\Delta^{-1}.$$

Then we compute their Whittaker realization similarly as Theorem 2.1, to get Theorem. By the definition of $\mathcal{C}_{+;0} \circ \mathcal{C}_{+;(+2)}\Delta^0$, the Whittaker realization of this vector satisfies the annihilator relations (1.2) and (1.3) obviously. Secondary, because the element $C - \chi_\Pi(C) \in Z(\mathfrak{g})$, we have $\{C - \chi_\Pi(C)\}\mathcal{C}_{+;(0)} \circ \mathcal{C}_{+;(+2)}\Delta^0 = \mathcal{C}_{+;(0)} \circ \mathcal{C}_{+;(+2)}\{C - \chi_\Pi(C)\}\Delta^0 = 0$. Passing to the Whittaker realization, this yields the equation (1.1) of Theorem 1.1. □

REMARK. Starting from (2.5) in place of (2.4), we have the same result.

The last theorem gives 4 linearly independent power series solutions for our holonomic system in Proposition 1.1, and Theorem 2.1 gives another 4 linearly independent solutions. Therefore we have 8 linear independent solutions. Here linear independence is obvious from the characteristic indices and the parity of non-vanishing monomials in the power series. We note here that our system has no solution with logarithmetic singularity along $y_1 = 0$ and $y_2 = 0$.

REMARK. The case of general l with $k = 2$ is handled in the same by our results in Chapter I. But we omit that, because the formulas are more complicated. For small k like $k = 4$, which is interesting from a geometric view point, we still can extend similar computations by force.

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