Coefficient inverse problems for partial differential equations in the viscoelasticity, the material science and population dynamics by Carleman estimates (カーレマン評価を用いた、粘弾性論・材料科学・個体群動態学における偏微分方程式系の係数決定逆問題について)

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Abstract

In this paper, we consider coefficient inverse problems in the viscoelasticity, the material science and the population studies and prove the stability of these problem by an *a priori* weighted L^2-norm estimate which is called a Carleman estimate.

In Chapter 1, an inverse problem of determining coefficients in a viscoelastic model which is called Kelvin-Voigt model is discussed. The data available to us is a Cauchy data on subboundary. We prove that with two appropriate measurements, we can obtain a Hölder stability estimate of the inverse problem.

In Chapter 2, we discuss the determination of a thermal conductivity and a mobility in the linearized phase field model with measurement of only one component in a small domain. Our result is the Lipschitz stability estimate of this problem.

In Chapter 3, we consider the coefficient inverse problem of the structured population model. In the structured population model, an age and an individual size as well as a spatial position and time are considered as indepenent variables and then the equation has a special form. We prove a Carleman estimate for this equation and obtain a stability estimate for the inverse problem.

Contents

Chapter 1

Inverse problems for some system of viscoelasticity via Carleman estimates

1.1 Introduction

In recent decades non-invasive measurement methods of the viscoelastic properties of the human body (so-called the elastography) have been developed. These properties give important information for a diagnosis of diseases such as cancers. The elastography finds the viscoelastic properties by measuring propagation of the mechanical wave in human body. In these points of view, it is very important to consider inverse problems for a viscoelasticity system, which is our main subject in this paper.

There are many kinds of the models for viscoelastic materials and the simplest model is so-called the Kelvin-Voigt model. This model is used frequently in elastography (e.g [4, 25]).

Here and henceforth, t and $x = (x_1, x_2, x_3)$ denote the time variable and spatial variable respectively and we write $x' := (x_2, x_3)$ in short. Derivatives with respect to t or x are denoted as follows:

$$\partial_t := \frac{\partial}{\partial t},\ \partial_j := \frac{\partial}{\partial x_j}.$$

Moreover for a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{N} \setminus \{0\})^3$, we set $\partial_x^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$. $\cdot^T$ denotes a transpose of a matrix.

We now state the equation of Kelvin-Voigt model. Let Ω be a domain in $\mathbb{R}^3$ with a smooth boundary and $T > 0$ fixed. First, in general, the governing equation of continuum is

$$\rho(x)\partial_t^2 u_j(x,t) - \sum_{k=1}^{3} \partial_k \sigma_{jk}(x,t) = \mathrm{f}(x,t)\ (x,t) \in \Omega \times (0,T) \tag{1.1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T \in \mathbb{R}^3$ is displacement and $(\sigma_{ik})_{i,k=1,2,3}$ is a stress tensor. In the Kelvin-Voigt model, we suppose that the stress tensor can be written as follows:

$$\sigma_{jk} = \sum_{l,m=1}^{3} \mu_{jlkm}\varepsilon_{lm} + \sum_{l,m=1}^{3} \eta_{jlkm}\partial_t\varepsilon_{lm} \tag{1.1.2}$$

where $(\varepsilon_{lm})_{l,m=1,2,3}$ is a strain tensor:

$$\varepsilon_{lm} = \frac{1}{2}\left(\partial_l u_m + \partial_m u_l\right) \tag{1.1.3}$$

and μ_{jklm} and η_{jklm} is called an elasticity tensor and a viscosity tensor. Relation (1.1.2) is called a constitutive relation. If we additionally assume that the material is isotropic, then we can write equation (1.1.2) as

$$\sigma_{jk} = 2\mu\varepsilon_{jk} + \lambda\delta_{jk}\sum_{l=1}^{3}\varepsilon_{ll} + 2\eta\partial_t\varepsilon_{jk} + \gamma\delta_{jk}\sum_{l=1}^{3}\partial_t\varepsilon_{ll}, \tag{1.1.4}$$

where

$$\delta_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

By (1.1.1), (1.1.3) and (1.1.4), we obtain the equation of the Kelvin-Voigt model:

$$\begin{aligned} &\rho(x)\partial_t^2 u_j(x,t) \\ &-\sum_{k=1}^{3}\partial_k\left[\mu(x)\varepsilon_{jk}(x,t) + \lambda(x)\delta_{jk}\nabla\cdot u(x,t) + \eta(x)\partial_t\varepsilon_{jk}(x,t) + \gamma(x)\delta_{jk}\nabla\cdot(\partial_t u)(x,t)\right] \\ &= f_j(x,t). \end{aligned} \tag{1.1.5}$$

The equation (1.1.5) can be written as the following shorter form:

$$P\mathbf{u}(x,t) := \rho(x)\partial_t^2\mathbf{u}(x,t) - L_{\lambda,\mu}\mathbf{u}(x,t) - L_{\eta,\gamma}(\partial_t\mathbf{u})(x,t) = \mathbf{f}(x,t), \tag{1.1.6}$$

where

$$L_{\lambda,\mu}\mathbf{u} := \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla\mathrm{div}\,\mathbf{u} + (\mathrm{div}\,\mathbf{u})\nabla\lambda + \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\nabla\mu \tag{1.1.7}$$

and $\nabla\mathbf{u} = (\partial_k u_j)_{1\leq j,k\leq 3}$.

We now consider the following initial value problem

$$\begin{cases} P\mathbf{u}(x,t) = \mathbf{f}(x,t) & (x,t) \in \Omega\times(-T,T) \\ u(x,0) = \mathbf{p}(x) & x \in \Omega \\ \partial_t u(x,0) = \mathbf{q}(x) & x \in \Omega \end{cases}. \tag{1.1.8}$$

Let $\mathbf{u} = \mathbf{u}(\lambda, \mu, \eta, \gamma, \rho; \mathbf{p}, \mathbf{q})$ satisfy (1.1.8).

For the problem (1.1.8), We formulate the inverse problems as follows:

Inverse Problem Let Γ be a open subset of $\partial\Omega$, and $\mathbf{p}^{(k)}, \mathbf{q}^{(k)}$ $1 \le k \le N$ be appropriately given. Then determine the coefficients $\lambda, \mu, \eta, \gamma, \rho$ by

$$\mathbf{u}(\lambda, \mu, \eta, \gamma, \rho; \mathbf{p}^{(k)}, \mathbf{q}^{(k)})|_{\Gamma\times(-T,T)}, \ \partial_\nu \mathbf{u}(\lambda, \mu, \eta, \gamma, \rho; \mathbf{p}^{(k)}, \mathbf{q}^{(k)})|_{\Gamma\times(-T,T)}.$$

To state main results, we give some notations. Let us assume $\overline{\Omega_0} \subset \Omega \cup \overline{\Gamma}$. For fixed smooth functions $\lambda_0, \mu_0, \gamma_0, \eta_0$, we set

$$\mathcal{W} := \Big\{(\lambda, \mu, \gamma, \eta, \rho) \in \{C^5(\overline{\Omega})\}^5 ; \lambda, \mu, \gamma, \eta, \rho > 0 \,\mathrm{in}\, \Omega$$
$$\rho = \rho_0, \lambda = \lambda_0, \mu = \mu_0, \gamma = \gamma_0, \eta = \eta_0 \text{ on } \Gamma\Big\}. \quad (1.1.9)$$

Let I_n be the $n \times n$ identity matrix and $\{\mathbf{a}\}_j$ denote the matrix or vector obtained from $\mathbf{a}$ by deleting the jth row. Especially for an $(n+1)\times n$ matrix A, $\det_j A$ mean $\det\{A\}_j$.

Now we are ready to state the main theorem:

Theorem 1.1.1. *Let $\omega \subset \overline{\Omega} \cup \Gamma$ be an arbitrary subdomain with* $\mathrm{dist}(\omega, \partial\Omega \setminus \Gamma) > 0$ *and $T > 0$ be arbitrary. For $\mathbf{p}^{(k)} = (p_1^{(k)}, p_2^{(k)}, p_3^{(k)})^T$ and $\mathbf{q}^{(k)} = (q_1^{(k)}, q_2^{(k)}, q_3^{(k)})(k = 1, 2)$, we assume that there exists $j_1, \ldots, j_4 \in \{1, 2, \ldots, 12\}$ such that for any point $x \in \Omega_0$,*

$$\begin{aligned}
&\det{}_{j_1}(\mathbf{a}(x), B(x), D(x), G(x), H(x)(A^2, x')^T) \neq 0,\\
&\det{}_{j_2}(\mathbf{a}(x), B(x), D(x), H(x), G(x)(A^2, x')^T) \neq 0,\\
&\det{}_{j_3}(\mathbf{a}(x), G(x), H(x), D(x), B(x)(A^2, x')^T) \neq 0,\\
&\det{}_{j_4}(\mathbf{a}(x), G(x), H(x), B(x), D(x)(A^2, x')^T) \neq 0,
\end{aligned} \quad (1.1.10)$$

where we denote

$$
\begin{cases}
\mathbf{a} := \begin{pmatrix} -\frac{1}{\rho}\left(L_{\mu,\lambda}\mathbf{p}^{(1)} + L_{\eta,\gamma}\mathbf{q}^{(1)}\right) \\ -\frac{1}{\rho}\left(L_{\mu,\lambda}\mathbf{p}^{(2)} + L_{\eta,\gamma}\mathbf{q}^{(2)}\right) \\ -\frac{1}{\rho}\left(L_{\mu,\lambda}\mathbf{q}^{(1)} + L_{\eta,\gamma}\mathbf{r}^{(1)}\right) \\ -\frac{1}{\rho}\left(L_{\mu,\lambda}\mathbf{q}^{(2)} + L_{\eta,\gamma}\mathbf{r}^{(2)}\right) \end{pmatrix}, \\
B := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = \begin{pmatrix} \operatorname{div}\mathbf{p}^{(1)} I_3 \\ \operatorname{div}\mathbf{p}^{(2)} I_3 \\ \operatorname{div}\mathbf{q}^{(1)} I_3 \\ \operatorname{div}\mathbf{q}^{(2)} I_3 \end{pmatrix}, D := (\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) = \begin{pmatrix} \operatorname{div}\mathbf{q}^{(1)} I_3 \\ \operatorname{div}\mathbf{q}^{(2)} I_3 \\ \operatorname{div}\mathbf{r}^{(1)} I_3 \\ \operatorname{div}\mathbf{r}^{(2)} I_3 \end{pmatrix}, \\
G := (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) = \begin{pmatrix} \nabla\mathbf{p}^{(1)} + (\nabla\mathbf{p}^{(1)})^T \\ \nabla\mathbf{p}^{(2)} + (\nabla\mathbf{p}^{(2)})^T \\ \nabla\mathbf{q}^{(1)} + (\nabla\mathbf{q}^{(1)})^T \\ \nabla\mathbf{q}^{(2)} + (\nabla\mathbf{q}^{(2)})^T \end{pmatrix}, H := (\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3) = \begin{pmatrix} \nabla\mathbf{q}^{(1)} + (\nabla\mathbf{q}^{(1)})^T \\ \nabla\mathbf{q}^{(2)} + (\nabla\mathbf{q}^{(2)})^T \\ \nabla\mathbf{r}^{(1)} + (\nabla\mathbf{r}^{(1)})^T \\ \nabla\mathbf{r}^{(2)} + (\nabla\mathbf{r}^{(2)})^T \end{pmatrix}, \\
\mathbf{r}^{(k)}(x) := \partial_t^2 \mathbf{u}^{(k)}(x, 0) = L_{\lambda,\mu}\mathbf{p}^{(k)} + L_{\gamma,\eta}\mathbf{q}^{(k)}.
\end{cases}
\tag{1.1.11}
$$

Set $\mathbf{u}_l^{(k)} := \mathbf{u}\left(\lambda_l, \mu_l, \eta_l, \gamma_l, \rho_l; \mathbf{p}^{(k)}, \mathbf{q}^{(k)}\right)(l = 1, 2,\ k = 1, 2)$. *Then there exist constants*

$$C = C(\mathcal{W}, \Gamma, \Omega, \omega, \lambda, \mu, \gamma, \eta, \rho) > 0$$

and

$$\kappa = \kappa(\mathcal{W}, \Gamma, \Omega, \omega, \lambda, \mu, \gamma, \eta, \rho) \in (0, 1)$$

such that

$$
\begin{aligned}
\|\lambda_2 - \lambda_1\|_{H^2(\omega)} + \|\mu_2 - \mu_1\|_{H^2(\omega)} + \|\gamma_2 - \gamma_1\|_{H^2(\omega)} + \|\eta_2 - \eta_1\|_{H^2(\omega)} + \|\rho_2 - \rho_1\|_{H^1(\omega)} \\
\le C\left(M^{1-\kappa}B^\kappa + B\right)
\end{aligned}
\tag{1.1.12}
$$

provided that $(\lambda_j, \mu_j, \gamma_j, \eta_j, \rho_j) \in \mathcal{W}$ *and*

$$M := \sum_{j=1,2} \sum_{k=1,2} \|\mathbf{u}_j^{(k)}\|_{H^5(-T,T;H^2(\Omega))} < \infty,$$

where we set

$$B := \sum_{k=1,2} \left(\|\mathbf{u}_2^{(k)} - \mathbf{u}_1^{(k)}\|_{H^5(-T,T;H^2(\Gamma))} + \|\mathbf{u}_2^{(k)} - \mathbf{u}_1^{(k)}\|_{H^6(-T,T;H^1(\Gamma))} \right)$$

In Theorem 2.1.1 we establish the local Hölder stability by two times measurements on the part Γ of boundary if the initial values can be chosen appropriately, that is, as satisfying the assumptions (1.1.10).

By changing the constitutive relation (1.1.2), we can consider many kinds of models for viscoelastic materials as well as the Kelvin-Voigt model. Especially, major one is the model which has the following constitutive relation:

$$\sigma_{jk}(x,t) = \sum_{l,m} \mu_{jklm}(x)\varepsilon_{lm}(x,t) + \int_\theta^t \sum_{l,m} \eta_{jklm}(x,\tau)\varepsilon_{lm}(x,t-\tau)\,d\tau$$

or a simplified hyperbolic integro-differential equation

$$\partial_t^2 u - \nabla\cdot(\mu\nabla u) - \int_0^t \eta(\cdot,\tau)u(x,t-\tau)\,d\tau = F(x,t).$$

We can refer as the inverse problems for the equation of this type to Cavaterra, Lorenzi and Yamamoto [5], Grasselli [10, 11], Janno and Wolfersdorf [17, 18], Lorenzi, Messina and Romanov [20] and Lorenzi, Ulekova, and Yakhno [21]. In these, [10, 11, 17, 18, 20, 21] are related to the reconstruction of the coefficients and the integral kernel K and [5] to the stability estimate of the inverse source problem via Carleman estimate. Our system is, however, essentially different since equation (1.1.6) does not have the integral term and has 3rd order differential term $\Delta\partial_t u$. Moreover, our proof is based upon the fact that we can derive a *parabolic* integro-differential equation from (1.1.6).

As far as inverse problems for system (1.1.5) with $\gamma\equiv 0$, $\eta\equiv 0$, that is, non-stationary Lamé systems is concerned, we notice that in [15, 16] Carleman estimates are established to prove the uniqueness and stability of the Lamé coefficients.

In this paper, we prove Theorem 2.1.1 as follows. First we prove a Carleman estimate of P and prove Theorem 2.1.1 when ω is a part of a paraboloid. Next, we prove the theorem for general $\omega$by using the stability estimate of unique continuation for the equation $P\mathbf{u} = \mathbf{f}$.

1.2 Proof of the theorem

1.2.1 Case I: ω is a paraboloid.

First we consider the case that ω is a part of a paraboloid.

We remark that we can assume that the axis of the paraboloid is parallel to x_1. Indeed, a rotation does not change a form of the operator P up to the coefficients μ,η,γ,λ and ρ.

Let us define the function

$$\psi(x,t) := x_1 + \frac{|x'|^2}{2A^2} + \frac{t^2}{2T^2} + \frac{1}{4}$$

and the domain

$$G_\delta := \left\{(x,t); \psi(x,t) < \frac{3}{4} - \delta\right\}, \quad \delta > 0.$$

and

$$\Omega_\delta := G_\delta \cap \{t = 0\}.$$

We assume $G_0 \subset \Omega \times (-T, T) \cup \Gamma$.

Under these notations, we state the Carleman estimate of the operator P.

Lemma 1.2.1. *There exist $s_0 > 0$, $\nu > 0$ and $C = C(s_0) > 0$ which are independent of $\mathbf{u}$ such that*

$$\begin{aligned}\int_{G_0} \Bigg[& s^4 \left(|\mathbf{u}|^2 + |\operatorname{div}\mathbf{u}|^2 + |\operatorname{rot}\mathbf{u}|^2\right) \\ & + s^2 \left(|\nabla \mathbf{u}|^2 + |\nabla \operatorname{div}\mathbf{u}|^2 + |\nabla \operatorname{rot}\mathbf{u}|^2\right) + \sum_{|\alpha|=2} |\partial_x^\alpha \mathbf{u}|^2 \Bigg] e^{2s\psi^{-\nu}}\, dx\, dt \\ & \qquad \le \int_{G_0} \left(|\mathbf{f}|^2 + |\operatorname{div}\mathbf{f}|^2 + |\operatorname{rot}\mathbf{f}|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\ & \qquad + Ce^{Cs}\left(\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\Gamma))} + \|\mathbf{u}\|^2_{H^2(-T,T;H^1(\Gamma))}\right). \end{aligned} \tag{1.2.1}$$

holds for any $s \ge s_0$ and for any $\mathbf{u} \in H^3(G_0)$ such that

$$P\mathbf{u} = \mathbf{f}, \quad \mathbf{u}(\cdot, 0) = \partial_t \mathbf{u}(\cdot, 0) = 0$$

and $\mathbf{u} = 0$ on $\partial G_0 \setminus (\partial\Omega \times (-T, T))$.

Proof.

We use the following Carleman estimate for the parabolic operator.

Lemma 1.2.2. *Let $\nu > 0$ be sufficiently large. Then there exist constants $s_0 > 0$ and $C, C_0 > 0$ such that for any $s > s_0$,*

$$\begin{aligned} & |\left(\rho(x)\partial_t - p(x)\Delta\right) w(x,t)|^2 e^{2s\psi^{-\nu}} \\ & \quad \le \frac{C_0}{s} \sum_{|\alpha|=2} |\partial_x^\alpha w(x,t)|^2 + C\left(s|\nabla w(x,t)|^2 + s^3 |w(x,t)|^2\right) e^{2s\psi^{-\nu}} \\ & \qquad\qquad + \left(\nabla U(x,t) + \partial_t V(x,t)\right) \quad (x,t) \in G_0 \end{aligned} \tag{1.2.2}$$

where the vector function (U, V) satisfies the estimate

$$|(U,V)| \le \frac{C_0}{s} \sum_{|\alpha|=2} |\partial_x^\alpha w(x,t)|^2 + Cs^3\left(|\partial_t w|^2 + |\nabla w|^2 + |w|^2\right) e^{2s\psi^{-\nu}}. \tag{1.2.3}$$

Moreover, this estimate also holds when $C_0 = 0$.

The proof of this theorem is referred to Klibanov and Timonov [19].

We first prove the Carleman estimate of an operator of the following type:

$$Lu = \rho \partial_t^2 u - p\Delta u - q\Delta \partial_t u$$

Lemma 1.2.3. *There exist $s_0 > 0$ and $C > 0$ such that for any $s > s_0$,*

$$\begin{aligned}&\int_{G_0} \left(s^4|u|^2 + s^2|\nabla u|^2\right) e^{2s\psi^{-\nu}}\,dx\,dt\\ &\le \int_{G_0} |Lu|^2 e^{2s\psi^{-\nu}}\,dx\,dt + Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\partial_t^2 u|^2 + |\nabla\partial_t u|^2 + |\partial_t u|^2\right) dS\end{aligned} \tag{1.2.4}$$

holds for any $u \in H^2(G_0)$ which vanishes on $\partial G_0 \setminus \partial\Omega$.
If $u \in H^3(G_0)$, then for any $s > s_0$,

$$\begin{aligned}&\int_{G_0} \left(s^4|u|^2 + s^2|\nabla u|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha u|^2\right) e^{2s\psi^{-\nu}}\,dx\,dt\\ &\le \int_{G_0} |Lu|^2 e^{2s\psi^{-\nu}}\,dx\,dt + Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\partial_t^2 u|^2 + \sum_{|\alpha|\le 2} |\partial_x^\alpha\partial_t u|^2\right) dS\end{aligned} \tag{1.2.5}$$

also holds.

Proof of Lemma 1.2.3 Let us set $F = Lu$. By setting $v = \partial_t u$, we have the integro-differential equation

$$\partial_t v - p\Delta u - \int_0^t q\Delta v(\cdot,\tau)\,d\tau = F. \tag{1.2.6}$$

We now introduce the new auxiliary function

$$w := pv + \int_0^t qv\,d\tau. \tag{1.2.7}$$

The simple calculation shows

$$\partial_t w = p\partial_t v + qv$$

and

$$\Delta w = p\Delta v + 2\nabla p\cdot\nabla v + v\Delta p + \int_0^t \left(q\Delta v + 2\nabla q\cdot\nabla v + v\Delta q\right)\,d\tau, \tag{1.2.8}$$

thus we see that w solves the equation

$$\partial_t w - p\Delta w = pF - L(v) \tag{1.2.9}$$

where

$$L(v) := 2\nabla p\cdot\nabla v + v\Delta p + \int_0^t \left(2\nabla q\cdot\nabla v + v\Delta q\right)\,d\tau. \tag{1.2.10}$$

By applying Lemma 1.2.2 to (1.2.9) and integrating on G_0, we obtain

$$\begin{aligned}\int_{G_0} \left(s|\nabla w|^2 + s^3|w|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\ \le C\int_{G_0} |pF|^2 e^{2s\psi^{-\nu}}\, dx\, dt + C\int_{G_0} |L(v)|^2 e^{2s\psi^{-\nu}}\, dx\, dt \\ + Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\nabla v|^2 + |\partial_t v|^2 + |v|^2\right) dS. \quad (1.2.11)\end{aligned}$$

Now we need to show the following lemma.

Lemma 1.2.4. *For all $f \in L^2(G_0)$,*

$$\int_{G_0} \left(\int_0^t |f(x,\tau)|\, d\tau\right)^2 e^{2s\psi^{-\nu}}\, dx\, dt \le \frac{C}{s}\int_{G_0} |f(x,\tau)|^2 e^{2s\psi^{-\nu}}\, dx\, dt. \quad (1.2.12)$$

Thanks to the factor $1/s$ in Lemma 1.2.4, we can estimate the integral term. We note that this lemma is proved in Cavaterra, Lorenzi and Yamamoto [5]. Lemma 1.2.4 will be proved in Appendix A.

By (1.2.7), We have

$$v = \frac{1}{p}w - \frac{q}{p}\int_0^t v\, d\tau.$$

Hence, by owing to Lemma (1.2.4), we have

$$\int_{G_0} |v|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le C\int_{G_0} |w|^2 e^{2s\psi^{-\nu}} dxdt + \frac{C}{s}\int_{G_0} |v|^2 e^{2s\psi^{-\nu}}\, d\tau.$$

For sufficiently large $s \ge s_0$, the second term on the right hand side is absorbed into the left hand side, and we obtain

$$\int_{G_0} |v|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le C\int_{G_0} |w|^2 e^{2s\psi^{-\nu}} dxdt \quad (1.2.13)$$

for any $s \ge s_0$. Similarly, we also have

$$\int_{G_0} |\nabla v|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le C\int_{G_0} \left(|\nabla w|^2 + |w|^2\right) e^{2s\psi^{-\nu}} dxdt \quad (1.2.14)$$

for any $s \ge s_0$. Moreover Lemma 1.2.4 yields the estimate of $L(v)$:

$$\int_{G_0} |L(v)|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le C\int_{G_0} \left(|\nabla v|^2 + |v|^2\right) e^{2s\psi^{-\nu}} dxdt. \quad (1.2.15)$$

By combining (1.2.11), (1.2.13), (1.2.14) and (1.2.15), we obtain

$$
\begin{aligned}
&\int_{G_0} \left(s|\nabla v|^2 + s^3|v|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\le \int_{G_0} \left(s|\nabla w|^2 + s^3|w|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\le C\int_{G_0} |F|^2 e^{2s\psi^{-\nu}}\, dx\, dt + \int_{G_0} |L(v)|^2 e^{2s\psi^{-\nu}} dxdt \\
&+ Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\nabla v|^2 + |\partial_t v|^2 + |v|^2\right) dS \\
&\le C\int_{G_0} |F|^2 e^{2s\psi^{-\nu}}\, dx\, dt + C\int_{G_0} \left(|\nabla v|^2 + |v|^2\right) e^{2s\psi^{-\nu}} dxdt \\
&+ Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\nabla v|^2 + |\partial_t v|^2 + |v|^2\right) dS.
\end{aligned}
$$

By taking $s > 0$ sufficient large, we absorb the second term on the right hand side into the left hand side, and obtain

$$
\begin{aligned}
&\int_{G_0} \left(s|\nabla v|^2 + s^3|v|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad \le C\int_{G_0} |F|^2 e^{2s\psi^{-\nu}}\, dx\, dt + Ce^{Cs}\int_{\Gamma\times[-T,T]} \left(|\nabla v|^2 + |\partial_t v|^2 + |v|^2\right) dS.
\end{aligned} \tag{1.2.16}
$$

By applying Lemma 1.2.4 to

$$
u(x,t) = \int_0^t v(x,\tau)\, d\tau,
$$

we have

$$
\int_{G_0} |u|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le \frac{C}{s}\int_{G_0} |v|^2 e^{2s\psi^{-\nu}} dxdt \tag{1.2.17}
$$

and

$$
\int_{G_0} |\nabla u|^2 e^{2s\psi^{-\nu}}\, dx\, dt \le \frac{C}{s}\int_{G_0} |\nabla v|^2 e^{2s\psi^{-\nu}}\, dx\, dt. \tag{1.2.18}
$$

Then we obtain

$$
\int_{G_0} \left(s^2|\nabla u|^2 + s^4|u|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \le \int_{G_0} \left(s|\nabla u|^2 + s^3|u|^2\right) e^{2s\psi^{-\nu}} dxdt. \tag{1.2.19}
$$

Combining (1.2.16) with (1.2.19), we complete the proof of the first inequality of Lemma 1.2.3. The second inequality of Lemma 1.2.3 can be proven in similar ways.

We now prove Lemma 1.2.1. By setting $v = \operatorname{div}\mathbf{u}$ and $\mathbf{w} = \operatorname{rot}\mathbf{u}$, we have the following system of equations of $\mathbf{u}$, v, $\mathbf{w}$:

$$\rho\partial_t^2\mathbf{u} - \mu\Delta\mathbf{u} - \eta\Delta\partial_t\mathbf{u} + L_1(\mathbf{u}, v, \partial_t\mathbf{u}, \partial_t v) = \mathbf{f}$$
$$\rho\partial_t^2 v - (\lambda + 2\mu)\Delta v - (\gamma + 2\eta)\Delta\partial_t v + L_2(\mathbf{u}, v, \mathbf{w}, \partial_t\mathbf{u}, \partial_t v, \partial_t\mathbf{w}) = \operatorname{div}\mathbf{f}$$
$$\rho\partial_t^2\mathbf{w} - \mu\Delta\mathbf{w} - \eta\Delta\partial_t\mathbf{w} + L_1(\mathbf{u}, v, \mathbf{w}, \partial_t\mathbf{u}, \partial_t v, \partial_t\mathbf{w}) = \operatorname{rot}\mathbf{f}$$

where L_1,L_2,L_3 represent linear combination with $L^\infty(G_0)$-coefficients. We apply Lemma 1.2.3 to this system and obtain

$$\int_{G_0}\Bigg[s^4\left(|\mathbf{u}|^2 + |\operatorname{div}\mathbf{u}|^2 + |\operatorname{rot}\mathbf{u}|^2\right)$$
$$+s^2\left(|\nabla\mathbf{u}|^2 + |\nabla\operatorname{div}\mathbf{u}|^2 + |\nabla\operatorname{rot}\mathbf{u}|^2\right) + \sum_{|\alpha|=2}|\partial_x^\alpha\mathbf{u}|^2\Bigg]e^{2s\psi^{-\nu}}\,dx\,dt$$
$$\le \int_{G_0}\left(|\mathbf{f}|^2 + |\operatorname{div}\mathbf{f}|^2 + |\operatorname{rot}\mathbf{f}|^2\right)e^{2s\psi^{-\nu}}\,dx\,dt$$
$$+ Ce^{Cs}||\mathbf{u}||^2_{H^3(\Gamma\times[-T,T])}. \quad (1.2.20)$$

Then the proof of Theorem 1.2.1 is complete.

Proof of Theorem 2.1.1 when ω is a paraboloid

Here and henceforth, we set

$$\mathbf{u}_j^{(k)} = \mathbf{u}(\lambda_j, \mu_j, \eta_j, \gamma_j, \rho_j; \mathbf{p}^{(k)}, \mathbf{q}^{(k)}), \quad j = 1, 2,\ k = 1, 2,$$

and

$$\widetilde{\mathbf{u}}^{(k)} = \mathbf{u}_2^{(k)} - \mathbf{u}_1^{(k)},\ \widetilde{\rho} = \rho_2 - \rho_1,\ \widetilde{\lambda} = \lambda_2 - \lambda_1,\ \widetilde{\mu} = \mu_2 - \mu_1,$$
$$\widetilde{\eta} = \eta_2 - \eta_1,\ \widetilde{\gamma} = \gamma_2 - \gamma_1.$$

Then we have

$$\rho_2\partial_t^2\widetilde{\mathbf{u}}^{(k)} = L_{\lambda_2,\mu_2}\widetilde{\mathbf{u}}^{(k)} + L_{\eta_2,\gamma_2}\partial_t\widetilde{\mathbf{u}}^{(k)} + G\mathbf{u}_1^{(k)}, \quad (x,t) \in \Omega\times(-T,T) \quad (1.2.21)$$
$$\widetilde{\mathbf{u}}^{(k)}(x,0) = \partial_t\widetilde{\mathbf{u}}^{(k)}(x,0) = 0, \quad x \in \Omega \quad (1.2.22)$$

where

$$G\mathbf{u}_1^{(k)} := -\widetilde{\rho}\partial_t^2\mathbf{u}_1^{(k)} + L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{u}_1^{(k)} + L_{\widetilde{\eta},\widetilde{\gamma}}\partial_t\mathbf{u}_1^{(k)}.$$

In order to apply Theorem 1.2.1 to the equation (1.2.21) and (1.2.22), we introduce the cut-off function $\chi \in C_0^\infty(\Omega)$ which satisfies

$$0 \le \chi(x,t) \le 1 \text{ on } G_0, \quad \chi(x,t) = \begin{cases} 1 & (x,t) \in G_{2\delta} \\ 0 & (x,t) \in G_0 \setminus G_\delta \end{cases}.$$

Calculating $P(\chi\partial_t^j\widetilde{\mathbf{u}}^{(k)})$ $(j = 2,3,4)$, we obtain from (1.2.21) the equation

$$
\begin{aligned}
&\rho_2\partial_t^2(\chi\partial_t^j\widetilde{\mathbf{u}}) = \\
&\qquad L_{\lambda_2,\mu_2}(\chi\partial_t^j\widetilde{\mathbf{u}}) + L_{\eta_2,\gamma_2}\partial_t(\chi\partial_t^j\widetilde{\mathbf{u}}) - (\partial_t\chi)L_{\eta_2,\gamma_2}(\partial_t^j\widetilde{\mathbf{u}}) \\
&\quad - P_1(\chi)\nabla\partial_t^{j+1}\widetilde{\mathbf{u}} - P_0(\chi)\partial_t^{j+1}\widetilde{\mathbf{u}} - Q_1(\chi)\nabla\partial_t^j\widetilde{\mathbf{u}} - Q_0(\chi)\partial_t^j\widetilde{\mathbf{u}} \\
&\qquad + 2\rho_2(\partial_t\chi)\partial_t^{j+1}\widetilde{\mathbf{u}} + \rho_2(\partial_t^2\chi)(\partial_t^j\widetilde{\mathbf{u}}) + \chi G(\partial_t^j\mathbf{u}_1). \quad (1.2.23)
\end{aligned}
$$

Here $\widetilde{\mathbf{u}}$ denotes either $\widetilde{\mathbf{u}}^{(1)}$ or $\widetilde{\mathbf{u}}^{(2)}$ and P_0, P_1, Q_0, Q_1 are at most first order linear differential operators with L^∞-coefficients. Then Lemma (1.2.1) yields

$$
\begin{aligned}
&\sum_{j=2}^{4}\int_{G_0}\left(s^4|\partial_t^j\widetilde{\mathbf{u}}|^2 + s^2|\nabla\partial_t^j\widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2}|\partial_x^\alpha\partial_t^j\widetilde{\mathbf{u}}|^2\right)\chi^2 e^{2s\psi^{-\nu}}\,dx\,dt \\
&\le C\sum_{j=2}^{4}\left(|\nabla G(\partial_t^j\mathbf{u}_1)|^2 + |G(\partial_t^j\mathbf{u}_1)|^2\right)\chi^2 e^{2s\psi^{-\nu}}\,dx\,dt \\
&\quad + C\int_{G_0}\left(\sum_{|\alpha|\le 2}(|\partial_x^\alpha\partial_t\chi|^2 + |\partial_x^\alpha\chi|^2)\right) \\
&\qquad \times\left(\sum_{j=2}^{5}(|\mathrm{div}\,(\partial_t^j\widetilde{\mathbf{u}})|^2 + |\mathrm{rot}\,(\partial_t^j\widetilde{\mathbf{u}})|^2 + |\partial_t^j\widetilde{\mathbf{u}}|^2)\right)e^{2s\psi^{-\nu}}\,dx\,dt \\
&\quad + C\int_{G_0}\left(\sum_{|\alpha|\le 2}(|\partial_x^\alpha\partial_t\chi|^2 + |\partial_x^\alpha\chi|^2)\right)\left(\sum_{j=2}^{5}\sum_{|\alpha|=2}|\partial_x^\alpha\partial_t^j\widetilde{\mathbf{u}}|^2 + |\nabla\partial_t^j\widetilde{\mathbf{u}}|^2)\right)e^{2s\psi^{-\nu}}\,dx\,dt \\
&\quad + Ce^{Cs}\mathcal{B}.
\end{aligned}
$$

Here we write

$$\mathcal{B} := \|\widetilde{\mathbf{u}}\|^2_{H^5(-T,T;H^2(\Gamma))} + \|\widetilde{\mathbf{u}}\|^2_{H^6(-T,T;H^1(\Gamma))}.$$

Since $\partial_x^\alpha\partial_t\chi \ne 0$ only on $G_\delta \setminus G_{2\delta}$ and

$$\psi^{-\nu} \le \left(\frac{3}{4} - 2\delta\right)^{-\nu} \quad \text{on } G_\delta \setminus G_{2\delta},$$

we obtain

$$
\begin{aligned}
&\sum_{j=2}^{4}\int_{G_0}\left(s^4|\partial_t^j\widetilde{\mathbf{u}}|^2 + s^2|\nabla\partial_t^j\widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2}|\partial_x^\alpha\partial_t^j\widetilde{\mathbf{u}}|^2\right)\chi^2 e^{2s\psi^{-\nu}}\,dx\,dt \\
&\le C\int_{G_0}\sum_{j=2}^{4}\left(|\nabla G(\partial_t^j\mathbf{u}_1)|^2 + |G(\partial_t^j\mathbf{u}_1)|^2\right)\chi^2 e^{2s\psi^{-\nu}}\,dx\,dt + Ce^{2s(3/4-2\delta)^{-\nu}}\mathcal{M} + Ce^{Cs}\mathcal{B},
\end{aligned}
\tag{1.2.24}
$$

where

$$\mathcal{M} := \|\widetilde{\mathbf{u}}\|^2_{H^5(-T,T;H^2(\Omega))}.$$

From $\chi = 1$ on $G_{2\delta}$ and from $0 \le \chi \le 1$, it follows that

$$\begin{aligned}
&\sum_{j=2}^{4} \int_{G_0} \left(s^4 |\partial_t^j \widetilde{\mathbf{u}}|^2 + s^2 |\nabla \partial_t^j \widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}|^2 \right) e^{2s\psi^{-\nu}} \, dx\, dt \\
&\le \sum_{j=2}^{4} \left(\int_{G_0 \setminus G_{2\delta}} + \int_{G_{2\delta}} \right) \left(s^4 |\partial_t^j \widetilde{\mathbf{u}}|^2 + s^2 |\nabla \partial_t^j \widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}|^2 \right) e^{2s\psi^{-\nu}} \, dx\, dt \\
&\le \sum_{j=2}^{4} \int_{G_0} \left(s^4 |\partial_t^j \widetilde{\mathbf{u}}|^2 + s^2 |\nabla \partial_t^j \widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}|^2 \right) \chi^2 e^{2s\psi^{-\nu}} \, dx\, dt \\
&\quad + C s^4 e^{2s(3/4-2\delta)^{-\nu}} \sum_{j=2}^{4} \int_{G_0} \left(|\partial_t^j \widetilde{\mathbf{u}}|^2 + |\nabla \partial_t^j \widetilde{\mathbf{u}}|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}|^2 \right) e^{2s\psi^{-\nu}} \, dx\, dt \\
&\le C \int_{G_0} \sum_{j=2}^{4} \left(|\nabla G(\partial_t^j \mathbf{u}_1)|^2 + |G(\partial_t^j \mathbf{u}_1)|^2 \right) \chi^2 e^{2s\psi^{-\nu}} \, dx\, dt \\
&\quad + C s^4 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + C e^{Cs} \mathcal{B}.
\end{aligned} \tag{1.2.25}$$

By definition of G, we have

$$\begin{aligned}
&\int_{G_0} \sum_{j=2}^{4} \left(|\nabla G(\partial_t^j \mathbf{u}_1)|^2 + |G(\partial_t^j \mathbf{u}_1)|^2 \right) \chi^2 e^{2s\psi^{-\nu}} \, dx\, dt \\
&\le \int_{G_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}} \, dx\, dt.
\end{aligned} \tag{1.2.26}$$

Then (1.2.24) and (1.2.26) yields

$$
\begin{aligned}
&\int_{\Omega_0} |\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}(x,0)|^2 e^{2s\psi(x,0)^{-\nu}}\, dx \\
&= \int_{-T}^{0} \partial_\tau \left(\int_{G_0\cap\{t=\tau\}} |\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}(x,\tau)|^2 \chi(x,\tau)^2 e^{2s\psi(x,\tau)^{-\nu}}\, dx \right) d\tau \\
&= \int_{G_0\cap\{t<0\}} 2(\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}) \cdot (\partial_x^\alpha \partial_t^3 \widetilde{\mathbf{u}}) \chi^2 e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + 2s \int_{G_0\cap\{t<0\}} |\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}|^2 \chi^2 (\partial_t(\psi^{-\nu})) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + \int_{G_0\cap\{t<0\}} |\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}|^2 \partial_t(\chi^2) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\le Cs \int_{G_0} (|\partial_x^\alpha \partial_t^2 \widetilde{\mathbf{u}}|^2 + |\partial_x^\alpha \partial_t^3 \widetilde{\mathbf{u}}|^2) \chi^2 e^{2s\psi^{-\nu}}\, dx\, dt + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} \\
&\le Cs \int_{G_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Cs^4 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + Ce^{Cs}\mathcal{B}.
\end{aligned} \tag{1.2.27}
$$

for any $|\alpha| = 2$. Similarly, we have

$$
\begin{aligned}
&\int_{\Omega_0} |\nabla \partial_t^2 \widetilde{\mathbf{u}}(x,0)|^2 e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \frac{C}{s} \int_{G_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + Ce^{Cs}\mathcal{B}
\end{aligned} \tag{1.2.28}
$$

and

$$
\begin{aligned}
&\int_{\Omega_0} |\partial_t^2 \widetilde{\mathbf{u}}(x,0)|^2 e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \frac{C}{s^3} \int_{G_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + Ce^{Cs}\mathcal{B}.
\end{aligned} \tag{1.2.29}
$$

Now we find first-order differential equations of $\widetilde{\mu}$, $\widetilde{\lambda}$, $\widetilde{\eta}$ and $\widetilde{\gamma}$. From (1.2.21), we have

$$
\rho_2 \partial_t^2 \widetilde{\mathbf{u}}^{(k)}(x,0) = G\mathbf{u}_1^{(k)}(x,0) \tag{1.2.30}
$$

$$
\rho_2 \partial_t^3 \widetilde{\mathbf{u}}^{(k)}(x,0) = L_{\eta_2,\gamma_2}\mathbf{r}^{(k)}(x) + G(\partial_t \mathbf{u}_1^{(k)})(x,0). \tag{1.2.31}
$$

we can rewrite (1.2.30) and (1.2.31) as

$$\mathbf{a}\widetilde{\rho} + B\nabla\widetilde{\lambda} + D\nabla\widetilde{\gamma} + G\nabla\widetilde{\mu} = \mathbf{K} - \mathbf{h}_1\partial_1\widetilde{\eta} - \mathbf{h}_2\partial_2\widetilde{\eta} - \mathbf{h}_3\partial_3\widetilde{\eta}. \tag{1.2.32}$$

where

$$\mathbf{K} := \begin{pmatrix} \widetilde{\rho}\partial_t^2(x,0) - \widetilde{\mu}\Delta\mathbf{p}^{(1)} - (\widetilde{\mu}+\widetilde{\lambda})\nabla(\operatorname{div}\mathbf{p}^{(1)}) - \widetilde{\gamma}\Delta\mathbf{q}^{(1)} - (\widetilde{\gamma}+\widetilde{\eta})\nabla(\operatorname{div}\mathbf{q}^{(1)}) \\ \widetilde{\rho}\partial_t^2(x,0) - \widetilde{\mu}\Delta\mathbf{p}^{(2)} - (\widetilde{\mu}+\widetilde{\lambda})\nabla(\operatorname{div}\mathbf{p}^{(2)}) - \widetilde{\gamma}\Delta\mathbf{q}^{(2)} - (\widetilde{\gamma}+\widetilde{\eta})\nabla(\operatorname{div}\mathbf{q}^{(2)}) \\ \widetilde{\rho}\partial_t^3(x,0) - \widetilde{\mu}\Delta\mathbf{q}^{(1)} - (\widetilde{\mu}+\widetilde{\lambda})\nabla(\operatorname{div}\mathbf{q}^{(1)}) - \widetilde{\gamma}\Delta\mathbf{r}^{(1)} - (\widetilde{\gamma}+\widetilde{\eta})\nabla(\operatorname{div}\mathbf{r}^{(1)}) \\ \widetilde{\rho}\partial_t^2(x,0) - \widetilde{\mu}\Delta\mathbf{q}^{(2)} - (\widetilde{\mu}+\widetilde{\lambda})\nabla(\operatorname{div}\mathbf{q}^{(2)}) - \widetilde{\gamma}\Delta\mathbf{r}^{(2)} - (\widetilde{\gamma}+\widetilde{\eta})\nabla(\operatorname{div}\mathbf{r}^{(2)}) \end{pmatrix}.$$

Hence for any index $1 \le j_1 \le 12$, we have

$$\{\mathbf{a}\}_{j_1}\widetilde{\rho} + \{B\}_{j_1}\nabla\widetilde{\lambda} + \{D\}_{j_1}\nabla\widetilde{\gamma} + \{G\}_{j_1}\nabla\widetilde{\mu} = \{\mathbf{K}\}_{j_1} - \{\mathbf{h}_1\}_{j_1}\partial_1\widetilde{\eta} - \{\mathbf{h}_2\}_{j_1}\partial_2\widetilde{\eta} - \{\mathbf{h}_3\}_{j_1}\partial_3\widetilde{\eta}.$$

This system has ten unknowns $\widetilde{\rho}$, $(\partial_k\widetilde{\lambda})_{k=1,2,3}$, $(\partial_k\widetilde{\gamma})_{k=1,2,3}$, $(\partial_k\widetilde{\mu})_{k=1,2,3}$ and in order to exist a solution, we need the following conditions:

$$\det{}_{j_1}(\mathbf{a}, B, D, G, \mathbf{K} - \mathbf{h}_1\partial_1\widetilde{\eta} - \mathbf{h}_2\partial_2\widetilde{\eta} - \mathbf{h}_3\partial_3\widetilde{\eta}) = 0, \tag{1.2.33}$$

that is,

$$\sum_{k=1}^{3} \det{}_{j_1}(\mathbf{a}, B, D, G, \mathbf{h}_k)\partial_k\widetilde{\eta} = \det{}_{j_1}(\mathbf{a}, B, D, G, \mathbf{K}).$$

Here we used the linearity of the determinant. We regard (1.2.33) as a first order partial differential equation.

In order to estimate the coefficients, we need the following estimate for a first order differential operator.

Lemma 1.2.5. *Let $(Pg)(x) := B(x)\cdot\nabla g(x) + B_0(x)g(x)$ be a first order differential operator with $B = (b_1, b_2, b_3) \in \{W^{2,\infty}(\Omega_0)\}^3$ and $B_0 \in W^{2,\infty}$. We assume*

$$\left| B(x)\cdot\begin{pmatrix} 1 \\ (1/A^2)x' \end{pmatrix} \right| > 0, \quad x \in \overline{\Omega_0}. \tag{1.2.34}$$

Then there exists a constant $\nu > 0$, $s_0 > 0$ and $C > 0$ such that for all $s > s_0$,

$$s^2 \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha g(x)|^2 \right) e^{2s\psi(x,0)^{-\nu}}\,dx \le C \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} |\partial_x^\alpha (Pg)(x)|^2 \right) e^{2s\psi(x,0)^{-\nu}}\,dx \tag{1.2.35}$$

for all $g \in H_0^3(\Omega_0)$.

We will prove it in Appendix B.

Noting that $\widetilde{\eta} = \eta_1 - \eta_2 = 0$ on Γ, we apply Lemma 1.2.5:

$$
\begin{aligned}
& s^2 \int_{\Omega_{2\delta}} \sum_{|\alpha|\le 2} |\partial_x^\alpha \widetilde{\eta}|^2 e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \le s^2 \int_{\Omega_0} \sum_{|\alpha|\le 2} |\partial_x^\alpha \widetilde{\eta}|^2 \chi^2 e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \le C \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2 \right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \quad + C \int_{\Omega_0} \sum_{j=2,3} \sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}^{(1)}(x,0)|^2 + |\partial_x^\alpha \partial_t^j \widetilde{\mathbf{u}}^{(2)}(x,0)|^2 \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \le C \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2 \right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \quad + Cs \int_{G_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2 \right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
& \quad + Cs^4 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + Ce^{Cs}\mathcal{B}.
\end{aligned} \tag{1.2.36}
$$

Similar arguments hold for $\widetilde{\mu}$, $\widetilde{\lambda}$ and $\widetilde{\gamma}$. Hence we have

$$
\begin{aligned}
& \int_{\Omega_{2\delta}} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \le \frac{C}{s^2} \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2 \right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
& \quad + \frac{C}{s} \int_{G_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2 \right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
& \quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B}
\end{aligned} \tag{1.2.37}
$$

for arbitrary $s > s_0$. By taking sufficiently large s, we can absorb the first term of the right-hand side into the left-hand side:

$$\begin{aligned}
&\int_{\Omega_{2\delta}} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \frac{C}{s^2} \int_{\Omega_0} \left(|\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + \frac{C}{s} \int_{G_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s} \mathcal{B}.
\end{aligned} \tag{1.2.38}$$

Finally, we give the estimate of $\widetilde{\rho}$. By (1.2.32), we have

$$\mathbf{a}\widetilde{\rho} = -B\nabla\widetilde{\lambda} - D\nabla\widetilde{\gamma} - G\nabla\widetilde{\mu} + \mathbf{K} - H\nabla\widetilde{\eta}$$

It follows from the assumptions that $|\mathbf{a}(x)| > 0$ holds for $\Omega_0 \setminus \Gamma$, then

$$\widetilde{\rho} = K_1\mathbf{K} + K_2(\nabla\widetilde{\lambda}, \nabla\widetilde{\gamma}, \nabla\widetilde{\mu}, \nabla\widetilde{\eta})$$

where K_1, K_2 is first-order differential operators with $W^{1,\infty}$-coefficients. Then we have

$$\begin{aligned}
|\nabla\widetilde{\rho}(x)| \le &\sum_{j=2}^{3} (|\nabla(\partial_t^j \widetilde{\mathbf{u}})(x,0)| + |\partial_t^j \widetilde{\mathbf{u}}(x,0)|) \\
&+ \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\lambda}(x)| + |\partial_x^\alpha \widetilde{\gamma}(x)| + |\partial_x^\alpha \widetilde{\mu}(x)| + |\partial_x^\alpha \widetilde{\eta}(x)|).
\end{aligned} \tag{1.2.39}$$

and

$$\begin{aligned}
|\widetilde{\rho}(x)| \le &\sum_{j=2}^{3} |\partial_t^j \widetilde{\mathbf{u}}(x,0)| \\
&+ \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\lambda}(x)| + |\partial_x^\alpha \widetilde{\gamma}(x)| + |\partial_x^\alpha \widetilde{\mu}(x)| + |\partial_x^\alpha \widetilde{\eta}(x)|).
\end{aligned} \tag{1.2.40}$$

In order that $\widetilde{\rho} = 0$ on $\Omega_0 \setminus \Gamma$, we obtain

$$\begin{aligned}
&\int_{\Omega_0} (|\widetilde{\rho}(x)|^2 + |\nabla\widetilde{\rho}(x)|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\lambda}(x)| + |\partial_x^\alpha \widetilde{\gamma}(x)| + |\partial_x^\alpha \widetilde{\mu}(x)| + |\partial_x^\alpha \widetilde{\eta}(x)|) \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + \int_{\Omega_0} \left(\sum_{j=2}^{3} (|\partial_t^j \widetilde{\mathbf{u}}(x,0)|^2 + |\nabla(\partial_t^j \widetilde{\mathbf{u}})(x,0)|^2 \right) e^{2s\psi(x,0)^{-\nu}}\, dx.
\end{aligned} \tag{1.2.41}$$

By substituting (1.2.29), (1.2.28) into (1.2.41), we obtain

$$\begin{aligned}
&\int_{\Omega_0} (|\widetilde{\rho}(x)|^2 + |\nabla \widetilde{\rho}(x)|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \int_{\Omega_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\lambda}(x)| + |\partial_x^\alpha \widetilde{\gamma}(x)| + |\partial_x^\alpha \widetilde{\mu}(x)| + |\partial_x^\alpha \widetilde{\eta}(x)|) \right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + Cs \int_{G_0} \left(\sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + Ce^{Cs} \mathcal{B}.
\end{aligned} \tag{1.2.42}$$

By using the estimate

$$\begin{aligned}
&\int_{G_0} g(x) e^{2s\psi(x,t)^{-\nu}}\, dx\, dt \\
&= \int_{\Omega_0} g(x) e^{2s\psi(x,0)^{-\nu}} \left(\int_{-l(x)}^{l(x)} e^{2s(\psi(x,t)^{-\nu} - \psi(x,0)^{-\nu})}\, dt \right) dx \\
&\le \int_{\Omega_0} g(x) e^{2s\psi(x,0)^{-\nu}} \left(\int_{-\infty}^{\infty} e^{-(3/4)^{-\nu} st^2/T^2}\, dt \right) dx \\
&\le \frac{C}{\sqrt{s}} \int_{\Omega_0} g(x) e^{2s\psi(x,0)^{-\nu}}\, dx
\end{aligned} \tag{1.2.43}$$

for arbitrary function $g(x)$ which is positive on Ω_0, (1.2.38) yields

$$\begin{aligned}
&\int_{\Omega_0} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\le \int_{\Omega_{2\delta}} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{E} \\
&\le \frac{C}{s^2} \int_{\Omega_0} \left(|\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + \frac{C}{s} \int_{G_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2 \right) e^{2s\psi^{-\nu}}\, dx\, dt \\
&\quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s} \mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{E} \\
&\le \frac{C}{s\sqrt{s}} \int_{\Omega_0} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + \frac{C}{s^2} \int_{\Omega_0} \left(|\widetilde{\rho}|^2 + |\nabla \widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\
&\quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s} \mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}} \mathcal{E}
\end{aligned} \tag{1.2.44}$$

where we set

$$\mathcal{E} := \int_{\Omega_0} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2)\, dx.$$

Then for sufficiently large $s > 0$, we obtain

$$\begin{aligned}&\int_{\Omega_0} \sum_{|\alpha|\le 2} (|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2) e^{2s\psi(x,0)^{-\nu}}\, dx \\ &\le Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}}\mathcal{E}.\end{aligned} \tag{1.2.45}$$

As for the estimate of ρ, combining (1.2.42) and (1.2.45) yields

$$\begin{aligned}&\int_{\Omega_0} \left(|\nabla\widetilde{\rho}|^2 + |\widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\ &\le Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}}\mathcal{E}\end{aligned} \tag{1.2.46}$$

Here with (1.2.38), we obtain

$$\begin{aligned}&\int_{\Omega_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla\widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\ &\le \frac{C}{s}\int_{G_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla\widetilde{\rho}|^2\right) e^{2s\psi^{-\nu}}\, dx\, dt \\ &\quad + Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}}\mathcal{E}.\end{aligned} \tag{1.2.47}$$

We have

$$\begin{aligned}&\int_{\Omega_0} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla\widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx \\ &\le Cs^2 e^{2s(3/4-2\delta)^{-\nu}} \mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B} + Ce^{2s(3/4-2\delta)^{-\nu}}\mathcal{E}\end{aligned} \tag{1.2.48}$$

for any large $s > 0$. From definition of ψ,

$$\psi^{-\nu} \ge \left(\frac{3}{4} - 3\delta\right)^{-\nu} \quad \text{on } G_{3\delta}$$

holds, and then

$$\begin{aligned}&e^{2s(3/4-3\delta)^{-\nu}}\mathcal{E} \\ &\le \int_{\Omega_{3\delta}} \left(\sum_{|\alpha|\le 2} \left(|\partial_x^\alpha \widetilde{\mu}|^2 + |\partial_x^\alpha \widetilde{\lambda}|^2 + |\partial_x^\alpha \widetilde{\eta}|^2 + |\partial_x^\alpha \widetilde{\gamma}|^2\right) + |\widetilde{\rho}|^2 + |\nabla\widetilde{\rho}|^2\right) e^{2s\psi(x,0)^{-\nu}}\, dx.\end{aligned} \tag{1.2.49}$$

By combining (1.2.48) and (1.2.49), we obtain

$$\left(e^{2s(3/4-3\delta)^{-\nu}} - Ce^{2s(3/4-2\delta)^{-\nu}}\right)\mathcal{E} \le Cs^2 e^{2s(3/4-2\delta)^{-\nu}}\mathcal{M} + \frac{Ce^{Cs}}{s}\mathcal{B}.$$

By taking sufficiently large $s > 0$ such that

$$e^{2s(3/4-3\delta)^{-\nu}} - Ce^{2s(3/4-2\delta)^{-\nu}} \le \frac{1}{2}e^{2s(3/4-3\delta)^{-\nu}},$$

we have

$$\begin{aligned}\mathcal{E} &\le Cs^2 e^{2s\{(3/4-2\delta)^{-\nu}-(3/4-3\delta)^{-\nu}\}}\mathcal{M} + \frac{Ce^{Cs-2s(3/4-3\delta)^{-\nu}}}{s}\mathcal{B}.\\ &\le Ce^{-2\varepsilon s}\mathcal{M} + Ce^{Cs}\mathcal{B}\end{aligned}$$

for suitable $C > 0$ and $\epsilon > 0$. Assume that $\mathcal{M} > \mathcal{B}$. By taking $s > \frac{1}{C+2\varepsilon}\log\frac{\mathcal{M}}{\mathcal{B}} > 0$, we obtain

$$\mathcal{E} \le C\mathcal{M}^{C/(C+2\varepsilon)}\mathcal{B}^{2\varepsilon/(C+2\varepsilon)}.$$

If $\mathcal{M} \le \mathcal{B}$, the proof is already complete. Thus we have proven Theorem 2.1.1 when ω is a part of a paraboloid.

1.2.2 Case II: a general case

Now we prove Theorem 2.1.1 for general ω. Our proof consists the following steps:
(1) First we prove the Hölder type estimate of unique continuation. Thanks to this estimate, the stability estimate on G_δ can be extended to the estimate on wider domain $\Omega_\delta \times (-T, T)$.
(2) We prove Thorem 2.1.1 on a domain which is a union of paraboloids.
(3) We take the covering of ω by domains in (2). By compactness of ω, we obtain Theorem 2.1.1 for general ω.

Now we will state the Hölder estimates of unique continuation.

Lemma 1.2.6. *Let $\omega \subset \Omega \cap \overline{\Gamma}$ be a part of a paraboloid. Then for any $\varepsilon_0 > 0$, there exist constants*

$$C = C(\Omega, \omega, \varepsilon_0, \mu, \eta, \gamma, \lambda, \rho) > 0$$

and

$$\kappa = \kappa(\Omega, \omega, \varepsilon_0, \mu, \eta, \gamma, \lambda, \rho) \in (0, 1)$$

such that

$$\|\mathbf{u}\|_{L^2(-T+\varepsilon_0, T-\varepsilon_0; H^2(\omega))} \le C\left(\|\mathbf{u}\|^{1-\kappa}_{H^1(-T,T;H^2(\Omega))}\|\mathbf{u}\|^{\kappa}_{H^3(\Gamma\times(-T,T))} + \|\mathbf{u}\|_{H^3(\Gamma\times(-T,T))}\right) \tag{1.2.50}$$

holds for any $\mathbf{u} \in H^2(\Omega \times (-T, T))$ which satisfies $P\mathbf{u} = 0$.

Proof. Let us introduce a cut-off function $\chi \in C_0^\infty(G_0)$ such that

$$\chi = \begin{cases} 1 & \text{on} \quad G_{2\delta} \\ 0 & \text{on} \quad G_0 \setminus G_\delta \end{cases} \tag{1.2.51}$$

and $0 \le \chi \le 1$ on G_0. Calculating $P(\chi\mathbf{u})$, we obtain

$$\begin{aligned} \rho\partial_t^2(\chi\mathbf{u}) &= L_{\lambda,\mu}(\chi\mathbf{u}) + L_{\eta,\gamma}\partial_t(\chi\mathbf{u}) - (\partial_t\chi)L_{\eta,\gamma}\mathbf{u} \\ &= -P_1(\chi)\nabla\partial_t\mathbf{u} - P_0(\chi)\partial_t\mathbf{u} - Q_1(\chi)\nabla\mathbf{u} - Q_0(\chi)\mathbf{u} \\ &\quad + 2\rho(\partial_t\chi)\partial_t\mathbf{u} + \rho(\partial_t^2\chi)\mathbf{u}. \end{aligned} \tag{1.2.52}$$

Applying Theorem 1.2.1 to 1.2.52, we have

$$\begin{aligned} &\int_{G_0} \left[s^4|\chi\mathbf{u}|^2 + s^2|\nabla(\chi\mathbf{u})|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha(\chi\mathbf{u})|^2 \right] e^{2s\psi^{-\nu}}\, dx\, dt \\ &\le \int_{G_0} \left(\sum_{j=0}^{2} \sum_{|\alpha|\le 2} |\partial_x^\alpha \partial_t^j \chi|^2 \right) \left(\sum_{\alpha\le 2} (|\partial_x^\alpha\partial_t\mathbf{u}|^2 + |\partial_x^\alpha\mathbf{u}|^2) \right) e^{2s\psi^{-\nu}}\, dx\, dt \\ &\quad + Ce^{Cs}\|\mathbf{u}\|^2_{H^3(\Gamma\times(-T,T))} \end{aligned} \tag{1.2.53}$$

for any $s \ge s_0$. Thus

$$\begin{aligned} &e^{2(1/4)^{-\nu}s} \sum_{|\alpha|\le 2} \|\partial_x^\alpha\mathbf{u}\|^2_{L^2(G_0)} \\ &\le \int_{G_0} \left[|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2 + \sum_{\alpha=2} |\partial_x^\alpha\mathbf{u}|^2 \right] e^{2s\psi^{-\nu}}\, dx\, dt \\ &\le \left(\int_{G_{2\delta}} + \int_{G_0\setminus G_{2\delta}} \right) \left[|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2 + \sum_{\alpha=2} |\partial_x^\alpha\mathbf{u}|^2 \right] e^{2s\psi^{-\nu}}\, dx\, dt \\ &\le \int_{G_0} \left[s^4|\chi\mathbf{u}|^2 + s^2|\nabla(\chi\mathbf{u})|^2 + \sum_{|\alpha|=2} |\partial_x^\alpha(\chi\mathbf{u})|^2 \right] e^{2s\psi^{-\nu}}\, dx\, dt \\ &\quad + e^{2(3/4-2\delta)^{-\nu}s}\|\mathbf{u}\|^2_{L^2(-T,T;H^2(\Omega))} \\ &\le Ce^{2(3/4-2\delta)^{-\nu}s}\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\Omega))} + Ce^{Cs}\|\mathbf{u}\|^2_{H^3(\Gamma\times(-T,T))}. \end{aligned} \tag{1.2.54}$$

Dividing it into $e^{2(1/4)^{-\nu}s}$ and taking sufficient large C, we have

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha\mathbf{u}\|^2_{L^2(G_0)} \le Ce^{-\varepsilon s}\|\mathbf{u}\|^2_{H^1(-T,T;H^2(\Omega))} + Ce^{Cs}\|\mathbf{u}\|^2_{H^3(\Gamma\times(-T,T))} \tag{1.2.55}$$

If $\|\mathbf{u}\|_{H^1(-T,T;H^2(\Omega))} > \|\mathbf{u}\|_{H^3(\Gamma\times(-T,T))}$, choosing

$$s = \frac{1}{C+\varepsilon}\left(\log\|\mathbf{u}\|_{H^1(-T,T;H^2(\Omega))} - \log\|\mathbf{u}\|_{H^3(\Gamma\times(-T,T))} \right) > 0,$$

we obtain

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(G_0)} \le C\|\mathbf{u}\|^{2C/(C+\varepsilon)}_{H^1(-T,T;H^2(\Omega))}\|\mathbf{u}\|^{2\varepsilon/(C+\varepsilon)}_{H^3(\Gamma\times(-T,T)}. \tag{1.2.56}$$

If $\|\mathbf{u}\|_{H^1(-T,T;H^2(\Omega))} \le \|\mathbf{u}\|_{H^3(\Gamma\times(-T,T))}$, then

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(G_0)} \le C\|\mathbf{u}\|^2_{H^3(\Gamma\times(-T,T))} \tag{1.2.57}$$

holds. Combining (1.2.56) and (1.2.57), we obtain

$$\sum_{|\alpha|\le 2} \|\partial_x^\alpha \mathbf{u}\|^2_{L^2(G_0)} \le C\left(\|\mathbf{u}\|^{2(1-\kappa)}_{H^1(-T,T;H^2(\Omega))}\|\mathbf{u}\|^{2\kappa}_{H^3(\Gamma\times(-T,T)} + \|\mathbf{u}\|^2_{H^3(\Gamma\times(-T,T)}\right). \tag{1.2.58}$$

By definition of G_0, there exists $\delta > 0$ such that $\Omega_\delta \times (-\varepsilon_0, \varepsilon_0)$ is contained in G_0. Thus the same estimate as (1.2.58) holds for $\|\mathbf{u}\|_{L^2(-\varepsilon_0,\varepsilon_0;H^2(\Omega_\delta))}$. Since we can cover $\Omega_\delta \times (-T+\varepsilon_0, T-\varepsilon_0)$ with finite intervals $\Omega_\delta \times (n\varepsilon_0, (n+1)\varepsilon_0)$ $(n = -N, \dots, N-1)$, we conclude the proof of Lemma 1.2.6.

Now we will prove Theorem 2.1.1 for general ω which is compactly embedded in $\Omega \cup \overline{\Gamma}$. Let $x \in \omega \setminus \Gamma$ and $x_0 \in \Gamma$ be arbitrary fixed and take a polygonal line L in ω which connects x and x_0. And let L consist the finite line segment $\{l_j\}_{j=1}^N$ such that l_j connects x_{j-1} and x_j, where $x_N := x$. Thus we can construct open sets $\{\omega_j\}_{j=1}^N$ which satisfy the following properties:

- Each ω_j is a part of a paraboloid whose axis is l_j.
- Each x_j is contained in ω_j.
- $\omega_1 \subset \Omega \cup \overline{\Gamma}$ and for any $j \ge 2$,

$$\omega_j \subset \Omega \setminus \left(\overline{\Gamma} \cup \left(\bigcup_{k=1}^{j-1} \omega_k\right)\right).$$

In fact, The assumption $\operatorname{dist}(\omega, \Omega \setminus \overline{\Gamma}) > 0$ means that $\operatorname{dist}(l_j, \Omega \setminus \overline{\Gamma}) > 0$ for any $j \ge 2$. By taking the sufficiently small width of paraboloid, we can construct $\{\omega_j\}$.

Since ω_1 is a part of paraboloid, we can apply the estimate (1.1.12) to ω_1 and obtain

$$\begin{aligned}&\|\lambda_2 - \lambda_1\|_{H^2(\omega_1)} + \|\mu_2 - \mu_1\|_{H^2(\omega_1)}\\ &+ \|\gamma_2 - \gamma_1\|_{H^2(\omega_1)} + \|\eta_2 - \eta_1\|_{H^2(\omega_1)} + \|\rho_2 - \rho_1\|_{H^1(\omega_1)} \le C\left(M^{1-\kappa}B_1^\kappa + B_1\right)\end{aligned} \tag{1.2.59}$$

for some $C > 0$ and $\kappa > 0$, where

$$M := \sum_{k=1,2} \sum_{j=1,2} \left(\|\mathbf{u}_j^{(k)}\|_{H^6(-T,T;H^2(\Omega))} + \|\mathbf{u}_j^{(k)}\|_{H^7(-T,T;H^1(\Omega))} \right)$$

$$B_1 := \sum_{k=1,2} \left(\|\widetilde{\mathbf{u}}^{(k)}\|_{H^5(-T,T;H^2(\Gamma))} + \|\widetilde{\mathbf{u}}^{(k)}\|_{H^6(-T,T;H^1(\Gamma))} \right).$$

Moreover, Lemma 1.2.6 implies that for arbitrary fixed $\varepsilon > 0$, we have

$$\begin{aligned} &\|\mathbf{u}\|_{H^4(-T+\varepsilon,T-\varepsilon;H^2(\omega_1))} \\ &\le C\Big\{ \|\mathbf{u}\|^{1-\kappa}_{H^5(-T,T;H^2(\omega_1))} \left(\|\mathbf{u}\|_{H^5(-T,T;H^2(\Gamma))} + \|\mathbf{u}\|_{H^6(-T,T;H^1(\Gamma))} \right)^{\kappa} \\ &\qquad + \left(\|\mathbf{u}\|_{H^5(-T,T;H^2(\Gamma))} + \|\mathbf{u}\|_{H^6(-T,T;H^1(\Gamma))} \right) \Big\} \end{aligned} \tag{1.2.60}$$

for some $C > 0$ by changing $\kappa > 0$ if necessary.

Now we divide $\partial\omega_2$ into the following two surfaces:

$$\sigma_{2,0} := \partial\omega_2 \cap \partial\omega_1,$$
$$\sigma_{2,1} := \partial\omega_2 \setminus \partial\omega_1.$$

Then we can apply Theorem 2.1.1 to ω_2 and obtain

$$\begin{aligned} &\|\lambda_2 - \lambda_1\|_{H^2(\omega_2)} + \|\mu_2 - \mu_1\|_{H^2(\omega_2)} \\ &\quad + \|\gamma_2 - \gamma_1\|_{H^2(\omega_2)} + \|\eta_2 - \eta_1\|_{H^2(\omega_2)} + \|\rho_2 - \rho_1\|_{H^1(\omega_2)} \\ &\le C\left(M^{1-\kappa} B_2^{\kappa} + B_2 \right), \end{aligned} \tag{1.2.61}$$

where

$$B_2 := \sum_{k=1,2} \left(\|\widetilde{\mathbf{u}}^{(k)}\|_{H^5(-T+\varepsilon,T-\varepsilon;H^2(\sigma_{2,0}))} + \|\widetilde{\mathbf{u}}^{(k)}\|_{H^6(-T+\varepsilon,T-\varepsilon;H^1(\sigma_{2,0}))} \right).$$

From (1.2.60), (1.2.61) and Sobolev's interpolation theorem, it follows that

$$\begin{aligned} B_2 &\le \sum_{k=1,2} C \Big(\|\widetilde{\mathbf{u}}^{(k)}\|^{1/6}_{L^2(-T+\varepsilon,T-\varepsilon;H^2(\omega_1))} \|\widetilde{\mathbf{u}}^{(k)}\|^{5/6}_{H^6(-T+\varepsilon,T-\varepsilon;H^2(\omega_1))} \\ &\qquad + \|\widetilde{\mathbf{u}}^{(k)}\|^{1/7}_{L^2(-T+\varepsilon,T-\varepsilon;H^1(\omega_1))} \|\widetilde{\mathbf{u}}^{(k)}\|^{6/7}_{H^7(-T+\varepsilon,T-\varepsilon;H^1(\omega_1))} \Big) \\ &\le \sum_{k=1,2} C \Big[\left(C\left(M^{1-\kappa} B_1^{\kappa} + B_1 \right) \right)^{1/6} \|\widetilde{\mathbf{u}}^{(k)}\|^{5/6}_{H^6(-T+\varepsilon,T-\varepsilon;H^2(\omega_1))} \\ &\qquad + \left(C\left(M^{1-\kappa} B_1^{\kappa} + B_1 \right) \right)^{1/7} \|\widetilde{\mathbf{u}}^{(k)}\|^{6/7}_{H^7(-T+\varepsilon,T-\varepsilon;H^1(\omega_1))} \Big] \\ &\le C\left(M^{1-\kappa/6} B_1^{\kappa/6} + M^{5/6} B_1^{1/6} + M^{1-\kappa/7} B_1^{\kappa/7} + M^{6/7} B_1^{1/7} \right) \\ &\le C\left(M^{1-\kappa_1} B_1^{\kappa_1} + M^{1-\kappa_2} B_1^{\kappa_2} \right) \end{aligned} \tag{1.2.62}$$

where we define

$$\kappa_1 := \min\left\{1-\frac{\kappa}{6}, \frac{5}{6}, 1-\frac{\kappa}{7}, \frac{6}{7}\right\},$$
$$\kappa_2 := \max\left\{1-\frac{\kappa}{6}, \frac{5}{6}, 1-\frac{\kappa}{7}, \frac{6}{7}\right\}.$$

(1.2.61) and (1.2.62) yields

$$\begin{aligned}
&\|\lambda_2-\lambda_1\|_{H^2(\omega_2)} + \|\mu_2-\mu_1\|_{H^2(\omega_2)} \\
&\quad + \|\gamma_2-\gamma_1\|_{H^2(\omega_2)} + \|\eta_2-\eta_1\|_{H^2(\omega_2)} + \|\rho_2-\rho_1\|_{H^1(\omega_2)} \\
&\le C\left(M^{1-\kappa}\left(M^{1-\kappa_1}B_1^{\kappa_1} + M^{1-\kappa_2}B_1^{\kappa_2}\right)^{\kappa} + M^{1-\kappa_1}B_1^{\kappa_1} + M^{1-\kappa_2}B_1^{\kappa_2}\right) \\
&\le C\left(M^{1-\kappa_3}B_1^{\kappa_3} + M^{1-\kappa_4}B_1^{\kappa_4}\right)
\end{aligned} \tag{1.2.63}$$

for some $\kappa_3, \kappa_4 \in (0,1)$.

We can repeat this argument for $\omega_3, \dots, \omega_N$ and thus we see that there exists the constants $C(x) > 0$ and $\kappa(x), \kappa'(x) > 0$ such that

$$\begin{aligned}
&\|\lambda_2-\lambda_1\|_{H^2(\omega(x))} + \|\mu_2-\mu_1\|_{H^2(\omega(x))} \\
&\quad + \|\gamma_2-\gamma_1\|_{H^2(\omega(x))} + \|\eta_2-\eta_1\|_{H^2(\omega(x))} + \|\rho_2-\rho_1\|_{H^1(\omega(x))} \\
&\le C(x)\left(M^{1-\kappa(x)}B_1^{\kappa(x)} + M^{1-\kappa'(x)}B_1^{\kappa'(x)}\right),
\end{aligned} \tag{1.2.64}$$

where $\omega(x) := \bigcup_{k=1}^{N} \omega_k$. $\{\omega(x)\}_{x\in\omega}$ covers ω and by assumptions ω is a relatively compact set in Ω. Hence, we can choose a finite sub-covering $\{\omega(x_1), \dots, \omega(x_l)\}$ of ω from $\{\omega(x)\}_{x\in\omega}$. By taking $\kappa, \kappa' \in (0,1)$ and $C > 0$ as

$$\begin{aligned}
\kappa &:= \min\{\kappa(x_1), \dots, \kappa(x_l), \kappa'(x_1), \dots, \kappa'(x_l)\}, \\
\kappa' &:= \max\{\kappa(x_1), \dots, \kappa(x_l), \kappa'(x_1), \dots, \kappa'(x_l)\}, \\
C &:= \max\{C(x_1), \dots, C(x_l)\},
\end{aligned}$$

we obtain from (1.2.64)

$$\begin{aligned}
&\|\lambda_2-\lambda_1\|_{H^2(\omega))} + \|\mu_2-\mu_1\|_{H^2(\omega))} \\
&\quad + \|\gamma_2-\gamma_1\|_{H^2(\omega))} + \|\eta_2-\eta_1\|_{H^2(\omega))} + \|\rho_2-\rho_1\|_{H^1(\omega))} \\
&\le C\left(M^{1-\kappa)}B_1^{\kappa)} + M^{1-\kappa'}B_1^{\kappa'}\right).
\end{aligned} \tag{1.2.65}$$

and thus the proof is complete.

Appendix A: The Proof of Lemma 1.2.4

By Hölder's inequality, we have

$$\left(\int_0^t |f(x,\tau)|\,d\tau\right)^2 \le t\int_0^t |f(x,\tau)|^2 d\tau.$$

By Setting

$$l(x) = T\sqrt{1-\left(2x_1+\frac{|x_2|^2}{A^2}\right)},$$

we have

$$\int_{G_0}\left(\int_0^t |f(x,\tau)|\,d\tau\right)^2 e^{2s\psi^{-\nu}}\,dx\,dt = \int_{\Omega_0}\left(\int_{-l(x)}^{l(x)}\left(\int_0^t |f(x,\tau)|\,d\tau\right)^2 e^{2s\psi^{-\nu}}dt\right)dx. \tag{1.2.66}$$

Since

$$\partial_t\left(e^{2s\psi(x,t)^{-\nu}}\right) = -\frac{4\nu st}{T^2\psi(x,t)^{\nu+1}}e^{2s\psi(x,t)^{-\nu}},$$

we estimate

$$\begin{aligned}
&\int_0^{l(x)}\left(\int_0^t |f(x,\tau)|\,d\tau\right)^2 e^{2s\psi^{-\nu}}dt\\
&\le \int_0^{l(x)}\left(\int_0^t |f(x,\tau)|^2\,d\tau\right)e^{2s\psi^{-\nu}}dt\\
&\le \int_0^{l(x)} t\left(\int_0^t |f(x,\tau)|^2\,d\tau\right)e^{2s\psi^{-\nu}}dt \qquad (1.2.67)\\
&\le -\frac{T^2}{4\nu s}\int_0^{l(x)}\psi^{\nu+1}\left(\int_0^t |f(x,\tau)|^2\,d\tau\right)\partial_t\left(e^{2s\psi^{-\nu}}\right)dt\\
&\le -\frac{T^2}{4\nu s}\left(x_1+\frac{|x'|^2}{A^2}\right)^{\nu+1}\int_0^{l(x)}\partial_t\left(e^{2s\psi^{-\nu}}\right)\left(\int_0^t |f(x,\tau)|^2\,d\tau\right)dt
\end{aligned}$$

where in the last equation, we use the fact that $\psi(x,t)$ is decreasing on $[0,l(x)]$ as a function of t for arbitrary fixed $x\in\Omega_0$. An integration by parts yields

$$\begin{aligned}
&\int_0^{l(x)}\partial_t\left(e^{2s\psi^{-\nu}}\right)\left(\int_0^t |f(x,\tau)|^2\,d\tau\right)dt\\
&= e^{2s\psi(x,l(x))^{-\nu}}\int_0^{l(x)}|f(x,t)|^2dt - \int_0^{l(x)}|f(x,t)|^2e^{2s\psi(x,t)^{-\nu}}dt \qquad (1.2.68)\\
&= e^{2^{\nu+1}s}\int_0^{l(x)}|f(x,t)|^2dt - \int_0^{l(x)}|f(x,t)|^2e^{2s\psi(x,t)^{-\nu}}dt.
\end{aligned}$$

By (1.2.67) and (1.2.68), we obtain

$$
\begin{aligned}
&\int_0^{l(x)} \left(\int_0^t |f(x,\tau)|\, d\tau \right)^2 e^{2s\psi^{-\nu}} dt \\
&\leq \frac{T^2}{4\nu s} \left(-e^{2^{\nu+1}s} \int_0^{l(x)} |f(x,t)|^2 dt + \int_0^{l(x)} |f(x,t)|^2 e^{2s\psi(x,t)^{-\nu}} dt \right) \\
&\leq \frac{T^2}{4\nu s} \left(x_1 + \frac{|x_2|^2}{A^2} \right)^{\nu+1} \int_0^{l(x)} |f(x,t)|^2 e^{2s\psi^{-\nu}} dt \\
&\leq \frac{T^2}{2^{\nu+3}s} \int_0^{l(x)} |f(x,t)|^2 e^{2s\psi^{-\nu}} dt.
\end{aligned}
$$

Similarly, we obtain

$$
\int_{-l(x)}^0 \left(\int_0^t |f(x,\tau)|\, d\tau \right)^2 e^{2s\psi^{-\nu}} dt \leq \frac{T^2}{2^{\nu+3}s} \int_{-l(x)}^0 |f(x,t)|^2 e^{2s\psi^{-\nu}} dt. \tag{1.2.69}
$$

Therefore

$$
\begin{aligned}
\int_{G_0} \left(\int_0^t |f(x,\tau)|\, d\tau \right)^2 e^{2s\psi^{-\nu}}\, dx\, dt &\leq \int_{\Omega_0} \left(\frac{T^2}{2^{\nu+3}s} \int_{-l(x)}^{l(x)} |f(x,t)|^2 e^{2s\psi^{-\nu}} dt \right) dx \\
&= \frac{T^2}{2^{\nu+3}s} \int_{G_0} |f(x,t)|^2 e^{2s\psi^{-\nu}}\, dx\, dt.
\end{aligned}
$$

The proof of Lemma 1.2.4 is completed.

Appendix B: The Proof of Lemma 1.2.5

Set $h(x) := g(x) e^{s\psi(x,0)^{-\nu}}$. Simple calculation shows

$$
\begin{aligned}
e^{s\psi^{-\nu}} Pg &= e^{s\psi^{-\nu}} P \left(h e^{-s\psi^{-\nu}} \right) \\
&= B \cdot \nabla h + \left[B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right] h.
\end{aligned} \tag{1.2.70}
$$

Then

$$
\begin{aligned}
\int_{\Omega_0} |Pg|^2 e^{2s\psi^{-\nu}}\, dx &\geq \int_{\Omega_0} \left(B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right)^2 |h|^2\, dx \\
&\quad + \int_{\Omega_0} \left(B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right) \left(B \cdot \nabla(|h|^2) \right) dx \\
&= \int_{\Omega_0} \left(B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right)^2 |h|^2\, dx \\
&\quad + \int_{\Omega_0} B \cdot \nabla \left(B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right) |h|^2\, dx \\
&\quad + \int_{\Omega_0} \left(B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \right) (\nabla \cdot B) |h|^2\, dx.
\end{aligned} \tag{1.2.71}
$$

By assumption, we can take sufficient large $s > 0$ such that

$$B_0 + s\nu\psi^{-\nu-1}(B \cdot \nabla\psi) \geq Cs, \quad x \in \Omega_0$$

for some constant $C > 0$. Hence we obtain for large $s > 0$,

$$\int_{\Omega_0} |Pg|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx \geq C \int_{\Omega_0} |g|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx. \tag{1.2.72}$$

Next we have

$$P(\partial_j g) = \partial_j(Pg) - (\partial_j B) \cdot \nabla g - (\partial_j B_0) g.$$

Then (1.2.72) yields

$$\begin{aligned} s^2 \int_{\Omega_0} |\partial_j g|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx &\leq C \int_{\Omega_0} |P(\partial_j g)|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx \\ &\leq C \int_{\Omega_0} |\partial_j(Pg)|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx \\ &\quad + C \int_{\Omega_0} (|\nabla g|^2 + |g|^2) e^{2s\psi(\cdot,0)^{-\nu}}\, dx. \end{aligned} \tag{1.2.73}$$

From (1.2.72) and (1.2.73), it follows that

$$s^2 \int_{\Omega_0} (|\nabla g|^2 + |g|^2) e^{2s\psi(\cdot,0)^{-\nu}}\, dx \leq C \int_{\Omega_0} \left(\sum_{|\alpha| \leq 1} |P(\partial_x^\alpha g)|^2 \right) e^{2s\psi(\cdot,0)^{-\nu}}\, dx. \tag{1.2.74}$$

Similarly, we have

$$\begin{aligned} P(\partial_j \partial_k g) = \partial_j \partial_k(Pg) - (\partial_j B) \cdot \nabla(\partial_k g) - (\partial_k B) \cdot \nabla(\partial_j g) - (\partial_j \partial_k B) g \\ - (\partial_j B_0)(\partial_k g) - (\partial_k B_0)(\partial_j g) - (\partial_j \partial_k B_0) g, \end{aligned} \tag{1.2.75}$$

so that from (1.2.74), we obtain

$$\begin{aligned} s^2 \int_{\Omega_0} |\partial_j \partial_k g|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx &\leq C \int_{\Omega_0} |\partial_j \partial_k(Pg)|^2 e^{2s\psi(\cdot,0)^{-\nu}}\, dx \\ &\quad + C \int_{\Omega_0} \left(\sum_{|\alpha| \leq 2} |\partial_x^\alpha g|^2 \right) e^{2s\psi(\cdot,0)^{-\nu}}\, dx. \end{aligned} \tag{1.2.76}$$

By combining (1.2.74) and (1.2.76), the proof is complete.

Chapter 2

Inverse problem for the phase field system by measurements of one component

2.1 Introduction

The phase field model, which was first introduced by Caginalp [3] and Fix [8], is used as the model of phase transition phenomena, such as crystallization. In the phase field model, a state variable is introduced in order to describe two phases, and the phase transition is considered as the system of partial differential equations of the state variable and other conservative quantities (for instance, temperature). For this reason, an interface between two phases are described as "diffuse interface" which is a narrow region covering the actual interface.

The phase field models are derived from the free energy functional which contains an interfacial energy and a latent heat and so on. (see Penrose and Fife [23].)

In this paper, we consider the linearized phase field model which describes the phase transition of two phases and analyze the inverse problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and $T > 0$. We consider the following generalized system:

$$\partial_t u + l\partial_t v = \nabla \cdot (K\nabla u) + a_{11}u + a_{12}v + A_{11} \cdot \nabla u + A_{12} \cdot \nabla v + f_1 \quad \text{on } \Omega_T := (0,T) \times \Omega \quad (2.1.1)$$

$$\partial_t v = \nabla \cdot (M \nabla v) + a_{21} u + a_{22} v + A_{21} \cdot \nabla u + A_{22} \cdot \nabla v + f_2 \quad \text{on } \Omega_T \quad (2.1.2)$$

with boundary conditions

$$u = h_1, \quad v = h_2 \text{ on } \Sigma_T := (0,T) \times \partial\Omega, \qquad (2.1.3)$$

where K, M and $a_{jk} (j,k = 1,2)$ are scalar functions and $A_{jk} (j,k = 1,2)$ are vectorial fields both defined on Ω. Let us fix the Dirichlet data h_1, h_2 and the source term f_j.

We will explain the physical meaning of our system. u describes a tempreture and K a thermal conductivity coefficient and l a latent heat. Meanwhile, v is a state variable and M is called a mobility coefficient, which represents how easily the interface can move.

Now our inverse problem is formulated as follows:

Inverse Problem Let $\omega \subset \Omega$ be a non-empty subdomain and fix $\theta \in (0,T)$. Determine the thermal conductivity M and the mobility K from the measurements

$$u|_{(0,T)\times\omega}, \quad (u,v)\,|_{\{\theta\}\times\Omega}.$$

We note that we have the extra data on ω only for u. This means that we can measure only the tempreture continuously in time on small domain and this inverse problem is whether M and K can be determined simultaneously in such a limited observation (if you can obtain the whole profile (u,v) at time θ). In practical viewpoint, this inverse problem is desirable, because the tempreture u can be measured much more easily than the state variable v.

To state our result, we prepare some notations. Let (K_1, M_1) and (K_2, M_2) be two pairs of mobility and thermal conductivity coefficients and let (u_j, v_j) solve (2.1.1), (2.1.2) and (2.1.3) with $K = K_j$ and $M = M_j$ for $j = 1,2$. We set $\omega_T := (0,T) \times \omega$. By $W^{m,p}(\Omega)$, we denotes usual Sobolev spaces for $m \in \mathbb{N}$ and $1 \le p \le \infty$. For any Banach space X, We denotes by $L^p(0,T;X)$ the space of X-valued p-Bochner integrable functions. We write $L^p(\Omega) := W^{0,p}(\Omega)$ and $H^m(\Omega) := W^{m,2}(\Omega)$.

Let $d \in C^2\left(\overline{\Omega}\right)$ satisfy

$$d > 0 \text{ in } \Omega, \quad |\nabla d| > 0 \text{ on } \overline{\Omega}, \quad \nabla d \cdot \nu \le 0 \text{ on } \partial\Omega \setminus \gamma.$$

The existence of d is guaranteed in [12].

We pose the following assumptions:

- $\|a_{jk}\|_{L^\infty(\Omega)}, \|K_j\|_{W^{1,\infty}(\Omega)}, \|M_j\|_{W^{1,\infty}(\Omega)} \le R \ (j,k = 1,2)$,
- $K_1 = K_2$ and $M_1 = M_2$ in ω and a neighborhood of $\partial\Omega$,
- $\|A_{jk}\|_{(L^\infty(\Omega))^n} \le R \ (j,k = 1,2)$,

- $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$, where $|\gamma|$ is a measure of γ. and $\partial\omega$ is of class C^2,

- $|A_{12} \cdot \nu| \neq 0$ on γ,

- $|\nabla u_2(\theta, \cdot) \cdot \nabla d|, |\nabla v_2(\theta, \cdot) \cdot \nabla d| > \delta_0$ on $\overline{\Omega}$ with some $\delta_0 > 0$,

- $\|u_2\|_{C(\overline{\Omega_T})}, \|v_2\|_{C(\overline{\Omega_T})} \leq R$.

We can now formulate our main results as follows:

Theorem 2.1.1. *Let $\theta \in (0, T)$ be fixed. Suppose that assumptions listed above are satisfied and $u_1(\theta, \cdot) = u_2(\theta, \cdot)$ and $v_1(\theta, \cdot) = v_2(\theta, \cdot)$ in Ω. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\|K_1 - K_2\|_{L^2(\Omega)} + \|M_1 - M_2\|_{L^2(\Omega)} \\ &\qquad \leq C\left(\|u_1 - u_2\|_{H^3(0,T;L^2(\omega))} + \|u_1 - u_2\|_{H^2(0,T;H^2(\omega))}\right). \end{aligned} \tag{2.1.4}$$

The key gradient of Theorem 2.1.1 is a global Carleman estimate for system (2.1.1), (2.1.2) and (2.1.3).

Carleman estimate is successfully used for the inverse problems since Bukhgeim-Klibanov [2]. As far as parabolic equations is concerned, the uniqueness and the stability in determining coefficients are proved in many situations in terms of Carleman estimate. See [13, 28, 29] and the reference therein. Especially for a 2×2 parabolic system, the stability estimate for determining coefficients by means of only one component is established in Cristofol, Gaitan and Ramoul [6] and Benabdallah, Critofol, Gaitan and Yamamoto [1]. Compared with these works, however, our equation has the term $l\partial_t v$ in (2.1.1) and we need some modification in the proof.

This paper is organized as follows. In Section 2, we prove a Carleman estimate for the system (2.1.1), (2.1.2) and (2.1.3). In Section 3, we prove the stability estimate by means of the Carleman estimate proved in Section 2 and Bukhgeim-Klibanov method.

2.2 Carleman Estimate

In order to prove the Carleman estimate for equations (2.1.1) and (2.1.2), we impose the following assumption

- $\omega \subset \Omega$ with $\partial\omega \cap \partial\Omega = \gamma$ and $|\gamma| \neq 0$.

- $|A_{12} \cdot \nu| \neq 0$ on $\gamma_T := (0, T) \times \gamma$.

- $\|A_{12}\|_{C^2(\overline{\omega_T})^n}, \|a_{12}\|_{C^2(\overline{\omega_T})}, \|A_{11}\|_{C^2(\overline{\omega_T})^n} \leq M$ for some constant $M > 0$.

We define the weight function as

$$\varphi(t,x) := \frac{e^{\lambda d(x)}}{t\,(T-t)}, \quad \alpha(t,x) := \frac{e^{\lambda d(x)} - e^{2\lambda\|d\|_{C(\overline{\Omega})}}}{t(T-t)}.$$

Under these assumptions and notations, we prove the following Carleman estimate.

Lemma 2.2.1. *Let ω be a subdomain of Ω and the assumption is fulfilled. Then there exist $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$, there exists $s_0(\lambda) > 0$ such that for any $s > s_0$, the estimate*

$$\begin{aligned}
&\int_{\Omega_T} \left(\frac{1}{s\varphi} |\partial_t u|^2 + |\partial_t v|^2 + \frac{1}{s\varphi} |\Delta u|^2 + |\Delta v|^2 \right. \\
&\left. + s\lambda^2\varphi\,|\nabla u|^2 + s^2\lambda^2\varphi^2\,|\nabla v|^2 + s^3\lambda^4\varphi^3\,|u|^2 + s^4\lambda^4\varphi^4\,|v|^2 \right) e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\leq C \int_{\Omega_T} \left(|f_1|^2 + s\varphi\,|f_2|^2 \right) e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + Ce^{Cs} s^4\lambda^4\varphi^4 \left(\|f_1\|^2_{L^2(\omega'_T)} + \|u\|^2_{H^1(0,T;L^2(\omega'))} + \|u\|^2_{L^2(0,T;H^2(\omega'))} \right),
\end{aligned} \tag{2.2.1}$$

holds for any u and v which solve (2.1.1) *and* (2.1.2) *with boundary condition $u = v = 0$ on Σ_T.*

We will remark some points. Lemma 3.2.2 is a Carleman estimate without extra data in ω_T of $\widetilde{v}$. Benabdallah, Cristofol, Gaitan and Yamamoto proved a similar estimate in [1] for the case when $l = 0$ on Ω. Our proof of Lemma 3.2.2 is done similarly as [1]. In our case, however, $\partial_t v$ term in (2.1.1) does not vanish, namely $l \neq 0$ on Ω generally. Therefore we need some modification.

In order to prove Lemma 3.2.2, we need the following lemma.

Lemma 2.2.2. *Consider the equation*

$$Pu(t,x) := p_0(t,x)\partial_t u(t,x) + \sum_{j=1}^{n} p_j(t,x)\partial_j u(t,x) + q(t,x)u(t,x) = f, \tag{2.2.2}$$

with the boundary condition $u|_{(0,T)\times\gamma} = 0$. We additionally assume that

$$\left| \sum_{j=1}^{n} p_j(t,x)\nu_j(x) \right| \neq 0 \ \ (t,x) \in (0,T) \times \overline{\gamma},$$

where $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ is the unit outward normal vector to $\partial\omega$ at x. Then there exist a subdomain $\omega' \subset \omega$ and a constant $C > 0$ which is independent of p_j, q and f such that

$$\|u\|_{L^2(\omega'_T)} \leq C\,\|f\|^2_{L^2(\omega'_T)}.$$

Here we denote $\omega'_T := (0,T)\times\omega'$.

Proof. We will modify the argument in [1] to prove it. Without loss of generality, we may assume that ω is written as

$$\omega = \{(x',x_n)\,;\, h_1(x') \le x_n \le h_2(x'),\ |x'| \le \rho\}$$

where $h_1, h_2 \in C^1(\{|x'| \le \rho\})$ with $h_1 = h_2$ on $\{|x'| = \rho\}$ and $\gamma = \{(x',x_n)\,;\, x_n = h_1(x'),\ |x'| \le \rho\}$. We introduce the independent variables (y', y_n) defined as

$$y' = x', \quad y_n = x_n - h_1(x').$$

Then ω is transformed into

$$\widetilde{\omega} = \{(y',y_n)\,;\, 0 \le y_n \le h_2(y') - h_1(y'),\ |y'| \le \rho\}.$$

Let us set

$$\widetilde{\Gamma}_1 := \{(y',0)\,;\, |y'| \le \rho\}$$
$$\widetilde{\Gamma}_2 := \{(y',y_n)\,;\, y_n = h_2(y') - h_1(y'),\ |y'| \le \rho\}$$

and note that $\partial\widetilde{\omega} = \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$.

By the coordinate change, P is transformed into

$$\begin{aligned}
&\widetilde{P}\widetilde{u}(t,y)\\
&:= \widetilde{p}_0(t,y)\partial_t\widetilde{u}(t,y) + \sum_{j=1}^{n-1}\widetilde{p}_j(t,y)\partial_{y_j}\widetilde{u}(t,y) + \widetilde{r}(t,y)\partial_{y_n}\widetilde{u}(t,y) + \widetilde{q}(t,y)\widetilde{u}(t,y)\\
&\qquad = \widetilde{f}(t,y), \quad (2.2.3)
\end{aligned}$$

where $\widetilde{p}_0(t,y) = p_0(t,x)$, $\widetilde{p}_j(t,y) = p_j(t,x)$, $\widetilde{q}(t,y) = q(t,x)$, $\widetilde{u}(t,y) = u(t,x)$, $\widetilde{f}(t,y) = f(t,x)$ and

$$\widetilde{r}(t,y) = \widetilde{p}_n(t,y) - \sum_{j=1}^{n-1}\widetilde{p}_j(t,y)\partial_{y_j}h_1(t,y).$$

Moreover the boundary condition $u|_{(0,T)\times\gamma} = 0$ is interpreted as

$$\widetilde{u}|_{(0,T)\times\widetilde{\Gamma}_1} = 0. \qquad (2.2.4)$$

The unit outward normal vector $\nu(x)$ to $\partial\omega$ is parallel to $(\partial_1 h_1(x'), \dots \partial_{n-1}h_1(x'), -1)$ on $\{(x',x_n)\,;\, x_n = h_1(x'),\ |x'| < \rho\}$. Therefore the assumption $|p(t,x)\cdot\nu(x)| \neq 0$ on $(t,x) \in (0,T)\times\overline{\gamma}$ implies that we may assume that $\widetilde{r}(t,y',0) > 2\delta$ for $|y'| < \rho$ and $0 < t < T$. By choosing $\rho > 0$ sufficiently small, we also assume that

$$\widetilde{r}(t,y) > \delta, \quad (t,y) \in (0,T)\times\widetilde{\omega}. \qquad (2.2.5)$$

Let $\widetilde{\nu}(y)$ be the unit outward normal vector to $\partial\widetilde{\omega}$. $\widetilde{\nu}(y)$ is parallel to $(0,\dots,0,-1)$ for $y \in \widetilde{\Gamma}_1$ and $\left(-\partial_{y_1}(h_2-h_1),\dots,-\partial_{y_{n-1}}(h_2-h_1),1\right)$ for $y \in \widetilde{\Gamma}_2$. Then by using (2.2.5),

$$\sum_{j=1}^{n-1} \widetilde{p}_j(t,y)\widetilde{\nu}_j(y) + \widetilde{r}(t,y)\widetilde{\nu}_n(y) = -C\widetilde{r}(t,y) < 0 \text{ if } y \in \widetilde{\Gamma}_1. \tag{2.2.6}$$

Moreover taking h_1, h_2 such that $\|h_2 - h_1\|_{C^1(\{|y'|\le\rho\})}$ is suffciently small if necessary, we have

$$\sum_{j=1}^{n-1} \widetilde{p}_j(t,y)\widetilde{\nu}_j(y) + \widetilde{r}(t,y)\widetilde{\nu}_n(y) \le 0 \text{ if } y \in \widetilde{\Gamma}_2. \tag{2.2.7}$$

In order to prove the lemma, we prove a Carleman estimate for (2.2.2). We set

$$\widetilde{P}_0\widetilde{u} := \widetilde{P}\widetilde{u} - \widetilde{q}\widetilde{u} = \widetilde{p}_0(t,y)\partial_t\widetilde{u}(t,y) + \sum_{j=1}^{n-1}\widetilde{p}_j(t,y)\partial_{y_j}\widetilde{u}(t,y) + \widetilde{r}(t,y)\partial_{y_n}\widetilde{u}(t,y)$$

and $w = \widetilde{u}e^{s(y_n+\psi_0(t))}$ where $\psi_0(t) = -A(t-\theta)^2$. We will choose the constant $A > 0$ later. Then

$$\begin{aligned} e^{s(y_n+\psi_0(t))}\widetilde{P}_0\left(e^{-s(y_n+\psi_0(t))}w\right) &= \widetilde{P}_0 w - s\widetilde{r}(t,y)w - s2A\,(\theta-t)\,\widetilde{p}_0 w \\ &=: \widetilde{P}_0 w - s g_0 w \end{aligned}$$

By integration by parts, we obtain

$$
\begin{aligned}
&\int_{\widetilde{\omega}_T} \left|\widetilde{P}_0\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \qquad (2.2.8)\\
&= \int_{\widetilde{\omega}_T} \left|e^{s(y_n+\psi_0(t))}\widetilde{P}_0\left(e^{-s(y_n+\psi_0(t))}w\right)\right|^2 \mathrm{d}t\,\mathrm{d}x\\
&= \int_{\widetilde{\omega}_T} \left|\widetilde{P}_0 w\right|^2 \mathrm{d}t\,\mathrm{d}x + \int_{\widetilde{\omega}_T} s^2 g_0^2 w^2\,\mathrm{d}t\,\mathrm{d}x\\
&\quad - 2\int_{\widetilde{\omega}_T} s g_0 w \left(\widetilde{p}_0\partial_t w + \sum_{j=1}^{n-1}\widetilde{p}_j\partial_{y_j} w + \widetilde{r}\partial_{y_n} w\right)\mathrm{d}t\,\mathrm{d}x\\
&\geq \int_{\widetilde{\omega}_T} s^2 g_0^2 w^2\,\mathrm{d}t\,\mathrm{d}x\\
&\quad - \left(\int_{(0,T)\times\widetilde{\Gamma}_1} + \int_{(0,T)\times\widetilde{\Gamma}_2}\right) s g_0 \left(\sum_{j=1}^{n-1}\widetilde{p}_j\widetilde{\nu}_j + \widetilde{r}\widetilde{\nu}_n\right) w^2\,\mathrm{d}t\,\mathrm{d}S\\
&\quad + \int_{\widetilde{\omega}} s\left(\widetilde{p}_0(0,y)g_0(0,y)\left(w(0,y)\right)^2 - \widetilde{p}_0(T,y)g_0(T,y)\left(w(T,y)\right)^2\right)\mathrm{d}x\\
&\quad + \int_{\widetilde{\omega}_T} s\partial_t\left(\widetilde{p}_0 g_0\right) w^2\,\mathrm{d}t\,\mathrm{d}x + \int_{\Omega} s\left(\sum_{j=1}^{n-1}\partial_{y_j}\left(\widetilde{p}_j g_0\right) + \partial_{y_n}\left(\widetilde{r} g_0\right)\right) w^2\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
\tag{2.2.9}
$$

Then we choose $A > 0$ such that the following inequalities are fulfilled:

$$\widetilde{p}_0(0,y)g_0(0,y) \geq 0 \tag{2.2.10}$$

$$\widetilde{p}_0(T,y)g_0(T,y) \leq 0. \tag{2.2.11}$$

Indeed by definition of $g_0 := \widetilde{r} + A(T-2t)\widetilde{p}_0$, we can rewrite these condition as

$$
\begin{aligned}
A\theta &\geq -\frac{\widetilde{r}(0,y)}{\widetilde{p}_0(0,y)}\\
-A\left(T-\theta\right) &\leq -\frac{\widetilde{r}(T,y)}{\widetilde{p}_0(T,y)}.
\end{aligned}
$$

Since we assume that $|\widetilde{p}_0| \neq 0$ in $\widetilde{\omega}$, we have

$$\left|\frac{\widetilde{r}(0,y)}{\widetilde{p}_0(0,y)}\right|, \left|\frac{\widetilde{r}(T,y)}{\widetilde{p}_0(T,y)}\right| \leq \kappa$$

for some constant $\kappa > 0$. Then if we choose A as $A \geq \max\left\{\kappa/\theta, \kappa/(T-\theta)\right\}$, the conditions (2.2.10) and (2.2.11) are satisfied.

Hence by (2.2.9) and (2.2.16)–(2.2.7), we obtain

$$\int_{\widetilde{\omega}_T} s^2\widetilde{u}^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \leq \left|\widetilde{P}_0\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x.$$

By definition of $\widetilde{P}_0$, we have

$$\begin{aligned}
&\int_{\widetilde{\omega}_T} s^2\widetilde{u}^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \\
&\le C\int_{\widetilde{\omega}_T} \left|\widetilde{P}_0\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \\
&\le C\int_{\widetilde{\omega}_T} \left|\widetilde{P}\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x + C\int_{\widetilde{\omega}_T} |\widetilde{q}\widetilde{u}|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \\
&\le C\int_{\widetilde{\omega}_T} \left|\widetilde{P}\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x + C\,\|\widetilde{q}\|^2_{C(\widetilde{\omega}_T)}\int_{\widetilde{\omega}_T} |\widetilde{u}|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

By choosing sufficiently large s, we finally obtain

$$\int_{\widetilde{\omega}_T} s^2\widetilde{u}^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x \le C\int_{\widetilde{\omega}_T} \left|\widetilde{P}\widetilde{u}\right|^2 e^{2s(y_n+\psi_0(t))}\,\mathrm{d}t\,\mathrm{d}x$$

for large $s>0$. Since $1 \le e^{2s(y_n+\psi_0(t))} \le e^{2Cs}$ in $\widetilde{\omega}_T$ for some $C>0$, we have

$$s^2\int_{\widetilde{\omega}_T} \widetilde{u}^2\,\mathrm{d}t\,\mathrm{d}x \le Ce^{2Cs}\int_{\widetilde{\omega}_T} \left|\widetilde{P}\widetilde{u}\right|^2\,\mathrm{d}t\,\mathrm{d}x.$$

By fixing large $s>0$, we conclude the lemma. □

We now return to the proof of Lemma 3.2.2. We can rewrite (2.1.1) as

$$A_{12}\cdot\nabla v + a_{12}v - l(x)\partial_t v = \partial_t u - \nabla\cdot(K(x)\nabla u) - a_{11}u - A_{11}\cdot\nabla u - f_1.$$

Applying Lemma 2.2.2 to it yields

$$\begin{aligned}
\int_{\omega'_T} |v|^2\,\mathrm{d}t\,\mathrm{d}x &\le C\int_{\omega'_T} |\partial_t u - \nabla\cdot(K(x)\nabla u) - a_{11}u - A_{11}\cdot\nabla u - f_1|^2\,\mathrm{d}t\,\mathrm{d}x \\
&\le C\left(\|f_1\|^2_{L^2(\omega'_T)} + \|u\|^2_{H^1(0,T;L^2(\omega'))} + \|u\|^2_{L^2(0,T;H^2(\omega'))}\right).
\end{aligned} \tag{2.2.12}$$

By [9] and [12], there exists $\lambda_0>0$ such that for all $\lambda\ge\lambda_0$, there exists $s_0>0$ such that

$$\begin{aligned}
&\int_{\Omega_T}\left(|\partial_t v|^2 + |\Delta v|^2 + s^2\lambda^2\varphi^2|\nabla v|^2 + s^4\lambda^4\varphi^4|v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\le C\int_{\Omega_T} s\varphi\left(|f_2|^2+|u|^2+|\nabla u|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x + C\int_{\omega'_T} s^4\lambda^4\varphi^4|v|^2e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x
\end{aligned} \tag{2.2.13}$$

and

$$\int_{\Omega_T}\left(\frac{1}{s\varphi}|\partial_t u|^2+\frac{1}{s\varphi}|\Delta u|^2+s\lambda^2\varphi|\nabla u|^2+s^3\lambda^4\varphi^3|u|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq C\int_{\Omega_T}\left(|f_1|^2+|\partial_t v|^2+|v|^2+|\nabla v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x+C\int_{\omega'_T}s^3\lambda^4\varphi^3|u|^2e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \tag{2.2.14}$$

for all $s\geq s_0$.

Combining (2.2.13) and (2.2.14) and taking sufficiently large $s,\lambda\geq 0$, we obtain

$$\int_{\Omega_T}\left(\frac{1}{s\varphi}|\partial_t u|^2+|\partial_t v|^2+\frac{1}{s\varphi}|\Delta u|^2+|\Delta v|^2\right.$$
$$\left.+s\lambda^2\varphi|\nabla u|^2+s^2\lambda^2\varphi^2|\nabla v|^2+s^3\lambda^4\varphi^3|u|^2+s^4\lambda^4\varphi^4|v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq C\int_{\Omega_T}\left(|f_1|^2+s\varphi|f_2|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x+C\int_{\omega'_T}\left(s^3\lambda^4\varphi^3|u|^2+s^4\lambda^4\varphi^4|v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x. \tag{2.2.15}$$

Since $\left|e^{2s\alpha}\right|\leq e^{Cs}$ on $\overline{\Omega_T}$, we have

$$\int_{\omega'_T}\left(s^3\lambda^4\varphi^3|u|^2+s^4\lambda^4\varphi^4|v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x\leq Ce^{Cs}s^4\lambda^4\varphi^4\left(\|u\|^2_{L^2(\omega'_T)}+\|v\|^2_{L^2(\omega'_T)}\right). \tag{2.2.16}$$

Then combining (2.2.12), (3.2.3) and (2.2.16), we deduce that

$$\int_{\Omega_T}\left(\frac{1}{s\varphi}|\partial_t u|^2+|\partial_t v|^2+\frac{1}{s\varphi}|\Delta u|^2+|\Delta v|^2\right. \tag{2.2.17}$$
$$\left.+s\lambda^2\varphi|\nabla u|^2+s^2\lambda^2\varphi^2|\nabla v|^2+s^3\lambda^4\varphi^3|u|^2+s^4\lambda^4\varphi^4|v|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \tag{2.2.18}$$
$$\leq C\int_{\Omega_T}\left(|f_1|^2+s\varphi|f_2|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \tag{2.2.19}$$
$$+Ce^{Cs}s^4\lambda^4\varphi^4\left(\|f_1\|^2_{L^2(\omega'_T)}+\|u\|^2_{H^1(0,T;L^2(\omega'))}+\|u\|^2_{L^2(0,T;H^2(\omega'))}\right), \tag{2.2.20}$$

which prove the lemma.

2.3 Proof of Theorem 2.1.1

Let (u_j,v_j) for $j=1,2$ satisfy (2.1.1) and (2.1.2).

We set $U = u_1 - u_2$ and $V = v_1 - v_2$. U and V solves

$$\partial_t U + l(x)\partial_t V = \nabla\cdot(K_1(x)\nabla U) + a_{11}U + a_{12}V + A_{11}\cdot\nabla U + A_{12}\cdot\nabla V + \nabla\cdot\left(\widetilde{K}(x)\nabla u_2\right) \quad (2.3.1)$$

$$\partial_t V = \nabla\cdot(M_1(x)\nabla V) + a_{21}U + a_{22}V + A_{21}\cdot\nabla U + A_{22}\cdot\nabla V + \nabla\cdot\left(\widetilde{M}(x)\nabla v_2\right). \quad (2.3.2)$$

and $U = V = 0$ on Σ_T.

Set $y_j := \partial_t^j U$ and $z_j := \partial_t^j V$ $(j = 1, 2)$. Then we obtain

$$\partial_t y_j + l_1\partial_t z_j = \nabla\cdot(K_1\nabla y_j) + a_{11}y_j + a_{12}z_j + A_{11}\cdot\nabla y_j + A_{12}\cdot\nabla z_j + \nabla\cdot\left(\widetilde{K}\nabla\partial_t^j u_2\right) \quad (2.3.3)$$

$$\partial_t z_j = \nabla\cdot(M_1\nabla z_j) + a_{21}y_j + a_{22}z_j + A_{21}\cdot\nabla y_j + A_{22}\cdot\nabla z_j + \nabla\cdot\left(\widetilde{M}\nabla\partial_t^j v_2\right) \quad (2.3.4)$$

and boundary conditions $y_j = z_j = 0$ on Σ_T.

Step 1 We will estimate the L^2-norm of z in a subdomain of Ω in terms of y. By using $\widetilde{u} = \int_\theta^t y(\tau,\cdot)\,\mathrm{d}\tau$ and $\widetilde{v} = \int_\theta^t z(\tau,\cdot)\,\mathrm{d}\tau$, we rewrite (2.3.3) as

$$\begin{aligned} &-l_1\partial_t z_j + A_{12}\cdot\nabla z_j + a_{12}z_j \\ &\quad = \partial_t y_j - \nabla\cdot(K_1\nabla y_j) - a_{11}y_j - A_{11}\cdot\nabla y_j - \nabla\cdot\left(\widetilde{K}\nabla\partial_t^j u_2\right) \\ &\quad =: Q(y_j)(t,x) - \nabla\cdot\left(\widetilde{K}\nabla\partial_t^j u_2\right). \quad (2.3.5)\end{aligned}$$

By Lemma 2.2.2, we have

$$\begin{aligned} &\int_{\omega'_T} s^2\,|z_j|^2\,e^{2s\left(\varphi_0 - A(t-\theta)^2\right)}\,\mathrm{d}t\,\mathrm{d}x \\ &\le C\int_{\omega'_T} |Q(y_j)|^2\,e^{2s\left(\varphi_0 - A(t-\theta)^2\right)}\,\mathrm{d}t\,\mathrm{d}x + C\int_{\omega'_T} |y_j|^2\,e^{2s\left(\varphi_0 - A(t-\theta)^2\right)}\,\mathrm{d}t\,\mathrm{d}x \\ &\qquad + C\int_{\omega'_T} \left|\nabla\cdot\left(\widetilde{K}\nabla\partial_t^j u_2\right)\right|^2 e^{2s\left(\varphi_0 - A(t-\theta)^2\right)}\,\mathrm{d}t\,\mathrm{d}x. \quad (2.3.6)\end{aligned}$$

Since

$$\int_{\omega'_T} |Q(y_j)|^2\,e^{2s\left(\varphi_0 - A(t-\theta)^2\right)}\,\mathrm{d}t\,\mathrm{d}x \le Ce^{Cs}\left(\|y_j\|^2_{H^1(0,T;L^2(\omega'))} + \|y_j\|^2_{L^2(0,T;H^2(\omega'))}\right),$$

and $\widetilde{K}=0$ on ω, we obtain

$$\int_{\omega_T'} s^2 |z_j|^2 e^{2s(\varphi_0 - A(t-\theta)^2)} \,\mathrm{d}t\,\mathrm{d}x \le Ce^{Cs}\left(\|y_j\|^2_{H^1(0,T;L^2(\omega'))} + \|y_j\|^2_{L^2(0,T;H^2(\omega'))}\right) \tag{2.3.7}$$

By fixing sufficiently large $s>0$, (2.3.7) gives

$$\|z_j\|^2_{L^2(\omega_T')} \le Ce^{Cs}\left(\|y_j\|^2_{H^1(0,T;L^2(\omega'))} + \|y_j\|^2_{L^2(0,T;H^2(\omega'))}\right). \tag{2.3.8}$$

Step 2 We apply Theorem to (2.3.3) and (2.3.4) for $(\delta, T-\delta)\times\Omega$ with

$$\alpha = \frac{e^{\lambda d} - e^{2\lambda\|d\|_{C(\overline{\Omega})}}}{(t-\delta)(T-\delta-t)},$$

and have

$$\begin{aligned}
&\int_{(\delta,T-\delta)\times\Omega} \left(\frac{1}{s\varphi}|\partial_t y_j|^2 + |\partial_t z_j|^2 + \frac{1}{s\varphi}|\Delta y_j|^2 + |\Delta z_j|^2\right.\\
&\left. + s\lambda^2\varphi|\nabla y_j|^2 + s^2\lambda^2\varphi^2|\nabla z_j|^2 + s^3\lambda^4\varphi^3|y_j|^2 + s^4\lambda^4\varphi^4|z_j|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x\\
&\le C\int_{(\delta,T-\delta)\times\Omega}\left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x\\
&\quad + C\int_{(\delta,T-\delta)\times\Omega} s\varphi\left(\left|\widetilde{M}\right|^2 + \left|\nabla\widetilde{M}\right|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x\\
&\quad + C(s,\lambda)\left(\|y_j\|^2_{H^1(0,T;L^2(\omega'))} + \|y_j\|^2_{L^2(0,T;H^2(\omega'))}\right).
\end{aligned} \tag{2.3.9}$$

We set the cut-off function $\chi \in C_0^\infty(0,T)$ defined as

$$0\le\chi\le 1,\quad \chi(t) = \begin{cases} 1 & 3\delta < t < T-3\delta \\ 0 & 0\le t\le 2\delta,\ T-2\delta\le t\le T. \end{cases}$$

Then we have

$$
\begin{aligned}
&\int_{\Omega} |\partial_t U(\theta,x)|^2 e^{2s\alpha(\theta,x)}\,\mathrm{d}x \\
&\le \int_{\theta}^{T-\delta} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega} |\partial_t U(t,x)|^2 \chi^2 e^{2s\alpha(t,x)}\,\mathrm{d}x\right)\,\mathrm{d}t \\
&\le 2\int_{(\theta,T-\delta)\times\Omega} (\partial_t U)\left(\partial_t^2 U\right)\chi^2 e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + 2s\int_{(\theta,T-\delta)\times\Omega} \frac{e^{\lambda\varphi}-e^{2\lambda\|\varphi\|_{C(\overline{\Omega})}}}{(t-\delta)^2(T-\delta-t)^2}(t-T+2\delta)\,|\partial_t U|^2\chi^2 e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + 2\int_{(\theta,T-\delta)\times\Omega} |\partial_t U|^2 \chi\,(\partial_t\chi)\,e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x. \\
&\le Cs\int_{(\delta,T-\delta)\times\Omega}\left(|y_1|^2+|y_2|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega} |\partial_t \nabla U(\theta,x)|^2 e^{2s\alpha(\theta,x)}\,\mathrm{d}x \\
&\le \int_{\theta}^{T-\delta} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega} |\partial_t \nabla U(t,x)|^2 \chi^2 e^{2s\alpha(t,x)}\,\mathrm{d}x\right)\,\mathrm{d}t \\
&\le 2\int_{(\theta,T-\delta)\times\Omega} (\partial_t \nabla U)\cdot\left(\partial_t^2 \nabla U\right)\chi^2 e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + 2s\int_{(\theta,T-\delta)\times\Omega} \frac{e^{\lambda\varphi}-e^{2\lambda\|\varphi\|_{C(\overline{\Omega})}}}{(t-\delta)^2(T-\delta-t)^2}(t-T+2\delta)\,|\partial_t \nabla U|^2\chi^2 e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + 2\int_{(\theta,T-\delta)\times\Omega} |\partial_t \nabla U|^2 \chi\,(\partial_t\chi)\,e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x. \\
&\le Cs\int_{(\delta,T-\delta)\times\Omega}\left(|\nabla y_1|^2+|\nabla y_2|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned}
$$

Similarly we can prove

$$
\begin{aligned}
&\int_{\Omega}\left(|\partial_t V(\theta,x)|^2+|\partial_t\nabla V(\theta,x)|^2\right)e^{2s\alpha(\theta,x)}\,\mathrm{d}x \\
&\qquad \le Cs\int_{(\delta,T-\delta)\times\Omega}\left(|z_1|^2+|\nabla z_1|^2+|z_2|^2+|\nabla z_2|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x. \qquad (2.3.10)
\end{aligned}
$$

By using (2.3.9), we have

$$
\begin{aligned}
&\int_{\Omega}\left(|\partial_t U(\theta,x)|^2+|\partial_t V(\theta,x)|^2+|\partial_t\nabla U(\theta,x)|^2+|\partial_t\nabla V(\theta,x)|^2\right)e^{2s\alpha(\theta,x)}\,\mathrm{d}x \\
&\le C(s,\lambda)\left(\|U\|^2_{H^3(0,T;L^2(\omega'))}+\|U\|^2_{H^2(0,T;H^2(\omega'))}\right) \\
&\quad + C\int_{(0,T)\times\Omega}\left(\left|\widetilde{K}\right|^2+\left|\nabla\widetilde{K}\right|^2+s\varphi\left|\widetilde{M}\right|^2+s\varphi\left|\nabla\widetilde{M}\right|^2\right)e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}
$$

Substituting $t = \theta$ into (2.3.1) and (2.3.2) yields

$$\nabla \cdot \left(\widetilde{K} \nabla u_2\right) = \partial_t U\left(\theta, \cdot\right) + l_1 \partial_t U\left(\theta, \cdot\right),$$
$$\nabla \cdot \left(\widetilde{M} \nabla v_2\right) = \partial_t U\left(\theta, \cdot\right).$$

Since l_1 is bounded in Ω, we finally obtain

$$\begin{aligned}
&\int_\Omega \left(\left|\nabla \cdot \left(\widetilde{K} \nabla u_2\right)\right|^2 + \left|\nabla \nabla \cdot \left(\widetilde{K} \nabla u_2\right)\right|^2 + \left|\nabla \cdot \left(\widetilde{M} \nabla v_2\right)\right|^2 + \left|\nabla \nabla \cdot \left(\widetilde{M} \nabla v_2\right)\right|^2\right) e^{2s\alpha(\theta,x)}\,\mathrm{d}x \\
&\le C \int_{(0,T)\times\Omega} \left(\left|\widetilde{K}\right|^2 + \left|\nabla \widetilde{K}\right|^2 + s\varphi \left|\widetilde{M}\right|^2 + s\varphi \left|\nabla \widetilde{M}\right|^2\right) e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + C(s,\lambda)\left(\|\partial_t U\|^2_{H^1(0,T;L^2(\omega'))} + \|\partial_t U\|^2_{L^2(0,T;H^2(\omega'))}\right). \qquad (2.3.11)
\end{aligned}$$

Step 3 In order to derive the estimate for $\widetilde{K}$ and $\widetilde{M}$, we need the following lemma:

Lemma 2.3.1. *Consider the operator*

$$Lf(x) := \nabla \cdot (f(x) \nabla b(x)).$$

Suppose that $b \in W^{2,\infty}(\Omega)$ and

$$|\nabla d(x) \cdot \nabla b(x)| \neq 0 \quad x \in \Omega.$$

Then there exists $\lambda_0 > 0$ such that for any $\lambda \ge \lambda_0$, there exists $s_0 > 0$ such that for any $s \ge s_0$,

$$\int_\Omega s^2 \lambda^2 \varphi\left(\theta, x\right)^2 \left(f^2 + |\nabla f|^2\right) e^{2s\varphi(\theta,\cdot)}\,\mathrm{d}x \le C \int_\Omega \left(|Lf|^2 + |\nabla\left(Lf\right)|^2\right) e^{2s\varphi(\theta,x)}\,\mathrm{d}x$$

for any $f \in H_0^2\left(\Omega\right)$.

Proof. Set $g = e^{s\varphi(\theta,\cdot)} f$. Then we have

$$e^{s\varphi(\theta,\cdot)} L\left(e^{-s\varphi(\theta,\cdot)} g\right) = \nabla \cdot (g \nabla b) - s\lambda\varphi(\theta,\cdot)\left(\nabla d \cdot \nabla b\right) g.$$

Integration by parts yields

$$
\begin{aligned}
&\left\|e^{s\varphi(\theta,\cdot)}Lf\right\|_{L^2(\Omega)}^2 \\
&\geq \int_\Omega s^2\lambda^2\varphi(\theta,\cdot)^2 |\nabla d\cdot\nabla b|^2 g^2 \,\mathrm{d}x - \int_\Omega 2s\lambda\varphi(\theta,\cdot)(\nabla d\cdot\nabla b) g\nabla\cdot(g\nabla b)\,\mathrm{d}x \\
&\geq \int_\Omega s^2\lambda^2\varphi(\theta,\cdot)|\nabla d\cdot\nabla b|^2 g^2\,\mathrm{d}x \\
&\quad - \int_\Omega s\lambda\varphi(\theta,\cdot)(\nabla d\cdot\nabla b)\left(\nabla\cdot\left(g^2\nabla b\right) - g^2\Delta b\right)\mathrm{d}x \\
&\geq \int_\Omega s^2\lambda^2\varphi(\theta,\cdot)|\nabla d\cdot\nabla b|^2 g^2\,\mathrm{d}x \\
&\quad + \int_\Omega s\lambda^2\varphi(\theta,\cdot)|\nabla d\cdot\nabla b|^2 g^2\,\mathrm{d}x + \int_\Omega s\lambda\varphi(\theta,\cdot)\nabla b\cdot\nabla(\nabla d\cdot\nabla b) g^2\,\mathrm{d}x \\
&\quad + \int_\Omega s\lambda\varphi(\theta,\cdot)(\Delta b)(\nabla d\cdot\nabla b) g^2\,\mathrm{d}x \\
&\geq \int_\Omega s^2\lambda^2\varphi(\theta,\cdot)|\nabla d\cdot\nabla b|^2 g^2\,\mathrm{d}x - C\int_\Omega s\lambda^2\varphi(\theta,\cdot) g^2\,\mathrm{d}x.
\end{aligned}
$$

Hence by assumption $|\nabla d\cdot\nabla b|\neq 0$, we have

$$
\begin{aligned}
&\int_\Omega s^2\lambda^2\varphi(\theta,x)^2 f^2 e^{2s\varphi(\theta,x)}\,\mathrm{d}x \\
&\qquad\leq C\int_\Omega |Lf|^2 e^{2s\varphi(\theta,x)}\,\mathrm{d}x + C\int_{\partial\Omega} s\lambda\varphi(\theta,\cdot) f^2 e^{2s\varphi(\theta,x)}\,\mathrm{d}S. \qquad (2.3.12)
\end{aligned}
$$

In order to estimate ∇f, we use the identity

$$
L(\partial_i f) = \partial_i(Lf) - \nabla\cdot(f\nabla\partial_i b).
$$

By applying (2.3.12) to this identity, we see that

$$
\begin{aligned}
&\int_\Omega s^2\lambda^2\varphi(\theta,x)^2|\nabla f|^2 e^{2s\varphi(\theta,\cdot)}\,\mathrm{d}x \\
&\quad\leq C\int_\Omega |\nabla(Lf)|^2 e^{2s\varphi(\theta,x)}\,\mathrm{d}x + C\int_{\partial\Omega} s\lambda\varphi(\theta,\cdot)|\nabla f|^2 e^{2s\varphi(\theta,x)}\,\mathrm{d}S \\
&\qquad\qquad + C\int_\Omega\left(|f|^2+|\nabla f|^2\right)e^{2s\varphi(\theta,x)}\,\mathrm{d}x. \qquad (2.3.13)
\end{aligned}
$$

By combining (2.3.12) and (2.3.13) and by taking sufficiently large $s,\lambda>0$, we complete the proof. □

Since $\widetilde{K}=\widetilde{M}=0$ near $\partial\Omega$ by assumption, Lemma 2.3.1 and (2.3.11) yields

$$\int_{\Omega} s^2\lambda^2\varphi^2 \left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2 + \left|\widetilde{M}\right|^2 + \left|\nabla\widetilde{M}\right|^2\right) e^{2s\alpha(\theta,x)}\,\mathrm{d}x$$
$$\leq C\int_{(0,T)\times\Omega} \left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2 + s\varphi\left|\widetilde{M}\right|^2 + s\varphi\left|\nabla\widetilde{M}\right|^2\right) e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x$$
$$+ C(s,\lambda)\left(\|U\|^2_{H^3(0,T;L^2(\omega'))} + \|U\|^2_{H^2(0,T;H^2(\omega'))}\right). \quad (2.3.14)$$

Since $\alpha(\theta,x) \geq \alpha(t,x)$ for any $t \in (0,T)$, we have

$$C\int_{(0,T)\times\Omega} \left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2 + s\varphi\left|\widetilde{M}\right|^2 + s\varphi\left|\nabla\widetilde{M}\right|^2\right) e^{2s\alpha}\,\mathrm{d}t\,\mathrm{d}x$$
$$\leq CT\int_{\Omega} \left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2 + s\varphi\left|\widetilde{M}\right|^2 + s\varphi\left|\nabla\widetilde{M}\right|^2\right) e^{2s\alpha}\,\mathrm{d}x. \quad (2.3.15)$$

Combining (2.3.14) and (2.3.15), and choosing sufficiently large $s > 0$ and $\lambda > 0$, we finally have

$$\int_{\Omega} s^2\lambda^2\varphi^2 \left(\left|\widetilde{K}\right|^2 + \left|\nabla\widetilde{K}\right|^2 + \left|\widetilde{M}\right|^2 + \left|\nabla\widetilde{M}\right|^2\right) e^{2s\alpha(\theta,x)}\,\mathrm{d}x$$
$$\leq C(s,\lambda)\left(\|U\|^2_{H^3(0,T;L^2(\omega'))} + \|U\|^2_{H^2(0,T;H^2(\omega'))}\right). \quad (2.3.16)$$

Now there exists $C > 0$ such that $e^{Cs} \leq e^{2s\alpha(\theta,x)}$ for $x \in \overline{\Omega}$, we have

$$s^2 e^{Cs}\left(\left\|\widetilde{K}\right\|^2_{H^1(\Omega)} + \left\|\widetilde{M}\right\|^2_{H^1(\Omega)}\right) \leq C(s,\lambda)\left(\|U\|^2_{H^3(0,T;L^2(\omega'))} + \|U\|^2_{H^2(0,T;H^2(\omega'))}\right).$$

By dividing it by $s^2 e^{Cs}$ and by fixing sufficiently large $s > 0$ and $\lambda > 0$, the proof is completed.

Chapter 3

Inverse problem of a structured population model

3.1 Introduction

Structured population models describes the change of distribution of individuals in a population. In these models, individuals are described by using several parameters– for example, age, size and so on– and a population density is considered as a function of not only time and spatial position but these individual parameters. In this meaning, we can say that structured population models describes "the detail structure of population." These models originated in Sharpe and Lotka [26] and McKendrik [22] and have been widely studied in the mathematical biology.

In this paper, we consider one of structured population models stated in Webb [27] in which age and size are considered as individual parameters. The model is described as follows: Let $\Omega \subset \mathbb{R}^3$ be an open set which represents an inhabited area and a_1, s_1, s_2, T be positive real constants. Then the main equation of our model is

$$\begin{aligned} &\partial_t u(t,x,a,s) + \partial_a u(t,x,a,s) + \partial_s \left(g(s)u(t,x,a,s)\right) \\ &= \alpha(x)\Delta u(t,x,a,s) - \nabla\cdot(\gamma(x)u(t,x,a,s)) - \mu\left(x,a,s,u(t,x,a,s)\right)u(t,x,a,s) \\ &\qquad (t,x,a,s) \in (0,T)\times\Omega\times(0,a_1)\times(s_1,s_2) \end{aligned} \tag{3.1.1}$$

with initial and boundary conditions

$$u(t,x,0,s) = \int_0^{a_1}\int_{s_1}^{s_2} \beta\left(x,a,s,\widehat{s}\right)u(t,x,a,\widehat{s})\,\mathrm{d}\widehat{s}\,\mathrm{d}a \qquad (t,x,s) \in (0,T)\times\Omega\times(s_1,s_2) \tag{3.1.2}$$

$$u(t,x,a,s_1) = 0 \quad (t,x,a) \in (0,T)\times\Omega\times(0,a_1) \tag{3.1.3}$$

$$u(0,x,a,s) = p(x,a,s) \quad (x,a,s) \in \Omega\times(0,a_1)\times(s_1,s_2) \tag{3.1.4}$$

$$\partial_\nu u = 0 \quad \text{on } (0,T) \times \partial\Omega \times (0,a_1) \times (s_1,s_2) \tag{3.1.5}$$

where

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$$

and ∂_s, ∂_a are defined similarly.

Interpretation of equation 3.1.1 is as follows: a is age of individual and s a size. $u(t,x,a,s)$ can be interpreted as the population density at time t, at position x, at age a and at size s. Moreover

- $\partial_a u(t,x,a,s)$ represents *aging effect.* (The coefficient is always exactly 1 because age increases exactly 1 per a year.)
- $\partial_s (g(s)u(t,x,a,s))$ represents *growth effect* with $g(s)$ a *growth modulus* (that is, $\int_s^{s'} 1/g(\sigma)\, d\sigma$ is a spending time to grow the individual from size s to size s').
- $\alpha(x)\Delta u(t,x,a,s)$ represents *diffusion* with $\alpha(x) > 0$ a diffusion coefficient.
- $-\nabla\cdot(\gamma(x)u(t,x,a,s))$ represents *taxis* with $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))^T$ a taxis coefficient.
- $-\mu(x,a,s,u(t,x,a,s))\, u(t,x,a,s)$ represents *mortality* with mortality rate $\mu(x,a,s,u)$.
- The initial condition (3.1.2) of $u(t,x,0,s)$ represents birth with birth rate β.

In Webb [27], the existence of the solution of the system (3.1.1)–(3.1.5) is considered and proved by the semigroup theory.

Our interest lies in inverse problems. To state the problems precisely, we have to make more preparation. First, although the original equation is non-linear, we consider the following "linearized" version for simplicity:

$$\begin{aligned} Lu(t,x,a,s) &:= \partial_t u(t,x,a,s) + \partial_a u(t,x,a,s) + \partial_s (g(s)u(t,x,a,s)) \\ &\quad - \alpha(x)\Delta u(t,x,a,s) + \gamma(x)\cdot\nabla u(t,x,a,s) + \mu(x)\, u(t,x,a,s) = 0 \\ &\qquad (t,x,a,s) \in Q_T := (0,T)\times\Omega\times(0,a_1)\times(s_1,s_2) \end{aligned} \tag{3.1.6}$$

Second we introduce more notation. Let $\Gamma \subset \partial\Omega$ be an open subboundary of Ω and set $\Sigma_T := (0,T)\times\Gamma\times(0,a_1)\times(s_1,s_2)$. $\theta \in (0,T)$ is fixed. Then our inverse problem is stated as follows:

Inverse problem (IP): Let $u(t,x,a,s)$ solve the equation (3.1.6) with g known and α, γ and μ unknown. Determine α, γ and μ from the (possibly several time) observations of

$$u|_{\Sigma_T}, \quad \partial_\nu u|_{\Sigma_T}, \quad u|_{t=\theta}.$$

In order to state the main theorem, we need same preparation. Let u_1 and u_2 satisfies (3.1.6) associated to the diffusion coefficients α_1 and α_2 respectively. $W^{m,p}$ and L^p denote the classical Sobolev and Lebesgue space. We write $H^m := W^{m,2}$. We denote

$$H^{l,m}(Q_T) := H^l((0,T)\times(0,a_1)\times(s_1,s_2);H^m(\Omega)).$$

$u_j(t,x,a,s;q)$ solves the following equations:

$$\begin{cases} \partial_t u_j + \partial_a u_j + \partial_s(gu_j) - \alpha_j \Delta u_j + \gamma_j \cdot \nabla u_j + \mu_j u_j = 0 & \text{in } Q_T \\ u_j(\theta,x,a,s;q) = q(x,a,s) & \text{on } \Omega \times (0,a_1)\times(s_1,s_2). \end{cases}$$

We take $p_j \in C^2(\Omega)$ $(j = 1,2,\dots,5)$ and pose the following assumptions:
Assumption:

1. $g \in L^\infty(s_1,s_2)$ and there exists $M > 0$ such that $\|g\|_{L^\infty(s_1,s_2)} \le M$.
2. $\alpha_1, \alpha_2, \mu_1, \mu_2 \in L^2(\Omega)$, $\gamma_1, \gamma_2 \in \left(L^2(\Omega)\right)^3$
3. $|\alpha_1|, |\alpha_2| \ne 0$ on Ω.
4. $\alpha_1 = \alpha_2$, $\gamma_1 = \gamma_2$, $\mu_1 = \mu_2$ on Γ.
5. The matrix

$$A := \begin{pmatrix} \Delta p_1 & \partial_1 p_1 & \partial_2 p_1 & \partial_3 p_1 & p_1 \\ \Delta p_2 & \partial_1 p_2 & \partial_2 p_2 & \partial_3 p_2 & p_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta p_5 & \partial_1 p_5 & \partial_2 p_5 & \partial_3 p_5 & p_5 \end{pmatrix}$$

   is invertible on $\overline{\Omega \times (0,a_1)\times(s_1,s_2)}$.

6. $\|u_j(\cdot,\cdot,\cdot,\cdot;p_k)\|_{C^2(Q_T)} \le M$.

Our main result is the stability estimate for (IP):

Theorem 3.1.1. *Let $\theta \in (0,T)$ be a fixed constant and suppose that Assumptions are satisfied. Let ω be an arbitrary domain which satisfies $\overline{\partial\omega \cap \partial\Omega} \subset \Gamma$ and $\operatorname{dist}\left(\omega, \overline{\partial\Omega \setminus \Gamma}\right) > 0$. Then there exist constants $C = C(T,\omega) > 0$ and $\kappa \in (0,1)$ such that*

$$\begin{aligned} &\|\alpha_1 - \alpha_2\|_{L^2(\Omega)} + \|\gamma_1 - \gamma_2\|_{(L^2(\Omega))^2} + \|\mu_1 - \mu_2\|_{L^2(\Omega)} \\ &\le CM^{1-\kappa}\left(\sum_{j=1}^{5}\left(\|u_1(p_j) - u_2(p_j)\|_{H^{0,2}(\Sigma_T)} + \|u_1(p_j) - u_2(p_j)\|_{H^{1,1}(\Sigma_T)}\right)\right)^\kappa. \end{aligned} \tag{3.1.7}$$

Example 1. We will give an example which satisfies the assumptions of Theorem 3.1.1. Let us take $p_1 = x_1^2$, $p_2 = x_1$, $p_3 = x_2$, $p_4 = x_3$, $p_5 = 1$. Then the matrix A is written as

$$A = \begin{pmatrix} 2 & 2x_1 & 0 & 0 & x_1^2 \\ 0 & 1 & 0 & 0 & x_1 \\ 0 & 0 & 1 & 0 & x_2 \\ 0 & 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and the determinant of A does not vanish ($\det A = 2$).

The key ingredient to prove Theorem 3.1.1 is Carleman estimate and Bukhgaim-Klibanov method.

Since Bukhgaim and Klibanov's pioneer work [2], Carleman estimates have been widely used to prove uniqueness and stability of inverse problems. Especially, for parabolic equations, see and [28] is a review for Carleman estimates and parabolic inverse problems.

In our case, however, The operator L in 3.1.6 is different from an ordinary parabolic operator, $\partial_t - \alpha\Delta$ for example, because L contains not only t-derivative but also a- and s-derivatives. For this reason, we have to make an original Carleman estimate for operator L.

We remark about assumption for Theorem (3.1.1). In order to prove the stability for the inverse problem by Carleman estimates, A positivity condition such as 5 is always needed. Moreover, we cannot choose $\theta = 0$ in our method and This case is a long standing open problem even for an ordinary parabolic equation.

The article is organized as follows: In Section 2, we establish the key Carleman estimate for operator L and in Section 3, we prove Theorem 3.1.1 by using the Carleman estimate and Bukhgeim-Klibanov method.

3.2 Carleman estimate

In this section, we prove a Carleman estimate for operator L. Let us write down again the definition of L:

$$\begin{aligned} Lu(t,x,a,s) &:= \partial_t u(t,x,a,s) + \partial_a u(t,x,a,s) + \partial_s \left(g(s)u(t,x,a,s)\right) \\ &\quad - \alpha(x)\Delta u(t,x,a,s) + \nabla \cdot (\gamma(x)u(t,x,a,s)) + \mu(x)u(t,x,a,s) \end{aligned} \tag{3.2.1}$$

To state the Carleman estimate, we set some notation. We denote

$$L_0 u := \partial_t u + \partial_a u + \partial_s \left(g(s)u\right).$$

we write in short

$$\nabla_{x,a,s,t} u := \left(\partial_t u, (\nabla u)^T, \partial_a u, \partial_s u\right)^T,$$

and similarly in $\nabla_{x,t}$, $\nabla_{a,s,t}$ and so on.

Next we define a weight function. We use the following lemma:

Lemma 3.2.1. *If Ω has a smooth boundary, then there exists $d \in C^2\left(\overline{\Omega}\right)$ such that*

$$d(x) > 0,\ x \in \Omega, \quad d(x) = 0,\ x \in \partial\Omega \setminus \Gamma, \quad |\nabla d(x)| > 0,\ x \in \overline{\Omega}. \tag{3.2.2}$$

For the proof of Lemma 3.2.1, see [28].

Now we state the Carleman estimate.

Lemma 3.2.2. *Let $a_0 \in (0, a_1)$ and $s_0 \in (s_1, s_2)$ be a fixed constant. Set*

$$\psi(x, a, s, t) := d(x) - \beta_1 |a - a_0|^2 - \beta_2 |s - s_0|^2 - \beta_3 |t - \theta|^2,$$

where $d(\cdot) \in C^2(\mathbb{R}^3)$ satisfies $|\nabla d| \neq 0$ in $\overline{\Omega}$, and $\varphi := e^{\lambda\psi}$. Then there exists a constant $\lambda_0 > 0$ such that for arbitrary $\lambda \geq \lambda_0$, we can choose a constant $\tau_0(\lambda) > 0$ satisfying: there exists a constant $C = C(\tau_0, \lambda_0) > 0$ such that

$$\begin{aligned}&\int_Q \left(\frac{1}{\tau\varphi}\left(|L_0 u|^2 + |\Delta u|^2\right) + \tau\lambda^2\varphi\,|\nabla u|^2 + \tau^3\lambda^4\varphi^3 u^2\right) e^{2\tau\varphi}\,dx\,da\,ds\,dt \\ &\leq C\int_Q |Lu|^2 e^{2\tau\varphi}\,dx\,da\,ds\,dt + Ce^{C(\lambda)s}\int_{\Sigma_T}\left(|\nabla_{x,a,s,t}u|^2 + |u|^2\right) dS\,da\,ds\,dt.\end{aligned} \tag{3.2.3}$$

for any $u \in H^2(Q)$ with $u = 0$ on $(\partial\Omega \setminus \Gamma) \times (0, a_1) \times (s_1, s_2) \times (0, T)$.

Proof. Let $u := e^{-\tau\varphi}v$ and $Pv := e^{\tau\varphi}L\left(e^{-\tau\varphi}v\right)$. Simple calculation yields

$$Pv = L_0 v - \alpha\Delta v - \tau^2\lambda^2\varphi^2\alpha\,|\nabla\psi|^2 v + \tau\lambda^2\varphi a_1(x,t)v + 2\tau\lambda\varphi\alpha\nabla\psi\cdot\nabla v. \tag{3.2.4}$$

Then we divide Pv as follows:

$$\begin{aligned}P_1 v &:= -\alpha\Delta v - \tau^2\lambda^2\varphi^2\alpha\,|\nabla\psi|^2 v + \tau\lambda^2\varphi a_1(x,t)v,\\ P_2 v &:= L_0 v + 2\tau\lambda\varphi\alpha\nabla\psi\cdot\nabla v.\end{aligned}$$

Since $(Lu)\,e^{\tau\varphi} = P_1 v + P_2 v$, we have

$$2\int_Q (P_1 v)(P_2 v)\,dx\,da\,ds\,dt + \|P_2 v\|^2_{L^2(Q)} \leq \int_Q |Lu|^2 e^{2\tau\varphi}\,dx\,da\,ds\,dt. \tag{3.2.5}$$

Next we will estimate $\int_Q P_1vP_2v\,dx\,da\,ds\,dt$. Expanding the product P_1vP_2v, we obtain

$$
\begin{aligned}
&\int_Q (P_1v)(P_2v)\;dx\,da\,ds\,dt \\
&= -\int_Q \alpha\,(\Delta v)\,(L_0v)\;dx\,da\,ds\,dt - \int_Q \tau^2\lambda^2\varphi^2\alpha\,|\nabla\psi|^2\,v\,(L_0v)\;dx\,da\,ds\,dt \qquad (3.2.6)\\
&\quad+\int_Q \tau^2\lambda^2\varphi^2a_1v\,(L_0v)\;dx\,da\,ds\,dt - \int_Q 2\tau\lambda\varphi\alpha^2\,(\Delta v)\,(\nabla\psi\cdot\nabla v)\;dx\,da\,ds\,dt\\
&\quad-\int_Q 2\tau^3\lambda^3\varphi^3\alpha^2\,|\nabla\psi|^2\,v\,(\nabla\psi\cdot\nabla v)\;dx\,da\,ds\,dt\\
&\quad+\int_Q 2\tau^2\lambda^3\varphi^2\alpha a_1v\,(\nabla\psi\cdot\nabla v)\;dx\,da\,ds\,dt\\
&=:\sum_{j=1}^{6} I_6. \qquad (3.2.7)
\end{aligned}
$$

$$
\begin{aligned}
|I_1| &= \left|-\int_Q \alpha\,(\Delta v)\,(L_0v)\;dx\,da\,ds\,dt\right|\\
&\le \left|\int_Q (\nabla\alpha\cdot\nabla v)\,(L_0v)\;dx\,da\,ds\,dt\right| + \left|\int_Q \alpha\nabla v\cdot L_0\,(\nabla v)\;dx\,ds\,da\,dt\right|\\
&\quad+\left|\int_{\widetilde{\Gamma}} \alpha\,(\partial_\nu v)\,(L_0v)\;dx\,da\,ds\,dt\right| \qquad (3.2.8)\\
&= \left|\int_Q (\nabla\alpha\cdot\nabla v)\,(L_0v)\;dx\,da\,ds\,dt\right| + \left|\frac{1}{2}\int_Q \alpha L_0\left(|\nabla v|^2\right)\;dx\,ds\,da\,dt\right|\\
&\quad+\left|\int_{\widetilde{\Gamma}} \alpha\,(\partial_\nu v)\,(L_0v)\;dx\,da\,ds\,dt\right| \qquad (3.2.9)\\
&\le C\int_Q |\nabla v|\,|L_0v|\;dx\,da\,ds\,dt + C\int_{\widetilde{\Gamma}} |\nabla_{x,a,s,t}v|^2\;dS\,da\,ds\,dt. \qquad (3.2.10)
\end{aligned}
$$

Next

$$
\begin{aligned}
|I_2| &= \left|-\int_Q \tau^2\lambda^2\varphi^2\alpha\,|\nabla\psi|^2\,v\,(L_0v)\;dx\,da\,ds\,dt\right|\\
&= \left|-\frac{1}{2}\int_Q \tau^2\lambda^2\varphi^2\alpha\,|\nabla\psi|^2\,(L_0v^2)\;dx\,da\,ds\,dt\right|\\
&\le C\int_Q \tau^2\lambda^3\varphi^2v^2\,dx\,da\,ds\,dt, \qquad (3.2.11)
\end{aligned}
$$

and similar argument yields

$$
|I_3| \le C\int_Q \tau^2\lambda^3\varphi^2v^2\,dx\,da\,ds\,dt. \qquad (3.2.12)
$$

Integration by parts yields

$$\begin{aligned}
I_4 &= -\int_Q \tau\lambda\varphi\alpha^2 \nabla\left(\nabla\psi \cdot |\nabla v|^2\right)\,dx\,da\,ds\,dt \\
&\geq \int_Q \tau\lambda^2\varphi\alpha^2 |\nabla\psi|^2 |\nabla v|^2\,dx\,da\,ds\,dt + 2\int_Q \tau\lambda\varphi\alpha\,(\nabla\alpha\cdot\nabla\psi)\,|\nabla v|^2\,dx\,da\,ds\,dt \\
&\quad + \int_Q \tau\lambda\varphi\alpha^2\,(\Delta\psi)\,|\nabla v|^2\,dx\,da\,ds\,dt - C\int_{\widetilde{\Gamma}} \tau\lambda\varphi\,|\nabla v|^2\,dx\,da\,ds\,dt \\
&\geq \int_Q \tau\lambda^2\varphi\alpha^2 |\nabla\psi|^2 |\nabla v|^2\,dx\,da\,ds\,dt - C\int_Q \tau\lambda\varphi\,|\nabla v|^2\,dx\,da\,ds\,dt \\
&- C\int_{\widetilde{\Gamma}} \tau\lambda\varphi\,|\nabla v|^2\,dS\,da\,ds\,dt, \qquad (3.2.13)
\end{aligned}$$

$$\begin{aligned}
I_5 &= -\int_Q \tau^3\lambda^3\varphi^3\alpha^2 |\nabla\psi|^2\left(\nabla\psi\cdot\nabla\left(v^2\right)\right)\,dx\,da\,ds\,dt \\
&\geq 3\int_Q \tau^3\lambda^4\varphi^3\alpha^2 |\nabla\psi|^4 v^2\,dx\,da\,ds\,dt \\
&\quad + \int_Q \tau^3\lambda^3\varphi^3\left(\nabla\left(\alpha^2|\nabla\psi|^2\right)\cdot\nabla\psi\right)v^2\,dx\,da\,ds\,dt \qquad (3.2.14) \\
&\quad + \int_Q \tau^3\lambda^3\varphi^3\alpha^2 |\nabla\psi|^2(\Delta\psi)\,v^2\,dx\,da\,ds\,dt - C\int_{\widetilde{\Gamma}} \tau^3\lambda^3\varphi^3 v^2\,dx\,da\,ds\,dt \\
&\geq 3\int_Q \tau^3\lambda^4\varphi^3\alpha^2 |\nabla\psi|^4 v^2\,dx\,da\,ds\,dt - C\int_Q \tau^3\lambda^3\varphi^3 v^2\,dx\,da\,ds\,dt \\
&\qquad (3.2.15) \\
&\quad - C\int_{\widetilde{\Gamma}} \tau^3\lambda^3\varphi^3 v^2\,dS\,da\,ds\,dt \qquad (3.2.16)
\end{aligned}$$

and

$$\begin{aligned}
|I_6| &= \left|\int_Q \tau^2\lambda^3\varphi^2\alpha a_1\left(\nabla\psi\cdot\nabla\left(v^2\right)\right)\,dx\,da\,ds\,dt\right| \\
&\leq \int_Q \tau^2\lambda^4\varphi^2 v^2\,dx\,da\,ds\,dt + C\int_{\widetilde{\Gamma}} \tau^2\lambda^3\varphi^2 v^2\,dS\,da\,ds\,dt. \qquad (3.2.17)
\end{aligned}$$

Hence combining (3.2.10)–(3.2.17) and (3.2.7), we finally obtain

$$
\begin{aligned}
&\int_Q (P_1 v)\,(P_2 v)\;dx\,da\,ds\,dt \\
&\geq 3\int_Q \tau^3\lambda^4\varphi^3\alpha^2\,|\nabla\psi|^4\,v^2\,dx\,da\,ds\,dt + \int_Q \tau\lambda^2\varphi\alpha^2\,|\nabla\psi|^2\,|\nabla v|^2\;dx\,da\,ds\,dt \qquad (3.2.18)\\
&\quad - C\int_Q |\nabla v|\,|L_0 v|\;dx\,da\,ds\,dt - C\int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) v^2\,dx\,da\,ds\,dt \\
&\quad + C\int_{\widetilde{\Gamma}} \tau^3\lambda^3\varphi^3\,|\nabla_{x,a,s,t} v|^2\;dS\,da\,ds\,dt. \qquad (3.2.19)
\end{aligned}
$$

Next we estimate the norm of $P_2 v$.

$$
\begin{aligned}
\|P_2 v\|^2_{L^2(Q)} &\geq \int_Q \frac{1}{\tau\varphi}\,|P_2 v|^2\;dx\,da\,ds\,dt \\
&= \int_Q \frac{1}{\tau\varphi}\,(L_0 v + 2\tau\lambda\varphi\alpha\nabla\psi\cdot\nabla v)^2\;dx\,da\,ds\,dt \\
&\geq \frac{1}{2}\int_Q \frac{1}{\tau\varphi}\,|L_0 v|^2\;dx\,da\,ds\,dt - C\int_Q \tau\lambda^2\varphi\,|\nabla v|^2\;dx\,da\,ds\,dt,
\end{aligned}
$$

where we used an inequality $|a+b|^2 \geq \frac{1}{2}a^2 - b^2$. This implies

$$
\epsilon\int_Q \frac{1}{\tau\varphi}\,|L_0 v|^2\;dx\,da\,ds\,dt \leq \|P_2 v\|^2_{L^2(Q)} + C\epsilon\int_Q \tau\lambda^2\varphi\,|\nabla v|^2\;dx\,da\,ds\,dt \qquad (3.2.20)
$$

for any $\epsilon > 0$. By (3.2.5), (3.2.19) and (3.2.20), we obtain

$$
\begin{aligned}
&\epsilon\int_Q \frac{1}{\tau\varphi}\,|L_0 v|^2\;dx\,da\,ds\,dt + 3\int_Q \tau^3\lambda^4\varphi^3\alpha^2\,|\nabla\psi|^4\,v^2\,dx\,da\,ds\,dt \\
&\quad + \int_Q \tau\lambda^2\varphi\alpha^2\,|\nabla\psi|^2\,|\nabla v|^2\;dx\,da\,ds\,dt - C\epsilon\int_Q \tau\lambda^2\varphi\,|\nabla v|^2\;dx\,da\,ds\,dt \\
&\quad \leq \int_Q |L_0 u|^2\,e^{2\tau\varphi}\,dx\,da\,ds\,dt + C\int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) v^2\,dx\,da\,ds\,dt \\
&\qquad + C\int_Q |\nabla v|\,|L_0 v|\;dx\,da\,ds\,dt + C\int_{\widetilde{\Gamma}} \tau^3\lambda^3\varphi^3\,|\nabla_{x,a,s,t} v|^2\;dS\,da\,ds\,dt. \qquad (3.2.21)
\end{aligned}
$$

Next we have to estimate $\int_Q |\nabla v|\,|L_0 v|\;dx\,da\,ds\,dt$. By Cauchy-Schwartz inequality, we have

$$
|\nabla v|\,|L_0 v| = \tau^{1/2}\lambda^{1/2}\varphi^{1/2}\,|\nabla v|\,\tau^{-1/2}\lambda^{-1/2}\varphi^{-1/2}\,|L_0 v| \leq \frac{1}{2}\tau\lambda\varphi\,|\nabla v|^2 + \frac{1}{2\tau\lambda\varphi}\,|L_0 v|^2 .
$$

Then taking sufficiently large $\lambda > 0$, (3.2.21) yields

$$\begin{aligned}
&\epsilon \int_Q \frac{1}{\tau\varphi} |L_0 v|^2 \, dx\, da\, ds\, dt + 3 \int_Q \tau^3 \lambda^4 \varphi^3 \alpha^2 |\nabla\psi|^4 v^2 \, dx\, da\, ds\, dt \\
&\quad + \int_Q \tau\lambda^2 \varphi \alpha^2 |\nabla\psi|^2 |\nabla v|^2 \, dx\, da\, ds\, dt - C\epsilon \int_Q \tau\lambda^2\varphi |\nabla v|^2 \, dx\, da\, ds\, dt \\
&\quad \le \int_Q |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + C \int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) v^2 \, dx\, da\, ds\, dt \\
&\qquad + C \int_{\widetilde{\Gamma}} \tau^3\lambda^3\varphi^3 |\nabla_{x,a,s,t} v|^2 \, dS\, da\, ds\, dt
\end{aligned}$$

for large $\lambda \ge \lambda_0$ and $\tau \ge \tau_0(\lambda)$. We choose sufficiently small $\epsilon > 0$ such that $\inf_Q \alpha^2 |\nabla\psi|^2 - C\epsilon > 0$ and then it follows that

$$\begin{aligned}
&\int_Q \frac{1}{\tau\varphi} |L_0 v|^2 \, dx\, da\, ds\, dt + \int_Q \tau^3\lambda^4\varphi^3 v^2 \, dx\, da\, ds\, dt + \int_Q \tau\lambda^2\varphi |\nabla v|^2 \, dx\, da\, ds\, dt \\
&\quad \le \int_Q |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + C \int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) v^2 \, dx\, da\, ds\, dt \\
&\qquad + C \int_{\widetilde{G}} \tau^3\lambda^3\varphi^3 |\nabla_{x,a,s,t} v|^2 \, dS\, da\, ds\, dt.
\end{aligned}$$

Substituting $v = e^{\tau\varphi} u$ into it, we have

$$\begin{aligned}
&\int_Q \frac{1}{\tau\varphi} |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + \int_Q \tau^3\lambda^4\varphi^3 u^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + \int_Q \tau\lambda^2\varphi |\nabla u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt \\
&\le \int_Q |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + C \int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) u^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt \\
&\qquad + C e^{C(\lambda)\tau} \int_{\widetilde{G}} \tau^3\lambda^3\varphi^3 |\nabla_{x,a,s,t} u|^2 \, dS\, da\, ds\, dt. \qquad (3.2.22)
\end{aligned}$$

Since $L = L_0 + (\partial_s g)\, u + \nabla \cdot (\gamma u)$, The estimate of Lu is given by

$$\int_Q |Lu|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt \le C \int_Q |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + C \int_Q \left(u^2 + |\nabla u|^2\right) e^{2\tau\varphi} \, dx\, da\, ds\, dt. \qquad (3.2.23)$$

Using (3.2.22) and (3.2.23), we obtain

$$\begin{aligned}
&\int_Q \frac{1}{\tau\varphi} |L_0 u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + \int_Q \tau^3\lambda^4\varphi^3 u^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + \int_Q \tau\lambda^2\varphi |\nabla u|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt \\
&\le \int_Q |Lu|^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt + C \int_Q \left(\tau^3\lambda^3\varphi^3 + \tau^2\lambda^4\varphi^2\right) u^2 e^{2\tau\varphi} \, dx\, da\, ds\, dt \\
&\qquad + C e^{C(\lambda)\tau} \int_{\widetilde{G}} \tau^3\lambda^3\varphi^3 |\nabla_{x,a,s,t} u|^2 \, dS\, da\, ds\, dt.
\end{aligned}$$

Finally we estimate the term Δu. . By (3.2.4), we have

$$|\alpha(x)\Delta v|^2 \le C\left(|Pv|^2 + |L_0 v|^2 + \tau^4\lambda^4\varphi^4|v|^2 + \tau^2\lambda^2\varphi^2|\nabla v|^2\right).$$

Then

$$\begin{aligned}\int_{Q_T} \frac{1}{\tau\varphi}|\alpha(x)\Delta v|^2 \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s &\le C\int_{Q_T}\left(\frac{1}{\tau\varphi}|L_0 v|^2 + \tau^3\lambda^4\varphi^3|v|^2 + \tau\lambda^2\varphi|\nabla v|^2\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s\\ &\quad + \int_{Q_T}|Pv|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s\\ &\le C\int_{Q_T}|Pv|^2\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s.\end{aligned}$$

Since $\alpha > 0$ in Ω, we complete the proof. □

3.3 The proof of Theorem 3.1.1

Let u_1 and u_2 solve the following problem:

$$\begin{cases} L_k u_k = 0 & \text{in } Q = \Omega\times(0,T)\times(0,a_1)\times(s_1,s_2) \\ u_k(\theta,\cdot,\cdot,\cdot) = p_k & \text{on } \Omega\times(0,a_1)\times(s_1,s_2)\end{cases} \quad (k=1,2)$$

where

$$L_k v := L_0 v - \alpha_k(x)\Delta v + \gamma_k(x)\cdot\nabla v - \mu_k(x)v,$$

$$L_0 v := \partial_t v + \partial_a v + \partial_s(g(s)v).$$

Set $y := u_1 - u_2$. y satisfies

$$L_0 y - \alpha_1(x)\Delta y + \gamma_1\cdot\nabla y - \mu_1(x,a)\,y = \widetilde{\alpha}(x)\Delta u_2 - \widetilde{\gamma}\cdot\nabla u_2 + \widetilde{\mu}(x)\,u_2. \quad (3.3.1)$$

Let us define $w = \chi L_0 y$. Then w_j satisfy the following equations:

$$\begin{aligned} L_0 w - \alpha_1\Delta w + \gamma_1\cdot\nabla w - \mu_1 w &= \chi\widetilde{\alpha}L_0\Delta u_2 - \chi\widetilde{\gamma}\cdot L_0\nabla u_2 + \chi\widetilde{\mu}L_0 u_2 + 2\widetilde{\alpha}\nabla\chi\cdot\nabla L_0 y\\ &\quad + (\partial_t\chi + \partial_a\chi + g_1\partial_s\chi - \alpha_1(\Delta\chi) + \gamma_1\cdot\nabla\chi)\,L_0 y.\end{aligned}$$

By applying Theorem, we obtain

$$\begin{aligned}&\int_Q\left(\frac{1}{\tau\varphi}|L_0 w|^2 + \tau\lambda^2\varphi|\nabla w|^2 + \tau^3\lambda^4\varphi^3|w|^2\right)e^{2\tau\varphi}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s\\ &\le C\int_Q\left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right)\chi^2 e^{2\tau\varphi}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s\\ &\quad + C\int_Q\left(|\nabla_{t,x,a,s}\chi|^2 + |\Delta\chi|^2\right)\left(\left|L_0^j y\right|^2 + \left|\nabla L_0^j y\right|^2\right)e^{2\tau\varphi}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s\\ &\quad + Ce^{C\tau}\int_\Sigma\left(|\nabla_{t,x,a,s}w_j|^2 + |w_j|^2\right)\mathrm{d}t\,\mathrm{d}S\,\mathrm{d}a\,\mathrm{d}s,\ (j=1,2).\end{aligned}$$

By $L_0 w = L_0^* \chi L_0 y + \chi L_0^2 y$ and $\nabla w = \nabla \chi L_0 y + \chi \nabla L_0 y$, we have

$$\begin{aligned}
&\int_Q \left(\frac{1}{\tau\varphi} \left|L_0^2 y\right|^2 + \tau\lambda^2\varphi \left|\nabla L_0 y\right|^2 + \tau^3\lambda^4\varphi^3 \left|L_0 y\right|^2 \right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\le C \int_Q \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\quad + C \int_Q \left(|\nabla_{t,x,a,s}\chi|^2 + |\Delta\chi|^2 \right) \left(|L_0 y|^2 + |\nabla L_0 y|^2 \right) e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s + Ce^{C\tau} F^2 \\
&\le C \int_Q \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\quad + Ce^{2\tau\exp(3\lambda\varepsilon)} \int_Q \left(|L_0 y|^2 + |\nabla L_0 y|^2 \right) \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s + Ce^{C\tau} F^2 \, (j = 1, 2).
\end{aligned}$$

By Stokes' theorem and $\chi = 0$ on $\Omega \times (0,T) \times (0, a_1) \times \{s_1, s_2\}$ and on $\Omega \times (0,T) \times \{0, a_1\} \times (s_1, s_2)$, we estimate

$$\begin{aligned}
&\int_{Q\cap\{t=\theta\}} |L_0 y(\theta, -)|^2 \chi^2 e^{2\tau\varphi}\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\le \int_Q L_0 \left(|L_0 y|^2 \chi^2 e^{2\tau\varphi} \right) \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \qquad (3.3.2) \\
&\le \int_Q \left(2\,(L_0 y)\,(L_0^2 y)\,\chi^2 + 2\,|L_0 y|^2 \chi L_0^* \chi + 2\tau\lambda\varphi L_0^* \psi\, |L_0 y|^2 \chi^2 \right) e^{2\tau\varphi} \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\le C \int_Q \tau^2\lambda^2\varphi^2 |L_0 y|^2 \chi^2 e^{2\tau\varphi} \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s + C \int_Q \frac{1}{\tau^2\lambda^2\varphi^2} \left|L_0^2 y\right|^2 \chi^2 e^{2\tau\varphi} \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\
&\le \frac{C}{\tau} \int_Q \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s + \frac{C}{\tau} e^{2\tau\exp(3\lambda\varepsilon)} M_1^2 + \frac{Ce^{C\tau}}{\tau} F^2. \qquad (3.3.3)
\end{aligned}$$

Substituting $t = \theta$ into (3.3.1), we have

$$L_0 y(\theta, \cdot, \cdot, \cdot) = \widetilde{\alpha}(x)\Delta p - \widetilde{\gamma} \cdot \nabla p + \widetilde{\mu}\,(x)\, p.$$

Then we can write this equation as the following matrix form:

$$A \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \\ \widetilde{\gamma}_3 \\ \widetilde{\mu} \end{pmatrix} = \mathbf{G},$$

where

$$\mathbf{G} := \begin{pmatrix} L_0 y(\theta, \cdot, \cdot, \cdot; p_1) \\ L_0 y(\theta, \cdot, \cdot, \cdot; p_2) \\ \vdots \\ L_0 y(\theta, \cdot, \cdot, \cdot; p_5) \end{pmatrix}$$

Hence, if $|\det A| \neq 0$, then we have

$$\begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\gamma}_1 \\ \widetilde{\gamma}_2 \\ \widetilde{\gamma}_3 \\ \widetilde{\mu} \end{pmatrix} = A^{-1}\mathbf{G}.$$

Then

$$\begin{aligned} &\int_{Q_T \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\ &\leq C \int_{Q_T \cap \{t=\theta\}} |\mathbf{G}|^2 \chi^2 e^{2\tau\varphi}\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\ &\leq \frac{C}{\tau} \int_{Q_T} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s + \frac{C}{\tau} e^{2\tau \exp(3\lambda\varepsilon)} M_1^2 + \frac{Ce^{C\tau}}{\tau} F^2. \end{aligned} \tag{3.3.4}$$

Since $\psi(\theta, x, a, s) \geq \psi(t, x, a, s)$ for any $(t, x, a, s) \in Q_T$, we obtain

$$\begin{aligned} &\int_Q \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) \chi^2 e^{2\tau\varphi} \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s \\ &\qquad \leq CT \int_{Q \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) \chi^2 e^{2\tau\varphi}\, \mathrm{d}x\, \mathrm{d}a\, \mathrm{d}s. \end{aligned} \tag{3.3.5}$$

Then by (3.3.4) and taking sufficiently large $\tau > 0$, we have

$$\int_{Q \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) \chi^2 e^{2\tau\varphi}\, dx da ds \leq \frac{C}{\tau} e^{2\tau \exp(3\lambda\varepsilon)} M_1^2 + \frac{Ce^{C\tau}}{\tau} F^2. \tag{3.3.6}$$

Since $\chi = 1$ on $Q_{4\varepsilon}$, we obtain

$$e^{2\tau \exp(4\lambda\varepsilon)} \int_{Q_{4\varepsilon} \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) dx da ds \leq \frac{C}{\tau} e^{2\tau \exp(3\lambda\varepsilon)} M_1^2 + \frac{Ce^{C\tau}}{\tau} F^2.$$

By dividing this inequality into $e^{2\tau \exp(4\lambda\varepsilon)}$, we have

$$\int_{Q_{4\varepsilon} \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2\right) dx da ds \leq \frac{C}{\tau} e^{-2\mu\tau} M_1^2 + \frac{Ce^{C\tau}}{\tau} F^2 \tag{3.3.7}$$

for any $\tau \geq \tau_0$, where $\mu = \exp(4\lambda\varepsilon) - \exp(3\lambda\varepsilon) > 0$. By replacing $Ce^{C\tau_0}$ with C, we obtain (3.3.7) for any $\tau \geq 0$. We choose τ as

$$\frac{C}{\tau} e^{-2\mu\tau} M_1^2 = \frac{Ce^{C\tau}}{\tau} F^2,$$

that is,

$$\tau = \frac{2}{2\mu + C} \log \frac{M_1}{F}.$$

Then

$$\int_{Q_{4\varepsilon} \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \, dx da ds \leq C M_1^{1-\kappa} F^{\kappa},$$

where $\kappa := \frac{4\mu}{2\mu + C}$.
Next we need to estimate $\|\widetilde{\alpha}\|^2_{L^2(5\varepsilon)} + \|\widetilde{\gamma}\|^2_{L^2(5\varepsilon)} + \|\widetilde{\mu}\|^2_{L^2(5\varepsilon)}$ by

$$\int_{Q_{4\varepsilon} \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \, dx \, da \, ds$$

By definition of Q_δ, we have

$$Q_{4\varepsilon} \cap \{t = \theta\} = \left\{ (x, a, s) \in Q_T \, ; \, d(x) - \beta_1 |a - a_0|^2 - \beta_2 |s - s_0|^2 \geq 4\varepsilon \right\}.$$

Then

$$\Omega_{5\varepsilon} \times (a_0 - \widetilde{\varepsilon}, a_0 + \widetilde{\varepsilon}) \times (s_0 - \widetilde{\varepsilon}, s_0 + \widetilde{\varepsilon}) \subset Q_{4\varepsilon} \cap \{t = \theta\},$$

where $\widetilde{\varepsilon} := \left(\frac{\varepsilon}{\beta_1 + \beta_2} \right)^{1/2}$. Hence

$$\begin{aligned} &4\widetilde{\varepsilon}^2 \left(\|\widetilde{\alpha}\|^2_{L^2(5\varepsilon)} + \|\widetilde{\gamma}\|^2_{L^2(5\varepsilon)} + \|\widetilde{\mu}\|^2_{L^2(5\varepsilon)} \right) \\ &\leq \int_{Q_{4\varepsilon} \cap \{t=\theta\}} \left(|\widetilde{\alpha}|^2 + |\widetilde{\gamma}|^2 + |\widetilde{\mu}|^2 \right) \, dx \, da \, ds \leq C M_1^{1-\kappa} F^{\kappa}. \end{aligned}$$

Then the proof is completed.

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