博士論文

Extensions between finite-dimensional simple modules over a generalized current Lie algebra

（一般化されたカレントリー代数上の有限次元単純加群の間の拡大）

小寺 諒介

Extensions between finite-dimensional simple modules over a generalized current Lie algebra

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Abstract

We calculate the first extension groups for finite-dimensional simple modules over an arbitrary generalized current Lie algebra, which includes the case of loop Lie algebras and their multivariable analogs.

1 Introduction

In this article we are concerned with finite-dimensional modules over a generalized current Lie algebra $A \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra defined over the complex number field $\mathbb{C}$ and A is a nonzero finitely generated commutative $\mathbb{C}$-algebra. This class of Lie algebras includes loop Lie algebras, current Lie algebras and their multivariable analogs. Since the category of finite-dimensional $A \otimes \mathfrak{g}$-modules is not semisimple in general, we need to study its homological properties. The purpose of this article is to give an answer to the following problem which naturally arises during the study.

Problem 1.1. Calculate $\mathrm{Ext}^1(V, V')$ for any finite-dimensional simple $A \otimes \mathfrak{g}$-modules V, V'.

Some results are known for special cases so far. Fialowski and Malikov [FM] give an answer for the case of loop Lie algebras or current Lie algebras under the assumption that V and V' are so-called evaluation modules. In fact they calculate $\mathrm{Ext}^i(V, V')$ for any finite-dimensional simple evaluation modules V, V' and all $i \geq 0$. The first extension groups for any finite-dimensional simple modules over a current Lie algebra are calculated by Chari and Greenstein [CG]. Our main theorem is regarded as a generalization of the result in [CG] although the approach is different. See Remark 3.5 and 3.7 for a more precise explanation on their result. Our approach is based on a work by Chari and Moura [CM], which determines the blocks of the category of finite-dimensional modules over a loop Lie algebra. One of the main tools used in [CM] is a family of the universal finite-dimensional highest weight modules called Weyl modules. In [CM] some knowledge on composition factors of Weyl modules is established (See Corollary 2.8) and they use it to determine the blocks. The notion of Weyl modules is generalized by Feigin and Loktev [FL] for an arbitrary generalized current Lie algebra. They also prove the properties of Weyl modules

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mentioned above in the general situation. Then techniques used in [CM] are applicable for the general case and in fact yield a stronger result than the block decomposition of the category.

Now we state the main result. For each maximal ideal $\mathfrak{m}$ of A and the finite-dimensional simple $\mathfrak{g}$-module $V(\lambda)$ with highest weight λ, the $A\otimes\mathfrak{g}$-module structure is defined on $V(\lambda)$ through the natural morphism $A\otimes\mathfrak{g}\to(A/\mathfrak{m})\otimes\mathfrak{g}\simeq\mathfrak{g}$. We denote by $V_{\mathfrak{m}}(\lambda)$ the resulting $A\otimes\mathfrak{g}$-module and call it the evaluation module associated with $V(\lambda)$ at $\mathfrak{m}$.

Theorem 1.2. *Let V, V' be finite-dimensional simple $A\otimes\mathfrak{g}$-modules.*

(i) *If* $\mathrm{Ext}^1(V,V')\neq 0$ *then*

$$V\simeq V_{\mathfrak{m}_1}(\lambda_1)\otimes\cdots\otimes V_{\mathfrak{m}_{r-1}}(\lambda_{r-1})\otimes V_{\mathfrak{m}_r}(\lambda_r)$$

and

$$V'\simeq V_{\mathfrak{m}_1}(\lambda_1)\otimes\cdots\otimes V_{\mathfrak{m}_{r-1}}(\lambda_{r-1})\otimes V_{\mathfrak{m}_r}(\lambda'_r)$$

for some nonnegative integer r, maximal ideals $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ of A, and dominant integral weights $\lambda_1,\dots,\lambda_r,\lambda'_r$.

(ii) *Suppose that*

$$V= V_{\mathfrak{m}_1}(\lambda_1)\otimes\cdots\otimes V_{\mathfrak{m}_{r-1}}(\lambda_{r-1})\otimes V_{\mathfrak{m}_r}(\lambda_r)$$

and

$$V'= V_{\mathfrak{m}_1}(\lambda_1)\otimes\cdots\otimes V_{\mathfrak{m}_{r-1}}(\lambda_{r-1})\otimes V_{\mathfrak{m}_r}(\lambda'_r)$$

where λ_r and λ'_r are possibly equal to zero.

If $\lambda_r\neq\lambda'_r$ then

$$\begin{aligned}\mathrm{Ext}^1(V,V')&\simeq \mathrm{Ext}^1(V_{\mathfrak{m}_r}(\lambda_r),V_{\mathfrak{m}_r}(\lambda'_r))\\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\lambda_r),V(\lambda'_r))\otimes \mathrm{Der}(A,A/\mathfrak{m}_r).\end{aligned}$$

If $\lambda_r=\lambda'_r$ then

$$\begin{aligned}\mathrm{Ext}^1(V,V')&\simeq \bigoplus_{i=1}^{r}\mathrm{Ext}^1(V_{\mathfrak{m}_i}(\lambda_i),V_{\mathfrak{m}_i}(\lambda_i))\\ &\simeq \bigoplus_{i=1}^{r}\big(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\lambda_i),V(\lambda_i))\otimes \mathrm{Der}(A,A/\mathfrak{m}_i)\big).\end{aligned}$$

Here $\mathrm{Der}(A,A/\mathfrak{m})$ *is the $\mathbb{C}$-vector space of derivations of A at $\mathfrak{m}$.*

By this result, it turns out that extensions between simple modules rely on the choice of a vector of the Zariski tangent space $\mathrm{Der}(A,A/\mathfrak{m})\simeq\mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2,\mathbb{C})$ at each point $\mathfrak{m}$ of $\mathrm{Specm}A$, the maximal spectrum of A. It should be noted that known results on the extension groups for $A\otimes\mathfrak{g}$-modules are limited so far to the case where $\mathrm{Der}(A,A/\mathfrak{m})$ is one-dimensional. Hence the appearance of $\mathrm{Der}(A,A/\mathfrak{m})$ in our work is a new feature.

The article is organized as follows. Section 2 is devoted to recalling some definitions and fundamental facts. It contains the definition of generalized current Lie algebras, the classification of finite-dimensional simple modules, various properties of Weyl modules and a reminder on some generality for the module category of a Hopf algebra with certain conditions. By using the notation introduced in Section 2, the main theorem is restated (Theorem 3.6) and proved in Section 3. In Section 4 we consider the block decomposition of the category of finite-dimensional modules over a generalized current Lie algebra. This generalizes the result by Chari and Moura [CM].

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2 Finite-dimensional modules over a generalized current Lie algebra

2.1 Semisimple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over the complex number field $\mathbb{C}$. We denote by $\mathfrak{h}$ a fixed Cartan subalgebra and $\mathfrak{n}$ the nilpotent radical of a fixed Borel subalgebra containing $\mathfrak{h}$. Let I be the index set of simple roots. We choose Chevalley generators e_i, h_i, f_i $(i \in I)$ of $\mathfrak{g}$.

We denote by P the weight lattice of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Q the root lattice. The set of dominant integral weights P^+ is defined by $P^+ = \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I\}$. For $\lambda, \mu \in P$ we say that $\lambda \geq \mu$ if $\lambda - \mu$ is expressed as a sum of simple roots whose coefficients are all nonnegative integers.

Let $V(\lambda)$ be the finite-dimensional simple $\mathfrak{g}$-module with highest weight $\lambda \in P^+$. The highest weight of the dual module $V(\lambda)^*$ of $V(\lambda)$ is denoted by λ^*.

2.2 Generalized current Lie algebras

Let $\mathfrak{a}$ be an arbitrary Lie algebra over $\mathbb{C}$. For a given nonzero finitely generated commutative $\mathbb{C}$-algebra A, we define the Lie algebra structure on the tensor product $A \otimes \mathfrak{a}$ by

$$[a \otimes x, b \otimes y] = ab \otimes [x, y]$$

for $a, b \in A$ and $x, y \in \mathfrak{a}$.

We call the Lie algebra $A \otimes \mathfrak{g}$ the *generalized current Lie algebra*. The most familiar examples in this class of Lie algebras are the *loop Lie algebra* for $A = \mathbb{C}[t, t^{-1}]$, the ring of Laurent polynomials in one variable and the *current Lie algebra* for $A = \mathbb{C}[t]$, the ring of polynomials in one variable.

2.3 Simple modules

We recall the classification of finite-dimensional simple $A\otimes\mathfrak{g}$-modules given by Chari, Fourier and Khandai [CFK]. For each maximal ideal $\mathfrak{m}$ of A, there exists the canonical isomorphism $A/\mathfrak{m}\simeq\mathbb{C}$ as $\mathbb{C}$-algebras since we assume that A is finitely generated. We denote by $a_\mathfrak{m}$ the image of an element $a\in A$ by the natural morphism $A\to A/\mathfrak{m}\simeq\mathbb{C}$ and all $\mathbb{C}$-vector spaces will be regarded as A-modules via this morphism. We define the *evaluation homomorphism* at $\mathfrak{m}$

$$\mathrm{ev}_\mathfrak{m}\colon A\otimes\mathfrak{g}\to\mathfrak{g}$$

by

$$\mathrm{ev}_\mathfrak{m}(a\otimes x)=a_\mathfrak{m}x$$

for $a\in A$ and $x\in\mathfrak{g}$. For a $\mathfrak{g}$-module V and a maximal ideal $\mathfrak{m}$ of A, we can define the $A\otimes\mathfrak{g}$-module structure on V through the Lie algebra homomorphism $\mathrm{ev}_\mathfrak{m}$. We call it the *evaluation module* associated with V at $\mathfrak{m}$ and denote it by $\mathrm{ev}_\mathfrak{m}^*(V)$. We denote by $V_\mathfrak{m}(\lambda)$ the evaluation module $\mathrm{ev}_\mathfrak{m}^*(V(\lambda))$. This module $V_\mathfrak{m}(\lambda)$ is simple since the evaluation homomorphism $\mathrm{ev}_\mathfrak{m}$ is surjective. The following proposition is proved in [CFK, 6.2].

Proposition 2.1. (i) *Suppose that $\lambda_1,\dots,\lambda_r\in P^+$ are nonzero. Then the module $\bigotimes_{i=1}^r V_{\mathfrak{m}_i}(\lambda_i)$ is simple if and only if $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ are all distinct.*

(ii) *Suppose that $\bigotimes_{i=1}^r V_{\mathfrak{m}_i}(\lambda_i)$ and $\bigotimes_{i=1}^s V_{\mathfrak{m}'_i}(\lambda'_i)$ are simple and $\lambda_1,\dots,\lambda_r,\lambda'_1,\dots,\lambda'_s\in P^+$ are nonzero. Then $\bigotimes_{i=1}^r V_{\mathfrak{m}_i}(\lambda_i)$ and $\bigotimes_{i=1}^s V_{\mathfrak{m}'_i}(\lambda'_i)$ are isomorphic if and only if $r=s$ and the tuples $((\mathfrak{m}_i,\lambda_i))_{1\le i\le r}$ and $((\mathfrak{m}'_i,\lambda'_i))_{1\le i\le r}$ are the same up to permutation.*

(iii) *Any finite-dimensional simple $A\otimes\mathfrak{g}$-module is of the form $\bigotimes_{i=1}^r V_{\mathfrak{m}_i}(\lambda_i)$.*

Let $\mathscr{P}$ be the set of all functions from $\operatorname{Specm}A$ to P^+ with finite supports, where $\operatorname{Specm}A$ denotes the set of all maximal ideals of A. For a given $\pi\in\mathscr{P}$, the isomorphism class of the simple module $\bigotimes_{\mathfrak{m}\in\operatorname{supp}\pi}V_\mathfrak{m}(\pi(\mathfrak{m}))$ depends only on π, not on the ordering of the factors of the tensor product. We denote by $\mathscr{V}(\pi)$ this simple module. Note that $\mathscr{V}(0)$ is the one-dimensional trivial $A\otimes\mathfrak{g}$-module by definition. Proposition 2.1 implies the classification of finite-dimensional simple $A\otimes\mathfrak{g}$-modules given in [CFK, 6.2 Proposition].

Theorem 2.2. *The assignment*

$$\pi\mapsto\mathscr{V}(\pi)$$

gives a one-to-one correspondence between $\mathscr{P}$ and the set of isomorphism classes of finite-dimensional simple $A\otimes\mathfrak{g}$-modules.

For $\pi\in\mathscr{P}$ we define $\pi^*\in\mathscr{P}$ by $\pi^*(\mathfrak{m})=\pi(\mathfrak{m})^*$ for $\mathfrak{m}\in\operatorname{Specm}A$. Recall that $\pi(\mathfrak{m})^*$ is the highest weight of the $\mathfrak{g}$-module $V(\pi(\mathfrak{m}))^*$. Then it is obvious that the dual module $\mathscr{V}(\pi)^*$ of $\mathscr{V}(\pi)$ is isomorphic to $\mathscr{V}(\pi^*)$.

2.4 Weyl modules

Definition 2.3. Let V be an $A \otimes \mathfrak{g}$-module. A nonzero element $v \in V$ is called a *highest weight vector* if v is annihilated by $A \otimes \mathfrak{n}$ and is a common eigenvector of $A \otimes \mathfrak{h}$. A module is called a *highest weight module* if it is generated by a highest weight vector. For a highest weight module V generated by a highest weight vector v, there exists $\Lambda \in (A \otimes \mathfrak{h})^*$ such that

$$xv = \langle x, \Lambda \rangle v$$

for every $x \in A \otimes \mathfrak{h}$. This Λ is called the *highest weight* of V.

Remark 2.4. The definition of highest weight modules above is consistent with the usual one for the case $A = \mathbb{C}$. They are called l-highest weight modules for the case $A = \mathbb{C}[t, t^{-1}]$ in the literature.

Any finite-dimensional simple $A \otimes \mathfrak{g}$-module is a highest weight module. Recall that such a module is of the form $\mathscr{V}(\pi)$ for some $\pi \in \mathscr{P}$. We use the same symbol π for the highest weight of $\mathscr{V}(\pi)$. In other words we regard $\mathscr{P}$ as a subset of $(A \otimes \mathfrak{h})^*$ via the classification of simple modules. To be explicit π is determined by

$$\langle a \otimes h, \pi \rangle = \sum_{\mathfrak{m} \in \operatorname{supp} \pi} a_{\mathfrak{m}} \langle h, \pi(\mathfrak{m}) \rangle$$

for $a \in A$ and $h \in \mathfrak{h}$. We identify $1 \otimes \mathfrak{h}$ with $\mathfrak{h}$. Then the restriction π to $1 \otimes \mathfrak{h}$ is identified with the element $\sum_{\mathfrak{m} \in \operatorname{supp} \pi} \pi(\mathfrak{m}) \in P^+$. We denote by $\pi|_{\mathfrak{h}}$ this element.

Definition 2.5. Let π be an element of $\mathscr{P}$. The *Weyl module* $\mathscr{W}(\pi)$ is the $A \otimes \mathfrak{g}$-module generated by a nonzero element v_π with the following defining relations:

$$(A \otimes \mathfrak{n}) v_\pi = 0,$$

$$x v_\pi = \langle x, \pi \rangle v_\pi$$

for $x \in A \otimes \mathfrak{h}$,

$$(1 \otimes f_i)^{\langle h_i, \pi|_{\mathfrak{h}} \rangle + 1} v_\pi = 0$$

for $i \in I$.

By the definition of the Weyl module $\mathscr{W}(\pi)$, any finite-dimensional highest weight module with highest weight π is a quotient of $\mathscr{W}(\pi)$. In particular the simple module $\mathscr{V}(\pi)$ is the unique simple quotient of $\mathscr{W}(\pi)$. We denote by $W_{\mathfrak{m}}(\lambda)$ the Weyl module which has the simple quotient $V_{\mathfrak{m}}(\lambda)$. The notion of Weyl modules for the case $A = \mathbb{C}[t, t^{-1}]$ is introduced by Chari and Pressley [CP] and the following fundamental results are proved. They are generalized by Feigin and Loktev [FL] for a general A.

Theorem 2.6. (i) *Any Weyl module is finite-dimensional.*

(ii) *We have an isomorphism*

$$\mathscr{W}(\pi) \simeq \bigotimes_{\mathfrak{m} \in \operatorname{supp} \pi} W_{\mathfrak{m}}(\pi(\mathfrak{m}))$$

for any $\pi \in \mathscr{P}$.

The following proposition is proved for the case $A = \mathbb{C}[t, t^{-1}]$ in [CM, Proposition 3.3 (ii)] and for the general case in [FL, Proposition 7].

Proposition 2.7. *For a sufficiently large k, we have*

$$(\mathfrak{m}^k \otimes \mathfrak{h}) W_{\mathfrak{m}}(\lambda) = 0.$$

Corollary 2.8. (i) *Any composition factor of $W_{\mathfrak{m}}(\lambda)$ is of the form $V_{\mathfrak{m}}(\mu)$ for some $\mu \in P^+$.*

(ii) *Any composition factor of $\mathscr{W}(\pi)$ is of the form $\mathscr{V}(\pi')$ such that* $\operatorname{supp} \pi' \subseteq \operatorname{supp} \pi$.

Proof. The assertion of (i) is deduced from the following fact: for distinct maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$, we have $\mathfrak{m}^k \not\subseteq \mathfrak{m}'$ for any k.

The assertion of (ii) is an immediate consequence of (i) and Theorem 2.6 (ii). □

Remark 2.9. The assertions of this corollary for the case $A = \mathbb{C}[t, t^{-1}]$ are proved in [CM, Proof of Proposition 3.3 (iii)] and used for the proof of vanishing of the extension groups for certain modules. We will also use it to prove vanishing of extension groups (Lemma 3.3) under an assumption slightly different from that in [CM].

2.5 Adjointness

We denote by Ext^1 the first Yoneda extension functor for finite-dimensional $A \otimes \mathfrak{g}$-modules. We recall an important fact (Corollary 2.11) which will be used repeatedly in the sequel. Let M be a finite-dimensional $A \otimes \mathfrak{g}$-module. Then the exact functor $M \otimes -$ is defined.

Proposition 2.10. *The functor $M^* \otimes -$ is a right and left adjoint functor of $M \otimes -$.*

The proposition immediately implies the following.

Corollary 2.11. *We have natural isomorphisms*

$$\mathrm{Ext}^1(V, M \otimes V') \simeq \mathrm{Ext}^1(M^* \otimes V, V'),$$

$$\mathrm{Ext}^1(M \otimes V, V') \simeq \mathrm{Ext}^1(V, M^* \otimes V')$$

for any finite-dimensional $A \otimes \mathfrak{g}$-modules V, V', M.

Remark 2.12. Proposition 2.10 and Corollary 2.11 are general facts which hold for the category of *finite-dimensional* modules over a Hopf algebra with an involutive antipode defined over a field. We give an explicit description of the morphisms in Corollary 2.11. The morphism

$$\mathrm{Ext}^1(M^* \otimes V, V') \to \mathrm{Ext}^1(V, M \otimes V')$$

is described as follows. Let

$$0 \longrightarrow V' \longrightarrow E \longrightarrow M^* \otimes V \longrightarrow 0$$

be an exact sequence which represents an extension class in $\mathrm{Ext}^1(M^* \otimes V, V')$. Then the corresponding element of $\mathrm{Ext}^1(V, M \otimes V')$ is represented by the first row of the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \otimes V' \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \otimes V' \arrow[r] & M \otimes E \arrow[r] & M \otimes M^* \otimes V \arrow[r] & 0 \end{tikzcd}$$

where E' is the fiber product which makes the right square cartesian. The other morphisms are obtained in similar ways.

3 Extensions between simple modules

The purpose of this section is to calculate $\mathrm{Ext}^1(V, V')$ for any finite-dimensional simple $A \otimes \mathfrak{g}$-modules V, V'.

3.1 Extensions between evaluation modules

A derivation of A into an A-module M is a $\mathbb{C}$-linear map $D\colon A \to M$ satisfying

$$D(ab) = aD(b) + bD(a)$$

for $a, b \in A$. We denote by $\mathrm{Der}(A, M)$ the $\mathbb{C}$-vector space of all derivations of A into M. There exists the canonical isomorphism

$$\mathrm{Der}(A, A/\mathfrak{m}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C})$$

for each $\mathfrak{m} \in \mathrm{Specm} A$. The following proposition is a special case of the main theorem and gives a generalization of the result by Fialowski and Malikov [FM, Theorem 5 (ii)].

Proposition 3.1. *We have an isomorphism*

$$\mathrm{Ext}^1(V_{\mathfrak{m}}(\lambda), V_{\mathfrak{m}}(\mu)) \simeq \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda), V(\mu)) \otimes \mathrm{Der}(A, A/\mathfrak{m}).$$

Proof. It is obvious that

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda), V(\mu)) \otimes \mathrm{Der}(A, A/\mathfrak{m})$$

is canonically isomorphic to

$$S = \{\varphi\colon A \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda), V(\mu)) \mid \varphi \text{ is } \mathbb{C}\text{-linear}, \varphi(ab) = a_{\mathfrak{m}}\varphi(b) + b_{\mathfrak{m}}\varphi(a)\}$$

as a $\mathbb{C}$-vector space. Hence it suffices to prove that

$$\mathrm{Ext}^1(V_{\mathfrak{m}}(\lambda), V_{\mathfrak{m}}(\mu)) \simeq S.$$

We prove the assertion by the following steps.

(Step 1) Define a map

$$\mathrm{Ext}^1(V_{\mathrm{m}}(\lambda), V_{\mathrm{m}}(\mu)) \to S.$$

(Step 2) Show that the map is bijective by constructing the inverse map.

(Step 3) Show that the map is $\mathbb{C}$-linear.

We start proving Step 1–3.
(Step 1) Suppose that an exact sequence

$$0 \longrightarrow V_{\mathrm{m}}(\mu) \xrightarrow{i} E \xrightarrow{p} V_{\mathrm{m}}(\lambda) \longrightarrow 0$$

is given. Take a splitting $j\colon V_{\mathrm{m}}(\lambda) \to E$ as $\mathfrak{g}$-modules. We identify $V_{\mathrm{m}}(\lambda)$ with $V(\lambda)$ and $V_{\mathrm{m}}(\mu)$ with $V(\mu)$ as $\mathfrak{g}$-modules by restriction. Then we can define the $\mathbb{C}$-linear map $\varphi_a\colon \mathfrak{g} \otimes V(\lambda) \to V(\mu)$ for each $a \in A$ via the action of $A \otimes \mathfrak{g}$ on E so that

$$(a \otimes x) j(u) = j(a_{\mathrm{m}} x u) + i(\varphi_a(x \otimes u))$$

for $x \in \mathfrak{g}$ and $u \in V(\lambda)$. Indeed, since

$$(a \otimes x) j(u) - j(a_{\mathrm{m}} x u)$$

belongs to $\mathrm{Ker}\, p = \mathrm{Im}\, i$, we can take the unique element in its preimage by i as $\varphi_a(x \otimes u)$. Note that $a \mapsto \varphi_a$ defines a $\mathbb{C}$-linear map and $\varphi_1 = 0$. We claim the following:

(*-1) φ_a does not depend on the choice of a splitting,

(*-2) φ_a depends only on the extension class of a given exact sequence.

To show (*-1), take another splitting j' and let φ'_a be the corresponding $\mathbb{C}$-linear map. Then we have

$$i((\varphi_a - \varphi'_a)(x \otimes u)) = (a \otimes x)(j - j')(u) - a_{\mathrm{m}} x (j - j')(u).$$

The right-hand side is equal to zero since $(j - j')(u) \in \mathrm{Ker}\, p = \mathrm{Im}\, i$. This shows (*-1). We show (*-2). Take two exact sequences which are equivalent:

$$\begin{tikzcd}
0 \arrow[r] & V_{\mathrm{m}}(\mu) \arrow[r, "i"] \arrow[d, equal] & E \arrow[r, "p"] \arrow[d, "\xi"] & V_{\mathrm{m}}(\lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & V_{\mathrm{m}}(\mu) \arrow[r, "i'"] & E' \arrow[r, "p'"] & V_{\mathrm{m}}(\lambda) \arrow[r] & 0.
\end{tikzcd}$$

Let φ_a, φ'_a be the corresponding maps. Splittings j of p and j' of p' can be taken so that $j' = \xi j$. We have

$$(a \otimes x) j(u) = j(a_{\mathrm{m}} x u) + i(\varphi_a(x \otimes u)),$$
$$(a \otimes x) j'(u) = j'(a_{\mathrm{m}} x u) + i'(\varphi'_a(x \otimes u))$$

by the definition of φ_a, φ'_a. We see that $\varphi_a = \varphi'_a$ by applying ξ to the both sides of the first equation and comparing it with the second one. The claim is proved.

We show that φ_a is a $\mathfrak{g}$-module homomorphism and the equation

$$\varphi_{ab} = a_{\mathrm{m}}\varphi_b + b_{\mathrm{m}}\varphi_a$$

holds. We have

$$(a\otimes x)(b\otimes y)j(u) = j(a_{\mathrm{m}}b_{\mathrm{m}}xyu) + a_{\mathrm{m}}i(x\varphi_b(y\otimes u)) + b_{\mathrm{m}}i(\varphi_a(x\otimes yu))$$

and hence

$$\begin{aligned}&(a\otimes x)(b\otimes y)j(u) - (b\otimes y)(a\otimes x)j(u)\\ &= j(a_{\mathrm{m}}b_{\mathrm{m}}[x,y]u) + a_{\mathrm{m}}i(x\varphi_b(y\otimes u) - \varphi_b(y\otimes xu)) + b_{\mathrm{m}}i(\varphi_a(x\otimes yu) - y\varphi_a(x\otimes u)).\end{aligned}$$

Compare the above with

$$(ab\otimes[x,y])j(u) = j(a_{\mathrm{m}}b_{\mathrm{m}}[x,y]u) + i(\varphi_{ab}([x,y]\otimes u))$$

and we obtain

$$\varphi_{ab}([x,y]\otimes u) = a_{\mathrm{m}}(x\varphi_b(y\otimes u) - \varphi_b(y\otimes xu)) + b_{\mathrm{m}}(\varphi_a(x\otimes yu) - y\varphi_a(x\otimes u)).$$

Consider the case $b = 1$. Then we obtain the equation

$$\varphi_a([x,y]\otimes u) = \varphi_a(x\otimes yu) - y\varphi_a(x\otimes u).$$

This proves that φ_a is a $\mathfrak{g}$-module homomorphism. Moreover we have

$$\varphi_{ab}([x,y]\otimes u) = a_{\mathrm{m}}\varphi_b([x,y]\otimes u) + b_{\mathrm{m}}\varphi_a([x,y]\otimes u)$$

and this implies that

$$\varphi_{ab} = a_{\mathrm{m}}\varphi_b + b_{\mathrm{m}}\varphi_a$$

since $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$.

As a result we obtain a $\mathbb{C}$-linear map $\varphi\colon A \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\lambda), V(\mu))$ satisfying

$$\varphi(ab) = a_{\mathrm{m}}\varphi(b) + b_{\mathrm{m}}\varphi(a).$$

This means that a map

$$\mathrm{Ext}^1(V_{\mathrm{m}}(\lambda), V_{\mathrm{m}}(\mu)) \to S$$

is defined.

(Step 2) Conversely if a $\mathbb{C}$-linear map $\varphi\colon A \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}\otimes V(\lambda), V(\mu))$ satisfying

$$\varphi(ab) = a_{\mathrm{m}}\varphi(b) + b_{\mathrm{m}}\varphi(a)$$

is given then we can define the $A\otimes\mathfrak{g}$-module structure on $E = V(\lambda)\oplus V(\mu)$ by

$$(a\otimes x)(u,v) = (a_{\mathrm{m}}xu, a_{\mathrm{m}}xv + \varphi(a)(x\otimes u))$$

for $u\in V(\lambda), v\in V(\mu)$. It is obvious that this gives the inverse of the map defined in Step 1.

(Step 3) We show that the bijective map is $\mathbb{C}$-linear. First we show that it is additive. Let

$$0 \longrightarrow V_{\mathrm{m}}(\mu) \xrightarrow{i_1} E_1 \xrightarrow{p_1} V_{\mathrm{m}}(\lambda) \longrightarrow 0,$$

$$0 \longrightarrow V_{\mathrm{m}}(\mu) \xrightarrow{i_2} E_2 \xrightarrow{p_2} V_{\mathrm{m}}(\lambda) \longrightarrow 0$$

be exact sequences and φ^1, φ^2 be the corresponding elements. The Baer sum of the classes of the above extensions is represented by

$$0 \longrightarrow V_{\mathrm{m}}(\mu) \xrightarrow{i} E \xrightarrow{p} V_{\mathrm{m}}(\lambda) \longrightarrow 0$$

where E is the quotient of the fiber product of p_1 and p_2 by $\mathrm{Im}(v \mapsto (i_1(v), -i_2(v)))$. Note that i is given by $v \mapsto (i_1(v), 0) = (0, i_2(v))$ in E and p by $(z_1, z_2) \mapsto p_1(z_1) = p_2(z_2)$. A splitting j of p as $\mathfrak{g}$-modules is given by $u \mapsto (j_1(u), j_2(u))$ if we take splittings j_1 of p_1 and j_2 of p_2. Then the equation

$$\begin{aligned}(a \otimes x) j(u) &= (j_1(a_{\mathrm{m}} x u) + i_1(\varphi_a^1(x \otimes u)), j_2(a_{\mathrm{m}} x u) + i_2(\varphi_a^2(x \otimes u))) \\ &= j(a_{\mathrm{m}} x u) + i((\varphi_a^1 + \varphi_a^2)(x \otimes u))\end{aligned}$$

in E holds for $a \in A$ and $x \in \mathfrak{g}$. This shows that the map under consideration is additive. Next we consider the multiplication by scalar. Take an exact sequence

$$0 \longrightarrow V_{\mathrm{m}}(\mu) \xrightarrow{i} E \xrightarrow{p} V_{\mathrm{m}}(\lambda) \longrightarrow 0$$

and let φ be the corresponding element. The action of $c \in \mathbb{C}$ on $\mathrm{Ext}^1(V_{\mathrm{m}}(\lambda), V_{\mathrm{m}}(\mu))$ is described by the diagram

$$\begin{tikzcd}
0 \arrow[r] & V_{\mathrm{m}}(\mu) \arrow[r, "i'"] \arrow[d, equal] & E' \arrow[r, "p'"] \arrow[d] & V_{\mathrm{m}}(\lambda) \arrow[r] \arrow[d, "c\mathrm{id}"] & 0 \\
0 \arrow[r] & V_{\mathrm{m}}(\mu) \arrow[r, "i"] & E \arrow[r, "p"] & V_{\mathrm{m}}(\lambda) \arrow[r] & 0
\end{tikzcd}$$

where E' is the fiber product of p and $c\,\mathrm{id}_{V_{\mathrm{m}}(\lambda)}$. Note that i' is given by $v \mapsto (i(v), 0)$ and p' by the second projection. A splitting j' of p' is given by $u \mapsto (cj(u), u)$ where j is a splitting of p. Then we obtain

$$\begin{aligned}(a \otimes x) j'(u) &= (c(j(a_{\mathrm{m}} x u) + i(\varphi_a(x \otimes u))), a_{\mathrm{m}} x u) \\ &= j'(a_{\mathrm{m}} x u) + i'(c\varphi_a(x \otimes u)).\end{aligned}$$

The proof is complete. $\square$

Remark 3.2. For the case $A = \mathbb{C}[t, t^{-1}]$, construction of the map from $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda), V(\mu))$ to $\mathrm{Ext}^1(V_{\mathrm{m}}(\lambda), V_{\mathrm{m}}(\mu))$ as in Step 2 appears in [FM, Remarks 7 (i)] and [CM, Proposition 3.4]. In such situations the space $\mathrm{Der}(A, A/\mathfrak{m})$ is one-dimensional and its contribution is not recognized explicitly.

3.2 Vanishing of extension groups

We prove some vanishing results for Ext^1 in this subsection. Lemma 3.4 is a key for proving the main theorem. The argument in the proof of the following lemma is similar to that in [CM, Lemma 5.2 (i)]. While they prove vanishing of Ext^1 for modules with different spectral characters (See Definition 4.2 for the definition of spectral characters), we show a slightly different statement.

Lemma 3.3. *Let π, π' be elements of $\mathscr{P}$ such that* $\operatorname{supp}\pi \cap \operatorname{supp}\pi' = \varnothing$. *If* $\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) \neq 0$ *then π or π' is equal to zero.*

Proof. We may assume that either of π or π' is nonzero since $\mathrm{Ext}^1(\mathscr{V}(0), \mathscr{V}(0)) = 0$ by Proposition 3.1. This assumption implies that $\pi \neq \pi'$.

Let

$$0 \longrightarrow \mathscr{V}(\pi') \longrightarrow E \xrightarrow{p} \mathscr{V}(\pi) \longrightarrow 0$$

be a nonsplit exact sequence. First we assume that $\pi'|_{\mathfrak{h}} \not> \pi|_{\mathfrak{h}}$. Let v_π be a highest weight vector of $\mathscr{V}(\pi)$ and take $v \in p^{-1}(v_\pi)$. By the assumption $\pi'|_{\mathfrak{h}} \not> \pi|_{\mathfrak{h}}$ and $\pi \neq \pi'$, this v is a highest weight vector. Hence the submodule of E generated by v is a highest weight module with highest weight π. This submodule is not isomorphic to $\mathscr{V}(\pi')$ since their highest weights are different. Then it follows that the submodule coincides with E since the length of E is two and the sequence does not split. This implies that E is a quotient of the Weyl module $\mathscr{W}(\pi)$. Therefore π' must be equal to zero by Corollary 2.8 and the assumption $\operatorname{supp}\pi \cap \operatorname{supp}\pi' = \varnothing$. Next assume that $\pi'|_{\mathfrak{h}} > \pi|_{\mathfrak{h}}$. In this case, take the dual of the exact sequence. Then we obtain the exact sequence

$$0 \longrightarrow \mathscr{V}(\pi^*) \longrightarrow E^* \longrightarrow \mathscr{V}((\pi')^*) \longrightarrow 0$$

and have $\pi^*|_{\mathfrak{h}} \not> (\pi')^*|_{\mathfrak{h}}$. This implies that π is equal to zero. □

Lemma 3.4. *Let π be an element of $\mathscr{P}$. We have* $\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(0)) = 0$ *and* $\mathrm{Ext}^1(\mathscr{V}(0), \mathscr{V}(\pi)) = 0$ *unless* $\#\operatorname{supp}\pi = 1$.

Proof. Assume that $\#\operatorname{supp}\pi \geq 2$. Take some $\mathrm{m} \in \operatorname{supp}\pi$ and define $\pi' \in \mathscr{P}$ by

$$\pi'(\mathrm{m}') = \begin{cases} \pi(\mathrm{m}') & \text{if } \mathrm{m}' \neq \mathrm{m} \\ 0 & \text{if } \mathrm{m}' = \mathrm{m} \end{cases}$$

for $\mathrm{m}' \in \operatorname{Specm} A$. Then the element π' is nonzero by the assumption $\#\operatorname{supp}\pi \geq 2$. We have $\mathscr{V}(\pi) \simeq V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes \mathscr{V}(\pi')$. Hence

$$\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(0)) \simeq \mathrm{Ext}^1(\mathscr{V}(\pi'), V_{\mathrm{m}}(\pi(\mathrm{m})^*))$$

and the right-hand side is equal to zero by Lemma 3.3. The assertion $\mathrm{Ext}^1(\mathscr{V}(0), \mathscr{V}(\pi)) = 0$ is proved by taking the dual.

The assertion $\mathrm{Ext}^1(\mathscr{V}(0), \mathscr{V}(0)) = 0$ is a consequence of Proposition 3.1. □

Remark 3.5. In fact, by Proposition 3.1, it is easy to prove a stronger result than the statement of Lemma 3.4. We state it without a proof since it is not used in the sequel. The following are equivalent for a finite-dimensional simple $A \otimes \mathfrak{g}$-module V:

- $\mathrm{Ext}^1(V, \mathscr{V}(0)) \neq 0$,
- $\mathrm{Ext}^1(\mathscr{V}(0), V) \neq 0$,
- $V \simeq V_{\mathfrak{m}}(\theta)$ for some $\mathfrak{m} \in \mathrm{Specm} A$ satisfying $\mathfrak{m}/\mathfrak{m}^2 \neq 0$ and the highest root θ of a simple component of $\mathfrak{g}$.

This result for the case $A = \mathbb{C}[t]$ is proved in [CG, 3.8 Proposition] by a different approach. They also prove that

$$\dim \mathrm{Ext}^1(V_{\mathfrak{m}}(\theta), \mathscr{V}(0)) = \dim \mathrm{Ext}^1(\mathscr{V}(0), V_{\mathfrak{m}}(\theta)) = 1$$

and deduce the following result:

$$\mathrm{Ext}^1(V, V') \simeq \bigoplus_{\mathfrak{m} \in \mathrm{Specm}\, \mathbb{C}[t]} \mathrm{Hom}_{\mathbb{C}[t] \otimes \mathfrak{g}}(\mathrm{ev}_{\mathfrak{m}}^*(\mathfrak{g}), V^* \otimes V')$$

holds for any finite-dimensional simple $\mathbb{C}[t] \otimes \mathfrak{g}$-modules V, V'.

3.3 Proof of the main theorem

The main theorem introduced as Theorem 1.2 is reformulated in terms of the set $\mathscr{P}$ as follows.

Theorem 3.6. *Let π, π' be elements of $\mathscr{P}$.*

(i) *If* $\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) \neq 0$ *then* $\#\{\mathfrak{m} \in \mathrm{Specm} A \mid \pi(\mathfrak{m}) \neq \pi'(\mathfrak{m})\} \leq 1$.

(ii) *If* $\#\{\mathfrak{m} \in \mathrm{Specm} A \mid \pi(\mathfrak{m}) \neq \pi'(\mathfrak{m})\} = 1$ *then*

$$\begin{aligned}\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) &\simeq \mathrm{Ext}^1(V_{\mathfrak{m}_0}(\pi(\mathfrak{m}_0)), V_{\mathfrak{m}_0}(\pi'(\mathfrak{m}_0))) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\pi(\mathfrak{m}_0)), V(\pi'(\mathfrak{m}_0))) \otimes \mathrm{Der}(A, A/\mathfrak{m}_0)\end{aligned}$$

where $\mathfrak{m}_0$ *is the unique element of* $\mathrm{Specm} A$ *such that* $\pi(\mathfrak{m}_0) \neq \pi'(\mathfrak{m}_0)$.

If $\pi = \pi'$ *then*

$$\begin{aligned}\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) &\simeq \bigoplus_{\mathfrak{m} \in \mathrm{supp}\, \pi} \mathrm{Ext}^1(V_{\mathfrak{m}}(\pi(\mathfrak{m})), V_{\mathfrak{m}}(\pi(\mathfrak{m}))) \\ &\simeq \bigoplus_{\mathfrak{m} \in \mathrm{supp}\, \pi} \big(\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\pi(\mathfrak{m})), V(\pi(\mathfrak{m}))) \otimes \mathrm{Der}(A, A/\mathfrak{m})\big).\end{aligned}$$

Remark 3.7. For the case $A = \mathbb{C}[t]$ it is proved in [CG] that

$$\mathrm{Ext}^1(V, V') \simeq \bigoplus_{\mathfrak{m} \in \mathrm{Specm}\, \mathbb{C}[t]} \mathrm{Hom}_{\mathbb{C}[t] \otimes \mathfrak{g}}(\mathrm{ev}_{\mathfrak{m}}^*(\mathfrak{g}), V^* \otimes V')$$

holds for any finite-dimensional simple $\mathbb{C}[t] \otimes \mathfrak{g}$-modules V, V' as explained in Remark 3.5. This implies our main theorem for the case $A = \mathbb{C}[t]$ after some calculation essentially same as the proof below.

Proof of Theorem 3.6. Recall that

$$\mathscr{V}(\pi) \simeq \bigotimes_{\mathrm{m}\in\operatorname{supp}\pi} V_{\mathrm{m}}(\pi(\mathrm{m}))$$

and

$$\mathscr{V}(\pi') \simeq \bigotimes_{\mathrm{m}\in\operatorname{supp}\pi'} V_{\mathrm{m}}(\pi'(\mathrm{m})).$$

We prove (i). Suppose that $\operatorname{Ext}^1(\mathscr{V}(\pi),\mathscr{V}(\pi')) \neq 0$. Put

$$\begin{aligned} S &= \operatorname{supp}\pi \cap \operatorname{supp}\pi', \\ T &= \operatorname{supp}\pi \setminus S, \\ T' &= \operatorname{supp}\pi' \setminus S. \end{aligned}$$

Let

$$V(\pi(\mathrm{m})) \otimes V(\pi'(\mathrm{m}))^* \simeq \bigoplus_{j_{\mathrm{m}}} V(v_{j_{\mathrm{m}}})$$

be a decomposition into a direct sum of simple $\mathfrak{g}$-modules. Note that $v_{j_{\mathrm{m}}} = 0$ for some j_{m} if and only if $\pi(\mathrm{m}) = \pi'(\mathrm{m})$. We have

$$\begin{aligned} &\operatorname{Ext}^1(\mathscr{V}(\pi),\mathscr{V}(\pi')) \\ &\simeq \operatorname{Ext}^1\Big(\bigotimes_{\mathrm{m}\in S}(V_{\mathrm{m}}(\pi(\mathrm{m}))\otimes V_{\mathrm{m}}(\pi'(\mathrm{m}))^*)\otimes \bigotimes_{\mathrm{m}\in T} V_{\mathrm{m}}(\pi(\mathrm{m}))\otimes \bigotimes_{\mathrm{m}\in T'} V_{\mathrm{m}}(\pi'(\mathrm{m}))^*, \mathscr{V}(0)\Big) \\ &\simeq \operatorname{Ext}^1\Big(\bigotimes_{\mathrm{m}\in S}\operatorname{ev}_{\mathrm{m}}^*(V(\pi(\mathrm{m}))\otimes V(\pi'(\mathrm{m}))^*)\otimes \bigotimes_{\mathrm{m}\in T} V_{\mathrm{m}}(\pi(\mathrm{m}))\otimes \bigotimes_{\mathrm{m}\in T'} V_{\mathrm{m}}(\pi'(\mathrm{m}))^*, \mathscr{V}(0)\Big) \\ &\simeq \bigoplus_{(j_{\mathrm{m}})_{\mathrm{m}\in S}} \operatorname{Ext}^1\Big(\bigotimes_{\mathrm{m}\in S} V_{\mathrm{m}}(v_{j_{\mathrm{m}}})\otimes \bigotimes_{\mathrm{m}\in T} V_{\mathrm{m}}(\pi(\mathrm{m}))\otimes \bigotimes_{\mathrm{m}\in T'} V_{\mathrm{m}}(\pi'(\mathrm{m}))^*, \mathscr{V}(0)\Big). \end{aligned}$$

There is a tuple $(j_{\mathrm{m}})_{\mathrm{m}\in S}$ such that

$$\operatorname{Ext}^1\Big(\bigotimes_{\mathrm{m}\in S} V_{\mathrm{m}}(v_{j_{\mathrm{m}}})\otimes \bigotimes_{\mathrm{m}\in T} V_{\mathrm{m}}(\pi(\mathrm{m}))\otimes \bigotimes_{\mathrm{m}\in T'} V_{\mathrm{m}}(\pi'(\mathrm{m}))^*, \mathscr{V}(0)\Big) \neq 0$$

by the assumption $\operatorname{Ext}^1(\mathscr{V}(\pi),\mathscr{V}(\pi')) \neq 0$. By Lemma 3.4, the number of nontrivial factors of the tensor product is exactly one. Hence one of the following three cases holds:

(*-1) $\pi(\mathrm{m}) = \pi'(\mathrm{m})$ for all $\mathrm{m} \in S$ but at most one element and $T = T' = \varnothing$,

(*-2) $\pi(\mathrm{m}) = \pi'(\mathrm{m})$ for $\mathrm{m} \in S$, $\#T = 1$ and $T' = \varnothing$,

(*-3) $\pi(\mathrm{m}) = \pi'(\mathrm{m})$ for $\mathrm{m} \in S$, $T = \varnothing$ and $\#T' = 1$.

The case (*-1) implies that $\#\{\mathrm{m} \in \operatorname{Specm} A \mid \pi(\mathrm{m}) \neq \pi'(\mathrm{m})\} \leq 1$ and the case (*-2) or (*-3) implies that $\#\{\mathrm{m} \in \operatorname{Specm} A \mid \pi(\mathrm{m}) \neq \pi'(\mathrm{m})\} = 1$. The proof of (i) is complete.

We prove (ii). Suppose that $\#\{\mathrm{m} \in \operatorname{Specm} A \mid \pi(\mathrm{m}) \neq \pi'(\mathrm{m})\} \leq 1$. Put $U = \{\mathrm{m} \in \operatorname{Specm} A \mid \pi(\mathrm{m}) = \pi'(\mathrm{m})\}$. We can write as

$$\mathscr{V}(\pi) \simeq \bigotimes_{\mathrm{m}\in U} V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes V_{\mathrm{m}_0}(\pi(\mathrm{m}_0))$$

and

$$\mathscr{V}(\pi') \simeq \bigotimes_{\mathrm{m}\in U} V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0))$$

for some m_0 where $\pi(\mathrm{m}_0)$ and $\pi'(\mathrm{m}_0)$ are possibly equal to zero. Again let

$$V(\pi(\mathrm{m})) \otimes V(\pi'(\mathrm{m}))^* \simeq \bigoplus_{j_{\mathrm{m}}} V(\nu_{j_{\mathrm{m}}})$$

be a decomposition into a direct sum of simple $\mathfrak{g}$-modules. Then we have

$$\begin{aligned}
&\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) \\
&\simeq \mathrm{Ext}^1(\bigotimes_{\mathrm{m}\in U}(V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes V_{\mathrm{m}}(\pi(\mathrm{m}))^*) \otimes (V_{\mathrm{m}_0}(\pi(\mathrm{m}_0)) \otimes V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0))^*), \mathscr{V}(0)) \\
&\simeq \bigoplus_{(j_{\mathrm{m}})_{\mathrm{m}\in U\cup\{\mathrm{m}_0\}}} \mathrm{Ext}^1(\bigotimes_{\mathrm{m}\in U} V_{\mathrm{m}}(\nu_{j_{\mathrm{m}}}) \otimes V_{\mathrm{m}_0}(\nu_{j_{\mathrm{m}_0}}), \mathscr{V}(0)).
\end{aligned}$$

By Lemma 3.4, the number of nontrivial factors of the tensor product is one in every nonzero summand. If we suppose that $\pi(\mathrm{m}_0) \neq \pi'(\mathrm{m}_0)$ then $V_{\mathrm{m}_0}(\pi(\mathrm{m}_0)) \otimes V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0))^*$ does not have a trivial direct summand. Hence

$$\begin{aligned}
\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) &\simeq \mathrm{Ext}^1(V_{\mathrm{m}_0}(\pi(\mathrm{m}_0)) \otimes V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0))^*, \mathscr{V}(0)) \\
&\simeq \mathrm{Ext}^1(V_{\mathrm{m}_0}(\pi(\mathrm{m}_0)), V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0))).
\end{aligned}$$

If $\pi = \pi'$, namely $U = \mathrm{Specm} A$ and $\pi(\mathrm{m}_0) = \pi'(\mathrm{m}_0) = 0$, then

$$\begin{aligned}
\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi)) &\simeq \mathrm{Ext}^1(\bigotimes_{\mathrm{m}\in\mathrm{supp}\,\pi} (V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes V_{\mathrm{m}}(\pi(\mathrm{m}))^*), \mathscr{V}(0)) \\
&\simeq \bigoplus_{\mathrm{m}\in\mathrm{supp}\,\pi} \mathrm{Ext}^1(V_{\mathrm{m}}(\pi(\mathrm{m})) \otimes V_{\mathrm{m}}(\pi(\mathrm{m}))^*, \mathscr{V}(0)) \\
&\simeq \bigoplus_{\mathrm{m}\in\mathrm{supp}\,\pi} \mathrm{Ext}^1(V_{\mathrm{m}}(\pi(\mathrm{m})), V_{\mathrm{m}}(\pi(\mathrm{m}))).
\end{aligned}$$

The proof of (ii) is complete together with Proposition 3.1, which asserts the second isomorphisms. □

Remark 3.8. We give a natural interpretation of the isomorphisms

$$\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi')) \simeq \mathrm{Ext}^1(V_{\mathrm{m}_0}(\pi(\mathrm{m}_0)), V_{\mathrm{m}_0}(\pi'(\mathrm{m}_0)))$$

and

$$\mathrm{Ext}^1(\mathscr{V}(\pi), \mathscr{V}(\pi)) \simeq \bigoplus_{\mathrm{m}\in\mathrm{supp}\,\pi} \mathrm{Ext}^1(V_{\mathrm{m}}(\pi(\mathrm{m})), V_{\mathrm{m}}(\pi(\mathrm{m})))$$

in Theorem 3.6 (ii). According to the proof, the above isomorphisms come from the composite of the morphisms

$$\mathrm{Ext}^1(V, V') \to \mathrm{Ext}^1(M^* \otimes M \otimes V, V') \simeq \mathrm{Ext}^1(M \otimes V, M \otimes V')$$

for appropriate modules V, V', M. This morphism coincides with the natural morphism

$$\mathrm{Ext}^1(V,V') \to \mathrm{Ext}^1(M\otimes V, M\otimes V')$$

obtained by applying the exact functor $M\otimes -$. This is proved as follows. Let

$$0 \longrightarrow V' \longrightarrow E \longrightarrow V \longrightarrow 0$$

be an exact sequence which represents an extension class in $\mathrm{Ext}^1(V,V')$. This element maps to the extension class represented by the first row of

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] & M^*\otimes M\otimes V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V' \arrow[r] & E \arrow[r] & V \arrow[r] & 0 \end{tikzcd}$$

by $\mathrm{Ext}^1(V,V') \to \mathrm{Ext}^1(M^*\otimes M\otimes V, V')$ and then maps to the class represented by the first row of

$$\begin{tikzcd} 0 \arrow[r] & M\otimes V' \arrow[r] \arrow[d, equal] & E'' \arrow[r] \arrow[d] & M\otimes V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M\otimes V' \arrow[r] & M\otimes E' \arrow[r] & M\otimes M^*\otimes M\otimes V \arrow[r] & 0 \end{tikzcd}$$

by $\mathrm{Ext}^1(M^*\otimes M\otimes V, V') \to \mathrm{Ext}^1(M\otimes V, M\otimes V')$, as explained in Remark 2.12. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M\otimes V' \arrow[r] \arrow[d, equal] & E'' \arrow[r] \arrow[d] & M\otimes V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M\otimes V' \arrow[r] \arrow[d, equal] & M\otimes E' \arrow[r] \arrow[d] & M\otimes M^*\otimes M\otimes V \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M\otimes V' \arrow[r] & M\otimes E \arrow[r] & M\otimes V \arrow[r] & 0. \end{tikzcd}$$

Since the composite of the right vertical maps $M\otimes V \to M\otimes M^*\otimes M\otimes V \to M\otimes V$ is identity, the first and the third rows are equivalent.

4 The block decomposition

We deduce the block decomposition of the category of finite-dimensional $A\otimes\mathfrak{g}$-modules from the results in Section 3. Although the argument is similar to that in [CM, Theorem 1, Proposition 2.3, Proposition 2.4], we give a proof for the sake of completeness.

4.1 Blocks of an abelian category

We briefly recall the notion of blocks of an abelian category. Let $\mathscr{C}$ be an abelian category and assume that any object of $\mathscr{C}$ has a composition series of finite length. First we introduce an equivalence relation for the set of isomorphism classes of simple objects of $\mathscr{C}$ as follows: two simple objects V, V' are equivalent if there exists a sequence $V = V_1, V_2, \dots, V_r = V'$ of simple objects satisfying $\mathrm{Ext}^1_{\mathscr{C}}(V_i, V_{i+1}) \neq 0$ or $\mathrm{Ext}^1_{\mathscr{C}}(V_{i+1}, V_i) \neq 0$ for all i. Next, for each equivalence class χ, we denote by $\mathscr{C}_\chi$ the full subcategory of $\mathscr{C}$ consisting of objects whose all composition factors belong to χ. Each $\mathscr{C}_\chi$ is called a *block* of $\mathscr{C}$.

Proposition 4.1. *We have $\mathscr{C} = \bigoplus_\chi \mathscr{C}_\chi$. Moreover each $\mathscr{C}_\chi$ cannot be decomposed into a direct sum of two nontrivial abelian full subcategories.*

The decomposition in Proposition 4.1 is called the *block decomposition* of the category.

4.2 The block decomposition

In the sequel *we assume that A is connected*, namely it is not isomorphic to a direct product of two nonzero $\mathbb{C}$-algebras for simplicity. Moreover we assume that $A \neq \mathbb{C}$ since the block decomposition is well known for the case $A = \mathbb{C}$ as completely reducibility of finite-dimensional $\mathfrak{g}$-modules. Let Ξ be the set of all functions from $\mathrm{Specm} A$ to P/Q with finite supports.

Definition 4.2. For each finite-dimensional simple $A \otimes \mathfrak{g}$-module $\mathscr{V}(\pi)$, we define its *spectral character* $\chi_\pi \in \Xi$ by

$$\chi_\pi(\mathfrak{m}) = \pi(\mathfrak{m}) \bmod Q$$

for $\mathfrak{m} \in \mathrm{Specm} A$. A finite-dimensional $A \otimes \mathfrak{g}$-module V is said to have the spectral character $\chi \in \Xi$ if $\chi = \chi_\pi$ for any composition factor $\mathscr{V}(\pi)$ of V.

Remark 4.3. The definition of spectral characters above is a straightforward generalization of that for the case $A = \mathbb{C}[t, t^{-1}]$ given in [CM, Definition 2.1].

We denote by $\mathscr{F}$ the category of finite-dimensional $A \otimes \mathfrak{g}$-modules. For each $\chi \in \Xi$ we define the full subcategory $\mathscr{F}_\chi$ of $\mathscr{F}$ whose objects have the spectral character χ.

Theorem 4.4. *We have the block decomposition $\mathscr{F} = \bigoplus_{\chi \in \Xi} \mathscr{F}_\chi$.*

It suffices to show the following proposition.

Proposition 4.5. (i) *Any finite-dimensional indecomposable $A \otimes \mathfrak{g}$-module has some spectral character.*

(ii) *Any finite-dimensional simple $A \otimes \mathfrak{g}$-modules which have the same spectral character belong to the same block.*

We need two lemmas.

Lemma 4.6. *Let V_1, V_2, V_1', V_2' be finite-dimensional simple $A \otimes \mathfrak{g}$-modules and suppose that V_1 and V_1' belong to the same block, and that V_2 and V_2' belong to the same block. Then $V_1 \otimes V_2$ and $V_1' \otimes V_2'$ belong to the same block.*

Proof. We may assume that $V_1 = V_1'$. Put $V = V_2, V' = V_2', M = V_1 = V_1'$ for the simplicity of notation. It suffices to show the following: if $\mathrm{Ext}^1(V,V') \neq 0$ then $\mathrm{Ext}^1(M \otimes V, M \otimes V') \neq 0$. As explained in Remark 3.8, the natural morphism

$$\mathrm{Ext}^1(V,V') \to \mathrm{Ext}^1(M \otimes V, M \otimes V')$$

coincides with

$$\mathrm{Ext}^1(V,V') \to \mathrm{Ext}^1(M^* \otimes M \otimes V, V') \simeq \mathrm{Ext}^1(M \otimes V, M \otimes V').$$

Therefore it suffices to show that

$$\mathrm{Ext}^1(V,V') \to \mathrm{Ext}^1(M^* \otimes M \otimes V, V')$$

is injective. This follows from the fact that the exact sequence

$$0 \longrightarrow \mathrm{Ker} \longrightarrow M^* \otimes M \longrightarrow \mathscr{V}(0) \longrightarrow 0$$

splits. □

The following lemma is proved in [CM, Proposition 1.2].

Lemma 4.7. *Let $\lambda, \mu \in P^+$ with $\lambda - \mu \in Q$. Then there exists a sequence $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_r = \mu$ in P^+ such that*

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V(\lambda_i), V(\lambda_{i+1})) \neq 0$$

for any i.

Proof of Proposition 4.5. The assertion of (i) immediately follows from Theorem 3.6.

We prove (ii). It suffices to show the assertion for the simple modules of the form $V_{\mathfrak{m}}(\lambda)$ by Lemma 4.6. By Proposition 3.1 and Lemma 4.7, we reduce to claim that $\mathrm{Der}(A, A/\mathfrak{m}) \neq 0$. This is deduced from the following well-known facts:

$$\mathrm{Der}(A, A/\mathfrak{m}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathfrak{m}/\mathfrak{m}^2, \mathbb{C})$$

and $\mathfrak{m}/\mathfrak{m}^2 = 0$ if and only if $A = \mathbb{C}$. □

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