Uniform Estimates for Distributions of Sums of i.i.d. Random Variables with Fat Tail (和訳：分布がファットテールをもつ場合の独立同分布の確率変数和の分布の一様評価について)

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第1章　序章

本論文では独立同分布の確率変数和の分布の一様評価について述べる．特に分布関数を $F(X)$, 上側分布関数 $\bar{F}(x) = 1 - F(x)$ とおくとき $\bar{F}(x)$ が指数 $-\alpha$ $(\alpha > 0)$ の regularly varying 関数と呼ばれるクラスに属する場合について述べる．これは大雑把にいうと $\bar{F}(x)$ が $x^{-\alpha}$ のオーダーで減衰する場合である．このような場合, 分布はファットテールを持つといわれる．このとき和の分布もファットテールを持ち, 平均値から大きく離れたところ (大偏差) では元の分布を用いて評価することができる．また分布に分散が存在するときには和の分布は平均値の近くでは正規分布で評価することができる (中心極限定理). 正規分布による近似がどの範囲まで有効かという問題については Cramér[4] による結果が有名である.

応用上, 和の分布の平均値の近くとも大偏差ともいえないところの値を評価したい場合がある．このときは和の分布の一様評価が必要となる．一様評価については, まず $\bar{F}(x)$ が指数 $-\alpha$ $(\alpha > 2)$ の regularly varying 関数の場合に A. Nagaev[10] によって示され, その後 S. Nagaev[13], Rozovskii[17] らによってより一般の場合に示された．A. Nagaev らの評価は収束の速さについては議論されていないが, 最近になって指数 $-\alpha$ $(\alpha > 2)$ の regularly varying 関数の場合に収束の速さを込めた評価が Fushiya-Kusuoka[7] によって示された.

本論文では $\bar{F}(x)$ が指数 -2 の regularly varying 関数の場合についての結果を示す. $\bar{F}(x)$ が指数 $-\alpha$ $(0 < \alpha < 2)$ の regularly varying 関数の場合は和の極限分布は指数 α の安定分布になり, $\bar{F}(x)$ が指数 $-\alpha$ $(\alpha \geq 2)$ の regularly varying 関数の場合には正規分布 (指数 2 の安定分布) になることが知られている．したがって指数 -2 の場合がちょうど境界の値となっており, そのため和の分布の評価を示すことも難しい.

本論文では第 1 章で全体を概観し, 第 2 章で $\bar{F}(x)$ が指数 -2 の regularly varying 関数で分散が存在する場合についての結果を示す．第 3 章で分散が必ずしも存在するとは限らない場合についての結果を示す.

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1.1 序

確率空間 $(\Omega, \mathcal{F}, P)$ 上に独立同分布な確率変数列 $X_1, X_2, \cdots$ で $E[X_1]=0$ となるものをとり、その和の分布について考える. また X_1 の分布 μ, 分布関数を F, 上側分布関数を $\bar{F}$ とおく. すなわち $F(x)=\mu((-\infty,x])$, $\bar{F}(x)=\mu((x,\infty))=1-F(x)$ とおく. このとき $E[X_1^2]<\infty$ ならば中心極限定理が成り立つ.

Theorem 1.1. **中心極限定理**

$\sigma=E[X_1^2]^{1/2}$ とおく. このとき

$$P(\sum_{k=1}^{n} X_k > \sigma n^{1/2} s) \to \Phi_0(s), \qquad n\to\infty,\ s\in\mathbb{R}$$

が成り立つ. ここで $\Phi_0(s)=\int_s^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$ とおいた.

以下 $E[X_1^2]=1$ とする. s が大きいところでは $P(\sum_{k=1}^n X_k > n^{1/2}s)$, $\Phi_0(s)$ は両方とも小さいから比の形の評価のほうが望ましい. また和の分布の正規分布による近似は平均値の周りでは精度は良いが平均値から大きく離れたところでは精度が悪くなる. 正規分布による近似がどの範囲まで有効に成り立つかという問題もよく研究されている. 例えば, どのような単調増加な正数列 $c_n\to\infty$ に対して

$$\sup_{s\in[1,c_n]}\left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{\Phi_0(s)}-1\right| \to 0, \qquad n\to\infty \tag{1.1}$$

となるかという問題が研究されている. Cramér[4] は次を示した.

Theorem 1.2. ある $c>0$ が存在して $E[\exp(c|X_1|)]<\infty$ が成り立つとする. このとき任意の $a\in(0,1/6)$ と任意の非増加正数列 $\varepsilon(n)\to 0$ に対して $c_n=n^a\varepsilon(n)$ とおくと式 (1.1) が成り立つ.

Linnik[9] は Cramér の結果を発展させて, 次を示した.

Theorem 1.3. ある $a\in(0,1/6)$ に対して $E[\exp(|X_1|^{4a/(2a+1)})]<\infty$ が成り立つとする. このとき任意の非増加正数列 $\varepsilon(n)\to 0$ に対して $c_n=n^a\varepsilon(n)$ とおくと式 (1.1) が成り立つ.

S. Nagaev[12] は X_1 の可積分性が弱いときに次を示した.

Theorem 1.4. ある $\delta>0$ に対して $E[|X_1|^{2+\delta}]<\infty$ が成り立つとする. このとき $c_n=(\delta\log n)^{1/2}$ として式 (1.1) が成り立つ.

一方で平均値から大きく離れたところ (大偏差) の和の分布を求めることは大偏差の問題と呼ばれ多くの研究がなされている. 特に分布関数が regularly varying 関数と呼ばれるクラスに属するときは極限の形がきれいにでることもあり, 多くの研究がなされている.

ここで regularly varying 関数の定義をしておく.

Definition 1.1. 関数 $f:\mathbb{R}\to\mathbb{R}$ が $x\to\infty$ で指数 $-\alpha$ の *regularly varying* とは 任意の $a>0$ に対して

$$\frac{f(ax)}{f(x)}\to a^{-\alpha}, \qquad x\to\infty$$

となるときいう.

特に指数 0 の *regularly varying* 関数を *slowly varying* 関数と呼ぶ.

以下では次を仮定する.

(A1) ある $\alpha>0$ に対して, $\bar{F}(x)$ は指数 $-\alpha$ の regularly varying 関数.

このとき $L(x)=x^{\alpha}\bar{F}(x)$ $(x\geq 1)$ とおくと定義より $L(x)$ は slowly varying 関数である.

独立な確率変数 Y_1,Y_2 の上側分布関数をそれぞれ $\bar{F}_1,\bar{F}_2$ とし $\bar{F}_1,\bar{F}_2$ は $x\to\infty$ で指数 $-\alpha$ の regularly varying 関数とする。このとき

$$\frac{P(Y_1+Y_2>s)}{\bar{F}_1(s)+\bar{F}_2(s)}\to 1, \qquad s\to\infty \tag{1.2}$$

が成り立つ. 特に任意の $n\geq 1$ に対して

$$\frac{P(\sum_{k=1}^{n}X_k>n^{1/2}s)}{n\bar{F}(n^{1/2}s)}\to 1, \qquad s\to\infty$$

が成り立つことがわかる. さらに次が成り立つことが知られている.

Theorem 1.5. $\alpha>1$ に対して (A1) を仮定する. このとき任意の $\gamma>0$ に対して

$$\sup_{s\geq\gamma n^{1/2}}\left|\frac{P(\sum_{k=1}^{n}X_k>n^{1/2}s)}{n\bar{F}(n^{1/2}s)}-1\right|\to 0, \qquad n\to\infty$$

が成り立つ.

この定理は $1<\alpha<2$ の場合に Heyde[8], $\alpha>2$ の場合に A. Nagaev[10],[11] が, 一般の場合には Cline-Hsing[3] が示した.

$F(-x)$,$\bar{F}(x)$ が共に $x\to\infty$ で指数 $\alpha\in(0,2)$ の regularly varying 関数となるときは和の極限分布は適当なスケーリングの下, 指数 α の安定分布となることが知られてい

る．また $\int_{-x}^{x} z^2\mu(dz)$ が $x \to \infty$ で slowly varying となるとき極限分布は正規分布 (指数 2 の安定分布) になることが知られている．特に $\int_{-\infty}^{\infty} z^2\mu(dz) < \infty$ ならば $\int_{-x}^{x} z^2\mu(dz)$ が $x \to \infty$ で slowly varying となる．さらに次を示すことができる (例えば Feller[6] を参照).

Proposition 1.1. $\int_{-x}^{x} z^2\mu(dz)$ が $x \to \infty$ で *slowly varying* となると仮定する．正数列 $\{a_n\}_{n=1}^{\infty}$ を $a_n \to \infty$ で

$$\frac{n\int_{-a_n}^{a_n} z^2\mu(dz)}{a_n^2} \to 1, \qquad n \to \infty \tag{1.3}$$

を満たすものとする．このとき

$$P(\sum_{k=1}^{n} X_k > a_n s) \to \Phi_0(s), \qquad n \to \infty,\ s \geq 1$$

が成り立つ.

特に $E[X_1^2] < \infty$ ならば $a_n = (E[X_1^2]n)^{1/2}$ とすれば a_n は式 (1.2) を満たすので, この命題は中心極限定理の拡張になっている.

中心極限定理と大偏差の問題を合わせて和の分布を実数軸上または $[1, \infty)$ 上で一様に評価する研究も行われている．応用上, 正規分布による評価が有効とも大偏差ともいえないところを評価したい場合があるので一様評価を示すことは重要である．(A1) に加えて次の仮定をおく.

(A2)　ある $\delta_0 > 0$ に対して $\int_{-\infty}^{0} |x|^{2+\delta_0}\mu(dx) < \infty$ が成り立つ.

A. Nagaev[10] と S. Nagaev[13] は独立に次を示した.

Theorem 1.6. $\alpha > 2$ に対して (A1), (A2) を仮定する．また $E[X_1^2] = 1$ とする. このとき

$$\sup_{s\geq 1} |\frac{P(\sum_{k=1}^{n} X_k > n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \to 0, \qquad n \to \infty \tag{1.4}$$

が成り立つ.

式 (1.4) に出てきた $\Phi_0(s) + n\bar{F}(n^{1/2}s)$ が s の範囲によって $\Phi_0(s)$ と $\bar{F}(n^{1/2}s)$ のどちらが大きいかを考えよう. 次のことを示すのは難しくない.

任意の $a \in (0, (\alpha-2)^{1/2})$ に対して $n \to \infty$ のとき

$$n\bar{F}(n^{1/2}s) = o(\Phi_0(s)), \qquad 1 \leq s < a(\log n)^{1/2}.$$

同様に任意の $b \in ((\alpha-2)^{1/2}, \infty)$ に対して $n \to \infty$ のとき

$$\Phi_0(s) = o(n\bar{F}(n^{1/2}s)), \qquad b(\log n)^{1/2} < s < \infty.$$

よって $\Phi_0(s)+n\bar{F}(n^{1/2}s)$ は $s\in[1,a(\log n)^{1/2})$ では $\Phi_0(s)$ が主要項で, $s\in(b(\log n)^{1/2},\infty)$ では $n\bar{F}(n^{1/2}s)$ が主要項であることがわかる.

分布の滑らかさを仮定したときはより詳細な評価を示すことができる. (A1), (A2) の他にさらに次を仮定する.

(A3) μ は密度関数 $\rho:\mathbb{R}\to[0,\infty)$ をもち, さらに ρ は右連続関数で有界な全変動をもつとする.

(A4) $|x|^{\alpha+2}F(x)\to 0,\ x\to-\infty$. ここで α は (A1) で定義されたものである.

(A5) ある $x_0>0$ が存在して $\bar{F}$ は (x_0,∞) で 2 階連続微分可能で

$$x^2\frac{d^2}{dx^2}\log\bar{F}(x)\to\alpha,\qquad x\to\infty.$$

Fushiya-Kusuoka[7] は大偏差の問題について次を示した.

Theorem 1.7. $\alpha>2$ に対して (A1), (A2)–(A5) を仮定する. また $\beta:\mathbb{N}\to(0,\infty)$ で

$$\frac{\beta(n)}{(\log n)^{1/2}}\to\infty,\qquad n\to\infty$$

となるものをとる. このとき

$$\sup_{s\geq\beta(n)}s^2\left|\frac{P(\sum_{k=1}^n X_k>n^{1/2}s)}{n\bar{F}(n^{1/2}s)}-(1+\frac{\alpha(\alpha+1)}{2s^2})\right|\to 0,\qquad n\to\infty$$

が成り立つ.

次の定理を述べるために記号を用意する.

$\Phi_k:\mathbb{R}\to\mathbb{R},k=1,2$ を $\Phi_1(x)=-\frac{d}{dx}\Phi_0(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $\Phi_2(x)=-\frac{d}{dx}\Phi_1(x)=\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $H_0(n,s)$ を

$$H_0(n,s)=\Phi_0(s)+n\int_{-\infty}^{s}\bar{F}((s-x)n^{1/2})\Phi_1(x)dx-\sum_{k=1}^{2}\frac{n^{-(k-2)/2}}{k!}\Phi_k(s)\int_0^\infty x^k\mu(dx)$$

とする. Fushiya-Kusuoka[7] は一様評価について次を示した.

Theorem 1.8. $\alpha>2$ に対して (A1), (A2)–(A5) を仮定する. このときある $C>0$, $\delta>0$, $n_0\geq 1$ が存在して

$$\sup_{s\in[1,\infty]}\left|\frac{P(\sum_{k=1}^n X_k>n^{1/2}s)}{H_0(n,s)}-1\right|\leq Cn^{-\delta},\qquad n\geq n_0$$

が成り立つ.

Theorem1.7 より特に

$$2(\log n)^2\left(\frac{P(\sum_{k=1}^n X_k > n^{1/2}\log n)}{\Phi_0(\log n)+\bar{F}((\log n)n^{1/2})}-1\right) \to \alpha(\alpha+1), \qquad n\to\infty$$

がわかる．よって $H_0(n,s)$ の方が $\Phi_0(s)+\bar{F}(n^{1/2}s)$ よりも $P(\sum_{k=1}^n X_k > n^{1/2}s)$ のよい近似となっていることがわかる．

1.2 本論文の主結果．$\alpha=2$ の場合の一様評価

本論文では仮定 (A1) で $\alpha=2$ のときについて議論する．この場合は必ずしも $E[X_1^2]<\infty$ ではないことに注意しておく．

$E[X_1^2]<\infty$ の場合の結果.

$v_n=\int_{-\infty}^{n^{1/2}} x^2\mu(dx)$, $t_n=v_n^{1/2}n^{1/2}$, $L(x)=x^2\bar{F}(x),\ x\geq 1$ とおき, $H_1:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ を

$$\begin{aligned} H_1(n,s) &= \Phi_0(s)+n\int_{-\infty}^{s}\bar{F}(t_n(s-x))\Phi_1(x)dx \\ &\quad -n\left(t_n^{-1}\Phi_1(s)\int_0^\infty x\mu(dx)+t_n^{-2}\frac{\Phi_2(s)}{2}\int_0^{n^{1/2}}x^2\mu(dx)\right) \end{aligned}$$

と定義する．$H_0(n,s)$ と異なり, 最後の項の積分区間が $[0,n^{1/2}]$ となっていることに注意しておく．

まず $E[X_1^2]<\infty$ の場合について考える．このとき次を示した．

Theorem 1.9. $\alpha=2$ に対して (A1), (A2), (A3) を仮定する．また $E[X_1^2]=1$ と仮定する．このとき任意の $\delta\in(0,1)$ に対してある $C>0$ が存在して

$$\sup_{s\in[1,\infty)}\left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H_1(n,v_n^{-1/2}s)}-1\right| \leq CL(n^{1/2})^{1-\delta}, \qquad n\geq 1 \tag{1.5}$$

が成り立つ. 特に

$$\sup_{s\in[1,\infty)}\left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{\Phi_0(v_n^{-1/2}s)+n\bar{F}(n^{1/2}s)}-1\right| \to 0, \qquad n\to\infty. \tag{1.6}$$

注意. 仮定 (A1), (A2) のもと $L(n^{1/2})\to 0\ (n\to\infty)$ が成り立つ.

この定理の証明は第 2 章において行う.

$\alpha>2$ のときと同様に $\Phi_0(v_n^{-1/2}s)+n\bar{F}(n^{1/2}s)$ が s の範囲によって $\Phi_0(v_n^{-1/2}s)$ と $\bar{F}(n^{1/2}s)$ のどちらが大きいかを考えよう．次のことを示すのは難しくない．

任意の $a \in (0,2)$ に対して $n \to \infty$ のとき

$$n(\bar{F}(n^{1/2}s)) = o(\Phi_0(v_n^{-1/2}s)), \qquad 1 < s < (-av_n \log L(n^{1/2}))^{1/2}.$$

同様に任意の $b \in (2,\infty)$ に対して $n \to \infty$ のとき

$$\Phi_0(v_n^{-1/2}s) = o(n\bar{F}(n^{1/2}s)), \qquad (-bv_n \log L(n^{1/2}))^{1/2} < s < \infty.$$

よって $\Phi_0(v_n^{-1/2}s) + n(\bar{F}(n^{1/2}s)$ は $s \in [1,(-av_n \log L(n^{1/2}))^{1/2})$ では $\Phi_0(v_n^{-1/2}s)$ が主要項で, $s \in ((-bv_n \log L(n^{1/2}))^{1/2},\infty)$ では $n\bar{F}(n^{1/2}s)$ が主要項であることがわかる.

また $\alpha = 2$ の場合には一般には式 (1.4) は成り立たない. 式 (1.4) が成り立つための条件として次を示した.

Theorem 1.10. $\alpha = 2$ に対して (A1), (A2) を仮定する. また $E[X_1^2] = 1$ と仮定する. このとき次が成り立つ.

(1) $\limsup_{n\to\infty}(1-v_n)\log \dfrac{1}{L(n^{1/2})} = 0$ ならば

$$\sup_{s\in[1,\infty)} \left|\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty \tag{1.7}$$

が成り立つ.

(2) $\limsup_{n\to\infty}(1-v_n)\log \dfrac{1}{L(n^{1/2})} > 0$ ならば式 (1.7) は成り立たない.

これより $\limsup_{n\to\infty}(1-v_n)\log \dfrac{1}{L(n^{1/2})} = 0$ のときは式 (1.4) は成り立ち, そうでなければ成り立たないことがわかる. Rosovskii[17] も同様の結果を示している. 条件 $\limsup_{n\to\infty}(1-v_n)\log \dfrac{1}{L(n^{1/2})} = 0$ は Rosovskii[17] の式 (56) に対応している. このことより [17] の Theorem 3b で $B_n = n^{1/2}$ とした評価は成り立たないことがわかる.

$E[X_1^2] < \infty$ とは限らない場合の結果.

Theorem1.9 では $E[X_1^2] = 1$ と仮定したが一般に $E[X_1^2] < \infty$ とは限らない場合にも同様の結果が成り立つ.

$\tilde{t}_n = \sup\{t > 0;\ \int_{-\infty}^{t} x^2\mu(dx) > t^2\}$, $\tilde{v}_n = \int_{-\infty}^{\tilde{t}_n} x^2\mu(dx)$ とおく. $\tilde{t}_n = \tilde{v}_n^{1/2}n^{1/2}$ となり, $\tilde{t}_n$ は Proposition1.1 の条件を満たしていることに注意しておく. また $H_2 : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ を

$$\begin{aligned} H_2(n,s) &= \Phi_0(s) + n\int_{-\infty}^{s} \bar{F}(\tilde{t}_n(s-x))\Phi_1(x)dx \\ &\quad -n\left(\tilde{t}_n^{-1}\Phi_1(s)\int_0^{\infty} x\mu(dx) + \tilde{t}_n^{-2}\frac{\Phi_2(s)}{2}\int_0^{\tilde{t}_n} x^2\mu(dx)\right) \end{aligned}$$

と定義する. 最後の項の積分区間が $[0,\tilde{t}_n]$ となっていることに注意しておく.

次が成り立つことを示した.

Theorem 1.11. $\alpha = 2$ に対して (A1), (A2), (A3) を仮定する. このとき任意の $\delta \in (0,1)$ に対してある $C > 0$ が存在して

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^n X_k > \tilde{t}_n s)}{H_2(n,s)} - 1| \le C(n\bar{F}(\tilde{t}_n))^{1-\delta}, \qquad n \ge 1 \tag{1.8}$$

が成り立つ. 特に

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^n X_k > \tilde{t}_n s)}{\Phi_0(s) + n\bar{F}(\tilde{t}_n s)} - 1| \to 0, \qquad n \to \infty. \tag{1.9}$$

注意. 仮定 (A1), (A2) のもと $n\bar{F}(\tilde{t}_n) = \dfrac{L(\tilde{t}_n)}{v_n} \to 0,\ n \to \infty$ が成り立つ.
この定理の証明は第 3 章において行う.

第2章 Uniform Estimates for Distributions of Sums of i.i.d. Random Variables with Fat Tail: Finite variance case.

We show uniform estimates of distributions of the sum of i.i.d. random variables with fat tail and finite variance. Rozovskii[17] showed several uniform estimates but the speeds of convergence were not shown. Our main uniform estimate has a speed of convergence. We also compare our estimates with Nagaev's estimate which is valid in the non-threshold case and we show a necessary and sufficient condition for holding Nagaev's estimate in this case.

2.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_n, n = 1, 2, \ldots,$ be independent identically distributed random variables whose probability law is μ. Let $F : \mathbb{R} \to [0, 1]$ and $\bar{F} : \mathbb{R} \to [0, 1]$ be given by $F(x) = \mu((-\infty, x])$ and $\bar{F}(x) = \mu((x, \infty))$, $\ x \in \mathbb{R}$. We assume the following.

(A1) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha \geq 2$, as $x \to \infty$, i.e., if we let

$$L(x) = x^{\alpha} \bar{F}(x)\ ,\ x \geq 1,$$

then $L(x) > 0$ for any $x \geq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1\ ,\ x \to \infty.$$

(A2) $\int_{-\infty}^{0} |x|^{\alpha+\delta_0} \mu(dx) < \infty$ for some $\delta_0 \in (0, 1)$, $\int_{\mathbb{R}} x^2 \mu(dx) = 1$ and $\int_{\mathbb{R}} x \mu(dx) = 0$

A. Nagaev[10] and S. Nagaev[13] proved the following theorem.

Theorem 2.1. *Assume* (A1) *for* $\alpha > 2$ *and* (A2). *Then we have*

$$\sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty. \tag{2.1}$$

Here $\Phi_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{y^2}{2})dy, \qquad x \in \mathbb{R}.$$

In this chapter, we assume (A1) for $\alpha = 2$ (threshold case), (A2) and the following.

(A3) The probability law μ is absolutely continuous and has a density function $\rho : \mathbb{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

Let us define $\Phi_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, 3,$ by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1}\frac{d^{k-1}}{dx^{k-1}}\Phi_1(x), \qquad k = 2, 3.$$

Let $v_n = \int_{-\infty}^{n^{1/2}} x^2\mu(dx)$ for $n \geq 1$ and let

$$\begin{aligned} H(n, s) \quad &= \quad \Phi_0(s) + n \int_{-\infty}^{s} \bar{F}((s - x)v_n^{1/2}n^{1/2})\Phi_1(x)dx \\ &- \left(v_n^{-1/2}n^{1/2}\Phi_1(s) \int_0^\infty x\mu(dx) + v_n^{-1}\frac{\Phi_2(s)}{2} \int_0^{n^{1/2}} x^2\mu(dx)\right). \end{aligned}$$

In this chapter, we show the following (Theorem 1.9).

Theorem 2.2. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a* $C > 0$ *such that*

$$\sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{H(n, v_n^{-1/2}s)} - 1\right| \leq CL(n^{1/2})^{1-\delta}, \qquad n \geq 1. \tag{2.2}$$

In particular, we have

$$\sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty. \tag{2.3}$$

Rozovskii [17] showed different types of uniform estimate. (see Theorem 1, 2 and 3b in [17].) The estimates in Theorem 1, 2 in [17] were proved under more general assumptions but the speeds of convergence were not shown. The estimate in Theorem 3b in [17] is strongly related to Equation (3) but does not completely include our result. The proof of uniform estimates in [17] is different from ours.

We also prove the following.

Theorem 2.3. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3).

If $\limsup_{n\to\infty}(1 - v_n)\log \dfrac{1}{L(n^{1/2})} = 0$, *then we have*

$$\sup_{s\in[1,\infty)} \left|\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty. \tag{2.4}$$

If $\limsup_{n\to\infty}(1 - v_n)\log \dfrac{1}{L(n^{1/2})} > 0$, *then* Equation (2.4) *does not hold.*

Combining with Theorem 2.2, Theorem 2.3 gives a necessary and sufficient condition for holding Equation (2.1). The condition $\limsup_{n\to\infty}(1 - v_n)\log \dfrac{1}{L(n^{1/2})} = 0$ is correspond to Equation (56) in [17]. Hence the estimate with $B_n = n^{1/2}$ in Theorem 3b in [17] is not valid under our assumptions.

We also prove the following to obtain Theorem 2.2.

Theorem 2.4. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a* $C > 0$ *such that*

$$|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2}s)| \le CL(n^{1/2})^{2-\delta}, \qquad s \ge 1.$$

Throughout this paper we assume (A1) for $\alpha = 2$, (A2) and (A3). Then we see that $L(t) \to 0$, $t \to \infty$ and $\dfrac{1 - v_n}{L(n^{1/2})} \to \infty$, $n \to \infty$ (see Equation (2.5), (2.6)).

2.2 Preliminary facts

We summarize several known facts (c.f. Fushiya-Kusuoka[7]).

Proposition 2.1. *We have*

$$\sup_{1/2\le a\le 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty,$$

and

$$\inf_{1/2\le a\le 2}\frac{L(ax)}{L(x)}\to 1,\qquad x\to\infty.$$

Proposition 2.2. *For any $\varepsilon\in(0,1)$, there is an $M(\varepsilon)\ge 1$ such that*

$$M(\varepsilon)^{-1}y^{-\varepsilon}\le\frac{L(yx)}{L(x)}\le M(\varepsilon)y^{\varepsilon}\qquad x,y\ge 1.$$

Proposition 2.3. (1) *For any $\beta<-1$,*

$$\frac{1}{t^{\beta+1}L(t)}\int_t^\infty x^\beta L(x)dx\to-\frac{1}{\beta+1},\qquad t\to\infty.$$

(2) *For any $\beta>-1$,*

$$\frac{1}{t^{\beta+1}L(t)}\int_1^t x^\beta L(x)dx\to\frac{1}{\beta+1},\qquad t\to\infty.$$

(3) *Let $f:[1,\infty)\to(0,\infty)$ be given by*

$$f(t)=\int_1^t x^{-1}L(x)dx\qquad t\ge 1.$$

Then f is slowly varying. Moreover if $\lim_{t\to\infty}f(t)<\infty$,we have

$$\frac{1}{L(t)}\int_t^\infty x^{-1}L(x)dx\to\infty,\qquad t\to\infty.$$

Proposition 2.4. *There is a $C_0>0$ such that*

$$|\Phi_k(x)|\le C_0(1+x)^{k-1}\Phi_1(x),\qquad x\ge 0,k=1,2,$$

and

$$C_0^{-1}\Phi_1(x)\le x\Phi_0(x)\le C_0\Phi_1(x),\qquad x\ge 1/2.$$

Proposition 2.5. (1) *For any $m\ge 1$, let $r_{e,m}:\mathbb{R}\to\mathbb{C}$ be given by*

$$r_{e,m}(t)=\exp(it)-(1+\sum_{k=1}^m\frac{(it)^k}{k!}),\qquad t\in\mathbb{R}.$$

Then we have

$$|r_{e,m}(t)|\le\frac{\min(|t|^{m+1},2(m+1)|t|^m)}{(m+1)!},\qquad t\in\mathbb{R}.$$

(2) *For any $m\ge 1$, let $r_{l,m}:\{z\in\mathbb{C};\ |z|\le 1/2\}\to\mathbb{C}$ be given by*

$$r_{l,m}(z)=\log(1+z)-\sum_{k=1}^m\frac{(-1)^{k-1}}{k}z^k,\qquad z\in\mathbb{C},|z|\le 1/2.$$

Then we have

$$|r_{l,m}(z)|\le 2|z|^{m+1},\qquad z\in\mathbb{C},|z|\le 1/2.$$

Let $\mu(t)$, $t > 0$, be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\mu(t)(A) = (1 - \bar{F}(t))^{-1}\mu(A \cap (-\infty, t])),$$

for any $A \in \mathcal{B}(\mathbb{R})$.

Let $\varphi(\cdot; \mu(t))$, $t > 0$, be the characteristic function of the probability measure $\mu(t)$, i.e.,

$$\varphi(\xi; \mu(t)) = \int_{\mathbb{R}} \exp(ix\xi)\mu(t)(dx), \qquad \xi \in \mathbb{R}.$$

Proposition 2.6. *There is a $c_0 > 0$ such that for any $t \geq 2$, $\xi \in \mathbb{R}$ and integer n, m with $n \geq m$,*

$$|\varphi(n^{-1/2}\xi, \mu(t))|^n \leq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

Proposition 2.7. *Let ν be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_{\mathbb{R}} x^2\nu(dx) < \infty$.Also,assume that there is a constant $C > 0$ such that the characteristic function $\varphi(\cdot, \nu) : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|\varphi(\xi; \nu)| \leq C(1 + |\xi|)^{-2}, \qquad \xi \in \mathbb{R}.$$

Then for any $x \in \mathbb{R}$ and $v > 0$

$$\nu((x, \infty)) = \Phi_0(v^{-1/2}x) + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{i\xi}(\varphi(\xi, \nu) - \exp(-\frac{v\xi^2}{2}))d\xi.$$

2.3 Estimate for moments

Let

$$\eta_k(t) = \int_{-\infty}^{t} x^k\mu(dx), \qquad t > 0,\ k = 1, 2,$$

and

$$\eta_3(t) = \int_{1}^{t} x^3\mu(dx), \qquad t > 1.$$

Then we see that

$$-\eta_1(t) = \int_{t}^{\infty} x\mu(dx) = \int_{t}^{\infty} \bar{F}(x)dx + t\bar{F}(t), \qquad t > 0,$$

$$1-\eta_2(t)=\int_t^\infty x^2\mu(dx)=2\int_t^\infty x\bar{F}(x)dx+t^2\bar{F}(t),\qquad t>0,$$

and

$$\eta_3(t)=\bar{F}(1)-t^3\bar{F}(t)+3\int_1^t x^2\bar{F}(x)dx\qquad t>1.$$

In particular, we see that

$$L(t)\le 1-\eta_2(t)\to 0,\quad t\to\infty, \tag{2.5}$$

$$\frac{1-\eta_2(t)}{L(t)}\to\infty,\quad t\to\infty. \tag{2.6}$$

For any $\delta>0$, let $t_n=n^{1/2}L(n^{1/2})^\delta$. Note that $n^{-1/2}t_n\to 0,\quad n\to\infty$.

Proposition 2.8. *For any $\varepsilon>0$, there is a $C>0$ such that*

$$\frac{L(t_n)}{L(n^{1/2})}\le CL(n^{1/2})^{-\varepsilon\delta}, \tag{2.7}$$

$$n\bar{F}(t_n)\le CL(n^{1/2})^{1-2\delta-\varepsilon\delta}, \tag{2.8}$$

$$\eta_2(n^{1/2})-\eta_2(t_n)\le CL(n^{1/2})^{1-2\varepsilon\delta}, \tag{2.9}$$

$$-n^{1/2}\eta_1(t_n)\le CL(n^{1/2})^{1-2\delta}, \tag{2.10}$$

$$n^{-1/2}\eta_3(t_n)\le CL(n^{1/2}). \tag{2.11}$$

for any $n\ge 1$.

Proof. From Proposition 2.2, there is an $M(\varepsilon)>0$ such that

$$\frac{L(t_n)}{L(n^{1/2})}=\frac{L(t_n)}{L(t_nL(n^{1/2})^{-\delta})}\le M(\varepsilon)L(n^{1/2})^{-\varepsilon\delta}.$$

Hence we have Inequality (2.7). Similarly, we see that

$$n\bar{F}(t_n)=L(n^{1/2})^{-2\delta}L(t_n)=L(n^{1/2})^{1-2\delta}\frac{L(t_n)}{L(n^{1/2})}$$

and

$$\begin{aligned}
\eta_2(n^{1/2})-\eta_2(t_n) &= L(t_n)-L(n^{1/2})+2\int_{t_n}^{n^{1/2}}\frac{L(z)}{z}dz\\
&= L(t_n)-L(n^{1/2})+2L(t_n)\int_1^{L(t_n)^{-\delta}}\frac{L(t_ny)}{L(t_n)y}dy\\
&\le L(t_n)-L(n^{1/2})+2L(t_n)M(\varepsilon)\int_1^{L(t_n)^{-\delta}}y^{-1+\varepsilon}dy\\
&\le L(t_n)-L(n^{1/2})+2\frac{M(\varepsilon)}{\varepsilon}L(t_n)(L(n^{1/2})^{-\varepsilon\delta}-1).
\end{aligned}$$

From Inequality (2.7), we have Inequalities (2.8) and (2.9).

Let

$$\varepsilon_1(t) = \frac{1}{t^{-1}L(t)} \int_t^\infty x^{-2}L(x)dx - 1$$

and

$$\varepsilon_3(t) = \frac{1}{tL(t)} \int_1^t L(x)dx - 1.$$

Then from Proposition 2.3 (1) and (2) we have $\varepsilon_1(t) \to 0$ and $\varepsilon_3(t) \to 0$ as $t \to \infty$.

Hence we see that

$$\begin{aligned} -n^{1/2}\eta_1(t_n) &= n^{1/2}\left(t_n\bar{F}(t_n) + \int_{t_n}^\infty \bar{F}(x)dx\right) = L(n^{1/2})^{-\delta}L(t_n)(2+\varepsilon_1(t_n)) \\ &= (2+\varepsilon_1(t_n))L(n^{1/2})^{1-\delta}\frac{L(t_n)}{L(n^{1/2})} \end{aligned}$$

and

$$\begin{aligned} n^{-1/2}\eta_3(t_n) &= n^{-1/2}\bar{F}(1) + (2+\varepsilon_3(t_n))L(n^{1/2})^{\delta}L(t_n) \\ &= n^{-1/2}\bar{F}(1) + (2+\varepsilon_3(t_n))L(n^{1/2})^{1+\delta}\frac{L(t_n)}{L(n^{1/2})}. \end{aligned}$$

From Inequality (2.7), we have Inequalities (2.10) and (2.11). □

2.4 Asymptotic expansion of characteristic functions

Remind that $v_n = \int_{-\infty}^{n^{1/2}} x^2\mu(dx)$ and $t_n = n^{1/2}L(n^{1/2})^{\delta}$.

In this section, we prove the following Lemma.

Lemma 2.1. *Let*

$$\begin{aligned} R_{n,0}(\xi) &= \exp(\frac{v_n}{2}\xi^2)\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - (1 + n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + \frac{v_n}{2}\xi^2), \\ R_{n,1}(\xi) &= \exp(\frac{v_n}{2}\xi^2)\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - 1, \\ R_{n,2}(\xi) &= \exp(\frac{v_n}{2}\xi^2)\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - 1. \end{aligned}$$

Then there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \le CL(n^{1/2})^{2-5\delta}|\xi| \tag{2.12}$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leq CL(n^{1/2})^{1-2\delta}|\xi| \tag{2.13}$$

for any $n \geq 8$ and $\xi \in \mathbb{R}$ with $|\xi| \leq L(n^{1/2})^{-\delta}$.

As a corollary to Lemma 2.1, we have the following.

Corollary 2.1. *Let*

$$\begin{aligned} \tilde{R}_0(n,s) &= \mu(t_n)^{*n}((sn^{1/2},\infty)) - \Phi_0(v_n^{-1/2}s) \\ &- \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\left(n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + \frac{v_n\xi^2}{2}\right)e^{-v_n\xi^2/2}d\xi \end{aligned}$$

$$\tilde{R}_{1,k}(n,s) = \mu(t_n)^{*(n-k)}((sn^{1/2},\infty)) - \Phi_0(v_n^{-1/2}s), \qquad k = 0, 1,$$

and

$$\tilde{R}_2(n,s) = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - e^{-\frac{v_n\xi^2}{2}}\right| d\xi.$$

Then there is a $C > 0$ such that for any $n \geq 1$, we have

$$|\tilde{R}_0(n,\xi)| \leq CL(n^{1/2})^{2-6\delta} \tag{2.14}$$

and

$$|\tilde{R}_{1,0}(n,\xi)| + |\tilde{R}_{1,1}(n,\xi)| + |\tilde{R}_2(n,\xi)| \leq CL(n^{1/2})^{1-4\delta}. \tag{2.15}$$

Proof. From Proposition 2.8, we see that

$$\begin{aligned} &\tilde{R}_0(n,s) \\ =& \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - \left(1 + n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + \frac{v_n\xi^2}{2}\right)e^{-\frac{v_n\xi^2}{2}}\right) d\xi \\ =& \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}R_{n,0}(\xi)e^{-v_n\xi^2/2}d\xi. \end{aligned}$$

By Lemma 2.1, there is a $C_0 > 0$ such that

$$\int_{|\xi|\leq L(n^{1/2})^{-\delta}} \frac{|R_{n,0}(\xi)|}{|\xi|}d\xi \leq C_0L(n^{1/2})^{2-6\delta}.$$

It is easy to see from Proposition 2.5 (1) that

$$n|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1| \leq \frac{n^{1/2}|\eta_1(t_n)||\xi|}{1 - \bar{F}(t_n)} + \frac{|\xi|^2}{2(1 - \bar{F}(t_n))}, \qquad \xi \in \mathbb{R}.$$

From the above inequality and Proposition 2.7, we see that for any $m \geq 2/\delta$, there is a $C_1 > 0$ such that for any $n \geq 2m$ and $\xi \in \mathbb{R}$ with $|\xi| \geq L(n^{1/2})^{-\delta}$,

$$|\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))|^n + \left|n(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1) + 1 + \frac{v_n\xi^2}{2}\right| e^{-\frac{v_n\xi^2}{2}} \leq C_1|\xi|^{-m}.$$

Hence we have

$$\begin{aligned} & \int_{|\xi|>L(n^{\frac{1}{2}})^{-\delta}} |\xi|^{-1} \left|\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))^n - \left(1 + n(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1) + \frac{v_n\xi^2}{2}\right) e^{-\frac{v_n\xi^2}{2}}\right| d\xi \\ \leq \;& 2C_1 \int_{L(n^{1/2})^{-\delta}}^{\infty} |\xi|^{-m-1} d\xi = \frac{2C_1}{m} L(n^{1/2})^{m\delta} \leq \frac{2C_1}{m} L(n^{1/2})^2. \end{aligned}$$

Therefore we have Inequation (2.14).

We see also that

$$\begin{aligned} \tilde{R}_{1,k}(n,s) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} \left(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))^n - e^{-\frac{v_n\xi^2}{2}}\right) d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} R_{n,1+k}(\xi) e^{-v_n\xi^2/2} d\xi, \\ \tilde{R}_2(n,s) &= \frac{1}{2\pi} \int_{\mathbb{R}} |R_{n,2}(\xi)| e^{-v_n\xi^2/2} d\xi. \end{aligned}$$

Similarly to the first equation, we have Inequation (2.15). □

We make some preparations to prove Lemma 2.1.

Let

$$R_0(n,\xi) = \varphi(n^{-1/2}\xi, \mu(t_n)) - (1 - v_n\frac{\xi^2}{2n}).$$

First we prove the following.

Proposition 2.9. *There is a constant $C > 0$ such that for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq L(n^{1/2})^{-\delta}$,*

$$|nR_0(n,\xi)| \leq CL(n^{1/2})^{1-2\delta}|\xi|$$

and

$$n|\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1| \leq CL(n^{1/2})^{-\delta}|\xi|.$$

In particular,

$$\sup\{|nR_0(n,\xi)|; |\xi| \leq L(n^{1/2})^{-\delta}\} \to 0, \qquad n \to \infty. \tag{2.16}$$

Proof. We can easily see that

$$
\begin{aligned}
\varphi(\xi;\mu(t)) &= \int_{\mathbb{R}} \exp(ix\xi)\mu(t)(dx) \\
&= 1+\eta_1(t)(i\xi)+\eta_2(t)\frac{(i\xi)^2}{2}+\int_{-\infty}^{1} r_{e,2}(\xi x)\mu(dx) \\
&\quad + \int_1^t r_{e,2}(\xi x)\mu(dx)+\frac{\bar{F}(t)}{1-\bar{F}(t)}\int_{-\infty}^{t} r_{e,0}(\xi x)\mu(dx).
\end{aligned}
$$

Hence we have that

$$
\begin{aligned}
R_0(n,\xi) &= n^{-1/2}\eta_1(t_n)(i\xi)+(\eta_2(t_n)-\eta_2(n^{1/2}))\frac{(i\xi)^2}{2n}+\int_{-\infty}^{1} r_{e,2}(n^{-1/2}\xi x)\mu(dx) \\
&\quad + \int_1^{t_n} r_{e,2}(n^{-1/2}\xi x)\mu(dx)+\frac{\bar{F}(t_n)}{1-\bar{F}(t_n)}\int_{-\infty}^{t} r_{e,0}(n^{-1/2}\xi x)\mu(dx).
\end{aligned}
$$

Then we see that

$$
\begin{aligned}
n|R_0(n,\xi)| &\le n^{\frac{1}{2}}|\eta_1(t_n)||\xi|+(\eta_2(n^{\frac{1}{2}})-\eta_2(t_n))\frac{|\xi|^2}{2}+n^{-\frac{\delta_0}{2}}\int_{-\infty}^{1}|x|^{2+\delta_0}\mu(dx)|\xi|^{2+\delta_0} \\
&\quad +\frac{1}{6}n^{-\frac{1}{2}}\eta_3(t_n)|\xi|^3+2n^{\frac{1}{2}}\bar{F}(t_n)\int_{\mathbb{R}}|x|\mu(dx)|\xi|, \qquad \xi\in\mathbb{R}, t\ge 2,
\end{aligned}
$$

where δ_0 is in (A2). Hence from Proposition 2.5, we see that there is a $C>0$ such that

$$
|nR_0(n,\xi)| \le C\left(L(n^{1/2})^{1-2\delta}|\xi|+L(n^{1/2})^{1-\delta}|\xi|^2+n^{-\delta_0/2}|\xi|^{2+\delta_0}+L(n^{1/2})|\xi|^3\right).
$$

Therefore we have the first inequality.

Since $n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1)=nR_0(n,\xi)-\eta_2(n^{1/2})\xi^2/2$, we have the second inequality. □

Let

$$
R_{1,k}(n,\xi)=(n-k)\log\varphi(n^{-1/2}\xi;\mu(t_n))-n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1), \quad k=0,1.
$$

Proposition 2.10. *There is a $C>0$ such that for any $\xi\in\mathbb{R}$ with $|\xi|\le L(n^{1/2})^{-\delta}$,*

$$
|R_{1,k}(n,\xi)|\le Cn^{-1}L(n^{1/2})^{-3\delta}|\xi|.
$$

In particular

$$
\sup\{|R_{1,k}(n,\xi)|;|\xi|\le L(n^{1/2})^{-\delta}\}\to 0, \qquad n\to\infty. \tag{2.17}
$$

Proof. First, we have

$$\log\varphi(\xi,\mu(t)) = \varphi(\xi,\mu(t)) - 1 + r_{l,1}(\varphi(\xi,\mu(t)) - 1), \quad |\xi| \le L(n^{1/2})^{-\delta}.$$

Hence we have

$$R_{1,k}(n,\xi) = -k\log\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) + nr_{l,1}(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1).$$

From Proposition 2.9, we see that there is a $C >$ such that

$$\begin{aligned} |R_{1,k}(n,\xi)| &\le |\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1| + 2n|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1|^2 \\ &\le Cn^{-1}L(n^{1/2})^{-3\delta}|\xi|, \qquad |\xi| \le L(n^{1/2})^{-\delta}. \end{aligned}$$

□

Let us prove Lemma 2.1. Note that for $k = 0, 1$

$$\log(e^{v_n\xi^2/2}\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-k}) = nR_0(n,\xi) + R_{1,k}(n,\xi).$$

We see that

$$e^{v_n\xi^2/2}\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-k} = \exp(nR_0(n,\xi) + R_{1,k}(n,\xi)).$$

Hence we see that

$$\begin{aligned} R_{n,0}(\xi) &= e^{v_n\xi^2/2}\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - (1 + nR_0(n,\xi)) \\ &= \exp(nR_0(n,\xi)) - (1 + nR_0(n,\xi)) + \exp(nR_0(n,\xi))(\exp(R_{1,0}(n,\xi)) - 1). \end{aligned}$$

From Inequality (2.16), we see that there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \le C\left(|nR_0(n,\xi)|^2 + |R_{1,0}(n,\xi)|\right).$$

Therefore we have Inequality (2.12) from Proposition 2.9 and 2.10.

Proof of Inequality (2.13) is similar to Inequality (2.12).

2.5 Proof of Theorem 2.4

Note that

$$P(\sum_{l=1}^{n} X_l > sn^{1/2}) = \sum_{k=0}^{n} I_k(n,s),$$

where

$$I_k(n,s) = P(\sum_{l=1}^n X_l > sn^{1/2}, \sum_{l=1}^n 1_{\{X_l > t_n\}} = k), \qquad k = 0, 1, \ldots, n.$$

Then we have

$$I_k(n,s) = \binom{n}{k} P(\sum_{l=1}^n X_l > sn^{1/2}, X_i > t_n, i = 1, \ldots, k, X_j \le t_n, j = k+1, \ldots, n),$$

for $k = 0, 1, \ldots, n$.

Let $\bar{F}_{n,0}(x) = P(X_1 > n^{1/2}x, X_1 \le t_n) = (1 - \bar{F}(t_n))\mu(t_n)((n^{1/2}x, \infty))$ and $\bar{F}_{n,1}(x) = P(X_1 > n^{1/2}x, X_1 > t_n)$. Note that $\bar{F}_{n,0}(x) + \bar{F}_{n,1}(x) = \bar{F}(n^{1/2}x)$.

Proposition 2.11. *There is a $C > 0$ such that*

$$\begin{aligned} & |I_0(n,s) - (1-n)\Phi_0(v_n^{-1/2}s) - \frac{1}{2}\Phi_2(v_n^{-1/2}s) - n\int_{\mathbb{R}} \bar{F}_{n,0}(s - v_n^{1/2}x)\Phi_1(x)dx| \\ \le\ & CL(n^{1/2})^{2-5\delta}, \qquad n \ge 1, s \ge 1. \end{aligned}$$

Proof. First, note that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s - v_n^{1/2}x)}{1 - \bar{F}(t_n)}\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s) \\ =\ & \int_s^\infty \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-ix\xi}(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1)e^{-\frac{v_n}{2}\xi^2}d\xi\right)dx \\ =\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1)e^{-\frac{v_n}{2}\xi^2}d\xi. \end{aligned}$$

Hence we have

$$\begin{aligned} & n\Big(\int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s - v_n^{1/2}x)}{1 - \bar{F}(t_n)}\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s)\Big) + \frac{1}{2}\Phi_2(v_n^{-1/2}s) \\ =\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\Big(n(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1) + \frac{v_n\xi^2}{2}\Big)e^{-v_n\xi^2/2}d\xi. \end{aligned}$$

By Corollary 2.1, we see that there is a $C > 0$ such that for any $n \ge 1$, we have

$$\begin{aligned} & |\mu(t_n)^{*n}((sn^{1/2}, \infty)) - (1-n)\Phi_0(v_n^{-1/2}s) - \frac{v_n^{-1/2}}{2}\Phi_2(s) \\ & \qquad -n\int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s - v_n^{1/2}x)}{1 - \bar{F}(t_n)}\Phi_1(x)dx| \\ \le\ & CL(n^{1/2})^{2-5\delta}. \end{aligned}$$

We decompose $I_0(n,s)$ into three parts, i.e.,

$$\begin{aligned} I_0(n,s) &= (1-n\bar{F}(t_n))^n \mu(t_n)^{*n}((sn^{1/2},\infty)) \\ &= I_{0,0}(n,s) + I_{0,1}(n,s) + I_{0,2}(n,s), \end{aligned}$$

where

$$\begin{aligned} I_{0,0}(n,s) &= (1-\bar{F}(t_n))\mu(t_n)^{*n}((sn^{1/2},\infty)), \\ I_{0,1}(n.s) &= -n\bar{F}(t_n)\mu(t_n)^{*n}((sn^{1/2},\infty)), \\ I_{0,2}(n,s) &= (1-n\bar{F}(t_n))^n - 1 + (n+1)\bar{F}(t_n))\mu(t_n)^{*n}((sn^{1/2},\infty)). \end{aligned}$$

Since

$$\begin{aligned} & (1-\bar{F}(t_n))\left((1-n)\Phi_0(v_n^{-1/2}s) + \frac{\Phi_2(v_n^{-1/2}s)}{2} + n\int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s-v_n^{1/2}x)}{1-\bar{F}(t_n)}\Phi_1(x)dx\right) \\ = \ & (1-n+n\bar{F}(t_n))\Phi_0(v_n^{-1/2}s) + \frac{\Phi_2(v_n^{-1/2}s)}{2} + n\int_{\mathbb{R}} \bar{F}_{n,0}(s-v_n^{1/2}x)\Phi_1(x)dx \\ & -\bar{F}(t_n)\Phi_0(v_n^{-1/2}s) - \bar{F}(t_n)\frac{\Phi_2(v_n^{-1/2}s)}{2}, \end{aligned}$$

we have

$$\begin{aligned} & |I_{0,0}(n,s) - (1-n+n\bar{F}(t_n))\Phi_0(v_n^{-\frac{1}{2}}s) - \frac{\Phi_2(v_n^{-\frac{1}{2}}s)}{2} - n\int_{\mathbb{R}} \bar{F}_{n,0}(s-v_n^{\frac{1}{2}}x)\Phi_1(x)dx| \\ \leq \ & CL(n^{1/2})^{2-5\delta}. \end{aligned}$$

By Corollary 2.1, we see that

$$|I_{0,1}(n,s) + n\bar{F}(t_n)\Phi_0(v_n^{-\frac{1}{2}}s)| \leq CL(n^{1/2})^{2-5\delta}.$$

Note that $|(1-x)^n - (1-nx)| \leq n^2x^2$ for any $x \in [0,1]$, $n \geq 1$. Hence we have

$$|I_{0,2}(n,s)| \leq Cn^2\bar{F}(t_n)^2 \leq CL(n^{1/2})^{2-5\delta}.$$

Therefore we have our assertion. □

Proposition 2.12. *There is a $C > 0$ such that*

$$|I_1(n,s) - n\int_{\mathbb{R}} \bar{F}_{n,1}(s-v_n^{1/2}x)\Phi_1(x)dx| \leq CL(n^{1/2})^{2-6\delta}, \qquad n \geq 1, s \geq 1.$$

Proof.

$$
\begin{aligned}
I_1(n,s) &= n(1-\bar{F}(t_n))^{n-1}\int_{\mathbb{R}} P(X_1+x>sn^{1/2}\ ,X_1>t_n)\mu(t_n)^{*(n-1)}(dx)\\
&= n(1-\bar{F}(t_n))^{n-1}\int_{\mathbb{R}} \bar{F}((sn^{1/2}-x)\vee t_n)\mu(t_n)^{*(n-1)}(dx)\\
&= nJ_0(n,s)+nJ_1(n,s)+nJ_2(n,s),
\end{aligned}
$$

where

$$
\begin{aligned}
J_0(n,s) &= \int_{-\infty}^{sn^{1/2}-t_n} \bar{F}(sn^{1/2}-x)\mu(t_n)^{*(n-1)}(dx),\\
J_1(n,s) &= \bar{F}(t_n)\int_{sn^{1/2}-t_n}^{\infty}\mu(t_n)^{*(n-1)}(dx)=\bar{F}(t_n)\mu(t_n)^{*(n-1)}(((s-L(n^{1/2})^{\delta})n^{1/2},\infty)),
\end{aligned}
$$

and

$$
J_2(n,s) = ((1-\bar{F}(t_n))^{n-1}-1)\int_{\mathbb{R}}\bar{F}((sn^{1/2}-x)\vee t_n)\mu(t_n)^{*(n-1)}(dx).
$$

We see that

$$
\begin{aligned}
&J_0(n,s)\\
=& \int_{-\infty}^{sn^{1/2}-t_n}\bar{F}(sn^{1/2}-x)\mu(t_n)^{*(n-1)}(dx)\\
=& \int_{-\infty}^{sn^{1/2}-t_n}\bar{F}(sn^{1/2}-x)\Big(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-in^{-1/2}x\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1}-e^{-\frac{v_n\xi^2}{2}}\right)d\xi\\
&\qquad +v_n^{-1/2}\Phi_1(n^{-1/2}v_n^{-1/2}x)\Big)n^{-1/2}dx\\
=& \int_{-\infty}^{s-L(n^{1/2})^{\delta}}\bar{F}((s-x)n^{1/2})\frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-ix\xi}(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1}-e^{-\frac{v_n\xi^2}{2}})d\xi\right)dx\\
&+\int_{-\infty}^{v_n^{-1/2}(s-L(n^{1/2})^{\delta})}\bar{F}((s-v_n^{1/2}x)n^{1/2})\Phi_1(x)dx.
\end{aligned}
$$

Hence from Corollary 2.1, we see that there is a $C>0$ such that

$$
\begin{aligned}
&n|J_0(n,s)-\int_{-\infty}^{v_n^{-1/2}(s-L(n^{1/2})^{\delta})}\bar{F}((s-v_n^{1/2}x)n^{1/2})\Phi_1(x)dx|\\
\leq\ & CL(n^{1/2})^{1-4\delta}n^{1/2}\int_{t_n}^{\infty}\bar{F}(x)dx\\
\leq\ & CL(n^{1/2})^{1-4\delta}n^{1/2}|\eta_1(t_n)|.
\end{aligned}
$$

We also see that

$$
\begin{aligned}
&n|J_1(n,s)-\bar{F}(t_n)\Phi_0(v_n^{-1/2}(s-L(n^{1/2})^{\delta}))|\\
\leq\ & Cn\bar{F}(t_n)|\mu(t_n)^{*(n-1)}(((s-L(n^{1/2})^{\delta})n^{1/2},\infty))-\Phi_0(v_n^{-1/2}(s-L(n^{1/2})^{\delta}))|\\
\leq\ & Cn\bar{F}(t_n)L(n^{1/2})^{1-4\delta}
\end{aligned}
$$

and

$$|nJ_2(n,s)| \leq \left(n\bar{F}(t_n)\right)^2 \leq L(n^{1/2})^{2-6\delta}.$$

Note that

$$\begin{aligned}
&\int_{-\infty}^{v_n^{-1/2}(s-L(n^{1/2})^\delta)} \bar{F}((s-v_n^{1/2}x)n^{1/2})\Phi_1(x)dx + \bar{F}(t_n)\Phi_0(v_n^{-1/2}(s-L(n^{1/2})^\delta)) \\
= \;& \int_{\mathbb{R}} \bar{F}_{n,1}(s-v_n^{1/2}x)\Phi_1(x)dx.
\end{aligned}$$

Therefore we have our assertion. $\square$

Let us prove Theorem 2.4.

From Proposition 2.11 and 2.12, we see that there is a $C>0$ such that

$$\begin{aligned}
&|I_0(n,s)+I_1(n,s) \\
&\quad -(1-n)\Phi_0(v_n^{-1/2}s) - \frac{1}{2}\Phi_2(v_n^{-1/2}s) - n\int_{\mathbb{R}} \bar{F}(n^{1/2}(s-v_n^{1/2}x))\Phi_1(x)dx| \\
\leq \;& CL(n^{1/2})^{2-6\delta}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}} \bar{F}(n^{\frac{1}{2}}(s-v_n^{1/2}x))\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s) \\
= \;& \int_{-\infty}^{v_n^{-1/2}s} \bar{F}(n^{\frac{1}{2}}(s-v_n^{1/2}x))\Phi_1(x)dx \\
&\quad + \int_{v_n^{-1/2}s}^{\infty} (\bar{F}(n^{\frac{1}{2}}(s-v_n^{1/2}x)) - 1_{\{v_n^{1/2}x>s\}})\Phi_1(x)dx \\
= \;& \int_{-\infty}^{v_n^{-1/2}s} \bar{F}(n^{\frac{1}{2}}v_n^{1/2}(v_n^{-\frac{1}{2}}s-x))\Phi_1(x)dx - \int_{v_n^{-1/2}s}^{\infty} F(n^{\frac{1}{2}}(s-v_n^{1/2}x))\Phi_1(x)dx
\end{aligned}$$

and

$$n\int_{v_n^{-1/2}s}^{\infty} F(n^{1/2}(s-v_nx))\Phi_1(x)dx = n^{1/2}\int_{-\infty}^{0} F(y)\Phi_1(v_n^{-1/2}s - n^{-1/2}v_n^{-1/2}y)v_n^{-1/2}dy.$$

Let $R(z,y) = \Phi_1(z-y) - \Phi_1(z) - \Phi_2(z)y$, for $z>0, y\leq 0$, then we see that there is a $C_1>0$ such that

$$|R(s,y)| \leq C_1|y|^{1+\delta_0},$$

where δ_0 is in (A2). Hence we have

$$
\begin{aligned}
& n\left|\int_{v_n^{-1/2}s}^{\infty} F(n^{1/2}(s-v_nx))\Phi_1(x)dx - \sum_{k=1}^{2} v_n^{-k/2}n^{-k/2}\Phi_k(v_n^{-1/2}s)\int_{-\infty}^{0} y^{k-1}F(y)dy\right| \\
= & \left|n^{1/2}\int_{-\infty}^{0} R(v_n^{-1/2}s, n^{-1/2}v_n^{-1/2}y)F(y)dy\right| \\
\leq & C_1 n^{-\delta_0/2} v_n^{-(1+\delta_0)/2} \int_{-\infty}^{0} |y|^{1+\delta_0} F(y)dy \\
\leq & Cn^{-\delta_0/2},
\end{aligned}
$$

where $C = C_1 v_1^{-(1+\delta_0)/2} \int_{-\infty}^{0} y^{1+\delta_1} F(y)dy < \infty$.

Since

$$
\int_{-\infty}^{0} F(y)dy = \int_{-\infty}^{0} y\mu(dy) = -\int_{0}^{\infty} y\mu(dy)
$$

and

$$
-\int_{-\infty}^{0} yF(y)dy = \frac{1}{2}\int_{-\infty}^{0} y^2\mu(dy) = \frac{v_n}{2} - \frac{1}{2}\int_{0}^{n^{1/2}} y^2\mu(dy),
$$

we see that

$$
\frac{1}{2}\Phi_2(v_n^{-1/2}s) + v_n^{-1}\Phi_2(v_n^{-1/2}s)\int_{-\infty}^{0} y^2F(y)dy = v_n^{-1}\frac{\Phi_2(v_n^{-1/2}s)}{2}\int_{0}^{n^{1/2}} y^2\mu(dy).
$$

Therefore we have

$$
\begin{aligned}
& \left|(1-n)\Phi_0(v_n^{-1/2}s) + \frac{1}{2}\Phi_2(v_n^{-1/2}s) + n\int_{\mathbb{R}} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx - H(n, v_n^{-1/2}s)\right| \\
\leq & Cn^{-\delta_0/2}.
\end{aligned}
$$

We also see that

$$
\begin{aligned}
\sum_{k=2}^{n} I_k(n,s) & \leq \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}\bar{F}(t_n)^k(1-\bar{F}(t_n))^{n-k} \leq \frac{n(n-1)}{2}\bar{F}(t_n)^2 \\
& \leq L(n^{1/2})^{2-5\delta}.
\end{aligned}
$$

This completes the proof of Theorem 2.4.

2.6 Some Estimations

Let

$$\begin{aligned}
\hat{F}_n(s) &= \int_{-\infty}^{s} \bar{F}((s-x)v_n^{1/2}n^{1/2})\Phi_1(x)dx, \\
A(n,s) &= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty x\mu(dx) - \frac{v_n^{-1}}{2}\Phi_2(s)\int_0^{n^{1/2}} x^2\mu(dx), \\
&= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty \bar{F}(x)dx \\
&\quad -v_n^{-1}\Phi_2(s)\left(\int_0^{n^{1/2}} x\bar{F}(x)dx - \frac{L(n^{1/2})}{2}\right).
\end{aligned}$$

Then we have

$$H(n,s) = \Phi_0(s) + A(n,s).$$

Let

$$H_0(n,s) = \Phi_0(s) + n\bar{F}(v_n^{1/2}n^{1/2}s).$$

In this section we prove the following Lemma.

Lemma 2.2.

$$\sup_{s\in[1,\infty)} \left|\frac{H(n,s)}{H_0(n,s)} - 1\right| \to 0, \qquad n\to\infty.$$

Let $u_n = v_n^{1/2}n^{1/2}$, $\alpha_n = L(u_n)^{1/3}$ and $\beta_n = L(u_n)^{-1/12}$.

Proposition 2.13. *For any $\varepsilon > 0$, there is a $C > 0$ such that*

$$\frac{1}{nF(u_ns)} \le CL(u_n)^{-1}s^{2+\varepsilon}, \qquad s\in[1,\infty).$$

In particular, for $s > \beta_n$ we have

$$\frac{1}{nF(u_ns)} \le Cs^{14+\varepsilon}.$$

Proof. From Proposition 2.8 we see that for any $\varepsilon > 0$ there is a $C > 0$ such that

$$\begin{aligned}
\frac{1}{nF(u_ns)} &= v_ns^2\frac{1}{L(u_n)}\frac{L(u_n)}{L(u_ns)} \\
&\le CL(u_n)^{-1}s^{2+\varepsilon}.
\end{aligned}$$

Since $L(u_n)^{-1} = \beta_n^{12} \le s^{12}$ for $s > \beta_n$, we have the second inequality. □

Let $n\hat{F}_n(s) = \sum_{k=1}^{4} I_k(n,s)$, where

$$\begin{aligned}
I_1(n,s) &= n\int_{s-\alpha_n}^{s} \bar{F}((s-x)u_n)\Phi_1(x)dx, \\
I_2(n,s) &= n\int_{\sqrt{7/8}s}^{s-\alpha_n} \bar{F}((s-x)u_n)\Phi_1(x)dx, \\
I_3(n,s) &= n\int_{-s}^{\sqrt{7/8}s} \bar{F}((s-x)u_n)\Phi_1(x)dx, \\
I_4(n,s) &= n\int_{-\infty}^{-s} \bar{F}((s-x)u_n)\Phi_1(x)dx.
\end{aligned}$$

Let

$$R(n,s,y) = \Phi_1(s - u_n^{-1}y) - (\Phi_1(s) + u_n^{-1}y\Phi_2(s)), \qquad for\ n \geq 1,\ \ s,y \in [1,\infty).$$

Proposition 2.14.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s)\int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \to 0,\ \ n\to\infty.$$

Proof. We see that

$$\begin{aligned}
& I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s)\int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy \\
=\ & nu_n^{-1}\int_0^{\alpha_n u_n} \bar{F}(y)\left(\Phi_1(s-u_n^{-1}y) - \Phi_1(s) - u_n^{-1}y\Phi_2(s)\right)dy \\
=\ & nu_n^{-1}\int_0^{\alpha_n u_n} \bar{F}(y)R(n,s,y)dy.
\end{aligned}$$

Note that for any $y \in [0,\alpha_n u_n]$,

$$\begin{aligned}
|R(n,s,y)| &\leq u_n^{-2}y^2 \sup_{z\in[s-\alpha_n,s]}|\Phi_3(z)| \\
&\leq C_0 n^{-1}y^2(1+s)^2\Phi_1(s-\alpha_n) \\
&\leq C_0^2 n^{-1}y^2(1+s)^3\Phi_0(s)\exp(\alpha_n s).
\end{aligned}$$

Hence for all $s \in [1,\infty)$

$$\begin{aligned}
& |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s)\int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \\
\leq\ & 8C_0\sup\{z^2\bar{F}(z);\ z\geq 0\}\alpha_n s^3\Phi_0(s)\exp(\alpha_n s).
\end{aligned}$$

Since $\alpha_n \beta_n^3 = L(u_n)^{1/12} \to 0$, $n \to \infty$, we have

$$\sup_{s \le \beta_n} \Phi_0(s)^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n u_n} y^{k-1} \bar{F}(y) dy| \to 0, \ n \to \infty.$$

From Proposition 2.13, we see that for any $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ such that

$$(n\bar{F}(u_n s))^{-1} \le C(\varepsilon) s^{14+\varepsilon}.$$

Hence we see that for $s > \beta_n$,

$$\begin{aligned} & (n\bar{F}(u_n s))^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n u_n} y^{k-1} \bar{F}(y) dy| \\ \le \ & 8C(\varepsilon) C_0^2 \sup\{z^2 \bar{F}(z); \ z \ge 0\} \alpha_n s^{17+\varepsilon} \Phi_0(s) \exp(\alpha_n s). \end{aligned}$$

Since $\sup_{n \ge 1} \sup_{s > \beta_n} s^{17+\varepsilon} \Phi_0(s) \exp(\alpha_n s) < \infty$, we have

$$\sup_{s > \beta_n} (n\bar{F}(u_n s))^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n u_n} y^{k-1} \bar{F}(y) dy| \to 0, \ n \to \infty.$$

Therefore we have our assertion. □

Proposition 2.15.

$$\sup_{s \in [1,\infty)} H_0(n,s)^{-1} |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n u_n}^{(1-\sqrt{7/8}) u_n s} y^{k-1} \bar{F}(y) dy| \to 0 \ , n \to \infty.$$

Proof. Similarly to Proposition 2.14, we see that

$$\begin{aligned} & |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n u_n}^{(1-\sqrt{7/8}) u_n s} y^{k-1} \bar{F}(y) dy| \\ \le \ & n u_n^{-1} \int_{\alpha_n u_n}^{(1-\sqrt{7/8}) u_n s} \bar{F}(y) |R(n,s,y)| dy \\ \le \ & n u_n^{-3} \bar{F}(u_n \alpha_n) C_0 (1+s)^2 (\sup_{z \in [\sqrt{7/8} s, s]} |\Phi_1(z)|) \int_{u_n \alpha_n}^{(1-\sqrt{7/8}) u_n s} y^2 dy \\ \le \ & 4 C_0 n \bar{F}(u_n \alpha_n) s^5 \Phi_1(\sqrt{7/8} s) \\ \le \ & 4 C_0^{1+7/8} n \bar{F}(u_n \alpha_n) s^6 \Phi_0(s)^{7/8}. \end{aligned}$$

Since $H_0(n,s)^{-1} \le \Phi_0(s)^{-6/7} (n\bar{F}(u_n s))^{-1/7}$, it is easy to see that for any $\varepsilon \in (0, 4/7)$, there is a $C_1 > 0$ such that

$$\begin{aligned} & H_0(n,s)^{-1} |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n u_n}^{(1-\sqrt{7/8}) u_n s} y^{k-1} \bar{F}(y) dy| \\ \le \ & C_1 s^{6+2/7+\varepsilon} \Phi_0(s)^{7/8-6/7} L(u_n)^{(1-\varepsilon)/3-1/7}. \end{aligned}$$

Since $\sup_{s\geq 1}\{s^{6+2/7+\varepsilon}\Phi_0(s)^{7/8-6/7}\} < \infty$ and $(1-\varepsilon)/3 - 1/7 > 0$, we have

$$\sup_{s\geq 1} H_0(n,s)^{-1}|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2}\Phi_k(s)\int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} y^{k-1}\bar{F}(y)dy| \to 0,\ n\to\infty.$$

$\square$

Proposition 2.16.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|I_3(n,s) - n\bar{F}(u_n s)| \to 0, \qquad n\to\infty.$$

Proof.

$$\begin{aligned} I_3(n,s) &= n\bar{F}(u_n s)\int_{-s}^{\sqrt{7/8}s} \frac{\bar{F}(u_n(s-x))}{\bar{F}(u_n s)}\Phi_1(x)dx \\ &= n\bar{F}(u_n s)\int_{-s}^{\sqrt{7/8}s}(1-\frac{x}{s})^{-2}\frac{L(u_n(s-x))}{L(u_n s)}\Phi_1(x)dx. \end{aligned}$$

It is easy to see that there is a $C_1 > 0$ such that

$$\begin{aligned} &\Phi_0(s)^{-1} \leq C_1 L(u_n)^{-2/3}, \qquad n\geq 1, s\in[1,(-\log L(u_n))^{1/2}], \\ &|\int_{-s}^{\sqrt{7/8}s}\frac{\bar{F}(u_n(s-x))}{\bar{F}(u_n s)}\Phi_1(x)dx| \leq C_1, \qquad n\geq 1, s\in[1,\infty). \end{aligned}$$

Then we have

$$\begin{aligned} \sup_{s\leq(-\log L(u_n))^{1/2}} H_0(n,s)^{-1}|I_3(n,s) - n\bar{F}(u_n s)| &\leq C_1(C_1+1)L(u_n)^{-2/3}n\bar{F}(u_n) \\ &\leq C_1(C_1+1)v_n^{-1}L(u_n)^{1/3} \to 0,\ n\to\infty. \end{aligned}$$

We take $M > 1$ arbitrarily, then $(-\log L(u_n))^{1/4} > M$ for sufficiently large n.

Hence we see that for $s > (-\log L(u_n))^{1/2}$

$$\begin{aligned} &\left|\int_{-s}^{\sqrt{7/8}s}(1-\frac{x}{s})^{-2}\frac{L(u_n(s-x))}{L(u_n s)}\Phi_1(x)dx - 1\right| \\ \leq &\left|\int_{-s}^{\sqrt{7/8}s}\{(1-\frac{x}{s})^{-2}-1\}\frac{L(u_n(s-x))}{L(u_n s)}\Phi_1(x)dx\right| \\ &+\left|\int_{-s}^{\sqrt{7/8}s}(\frac{L(u_n(s-x))}{L(u_n s)}-1)\Phi_1(x)dx\right| + \int_{[-s,\sqrt{7/8}s]^C}\Phi_1(x)dx \\ \leq &\ 2\left(\int_{-M}^{M}|(1-\frac{x}{s})^{-2}-1|\Phi_1(x)dx + 8\Phi_0(M)\right) \\ &+ \sup_{t>(-\log L(u_n))^{1/2}}\ \sup_{1-\sqrt{7/8}\leq a\leq 1}|\frac{L(at)}{L(t)}-1| + 2\Phi_0(\sqrt{7/8}s). \end{aligned}$$

Hence we have

$$\sup_{s>(-\log L(u_n))^{1/2}} |n\bar{F}(u_n s)|^{-1}|I_3(n,s) - n\bar{F}(u_n s)| \to 0, \qquad n \to \infty.$$

So we have our assertion. □

Proposition 2.17.

$$\sup_{s\in[1,\infty)} \frac{I_4(n,s)}{H_0(n,s)} \to 0, \qquad n \to \infty.$$

Proof. $|I_4(n,s)| \le n\bar{F}(2u_n s)\Phi_0(s)$. Hence we have

$$\Phi_0(s)^{-1}|I_4(n,s)| \le n\bar{F}(2u_n s) \le n\bar{F}(u_n) \to 0, \qquad n \to \infty.$$

□

Proposition 2.18.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|v_n^{-1/2}n^{1/2}\Phi_1(s)\int_{\sqrt{7/8}u_n s}^{\infty} \bar{F}(y)dy| \to 0, \qquad n \to \infty,$$

and

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|v_n^{-1}\Phi_2(s)\left(\int_{\sqrt{7/8}u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n \to \infty.$$

Proof. From Proposition 2.3 (2), we see that there is a $C_1 > 0$ such that

$$n^{1/2}\int_{(1-\sqrt{7/8})u_n s}^{\infty} \bar{F}(y)dy \le C_1 s^{-1}L((1-\sqrt{7/8})u_n s).$$

We can easily see that

$$\sup_{s\in[1,\beta_n)} \Phi_0(s)^{-1}n^{1/2}\Phi_1(s)\int_{(1-\sqrt{7/8})n^{1/2}s}^{\infty} \bar{F}(y)dy \to 0, \qquad n \to \infty,$$

and

$$\sup_{s\in[\beta_n,\infty)} (n\bar{F}(n^{1/2}s))^{-1}n^{1/2}\Phi_1(s)\int_{(1-\sqrt{7/8})n^{1/2}s}^{\infty} \bar{F}(y)dy \to 0, \qquad n \to \infty.$$

Also we see that for any $\varepsilon \in (0,1)$, there is a $C_2 > 0$ such that

$$\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy = \int_{(1-\sqrt{7/8})v_n^{1/2}s}^{1} \frac{L(n^{1/2}x)}{x}dx \le C_2 L(n^{1/2})s^{\varepsilon}.$$

Hence we can easily see that

$$\sup_{s\in[1,\beta_n)} \Phi_0(s)^{-1}|\Phi_2(s)\left(\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n\to\infty,$$

and

$$\sup_{s\in[\beta_n,\infty)} (n\bar{F}(n^{1/2}s))^{-1}|\Phi_2(s)\left(\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n\to\infty.$$

Therefore we have our assertion. $\square$

Now let us prove Lemma 2.2. Note that $H(n,s) - H_0(n,s) = A(n,s) - n\bar{F}(sn^{1/2})$. So Proposition 2.14, 15, 16, 17 and 18 imply Lemma 2.2.

2.7 Proof of Theorem 2.2

First we prove the following Lemma.

Lemma 2.3. *For any $\beta > 0$ and $\delta \in (0,1)$, there is a $C > 0$ such that*

$$\sup_{s>L(n^{1/2})^{-\beta}} |\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n, v_n^{-1/2}s)} - 1| \le CL(n^{1/2})^{1-\delta}$$

We make some preparations to prove Lemma 2.3. Similarly to Proposition 26 in [7], we can prove the following.

Proposition 2.19. (1) *For any $t,s>0$, and $n\ge 2$,*

$$P(\sum_{k=2}^n X_k 1_{\{X_k\le tn^{1/2}\}} > sn^{\frac{1}{2}}) \le \exp(\frac{6}{t^2} - \frac{s}{t}).$$

(2) *For any $s,t>0$, $\varepsilon\in(0,1)$ with $t<(1-\varepsilon)s$,*

$$|P(\sum_{k=1}^n X_k > sn^{\frac{1}{2}}) - nP(X_1 + \sum_{k=2}^n X_k 1_{\{X_k\le tn^{1/2}\}} > sn^{\frac{1}{2}}, \sum_{k=2}^n X_k 1_{\{X_k\le tn^{1/2}\}} \le \varepsilon sn^{\frac{1}{2}})|$$
$$\le 2n(n-1)\bar{F}(tn^{\frac{1}{2}})^2 + \exp(\frac{6}{t^2} - \frac{s}{t}) + n\bar{F}(tn^{\frac{1}{2}})\exp(\frac{6}{t^2} - \frac{\varepsilon s}{2t}).$$

Proof. We prove this proposition briefly. We see that

$$\begin{aligned} P(\sum_{k=2}^{n} X_k 1_{\{X_k \le tn^{1/2}\}} > sn^{1/2}) &\le \exp(-\frac{s}{t}) E[\exp(\frac{1}{tn^{1/2}} \sum_{k=2}^{n} X_k 1_{\{X_k \le tn^{1/2}\}})] \\ &\le \exp(-\frac{s}{t}) E[\exp(\frac{1}{tn^{1/2}} X_1 1_{\{X_1 \le tn^{1/2}\}})]^{n-1}. \end{aligned}$$

It is easy to see that $e^x \le 1 + x + x^2(1 \vee e^x)$ for any $x \in \mathbb{R}$. So we have

$$\begin{aligned} E[\exp(\frac{1}{tn^{1/2}} X_1 1_{\{X_1 \le tn^{1/2}\}})] &\le 1 + \frac{1}{tn^{1/2}} E[X_1 1_{\{X_1 \le tn^{1/2}\}}] + \frac{1}{t^2 n} E[X_1^2] \exp(1) \\ &\le 1 + \frac{6}{t^2 n}. \end{aligned}$$

Since $\log(1 + x) \le x$ for $x > 0$, we see that

$$(n-1) \log E[\exp(\frac{1}{tn^{1/2}} X_1 1_{\{X_1 \le tn^{1/2}\}})] \le \frac{6}{t^2}.$$

Hence we have the assertion (1).

Note that

$$P(\sum_{k=1}^{n} X_k > sn^{1/2}) = \sum_{m=0}^{n} I_m,$$

where

$$I_m = P(\sum_{k=1}^{n} X_k > sn^{1/2}, \sum_{k=1}^{n} 1_{\{X_k > tn^{1/2}\}} = m), \qquad m = 0, 1, \ldots, n.$$

Then we have

$$I_m = \binom{n}{m} P(\sum_{k=1}^{n} X_k > sn^{1/2}, X_i > tn^{1/2}, i = 1, \ldots, m,\ X_j \le tn^{1/2}, j = m+1, \ldots, n),$$

for $m = 0, 1, \ldots, n$. We can easily see that

$$\sum_{m=2}^{n} I_m \le \frac{n(n-1)}{2} \bar{F}(tn^{1/2})^2. \tag{2.18}$$

From (1), we have

$$I_0 \le \exp(\frac{6}{t^2} - \frac{s}{t}). \tag{2.19}$$

Let $A_1 = \{X_1 > tn^{1/2}\}$, $A_2 = \{X_k \le tn^{1/2}, k = 2, 3, \ldots, n\}$,
$B_1 = \{X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le tn^{1/2}\}} > sn^{1/2}\}$ and $B_2 = \{\sum_{k=2}^{n} X_k 1_{\{X_k \le tn^{1/2}\}} \le \varepsilon sn^{1/2}\}$.

Note that $B_1 \cap B_2 \subset A_1$, since $t < (1-\varepsilon)s$. So we see that

$$\begin{aligned}
& |P(B_1 \cap A_1 \cap A_2) - P(B_1 \cap B_2)| \\
\le\ & P(B_1 \cap B_2^c \cap A_1 \cap A_2) + P(B_1 \cap B_2 \cap A_1 \cap A_2^c) \\
\le\ & P(A_1)P(B_2^c) + P(A_1)P(A_2^c). \qquad (2.20)
\end{aligned}$$

Note that

$$P(A_2^c) \le \sum_{k=2}^{n} P(X_k > tn^{1/2}) = (n-1)\bar{F}(tn^{1/2}).$$

Also, by the assertion (1) we have

$$P(B_2^c) \le \exp(\frac{6}{t^2} - \frac{\varepsilon s}{2t}).$$

Since $I_1 = nP(B_1 \cap A_1 \cap A_2)$, we have the assertion (2) from Equations (2.18), (2.19) and (2.20). This completes the proof. $\square$

Also we prove the following for the proof of Lemma 2.3.

Proposition 2.20. *For any γ, δ, $\varepsilon \in (0,1)$ and $\beta > 0$, there is a $C > 0$ such that*

$$\begin{aligned}
& |P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} > sn^{1/2}, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} \le \varepsilon sn^{1/2}) \\
& - \int_{-\infty}^{\varepsilon v_n^{-1/2} s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \\
& \le C\bar{F}((1-\varepsilon)n^{1/2}s)L(n^{1/2})^{1-3\delta}, \qquad \text{for } s > L(n^{1/2})^{-\beta}.
\end{aligned}$$

Proof. Let $t_n = L(n^{1/2})^{\delta} n^{1/2}$.

Since

$$\begin{aligned}
& P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} > sn^{\frac{1}{2}}, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} \le \varepsilon sn^{\frac{1}{2}}) \\
=\ & P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} > sn^{\frac{1}{2}}, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} \le \varepsilon sn^{\frac{1}{2}}, \max(X_2, \ldots, X_n) > t_n) \\
& + P(X_1 + \sum_{k=2}^{n} X_k > sn^{\frac{1}{2}}, \sum_{k=2}^{n} X_k \le \varepsilon sn^{\frac{1}{2}}, X_2 \le t_n, \ldots, X_n \le t_n),
\end{aligned}$$

it is easy to see that there is a $C_1 > 0$ such that for $s > L(n^{1/2})^{-\beta}$

$$
\begin{aligned}
&|P(X_1 + \sum_{k=2}^n X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} > sn^{\frac{1}{2}}, \sum_{k=2}^n X_k 1_{\{X_k \le s^\gamma n^{1/2}\}} \le \varepsilon sn^{\frac{1}{2}}) \\
&\qquad -P(X_1 + \sum_{k=2}^n X_k > sn^{\frac{1}{2}}, \sum_{k=2}^n X_k \le \varepsilon sn^{\frac{1}{2}}, X_2 \le t_n, \ldots, X_n \le t_n)| \\
&\le C_1 \bar{F}((1-\varepsilon)n^{\frac{1}{2}}s)L(n^{\frac{1}{2}})^{1-3\delta}.
\end{aligned}
$$

We also see that

$$
\begin{aligned}
&P(X_1 + \sum_{k=2}^n X_k > sn^{1/2}, \sum_{k=2}^n X_k \le \varepsilon sn^{1/2}, X_2 \le t_n, \ldots, X_n \le t_n) \\
&= (1-\bar{F}(t_n))^{n-1} \int_{-\infty}^{\varepsilon s} \bar{F}(n^{1/2}(s-x))\mu(t_n)^{*(n-1)}(dx).
\end{aligned}
$$

Similarly to the proof of Proposition 2.12, we have our assertion. □

Now let us prove Lemma 2.3. Since

$$
\begin{aligned}
&H(n, v_n^{-1/2}s) - n\int_{-\infty}^{\varepsilon v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx \\
=\ & \Phi_0(v_n^{-1/2}s) - n\int_{\varepsilon v_n^{-1/2}s}^{v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx \\
&+v_n^{-1/2}n^{1/2}\Phi_1(v_n^{-1/2}s)\int_0^\infty x\mu(dx) + v_n^{-1}\frac{\Phi_2(v_n^{-1/2}s)}{2}\int_0^{n^{1/2}} x^2\mu(dx) \\
=\ & \Phi_0(v_n^{-1/2}s) + v_n^{-1}\frac{\Phi_2(v_n^{-1/2}s)}{2}\int_0^{n^{1/2}} x^2\mu(dx) \\
&-v_n^{-1/2}n^{1/2}\eta_1((1-\varepsilon)n^{1/2}s)\Phi_1(v_n^{-1/2}s) \\
&-v_n^{-1/2}n^{1/2}(\int_0^{(1-\varepsilon)n^{1/2}s} \bar{F}(z)(\Phi_1(v_n^{-1/2}s - n^{-1/2}v_n^{-1/2}z) - \Phi_1(v_n^{-1/2}s))dz),
\end{aligned}
$$

it is easy to see that there is a $C_1 > 0$ such that for $s \ge 1$

$$
|H(n, v_n^{-1/2}s) - n\int_{-\infty}^{\varepsilon v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \le C_1 s\Phi_1(\varepsilon v_n^{-1/2}s).
$$

Combining Proposition 2.19 (2) and 2.20, we see that there is a $C_2 > 0$ such that

$$
\begin{aligned}
&|P(\sum_{k=1}^n X_k > sn^{1/2}) - n\int_{-\infty}^{\varepsilon v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \\
\le\ & 2n(n-1)\bar{F}(s^\gamma n^{1/2})^2 + \exp(\frac{6}{s^{2\gamma}} - \frac{s}{s^\gamma}) + n\bar{F}(s^\gamma n^{1/2})\exp(\frac{6}{s^{2\gamma}} - \frac{\varepsilon s}{2s^\gamma}) \\
&+C_2\bar{F}((1-\varepsilon)n^{1/2}s)L(n^{1/2})^{1-\delta}.
\end{aligned}
$$

Hence we see that there is a $C > 0$ such that

$$\sup_{s > L(n^{1/2})^{-\beta}} (n\bar{F}(n^{1/2}s))^{-1} |P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2}s)| \leq CL(n^{1/2})^{1-\delta}.$$

Therefore by Lamma 2.2, we have our assertion.

Now let us prove Theorem 2.2.

By Theorem 2.4, we see that there is a $C_1 > 0$ such that

$$|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2}s)| \leq C_1 L(n^{1/2})^{2-\delta/2}, \qquad s \geq 1.$$

Note that for any $\varepsilon > 0$, there is a $C_2 > 0$ such that $n\bar{F}(n^{1/2}s) \geq C_2^{-1}s^{-3}L(n^{1/2}) \geq C_2^{-1}L(n^{1/2})^{1+\delta/2}$ for $s \leq L(n^{1/2})^{-\delta/6}$. Hence by Lemma 2.2, we see that there is a $C_3 > 0$ such that

$$H(n, v_n^{-1/2}s)^{-1} \leq C_3(n\bar{F}(n^{1/2}s))^{-1} \leq C_2C_3L(n^{1/2})^{-1-\delta/2}, \qquad s \leq L(n^{1/2})^{-\delta/6}.$$

So we have

$$\sup_{s \leq L(n^{1/2})^{-\delta/6}} |\frac{P(\sum_{k=1}^{n} X_k > sn^{1/2})}{H(n, v_n^{-1/2}s)} - 1| \leq C_1C_2C_3L(n^{1/2})^{1-\delta}.$$

From this and Lemma 2.3, we have Equation (2.2).

Equation (2.3) is an easy consequence of Equation (2.2) and Lemma 2.2.

2.8 Proof of Theorem 2.3

First let us assume $\limsup_{n\to\infty}(1 - v_n)\log\dfrac{1}{L(n^{1/2})} = 0$.

we see that

$$\begin{aligned}\Phi_0(s) - \Phi_0(v_n^{-1/2}s) = \int_s^{v_n^{-1/2}s} \Phi_1(z)dz &= \int_{s^2}^{v_n^{-1}s^2} \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{dy}{2\sqrt{y}} \\ &\leq \frac{s}{2v_1}(1 - v_n)\Phi_1(s) \\ &\leq C_0\frac{s^2}{2v_1}(1 - v_n)\Phi_0(s).\end{aligned}$$

Let $z_n = \dfrac{1}{L(n^{1/2})}$, then we have $\limsup_{n\to\infty}(1 - v_n)\log z_n = 0$. Hence we have

$$\sup_{s\in[1,\sqrt{3\log z_n})} |\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \leq \frac{3C_0}{2v_1}(1 - v_n)\log z_n \to 0, \qquad n \to \infty.$$

We also see that for $s > \sqrt{3\log z_n}$,

$$
\begin{aligned}
&|\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \le \frac{C_0}{2v_1}\frac{(1-v_n)s^2\Phi_0(s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} \\
\le\ & \frac{C_0}{2v_1}\frac{(1-v_n)s^4\Phi_0(s)}{L(n^{1/2}s)} \le \frac{C_0^2}{2\sqrt{2\pi}v_1}s^5\exp(-s^2/2)\frac{L(n^{1/2})}{L(n^{1/2}s)}z_n \\
\le\ & \frac{C_0^2}{2\sqrt{2\pi}v_1}s^6\exp(-s^2/2)z_n \\
\le\ & \frac{C_0^2}{2\sqrt{2\pi}v_1}\sup_{s\ge\sqrt{3\log z_n}} s^6\exp(-s^2/6) \to 0, \qquad n\to\infty.
\end{aligned}
$$

Hence we have $\sup_{s\in[1,\infty)}|\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \to 0$, $n\to 0$.

Next, we assume $\limsup_{n\to\infty}(1-v_n)\log\frac{1}{L(n^{1/2})} > 0$.

Let $y_n = (1-v_n)\log z_n$ and $s_n = \sqrt{\log z_n}$.

Then $\limsup_{n\to\infty} y_n > 0$. Hence we see that

$$
\begin{aligned}
\liminf_{n\to\infty}\Phi_0(s_n)^{-1}\Phi_0(v_n^{-1/2}s_n) &= \liminf_{n\to\infty} v_n^{1/2}\Phi_1(s_n)^{-1}\Phi_1(v_n^{-1/2}s_n) \\
&\le \liminf_{n\to\infty}\exp(-v_n^{-1}(1-v_n)s_n^2) = \exp(-\limsup_{n\to\infty} y_n) < 1
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_0(s_n)^{-1}n\bar{F}(n^{1/2}s_n) &\le C_0 s_n\Phi_1(s_n)^{-1}s_n^{-2}L(n^{1/2}s_n) \\
&\le \sqrt{2\pi}C_0 M(1)L(n^{1/2})^{1/2} \to 0, \qquad n\to\infty.
\end{aligned}
$$

Hence we have

$$
\liminf_{n\to\infty}\frac{\Phi_0(v_n^{-1/2}s_n) + n\bar{F}(n^{1/2}s_n)}{\Phi_0(s_n) + n\bar{F}(n^{1/2}s_n)} < 1.
$$

Therefore we have Theorem 2.3.

Example

We give an example in the rest of this section.

Let $x_0 \ge 1$ and $L : [x_0,\infty) \to (0,\infty)$ be a C^2 slowly varying function satisfying

$$
\int_{x_0}^\infty \frac{L(x)}{x}dx < \infty, \qquad L(x)\to 0, x\to\infty, \qquad \sup_{x\ge x_0}(|L'(x)| + |L''(x)|) < \infty.
$$

Then we can find $F : \mathbb{R} \to [0,1]$ non-decreasing C^2 function with $F(-\infty) = 0$, $F(\infty) = 1$, $\int_{\mathbb{R}}|F''(x)|dx < \infty$ and $F(x) = x^{-2}L(x)$ for sufficient large $x > 0$.

Let μ be a probability measure whose distribution function is F. Then we see that μ satisfies (A3).

Let $L(x) = (\log x)^{-1}(\log\log x)^{-1-b}$, $b > 0$ for sufficiently large $x > 0$. We can easily see that $L(x)$ satisfies the above condition. For sufficiently large $n \geq 1$, we see that

$$\begin{aligned} 1 - v_n &= \int_{n^{1/2}}^{\infty} x^2 \mu(dx) = L(n^{1/2}) + 2\int_{n^{1/2}}^{\infty} \frac{L(x)}{x} dx = L(n^{1/2}) + \frac{2}{b}\left(\log\log n - \log 2\right)^{-b} \\ &\sim \frac{2}{b}\left(\log\log n\right)^{-b}. \end{aligned}$$

Hence we have the following

Proposition 2.21. *Let* $L(x) = (\log x)^{-1}(\log\log x)^{-1-b}$, $b > 0$ *for sufficiently large* $x > 0$. *Then we have*

$$\limsup_{n\to\infty} (1 - v_n) \log \frac{1}{L(n^{1/2})} > 0, \qquad for\ b \in (0, 1]$$

and

$$\lim_{n\to\infty} (1 - v_n) \log \frac{1}{L(n^{1/2})} = 0, \qquad for\ b \in (1, \infty).$$

Therefore Equation (2.1) does not hold for $b \in (0, 1]$.

第3章 Uniform Estimates for Distributions of Sums of i.i.d. Random Variables with Fat Tail: Infinite variance case.

In the previous chapter, We showed uniform estimates of distributions of the sum of i.i.d. random variables with finite variance in the threshold case. In this chapter, we show a uniform estimate without variance condition in the threshold case.

3.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_n, n = 1, 2, \ldots$, be independent identically distributed random variables whose probability law is μ. Let $F : \mathbb{R} \to [0,1]$ and $\bar{F} : \mathbb{R} \to [0,1]$ be given by $F(x) = \mu((-\infty, x])$ and $\bar{F}(x) = \mu((x, \infty))$, $x \in \mathbb{R}$. We assume the following.

(A1) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha \geq 2$, as $x \to \infty$,i.e., if we let

$$L(x) = x^{\alpha} \bar{F}(x)\ ,\ x \geq 1,$$

then $L(x) > 0$ for any $x \geq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1\ ,\ x \to \infty.$$

(A2) $\int_{-\infty}^{0} |x|^{2+\delta_0} \mu(dx) < \infty$ for some $\delta_0 \in (0,1)$ and $\int_{\mathbb{R}} x\mu(dx) = 0$

(A3) The probability law μ is absolutely continuous and has a density function $\rho : \mathbb{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

Let us define $\Phi_k : \mathbb{R} \to \mathbb{R}, k = 0, 1, 2, 3$ by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{y^2}{2}) dy, \qquad x \in \mathbb{R},$$

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}} \Phi_1(x), \qquad k = 2, 3.$$

Let $t_n = \sup\{t > 0; n \int_{-\infty}^t x^2 \mu(dx) > t^2\}$. Then from (A1), (A2) we can see that

$$P(\sum_{k=1}^n X_k > t_n s) \to \Phi_0(s), \qquad n \to \infty, \quad s \geq 1.$$

Let $v_n = \int_{-\infty}^{t_n} x^2 \mu(dx)$ for $n \geq 1$. We also define $H : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} H(n, s) = \Phi_0(s) \quad &+ \quad n \int_{-\infty}^s \bar{F}(t_n(s - x)) \Phi_1(x) dx \\ &- \quad \left(v_n^{-1/2} n^{1/2} \Phi_1(s) \int_0^\infty x \mu(dx) + \frac{\Phi_2(s)}{2} v_n^{-1} \int_0^{t_n} x^2 \mu(dx) \right). \end{aligned}$$

In this chapter, we show the following (Theorem 1.11).

Theorem 3.1. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a* $C > 0$ *such that*

$$\sup_{s \in [1,\infty)} |\frac{P(\sum_{k=1}^n X_k > t_n s)}{H(n, s)} - 1| \leq C(n\bar{F}(t_n))^{1-\delta}, \qquad n \geq 1. \tag{3.1}$$

In particular,

$$\sup_{s \in [1,\infty)} |\frac{P(\sum_{k=1}^n X_k > t_n s)}{\Phi_0(s) + n\bar{F}(t_n s)} - 1| \to 0, \qquad n \to \infty.$$

We also prove the following to obtain Theorem 3.1.

Theorem 3.2. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a* $C > 0$ *such that*

$$|P(\sum_{k=1}^n X_k > t_n s) - H(n, s)| \leq C(n\bar{F}(t_n))^{2-\delta}, \qquad s \geq 1.$$

Throughout this paper we assume (A1) for $\alpha = 2$, (A2) and (A3). Then we see that $t_n = n^{1/2} v_n^{1/2}$ and $n\bar{F}(t_n) = \dfrac{L(t_n)}{v_n} \to 0, n \to \infty$ (see Equation (3.2)).

3.2 Estimate for moments and characteristic functions

Let

$$\eta_k(t) = \int_{-\infty}^{t} x^k \mu(dx), \qquad t > 0,\ k = 1, 2,$$

and

$$\eta_3(t) = \int_{1}^{t} x^3 \mu(dx), \qquad t > 1.$$

Then we see that

$$-\eta_1(t) = \int_t^{\infty} x\mu(dx) = \int_t^{\infty} \bar{F}(x)dx + t\bar{F}(t), \qquad t > 0,$$

$$\eta_3(t) = \bar{F}(1) - t^3\bar{F}(t) + 3\int_1^t x^2\bar{F}(x)dx \qquad t > 1.$$

and $\eta_2(t)$ is slowly varying.

Let $t_n = \sup\{t > 0; n\eta_2(t) > t^2\}$ and $v_n = \eta_2(t_n) = \int_{-\infty}^{1} x^2\mu(dx) - L(t_n) + L(1) + 2\int_1^{t_n} x^{-1}L(x)dx$.

Note that $t_n = n^{1/2}\eta_2(t_n)^{1/2} \geq n^{1/2}\eta_2(0) \to \infty, \quad n \to \infty$.

Let $a_n = n\bar{F}(t_n)$. Then for any $t_0 > 0$, we see that for $t > t_0$,

$$\frac{1}{L(t)}\int_1^t x^{-1}L(x)dx = \int_{1/t}^{1} \frac{L(tx)}{L(t)}\frac{dz}{z} \geq \int_{1/t_0}^{1} \frac{L(tx)}{L(t)}\frac{dz}{z} \to \int_{1/t_0}^{1}\frac{dz}{z} = \log t_0.$$

Since t_0 is arbitrary, we see that

$$a_n = \frac{L(t_n)}{v_n} \to 0, \quad n \to \infty. \tag{3.2}$$

Proposition 3.1. *There is a $C > 0$ such that*

$$-n\frac{\eta_1(t_n)}{t_n} \leq Ca_n, \tag{3.3}$$

$$n\frac{\eta_3(t_n)}{t_n^3} \leq Ca_n. \tag{3.4}$$

for any $n \geq 1$.

Proof. Let

$$\varepsilon_1(t) = \frac{1}{t^{-1}L(t)} \int_t^\infty x^{-2}L(x)dx - 1$$

and

$$\varepsilon_3(t) = \frac{1}{tL(t)} \int_1^t L(x)dx - 1.$$

Then from Proposition 2.3 (1) and (2) we have $\varepsilon_1(t) \to 0$ and $\varepsilon_3(t) \to 0$ as $t \to \infty$.

Hence we see that

$$-n\frac{\eta_1(t_n)}{t_n} = \frac{n}{t_n}\left(t_n\bar{F}(t_n) + \int_{t_n}^\infty \bar{F}(x)dx\right) = n\bar{F}(t_n)(2+\varepsilon_1(t_n))$$

and

$$\frac{n}{t_n^3}\eta_3(t_n) = n^{-1/2}v_n^{-3/2}\bar{F}(1) + (2+\varepsilon_3(t_n))n\bar{F}(t_n).$$

Hence we have our assertion. □

Proposition 3.2. *There is $c_1 > 0$ such that for any integer n, m with $n \geq m$ and $\xi \in \mathbb{R}$ with $|\xi| \geq a_n^{-\delta}$,*

$$|\varphi(t_n^{-1}\xi, \mu(t_n))|^n \leq (1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{mv_n}|\xi|^2)^{-m/4}.$$

In particular, there is $c_2 > 0$ such that for any integer n, m with $n \geq m$ and $\xi \in \mathbb{R}$ with $|\xi| \in (a_n^{-\delta}, t_n)$,

$$|\varphi(t_n^{-1}\xi, \mu(t_n))|^n \leq (1 + \frac{c_2}{m}|\xi|)^{-m/4}.$$

Proof. Let $t > 2$. We see that for $\xi \in (-t^{-1}, t^{-1})$,

$$\begin{aligned}
&|\varphi(\xi, \mu(t))|^2 \\
=\ & (1-\bar{F}(t))^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(i\xi(x-y))\rho(x)1_{(-\infty,t)}(x)\rho(y)1_{(-\infty,t)}(y)dxdy \\
\leq\ & 1 - \int_{\mathbb{R}}\int_{\mathbb{R}}(1-\cos(\xi(x-y)))\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy \\
\leq\ & 1 - \frac{|\xi|^2}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}(x-y)^2\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy.
\end{aligned}$$

Similarly we have for $\xi \in \mathbb{R}$ with $|\xi| > t^{-1}$,

$$\begin{aligned}
&|\varphi(\xi, \mu(t))|^2 \\
\leq\ & 1 - \frac{|\xi|^2}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}(x-y)^2\rho(x)1_{(-|\xi|^{-1},|\xi|^{-1})}(x)\rho(y)1_{(-|\xi|^{-1},|\xi|^{-1})}(y)dxdy.
\end{aligned}$$

We can easily see that

$$\eta_2(t)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}(x-y)^2\rho(x)1_{(-t,t)}(x)\rho(y)1_{(-t,t)}(y)dxdy \to 2, \qquad t\to\infty.$$

Hence we see that there is a $c_1 > 0$ such that for any $n \geq 2$ and $\xi \in \mathbb{R}$ with $|\xi| \geq a_n^{-\delta}$,

$$|\varphi(t_n^{-1}\xi, \mu(t_n))| \leq (1 - \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2)^{1/2} \leq (1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2)^{-1/4}.$$

It is easy to check that $(1+x/\beta)^\beta \geq 1+x$ for any $\beta \geq 1$ and $x \geq 0$. Therefore if $n \geq m$, we have

$$\left(1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{nv_n}|\xi|^2\right)^{n/m} \geq 1 + \frac{c_1\eta_2(t_n|\xi|^{-1})}{mv_n}|\xi|^2.$$

Since $\eta_2(t)$ is slowly varying, we see that for $\xi \in \mathbb{R}$ with $t_n \geq |\xi| \geq a_n^{-\delta}$,

$$\frac{\eta_2(t_n|\xi|^{-1})}{v_n} = \frac{\eta_2(t_n|\xi|^{-1})}{\eta_2(t_n|\xi|^{-1}|\xi|)} \geq M(1)^{-1}|\xi|^{-1}.$$

Therefore we have our assertion. □

3.3 Asymptotic expansion of characteristic functions

Remind that $t_n = n^{1/2}v_n^{1/2}$ and $a_n = n\bar{F}(t_n) = v_n^{-1}L(t_n)$.

In this section, we prove the following Lemma.

Lemma 3.1. *Let*

$$\begin{aligned}
R_{n,0}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - (1 + n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + \frac{\xi^2}{2}),\\
R_{n,1}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - 1,\\
R_{n,2}(\xi) &= \exp(\frac{\xi^2}{2})\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - 1.
\end{aligned}$$

Then there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \leq Ca_n^{2-5\delta}|\xi| \tag{3.5}$$

and

$$|R_{n,1}(\xi)| + |R_{n,2}(\xi)| \leq Ca_n^{1-2\delta}|\xi|, \tag{3.6}$$

for any $n \geq 8$ and $\xi \in \mathbb{R}$ with $|\xi| \leq a_n^{-\delta}$.

As a corollary to Lemma 3.1, we have the following.

Corollary 3.1. *Let*

$$\tilde{R}_0(n,s) = \mu(t_n)^{*n}((t_n s,\infty)) - \Phi_0(s) - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\left(n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1)+\frac{\xi^2}{2}\right)e^{-\xi^2/2}d\xi,$$

$$\tilde{R}_{1,k}(n,s) = \mu(t_n)^{*(n-k)}((t_n s,\infty)) - \Phi_0(s), \qquad k=0,1,$$

and

$$\tilde{R}_2(n,s) = \frac{1}{2\pi}\int_{\mathbb{R}}\left|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - e^{-\frac{\xi^2}{2}}\right|d\xi.$$

Then there is a $C>0$ such that for any $n \geq 1$ and $s \in \mathbb{R}$, we have

$$|\tilde{R}_0(n,s)| \leq C a_n^{2-6\delta} \tag{3.7}$$

and

$$|\tilde{R}_{1,0}(n,s)| + |\tilde{R}_{1,0}(n,s)| + |\tilde{R}_2(n,s)| \leq C a_n^{1-4\delta}. \tag{3.8}$$

Proof. From Proposition 2.7, we see that

$$\begin{aligned}
&\tilde{R}_0(n,s)\\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}} - \left(n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1)+\frac{\xi^2}{2}\right)e^{-\frac{\xi^2}{2}}\right)d\xi\\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}R_{n,0}(\xi)e^{-\xi^2/2}d\xi.
\end{aligned}$$

By Lemma 1, there is a $C_0 > 0$ such that

$$\int_{|\xi|\leq a_n^{-\delta}} \frac{|R_{n,0}(\xi)|}{|\xi|}d\xi \leq C_0 a_n^{2-6\delta}.$$

It is easy to see that

$$n|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1| \leq \frac{nt_n^{-1}|\eta_1(t_n)||\xi|}{1-\bar{F}(t_n)} + \frac{|\xi|^2}{2\eta_2(t_n)(1-\bar{F}(t_n))}, \qquad \xi\in\mathbb{R}.$$

From the above inequality and Proposition 2.6 and 3.2, we see that for any $m \geq 2/\delta$, there is a $C_1>0$ such that for any $n \geq 4m$

$$|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))|^n + \left|n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1)+1+\frac{\xi^2}{2}\right|e^{-\frac{\xi^2}{2}} \leq C_1|\xi|^{-m}, for\ |\xi|\in(a_n^{-\delta}, v_n^{1/2}a_n^{-\delta})$$

and

$$|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))|^n + \left|n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + 1 + \frac{\xi^2}{2}\right|e^{-\frac{\xi^2}{2}} \le C_1\left(\frac{|\xi|}{v_n^{1/2}}\right)^{-m},$$

for $|\xi| \ge v_n^{1/2}a_n^{-\delta}$. Hence we have

$$\begin{aligned}
&\int_{|\xi|>a_n^{-\delta}} |\xi|^{-1}\left|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}} - \left(n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1) + \frac{\xi^2}{2}\right)e^{-\frac{\xi^2}{2}}\right| d\xi \\
\le\ & 2C_1\int_{a_n^{-\delta}}^{v_n^{1/2}a_n^{-\delta}} |\xi|^{-m-1}v_n^{1/2}d\xi + 2C_1\int_{v_n^{1/2}a_n^{-\delta}}^{\infty}\left(\frac{|\xi|}{v_n^{1/2}}\right)^{-m-1}d\xi \\
=\ & \frac{4C_1}{m}a_n^{m\delta} \le \frac{4C_1}{m}a_n^2.
\end{aligned}$$

Therefore we have Equation (3.7).

We also see that

$$\begin{aligned}
\tilde{R}_{1,k}(n,s) &= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^n - e^{-\frac{\xi^2}{2}}\right)d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}R_{n,1+k}(\xi)e^{-\xi^2/2}d\xi, \\
\tilde{R}_2(n,s) &= \frac{1}{2\pi}\int_{\mathbb{R}}|R_{n,2}(\xi)|e^{-\xi^2/2}d\xi.
\end{aligned}$$

Similarly to Equation (3.7), we have Equation (3.8). □

We make some preparations to prove Lemma 3.1.

Let

$$R_0(n,\xi) = \varphi(t_n^{-1}\xi,\mu(t_n)) - (1 - \frac{\xi^2}{2n}).$$

First we prove the following.

Proposition 3.3. *There is a constant $C > 0$ such that for any $n \ge 1$, and $\xi \in \mathbb{R}$ with $|\xi| \le a_n^{-\delta}$,*

$$|nR_0(n,\xi)| \le Ca_n^{1-2\delta}|\xi|$$

and

$$n|\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n)) - 1| \ \le\ Ca_n^{-\delta}|\xi|.$$

In particular

$$\sup\{|nR_0(n,\xi)|; |\xi| \le a_n^{-\delta}\} \to 0, \qquad n \to \infty. \tag{3.9}$$

Proof. We can easily see that

$$\begin{aligned}\varphi(\xi;\mu(t)) &= \int_{\mathbb{R}} \exp(ix\xi)\mu(t)(dx) \\ &= 1+\eta_1(t)(i\xi)+\eta_2(t)\frac{(i\xi)^2}{2}+\int_{-\infty}^{1} r_{e,2}(\xi x)\mu(dx) \\ &\quad + \int_1^t r_{e,2}(\xi x)\mu(dx)+\frac{\bar{F}(t)}{1-\bar{F}(t)}\int_{-\infty}^{t} r_{e,0}(\xi x)\mu(dx).\end{aligned}$$

Hence we have

$$\begin{aligned}R_0(n,\xi) = \frac{\eta_1(t_n)}{t_n}(i\xi) \quad &+ \quad \int_{-\infty}^{1} r_{e,2}(t_n^{-1}\xi x)\mu(dx) \\ &+ \quad \int_1^t r_{e,2}(t_n^{-1}\xi x)\mu(dx)+\frac{\bar{F}(t_n)}{1-\bar{F}(t_n)}\int_{-\infty}^{t_n} r_{e,0}(t_n^{-1}\xi x)\mu(dx).\end{aligned}$$

Then we see that

$$\begin{aligned}n|R_0(n,\xi)| \le nt_n^{-1}|\eta_1(t_n)||\xi| \quad &+ \quad n^{-\delta_0/2}\eta_2(t_n)^{-1-\delta_0/2}\int_{-\infty}^{1}|x|^{2+\delta_0}\mu(dx)|\xi|^{2+\delta_0} \\ &+ \quad \frac{1}{6}n\frac{\eta_3(t_n)}{t_n^3}|\xi|^3+2n^{1/2}\bar{F}(t_n)\int_{\mathbb{R}}|x|\mu(dx)|\xi|, \qquad \xi\in\mathbb{R}, t\ge 2,\end{aligned}$$

where δ_0 is in (A2). Hence from Proposition 3.1, we see that there is a $C>0$ such that

$$|nR_0(n,\xi)| \le C\left(a_n|\xi|+n^{-\delta_0/2}|\xi|^{2+\delta_0}+a_n|\xi|^3\right).$$

Therefore we have first inequality.

Since $n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1)=nR_0(n,\xi)-\xi^2/2$, we have the second inequality. □

Let

$$R_{1,k}(n,\xi)=(n-k)\log\varphi(t_n^{-1}\xi;\mu(t_n))-n(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))-1), \quad k=0,1.$$

Proposition 3.4. *There is a $C>0$, such that for any $\xi\in\mathbb{R}$ with $|\xi|\le a_n^{-\delta}$,*

$$|R_{1,k}(n,\xi)|\le Cn^{-1}a_n^{-3\delta}|\xi|.$$

In particular

$$\sup\{|R_{1,k}(n,\xi)|;|\xi|\le a_n^{-\delta}\}\to 0, \qquad n\to\infty. \tag{3.10}$$

Proof. First, we have

$$\log \varphi(\xi, \mu(t)) = \varphi(\xi, \mu(t)) - 1 + r_{l,1}(\varphi(\xi, \mu(t)) - 1), \quad |\xi| \leq a_n^{-\delta}.$$

Hence we have

$$R_{1,k}(n, \xi) = -k \log \varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) + n r_{l,1}(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1).$$

From Proposition 3.3, we see that there is a $C >$ such that

$$\begin{aligned} |R_{1,k}(n, \xi)| &\leq |\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1| + 2n|\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1|^2 \\ &\leq C n^{-1} a_n^{-3\delta} |\xi|, \qquad |\xi| \leq a_n^{-\delta}. \end{aligned}$$

□

Therefore we have our assertion. Let us prove Lemma 3.1. Note that for $k = 0, 1$

$$\log(e^{\xi^2/2} \varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))^{n-k}) = nR_0(n, \xi) + R_{1,k}(n, \xi).$$

We see that

$$e^{\xi^2/2} \varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))^{n-k} = \exp(nR_0(n, \xi) + R_{1,k}(n, \xi)).$$

Hence we see that

$$\begin{aligned} R_{n,0}(\xi) &= e^{\xi^2/2} \varphi(n^{-\frac{1}{2}}\xi; \mu(t_n))^n - (1 + nR_0(n, \xi)) \\ &= \exp(nR_0(n, \xi)) - (1 + nR_0(n, \xi)) + \exp(nR_0(n, \xi))(\exp(R_{1,0}(n, \xi)) - 1) \end{aligned}$$

From Equation (3.9), we see that there is a $C > 0$ such that

$$|R_{n,0}(\xi)| \leq C \left(|nR_0(n, \xi)|^2 + |R_{1,0}(n, \xi)| \right).$$

Therefore we have Equation (3.5) from Proposition 3.3 and 3.4. Proof of Equation (3.6) is similar to Equation (3.5).

3.4 Proof of Theorem 3.2

Note that

$$P(\sum_{l=1}^{n} X_l > t_n s) = \sum_{k=0}^{n} I_k(n, s),$$

where

$$I_k(n,s) = P(\sum_{l=1}^n X_l > t_n s, \sum_{l=1}^n 1_{\{X_l > t_n\}} = k), \qquad k = 0, 1, \ldots, n.$$

Then we have

$$I_k(n,s) = \binom{n}{k} P(\sum_{l=1}^n X_l > t_n s, X_i > t_n, i = 1, \ldots, k, X_j \le t_n, j = k+1, \ldots, n),$$

for $k = 0, 1, \ldots, n$.

Let $\bar{F}_{n,0}(x) = P(X_1 > t_n x, X_1 \le t_n) = (1 - \bar{F}(t_n))\mu(t_n)((t_n^{-1}x, \infty))$ and $\bar{F}_{n,1}(x) = P(X_1 > t_n x, X_1 > t_n)$. Note that $\bar{F}_{n,0}(x) + \bar{F}_{n,1}(x) = \bar{F}(t_n x)$.

Proposition 3.5. *There is a $C > 0$ such that*

$$\begin{aligned} & |I_0(n,s) - (1-n)\Phi_0(s) - \frac{1}{2}\Phi_2(s) - n\int_{\mathbb{R}} \bar{F}_{n,0}(s-x)\Phi_1(x)dx| \\ \le \;& Ca_n^{2-5\delta}, \qquad n \ge 1, s \ge 1. \end{aligned}$$

Proof. First, note that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s-x)}{1-\bar{F}(t_n)}\Phi_1(x)dx - \Phi_0(s) \\ = \;& \int_s^\infty \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1)e^{-\frac{\xi^2}{2}}d\xi dx \\ = \;& \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1)e^{-\frac{\xi^2}{2}}d\xi. \end{aligned}$$

Hence we have

$$\begin{aligned} & n\Big(\int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s-x)}{1-\bar{F}(t_n)}\Phi_1(x)dx - \Phi_0(s)\Big) + \frac{1}{2}\Phi_2(s) \\ = \;& \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\Big(n(\varphi(n^{-\frac{1}{2}}\xi; \mu(t_n)) - 1) + \frac{\xi^2}{2}\Big)e^{-\xi^2/2}d\xi. \end{aligned}$$

By Corollary 3.1, we see that there is a $C > 0$ such that for any $n \ge 1$, we have

$$\begin{aligned} & |\mu(t_n)^{*n}((t_n s, \infty)) - (1-n)\Phi_0(s) - \frac{1}{2}\Phi_2(s) - n\int_{\mathbb{R}} \frac{\bar{F}_{n,0}(s-x)}{1-\bar{F}(t_n)}\Phi_1(x)dx| \\ \le \;& Ca_n^{2-5\delta}. \end{aligned}$$

We decompose $I_0(n,s)$ into three parts, i.e.,

$$\begin{aligned} I_0(n,s) &= (1 - n\bar{F}(t_n))^n \mu(t_n)^{*n}((t_n s, \infty)) \\ &= I_{0,0}(n,s) + I_{0,1}(n,s) + I_{0,2}(n,s), \end{aligned}$$

where

$$\begin{aligned}
I_{0,0}(n,s) &= (1-\bar{F}(t_n))\mu(t_n)^{*n}((t_n s,\infty)),\\
I_{0,1}(n.s) &= -n\bar{F}(t_n)\mu(t_n)^{*n}((t_n s,\infty)),\\
I_{0,2}(n,s) &= (1-n\bar{F}(t_n))^n-1+(n+1)\bar{F}(t_n))\mu(t_n)^{*n}((t_n s,\infty)).
\end{aligned}$$

Since

$$\begin{aligned}
&(1-\bar{F}(t_n))\left((1-n)\Phi_0(s)+\frac{\Phi_2(s)}{2}+n\int_{\mathbb{R}}\frac{\bar{F}_{n,0}(s-x)}{1-\bar{F}(t_n)}\Phi_1(x)dx\right)\\
=\ &(1-n-n\bar{F}(t_n))\Phi_0(s)+\frac{\Phi_2(s)}{2}+n\int_{\mathbb{R}}\bar{F}_{n,0}(s-x)\Phi_1(x)dx\\
&-\bar{F}(t_n)\Phi_0(s)-\bar{F}(t_n)\frac{\Phi_2(s)}{2},
\end{aligned}$$

we have

$$\begin{aligned}
&|I_{0,0}(n,s)-(1-n-n\bar{F}(t_n))\Phi_0(s)-\frac{\Phi_2(s)}{2}-n\int_{\mathbb{R}}\bar{F}_{n,0}(s-x)\Phi_1(x)dx|\\
\leq\ & Ca_n^{2-5\delta}.
\end{aligned}$$

By Corollary 3.1, we see that

$$|I_{0,1}(n,s)+n\bar{F}(t_n)\Phi_0(s)|\leq Ca_n^{2-5\delta}.$$

Note that $|(1-x)^n-(1-nx)|\leq n^2x^2$ for any $x\in[0,1]$, $n\geq 1$. Hence we have

$$|I_{0,2}(n,s)|\leq Cn^2\bar{F}(t_n)^2\leq Ca_n^2.$$

Therefore we have our assertion. $\square$

Proposition 3.6. *There is a $C>0$ such that*

$$|I_1(n,s)-n\int_{\mathbb{R}}\bar{F}_{n,1}(s-x)\Phi_1(x)dx|\leq Ca_n^{2-5\delta},\qquad n\geq 1, s\geq 1.$$

Proof.

$$\begin{aligned}
I_1(n,s) &= n(1-\bar{F}(t_n))^{n-1}\int_{\mathbb{R}}P(X_1+x>t_n s\ ,X_1>t_n)\mu(t_n)^{*(n-1)}(dx)\\
&= n(1-\bar{F}(t_n))^{n-1}\int_{\mathbb{R}}\bar{F}((t_n s-x)\vee t_n)\mu(t_n)^{*(n-1)}(dx)\\
&= nJ_0(n,s)+nJ_1(n,s)+nJ_2(n,s),
\end{aligned}$$

where

$$\begin{aligned} J_0(n,s) &= \int_{-\infty}^{t_n s - t_n} \bar{F}((t_n s - x))\mu(t_n)^{*(n-1)}(dx), \\ J_1(n,s) &= \bar{F}(t_n)\int_{t_n s - t_n}^{\infty} \mu(t_n)^{*(n-1)}(dx) = \bar{F}(t_n)\mu(t_n)^{*(n-1)}((t_n(s-1),\infty)), \end{aligned}$$

and

$$J_2(n,s) = ((1-\bar{F}(t_n))^{n-1} - 1)\int_{\mathbb{R}} \bar{F}((t_n s - x)\vee t_n)\mu(t_n)^{*(n-1)}(dx).$$

We see that

$$\begin{aligned} & J_0(n,s) \\ = & \int_{-\infty}^{t_n s - t_n} \bar{F}((t_n s - x))\mu(t_n)^{*(n-1)}(dx) \\ = & \int_{-\infty}^{t_n s - t_n} \bar{F}(t_n s - x)\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{-it_n^{-1}x\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - e^{-\frac{\xi^2}{2}}\right)d\xi + \Phi_1(t_n^{-1}x)\right)t_n^{-1}dx \\ = & \int_{-\infty}^{s-1} \bar{F}(t_n(s-x))\frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi}\left(\varphi(n^{-\frac{1}{2}}\xi;\mu(t_n))^{n-1} - e^{-\frac{\xi^2}{2}}\right)d\xi dx \\ & + \int_{-\infty}^{s-1} \bar{F}(t_n(s-x))\Phi_1(x)dx. \end{aligned}$$

Hence from Corollary 3.1, we see that there is a $C > 0$ such that

$$\begin{aligned} n|J_0(n,s) - \int_{-\infty}^{s-1} \bar{F}(t_n(s-x))\Phi_1(x)dx| &\leq Ca_n^{1-2\delta}nt_n^{-1}\int_{t_n}^{\infty}\bar{F}(x)dx \\ &\leq Ca_n^{1-2\delta}nt_n^{-1}|\eta_1(t_n)| \\ &\leq Ca_n^{2-2\delta}. \end{aligned}$$

We also see that

$$\begin{aligned} & n|J_1(n,s) - \bar{F}(t_n)\Phi_0(s-1)| \\ \leq & Cn\bar{F}(t_n)|\mu(t_n)^{*(n-1)}((t_n(s-1),\infty)) - \Phi_0(s-1)| \\ \leq & Ca_n^{2-2\delta} \end{aligned}$$

and

$$|nJ_2(n,s)| \leq \left(n\bar{F}(t_n)\right)^2 = a_n^2.$$

Note that

$$\begin{aligned} & \int_{-\infty}^{s-1} \bar{F}(t_n(s-x))\Phi_1(x)dx + \bar{F}(t_n)\Phi_0(s-1) \\ = & \int_{\mathbb{R}} \bar{F}_{n,1}(s-x)\Phi_1(x)dx. \end{aligned}$$

Therefore we have our assertion. □

Let us prove Theorem 3.2.

From Proposition 3.5 and 3.6, we see that there is a $C > 0$ such that

$$\begin{aligned} & |I_0(n,s) + I_1(n,s) - (1-n)\Phi_0(s) - \frac{1}{2}\Phi_2(s) - n\int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx| \\ \leq \; & Ca_n^{2-5\delta}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx - \Phi_0(s) \\ = \; & \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx + \int_{s}^{\infty} (\bar{F}(t_n(s-x)) - 1_{\{x>s\}})\Phi_1(x)dx \\ = \; & \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx - \int_{s}^{\infty} F(t_n(s-x))\Phi_1(x)dx \end{aligned}$$

and

$$n\int_{s}^{\infty} F((t_n(s-x))\Phi_1(x)dx = nt_n^{-1}\int_{-\infty}^{0} F(y)\Phi_1(s - t_n^{-1}y)dy.$$

Let $R(s,y) = \Phi_1(s-y) - \Phi_1(s) - \Phi_2(s)y$, for $s > 0$ and $y \leq 0$, then we see that there is a $C_1 > 0$ such that

$$|R(s,y)| \leq C_1|y|^{1+\delta_0}.$$

Hence we have

$$\begin{aligned} & n|\int_{s}^{\infty} F(t_n(s-x))\Phi_1(x)dx - \sum_{k=1}^{2} t_n^{-k}\Phi_k(s)\int_{-\infty}^{0} y^{k-1}F(y)dy| \\ = \; & nt_n^{-1}|\int_{-\infty}^{0} R(s,t_n^{-1}y)F(y)dy| \\ \leq \; & C_1 n^{-\delta_0/2}\eta_2(t_n)^{-(1+\delta_0)/2}\int_{-\infty}^{0} y^{1+\delta_0}F(y)dy \\ \leq \; & Cn^{-\delta_0/2}, \end{aligned}$$

where $C = C_1\eta_2(0)^{-(1+\delta_0)/2}\int_{-\infty}^{0} y^{1+\delta_0}F(y)dy < \infty$.

Since

$$\int_{-\infty}^{0} F(y)dy = \int_{-\infty}^{0} y\mu(dy) = -\int_{0}^{\infty} y\mu(dy)$$

and

$$\int_{-\infty}^{0} yF(y)dy = \frac{1}{2}\int_{-\infty}^{0} y^2\mu(dy) = \frac{\eta_2(t_n)}{2} - \frac{1}{2}\int_{0}^{t_n} y^2\mu(dy),$$

we see that

$$\frac{1}{2}\Phi_2(s) - nt_n^{-2}\Phi_2(s)\int_{-\infty}^{0} yF(y)dy = \frac{\Phi_2(s)}{2}\eta_2(t_n)^{-1}\int_{0}^{t_n} y^2\mu(dy).$$

Therefore we have

$$\begin{aligned} & |(1-n)\Phi_0(s) + \frac{1}{2}\Phi_2(s) + n\int_{\mathbb{R}} \bar{F}(t_n(s-x))\Phi_1(x)dx - H(n,s)| \\ \leq \ & Cn^{-\delta_0/2}. \end{aligned}$$

We also see that

$$\sum_{k=2}^{n} I_k(n,s) \leq \sum_{k=2}^{n} \frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}\bar{F}(t_n)^k(1-\bar{F}(t_n))^{n-k} \leq \frac{n(n-1)}{2}\bar{F}(t_n)^2 = a_n^2.$$

This completes the proof of Theorem 3.2.

3.5 Some Estimations

Let $v_n = \int_{-\infty}^{t_n} x^2\mu(dx)$. Let

$$\begin{aligned} \hat{F}_n(s) &= \int_{-\infty}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx, \\ A(n,s) &= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty x\mu(dx) - \frac{v_n^{-1}}{2}\Phi_2(s)\int_0^{t_n} x^2\mu(dx), \\ &= n\hat{F}_n(s) - v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty \bar{F}(x)dx - v_n^{-1}\Phi_2(s)\left(\int_0^{t_n} x\bar{F}(x)dx - \frac{L(t_n)}{2}\right), \\ H(n,s) &= \Phi_0(s) + A(n,s), \end{aligned}$$

and

$$H_0(n,s) = \Phi_0(s) + n\bar{F}(t_n s).$$

In this section, we prove the following Lemma.

Lemma 3.2.

$$\sup_{s\in[1,\infty)} \left|\frac{H(n,s)}{H_0(n,s)} - 1\right| \to 0, \qquad n\to\infty.$$

Let $\alpha_n = a_n^{1/3}$ and $\beta_n = a_n^{-1/12}$.

Proposition 3.7. *For any $\varepsilon > 0$, there is a $C > 0$ such that*

$$\frac{1}{nF(t_n s)} \leq C a_n^{-1} s^{2+\varepsilon}, \qquad s \in [1, \infty).$$

In particular, for $s > \beta_n$ we have

$$\frac{1}{nF(t_n s)} \leq C a_n^{-1} s^{2+\varepsilon} \leq C s^{14+\varepsilon}.$$

Proof. From Proposition 2.2, we see that for any $\varepsilon > 0$ there is a $C > 0$ such that

$$\begin{aligned}\frac{1}{nF(t_n s)} &= v_n s^2 \frac{1}{L(t_n)} \frac{L(t_n)}{L(t_n s)} \\ &\leq C a_n^{-1} s^{2+\varepsilon}.\end{aligned}$$

Since $a_n^{-1} = \beta_n^{12} \leq s^{12}$ for $s > \beta_n$, we have the second inequality. □

Let $n\hat{F}_n(s) = \sum_{k=1}^4 I_k(n,s)$, where

$$\begin{aligned}I_1(n,s) &= n \int_{s-\alpha_n}^{s} \bar{F}(t_n(s-x))\Phi_1(x)dx, \\ I_2(n,s) &= n \int_{\sqrt{7/8s}}^{s-\alpha_n} \bar{F}(t_n(s-x))\Phi_1(x)dx, \\ I_3(n,s) &= n \int_{-s}^{\sqrt{7/8s}} \bar{F}(t_n(s-x))\Phi_1(x)dx, \\ I_4(n,s) &= n \int_{-\infty}^{-s} \bar{F}(t_n(s-x))\Phi_1(x)dx.\end{aligned}$$

Let

$$R(n,s,y) = \Phi_1(s - t_n^{-1}y) - (\Phi_1(s) + t_n^{-1}y\Phi_2(s)), \qquad \textit{for } n \geq 1,\ s, y \in [1,\infty).$$

Proposition 3.8.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n t_n} y^{k-1}\bar{F}(y)dy| \to 0, \qquad n \to \infty.$$

Proof. We see that

$$\begin{aligned}& I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n t_n} y^{k-1}\bar{F}(y)dy \\ =\ & n t_n^{-1} \int_0^{\alpha_n t_n} \bar{F}(y)\left(\Phi_1(s - t_n^{-1}y) - \Phi_1(s) - t_n^{-1}y\Phi_2(s)\right)dy \\ =\ & n t_n^{-1} \int_0^{\alpha_n t_n} \bar{F}(y) R(n,s,y)dy.\end{aligned}$$

Note that for any $y \in [0, \alpha_n u_n]$,

$$
\begin{aligned}
|R(n,s,y)| &\leq t_n^{-2} y^2 \sup_{z \in [s-\alpha_n, s]} |\Phi_3(z)| \\
&\leq C_0 v_n^{-1} n^{-1} y^2 (1+s)^2 \Phi_1(s-\alpha_n) \\
&\leq C_0^2 v_n^{-1} n^{-1} y^2 (1+s)^3 \Phi_1(s) \exp(\alpha_n s).
\end{aligned}
$$

Hence for all $s \in [1, \infty)$

$$
\begin{aligned}
& |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n t_n} y^{k-1} \bar{F}(y) dy| \\
\leq\ & 8C_0 \sup\{z^2 \bar{F}(s);\ z \geq 0\} v_n^{-1} \alpha_n s^3 \Phi_0(s) \exp(\alpha_n s).
\end{aligned}
$$

Since $\alpha_n \beta_n^3 = a_n^{1/12} \to 0$, $n \to \infty$, we have

$$
\sup_{s \leq \beta_n} \Phi_0(s)^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n u_n} y^{k-1} \bar{F}(y) dy| \to 0,\ n \to \infty.
$$

From Proposition 3.7, we see that for any $\varepsilon > 0$ there is a $C(\varepsilon) > 0$ such that

$$
(n\bar{F}(t_n s))^{-1} \leq C(\varepsilon) s^{14+\varepsilon}.
$$

Hence we see that for $s > \beta_n$,

$$
\begin{aligned}
& (n\bar{F}(t_n s))^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n t_n} y^{k-1} \bar{F}(y) dy| \\
\leq\ & 8C(\varepsilon) C_0^2 \sup\{z^2 \bar{F}(z);\ z \geq 0\} v_n^{-1} \alpha_n s^{17+\varepsilon} \Phi_0(s) \exp(\alpha_n s).
\end{aligned}
$$

Since $\sup_{n \geq 1} \sup_{s > \beta_n} s^{17+\varepsilon} \Phi_0(s) \exp(\alpha_n s) < \infty$, we have

$$
\sup_{s > \beta_n} (n\bar{F}(t_n s))^{-1} |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2} \Phi_k(s) \int_0^{\alpha_n t_n} y^{k-1} \bar{F}(y) dy| \to 0,\ n \to \infty.
$$

Therefore we have our assertion. $\square$

Proposition 3.9.

$$
\sup_{s \in [1,\infty)} H_0(n,s)^{-1} |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n t_n}^{(1-\sqrt{7/8}) t_n s} y^{k-1} \bar{F}(y) dy| \to 0, \qquad n \to \infty.
$$

Proof. Similarly to previous proposition, we see that

$$\begin{aligned}
&|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n t_n}^{(1-\sqrt{7/8})t_n s} y^{k-1}\bar{F}(y)dy| \\
\leq\ & n t_n^{-1} \int_{\alpha_n t_n}^{(1-\sqrt{7/8})t_n s} \bar{F}(y)|R(n,s,y)|dy \\
\leq\ & n t_n^{-3} \bar{F}(t_n\alpha_n) C_0 (1+s)^2 (\sup_{z\in[\sqrt{7/8}s,s]} |\Phi_1(z)|) \int_{t_n\alpha_n}^{(1-\sqrt{7/8})t_n s} y^2 dy \\
\leq\ & 4C_0 n\bar{F}(t_n\alpha_n) s^5 \Phi_1(\sqrt{7/8}s) \\
\leq\ & 4C_0^{1+7/8} n\bar{F}(t_n\alpha_n) s^6 \Phi_0(s)^{7/8}.
\end{aligned}$$

Since $H_0(n,s)^{-1} \leq \Phi_0(s)^{-6/7}(n\bar{F}(t_n s))^{-1/7}$, it is easy to see that for any $\varepsilon \in (0,4/7)$, there is a $C_1 > 0$ such that

$$\begin{aligned}
&H_0(n,s)^{-1}|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n t_n}^{(1-\sqrt{7/8})t_n s} y^{k-1}\bar{F}(y)dy| \\
\leq\ & C_1 s^{6+2/7+\varepsilon} \Phi_0(s)^{7/8-6/7} a_n^{(1-\varepsilon)/3-1/7}.
\end{aligned}$$

Since $\sup_{s\geq 1}\{s^{6+2/7+\varepsilon}\Phi_0(s)^{7/8-6/7}\} < \infty$ and $(1-\varepsilon)/3 - 1/7 > 0$, we have

$$\sup_{s\geq 1} H_0(n,s)^{-1}|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n t_n}^{(1-\sqrt{7/8})t_n s} y^{k-1}\bar{F}(y)dy| \to 0,\ n\to\infty.$$

□

Proposition 3.10.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|I_3(n,s) - n\bar{F}(t_n s)| \to 0, \qquad n\to\infty.$$

Proof.

$$\begin{aligned}
I_3(n,s) &= n\bar{F}(t_n s) \int_{-s}^{\sqrt{7/8}s} \frac{\bar{F}(t_n(s-x))}{\bar{F}(t_n s)} \Phi_1(x)dx \\
&= n\bar{F}(t_n s) \int_{-s}^{\sqrt{7/8}s} (1-\frac{x}{s})^{-2} \frac{L(t_n(s-x))}{L(t_n s)} \Phi_1(x)dx.
\end{aligned}$$

It is easy to see that there is a $C_1 > 0$ such that

$$\Phi_0(s)^{-1} \leq C_1 a_n^{-1/2}, \qquad n\geq 1, s\in[1,(-\log a_n)^{1/2}],$$

$$|\int_{-s}^{\sqrt{7/8}s} \frac{\bar{F}(t_n(s-x))}{\bar{F}(t_n s)} \Phi_1(x)dx| \leq C_1, \qquad n\geq 1, s\in[1,\infty).$$

Then we have

$$\begin{aligned}\sup_{s\le(-\log a_n)^{1/2}} H_0(n,s)^{-1}|I_3(n,s)-n\bar{F}(t_ns)| &\le C_1(C_1+1)a_n^{-1/2}n\bar{F}(t_n) \\ &\le C_1(C_1+1)a_n^{1/2}\to 0, \qquad n\to\infty.\end{aligned}$$

We take $M>1$ arbitrarily, then $(-\log a_n)^{1/4}>M$ for sufficiently large n.

Hence we see that for $s>(-\log a_n)^{1/2}$

$$\begin{aligned}& \left|\int_{-s}^{\sqrt{7/8}s}(1-\frac{x}{s})^{-2}\frac{L(t_n(s-x))}{L(t_ns)}\Phi_1(x)dx-1\right| \\ \le\ & \left|\int_{-s}^{\sqrt{7/8}s}\{(1-\frac{x}{s})^{-2}-1\}\frac{L(t_n(s-x))}{L(t_ns)}\Phi_1(x)dx\right|+\left|\int_{-s}^{\sqrt{7/8}s}(\frac{L(u_n(s-x))}{L(u_ns)}-1)\Phi_1(x)dx\right| \\ & +\int_{[-s,\sqrt{7/8}s]^C}\Phi_1(x)dx \\ \le\ & 2\left(\int_{-M}^{M}|(1-\frac{x}{s})^{-2}-1|\Phi_1(x)dx+8\Phi_0(M)\right) \\ & +\sup_{s>(-\log a_n)^{1/2}}\ \sup_{1-\sqrt{7/8}\le a\le 1}|\frac{L(as)}{L(s)}-1|+2\Phi_0(\sqrt{7/8}s).\end{aligned}$$

Hence we have

$$\sup_{s>(-\log a_n)^{1/2}}|n\bar{F}(t_ns)|^{-1}|I_3(n,s)-n\bar{F}(t_ns)|\to 0, \qquad n\to\infty.$$

So we have our assertion. □

Proposition 3.11.

$$\sup_{s\in[1,\infty)}\frac{I_4(n,s)}{H_0(n,s)}\to 0, \qquad n\to\infty.$$

Proof. $|I_4(n,s)|\le n\bar{F}(2t_ns)\Phi_0(s)$. Hence we have

$$\Phi_0(s)^{-1}|I_4(n,s)|\le n\bar{F}(2t_ns)\le n\bar{F}(t_n)\to 0, \qquad n\to\infty.$$

□

Proposition 3.12.

$$\sup_{s\in[1,\infty)}H_0(n,s)^{-1}|v_n^{-1/2}n^{1/2}\Phi_1(s)\int_{\sqrt{7/8}t_ns}^{\infty}\bar{F}(y)dy|\to 0, \qquad n\to\infty,$$

and

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|v_n^{-1}\Phi_2(s)\left(\int_{\sqrt{7/8t_ns}}^{t_n} y\bar{F}(y)dy - L(t_n)\right)| \to 0, \qquad n\to\infty.$$

Proof. From Proposition 2.3 (2), we see that there is a $C_1>0$ such that

$$t_n\int_{(1-\sqrt{7/8})t_ns}^{\infty}\bar{F}(y)dy \le C_1 s^{-1}L((1-\sqrt{7/8})t_ns).$$

We can easily see that

$$\sup_{s\in[1,\beta_n)}\Phi_0(s)^{-1}v_n^{-1}t_n\Phi_1(s)\int_{(1-\sqrt{7/8})t_ns}^{\infty}\bar{F}(y)dy\to 0, \qquad n\to\infty,$$

and

$$\sup_{s\in[\beta_n,\infty)}(n\bar{F}(n^{1/2}s))^{-1}v_n^{-1}t_n\Phi_1(s)\int_{(1-\sqrt{7/8})t_ns}^{\infty}\bar{F}(y)dy\to 0, \qquad n\to\infty.$$

Also we see that for any $\varepsilon\in(0,1)$, there is a $C_2>0$ such that

$$v_n^{-1}\int_{(1-\sqrt{7/8})t_ns}^{t_n} y\bar{F}(y)dy = v_n^{-1}\int_{(1-\sqrt{7/8})s}^{1}\frac{L(t_nx)}{x}dx \le C_2a_ns^{\varepsilon}.$$

Hence we can easily see that

$$\sup_{s\in[1,\beta_n)}\Phi_0(s)^{-1}|v_n^{-1}\Phi_2(s)\left(\int_{(1-\sqrt{7/8})t_ns}^{t_n} y\bar{F}(y)dy - L(n^{1/2})\right)| \to 0, \qquad n\to\infty,$$

and

$$\sup_{s\in[\beta_n,\infty)}(n\bar{F}(t_ns))^{-1}|v_n^{-1}\Phi_2(s)\left(\int_{(1-\sqrt{7/8})t_ns}^{t_n} y\bar{F}(y)dy - L(t_n)\right)| \to 0, \qquad n\to\infty.$$

Therefore we have our assertion. □

Now let us prove Lemma 3.2. Note that $H(n,s)-H_0(n,s) = A(n,s)-n\bar{F}(t_ns)$. So Proposition 3.8, 9, 10, 11 and 12 imply Lemma 3.2.

3.6 Proof of Theorem 3.1

Recall that $t_n=\sup\{t>0; n\eta_2(t)>t^2\}$,$v_n=\eta_2(t_n)$ and $a_n=\frac{L(t_n)}{v_n}$. Let $v_n(t)=\eta_2(t_nt)$ for $t>0$.

First we prove the following Lemma.

Lemma 3.3. *For any $\beta > 0$ and $\delta \in (0,1)$, there is a $C > 0$ such that we have*

$$\sup_{s > a_n^{-\beta}} \left|\frac{P(\sum_{k=1}^n X_k > t_n s)}{H(n,s)} - 1\right| \le C a_n^{1-\delta}.$$

We make some preparations to prove Lemma 3.3. Similarly to Proposition 26 in [7] (see also the proof of Proposition 2.19), we can prove the following.

Proposition 3.13. (1) *For any $t, s > 0$, and $n \ge 2$,*

$$P(\sum_{k=2}^n X_k 1_{\{X_k \le tn^{1/2}\}} > sn^{\frac{1}{2}}) \le \exp(\frac{6}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}] - \frac{s}{t}).$$

(2) *For any $s, t > 0$, $\varepsilon \in (0,1)$ with $t < (1-\varepsilon)s$,*

$$|P(\sum_{k=1}^n X_k > sn^{\frac{1}{2}}) - nP(X_1 + \sum_{k=2}^n X_k 1_{\{X_k \le tn^{1/2}\}} > sn^{\frac{1}{2}}, \sum_{k=2}^n X_k 1_{\{X_k \le tn^{1/2}\}} \le \varepsilon sn^{\frac{1}{2}})|$$
$$\le 2n(n-1)\bar{F}(tn^{\frac{1}{2}})^2 + \exp(\frac{6}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}] - \frac{s}{t}) + n\bar{F}(tn^{\frac{1}{2}}) \exp(\frac{6}{t^2} E[X_1^2 1_{\{X_1 \le tn^{1/2}\}}] - \frac{\varepsilon s}{2t}).$$

We apply for Proposition 3.13 with $v_n^{1/2}s$, $v_n^{1/2}t$. Then we have

$$P(\sum_{k=2}^n X_k 1_{\{X_k \le t_n t\}} > st_n) \le 2\exp(\frac{6v_n(t)}{t^2 v_n} - \frac{s}{t}) \tag{3.11}$$

and

$$|P(\sum_{k=1}^n X_k > st_n) - nP(X_1 + \sum_{k=2}^n X_k 1_{\{X_k \le t_n t\}} > st_n, \sum_{k=2}^n X_k 1_{\{X_k \le t_n t\}} \le \varepsilon t_n s)|$$
$$\le \quad 2n(n-1)\bar{F}(t_n t)^2 + \exp(\frac{6\eta_2(t_n t)}{t^2 \eta_2(t_n)} - \frac{s}{t}) + 2n\bar{F}(t_n t)\exp(\frac{6\eta_2(t_n t)}{t^2\eta_2(t_n)} - \frac{\varepsilon s}{2t}). \tag{3.12}$$

Since $\eta_2(t)$ is slowly varying, we see that there is a $C > 0$ such that $\eta_2(t_n t)/\eta_2(t_n) \le Ct$ for $t \ge 1$. So we have

$$|P(\sum_{k=1}^n X_k > st_n) - nP(X_1 + \sum_{k=2}^n X_k 1_{\{X_k \le t_n t\}} > st_n, \sum_{k=2}^n X_k 1_{\{X_k \le t_n t\}} \le \varepsilon t_n s)|$$
$$\le 2n(n-1)\bar{F}(t_n t)^2 + \exp(\frac{6C}{t} - \frac{s}{t})$$
$$+2n\bar{F}(t_n t)\exp(\frac{6C}{t} - \frac{\varepsilon s}{2t}). \tag{3.13}$$

Also we prove the following for the proof of Lemma 3.3.

Proposition 3.14. *For any γ, δ, $\varepsilon \in (0,1)$ and $\beta > 0$, there is a $C > 0$ such that*

$$\begin{aligned}
&|P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} > st_n, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} \le \varepsilon st_n) \\
&- \int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\Phi_1(x)dx| \\
&\le C\bar{F}((1-\varepsilon)n^{1/2}s)a_n^{1-4\delta}, \qquad \textit{for } s > a_n^{-\beta}.
\end{aligned}$$

Proof. It is easy to see that there is a $C_1 > 0$ such that for $s > a_n^{-\beta}$

$$\begin{aligned}
&|P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} > st_n, \sum_{k=2}^{n} X_k 1_{\{X_k \le s^\gamma t_n\}} \le \varepsilon st_n) \\
&\quad -P(X_1 + \sum_{k=2}^{n} X_k > st_n, \sum_{k=2}^{n} X_k \le \varepsilon st_n, X_2 \le t_n, \ldots, X_n \le t_n)| \\
&\le C_1\bar{F}((1-\varepsilon)t_n s)a_n^{1-3\delta}.
\end{aligned}$$

We also see that

$$\begin{aligned}
&P(X_1 + \sum_{k=2}^{n} X_k > st_n, \sum_{k=2}^{n} X_k \le \varepsilon st_n, X_2 \le t_n, \ldots, X_n \le t_n) \\
&= (1-\bar{F}(t_n))^{n-1} \int_{-\infty}^{\varepsilon s} \bar{F}(t_n(s-x))\mu(t_n)^{*(n-1)}(dx).
\end{aligned}$$

Similarly to the proof of Proposition 3.6, we have our assertion. □

Now let us prove Lemma 3.3. Since

$$
\begin{aligned}
& H(n,s) - n\int_{-\infty}^{\varepsilon s}\bar{F}(t_n(s-x))\Phi_1(x)dx \\
= \ & \Phi_0(s) - n\int_{\varepsilon s}^{s}\bar{F}(t_n(s-x))\Phi_1(x)dx \\
& + v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^{\infty}x\mu(dx) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_0^{t_n}x^2\mu(dx) \\
= \ & \Phi_0(s) - v_n^{-1/2}n^{1/2}\eta_1((1-\varepsilon)t_n s)\Phi_1(s) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_{(1-\varepsilon)t_n s}^{t_n}x^2\mu(dx) \\
& - v_n^{-1/2}n^{1/2}(\int_0^{(1-\varepsilon)t_n s}\bar{F}(z)(\Phi_1(s-t_n^{-1}z) - \Phi_1(s) - t_n^{-1}z\Phi_2(s))dz) \\
& - v_n^{-1}\frac{L((1-\varepsilon)t_n s)}{(1-\varepsilon)s}\Phi_1(s) - v_n^{-1}L((1-\varepsilon)t_n s)\Phi_2(s) \\
= \ & \Phi_0(s) - v_n^{-1/2}n^{1/2}\eta_1((1-\varepsilon)t_n s)\Phi_1(s) + v_n^{-1}\frac{\Phi_2(s)}{2}\int_{(1-\varepsilon)t_n s}^{t_n}x^2\mu(dx) \\
& - v_n^{-1/2}n^{1/2}(\int_0^{(1-\varepsilon)t_n s}\bar{F}(z)(\Phi_1(s-t_n^{-1}z) - \Phi_1(s) - t_n^{-1}z\Phi_2(s))dz) \\
& - \frac{\eta_2((1-\varepsilon)st_n)}{(1-\varepsilon)s\eta_2(t_n)}\frac{L((1-\varepsilon)st_n)}{\eta_2((1-\varepsilon)st_n)}\Phi_1(s) - \frac{\eta_2((1-\varepsilon)st_n)}{\eta_2(t_n)}\frac{L((1-\varepsilon)st_n)}{\eta_2((1-\varepsilon)st_n)}\Phi_2(s),
\end{aligned}
$$

it is easy to see that there is a $C_1 > 0$ such that for $s \geq 1$

$$
|H(n,s) - n\int_{-\infty}^{\varepsilon s}\bar{F}(t_n(s-x))\Phi_1(x)dx| \leq C_1 s^3\Phi_1(\varepsilon s).
$$

Combining Equation (3.13) and Proposition 3.14, we see that there is a $C_1, C_2 > 0$ such that

$$
\begin{aligned}
& |P(\sum_{k=1}^{n}X_k > st_n) - n\int_{-\infty}^{\varepsilon s}\bar{F}(t_n(s-x))\Phi_1(x)dx| \\
\leq \ & 2n(n-1)\bar{F}(s^{\gamma}t_n)^2 + \exp(\frac{6C_1}{s^{\gamma}} - \frac{s}{s^{\gamma}}) + n\bar{F}(s^{\gamma}t_n)\exp(\frac{6C_1}{s^{\gamma}} - \frac{\varepsilon s}{2s^{\gamma}}) \\
& + C_2 n\bar{F}((1-\varepsilon)t_n s)a_n^{1-\delta}.
\end{aligned}
$$

Hence we see that there is a $C > 0$ such that

$$
\sup_{s > a_n^{-\beta}}(n\bar{F}(t_n s))^{-1}|P(\sum_{k=1}^{n}X_k > st_n) - H(n,s)| \leq Ca_n^{1-\delta}.
$$

Therefore by Lamma 3.2, we have our assertion.

Now let us prove Theorem 3.1. From Theorem 3.2, we see that there is a $C_1 > 0$ such that

$$|P(\sum_{k=1}^{n} X_k > st_n) - H(n,s)| \le C_1 a_n^{2-\delta/2}, \qquad s \ge 1.$$

Note that for any $\varepsilon > 0$, there is a $C_2 > 0$ such that $n\bar{F}(t_n s) \ge C_2^{-1} s^{-3} a_n \ge C_2^{-1} a_n^{1+\delta/2}$ for $s \le a_n^{-\delta/6}$. Hence by Lemma 3.2, we see that there is a $C_3 > 0$ such that

$$H(n,s)^{-1} \le C_3 (n\bar{F}(t_n s))^{-1} \le C_2 C_3 a_n^{-1-\delta/2}, \qquad s \le a_n^{-\delta/6}.$$

So we have

$$\sup_{s \le a_n^{-\delta/6}} |\frac{P(\sum_{k=1}^{n} X_k > st_n)}{H(n,s)} - 1| \le C_1 C_2 C_3 a_n^{1-\delta}.$$

From this inequality and Lemma 3.3, we have Equation (3.1).

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