Identification Problems for Partial Differential Equations of Hyperbolic Type

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Abstract

In this paper, we consider initial value-boundary value problems for partial differential equations of hyperbolic type whose coefficients are all independent of time t. We show a necessary and sufficient condition on initial values guaranteeing that values of the corresponding solutions over the whole spatial domain and $t \geqq t_0$ (given time), uniquely determine coefficients in the equation. Moreover for any given initial values, we specify a spatial subdomain where all coefficients can be determined from the above-mentioned values of the solutions. Finally we prove: in the case of the one spatial dimension, such a domain is maximal among subdomains where coefficients can be uniquely determined.

§1. Introduction

In this paper, we consider initial value-boundary value problems for partial differential equations of hyperbolic type:

$$(1.1)\qquad \begin{cases} \dfrac{\partial^2 u^m(x,t)}{\partial t^2} + \mathscr{A}^m u^m(x,t) = 0 & (x \in \Omega,\ t > 0) \\ u^m(x,0) = a^m(x)\ ,\quad \dfrac{\partial u^m}{\partial t}(x,0) = b^m(x) & (x \in \Omega) \\ u^m \text{ satisfies an appropriate boundary condition on } \partial\Omega. \\ (\text{e.g. } u^m(x,t) = 0 \quad (x \in \partial\Omega,\ t > 0)) . \end{cases}$$

$$(1.2)\quad \begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} + \mathscr{A}u(x,t) = 0 & (x \in \Omega,\ t>0) \\ u(x,0) = a(x)\,, \quad \dfrac{\partial u}{\partial t}(x,0) = b(x) & (x \in \Omega) \\ u \text{ satisfies an appropriate boundary condition on } \partial\Omega. \\ (\text{e.g. } u(x,t) = 0 \quad (x \in \partial\Omega,\ t>0))\,. \end{cases}$$

Here $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $-\mathscr{A}^m$ and $-\mathscr{A}$ are uniformly elliptic differential operators of the second order.

The purpose of this paper is to solve

PROBLEM. Let u^m and u be the solutions respectively to (1.1) and (1.2) (in a sense stated below). Then, does an equality

$$(1.3)\quad u(x,t) = u^m(x,t) \quad (x \in \Omega,\ t \geqq t_0\text{: a given non-negative number})$$

imply

$$(1.4)\quad \mathscr{A} = \mathscr{A}^m\,?$$

What informations on $\mathscr{A}$ can we derive from (1.3) for a given initial data?

Throughout this paper, we understand that the superscript "m" means quantities known as the model. In PROBLEM, we consider that $\mathscr{A}$, a, b and a boundary condition imposed on u should be determined (identified) from the condition (1.3), which means that a state u of an unknown system (1.2) is equal to the state u^m of the known model system (1.1).

Moreover, in (1.3), we note that the equality $u(x,t) = u^m(x,t)$ holds for each $x \in \Omega$ and for $t \geqq t_0$: a given non-negative number. This means that the "observation" of a state u of an unknown system is over the whole domain Ω and that the observation begins after a finite time passing.

This problem is one of what are called identification problems, which are important as a necessary step for modelling processes or dynamics. For such identification problems, we can refer to Courdesses, Polis and Amouroux [3], Kitamura and Nakagiri [9], Nakagiri [11], Nakagiri and Yamamoto [12], [13], [14], [15], Polis and Goodson [16]. In Nakagiri and Yamamoto [13], the identification problem is proved to be equivalent to a controllability problem and to an observability problem, and in [14], we apply results in [13] to identification problems for partial differential equations of parabolic type. Furthermore, in [15], the results in [13] are generalized so as to be applicable to our problem for partial differential equations of hyperbolic type. In this paper, we use the results in [15] and give answers to PROBLEM.

This paper is composed of four sections. In §2 we give an exact formulation of our identification problem and state our main results (Theorems 1–4). In §3 we state abstract theorems established in Nakagiri and Yamamoto [15], and in §4, by applying those theorems, we prove the main results.

§2. Formulation and Main Results

Let $\Omega\subset\mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $X=L^2(\Omega)$ be the ordinary L^2-space with the inner product $(\cdot,\cdot)_{L^2}$. Let us consider the following two elliptic differential operators $-\mathscr{A}^m$ and $-\mathscr{A}$ of the second order in Ω:

$$-\mathscr{A}^m=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(p_{ij}^m(x)\frac{\partial}{\partial x_j}\right)+\sum_{j=1}^n q_j^m(x)\frac{\partial}{\partial x_j}+r^m(x)\quad(x\in\Omega)\,. \tag{2.1}$$

$$-\mathscr{A}=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(p_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_{j=1}^n q_j(x)\frac{\partial}{\partial x_j}+r(x)\quad(x\in\Omega)\,. \tag{2.2}$$

Henceforth the coefficients $p_{ij}^m=p_{ji}^m$, $p_{ij}=p_{ji}$, q_j^m, q_j, r^m and r $(1\leqq i,\ j\leqq n)$ are real-valued and smooth on $\bar{\Omega}$, the closure of Ω, and the uniform ellipticity

$$\begin{cases}\displaystyle\sum_{i,j=1}^n p_{ij}^m(x)\xi_i\xi_j\geqq\delta^m\sum_{i=1}^n|\xi_i|^2\\ \displaystyle\sum_{i,j=1}^n p_{ij}(x)\xi_i\xi_j\geqq\delta\sum_{i=1}^n|\xi_i|^2\quad((\xi_1,\cdots,\xi_n)\in\mathbf{R}^n)\end{cases} \tag{2.3}$$

is assumed, δ^m and δ being positive constants.

Next we introduce boundary conditions. Let β^m and β be real-valued smooth functions on $\partial\Omega$, and let $\partial/\partial\nu_{\mathscr{A}^m}$ and $\partial/\partial\nu_{\mathscr{A}}$ denote the differentiations along the outer conormal $\nu_{\mathscr{A}^m}$ and $\nu_{\mathscr{A}}$, respectively:

$$\frac{\partial}{\partial\nu_{\mathscr{A}^m}}=\sum_{i,j=1}^n\nu_i(x)p_{ij}^m(x)\frac{\partial}{\partial x_j}\quad\text{and}\quad\frac{\partial}{\partial\nu_{\mathscr{A}}}=\sum_{i,j=1}^n\nu_i(x)p_{ij}(x)\frac{\partial}{\partial x_j}\,,$$

where $\nu(x)=(\nu_1(x),\cdots,\nu_n(x))$ is the outer unit normal on $\partial\Omega$. Then we define A^m and A_D^m by the realizations in $X=L^2(\Omega)$ of $\mathscr{A}^m$ with boundary conditions

$$\frac{\partial u}{\partial\nu_{\mathscr{A}^m}}+\beta^m u_{|\partial\Omega}=0 \tag{2.4}$$

and

$$u_{|\partial\Omega}=0\,, \tag{2.5}$$

respectively. Furthermore let us define A by the realization in $L^2(\Omega)$ of $\mathscr{A}$ with a boundary condition

$$\frac{\partial u}{\partial\nu_{\mathscr{A}}}+\beta u_{|\partial\Omega}=0 \tag{2.6}$$

or

$$u_{|\partial\Omega}=0\,. \tag{2.6$_\mathrm{D}$}$$

We consider the following systems:

(2.7) $$\frac{d^2u(t)}{dt^2}+A^m u(t)=0 \qquad (t\geqq 0)$$

or

(2.7)$_D$ $$\frac{d^2u(t)}{dt^2}+A_D^m u(t)=0 \qquad (t\geqq 0)$$

with

(2.8) $$u(0)=a^m\,, \quad \frac{du}{dt}(0)=b^m\,.$$

(2.9) $$\frac{d^2v(t)}{dt^2}+Av(t)=0 \qquad (t\geqq 0)$$

with

(2.10) $$v(0)=a\,, \quad \frac{dv}{dt}(0)=b\,.$$

As is known, $\sigma(A^m)$, $\sigma(A_D^m)$ and $\sigma(A)$ consist entirely of denumerable eigenvalues with finite multiplicities. Thus we can set $\sigma(A^m)=\{\lambda_i\}_{i\geqq 1}$ and $\sigma(A_D^m)=\{\lambda_{Di}\}_{i\geqq 1}$, and we have $\inf\{\mathrm{Re}\,\mu;\ \mu\in\sigma(A^m)\}>-\infty$, $\inf\{\mathrm{Re}\,\mu;\ \mu\in\sigma(A_D^m)\}>-\infty$ and $\inf\{\mathrm{Re}\,\mu;\ \mu\in\sigma(A)\}>-\infty$ (e.g. Agmon [2]). Let us fix a sufficiently large α such that

(2.11) $$\alpha>\inf\{\mathrm{Re}\,\mu;\ \mu\in\sigma(A^m)\cup\sigma(A_D^m)\cup\sigma(A)\}\,.$$

Then the fractional power $(A^m+\alpha)^{1/2}$ is well-defined (e.g. Tanabe [18]). For each $a^m\in\mathscr{D}(A^m)$ and $b^m\in\mathscr{D}((A^m+\alpha)^{1/2})$, there exists a unique strong solution $u(t)$ to (2.7) and (2.8) satisfying

(2.12) $$\begin{aligned}&u\in C^1([0,\infty);\ \mathscr{D}((A^m+\alpha)^{1/2}))\cap C^2((0,\infty);\ L^2(\Omega))\\ &A^m u\in C((0,\infty);\ L^2(\Omega))\end{aligned}$$

(e.g. Fattorini [4]). Henceforth we denote the solution to (2.7) and (2.8) by $u(t;\ A^m, a^m, b^m)$. For the unique existence of solution to (2.7)$_D$ with (2.8), or to (2.9) with (2.10), similar results hold true and we denote the solutions by $u(t;\ A_D^m, a^m, b^m)$ and $u(t;\ A, a, b)$, respectively.

REMARK 1. By Fujiwara [5], we get an isomorphic relation:

(2.13) $$\mathscr{D}((A_D^m+\alpha)^{1/2})=H_0^1(\Omega)\,.$$

The adjoint operators $(A^m)^*$ and $(A_D^m)^*$ of A^m and A_D^m in the Hilbert space $L^2(\Omega)$, are given by (2.14) and (2.15), respectively:

$$
(2.14)\quad \begin{cases} ((A^m)^*u)(x) \\ \qquad = -\sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(p_{ij}^m(x)\dfrac{\partial u(x)}{\partial x_j}\right) + \sum_{i=1}^{n} \dfrac{\partial}{\partial x_j}(q_j^m(x)u(x)) - r^m(x)u(x) \quad (x \in \Omega) \\ \mathscr{D}((A^m)^*) = \left\{u \in H^2(\Omega);\ \dfrac{\partial u}{\partial \nu_{\mathscr{A}^m}} + (\beta^m - \gamma^m)u_{|\partial\Omega} = 0\right\}, \end{cases}
$$

where $\gamma^m(x) = \sum_{i=1}^n \nu_j(x)q_j^m(x)$ $(x \in \partial\Omega)$.

$$
(2.15)\quad \begin{cases} ((A_D^m)^*u)(x) \\ \qquad = -\sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\left(p_{ij}^m(x)\dfrac{\partial u(x)}{\partial x_j}\right) + \sum_{j=1}^{n} \dfrac{\partial}{\partial x_j}(q_j^m(x)u(x)) - r^m(x)u(x) \quad (x \in \Omega) \\ \mathscr{D}((A_D^m)^*) = \{u \in H^2(\Omega);\ u_{|\partial\Omega} = 0\}. \end{cases}
$$

Then we know that $\sigma((A^m)^*) = \{\bar{\lambda}_i\}_{i\geq 1}$, $\sigma((A_D^m)^*) = \{\overline{\lambda_{Di}}\}_{i\geq 1}$ and $\dim \mathrm{Ker}(\lambda_i - A^m) = \dim \mathrm{Ker}(\bar{\lambda}_i - (A^m)^*)$, $\dim \mathrm{Ker}(\lambda_{Di} - A_D^m) = \dim \mathrm{Ker}(\overline{\lambda_{Di}} - (A_D^m)^*)$ (cf. Kato [8]). Throughout this paper, for $\alpha \in \mathbf{C}$ and a set $D \subset \mathbf{R}^n$, we denote the complex conjugate of α and the closure of D respectively by $\bar{\alpha}$ and $\bar{D}$. Henceforth let $\{\psi_{ij}^m\}_{1\leq j\leq d_i^m}$ and $\{\psi_{Dij}^m\}_{1\leq j\leq d_{Di}^m}$ denote bases of $\mathrm{Ker}(\bar{\lambda}_i - (A^m)^*)$ and $\mathrm{Ker}(\overline{\lambda_{Di}} - (A_D^m)^*)$, respectively.

We regard A^m or A_D^m as a model (that is, p_{ij}^m, q_j^m $(1\leq i, j\leq n)$, r^m and β^m are assumed to be known), whereas we consider A as an unknown operator except for the assumption on the smoothness of p_{ij}, q_j $(1\leq i, j\leq n)$, r, β and the uniform ellipticity.

The identification problem for hyperbolic equations is formulated as follows;

PROBLEM. (I) Let $a_k^m \in \mathscr{D}(A^m)$, $b_k^m \in \mathscr{D}((A^m+\alpha)^{1/2})$ $(1\leq k\leq N)$ be given and let $t_0 \geq 0$. Under what conditions on a_k^m, b_k^m $(1\leq k\leq N)$, do equalities

$$
(2.16)\quad \begin{aligned} &u(t;\ A, a_k, b_k) = u(t;\ A^m, a_k^m, b_k^m) \quad \text{in}\ L^2(\Omega)\ (t\geq t_0) \\ &\text{for}\quad a_k \in \mathscr{D}(A),\ b_k \in \mathscr{D}(A+\alpha)^{1/2}) \qquad (1\leq k\leq N) \end{aligned}
$$

imply the following (2.17)-(2.19)?

$$
(2.17)\quad p_{ij}(x) = p_{ij}^m(x),\ q_j(x) = q_j^m(x),\ r(x) = r^m(x) \qquad (x \in \bar{\Omega},\ 1\leq i,\ j\leq n).
$$

$$
(2.18)\quad \beta(x) = \beta^m(x) \qquad (x \in \partial\Omega).
$$

$$
(2.19)\quad a_k(x) = a_k^m(x),\ b_k(x) = b_k^m(x) \qquad (x \in \bar{\Omega},\ 1\leq k\leq N).
$$

(II) Let $a_k^m \in \mathscr{D}(A_D^m)$, $b_k^m \in \mathscr{D}((A_D^m+\alpha)^{1/2})$ $(1\leq k\leq N)$ be given and let $t_0\geq 0$. Under what conditions on a_k^m, b_k^m $(1\leq k\leq N)$, do equalities

$$
(2.16)_{\mathrm{D}}\quad \begin{aligned} &u(t;\ A, a_k b_k) = u(t;\ A_D^m, a_k^m, b_k^m) \quad \text{in}\ L^2(\Omega)\ (t\geq t_0) \\ &\text{for}\quad a_k \in \mathscr{D}(A),\ b_k \in \mathscr{D}((A+\alpha)^{1/2}) \qquad (1\leq k\leq N) \end{aligned}
$$

imply (2.17), (2.19) and $\mathscr{D}(A) = \mathscr{D}(A_D^m)$?

Now we can state the first of our main results which is a modification of Theorem 5.2 in Nakagiri [11].

THEOREM 1. (I) *Let us consider* A^m *given by* (2.1) *and* (2.4) *as a model. For given* $a_k^m \in \mathscr{D}(A^m)$, $b_k^m \in \mathscr{D}((A^m+\alpha)^{1/2})$ $(1 \leqq k \leqq N)$, *we assume that*

$$\text{rank}\, L_i^{\pm} = d_i^m \quad \text{for each} \quad i \geqq 1 \,, \tag{2.20}$$

where L_i^+ *and* L_i^- *are* $N \times d_i^m$ *matrices given by*

$$\begin{cases} L_i^+ = (\sqrt{-\lambda_i}(a_k^m, \phi_{ij}^m)_{L^2} + (b_k^m, \phi_{ij}^m)_{L^2})_{1 \leqq k \leqq N, 1 \leqq j \leqq d_i^m} \\ L_i^- = (-\sqrt{-\lambda_i}(a_k^m, \phi_{ij}^m)_{L^2} + (b_k^m, \phi_{ij}^m)_{L^2})_{1 \leqq k \leqq N, 1 \leqq j \leqq d_i^m} \,. \end{cases} \tag{2.21}$$

If A *and* $a_k \in \mathscr{D}(A)$, $b_k \in \mathscr{D}((A+\alpha)^{1/2})$ $(1 \leqq k \leqq N)$ *satisfy the equalities* (2.16) *for some* $t_0 \geqq 0$, *then* A *is the realization of* $\mathscr{A}$ *with* (2.6), *and* (2.17)–(2.19) *hold.*

(II) *Let us consider* A_D^m *given by* (2.1) *and* (2.5) *as a model. For given* $a_k^m \in \mathscr{D}(A_D^m)$, $b_k^m \in \mathscr{D}((A_D^m+\alpha)^{1/2})$ $(1 \leqq k \leqq N)$, *we assume that*

$$\text{rank}\, L_{Di}^{\pm} = d_{Di}^m \quad \text{for each} \quad i \geqq 1 \,, \tag*{(2.20)$_\mathrm{D}$}$$

where L_{Di}^+ *and* L_{Di}^- *are* $N \times d_{Di}^m$ *matrices given by*

$$\begin{cases} L_{Di}^+ = (\sqrt{-\lambda_{Di}}(a_k^m, \phi_{Dij}^m)_{L^2} + (b_k^m, \phi_{Dij}^m)_{L^2})_{1 \leqq k \leqq N, 1 \leqq j \leqq d_{Di}^m} \\ L_{Di}^- = (-\sqrt{-\lambda_{Di}}(a_k^m, \phi_{Dij}^m)_{L^2} + (b_k^m, \phi_{Dij}^m)_{L^2})_{1 \leqq k \leqq N, 1 \leqq j \leqq d_{Di}^m} \,. \end{cases} \tag*{(2.21)$_\mathrm{D}$}$$

If A *and* $a_k \in \mathscr{D}(A)$, $b_k \in D((A+\alpha)^{1/2})$ $(1 \leqq k \leqq N)$ *satisfy the equalities* (2.16)$_\mathrm{D}$ *for some* $t_0 \geqq 0$, *then* A *is the realization of* $\mathscr{A}$ *with* (2.6)$_\mathrm{D}$, *and* (2.17) *and* (2.19) *hold.*

REMARK 2. From Remark 7 in §3 (cf. Nakagiri and Yamamoto [15]), we see that the conditions (2.20) are equivalent to the following conditions (2.20)′ considered at an arbitrary time t:

$$\text{rank}\, L_i^{\pm}(t) = d_i^m \quad \text{for some} \quad t \geqq 0 \quad \text{and each} \quad i \geqq 1 \tag*{(2.20)$'$}$$

where we set $u_k^m(t) = u(t;\, A^m, a_k^m, b_k^m)$ and

$$\begin{cases} L_i^+(t) = \left(\sqrt{-\lambda_i}(u_k^m(t), \phi_{ij}^m)_{L^2} + \left(\dfrac{du_k^m(t)}{dt}, \phi_{ij}^m\right)_{L^2}\right)_{1 \leqq k \leqq N, 1 \leqq j \leqq d_i^m} \\ L_i^-(t) = \left(-\sqrt{-\lambda_i}(u_k^m(t), \phi_{ij}^m)_{L^2} + \left(\dfrac{du_k^m(t)}{dt}, \phi_{ij}^m\right)_{L^2}\right)_{1 \leqq k \leqq N, 1 \leqq j \leqq d_i^m} \,. \end{cases}$$

For A_D^m, a similar equivalence holds.

REMARK 3. This theorem corresponds to Theorem 1 in Nakagiri and Yamamoto [14], where similar identification problems are considered for partial differential equations of parabolic type.

In Theorem 1, in order to identify a finite number of coefficients p_{ij}, q_j

$(1 \leqq i, j \leqq n)$, r, β and initial values a_k, b_k $(1 \leqq k \leqq N)$, we require denumerable conditions (2.20) or $(2.20)_D$ on the initial data a_k^m, b_k^m $(1 \leqq k \leqq N)$. Thus it is natural to discuss the determination of coefficients in terms of arbitrarily given initial values a_k^m, b_k^m $(1 \leqq k \leqq N)$. To this end, we introduce some notation. Let q denote the natural number:

$$q = \frac{(n+1)(n+2)}{2} . \tag{2.22}$$

For $\phi \in C^2(\Omega)$ and $x \in \Omega$, we define a q-dimensional column vector $\vec{\omega}(\phi; x)$ whose components are partial derivatives of order $\leqq 2$ with respect to x_i $(1 \leqq i \leqq n)$:

$$\begin{aligned} \vec{\omega}(\phi; x) = {}^t\Big(&\frac{\partial^2 \phi(x)}{\partial x_1^2}, \cdots, \frac{\partial^2 \phi(x)}{\partial x_n^2}, 2\frac{\partial^2 \phi(x)}{\partial x_1 \partial x_2}, \cdots, 2\frac{\partial^2 \phi(x)}{\partial x_1 \partial x_n}, \\ &2\frac{\partial^2 \phi(x)}{\partial x_2 \partial x_3}, \cdots, 2\frac{\partial^2 \phi(x)}{\partial x_2 \partial x_n}, \cdots, 2\frac{\partial^2 \phi(x)}{\partial x_{n-1} \partial x_n}, \frac{\partial \phi(x)}{\partial x_1}, \cdots, \frac{\partial \phi(x)}{\partial x_n}, \phi(x)\Big) , \end{aligned} \tag{2.23}$$

where ${}^t\cdot$ denotes the transpose of the vector under consideration. Henceforth we denote the eigenprojection and the nilpotent of A^m for λ_i by P_i^m and D_i^m:

$$\begin{cases} P_i^m = P_{\lambda_i}(A^m) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\Gamma_{\lambda_i}} (z - A^m)^{-1} dz \\ D_i^m = D_{\lambda_i}(A^m) = \dfrac{1}{2\pi\sqrt{-1}} \displaystyle\int_{\Gamma_{\lambda_i}} (z - \lambda_i)(z - A^m)^{-1} dz , \end{cases} \tag{2.24}$$

where Γ_{λ_i} is a sufficiently small circle around λ_i $(i \geqq 1)$. We denote the eigenprojection and the nilpotent of A_D^m for λ_{Di} $(i \geqq 1)$ respectively by the same notations P_i^m and D_i^m, for simplicity. Since λ_i is an eigenvalue of A^m with finite multiplicity m_i, the point $\lambda_i \in \mathbf{C}$ is a pole of the function $(z - A^m)^{-1}$ of order $k_i + 1$ and

$$(D_i^m)^{k_i+1} = 0 \qquad (i \geqq 1) . \tag{2.25}$$

Moreover we put $(D_i^m)^0 = P_i^m$.

For the determination of coefficients in terms of N-pairs of general initial values, we have

THEOREM 2. *Let* $a_k^m \in \mathscr{D}(A^m)$, $b_k^m \in \mathscr{D}((A^m + \alpha)^{1/2})$ $(1 \leqq k \leqq N)$ *be given and let* $t_0 \geqq 0$. *We set*

$$\begin{aligned} \Omega_{ID}(t_0) = \Big\{ x \in \Omega;\ \dim \mathrm{Span}\Big\{ &\vec{\omega}((D_i^m)^j u(t_0; A^m, a_k^m, b_k^m); x) , \\ &\vec{\omega}\Big((D_i^m)^j \frac{du}{dt}(t_0; A^m, a_k^m, b_k^m); x\Big);\ i \geqq 1, 0 \leqq j \leqq k_i, 1 \leqq k \leqq N \Big\} = q \Big\} . \end{aligned} \tag{2.26}$$

Then:

(i) *$\Omega_{ID}(t_0)$ is well-defined and open for each $t_0 \geqq 0$. Moreover $\Omega_{ID}(t_0)$ is invariant with respect to t_0, that is,* (2.27) $$\Omega_{ID}(t_0)=\Omega_{ID}=\{x\in\Omega;\ \dim\mathrm{Span}\{\vec{\omega}((D_i^m)^j a_k^m;\ x),\ \vec{\omega}((D_i^m)^j b_k^m;\ x);\ i\geqq 1,\ 0\leqq j\leqq k_i,\ 1\leqq k\leqq N\}=q\}\quad\text{for}\quad t_0\geqq 0\,.$$

(ii) *The equalities* (2.16) *imply*

(2.28) $$p_{ij}(x)=p_{ij}^m(x),\ q_j(x)=q_j^m(x),\ r(x)=r^m(x)\qquad (x\in\overline{\Omega_{ID}},\ 1\leqq i,\ j\leqq n)\,.$$

In the case where a model is given by A_D^m, *similar results hold.*

For the determination of boundary conditions, we have

THEOREM 3. *Let us assume that* $\partial\Omega\cap\overline{\Omega_{ID}}$ *contains an arc.*

(i) *Let* $a_k^m\in\mathscr{D}(A^m)$, $b_k^m\in\mathscr{D}((A^m+\alpha)^{1/2})$ $(1\leqq k\leqq N)$. *Then the equalities* (2.16) *imply that* A *is an operator with a boundary condition* (2.6) *and*

(2.29) $$\beta(x)=\beta^m(x)\qquad (x\in\Gamma:\ \textit{any arc in}\ \partial\Omega\cap\overline{\Omega_{ID}})\,.$$

In particular, in the case of $\overline{\Omega_{ID}}=\bar{\Omega}$, *we get* (2.18).

(ii) *Let* $a_k^m\in\mathscr{D}(A_D^m)$, $b_k^m\in\mathscr{D}((A_D^m+\alpha)^{1/2})$ $(1\leqq k\leqq N)$. *Then the equalities* (2.16) *imply that* A *is an operator with the boundary condition* $(2.6)_D$.

For given $a_k^m\in\mathscr{D}(A^m)$, $b_k^m\in\mathscr{D}((A^m+\alpha)^{1/2})$ $(1\leqq k\leqq N)$, the set Ω_{ID} defined by the right hand side of (2.27) is an "identifiability domain" where the coefficients of $\mathscr{A}$ can be identified, but in $\Omega\backslash\Omega_{ID}$, it is not certain whether we can identify those. Therefore if $\overline{\Omega_{ID}}=\bar{\Omega}$, then (2.16) or $(2.16)_D$ implies (2.17) and (2.19). Moreover, by Theorem 2 (i), this identifiability domain is invariant with respect to time.

For $\mathscr{A}^m$ with real analytic coefficients, we have

COROLLARY 1. *Let us assume that all the coefficients* p_{ij}^m, q_{ij}^m, r $(1\leqq i,\ j\leqq n)$ *in* $\mathscr{A}^m$ *are real analytic. Then we have either* $\overline{\Omega_{ID}}=\bar{\Omega}$ *or* $\Omega_{ID}=\varnothing$.

Here we do not assume the analyticity of unknown coefficients p_{ij}, q_j, r $(1\leqq i,\ j\leqq n)$. Corollory 1 means that we can determine coefficients completely (i.e. on the whole domain) or not at all (i.e. $\Omega_{ID}=\varnothing$).

REMARK 4. Theorem 2 corresponds to Proposition 1 in [14] where we give a result on the identification problem for parabolic equations in terms of N-numbers of general initial values.

Similarly to the identification problem for the parabolic equations ([14]), we will consider:
What is the condition on a_k^m, b_k^m $(1\leqq k\leqq N)$ assuring $\overline{\Omega_{ID}}=\bar{\Omega}$? In order that $\overline{\Omega_{ID}}=\bar{\Omega}$, it is necessary that

(2.30) $$\dim\mathrm{Span}\{(D_i^m)^j a_k^m,\ (D_i^m)^j b_k^m;\ i\geqq 1\ 0\leqq j\leqq k_i,\ 1\leqq k\leqq N\}\geqq q\,.$$

In fact, if (2.30) eoes not hold, then $\Omega_{ID}=\varnothing$ follows from the definition of Ω_{ID}.

Conversely, does the condition (2.30) imply $\bar{\Omega}=\overline{\Omega_{ID}}$? As in the the parabolic case, for the spatial dimension $n\geqq 2$, the answer is negative. That is, the set Ω_{ID} may be empty even if (2.30) is fulfilled. Actually we can give the following example:

$$\begin{cases} n=2 \text{ (i.e. } q=6), \ \Omega=\{(x_1, x_2);\ 0<x_1, x_2<1\}\ , \\ -A^m=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2} \text{ with the Dirichlet boundary condition} \\ \text{and} \\ a_1^m(x_1, x_2)=\sin \pi x_1 \sin 18\pi x_2,\ b_1^m(x_1\ x_2)=\sin 6\pi x_1 \sin 17\pi x_2 \\ a_2^m(x_1, x_2)=\sin 10\pi x_1 \sin 15\pi x_2,\ b_2^m(x_1, x_2)=\sin 15\pi x_1 \sin 10\pi x_2 \\ a_3^m(x_1, x_2)=\sin 17\pi x_1 \sin 6\pi x_2,\ b_3^m(x_1, x_2)=\sin 18\pi x_1 \sin \pi x_2\ . \end{cases}$$

Then $\sigma(A^m)=\{\lambda_i\}_{i\geqq 1}=\{(m^2+n^2)\pi^2\}_{m,n\geqq 1}$ and let i_0 be the number such that $\lambda_{i_0}=325\pi^2$, and let us set $N=3$. Then we have $\dim \mathrm{Span}\{(D_i^m)^j a_k^m,\ (D_i^m)^j b_k^m;\ i\geqq 1, j\geqq 0, 1\leqq k\leqq 3\}=\dim \mathrm{Span}\{P_{i_0}^m a_k^m,\ P_{k_0}^m b_k^m;\ 1\leqq k\leqq 3\}=\dim \mathrm{Span}\{a_k^m,\ b_k^m;\ 1\leqq k\leqq 3\}=6\ (=q)$, while by direct computations, we can see $\dim \mathrm{Span}\{\vec{\omega}((D_i^m)^j a_k^m;\ x),\ \vec{\omega}((D_i^m)^j b_k^m;\ x);\ i\geqq 1,\ 0\leqq j\leqq k_i,\ 1\leqq k\leqq 3\}\leqq 5<q$, that is $\Omega_{ID}=\emptyset$.

Furthermore, Theorem 2 (ii) can be rewritten as follows.

$$\overline{\Omega_{ID}}\subset\{x\in\bar{\Omega};\ \text{the equalities (2.16) imply } p_{ij}(x)=p_{ij}^m(x)\ ,\ q_j(x)=q_j^m(x),\ r(x)=r^m(x)\quad (1\leqq i, j\leqq n)\}\ . \tag{2.31}$$

In general, it is not certain that the equality holds in the inclusion in (2.31).

If $n=1$, namely, $\Omega\subset\mathbf{R}$ is an arbitrary finite interval, then we can prove that $\overline{\Omega_{ID}}=\bar{\Omega}$ is equivalent to the condition (2.30) and that the both hand sides in (2.31) coincide.

THEOREM 4. *Let $\Omega=(0, 1)$ and let us define operators $-A^m$, $-A_D^m$ and $-A$ respectively by*

$$\begin{cases} (-A^m u)(x)=p^m(x)\dfrac{d^2u(x)}{dx^2}+q^m(x)\dfrac{du(x)}{dx}+r^m(x)u(x) \quad (0<x<1) \\ \mathscr{D}(A^m)=\left\{u\in H^2(0, 1);\ \dfrac{du}{dx}(0)+\beta_0^m u(0)=\dfrac{du}{dx}(1)+\beta_1^m u(1)=0\right\}\ , \end{cases} \tag{2.32}$$

$$\begin{cases} (-A_D^m u)(x)=p^m(x)\dfrac{d^2u(x)}{dx^2}+q^m(x)\dfrac{du(x)}{dx}+r^m(x)u(x) \quad (0<x<1) \\ \mathscr{D}(A_D^m)=\{u\in H^2(0, 1);\ u(0)=u(1)=0\} \end{cases} \tag{2.33}$$

and

$$\begin{cases} (-Av)(x)=p(x)\dfrac{d^2v(x)}{dx^2}+q(x)\dfrac{dv(x)}{dx}+r(x)v(x) \quad (0<x<1) \\ \mathscr{D}(A)=\left\{v\in H^2(0, 1);\ \dfrac{dv}{dx}(0)+\beta_0 v(0)=\dfrac{dv}{dx}(1)+\beta_1 v(1)=0\right\}\ , \\ \text{or} \\ \mathscr{D}(A)=\{v\in H^2(0, 1);\ v(0)=v(1)=0\}\ . \end{cases} \tag{2.34}$$

Here p^m, q^m, r^m, p, q, r *are real-valued smooth functions ond* $p^m(x)>0$, $p(x)>0$ $(0\leqq x\leqq 1)$, *and* β_0^m, β_1^m, β_0, β_1 *are real constants.*

(I) *Let* $a_k^m\in\mathscr{D}(A^m)$, $b_k^m\in\mathscr{D}((A^m+\alpha)^{1/2})$ $(1\leqq k\leqq N)$ *be given and* $t_0\geqq 0$. *Then*

(2.35) $$\overline{\Omega_{ID}}=\{x\in[0,1];\ \textit{the equalities (2.16) imply } p(x)=p^m(x),\ q(x)=q^m(x),\ r(x)=r^m(x) \textit{ and } a_k(x)=a_k^m(x),\ b_k(x)=b_k^m(x)\quad (1\leqq k\leqq N)\}\,.$$

Moreover

(a) $\Omega_{ID}=\emptyset$ *if and only if*

(2.36) $$\dim \mathrm{Span}\{P_i^m a_k^m,\ P_i^m b_k^m;\ i\geqq 1,\ 1\leqq k\leqq N\}\leqq 2\,.$$

(b) $\overline{\Omega_{ID}}=\bar{\Omega}$ *if and only if*

(2.37) $$\dim \mathrm{Span}\{P_i^m a_k^m,\ P_i^m b_k^m;\ i\geqq 1,\ 1\leqq k\leqq N\}\geqq 3\,.$$

That is, in order that the equalities (2.16) *imply*

(2.38) $$\begin{gathered} p(x)=p^m(x),\ q(x)=q^m(x),\ r(x)=r^m(x) \\ a_k(x)=a_k^m(x),\ b_k(x)=b_k^m(x)\quad (0\leqq x\leqq 1,\ 1\leqq k\leqq N) \end{gathered}$$

and

(2.39) $$\beta_0=\beta_0^m,\ \beta_1=\beta_1^m\,,$$

the condition (2.37) *is necessary and sufficient.*

(II) *Let* $a_k^m\in H^2(0,1)\cap H_0^1(0,1)$, $b_k^m\in H_0^1(0,1)$ $(1\leqq k\leqq N)$ *be given and* $t_0\geqq 0$. *Then the relation* (2.35) *and the statements* (a), (b) *in* (I) *hold true. Moreover* (2.37) *implies* $\mathscr{D}(A)=H^2(0,1)\cap H_0^1(0,1)$.

REMARK 5. Let us consider one-dimensional wave equations:

(2.40) $$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2}=\dfrac{\partial^2 u(x,t)}{\partial x^2}-r^m(x)u(x,t) & (0<x<1,\ t>0) \\ u(x,0)=a^m(x),\ \dfrac{\partial u}{\partial t}(x,0)=b^m(x) & (0\leqq x\leqq 1) \\ \dfrac{\partial u}{\partial x}(0,t)+\beta_0^m u(0,t)=\dfrac{\partial u}{\partial x}(1,t)+\beta_1^m u(1,t)=0 & (t>0) \end{cases}$$

and

(2.41) $$\begin{cases} \dfrac{\partial^2 v(x,t)}{\partial t^2}=\dfrac{\partial^2 v(x,t)}{\partial x^2}-r(x)v(x,t) & (0<x<1,\ t>0) \\ v(x,0)=a(x),\ \dfrac{\partial v}{\partial t}(x,0)=b(x) & (0\leqq x\leqq 1) \\ \dfrac{\partial v}{\partial x}(0,t)+\beta_0 v(0,t)=\dfrac{\partial v}{\partial x}(1,t)+\beta_1 v(1,t)=0 & (t>0)\,. \end{cases}$$

Let $-A^m$ denote the realization in $L^2(0,1)$ of the operator $\frac{d^2}{dx^2}-r^m(x)$ with the boundary condition in (2.40). Let $\sigma(A^m)=\{\lambda_i\}_{i\geqq 1}$ and let $\{\phi_i^m\}_{i\geqq 1}$ be a system of all eigenfunctions of A^m. By Suzuki [17], the following result is proved:

The "observation condition"

$$v(0,t)=u(0,t),\ v(1,t)=u(1,t)\quad (t\geqq t_0) \tag{2.42}$$

implies

$$r(x)=r^m(x),\ a(x)=a^m(x),\ b(x)=b^m(x)\quad (0\leqq x\leqq 1),\quad \beta_0=\beta_0^m,\ \beta_1=\beta_1^m, \tag{2.43}$$

provided that the initial data $(a^m, b^m)\in H^1(0,1)\times L^2(0,1)$ satisfies

$$\int_0^1 a^m(x)\phi_i^m(x)dx\neq 0,\ \int_0^1 b^m(x)\phi_i^m(x)dx\neq 0\quad \text{for each}\quad i\geqq 1\,. \tag{2.44}$$

The observation condition (2.42) in [17] is restricted to the end points $x=0, 1$, and the condition (2.44) is necessary for the identification. Furthermore "only one" coefficient can be identified. If (a^m, b^m) satisfies (2.44), then the linear space $\mathrm{Span}\{P_i^m a^m, P_i^m b^m;\ i\geqq 1\}$ is infinitely dimensional. Thus the condition required in [17] is much stricter than (2.37).

In our problem, we impose the stronger observation (2.16) or $(2.16)_D$ for the identification, so that on the much weaker assumption (2.37) on the initial data, we can completely identify "all three" coefficients p, q, r, initial data a, b and all boundary conditions. For $n=1$, we have $D_i^m a_k=0$ $(i\geqq 1, 1\leqq k\leqq N)$, as is well-known (e.g. Lemma 1 in [14]), so that the condition (2.30) is actually equivalent to (2.37).

By a way similar to the one in [14], we can derive the following corollary from Theorem 4.

COROMLARY 2. (I) *In order that* (2.16) *implies* (2.38) *and* (2.39), *it is sufficient that*

$$\dim \mathrm{Span}\{a_k^m, b_k^m;\ 1\leqq k\leqq N\}\geqq 3\,. \tag{2.45}$$

(II) *In order that* $(2.16)_D$ *implies* (2.38) *and* $\mathscr{D}(A)=H^2(0,1)\cap H_0^1(0,1)$, *the condition* (2.45) *is sufficient.*

REMARK 6. Let $t_0=0$ and a_k^m, b_k^m $(1\leqq k\leqq N)$ be all sufficiently smooth and satisfy appropriate compatibility conditions. Then we can let t tend to 0 in equalities $\frac{d^j u}{dt^j}(t;\ A, a_k^m, b_k^m)=\frac{d^j u}{dt^j}(t;\ A^m, a_k^m, b_k^m)$ in $L^2(0,1)$ $(1\leqq k\leqq N,\ j=2, 3)$, so that we can derive $Aa_k^m=A^m a_k^m$, $Ab_k^m=A^m b_k^m$ $(1\leqq k\leqq N)$. Thus, not using the results in §3, we can directly prove this corollary only by discussions on the Wronskian. However, in this paper, on the smoothness of a_k^m, b_k^m, we assume only that $a_k^m\in\mathscr{D}(A^m)$ and $b_k^m\in\mathscr{D}((A^m+\alpha)^{1/2})$ $(1\leqq k\leqq N)$ and do not assume $t_0=0$. For A_D^m, we have a similar remark.

§ 3. Abstract Results

In this section, for the proof of Theorems 1-4 in § 2, we state Theorems A and B from the viewpoint of the operator theory. For those proofs, we can refer to Nakagiri and Yamamoto [15].

Let X be a Hilbert space over C with an inner product $(\cdot, \cdot)$. Let us assume that both $-A^m$ and $-A$ are generators of C_0-semigroups on X.

Henceforth $\sigma(A^m)$ and $\rho(A^m)=\mathrm{C}\backslash\sigma(A^m)$ denote the spectrum and the resolvent set of a closed linear operator A^m in X, respectively. Let λ be an isolated point of $\sigma(A^m)$. Then there exists a circle Γ_λ with center λ such that its interior and Γ_λ contain no points of $\sigma(A^m)$ except for λ. Let us put

$$P_\lambda^m=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\lambda}(z-A^m)^{-1}dz \tag{3.1}$$

and

$$D_\lambda^m=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\lambda}(z-\lambda)(z-A^m)^{-1}dz\,. \tag{3.2}$$

As is known (e.g. Kato [8]), P_λ^m and D_λ^m are bounded linear operators on X satisfying $(P_\lambda^m)^2=P_\lambda^m$, $P_\lambda^m X\subset\mathscr{D}(A^m)$ and $(D_\lambda^m)^j X\subset\mathscr{D}(A^m)$ $(j\geqq 1)$.

THEOREM A ([15]). *Let us assume that λ is an isolated point of $\sigma(A^m)$ satisfying*

(3.3) *there exists an arc C_λ which joints λ and z_0 with* $\mathrm{Re}\, z_0<\inf\{\mathrm{Re}\, z;\ z\in\sigma(A^m)\}$ *and* $C_\lambda\backslash\{\lambda\}\subset\rho(A^m)$.

Let a, $a^m\in X$ and $t_0\geqq 0$ be given. Then a relation

$$e^{-tA}a=e^{-tA^m}a^m \qquad (t\geqq t_0) \tag{3.4}$$

implies

$$(D_\lambda^m)^j a^m\in\mathscr{D}(A^l) \qquad (j\geqq 0,\ l\geqq 1) \tag{3.5}$$

and

$$A(D_\lambda^m)^j a^m=A^m(D_\lambda^m)^j a^m \qquad (j\geqq 0)\,. \tag{3.6}$$

Here and henceforth we set

$$(D_\lambda^m)^0=P_\lambda^m\,. \tag{3.7}$$

In the identification of A, it is essetial to search for a set U where $A_{|U}=A^m{}_{|U}$, and this theorem asserts that such a U is $\mathrm{Span}\{(D_\lambda^m)^j a^m;\ j\geqq 0\}$.

Henceforth $\bar{Z}$ denotes the closure in X of $Z\subset X$, and $\cdot^*$ denotes the adjoint of an operator under consideration.

Next we show Theorem B which asserts the unique determination of an unknown operator under stronger assumptions on A^m.

THEOREM B ([15]). *Let us assume that* $\sigma(A^m)$ *consists entirely of isolated eigenvalues* λ_i $(i\geqq 1)$ *with finite multiplicities and that the system of the generalized eigenvectors of* A^m *is complete in* X, *that is,*

$$\overline{\mathrm{Span}\{P^m_{\lambda_i}X;\ i\geqq 1\}}=X\,. \tag{3.8}$$

Moreover we assume

$$\overline{\mathrm{Span}\{(P^m_{\lambda_i})^*X;\ i\geqq 1\}}=X\,. \tag{3.9}$$

Let $\{\psi^m_{ij}\}_{1\leqq j\leqq d^m_i}$ *be a basis of* $\mathrm{Ker}(\bar\lambda_i-(A^m)^*)$.

(I) *For a generator* $-A$ *of any* C_0*-semigroup, equalities*

$$e^{-tA}a_k=e^{-tA^m}a^m_k \quad (t\geqq t_0,\ 1\leqq k\leqq N) \quad \textit{for any fixed } t_0\geqq 0 \tag{3.10}$$

imply $A=A^m$ *and* $a_k=a^m_k$ $(1\leqq k\leqq N)$ *if and only if for each* $i\geqq 1$, *the condition*

$$\mathrm{rank}((a^m_k,\ \psi^m_{ij}))_{1\leqq j\leqq N,1\leqq j\leqq d^m_i}=d^m_i \tag{3.11}$$

holds.

In this theorem, we note that $\sigma((A^m)^*)=\{\bar\lambda_i\}_{i\geqq 1}$ and $\dim \mathrm{Ker}(\lambda_i-A^m)=\dim \mathrm{Ker}(\bar\lambda_i-(A^m)^*)$ by $\sigma(A^m)=\{\lambda_i\}_{i\geqq 1}$.

REMARK 7. In [15], the following is proved: the rank of the $N\times d^m_i$ matrix $((e^{-tA^m}a^m_k,\ \psi^m_{ij}))_{1\leqq k\leqq N,1\leqq 1\leqq d^m_i}$ is invariant with respect to $t\geqq 0$. That is, in Theorem B, we can replace (3.11) by

$$\mathrm{rank}((e^{-t_0A^m}a^m_k,\ \psi^m_{ij}))_{1\leqq k\leqq N,1\leqq j\leqq d^m_i}=d^m_i\ , \tag{3.11$'$}$$

which is described by quantities at the time t_0 when the observation begins.

Theorem B asserts that the rank conditions (3.11) for each $i\geqq 1$ are necessary and sufficient for the unique determination of A and initial values *within generators of* C_0*-semigroups*, under the assumptions (3.8) and (3.9).

Theorems 2–4 in §2 are proved on the basis of Theorem A, while Theorem 1 is verified by means of Theorem B. These proofs are carried out in §4.

§4. Proof of Theorems 1–4

For the proof, we apply Theorems A and B. To this end, we reduce (2.7), $(2.7)_D$ and (2.9) of the second order with respect to t to equations of the first order. The reduction for (2.7) and (2.9) is done as follows, for example, according to Fattorini [4].

We introduce a Hilbert space $\tilde X$ with an inner product by

$$(4.1)\qquad \begin{cases} \tilde{X}=\mathscr{D}((A^m+\alpha)^{1/2})\times L^2(\Omega) \\ (\tilde{u},\tilde{v})_{\tilde{X}}=((u_1,v_1))+(u_2,v_2)_{L^2} \\ \text{for } \tilde{u}=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\quad \tilde{v}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\in\tilde{X}. \end{cases}$$

Here the space $\mathscr{D}((A^m+\alpha)^{1/2})$ is considered as a Hilbert space with an inner product $((u, v))=(A^m+\alpha)^{1/2}u,\ ((A^m+\alpha)^{1/2}v)_{L^2}$. Furthermore we define operators $\tilde{A}^m$ and $\tilde{A}$ in $\tilde{X}$ by

$$(4.2)\qquad \tilde{A}^m=\begin{pmatrix} 0 & -1 \\ A^m & 0 \end{pmatrix},\ \mathscr{D}(\tilde{A}^m)=\left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix};\ u_1\in\mathscr{D}(A^m),\ u_2\in\mathscr{D}((A^m+\alpha)^{1/2})\right\}$$

and

$$(4.3)\qquad \tilde{A}=\begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix},\ \mathscr{D}(\tilde{A})=\left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix};\ u_1\in\mathscr{D}(A),\ u_2\in\mathscr{D}((A+\alpha)^{1/2})\right\}.$$

Then, both $-\tilde{A}^m$ and $-A$ generate C_0-semigroups in $\tilde{X}$, and moreover the problem (2.7) with (2.8) and the one (2.9) with (2.10) are equivalent to the following (4.4) and (4.5), respectively:

$$(4.4)\qquad \frac{d\tilde{u}(t)}{dt}+\tilde{A}^m\tilde{u}(t)=0\ (t\geqq 0)\ \text{ and }\ \tilde{u}(0)=\begin{pmatrix} a^m \\ b^m \end{pmatrix},$$
$$\text{where } \tilde{u}(t)=\begin{pmatrix} u(t) \\ \dfrac{du(t)}{dt} \end{pmatrix}\quad (t\geqq 0).$$

$$(4.5)\qquad \frac{d\tilde{v}(t)}{dt}+\tilde{A}\tilde{v}(t)=0\ (t\geqq 0)\ \text{ and }\ \tilde{v}(0)=\begin{pmatrix} a \\ b \end{pmatrix},$$
$$\text{where } \tilde{v}(t)=\begin{pmatrix} v(t) \\ \dfrac{dv(t)}{dt} \end{pmatrix}\quad (t\geqq 0).$$

We can similarly reduce the problem $(2.7)_D$ with (2.8) to an equation of the first order.

In applying Theorems A and B, it is necessary to calculate $(\tilde{D}_\lambda^m)^j$ $(j\geqq 0)$, where

$$\begin{cases} \tilde{P}_\lambda^m=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\lambda}(z-\tilde{A}^m)^{-1}dz \\ \tilde{D}_\lambda^m=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\lambda}(z-\lambda)(z-\tilde{A}^m)^{-1}dz \end{cases}$$

($\Gamma_\lambda\subset\rho(\tilde{A}^m)$: a sufficiently small circle around λ). To this end, we show the following lemma, whose proof is given at the end of this section.

LEMMA 1. *Let L be a densely-defined closed linear operator in a Banach space Y over $\mathbf{C}$ and let us define an operator $\tilde{L}$ in $Y\times Y$ by*

$$\tilde{L}=\begin{pmatrix}0 & -1\\ L & 0\end{pmatrix},\quad \mathscr{D}(\tilde{L})=\mathscr{D}(L)\times Y\,. \tag{4.7}$$

For isolated points λ and μ of $\sigma(L)$ and $\sigma(\tilde{L})$, we set

$$\begin{cases} P_\lambda(L)=P_\lambda=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\lambda}(z-L)^{-1}dz\\ D_\lambda(L)=D_\lambda=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\lambda}(z-\lambda)(z-L)^{-1}dz\end{cases} \tag{4.8}$$

and

$$\begin{cases} \tilde{P}_\mu(\tilde{L})=\tilde{P}_\mu=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\mu}(z-\tilde{L})^{-1}dz\\ \tilde{D}_\mu(\tilde{L})=\tilde{D}_\mu=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\mu}(z-\mu)(z-\tilde{L})^{-1}dz\,.\end{cases} \tag{4.9}$$

Here Γ_λ and Γ_μ are sufficiently small circles around λ and μ, respectively. Then for an isolated point $-\mu^2$ of $\sigma(L)$, the following relations hold:

$$\pm\mu \text{ are isolated points of } \sigma(\tilde{L})\,. \tag{4.10}$$

$$\tilde{P}_\mu=\begin{pmatrix}Q_{1,\mu} & Q_{2,\mu}\\ Q_{3,\mu} & Q_{1,\mu}\end{pmatrix}. \tag{4.11}$$

$$\tilde{P}_{-\mu}=\begin{pmatrix}Q_{1,\mu} & -Q_{2,\mu}\\ -Q_{3,\mu} & Q_{1,\mu}\end{pmatrix} \tag{4.12}$$

$$(\tilde{D}_\mu)^j=\begin{pmatrix}R^{(j)}_{1,\mu} & R^{(j)}_{2,\mu}\\ R^{(j)}_{3,\mu} & R^{(j)}_{1,\mu}\end{pmatrix}\quad (j\geqq 1)\,. \tag{4.13}$$

$$(\tilde{D}_{-\mu})^j=\begin{pmatrix}(-1)^jR^{(j)}_{1,\mu} & (-1)^{j+1}R^{(j)}_{2,\mu}\\ (-1)^{j+1}R^{(j)}_{3,\mu} & (-1)^jR^{(j)}_{1,\mu}\end{pmatrix}\quad (j\geqq 1)\,. \tag{4.14}$$

Here we set

$$(D_{-\mu^2})^0=P_{-\mu^2}\,, \tag{4.15}$$

$$Q_{1,\mu}=\begin{cases}\dfrac{1}{2}P_{-\mu^2}\,, & \text{if } \mu\neq 0\\ P_0\,, & \text{if } \mu=0\,.\end{cases} \tag{4.16}$$

$$Q_{2,\mu}=\begin{cases}-(2\mu)^{-1}P_{-\mu^2}-\displaystyle\sum_{i=1}^{\infty}\frac{(2i)!\,(2\mu)^{-2i-1}}{(i!)^2}(D_{-\mu^2})^i\,, & \text{if } \mu\neq 0\\ 0\,, & \text{if } \mu=0\,.\end{cases} \tag{4.17}$$

$$Q_{3,\mu}=\begin{cases} -\dfrac{\mu}{2}P_{-\mu^2}+\sum_{i=1}^{\infty}\dfrac{(2i-2)!(2\mu)^{-2i+1}}{2i!\,(i-1)!}(D_{-\mu^2})^i\,, & \text{if } \mu\neq 0 \\ 0\,, & \text{if } \mu=0\,. \end{cases} \tag{4.18}$$

$$R_{1,\mu}^{(j)}=\begin{cases} \sum_{i=j}^{\infty}\dfrac{(-1)^j j(2i-j-1)!(2\mu)^{-2i+j}}{2(i-j)!\,i!}(D_{-\mu^2})^i\,, & \text{if } \mu\neq 0 \\ (-1)^{j/2}D_0^{j/2} & \text{if } \mu=0 \text{ and } j \text{ is even} \\ 0\,, & \text{if } \mu=0 \text{ and } j \text{ is odd}\,. \end{cases} \tag{4.19}$$

$$R_{2,\mu}^{(j)}=\begin{cases} \sum_{i=j}^{\infty}\dfrac{(-1)^{j+1}(2i-j)!(2\mu)^{-2i+j-1}}{(i-j)!\,i!}(D_{-\mu^2})^i\,, & \text{if } \mu\neq 0 \\ 0\,, & \text{if } \mu=0 \text{ and } j \text{ is even} \\ (-1)^{(j+1)/2}D_0^{(j-1)/2}\,, & \text{if } \mu=0 \text{ and } j \text{ is odd}\,. \end{cases} \tag{4.20}$$

$$R_{3,\mu}^{(j)}=\begin{cases} \sum_{i=j}^{\infty}\dfrac{(-1)^{j+1}(j^2+j-2i)(2i-j-2)!(2\mu)^{-2i+j+1}}{4i!\,(i-j)!}(D_{-\mu^2})^i\,, & \text{if } \mu\neq 0 \\ 0\,, & \text{if } \mu=0 \text{ and } j \text{ is even} \\ (-1)^{(j-1)/2}D_0^{(j+1)/2}\,, & \text{if } \mu=0 \text{ and } j \text{ is odd}\,. \end{cases} \tag{4.21}$$

Proof of Theorem 1. We will prove this theorem by applying Theorem B in §3 to reduced equations (4.4) and (4.5) of the first order. It is sufficient to prove the part (I). Without loss of generality, we may assume that $0\notin\sigma(A^m)\equiv\{\lambda_i\}_{i\geq 1}$. We see by Lemma 1that $\pm\sqrt{-\lambda_i}$ $(i\geq 1)$ are eigenvalues of $\tilde{A}^m$. Then we define the eigenprojection for $\pm\sqrt{-\lambda_i}$ by a way similar to (4.9):

$$\tilde{P}_{\pm i}^m=\tilde{P}_{\pm\sqrt{-\lambda_i}}(\tilde{A}^m)=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_{\pm\sqrt{-\lambda_i}}}(z-\tilde{A}^m)^{-1}dz\,, \tag{4.22}$$

where $\Gamma_{\sqrt{-\lambda_i}}$ and $\Gamma_{-\sqrt{-\lambda_i}}$ denote sufficiently small circles around $\sqrt{-\lambda_i}$ and $-\sqrt{-\lambda_i}$, respectively.

Since the equalities (2.16) are equivalent to

$$e^{-t\tilde{A}}\begin{pmatrix} a_k \\ b_k \end{pmatrix}=e^{-t\tilde{A}^m}\begin{pmatrix} a_k^m \\ b_k^m \end{pmatrix} \text{ in } \tilde{X} \quad (t\geq t_0,\ 1\leq k\leq N)\,,$$

in view of Theorem B, for the proof, we have only to verify

$$\overline{\mathrm{Span}\{\tilde{P}_{\pm i}^m\tilde{X};\ i\geq 1\}}=\tilde{X}\,. \tag{4.23}$$

$$\overline{\mathrm{Span}\{(\tilde{P}_{\pm i}^m)^*\tilde{X};\ i\geq 1\}}=\tilde{X}\,. \tag{4.24}$$

(4.25) The rank condition (3.11) for $\tilde{A}^m$ is equivalent to (2.20) .

Verification of (4.23). Assuming that $\binom{u}{v}\in\tilde{X}$ satisfies

(4.26)
$$\left(\tilde{P}^m_{\pm i}\begin{pmatrix} x \\ y \end{pmatrix},\ \begin{pmatrix} u \\ v \end{pmatrix}\right)_{\tilde{X}}=0$$

for each $\begin{pmatrix} x \\ y \end{pmatrix}\in\tilde{X}$ and $i\geqq 1$, we have only to prove $u=v=0$. Firstly let us set

$$Q_{1,i}=\frac{1}{2}P^m_i ,$$

$$Q_{2,i}=-\frac{1}{2\sqrt{-\lambda_i}}P^m_i-\sum_{l=1}^{k_i}\frac{(2l)!\,(2\sqrt{-\lambda_i})^{-2l-1}}{(l!)^2}(D^m_i)^l$$

and

$$Q_{3,i}=-\frac{\sqrt{-\lambda_i}}{2}P^m_i+\sum_{l=1}^{k_i}\frac{(2l-2)!\,(2\sqrt{-\lambda_i})^{-2l+1}}{2l!\,(l-1)!}(D^m_i)^l .$$

The relations (4.26) imply $((Q_{1,i}x+Q_{2,i}y,\ u))+(Q_{3,i}x+Q_{1,i}y,\ v)_{L^2}=0$ and $((Q_{1,i}x-Q_{2,i}y,\ u))+(-Q_{3,i}x+Q_{1,i}y,\ v)_{L^2}=0$, so that $((Q_{1,i}x,\ u))=(Q_{1,i}y,\ v)_{L^2}=0$, namely, by Lemma 1, $(P^m_i(A^m+\alpha)^{1/2}x,\ (A^m+\alpha)^{1/2}u)_{L^2}=(P^m_i y,\ v)_{L^2}=0$ for each $x\in\mathscr{D}((A^m+\alpha)^{1/2})$ and $y\in X$. By Theorems 15.4 and 16.5 in Agmon [2], we have $\overline{\mathrm{Span}\{P^m_i X;\ i\geqq 1\}}=X$. Therefore $(A^m+\alpha)^{1/2}u=v=0$ follows, which verifies (4.23).

Verification of (4.24). We have $P^m_i(A^m+\alpha)^{1/2}=(A^m+\alpha)^{1/2}P^m_i$, $(D^m_i)^j(A^m+\alpha)^{1/2}=(A^m+\alpha)^{1/2}(D^m_i)^j$ ($x\in\mathscr{D}((A^m+\alpha)^{1/2})$, $j\geqq 1$) and $(A^m+\alpha)^{1/2}P^m_i$, $(A^m+\alpha)^{1/2}(D^m_i)^j$ ($j\geqq 1$) are bounded, so that we can see

LEMMA 2. *The adjoint of $\tilde{P}^m_{\pm i}$ in $\tilde{X}$ is given by:*

$$(\tilde{P}^m_{\pm i})^*=\begin{pmatrix} Q_{1,i} & \pm Q_{2,i} \\ \pm Q_{3,i} & Q_{1,i}\end{pmatrix}^*$$
$$=\begin{pmatrix} (A^m+\alpha)^{1/2}Q^*_{1,i}(A^m+\alpha)^{1/2} & \pm(A^m+\alpha)^{-1/2}((A^m)^*+\alpha)^{1/2}Q^*_{3,i} \\ \pm((A^m+\alpha)^{1/2}Q_{2,i})^*(A^m+\alpha)^{1/2} & Q^*_{1,i}\end{pmatrix}$$

In the last expression, $\cdot^*$ denotes the adjoints of operators in X.

Next we will complete the verification of (4.24). Assuming that $\begin{pmatrix} u \\ v \end{pmatrix}\in\tilde{X}$ satisfies $\left(\begin{pmatrix} u \\ v \end{pmatrix},\ (\tilde{P}^m_{\pm i})^*\begin{pmatrix} x \\ y \end{pmatrix}\right)_{\tilde{X}}=0$ for each $\begin{pmatrix} x \\ y \end{pmatrix}\in\tilde{X}$ and $i\geqq 1$, we have only to prove $u=v=0$. Similarly to (4.23), by Lemma 2 we can get $((A^m+\alpha)^{1/2}u,\ (P^m_i)^*(A^m+\alpha)^{1/2}x)_{L^2}=0$ and $(v,\ (P^m_i)^*y)_{L^2}=0$ for each $i\geqq 1$ and $x\in\mathscr{D}((A^m+\alpha)^{1/2})$, $y\in X$. By Theorems 15.4 and 16.5 in [2], also system of the generalized eigenvectors of $(A^m)^*$ is complete in $L^2(\Omega)$, so that $(A^m+\alpha)^{1/2}u=v=0$, namely, $u=v=0$. Thus the verification of (4.24) is complete.

Verification of (2.45). We have to give eigenvectors of $(\tilde{A}^m)^*$, the adjoint operator of $\tilde{A}^m$ in the Hilbert space $\tilde{X}$. Since $\sigma(\tilde{A}^m)=\{\pm(-\lambda_i)^{1/2}\}_{i\geqq 1}$, we have

(4.27)
$$\sigma((\tilde{A}^m)^*)=\{\pm(-\bar{\lambda}_i)^{1/2}\}_{i\geqq 1}$$

(e.g. [8]). We will prove

LEMMA 3. *A basis of* $\mathrm{Ker}(\pm(-\overline{\lambda_i})^{1/2}-(\tilde{A}^m)^*)$ *is given by*

$$\tilde{\phi}^m_{\pm ij}=\begin{pmatrix}\mp(-\overline{\lambda_i})^{1/2}(A^m+\alpha)^{-1/2}((A^m)^*+\alpha)^{-1/2}\phi^m_{ij}\\ \phi^m_{ij}\end{pmatrix}\quad(1\leqq j\leqq d^m_i)\,. \tag{4.28}$$

Proof of Lemma 3. As is easily seen, $\{\tilde{\phi}^m_{\pm ij}\}_{1\leqq j\leqq d^m_i}$ is linearly independent in $\tilde{X}$. As is easily checked, $\dim\mathrm{Ker}(\pm(-\lambda_i)^{1/2}-\tilde{A}^m)=d^m_i$, and hence, $\dim\mathrm{Ker}(\pm(-\overline{\lambda_i})^{1/2}-\tilde{A}^m)^*)=d^m_i$. Thus we have only to show that each $\tilde{\phi}^m_{\pm ij}$ belongs to $\mathrm{Ker}(\pm(-\overline{\lambda_i})^{1/2}-(\tilde{A}^m)^*)$. To this end, it is sufficient to prove that

$$(\tilde{A}^m\tilde{u},\ \tilde{\phi}^m_{\pm ij})_{\tilde{X}}=(\tilde{u},\ \pm(-\overline{\lambda_i})^{1/2}\tilde{\phi}^m_{\pm ij})_{\tilde{X}}\quad\text{for each}\quad \tilde{u}=\begin{pmatrix}u_1\\ u_2\end{pmatrix}\in\mathscr{D}(\tilde{A}^m)\,.$$

This is proved as follows; we have

$$\begin{aligned}(\tilde{A}^m\tilde{u},\ \tilde{\phi}^m_{\pm ij})_{\tilde{X}}&=\left(\begin{pmatrix}-u_2\\ A^m u_1\end{pmatrix},\ \begin{pmatrix}\mp(-\overline{\lambda_i})^{1/2}(A^m+\alpha)^{-1/2}((A^m)^*+\alpha)^{-1/2}\phi^m_{ij}\\ \phi^m_{ij}\end{pmatrix}\right)_{\tilde{X}}\\ &=((A^m+\alpha)^{1/2}u_2,\ \pm(-\overline{\lambda_i})^{1/2}((A^m)^*+\alpha)^{1/2}\phi^m_{ij})_{L^2}+(A^m u_1,\ \phi^m_{ij})_{L^2}\\ &=((A^m+\alpha)^{-1/2}(A^m+\alpha)^{1/2}u_2,\ \pm(-\overline{\lambda_i})^{1/2}\phi^m_{ij})_{L^2}+(u_1\ (A^m)^*\phi^m_{ij})_{L^2}\\ &=\overline{\pm(-\overline{\lambda_i})^{1/2}}\\ &\quad\times((A^m+\alpha)^{1/2}u_1,\ \mp(A^m+\alpha)^{1/2}(-\overline{\lambda_i})^{1/2}(A^m+\alpha)^{-1/2}((A^m)^*+\alpha)^{-1/2}\phi^m_{ij})_{L^2}\\ &\quad+(\overline{\pm(-\overline{\lambda_i})^{1/2}})(u_2\ \phi^m_{ij})_{L^2}\\ &\qquad\qquad(\text{by }(A^m)^*\phi^m_{ij}=\overline{\lambda_i}\phi^m_{ij}\text{ and }u_1\in\mathscr{D}(A^m)\subset\mathscr{D}((A^m+\alpha)^{1/2}))\\ &=\pm\overline{(-\lambda_i)}^{1/2}(\tilde{u},\ \tilde{\phi}^m_{\pm ij})_{\tilde{X}}\,.\end{aligned}$$

This prove this lemma.

Now we return to the calculation of the left hand side of (3.11). We have

$$\begin{aligned}&\left(\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix},\ \tilde{\phi}^m_{\pm ij}\right)_{\tilde{X}}\\ &\quad=((A^m+\alpha)^{1/2}a^m_k,\ (A^m+\alpha)^{1/2}(\mp(-\overline{\lambda_i})^{1/2})(A^m+\alpha)^{-1/2}((A^m)^*+\alpha)^{-1/2}\phi^m_{ij})_{L^2}\\ &\quad\quad+(b^m_k,\ \phi^m_{ij})_{L^2}\qquad\qquad(\text{by }(4.28))\\ &\quad=\mp(-\lambda_i)^{1/2}(a^m_k,\ \phi^m_{ij})_{L^2}+(b^m_k,\ \phi^m_{ij})_{L^2}\,.\end{aligned}$$

This verifies (4.25).

Proof of Theorem 2.

Proof of (*i*). Let $a\in L^2(\Omega)$ be given. Since $(D^m_i)^j a\in P^m_i X$ $(j\geqq 0)$ and $P^m_i X\subset\mathscr{D}((A^m)^l)$ for each $l\geqq 1$, we get $(D^m_i)^j a\in\mathscr{D}((A^m)^l)\subset H^{2l}(\Omega)$ for each $l\geqq 1$, so that the Sobolev imbedding theorem (e.g. Adams [1]) implies

$$(D^m_i)^j a\in C^\infty(\Omega)\quad(i\geqq 1,\ 0\leqq j\leqq k_i)\,. \tag{4.29}$$

Therefore $\vec{\omega}((D^n_i)^j a;\ x)$ is well-defined for any $a\in L^2(\Omega)$ and so $\Omega_{ID}(t_0)$ is well-defined for each $t_0\geqq 0$. Now we recall that $k_i\in\mathbf{N}$ satisfies (2.25).

Next we show

$$\text{(4.30)}\qquad \begin{pmatrix} (D_i^m)^j u(t;\, A^m,\, a_k^m,\, b_k^m) \\ (D_i^m)^j \dfrac{du}{dt}(t;\, A^m,\, a_k^m,\, b_k^m) \end{pmatrix} = \begin{pmatrix} u(t;\, A^m,\, (D_i^m)^j a_k^m,\, (D_i^m)^j b_k^m) \\ \dfrac{du}{dt}(t;\, A^m,\, (D_i^m)^j a_k^m,\, (D_i^m)^j b_k^m) \end{pmatrix}$$
$$(t \geqq 0,\ i \geqq 1,\ 0 \leqq j \leqq k_i,\ 1 \leqq k \leqq N)\,.$$

In fact, since $u(\cdot\,;\, A^m, a_k^m, b_k^m) \in C^1([0, \infty);\, \mathscr{D}((A^m+\alpha)^{1/2}))$ satisfies

$$\frac{d^2u(t;\, A^m,\, a_k^m,\, b_k^m)}{dt^2} + A^m u(t;\, A^m,\, a_k^m,\, b_k^m) = 0 \quad (t \geqq 0)\,,$$

$u(0;\, A^m, a_k^m, b_k^m) = a_k^m$ and $\dfrac{du}{dt}(0;\, A^m, a_k^m, b_k^m) = b_k^m$, we have

$$\text{(4.31)}\qquad \begin{cases} \dfrac{d^2}{dt^2}((D_i^m)^j u(t;\, A^m,\, a_k^m,\, b_k^m)) + (A^m P_i^m)((D_i^m)^j u(t;\, A^m,\, a_k^m,\, b_k^m)) = 0 \quad (t \geqq 0) \\ (D_i^m)^j u(0;\, A^m,\, a_k^m,\, b_k^m) = (D_i^m)^j a_k^m \\ \dfrac{d}{dt}((D_i^m)^j u(0;\, A^m,\, a_k^m,\, b_k^m)) = (D_i^m)^j b_k^m\,, \end{cases}$$

by $(D_i^m)^j A^m = (A^m P_i^m)(D_i^m)^j$, so that the uniqueness of solutions to (4.31) implies $(D_i^m)^j u(t;\, A^m, a_k^m, b_k^m) = u(t;\, A^m, (D_i^m)^j a_k^m, (D_i^m)^j b_k^m)$ $(t \geqq 0, i \geqq 1, 0 \leqq j \leqq k_i, 1 \leqq k \leqq N)$, that is, (4.30) is proved.

In order to prove that $\Omega_{ID}(t_0)$ is invariant with respect to t_0, noting the definition (2.23), we have only to show

$$\text{(4.32)}\qquad \mathrm{Span}\left\{(D_i^m)^j u(t_0;\, A^m,\, a_k^m,\, b_k^m),\ (D_i^m)^j \frac{du}{dt}(t_0;\, A^m,\, a_k^m,\, b_k^m);\ 0 \leqq j \leqq k_i\right\}$$
$$= \mathrm{Span}\{(D_i^m)^j a_k^m,\, (D_i^m)^j b_k^m;\ 0 \leqq j \leqq k_i\} \quad \text{for } t_0 \geqq 0 \text{ and each } i \geqq 1\,.$$

Proof of (4.32). Let us denote the left hand side and right hand side of (4.32) by $M(t_0)$ and $M(0)$, respectively, and let us set $A_i^m = A^m P_i^m$, which is a bounded operator. By (4.4) and (4.30), (4.31), we have

$$\text{(4.33)}\qquad \begin{pmatrix} (D_i^m)^j u(t;\, A^m,\, a_k^m,\, b_k^m) \\ (D_i^m)^j \dfrac{du}{dt}(t;\, A^m,\, a_k^m,\, b_k^m) \end{pmatrix} = \exp\left(-\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix} t\right) \begin{pmatrix} (D_i^m)^j a_k^m \\ (D_i^m)^j b_k^m \end{pmatrix} \quad (t \geqq 0)\,.$$

Since the operator $\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix}$ is bounded on $X \times X$, we get

$$\exp\left(-\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix} t_0\right) \begin{pmatrix} (D_i^m)^j a_k^m \\ (D_i^m)^j b_k^m \end{pmatrix} = \sum_{l=0}^{\infty} \frac{t_0^l}{l!} \begin{pmatrix} 0 & -1 \\ -A_i^m & 0 \end{pmatrix}^l \begin{pmatrix} (D_i^m)^j a_k^m \\ (D_i^m)^j b_k^m \end{pmatrix}$$
$$= \sum_{l=0}^{\infty} \left(\frac{t_0^{2l}}{(2l)!} \begin{pmatrix} (-A_i^m)^l & 0 \\ 0 & (-A_i^m)^l \end{pmatrix} + \frac{t_0^{2l+1}}{(2l+1)!} \begin{pmatrix} 0 & (-A_i^m)^l \\ (-A_i^m)^{l+1} & 0 \end{pmatrix} \right) \begin{pmatrix} (D_i^m)^j a_k^m \\ (D_i^m)^j b_k^m \end{pmatrix}$$
$$= \sum_{l=0}^{\infty} \begin{pmatrix} \dfrac{t_0^{2l}}{(2l)!}(-A_i^m)^l (D_i^m)^j a_k^m + \dfrac{t_0^{2l+1}}{(2l+1)!}(-A_i^m)^l (D_i^m)^j b_k^m \\ \dfrac{t_0^{2l}}{(2l)!}(-A_i^m)(D_i^m)^j b_k^m + \dfrac{t_0^{2l+1}}{(2l+1)!}(-A_i^m)^{l+1} (D_i^m)^j a_k^m \end{pmatrix}.$$

Here we use

$$(4.34)\qquad \begin{pmatrix} 0 & 1 \\ -A_i^m & 0 \end{pmatrix}^l = \begin{cases} \begin{pmatrix} (-A_i^m)^{l/2} & 0 \\ 0 & (-A_i^m)^{l/2} \end{pmatrix}, & \text{if } l \text{ is even} \\ \begin{pmatrix} 0 & (-A_i^m)^{(l-1)/2} \\ (-A_i^m)^{(l+1)/2} & 0 \end{pmatrix}, & \text{if } l \text{ is odd}, \end{cases}$$

which is seen by induction.

Since the equality

$$(4.35)\qquad A_i^m = \lambda_i P_i^m + D_i^m$$

holds (e.g. [8]), noting (2.25), we have

$$\begin{pmatrix} (D_i^m)^j u(t_0;\ A^m, a_k^m, b_k^m) \\ (D_i^m)^j \dfrac{du}{dt}(t_0;\ A^m, a_k^m, b_k^m) \end{pmatrix} \in M(0)\times M(0) \text{ by } (4.33)\,.$$

Therefore we can see that $M(t_0)\subset M(0)$.

Next we have to prove that $M(0)\subset M(t_0)$. Since $\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix}$ is bound on $X\times X$, we see that

$$\begin{pmatrix} P_i^m & 0 \\ 0 & P_i^m \end{pmatrix} = \exp\left(\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix} t_0\right) \exp\left(-\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix} t_0\right).$$

Thus we have

$$\begin{aligned} \begin{pmatrix} P_i^m & 0 \\ 0 & P_i^m \end{pmatrix}\begin{pmatrix} (D_i^m)^j a_k^m \\ (D_i^m)^j b_k^m \end{pmatrix} &= \exp\left(\begin{pmatrix} 0 & -1 \\ A_i^m & 0 \end{pmatrix} t_0\right)\begin{pmatrix} (D_i^m)^j u(t_0;\ A^m, a_k^m, b_k^m) \\ (D_i^m)^j \dfrac{du}{dt}(t_0;\ A^m, a_k^m, b_k^m) \end{pmatrix} \\ & \hspace{20em} (\text{by } (4.33)) \\ &= \sum_{l=0}^{\infty}\begin{pmatrix} \dfrac{t_0^{2l}}{(2l)!}(-A_i^m)^l (D_i^m)^j u(t_0;\ A^m, a_k^m, b_k^m) \\ \dfrac{t_0^{2l}}{(2l)!}(-A_i^m)^l (D_i^m)^j \dfrac{du}{dt}(t_0;\ A^m, a_k^m, b_k^m) \end{pmatrix} \\ &\quad -\sum_{l=0}^{\infty}\begin{pmatrix} \dfrac{t_0^{2l+1}}{(2l+1)!}(-A_i^m)^l (D_i^m)^j \dfrac{du}{dt}(t_0;\ A^m, a_k^m, b_k^m) \\ \dfrac{t_0^{2l+1}}{(2l+1)!}(-A_i^m)^{l+1} (D_i^m)^j u(t_0;\ A^m, a_k^m, b_k^m) \end{pmatrix} \in M(t_0)\times M(t_0) \\ & \hspace{20em} (\text{by } (4.35))\,, \end{aligned}$$

which means $M(0)\subset M(t_0)$. This completes the proof of (4.32).

Finally we prove the openness of Ω_{ID}. We have $\Omega_{ID}=\cup\,\Omega_{ID}(i(1), \cdots, i(q_1), j(1), \cdots, j(q_2), k(1), \cdots, k(q_3))$, where the sum of the sets is taken over all $(i(1), \cdots, i(q_1))\in \mathbf{N}^{q_1}$, $(j(1), \cdots, j(q_2))\in(\mathbf{N}\cup\{0\})^{q_2}$, $(k(1), \cdots, k(q_3))\in\{1, \cdots, 2N\}^{q_3}$, q_1,

$q_2\ q_3 \in \mathbf{N}$, $q_1+q_2+q_3=q$, and we set $a^m_{N+j}=b^m_j$ $(1\leqq j\leqq N)$ and $\Omega_{ID}(i(1), \cdots, i(q_1), j(q_1), \cdots, j(q_2), k(1), \cdots, k(q_3))=\{x\in\Omega;\ \det(\vec{\omega}((D^m_{i(r_1)})^{j(r_2)}a^m_{k(r_3)};x))_{1\leqq r_1\leqq q_1, 1\leqq r_2\leqq q_2, 1\leqq r_3\leqq q_3}\neq 0\}$. Since each $\Omega_{ID}(i(1), \cdots, i(q_1), j(1), \cdots, j(q_2), k(1), \cdots, k(q_3))$ is open, we can see the openness of Ω_{ID}. Thus we complete the proof of (i).

Proof of (ii). Without loss of generality, we may assume that $0\notin\sigma(A^m)$. Let $\sigma(A^m)=\{\lambda_i\}_{i\geqq 1}$. We see by Lemma 1 that $\pm\sqrt{-\lambda_i}$ $(i\geqq 1)$ are eigenvalues of $\tilde{A}^m$. Then we define the nilpotent for $\pm\sqrt{-\lambda_i}$ $(i\geqq 1)$:

$$(4.36)\qquad \tilde{D}^m_{\pm i}=\tilde{D}_{\pm\sqrt{-\lambda_i}}(\tilde{A}^m)=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_{\pm\sqrt{-\lambda_i}}}(z-(\pm\sqrt{-\lambda_i}))(z-\tilde{A}^m)^{-1}dz\ .$$

We recall that the eigenprojection $\tilde{P}^m_{\pm i}$ are defined by (4.22). Then we have

$$(4.37)\qquad (\tilde{D}^m_{\pm i})^j=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_{\pm\sqrt{-\lambda_i}}}(z-(\pm\sqrt{-\lambda_i}))^j(z-\tilde{A}^m)^{-1}dz\ .$$

(e.g. Kato [8]).

Since the equalities (2.16) mean

$$e^{-t\tilde{A}}\begin{pmatrix}a_k\\ b_k\end{pmatrix}=e^{-t\tilde{A}^m}\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\quad (t\geqq t_0,\ 1\leqq k\leqq N)\ ,$$

we get

$$\tilde{A}(\tilde{D}^m_{\pm i})^j\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}=\tilde{A}^m(D^m_{\pm i})^j\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\quad (i\geqq 1,\ j\geqq 0,\ 1\leqq k\leqq N)\ ,$$

by Theorem A, that is,

$$(4.38)\qquad \tilde{A}\left((\tilde{D}^m_{\pm i})^j\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\right)_1=\tilde{A}^m\left((\tilde{D}^m_{\pm i})^j\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\right)_1\quad (i\geqq 1,\ j\geqq 0,\ 1\leqq k\leqq N)\ .$$

Here and henceforth let us denote the first component u_1 of $\begin{pmatrix}u_1\\ u_2\end{pmatrix}\in X\times X$ by $\begin{pmatrix}u_1\\ u_2\end{pmatrix}_1$. Then we will prove that (4.38) implies

$$(4.39)\qquad A(D^m_i)^j a^m_k=A^m(D^m_i)^j a^m_k,\ A(D^m_i)^j b^m_k=A^m(D^m_i)^j b^m_k\quad (i\geqq 1,\ 0\leqq j\leqq k_i,\ 1\leqq k\leqq N)\ .$$

For the proof, we calculate $\left((\tilde{D}^m_{\pm i})^j\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\right)_1$ by using Lemma 1. Noting (2.25), we have

$$(4.40)\qquad \left(\tilde{P}^m_{\pm i}\begin{pmatrix}a^m_k\\ b^m_k\end{pmatrix}\right)_1$$
$$=\frac{1}{2}P^m_i a^m_k\mp\left(\frac{1}{2\sqrt{-\lambda_i}}P^m_i b^m_k+\sum_{l=1}^{k_i}\frac{(2l)!\ (2\sqrt{-\lambda_i})^{-2l-1}(D^m_i)^l b^m_k}{(l!)^2}\right),$$

and

$$(4.41)\quad (-1)^j\left((\tilde{D}_i^m)^j\begin{pmatrix} a_k^m \\ b_k^m \end{pmatrix}\right)_1$$
$$=\sum_{l=j}^{k_i}\frac{j(2l-j-1)!\,(2\sqrt{-\lambda_i})^{-2l+j}(D_i^m)^l a_k^m}{2(l-j)!\,l!}$$
$$-\sum_{l=j}^{k_i}\frac{(2l-j)!\,(2\sqrt{-\lambda_i})^{-2l+j-1}(D_i^m)^l b_k^m}{(l-j)!\,l!}\quad (1\leqq j\leqq k_i)\,,$$

$$(4.42)\quad \left((\tilde{D}_{-i}^m)^j\begin{pmatrix} a_k^m \\ b_k^m \end{pmatrix}\right)_1$$
$$=\sum_{l=j}^{k_i}\frac{j(2l-j-1)!\,(2\sqrt{-\lambda_i})^{-2l+j}(D_i^m)^l a_k^m}{2(l-j)!\,l!}$$
$$+\sum_{l=j}^{k_i}\frac{(2l-j)!\,(2\sqrt{-\lambda_i})^{-2l+j-1}(D_i^m)^l b_k^m}{(l-j)!\,l!}\quad (1\leqq j\leqq k_i)\,.$$

Therefore (4.38) implies

$$(4.43)\quad AP_i^m a_k^m = A^m P_i^m a_k^m\,,$$

$$(4.44)\quad \sum_{l=j}^{k_i}\frac{j(2l-j-1)!\,(2\sqrt{-\lambda_i})^{-2l+j}}{2(l-j)!\,l!}\{A(D_i^m)^l a_k^m - A^m(D_i^m)^l a_k^m\}=0\quad (1\leqq j\leqq k_i)\,,$$

$$(4.45)\quad \frac{1}{2\sqrt{-\lambda_i}}(AP_i^m b_k^m - A^m P_i^m b_k^m)$$
$$+\sum_{l=1}^{k_i}\frac{(2l)!\,(2\sqrt{-\lambda_i})^{-2l-1}}{(l!)^2}\{A(D_i^m)^l b_k^m - A^m(D_i^m)^l b_k^m\}=0$$

and

$$(4.46)\quad \sum_{l=j}^{k_i}\frac{(2l-j)!\,(2\sqrt{-\lambda_i})^{-2l+j-1}}{(l-j)!\,l!}\{A(D_i^m)^l b_k^m - A^m(D_i^m)^l b_k^m\}=0\quad (1\leqq j\leqq k_i)\,.$$

In (4.44) and (4.46), putting $j=k_i$, we get

$$(4.47)\quad A(D_i^m)^{k_i}a_k^m = A^m(D_i^m)^{k_i}a_k^m,\ A(D_i^m)^{k_i}b_k^m = A^m(D_i^m)^{k_i}b_k^m\,.$$

Next, putting $j=k_i-1$ in (4.44) and (4.46), and substituting (4.47), we obtain $A(D_i^m)^{k_i-1}a_k^m = A^m(D_i^m)^{k_i-1}a_k^m$ and $A(D_i^m)^{k_i-1}b_k^m = A^m(D_i^m)^{k_i-1}b_k^m$. Continuing this procedure, we reach

$$(4.48)\quad A(D_i^m)^j a_k^m = A^m(D_i^m)^j a_k^m,\ A(D_i^m)^j b_k^m = A^m(D_i^m)^j b_k^m$$

for $i\geqq 1$, $1\leqq j\leqq k_i$ and $1\leqq k\leqq N$. Moreover, applying (4.48) for $1\leqq j\leqq k_i$ in (4.45), we see by (4.43) that (4.48) holds also for $j=0$. Thus we reach (4.39).

In view of (4.29), the equalities (4.39) are rewritten as

$$(4.49)\quad \vec{\omega}((D_i^m)^j a_k^m;\,x)\vec{\Phi}(x)=0 \quad\text{and}\quad \vec{\omega}((D_i^m)^j b_k^m;\,x)\vec{\Phi}(x)=0$$
$$(x\in\Omega,\ i\geqq 1,\ 0\leqq j\leqq k_i,\ 1\leqq k\leqq N)\,.$$

Here we define a q-dimensional column vector $\vec{\Phi}(x)$ by

$$(4.50)\qquad \vec{\Phi}(x)={}^t\Big(p_{11}(x)-p^m_{11}(x),\cdots,p_{nn}(x)-p^m_{nn}(x)\,,\ p_{12}(x)-p^m_{12}(x),\cdots,p_{1n}(x)-p^m_{1n}(x),\ p_{23}(x)-p^m_{23}(x)\,,\ \cdots,p_{2n}(x)-p^m_{2n}(x),\cdots,p_{nn-1}(x)-p^m_{nn-1}(x)\,,\ q_1(x)-q^m_1(x)+\sum_{i=1}^n\frac{\partial(p_{i1}(x)-p^m_{i1}(x))}{\partial x_i},\cdots,\ q_n(x)-q^m_n(x)+\sum_{i=1}^n\frac{\partial(p_{in}(x)-p^m_{in}(x))}{\partial x_i},\ r(x)-r^m(x)\Big)\,.$$

Now we will complete the proof of the part (ii) of Theorem 2. To this end, assuming $x\in\overline{\Omega_{ID}}$ and (2.16), we have to prove

$$(4.51)\qquad p_{ij}(x)=p^m_{ij}(x),\ q_j(x)=q^m_j(x),\ r(x)=r^m(x)\quad(1\leqq i,j\leqq n)\,.$$

By $x\in\overline{\Omega_{ID}}$, for any $\varepsilon>0$, we can take an $x_\varepsilon\in\Omega_{ID}$ such that $|x_\varepsilon-x|<\varepsilon$. Moreover, there exist q-dimensional column vectors $\tilde{a}_l(x_\varepsilon)$ $(1\leqq l\leqq q)$ each of which is of the form $\vec{\omega}((D^m_i)^j a^m_k;\,x_\varepsilon)$ or $\vec{\omega}((D^m_i)^j b^m_k;\,x_\varepsilon)$ for some $i\geqq 1$, $0\leqq j\leqq k_i$ $1\leqq k\leqq N$, and $\dim\mathrm{Span}\{\tilde{a}_l(x_\varepsilon);\,1\leqq l\leqq q\}=q$, namely, $\det(\tilde{a}_1(x_\varepsilon),\cdots,\tilde{a}_q(x_\varepsilon))\neq 0$. Therefore we have $\det(\tilde{a}_1(y),\cdots,\tilde{a}_q(y))\neq 0$ for each y in a neighborhood U of x_ε, so that (4.49) implies $\vec{\Phi}(y)=0$ $(y\in U)$. Hence, by the definition (4.50), we obtain $p_{ij}(x_\varepsilon)=p^m_{ij}(x_\varepsilon)$, $q_j(x_\varepsilon)=q^m_j(x_\varepsilon)$, $r(x_\varepsilon)=r^m(x_\varepsilon)$ $(1\leqq i,j\leqq n)$. We can let $x_\varepsilon\to x$, so that by the continuity of p_{ij}, q_j, r, p^m_{ij}, q^m_j, r^m, we reach (4.51).

Proof of Theorem 3. By the equalities (2.16) and Theorem A, we have $P^m_i a^m_k$, $P^m_i b^m_k\in\mathscr{D}(A)\cap\mathscr{D}(A^m)$ $(i\geqq 1,\ 1\leqq k\leqq N)$ Without loss of generality, we can assume

$$(4.52)\qquad P^m_1 a^m_1\neq 0\,.$$

(In fact, let $P^m_i a^m_k=P^m_i b^m_k=0$ for each $i\geqq 1$ and $1\leqq k\leqq N$. Then, by Theorems 15.4 and 16.5 in [2], we get $a^m_k=b^m_k=0$ $(1\leqq k\leqq N)$, so that $\Omega_{ID}=\Omega_{ID}(t_0)=\emptyset$ by the definition (2.26). Therefore our conclusion (2.29) is trivial.)

(i) We consider the case of the model with the boundary condition (2.4). Since $(\mathscr{A}^m-\lambda_1)^{k_1+1}P^m_1 a^m_1=(D^m_1)^{k_1+1}P^m_1 a^m_1=0$ by (2.25), we get

$$(4.53)\qquad (\mathscr{A}^m-\lambda_1)((\mathscr{A}^m-\lambda_1)^{k_1}P^m_1 a^m_1)(x)=0\quad(x\in\Omega)\,.$$

By (3.5) in Theorem A, we have $(\tilde{D}^m_{\pm i})^j a^m_k$, $(\tilde{D}^m_{\pm i})^j b^m_k\in\mathscr{D}(\tilde{A}^l)\cap\mathscr{D}((\tilde{A}^m)^l)$ $(i\geqq 1,\ j\geqq 0,\ 1\leqq k\leqq N,\ l\geqq 1)$, so that by the relations (4.40)–(4.42) we get $(\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_1=(A^m-\lambda_1)^j P^m_1 a^m_1=(D^m_1)^j P^m_1 a^m_1\in\mathscr{D}(A^l)\cap\mathscr{D}((A^m)^l)$ $(i\geqq 1,\ j\geqq 0,\ l\geqq 1)$, that is,

$$(4.54)\qquad \begin{cases}\dfrac{\partial(\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_1}{\partial\nu_{\mathscr{A}^m}}+\beta^m(\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_{1|\partial\Omega}=0\\[2ex] \dfrac{\partial(\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_1}{\partial\nu_{\mathscr{A}}}+\beta(\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_{1|\partial\Omega}=0\quad\text{or}\\[2ex] (\mathscr{A}^m-\lambda_1)^j P^m_1 a^m_{1|\partial\Omega}=0\,.\end{cases}$$

Then we can show

(4.55) The set $\{\xi \in \partial\Omega;\ ((\mathscr{A}^m-\lambda_1)^j P_1^m a_1^m)(\xi)=0\}$ has no interior points for some $0 \leqq j_0 \leqq k_1$.

In fact, assume contrarily that for each j, the above set has some interior point. By the condition (4.54), we see that the set

$$\left\{\xi \in \partial\Omega;\ \frac{\partial(\mathscr{A}^m-\lambda_1)^j P_1^m a_1^m}{\partial\nu_{\mathscr{A}^m}}(\xi)=((\mathscr{A}^m-\lambda_1)^j P_1^m a_1^m)(\xi)=0\right\}$$

has some interior point for each $j \geqq 0$. Setting $j=k_1$, for $(\mathscr{A}^m-\lambda_1)^{k_1} P_1^m a_1^m$ in (4.53), we apply the unique continuation theorem (e.g. Mizohata [10]), so that $((\mathscr{A}^m-\lambda_1)^{k_1} P_1^m a_1^m)(x)=0$ $(x \in \Omega)$. Now, noting $(\mathscr{A}^m-\lambda_1)((\mathscr{A}^m-\lambda_1)^{k_1-1} P_1^m a_1^m)(x)=0$ $(x \in \Omega)$ and applying the unique continuation theorem for $(\mathscr{A}^m-\lambda_1)^{k_1-1} P_1^m a_1^m$, we obtain $((\mathscr{A}^m-\lambda_1)^{k_1-1} P_1^m a_1^m)(x)=0$ $(x \in \Omega)$. We continue this argument to get $(\mathscr{A}^m-\lambda_1)^{k_1-1} P_1^m a_1^m=(\mathscr{A}^m-\lambda_1)^{k_1-2} P_1^m a_1^m=\cdots=P_1^m a_1^m=0$, which contradicts (4.52). Thus (4.55) is proved.

We will complete the proof of (i) of this theorem. By (4.54) and (4.55), A is an operator with the boundary condition (2.6). Next we have to verify (2.29). Let Γ be any arc in $\partial\Omega \cap \overline{\Omega_{ID}}$. By Theorem 2, we have $p_{ij}(\xi)=p_{ij}^m(\xi)$ $(\xi \in \Gamma,\ 1 \leqq i, j \leqq n)$, so that

$$\frac{\partial(\mathscr{A}^m-\lambda_1)^{j_0} P_1^m a_1^m}{\partial\nu_{\mathscr{A}^m}}(\xi)=\frac{\partial(\mathscr{A}^m-\lambda_1)^{j_0} P_1^m a_1^m}{\partial\nu_{\mathscr{A}}}(\xi) \quad (\xi \in \Gamma)\ .$$

Therefore the condition (4.54) with $j=j_0$, implies

$$(\beta(\xi)-\beta^m(\xi))((\mathscr{A}^m-\lambda_1)^{j_0} P_1^m a_1^m)(\xi)=0 \quad (\xi \in \Gamma)\ .$$

Since $\{\xi \in \Gamma;\ ((\mathscr{A}^m-\lambda_1)^{j_0} P_1^m a_1^m)(\xi)=0\}$ has no interior points by (4.55), also noting the continuity of β and β^m, we obtain $\beta(\xi)=\beta^m(\xi)$ $(\xi \in \Gamma)$. This completes the proof of the part (i). For the part (ii), we can proceed similarly, and we omit the proof.

Proof of Corollary 1. First

(4.56) $(D_i^m)^j a$ are real analytic in Ω for $a \in X$ $(i \geqq 1,\ 0 \leqq j \leqq k_i)$.

In fact, since $(\mathscr{A}^m-\lambda_i)((\mathscr{A}^m-\lambda_i)^{k_i} P_i^m a)=0$ by (2.25), we see by a result on the analyticity of solutions of elliptic equations (e.g. Hörmander [6, p. 178]) that $b \equiv (\mathscr{A}^m-\lambda_i)^{k_i} P_i^m a=(D_i^m)^{k_i} a$ is real analytic. Since we have $(\mathscr{A}^m-\lambda_i) \times ((\mathscr{A}^m-\lambda_i)^{k_i-1} P_i^m a)=b$ and b is real analytic, also $(D_i^m)^{k_i-1} a=(\mathscr{A}^m-\lambda_i)^{k_i-1} P_i^m a$ is real analytic by [6]. Continuing this procedure, we see that $P_i^m a$ is real analytic.

Now we will complete the proof of this corollary. To this end, assume that $\Omega_{ID} \neq \emptyset$. Then we have only to prove that $\overline{\Omega_{ID}}=\overline{\Omega}$. By $\Omega_{ID} \neq \emptyset$, there exist q-dimensional column vectors $\tilde{a}_l(x)$ $(1 \leqq l \leqq q)$ of the forms $\vec{\omega}((D_i^m)^j a_k^m;\ x)$

or $\vec{\omega}((D_i^m)^j b_k^m; x)$ $(i\geqq 1, 0\leqq j\leqq k_i, 1\leqq k\leqq N)$ such that $W(x)=\det(\tilde{a}_1(x), \cdots, \tilde{a}_q(x))$ never vanishes in some open set in Ω. By (4.56), $W(x)$ is real analytic in Ω and $W(x)\neq 0$, so that $\{x\in\Omega;\ W(x)=0\}$ is a finite set. Noting that $\{x\in\Omega;\ W(x)\neq 0\}\subset\Omega_{ID}$, we get $\overline{\Omega_{ID}}=\bar{\Omega}$.

Proof of Theorem 4. We will prove the part (I), because we can similarly do (II). Since the spatial dimension n is one, we see $H^1(0,1)\subset C^0[0,1]$ by the Sobolev imbedding (e.g. [1]). Hence we note that $a_k^m, b_k^m\in\mathscr{D}((A^m+a)^{1/2})\subset H^1(0,1)$ are continuous on $[0,1]$.

First we will verify the equivalences in the statements (a) and (b) of this theorem. Next we will prove that the set

$$(4.57)\qquad \begin{aligned} M=\{x\in[0,1];\ &\text{the equalities (2.16) imply}\\ &p(x)=p^m(x),\ q(x)=q^m(x),\ r(x)=r^m(x)\ \text{ and}\\ &a_k^m(x)=a_k(x),\ b_k^m(x)=b_k(x)\quad (1\leqq k\leqq N)\}\end{aligned}$$

is empty if (2.36) holds. Then, since $\Omega_{ID}=\emptyset$ or $\overline{\Omega_{ID}}=\bar{\Omega}$ by (a), (b) and $M\supset\overline{\Omega_{ID}}$ always holds by (2.31), we will complete the proof of (2.35). As for the determination of boundary conditions, we can see the conclusion by Theorem 3.

Proof of the "if" part in (a). Since the spatial dimension is one, we have $D_i^m=0$ $(i\geqq 1)$ (e.g. Lemma 1 in [14]). Therefore, by the definition, we get

$$(4.58)\qquad \begin{aligned}\Omega_{ID}=\{x\in(0,1);&\\ &\dim\operatorname{Span}\{\vec{\omega}(P_i^m a_k^m;x), \vec{\omega}(P_i^m b_k^m;x);\ i\geqq 1, 1\leqq k\leqq N\}=3\}\,.\end{aligned}$$

Now assume (2.36). Then, as is easily checked, we have

$$\dim\operatorname{Span}\{\vec{\omega}(P_i^m a_k^m;x), \vec{\omega}(P_i^m b_k^m;x);\ i\geqq 1, 1\leqq k\leqq N\}<3$$

for each $x\in\Omega$. Therefore (4.58) implies $\Omega_{ID}=\emptyset$.

Proof of the "if" part in (b). We can take a linearly independent subset $\{\phi_1, \phi_2, \phi_3\}$ of $\operatorname{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}$. Then, in view of the proof of Theorem 2, we have only to prove that the set

$$\left\{x\in(0,1);\ \det\begin{pmatrix}\phi_1(x) & \phi_2(x) & \phi_3(x)\\ d\phi_1(x)/dx & d\phi_2(x)/dx & d\phi_3(x)/dx\\ d^2\phi_1(x)/dx^2 & d^2\phi_2(x)/dx^2 & d^2\phi_3(x)/dx^2\end{pmatrix}=0\right\}$$

has no interior points. Its proof is carried out by the same way as the proof of Theorem 2 in [14], and so we omit its proof.

Now the "only if" parts in (a) and (b) are directly seen. In fact, assume that $\Omega_{ID}=\emptyset$. If (2.37) held, then we would reach $\overline{\Omega_{ID}}=\bar{\Omega}$ by the "if" part in (b), which contradicts $\Omega_{ID}=\emptyset$. Hence (2.37) cannot hold, that is we see the "only if" part in (a). In (b), we can similarly proceed. Thus we complete he proof of the equivalences.

Proof of "(2.36)$\Rightarrow M=\emptyset$". Assume that (2.36) holds. Then we will prove: if $\dim \mathrm{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}=2$, then for some p, q, r and β_0, β_1 such that $p\neq p^m$ or $q\neq q^m$ or $r\neq r^m$, we have

$$u(t;\ A, a_k^m, b_k^m)=u(t;\ A^m, a_k^m, b_k^m) \quad \text{in} \quad L^2(0,1) \quad (t\geqq 0, 1\leqq k\leqq N)\ . \tag{4.59}$$

For the case of $\dim \mathrm{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}=1$, we can proceed similarly and, if $\dim \mathrm{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}=0$, then $a_k^m=b_k^m=0$ $(1\leqq k\leqq N)$ by the completeness of the eigenvectors, so that the proof is trivial.

Let $\{\phi_1, \phi_2\}$ be a basis of $\mathrm{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}$. Then, by the same way as (4.29), we can see

$$\phi_k\in C^\infty(\Omega)\ . \tag{4.60}$$

Moreover, we have $A^m\phi_k=\lambda_{i(k)}\phi_k$ for some $i(k)\geqq 1$ $(k=1,2)$. (e.g. Lemma 1 in [14]). Henceforth, without loss of generality, we may assume that $i(1)=1$ and $i(2)=2$. That, is,

$$A^m\phi_k=\lambda_k\phi_k \quad (k=1,2)\ . \tag{4.61}$$

Since $p^m(x)>0$ $(0\leqq x\leqq 1)$, if we take a real constant ε such that $|\varepsilon|$ is sufficiently small, then

$$p^m(x)+\varepsilon\left(\phi_1(x)\frac{d\phi_2(x)}{dx}-\frac{d\phi_1(x)}{dx}\phi_2(x)\right)>0 \quad (0\leqq x\leqq 1)\ . \tag{4.62}$$

Let us fix such an ε. We set

$$p_1(x)=p^m(x)+\varepsilon\left(\phi_1(x)\frac{d\phi_2(x)}{dx}-\frac{d\phi_1(x)}{dx}\phi_2(x)\right) \tag{4.63}$$

$$q_1(x)=q^m(x)+\varepsilon\left(\frac{d^2\phi_1(x)}{dx^2}\phi_2(x)-\phi_1(x)\frac{d^2\phi_2(x)}{dx^2}\right) \tag{4.64}$$

$$r_1(x)=r^m(x)+\varepsilon\left(\frac{d\phi_1(x)}{dx}\frac{d^2\phi_2(x)}{dx^2}-\frac{d^2\phi_1(x)}{dx^2}\frac{d\phi_2(x)}{dx}\right) \quad (0\leqq x\leqq 1)\ . \tag{4.65}$$

By (4.60) and (4.63)-(4.65), we see that p_1, q_1, r_1 are real-valued smooth functions on $[0,1]$ and $p_1(x)>0$ $(0\leqq x\leqq 1)$. Moreover, since $\phi_1(x)\dfrac{d\phi_2(x)}{dx}-\dfrac{d\phi_1(x)}{dx}\phi_2(x)$ is nothing but the Wronskian of ϕ_1 and ϕ_2, by the property of the Wronskian (e.g. Ince [7]), it follows from the linear independence of ϕ_1 and ϕ_2 that

$$p_1(x_0)\neq p^m(x_0) \quad \text{for some} \quad x_0\in[0,1]\ . \tag{4.66}$$

We define an operator $-A_1$ in $X=L^2(0,1)$ by

$$\begin{cases}(-A_1u)(x)=p_1(x)\dfrac{d^2u(x)}{dx^2}+q_1(x)\dfrac{du(x)}{dx}+r_1(x)u(x) \quad (0<x<1)\\ \mathscr{D}(A_1)=\left\{u\in H^2(0,1);\ \dfrac{du}{dx}(0)+\beta_0^m u(0)=\dfrac{du}{dx}(1)+\beta_1^m u(1)=0\right\}.\end{cases}$$

Then, for $A=A_1$, we will prove (4.59). Noting (4.63)-(4.65), we have

$$A_1\phi_1=A^m\phi_1\,,\quad A_1\phi_2=A^m\phi_2 \tag{4.67}$$

by direct computations. Noting that the system of the eigenvectors of A^m is complete in $L^2(0,1)$, we have:

$$a_k^m,\ b_k^m\in\overline{\mathrm{Span}\{P_i^m a_k^m,\ P_i^m b_k^m;\ i\geqq 1,\ 1\leqq k\leqq N\}}\,,$$

and since $\{\phi_1,\phi_2\}$ is a basis of $\mathrm{Span}\{P_i^m a_k^m, P_i^m b_k^m;\ i\geqq 1, 1\leqq k\leqq N\}$, there exist $a_{kj}, b_{kj}\in\mathbf{R}$ $(1\leqq k\leqq N, j=1,2)$ such that

$$a_k^m=a_{k1}\phi_1+a_{k2}\phi_2,\ b_k^m=b_{k1}\phi_1+b_{k2}\phi_2 \quad (1\leqq k\leqq N)\,. \tag{4.68}$$

Since the restriction A_0^m of A^m on $\mathrm{Span}\{\phi_1,\phi_2\}$, is bounded and $A_0^m\phi_k=\lambda_k\phi_k$ $(k=1,2)$ by (4.61), $\tilde{A}_0^m=\begin{pmatrix}0 & -1\\ A_0^m & 0\end{pmatrix}$ is a bounded operator on $\{\mathrm{Span}\{\phi_1,\phi_2\}\}^2$ to itself. Therefore we have

$$\begin{aligned}\begin{pmatrix}u(t;A^m,a_k^m,b_k^m)\\ \dfrac{du}{dt}(t;A^m,a_k^m,b_k^m)\end{pmatrix}&=e^{-\tilde{A}_0^m t}\begin{pmatrix}a_k^m\\ b_k^m\end{pmatrix}\\ &=\sum_{j=1}^{2}e^{-\tilde{A}_0^m}\begin{pmatrix}a_{kj}\phi_j\\ b_{kj}\phi_j\end{pmatrix} \qquad \text{(by (4.68))}\\ &=\sum_{j=1}^{2}\sum_{l=0}^{\infty}\frac{t^l}{l!}(-\tilde{A}_0^m)^l\begin{pmatrix}a_{kj}\phi_j\\ b_{kj}\phi_j\end{pmatrix}.\end{aligned}$$

Since we can calculate $(-\tilde{A}_0^m)^l$ by a way similar to (4.34), we get

$$\begin{aligned}&\begin{pmatrix}u(t;A^m,a_k^m,b_k^m)\\ \dfrac{du}{dt}(t;A^m,a_k^m,b_k^m)\end{pmatrix}\\ &\quad=\sum_{j=1}^{2}\sum_{l=0}^{\infty}\left(\frac{t^{2l}}{(2l)!}\begin{pmatrix}a_{kj}(-A_0^m)^l\phi_j\\ b_{kj}(-A_0^m)^l\phi_j\end{pmatrix}+\frac{t^{2l+1}}{(2l+1)!}\begin{pmatrix}b_{kj}(-A_0^m)^l\phi_j\\ a_{kj}(-A_0^m)^{l+1}\phi_j\end{pmatrix}\right).\end{aligned}$$

Thus by (4.61), we obtain

$$\begin{aligned}&u(t;A^m,a_k^m,b_k^m)\\ &\quad=\sum_{j=1}^{2}\left(\sum_{l=0}^{\infty}\frac{(-\lambda_j)^l t^{2l}}{(2l)!}a_{kj}+\frac{(-\lambda_j)^l t^{2l+1}}{(2l+1)!}b_{kj}\right)\phi_j \quad (t\geqq 0)\,.\end{aligned} \tag{4.69}$$

Consequently we have for $t\geqq 0$,

$$\begin{aligned}\frac{d^2u(t)}{dt^2}&=-\sum_{j=1}^{2}\left(\sum_{l=0}^{\infty}\frac{(-\lambda_j)^l t^{2l}}{(2l)!}a_{kj}+\frac{(-\lambda_j)^l t^{2l+1}}{(2l+1)!}b_{kj}\right)\lambda_j\phi_j\\&=-\sum_{j=1}^{2}\left(\sum_{l=0}^{\infty}\frac{(-\lambda_j)^l t^{2l}}{(2l)!}a_{kj}+\frac{(-\lambda_j)^l t^{2l+1}}{(2l+1)!}b_{kj}\right)A^m\phi_j \qquad \text{(by (4.61))}\\&=-A_1\left(\sum_{j=1}^{2}\left(\sum_{l=0}^{\infty}\frac{(-\lambda_j)^l t^{2l}}{(2l)!}a_{kj}+\frac{(-\lambda_j)^l t^{2l+1}}{(2l+1)!}b_{kj}\right)\phi_j\right) \qquad \text{(by (4.67))}\\&=-A_1u(t;\ A^m,a_k^m,b_k^m) \qquad \text{(by (4.69))}\,.\end{aligned}$$

That is, $u(t;\ A^m, a_k^m, b_k^m)$ is a solution to the Cauchy problem $\frac{d^2u(t)}{dt^2}+A_1u(t)=0$ $(t\geqq 0)$ with $u(0)=a_k^m$ and $\frac{du}{dt}(0)=b_k^m$, so that the uniqueness to the problem implies (4.59). This completes the proof of Theorem 4.

Now we conclude this section with proof of Lemma 1.

Proof of Lemma 1. As is easily seen, if $z\neq\mu$ and $|z-\mu|$ is sufficiently small, then

$$(z-\tilde{L})^{-1}=\begin{pmatrix} z & -1\\ L & z\end{pmatrix}(z^2+L)^{-1} \tag{4.70}$$

and $(z-\tilde{L})^{-1}$ is a bounded linear operator. This means (4.10). We prove only (4.11) and (4.13), because proofs of (4.12) and (4.14) are similar.

Proof of (4.11). By the definitions (4.9) and (4.70), we have

$$Q_{1,\mu}=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu} z(z^2+L)^{-1}dz\,, \tag{4.71}$$

$$Q_{2,\mu}=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu} (z^2+L)^{-1}dz\,, \tag{4.72}$$

and

$$Q_{3,\mu}=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu} L(z^2+L)^{-1}dz\,. \tag{4.73}$$

Since μ is an isolated point of $\sigma(\tilde{L})$, we have the Laurent expansion at $z=-\mu^2(\equiv\lambda)$:

$$(z^2+L)^{-1}=\frac{P_\lambda}{z^2+\lambda}+\sum_{i=1}^{\infty}\frac{(-1^i D_\lambda^i}{(z^2+\lambda)^{i+1}}+S_\lambda(z^2) \tag{4.74}$$

$$(|z^2+\lambda|\neq 0:\ \text{sufficiently small})\,,$$

where $S_\lambda(z^2)$ is analytic at $z=\mu$ and $P_\lambda=P_\lambda(L)$, $D_\lambda=D_\lambda(L)$ are defined by (4.8) (e.g. Kato [8]). Substituting (4.74) into (4.71), we get

$$\begin{aligned}Q_{1,\mu}=&\left(\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}\frac{z}{z^2+\lambda}dz\right)P_\lambda+\sum_{i=1}^{\infty}(-1)^i\left(\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}\frac{z}{(z^2+\lambda)^{i+1}}dz\right)D_\lambda^i\\&+\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu} zS_\lambda(z^2)dz\,.\end{aligned}$$

Here the third term at the right hand side vanishes by the analyticity of $zS_\lambda(z^2)$ at $z=\mu$. Henceforth we denote the residue of a function f at $z=a$ by $\mathrm{Res}(f;\, a)$. By the residue theorem, we have

$$Q_{1,\mu}=\mathrm{Res}\left(\frac{z}{z^2-\mu^2};\ \mu\right)P_\lambda+\sum_{i=1}^{\infty}(-1)^i\,\mathrm{Res}\left(\frac{z}{(z^2-\mu^2)^{i+1}};\ \mu\right)D_\lambda^i\,. \tag{4.75}$$

First let $\mu\neq 0$. Since μ is the pole of order $i+1$ of the function $\dfrac{z}{(z^2-\mu^2)^{i+1}}$, we have

$$\begin{aligned}\mathrm{Res}\left(\frac{z}{(z^2-\mu^2)^{i+1}};\ \mu\right)&=\frac{1}{i!}\lim_{z\to\mu}\frac{d^i}{dz^i}\left(\frac{z}{(z+\mu)^{i+1}}\right)\\&=\frac{(-1)^i}{i!}\left(\frac{(2i-1)!}{(i-1)!}(z+\mu)^{-2i}|_{z=\mu}-\mu\frac{(2i)!}{i!}(z+\mu)^{-2i-1}|_{z=\mu}\right)\\&=\begin{cases}\frac{1}{2} & (i=0)\\ 0 & (i\geqq 1)\,.\end{cases}\end{aligned}$$

Second let $\mu=0$. Since μ is the pole of order $2i+1$ of the function $\dfrac{z}{(z^2-\mu^2)^{i+1}}=\dfrac{1}{z^{2i+1}}$, we have

$$\mathrm{Res}\left(\frac{z}{(z^2-\mu^2)^{i+1}};\ \mu\right)=\mathrm{Res}\left(\frac{1}{z^{2i+1}};\ 0\right)=\begin{cases}1 & (i=0)\\ 0 & (i\geqq 1)\,.\end{cases}$$

Therefore we obtain (4.16).

Similarly, by (4.74) and (4.72), we can see (4.17). To prove (4.18), in (4.73), we note

$$\begin{aligned}\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}L(z^2+L)^{-1}dz&=\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}(1-z^2(z^2+L)^{-1})dz\\&=-\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}z^2(z^2+L)^{-1}dz\,,\end{aligned}$$

so that we have only to use (4.74) and the residue theorem.

Proof of (4.13). We have $(\tilde{D}_\mu)^j=\dfrac{1}{2\pi\sqrt{-1}}\displaystyle\int_{\Gamma_\mu}(z-\mu)^j(z-\tilde{L})^{-1}dz$. In order to prove (4.13), we substitute (4.70) and (4.74) and by the residue theorem we calculate the complex integrations:

$$\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma_\mu}z^l(z+\mu)^{-i-1}(z-\mu)^{-i-1+j}dz\quad(i\geqq 0,\ j\geqq 1,\ l=0,1,2)\,.$$

We can similarly proceed, and so we omit details.

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