Admissible Controllability for Linear Time-delay Systems in Banach Spaces — A problem in game theory

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Abstract

We consider a linear control system with time-delays in a reflexive Banach space X:

$$\begin{cases} \dfrac{dx(t)}{dt}=A_0x(t)+\displaystyle\int_{-h}^{0} d\eta(s)x(t+s)+B(t)u(t) \quad a.e.\ t>0. \\ x(0)=g^0,\ x(s)=g^1(s) \quad a.e.\ s\in[-h,\ 0), \end{cases}$$

where $(g^0,\ g^1)\in X\times L_p([-h,\ 0];\ X)$, $u\in L_q^{loc}(\boldsymbol{R}^+;\ U)$, U is a reflexive Banach space, $p,\ q\in(1,\ \infty)$, $B(t)$ is a family of bounded linear operators on U to X and A_0 generates a C_0-semigroup, η is a Stieltjes measure. Moreover g^1 and u are assumed to be restricted in $\{g^1;\ \|g^1\|_{L_p([-h,0];\,X)}\leqq\rho\}$ and $\{u;\ \|u\|_{L_q([0,T];\,U)}\leqq\delta\}$ $(\rho,\ \delta>0)$. For

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given x^0, $g^0 \in X$ and a given time $T>0$, we discuss admissible controllability problems:

(1) to determine independently $g^1(\cdot)$ and $u(\cdot)$ such that $x(0)=g^0$ and $x(T)=x^0$ or $\|x(T)-x^0\| \leqq \varepsilon$ (ε: a given error). (the cooperative type)

(2) to determine $u(\cdot)$ for a given $g^1(\cdot)$ such that $x(0)=g^0$ and $\|x(T)-x^0\| \leqq \varepsilon$. (the noncooperative type)

In this paper, for the both types, we establish necessary and sufficient conditions involving ρ and δ, in order that we can find such g^1 and u. These conditions are expressed in terms of the fundamental solution of the homogeneous system with time-delays.

§ 1. Introduction

For linear systems in Banach spaces, the concept of controllability for free (unconstrained) controls as well as observability and identifiability has been studied extensively. We refer to Curtain and Pritchard [3], Fattorini [4], Nakagiri and Yamamoto [10], Suzuki and Yamamoto [15] for systems without time-delays, and to Manitius and Triggiani [6], Nakagiri and Yamamoto [11], [12], Salamon [14] for time-delay systems. However, with respect to time-delay systems, there has appeared little work for controllability with constraint (e.g. Chan and Li [1]).

In this paper we study admissible controllability problems for a linear time-delay system in a Banach space:

Forcing functions and initial functions operate as controls in the system and those controls are assumed to belong to some norm-bounded constraint sets. Then, in such constraint sets, we find controls steering a given initial state to a desired state.

These are problems in game theory, where in order to obtain a desired state, two persons (called players) can move respective controls in a linear time-delay system; a forcing function and an initial function correspond to two players' controls. For these problems, there are two types (the cooperative type and the noncooperative type), according to choices of two players' controls. The purpose of this paper is to derive necessary and sufficient conditions for the two types of admissible controllability. For the cooperative type and the noncooperative type, these conditions are stated respectively in Theorems 1 and 2 (§ 2), and Theorem 3 (§ 3). Moreover in § 4, we give a simple example explaining Theorem 1.

Now, in this secton, we give exact description of a linear time-delay control system in a Banach space. Let X and U be reflexive Banach spaces over $\boldsymbol{C}$ or $\boldsymbol{R}$, with norms $\|\cdot\|$ and $\|\cdot\|_U$, respectively. Consider an abstract control system (1.1) on X with time-delays:

$$(1.1)\quad \begin{cases} \dfrac{dx(t)}{dt}=A_0x(t)+\displaystyle\int_{-h}^{0} d\eta(s)x(t+s)+B(t)u(t) \quad a.e.\ t>0 \\ x(0)=g^0,\ x(s)=g^1(s) \quad a.e.\ s\in[-h,\ 0), \end{cases}$$

where $g=(g^0,\ g^1)\in X\times L_p([-h,\ 0];\ X)$, $u\in L_q^{loc}(\boldsymbol{R}^+;\ U)$, $p,\ q\in(1,\ \infty)$, $\{B(t);\ t\geqq 0\}\subset \mathcal{L}(U,\ X)$ is a strongly continuous family of bounded operators from U into X, A_0 generates a C_0-semigroup $\{T(t);\ t\geqq 0\}$ on X (cf. Tanabe [16]) and η is a Stieltjes measure given by

$$(1.2)\qquad \eta(s)=-\sum_{r=1}^{m}\chi_{(-\infty,\ -h_r]}(s)A_r-\int_s^0 A_I(\xi)d\xi, \quad s\in[-h,\ 0].$$

In (1.2), χ_E denotes the characteristic function of E and it is assumed that $0<h_1<\ldots<h_m\equiv h$, $A_r\in\mathcal{L}(X)$ $(r=1,\ldots,\ m)$ and $A_I(\cdot)\in L_1([-h,\ 0];\ \mathcal{L}(X))$. Here and henceforth $\mathcal{L}(U,\ X)$ denotes the set of all bounded linear operators on U into X and also $\mathcal{L}(X)\equiv\mathcal{L}(X,\ X)$ is defined similarly. Then the delayed term in (1.1) is written by

$$\sum_{r=1}^{m} A_r x(t-h_r)+\int_{-h}^{0} A_I(s)x(t+s)ds.$$

Let $W(t)$ be the fundamental solution of (1.1), which is a unique solution of the equation

$$W(t)=\begin{cases} T(t)+\displaystyle\int_0^t T(t-s)\int_{-h}^{0} d\eta(\xi)W(\xi+s)ds, & t\geqq 0 \\ 0, & t<0. \end{cases}$$

Then $W(t)\in\mathcal{L}(X)$ for each $t\geqq 0$ and $W(t)$ is strongly continuous in $\boldsymbol{R}^+$ (e.g. Nakagiri [8]).

If the condition

$$(1.3)\qquad A_I(\cdot)\in L_{p'}([-h,\ 0];\ \mathcal{L}(X)), \quad 1/p+1/p'=1$$

is satisfied, then for each $t\geqq 0$, the operator valued function $U_t(\cdot)$ given by

$$(1.4)\qquad U_t(s)=\int_{-h}^{s} W(t-s+\xi)d\eta(\xi) \quad a.e.\ s\in[-h,\ 0]$$

belongs to $L_{p'}([-h,\ 0];\ \mathcal{L}(X))$. This follows from the Hausdorff-Young inequality. Hence the function

$$(1.5)\qquad x(t;\ g,\ u)=\begin{cases} W(t)g^0+\displaystyle\int_{-h}^{0} U_t(s)g^1(s)ds+\int_0^t W(t-s)B(s)u(s)ds, & t\geqq 0 \\ g^1(t) \quad a.e.\ t\in[-h,\ 0) \end{cases}$$

is well-defined and is an element of $C(\boldsymbol{R}^+;\ X)$. Moreover it is proved in [8] that under the condition (1.3), the function $x(t)=x(t;\ g,\ u)$ is a unique solution of the integrated form of (1.1) by $T(t)$, i.e.,

$$x(t)=T(t)g^0+\int_0^t T(t-s)B(s)u(s)ds+\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)x(s+\xi)ds, \quad t\geqq 0. \tag{1.6}$$

In this sense, this function $x(t)$ is called the mild solution of (1.1). In the system (1.1), $u(t)$ and $g^1(s)$ are called a forcing function control and an initial function control, respectively. Here we note that $g^0\equiv x(0)$ is not considered as a control. We will study the admissible controllability by means of mild solutions. Admissible controllability problems for linear time-delay systems can be solved also by directly applying the semigroup theory (e.g. Example 5.1 in [1]). In this paper, however, we will use the fundamental solution developed in Nakagiri [7], [8] rather than the semigroup associated with (1.1), so that we can obtain results which are more useful than those in [1].

§ 2. Admissible controllability of the cooperative type

For each $t>0$, $\delta\geqq 0$, $\rho\geqq 0$ and p, $q\in(1, \infty)$, we define the constraint sets U_δ^t and G_ρ^1 by

$$\{u\in L_q([0, t]; U); \|u\|_q\equiv\left(\int_0^t \|u(s)\|_U^q ds\right)^{1/q}\leqq\delta\} \tag{2.1}$$

and

$$\{g^1\in L_p([-h, 0]; X); \|g^1\|_p\equiv\left(\int_{-h}^0 \|g^1(s)\|^p ds\right)^{1/p}\leqq\rho\}, \tag{2.2}$$

respectively. The sets U_δ^t and G_ρ^1 are convex and closed in $L_q([0, t]; U)$ and $L_p([-h, 0]; X)$, respectively. We put $Y_{\delta,\rho}^t=U_\delta^t\times G_\rho^1$ and define reachable sets $\mathcal{R}_t(g^0, Y_{\delta,\rho}^t)$ and $\mathcal{R}_\infty(g^0, \delta, \rho)$ by

$$\mathcal{R}_t(g^0, Y_{\delta,\rho}^t)=\{x\in X; x=x(t; (g^0, g^1), u) \quad \textit{where } (u, g^1)\in Y_{\delta,\rho}^t\} \tag{2.3}$$

and

$$\mathcal{R}_\infty(g^0, \delta, \rho)=\bigcup_{t>0}\mathcal{R}_t(g^0, Y_{\delta,\rho}^t). \tag{2.4}$$

For any $t>0$, δ, $\rho\geqq 0$ and p, $q\in(1, \infty)$, by the definition, $\mathcal{R}_t(g^0, Y_{\delta,\rho}^t)$ is convex. Moreover we have

LEMMA 1. *For any $t>0$, δ, $\rho\geqq 0$, p, $q\in(1, \infty)$ and $g^0\in X$, the set $\mathcal{R}_t(g^0, Y_{\delta,\rho}^t)$ is closed.*

We can see this lemma by the reflexiveness of U and X, and the proof is given in Appendix, for convenience.

In this section, we consider the admissible controllability of the cooperative type:

DEFINITION 1. The system (1.1) is said to be

(i) admissibly (δ, ρ)-controllable on $[0, t]$ (resp. in finite time) with respect to g^0, x^0 if $x^0 \in \mathcal{R}_t(g^0, Y^t_{\delta,\rho})$ (resp. $x^0 \in \mathcal{R}_\infty(g^0, \delta, \rho)$),

(ii) admissibly (δ, ρ)-controllable on $[0, t]$ with respect to g^0, $B(x^0; \varepsilon)$ if $B(x^0; \varepsilon) \cap \mathcal{R}_t(g^0, Y^t_{\delta,\rho}) \neq \emptyset$, where we set $B(x^0; \varepsilon) = \{x \in X;\ \|x - x^0\| \leqq \varepsilon\}$.

Here g^0 and x^0 are assumed to be an initial state and a desired state (a target point), respectively, and $B(x^0; \varepsilon)$ is a target set. Henceforth $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X and its adjoint space X^*.

Now we are ready to state Theorems 1 and 2, which generalize Theorems 4.1 and 4.2 in Conti [2] in two directions, to infinite dimensional spaces and to differential equations with time-delays.

THEOREM 1. *The system* (1.1) *is admissibly* (δ, ρ)*-controllable on* $[0, t]$ *(resp. in finite time) with respect to* g^0, x^0 *if and only if (resp. if and only if there exists a* $t>0$ *such that)*

$$(2.5)\qquad |\langle x^0 - W(t)g^0, x^* \rangle| \leqq \delta\Big(\int_0^t \|B^*(s)W^*(t-s)x^*\|_{U^*}^{q'} ds\Big)^{1/q'} + \rho\Big(\int_{-h}^0 \|U_t^*(s)x^*\|_*^{p'} ds\Big)^{1/p'} \quad \textit{for each } x^* \in X^*,$$

where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, *the operator* $U_t^*(s)$ *is given by* $U_t^*(s) = \sum_{r=1}^m \chi_{[-h_r, 0]}(s) A_r^* W^*(t-s-h_r) + \int_{-h}^s A_1^*(\xi) W^*(t-s+\xi) d\xi$ *a.e.* $s \in [-h, 0)$ *and* $\|\cdot\|_{U^*}$, $\|\cdot\|_*$ *denote the norms in* U^*, X^*, *respectively.*

With respect to fixed g^0 and x^0, the (δ, ρ)-controllability depends heavily upon δ and ρ, the radii of constraint sets of controls. For example, by the definition, the (δ, ρ)-controllability implies the (δ', ρ')-controllability if $\delta \leqq \delta'$ and $\rho \leqq \rho'$. Theorem 1 states how the admissible controllability depends upon δ and ρ.

THEOREM 2. *We assume that*

$$(2.6)\qquad T(t) \textit{ is compact for all } t>0.$$

Then the system (1.1) *is admissibly* (δ, ρ)*-controllable on* $[0, t]$ *with respect to* g^0, $B(x^0; \varepsilon)$ *if and only if*

$$(2.7)\qquad |\langle x^0 - W(t)g^0, x^* \rangle| - \varepsilon\|x^*\|_* \leqq \delta\Big(\int_0^t \|B^*(s)W^*(t-s)x^*\|_{U^*}^{q'} ds\Big)^{1/q'} + \rho\Big(\int_{-h}^0 \|U_t^*(s)x^*\|_*^{p'} ds\Big)^{1/p'} \quad \textit{for each } x^* \in X^*.$$

In Theorem 2, we note that the condition (2.7) is weaker than (2.5), in proportion to ε, the radius of the target set. Moreover Theorem 1 is true also for functional differential equations with unbounded delay terms studied in Tanabe [17].

Proof of Theorem 2. We recall the strict separating hyperplane theorem (Reed and Simon [p. 130, 13]).

LEMMA 2. *Let E and F be disjoint closed and convex sets in X. If E or F is compact, then they can be strictly separated by a hyperplane, in other words, there exists an $x_s^* \in X^*$ such that* $\sup_{x\in E}\langle x, x_s^*\rangle < \inf_{x\in F}\langle x, x_s^*\rangle$.

For each $t>0$, we define two operators $\mathcal{B}_t: L_q([0, t]; U) \longrightarrow X$ and $\mathcal{G}_t: L_p([-h, 0]; X) \longrightarrow X$ by

$$\mathcal{B}_t u = \int_0^t W(t-s)B(s)u(s)ds \tag{2.8}$$

and

$$\mathcal{G}_t g^1 = \int_{-h}^0 U_t(s)g^1(s)ds, \tag{2.9}$$

respectively.

We prove the "if" part by contradiction. Suppose that (1.1) is not admissibly (δ, ρ)-controllable on $[0, t]$ with respect to g^0, $B(x^0; \varepsilon)$. Then by Definition 1, $B(x^0; \varepsilon)\cap \mathcal{R}_t(g^0, Y^t_{\delta,\rho})=\emptyset$. This implies by (1.5) that the two convex sets $E\equiv\mathcal{R}_t(0, Y^t_{\delta,\rho})=\mathcal{B}_t(U^t_\delta)+\mathcal{G}_t(G^1_\rho)$ and $F\equiv B(x^0-W(t)g^0; \varepsilon)$ are disjoint. By Lemma 1, E is closed. Hence both E and F are closed. Furthermore, on the assumption (2.6), $\mathcal{B}_t$ and $\mathcal{G}_t$ are compact for all $t>0$ (Nakagiri [Lemma 3.3, 9]), so that E is compact in X. Therefore it follows by Lemma 2 that there exists an $x_s^*\in X^*$ such that

$$\inf_{x\in F}\langle x, x_s^*\rangle > d_0 \equiv \sup_{x\in E}\langle x, x_s^*\rangle. \tag{2.10}$$

Note that d_0 is finite (E is compact!). On the other hand,

$$\begin{aligned}\inf_{x\in F}\langle x, x_s^*\rangle &= \inf_{\|y\|\le\varepsilon}\langle x^0-W(t)g^0+y, x_s^*\rangle \\ &=\langle x^0-W(t)g^0, x_s^*\rangle-\varepsilon \sup_{\|z\|\le 1}\langle z, x_s^*\rangle \\ &=\langle x^0-W(t)g^0, x_s^*\rangle-\varepsilon\|x_s^*\|_*\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}d_0 &= \sup_{x\in E}\langle x, x_s^*\rangle \\ &= \sup\left\{\left\langle\int_0^t W(t-s)B(s)u(s)ds, x_s^*\right\rangle ; u\in U^t_\delta\right\}\end{aligned} \tag{2.12}$$

$$+\sup\left\{\left\langle\int_{-h}^{0} U_t(s)g^1(s)ds,\ x_s^*\right\rangle;\ g^1\in G_\rho^1\right\}$$

$$=\sup\left\{\int_0^t \langle u(s),\ B^*(s)W^*(t-s)x_s^*\rangle_{U,U^*}ds;\ \|u\|_q\leqq\delta\right\}$$

$$+\sup\left\{\int_{-h}^0 \langle g^1(s),\ U_t^*(s)x_s^*\rangle ds;\ \|g^1\|_p\leqq\rho\right\}$$

$$=\delta\left(\int_0^t \|B^*(s)W^*(t-s)x_s^*\|_{U^*}^{q'}ds\right)^{1/q'}$$

$$+\rho\left(\int_{-h}^0 \|U_t^*(s)x_s^*\|_*^{p'}ds\right)^{1/p'},$$

where $\langle\cdot,\ \cdot\rangle_{U,U^*}$ is the duality pairing between U and U^*. Then by (2.10)–(2.12), we have

$$\langle x^0-W(t)g^0,\ x_s^*\rangle-\varepsilon\|x_s^*\|_*>\delta\left(\int_0^t \|B^*(s)W^*(t-s)x_s^*\|_{U^*}^{q'}ds\right)^{1/q'}$$

$$+\rho\left(\int_{-h}^0 \|U_t^*(s)x_s^*\|_*^{p'}ds\right)^{1/p'},$$

which contradicts (2.7).

Next we will prove the "only if" part. Assume that (1.1) is admissibly $(\delta,\ \rho)$-controllable. Then there exist $\bar{x}^1\in X$, $\bar{u}\in U_\delta^t$ and $\bar{g}^1\in G_\rho^1$ such that

$$\bar{x}^1=\int_0^t W(t-s)B(s)\bar{u}(s)ds+\int_{-h}^0 U_t(s)\bar{g}^1(s)ds\in B(x^0-W(t)g^0;\ \varepsilon).$$

Hence for any $x^*\in X^*$, we have

$$|\langle x^0-W(t)g^0,\ x^*\rangle|-\varepsilon\|x^*\|_*$$

$$\leqq|\langle x^0-W(t)g^0,\ x^*\rangle|-|\langle x^0-W(t)g^0-\bar{x}^1,\ x^*\rangle|\leqq|\langle\bar{x}^1,\ x^*\rangle|$$

$$\leqq\left|\left\langle\int_0^t W(t-s)B(s)\bar{u}(s)ds,\ x^*\right\rangle\right|+\left|\left\langle\int_{-h}^0 U_t(s)\bar{g}^1(s)ds,\ x^*\right\rangle\right|$$

$$\leqq\delta\left(\int_0^t \|B^*(s)W^*(t-s)x^*\|_{U^*}^{q'}ds\right)^{1/q'}+\rho\left(\int_{-h}^0 \|U_t^*(s)x^*\|_*^{p'}ds\right)^{1/p'}$$

(by Hölder's inequality).

This proves (2.7).

Proof of Theorem 1. In this theorem, the compactness of $\mathcal{B}_t$ and $\mathcal{G}_t$ is not necessary, since $F=B(x^0-W(t)g^0;\ 0)=\{x^0-W(t)g^0\}$ is compact (one point). Thus we readily see this theorem from the proof of Theorem 2. The admissible $(\delta,\ \rho)$-controllability in finite time can be proved similarly.

§ 3. Admissible controllability of the noncooperative type

In this section, we discuss an admissible controllability problem which is noncooperative in the sense that against one player's control, the other player may select an appropriate control. That is,

DEFINITION 2. The system (1.1) is said to be admissibly (δ, ρ)-controllable on $[0, t]$ in the noncooperative sense with respect to g^0, $B(x^0; \varepsilon)$ if for each initial function control $g^1 \in G^1_\rho$, there exists a forcing function control $u \in U^t_\delta$ such that $x(t; (g^0, g^1), u) \in B(x^0; \varepsilon)$.

In game theory, g^1 and u correspond to an evader's control and a pursuer's control, respectively (cf. Hajek [5]).

Our result is

THEOREM 3. *The system* (1.1) *is admissibly* (δ, ρ)*-controllable on* $[0, t]$ *in the noncooperative sense with respect to* g^0, $B(x^0; \varepsilon)$ *if and only if*

$$|\langle x^0 - W(t)g^0, x^*\rangle| - \varepsilon\|x^*\|_* \leqq \delta\Big(\int_0^t \|B^*(s)W^*(t-s)x^*\|_{U^*}^{q'} ds\Big)^{1/q'} - \rho\Big(\int_{-h}^0 \|U_t^*(s)x^*\|_*^{p'} ds\Big)^{1/p'} \quad \textit{for each } x^* \in X^*. \tag{3.1}$$

The last term in (3.1) corresponding to the evader's controls is negative by the noncooperativeness.

Proof. First we show

LEMMA 3. *Let E and F be closed convex sets in X. Then $E \subset F$ if and only if*

$$\sup_{x\in E} \langle x, x^*\rangle \leqq \sup_{x\in F} \langle x, x^*\rangle \quad \textit{for all } x^* \in X^*. \tag{3.2}$$

Proof of Lemma 3. The "only if" part is obvious. We shall show the "if" part by contradiction. Let (3.2) be satisfied and suppose that E is not contained in F. Then there exists an element $x_0 \in E \backslash F$. Since $\{x_0\}$ is compact and $\{x_0\} \cap F = \emptyset$, it follows by Lemma 2 that $\{x_0\}$ and F can be separated strictly, i.e.,

$$\sup_{x\in F} \langle x, x_s^*\rangle < \langle x_0, x_s^*\rangle \leqq \sup_{x\in E} \langle x, x_s^*\rangle \tag{3.3}$$

for some $x_s^* \in X^*$. Clearly (3.3) contradicts (3.2), and hence the proof is finished.

Now we return to the proof of Theorem 3. By Definition 2, the system (1.1) is admissibly (δ, ρ)-controllable on $[0, t]$ in the noncooperative sense with respect to g^0, $B(x^0; \varepsilon)$ if and only if

$$W(t)g^0+\mathcal{G}_t(G^1_\rho)\subset-\mathcal{B}_t(U^t_\delta)+B(x^0;\varepsilon). \tag{3.4}$$

By Lemma 1, $\mathcal{G}_t(G^1_\rho)=\mathcal{R}_t(0, Y^t_{0,\rho})$ is closed. Hence the set $W(t)g^0+\mathcal{G}_t(G^1_\rho)$ is closed in X. Moreover $-\mathcal{B}_t(U^t_\delta)+B(x^0;\varepsilon)=\mathcal{B}_t(U^t_\delta)+B(x^0;\varepsilon)$ is closed. (In fact, let $u_n=y_n+z_n$, $y_n\in\mathcal{B}_t(U^t_\delta)$ and $z_n\in B(x^0;\varepsilon)$ $(n\geqq 1)$ and let u_n converge to u strongly in X. Then by the reflexiveness of X, there exists a subsequence $\{z_{n'}\}$ such that $z_{n'}\longrightarrow z$ weakly for some $z\in X$ (e.g. [p. 141, 18]). Thus $y_{n'}=u_{n'}-z_{n'}$ converge weakly to $y\equiv u-z$. Since the convex set $\mathcal{B}_t(U^t_\delta)=\mathcal{R}_t(0, Y^t_{\delta,0})$ is closed by Lemma 1, we have $y\in\mathcal{B}_t(U^t_\delta)$ (e.g. [p. 120, 18]). On the other hand, since $z_{n'}-x^0\longrightarrow z-x^0$ weakly, we have $\|z-x^0\|\leqq\liminf_{n'\to\infty}\|z_{n'}-x^0\|\leqq\varepsilon$ (e.g. [p. 120, 18]), namely, $z\in B(x^0;\varepsilon)$. Therefore $u\in\mathcal{B}_t(U^t_\delta)+B(x^0;\varepsilon)$.)
Obviously both $W(t)g^0+\mathcal{G}_t(G^1_\rho)$ and $\mathcal{B}_t(U^t_\delta)+B(x^0;\varepsilon)$ are convex. Thus we can apply Lemma 3, so that we see that (3.4) is equivalent to

$$\begin{aligned}&\sup\{\langle W(t)g^0+\mathcal{G}_tg^1, x^*\rangle;\ g^1\in G^1_\rho\}\\ &\leqq\sup\{\langle\mathcal{B}_tu+y, x^*\rangle;\ u\in U^t_\delta,\ y\in B(x^0;\varepsilon)\}\end{aligned} \tag{3.5}$$

for each $x^*\in X^*$.

Then as in the proof of Theorem 2, we can verify that (3.5) is equivalent to that

$$\begin{aligned}&\langle W(t)g^0, x^*\rangle+\rho\Big(\int_{-h}^0\|U^*_t(s)x^*\|^{p'}_*\,ds\Big)^{1/p'}\\ &\leqq\delta\Big(\int_0^t\|B^*(s)W^*(t-s)x^*\|^{q'}_{Y^*}\,ds\Big)^{1/q'}+\langle x^0, x^*\rangle+\varepsilon\|x^*\|_*.\end{aligned} \tag{3.6}$$

Replacing x^* by $-x^*$ in (3.6), we obtain the condition (3.1). This completes the proof.

§ 4. An example

In this section, we consider the following simple control system on $X=\boldsymbol{R}$ with one time-delay and we express the condition (2.5) for the admissible (δ,ρ)-controllability.

$$\begin{cases}\dfrac{dx(t)}{dt}=x(t)+x(t-1)+u(t) & a.e.\ t>0\\ x(0)=g^0,\ x(s)=g^1(s) & a.e.\ s\in[-1, 0),\end{cases} \tag{4.1}$$

where $g=(g^0, g^1)\in\boldsymbol{R}\times L_2([-1, 0];\boldsymbol{R})$ and $u\in L_2^{\mathrm{loc}}(\boldsymbol{R}^+;\boldsymbol{R})$.

As is easily checked, the fundamental solution $W(t)$ of (4.1) is given by

$$W(t)=\begin{cases}e^t & (0\leqq t\leqq 1)\\ e^t+(t-1)e^{t-1} & (1\leqq t\leqq 2),\ \text{etc.}\end{cases} \tag{4.2}$$

Thus, by Theorem 1, for $1 \leqq t \leqq 2$, the system (4.1) is admissibly (δ, ρ)-controllable on $[0, t]$ with respect to g^0, $x^0 \in \boldsymbol{R}$ if and only if

$$(4.3) \qquad |x^0 - W(t)g^0| \leqq \delta\left(\int_0^t W(s)^2 ds\right)^{1/2} + \rho\left(\int_{t-1}^t W(s)^2 ds\right)^{1/2},$$

namely,

$$(4.4) \qquad \begin{aligned} &2|x^0 - (e^t + (t-1)e^{t-1})g^0| \\ &\leqq \delta(2e^{2t} + 2e - 3 + (2t^2 - 6t + 5)e^{2t-2} + (4t-6)e^{2t-1})^{1/2} \\ &+ \rho(2e^{2t} + 2e - 1 + (2t^2 - 6t + 3)e^{2t-2} + (4t-6)e^{2t-1})^{1/2}. \end{aligned}$$

Here we note that $W(t) = W(t)^*$ $(t \geqq 0)$.

Appendix. Proof of Lemma 1

Let $x(t; (g^0, g_n^1), u_n)$ strongly converge to some $x_0 \in X$ as $n \longrightarrow \infty$ for $(u_n, g_n^1) \in Y_{\delta,\rho}^t$. Then we have to prove that $x_0 = x(t; (g^0, g^1), u)$ for some $(u, g^1) \in Y_{\delta,\rho}^t$. The reflexiveness of U and X, and p, $q \in (1, \infty)$ imply the reflexiveness of $L_q([0, t]; U)$ and $L_p([-h, 0]; X)$. Since $Y_{\delta,\rho}^t$ is bounded in the reflexive Banach space $L_q([0, t]; U) \times L_p([-h, 0]; X)$, there exists a subsequence $\{(u_n, g_n^1)\}$ weakly convergent to (u, g^1) (e.g. [p. 141, 18]). Furthermore, by $\|u\|_q \leqq \liminf_{n\to\infty} \|u_n\|_q$ and $\|g^1\|_p \leqq \liminf_{n\to\infty} \|g_n^1\|_p$ (e.g. [p. 120, 18]), we see $(u, g^1) \in Y_{\delta,\rho}^t$. By Mazur's theorem (e.g. [p. 120, 18]), a sequence consisting of appropriate convex combinations of (u_n, g_n^1) $(n \geqq 1)$, strongly converges to (u, g^1), say, $(v_n, h_n^1) = \sum_{k=1}^{N_n} \alpha_{nk}(u_k, g_k^1)$ $(\sum_{k=1}^{N_n} \alpha_{nk} = 1, \alpha_{nk} > 0)$, we have $\|v_n - u\|_q$, $\|h_n^1 - g^1\|_p \longrightarrow 0$ as $n \longrightarrow \infty$. On the other hand, by the strong continuity of $B(\cdot)$, we see that $\{\|B(s)u\|;\ 0 \leqq s \leqq t\}$ is bounded for each $u \in U$, so that $\sup_{0 \leqq s \leqq t} \|B(s)\|_{\mathcal{L}(U,X)} < \infty$ by the resonance theorem (e.g. [p. 69, 18]). Therefore by the linearity of (1.1) and an estimate of solutions ([Theorem 2.1, 8]), we see: $\sum_{k=1}^{N_n} \alpha_{nk} x(t; (g^0, g_k^1), u_k) = x(t; (g^0, h_n^1), v_n) \longrightarrow x(t; (g^0, g^1), u)$ strongly in X. This implies that $x_0 = x(t; (g^0, g^1), u)$, which completes the proof of Lemma 1.

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