Control of Sylow 2-Intersections in Groups of Chev(2) Type and Groups of Alternating Type

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Introduction

Let p be a prime, G be a group of order divisible by p, and $\mathscr{F}$ be a normal set of subgroups of G. For $S \in \mathrm{Syl}_p(G)$, let $\mathscr{F}(S)$ be the set of all $X \in \mathscr{F}$ such that $S \cap X \in \mathrm{Syl}_p(X)$. The set $\mathscr{F}$ is said to *control* Sylow p-intersections in G if for each pair S, T of distinct Sylow p-subgroups of G such that $S \cap T \neq 1$, there exist Sylow p-subgroups $S_0, S_1, \cdots, S_n$ of G, elements $X_1, \cdots, X_n$ of $\mathscr{F}$, and an element $x_i \in X_i$ for each i satisfying the following conditions:

(1) $S_0 = S$ and $S_n = T$;

(2) $X_i \in \mathscr{F}(S_{i-1}) \cap \mathscr{F}(S_i)$ for each i;

(3) $S_i^{x_i} = S_{i-1}$ for each i;

(4) $S \cap T \leqq S_i \cap X_i$ for each i.

As is shown in [**6**, Theorem 1.4], if the set $\mathscr{F}$ controls Sylow p-intersections in G, then it controls the p-fusion in G. That is, whenever A is a subset $\neq 1$ of $S \in \mathrm{Syl}_p(G)$ and g is an element of G such that $A^g \leqq S$, there exist elements $Y_1, \cdots, Y_n$ of $\mathscr{F}(S)$, an element $y_i \in Y_i$ for each i, and an element $y \in N_G(S)$ such that $A^g = A^{y_1 \cdots y_n y}$ and $A^{y_1 \cdots y_i} \leqq S \cap Y_i$ for each i. Let $\mathscr{H}_0 = \mathscr{H}_{0,p,G}$ be the set of all nonidentity p-subgroups H of G such that $N_G(H)/H$ is p-isolated in the sense of Goldschmidt [**4**]. For $H \in \mathscr{H}_0$, let $N_G^*(H)$ be the subgroup of $N_G(H)$ containing H such that $N_G^*(H)/H$ is the unique minimal subnormal subgroup of $N_G(H)/H$ of order divisible by p. A useful sufficient condition for the set $\mathscr{F}$ to control Sylow p-intersections in G is that, for each $H \in \mathscr{H}_0$, $N_G^*(H)$ is contained in some member of $\mathscr{F}$ [**6**, Theorem 1.7]. Thus, in studying the conjugacy of subsets of $S \in \mathrm{Syl}_p(G)$, one is naturally led to the problem of finding a suitable set $\mathscr{F}$ in G satisfying this sufficient condition.

In this paper, we consider the above problem in the special case where $p=2$ and G has a normal subgroup N such that G/N is a 2-group and N is a central product of conjugates in G of a quasisimple subgroup L. We will show that if $L \in \mathrm{Chev}(2)$ or $L \cong A_n$ then we can find, in most cases, a normal set $\mathscr{F}$ in

G consisting of almost 2-constrained core-free subgroups of odd index such that, for each $H \in \mathcal{H}_{0,2,G}$, $N_G^*(H)$ is contained in some member of $\mathcal{F}$. Here, Chev(2) denotes the collection of all quasisimple groups L with $O_2(L)=1$ such that $L/Z(L)$ is isomorphic to a finite simple group of Lie type and of characteristic 2.

The exact statement of the above result and its proof will be presented in Section 4 (4.2 and 4.3). The proof requires precise information about the set $\mathcal{H}_{0,2}$ in the automorphism groups of the groups in Chev(2) and the alternating groups. Sections 2 and 3 are intended for this topic. For instance, we show in Section 3 that if $G=\Sigma_n$, n even, and $H \in \mathcal{H}_{0,2,G}$, then $N_G(H)$ is contained in a subgroup of G isomorphic to $\Sigma_2 \int \Sigma_{n/2}$ or $\Sigma_4 \int \Sigma_{n/4}$ or $\Sigma_4 \int \Sigma_{(n-2)/4} \times \Sigma_2$. Section 1 contains a list of properties of the set $\mathcal{H}_{0,2,G}$ and a related set $\mathcal{H}_{2,G}$.

The main results of this paper (3.3, 4.2, and 4.3) are needed in the final section of [**6**], where we prove a theorem generalizing the $C(G, T)$-theorem of Aschbacher [**1**] and a 2-fusion theorem due to P. McBride.

1. The Sets $\mathcal{H}_{2,G}$ and $\mathcal{H}_{0,2,G}$

In this section, G is an arbitrary finite group. Let $\mathcal{H}_G=\mathcal{H}_{2,G}$ be the set of all 2-subgroups H of G such that $H=O_2(N_G(H))$, and let $\mathcal{H}_{0,G}=\mathcal{H}_{0,2,G}$ be the set of all nonidentity 2-subgroups H of G such that $N_G(H)/H$ is 2-isolated of even order. Clearly, $\mathcal{H}_{0,G}$ is a subset of $\mathcal{H}_G$. For $H \in \mathcal{H}_{0,G}$, let $N_G^*(H)$ be the subgroup of $N_G(H)$ containing H such that $N_G^*(H)/H$ is the unique minimal subnormal subgroup of $N_G(H)/H$ of even order. The $N_G^*(H)$ is well-defined (see [**6**, Definition 1.6]).

1.1. *If $H \in \mathcal{H}_G$, then H contains all $N_G(H)$-invariant 2-subgroups of G.*

Proof. Let X be an $N_G(H)$-invariant 2-subgroup of G. Then $N_{XH}(H)$ is a normal 2-subgroup of $N_G(H)$; so $N_{XH}(H)=H$ and $X \leqq H$.

1.2. *Let Z be a subgroup of $Z(G)$ and let bars denote images in G/Z. Then $\mathcal{H}_{\bar{G}}=\overline{\mathcal{H}_G}$.*

Proof. In view of 1.1, we assume that Z has odd order. If H is a 2-subgroup of G, then $N_{\bar{G}}(\bar{H})=\overline{N_G(H)}$ and $O_2(N_{\bar{G}}(\bar{H}))=\overline{O_2(N_G(H))}$, whence the assertion follows.

1.3. *If G is a central product of subgroups $G_1, G_2, \cdots, G_n$ and if $H \in \mathcal{H}_G$, then $H=H_1H_2\cdots H_n$, where $H_i \in \mathcal{H}_{G_i}$ for each i.*

Proof. In view of 1.2, we assume $G=G_1 \times G_2 \times \cdots \times G_n$. Let H_i be the projection of H into G_i for each i. Then $H_i \leqq H$ for each i by 1.1; so $H=H_1H_2\cdots H_n$ and, since $N_G(H)$ is the product of the $N_{G_i}(H_i)$, it follows that $H_i \in \mathcal{H}_{G_i}$ for each i.

1.4. *If A is a group acting on G and X is an A-invariant 2-subgroup of G, then there is an A-invariant element $H \in \mathcal{H}_G$ such that $X \leqq H$ and $N_G(X) \leqq N_G(H)$.*

Proof. Recursively, define $H_1 = X$ and $H_n = O_2(N_G(H_{n-1}))$ for $n = 2, 3, \cdots$. Then $H_n \leqq H_{n+1}$ and $N_G(H_n) \leqq N_G(H_{n+1})$ for each n. Choose n so that $H_n = H_{n+1}$. Then $H = H_n$ meets all the requirements.

1.5 LEMMA. *Let N be a 2-isolated finite group and M be the unique minimal subnormal subgroup of N of even order. Then any subgroup of N containing M is 2-isolated. In particular, subnormal subgroups of N of even order are 2-isolated.*

Proof. Let L be a strongly embedded subgroup of N. Then $N = ML$ and so if $M \leqq X \leqq N$ then $L \cap X$ is a strongly embedded subgroup of X.

1.6. *If $H \in \mathcal{H}_{0,G}$ and $N_G^*(H) \leqq X \leqq G$, then $H \in \mathcal{H}_{0,X}$ and $N_X^*(H) = N_G^*(H)$.*

Proof. This follows from 1.5 applied to $N_G(H)/H$ and from the definition of $N_G^*(H)$.

1.7 LEMMA. *Suppose a group H acts on $G = G_1 \times G_2 \times \cdots \times G_n$, and let $H_i = N_H(G_i)$. If H permutes the G_i, $1 \leqq i \leqq n$, transitively, and $C_{G_i}(H_i) \neq 1$ for each i, then the following holds:*

(1) *if $x \in C_{G_1}(H_1)$, then $C_{G_i}(H_i)$ contains a unique conjugate, say, x_i of x under the action of H, $1 \leqq i \leqq n$;*

(2) $C_G(H) = \{x_1 x_2 \cdots x_n\,;\, x \in C_{G_1}(H_1)\}$;

(3) $C_G(H) \cong C_{G_1}(H_1)$.

Proof. Since $H_i \leqq H$, it follows that $C_G(H)$ is contained in the product of the $C_{G_i}(H_i)$. By assumption, $C_{G_i}(H_i) \neq 1$, whence $N_H(C_{G_i}(H_i)) = H_i$ for each i. Therefore, it suffices to consider the case where H_i centralizes G_i for each i. Let $1 \neq x \in G_1$ and assume $x^g, x^h \in G_i$ with $g, h \in H$. Then $1 \neq x^g \in G_i \cap G_i^{h^{-1}g}$; so $h^{-1}g \in H_i = C_H(G_i)$ and $x^g = (x^h)^{h^{-1}g} = x^h$. Thus if $x \in G_1$, $x^H \cap G_i$ consists of a single element, say, x_i. If $h \in H$ and $G_1{}^h = G_i$, then $x^h = x_i$ for each $x \in G_1$. Hence it follows that $C_G(H) = \{x_1 x_2 \cdots x_n\,;\, x \in G_1\}$ and that the mapping $x \to x_1 x_2 \cdots x_n$, $x \in G_1$, is an isomorphism of G_1 onto $C_G(H)$.

1.8 LEMMA. *Suppose a 2-group H acts on $G = G_1 \times G_2 \times \cdots \times G_n$, where $|G_i|$ is even for each i and H permutes the G_i. If furthermore $C_G(H)$ is 2-isolated, then the G_i are all conjugate under the action of H and $C_{G_1}(H_1)$ is 2-isolated, where $H_1 = N_H(G_1)$.*

Proof. Since $C_G(H)$ has a unique minimal subnormal subgroup of even order, it follows that the G_i are all conjugate under the action of H, whence $C_{G_1}(H_1)\cong C_G(H)$ by 1.7.

2. The Set $\mathscr{H}_{0,2,G}$ and Groups in Chev(2).

Let Chev(2) be the collection of all quasisimple groups L with $O_2(L)=1$ such that $L/Z(L)$ is isomorphic to a finite simple group of Lie type and of characteristic 2. We consider $A_6\cong Sp_4(2)'$, $SU_3(3)\cong G_2(2)'$, and ${}^2F_4(2)'$ to be of Lie type and of characteristic 2. Thus the 3-fold covering group $\hat{A}_6$ of A_6 is a member of Chev(2). Let

$$\mathrm{Chev}(2)^+=\mathrm{Chev}(2)\cup\{Sp_4(2),\ G_2(2),\ {}^2F_4(2)\},$$

and let Chev(2)$^-$ be the collection of all $L\in$Chev(2) such that $L/Z(L)\not\cong A_6$, $SU_3(3)$ or ${}^2F_4(2)'$.

For $L\in$Chev(2)$^+$, a *Borel subgroup* of L is a Sylow 2-normalizer of L, and a *parabolic subgroup* of L is a subgroup containing a Borel subgroup. Borel subgroups and L itself are called the *trivial* parabolic subgroups. Notice that if P is a parabolic subgroup of L and U is a Sylow 2-subgroup of L contained in P then $N_L(U)\leqq P$ by Sylow's theorem.

For the definition and fundamental properties of the groups of Lie type, the reader is referred to Steinberg's lecture note [8].

2.1 LEMMA. (1) *If $L\in$Chev(2)$^+$, then a maximal 2-local subgroup of L is a proper parabolic subgroup of L.*

(2) *Suppose $L=A_6$, $SU_3(3)$ or ${}^2F_4(2)'$ and L is embedded in $G=Sp_4(2)$, $G_2(2)$ or ${}^2F_4(2)$, as the case may be. Then a Sylow 2-subgroup U of L is contained in a unique Sylow 2-subgroup T of G. The maximal 2-local subgroups of L containing U are the $P_i\cap L$, where P_i, $i=1,2$, are the nontrivial parabolic subgroups of G containing T, and $P_i\cap L/O_2(P_i\cap L)\cong P_i/O_2(P_i)$ for each i.*

Proof. We may assume $Z(L)=1$. The assertion follows from a theorem of Borel and Tits [3] except in the case $L=A_6$, $SU_3(3)$ or ${}^2F_4(2)'$, where the argument is as follows. Embed L in $G=Sp_4(2)$, $G_2(2)$ or ${}^2F_4(2)$, as the case may be. Let M be a maximal 2-local subgroup of L and N be a maximal 2-local subgroup of G containing M. Then N is a proper parabolic subgroup of G by [3], and moreover N is not Borel in G as M is not Borel in L (this is seen, when $L=SU_3(3)$ or ${}^2F_4(2)'$, from the structure of the centralizer of a central involution of L). So $N/O_2(N)\cong SL_2(2)$ or $Sz(2)$, $Sz(2)$ occuring only when $G={}^2F_4(2)$. This forces $O_2(M)\leqq O_2(N)$, whence $M=N\cap L$ and $O_2(M)=O_2(N)\cap L$. Moreover, $O_2(N)\not\leqq L$ (when $N/O_2(N)\cong Sz(2)$, this follows from Section 4 of [9]), whence $M/O_2(M)\cong N/O_2(N)$. Now let T be a Sylow 2-subgroup of G contained in N and set $U=T\cap L$, then $U\in\mathrm{Syl}_2(L)$ and $U\leqq M$. As $N_L(U)=U$, M is a parabolic subgroup of L. By the same reason, T is a unique Sylow 2-subgroup of G containing U. The assertions (1) and (2) now follow from the above discussion.

In A_6, a Sylow 2-subgroup is contained in precisely two nontrivial parabolic subgroups and they are 2-local subgroups (this is proved by using 2.1, Sylow's theorem, and Burnside's $p^a q^b$-theorem). The same is true of $SU_3(3)$ and ${}^2F_4(2)'$ (but this fact is irrelevant to the present work). In view of these facts, we define the ranks of these three groups and their central extensions to be 2. The rank of a remaining group in Chev$(2)^+$ is the ordinary one; that is, the rank of its split BN-pairs at characteristic 2.

2.2 LEMMA. *If* $L \in \mathrm{Chev}(2)^+$ *and* $H \in \mathcal{H}_L$, *then* $N_L(H)$ *is a parabolic subgroup of* L.

Proof. We argue by induction on the rank of L. In view of 1.2, we assume $Z(L)=1$. We also assume $H \neq 1$ and $H \notin \mathrm{Syl}_2(L)$. Let M be a maximal 2-local subgroup of L containing $N_L(H)$. By 2.1, M is a nontrivial parabolic subgroup of L and, if $K=O^{2'}(M)$, then $\bar{K}=K/O_2(M)$ is a nonempty central product of subgroups $\bar{K}_i$, $i=1,2,\cdots,k$, in the family

$$\mathrm{Chev}(2)^+ \cup \{SL_2(2),\ Sz(2),\ (P)SU_3(2)\}.$$

Now $O_2(M) \leqq H$ by 1.1. So $\bar{H} \in \mathcal{H}_{\bar{K}}$ and, by 1.3, $\bar{H}=\bar{H}_1\bar{H}_2\cdots\bar{H}_k$ with $\bar{H}_i \in \mathcal{H}_{\bar{K}_i}$ for each i. By the induction hypothesis when $\bar{K}_i \in \mathrm{Chev}(2)^+$ and by direct check when $\bar{K}_i \in \{SL_2(2), Sz(2), (P)SU_3(2)\}$, we get that $N_{\bar{K}_i}(\bar{H}_i)$ is a parabolic subgroup of $\bar{K}_i$. Furthermore, $\bar{H}_i$ is a Sylow 2-intersection in $N_{\bar{K}_i}(\bar{H}_i)$. Thus, there is a Sylow 2-subgroup U of L such that $U \leqq N_L(H)$ and $H=U \cap U^x$ for some $x \in L$. We assume $L \in \mathrm{Chev}(2)^-$, for U is Borel in L in the contrary case. Take a BN-pair of L such that $B=N_L(U)$. Then $x \in BnB$ for some $n \in N$; so $H=(U \cap U^n)^b$ for some $b \in B$, and H is normalized by $\langle U, (B \cap N)^b \rangle = B$. The proof is complete.

2.3 LEMMA. $\mathrm{Aut}({}^2F_4(2)') \cong \mathrm{Aut}({}^2F_4(2)) \cong {}^2F_4(2)$ *and* $\mathrm{Aut}(SU_3(3)) \cong \mathrm{Aut}(G_2(2)) \cong G_2(2)$.

We present only a proof of $\mathrm{Aut}({}^2F_4(2)') \cong {}^2F_4(2)$. The remaining parts of the lemma are well-known and their proofs are available in [8]. Let $G={}^2F_4(2)$, $L={}^2F_4(2)'$, and $A=\mathrm{Aut}(L)$. Embed G in A. Then since $\mathrm{Aut}(G) \cong G$, it follows that $N_A(G)=G$. Therefore, the proof will be completed once we show that A/L is a 2-group.

Let P and Q be the maximal 2-local subgroups of L containing $U \in \mathrm{Syl}_2(L)$, and set $R=O_2(P)$ and $S=O_2(Q)$. Choose notation so that $P/R \cong SL_2(2)$ and $Q/S \cong Sz(2)$ (see 2.1). We follow the notation used in Section 4 of [9], where Tits gives generators and relations for G. We may assume $U=U^+ \cap L$. Then $P=\langle U^+, r_8 \rangle \cap L$, $Q=\langle U^+, r_1 \rangle \cap L$, $R=\langle u_2, u_4, u_6, u_iu_j\,;\, i,j \in \{1,3,5,7\} \rangle$, and $S=\langle u_2, u_4, u_6, u_8, u_iu_j\,;\, i,j \in \{3,5,7\} \rangle$. Using relations given in [9], we can show that Q has, other than S and 1, precisely two normal 2-subgroups S_1 and S_2, where $S_1=\langle u_3^2, u_4, u_5^2, u_6, u_7^2 \rangle$ and $S_2=\langle u_5^2 \rangle$. Using S, S_1, and S_2, we define five normal subgroups of P:

$$R_1=<S_1{}^P>,\ R_2=\bigcap_{x\in P}S^x,\ R_3=<(S_1\cap R_2)^P>,$$

$$R_4=\bigcap_{x\in P}S_1{}^x,\ R_5=<S_2{}^P>.$$

We can verify the following:

$$R_1=<u_1{}^2, u_2, u_3{}^2, u_4, u_5{}^2, u_6, u_7{}^2, u_3u_5>;$$
$$R_2=<u_2, u_3{}^2, u_4, u_5{}^2, u_6, u_3u_5>;$$
$$R_3=<u_2, u_3{}^2, u_4, u_5{}^2, u_6>;$$
$$R_4=<u_3{}^2, u_4, u_5{}^2>;$$
$$R_5=<u_3{}^2, u_5{}^2>.$$

Thus, we have obtained a descending series $R_0=R, R_1, \cdots, R_5, R_6=1$ of normal 2-subgroups of P. The above equalities also show that if $i=3$ or 5 the factor group $V_i=R_{i-1}/R_i$ is of order 2, while if $i=1,2,4$ or 6 V_i is an elementary abelian group of order 4 admitting a faithful action of P/R.

Now let $N=N_A(P)$ and $C=C_N(P/R)$. Then $A=LN$ and $L\cap N=P$ as $N_A(U)\leqq N$ and $N_L(P)=P$. Also, $N=PC$ and $P\cap C=R$ as $\mathrm{Aut}(SL_2(2))\cong SL_2(2)$. Thus $A/L\cong C/R$ and it suffices to show that C is a 2-group. Now $C\leqq N_A(U)\leqq N_A(Q)$ and so the uniqueness of the S_i implies that C normalizes the S_i as well as S. The definitions of C and the R_i then show that C normalizes the R_i. So $N=PC$ acts on V_i for each i. If $i=1,2,4$ or 6 then $P/R\hookrightarrow N/C_N(V_i)\hookrightarrow \mathrm{Aut}(V_i)\cong SL_2(2)$, whence $N=PC_N(V_i)$ and $C_P(V_i)=R$. This implies that $C_N(V_i)=C$ for $i=1,2,4$ or 6. We conclude that C centralizes each V_i, $1\leqq i\leqq 6$, as well as U/R. Therefore $O^2(C)$ centralizes U and, by [6, Lemma 4.9], $O^2(C)=1$. We have shown that C is a 2-group, as required.

2.4 LEMMA. *Suppose a 2-group H acts on a group L, where $L\in\mathrm{Chev}(2)^+$ or $L\cong A_n$, $n\geqq 5$. If $C_L(H)$ is 2-isolated, then one of the following holds*:

(1) $L\cong SL_2(2^m)$, $(P)SU_3(2^m)$, *or* $Sz(2^{2m-1})$, $m\geqq 2$;

(2) $L\cong(P)SL_3(2^m)$, $Sp_4(2^m)$, A_6, *or* $\hat{A}_6$, *and if U is an H-invariant Sylow 2-subgroup of L, some element of H interchanges the two nontrivial parabolic subgroups of L containing U.*

Proof. Suppose $L\cong A_n$, $n\geqq 7$. Then $L\neq C_L(H)$ and, as $O_2(C_L(H))=1$, every element of $H-C_H(L)$ induces the same outer automorphism α of order 2 on L. We may assume that the action of α is the conjugation by the element $\beta=(1,2)(3,4)\cdots(2s-1,2s)$ of Σ_n. Since $O_2(C_L(\beta))=1$, it follows that $s=1$. But then $C_L(\beta)\cong\Sigma_{n-2}$ is not 2-isolated, a contradiction.

Assume therefore $L\in\mathrm{Chev}(2)^+$. In this case, we argue by induction on the order of L. The knowledge of the Schur multipliers of the relevant groups in $\mathrm{Chev}(2)^+$ permits us to assume $Z(L)=1$ (see [7]). We may also assume that H is faithful and nontrivial on L. Embed H and L in $A=\mathrm{Aut}(L)$, and let z be an involution in $Z(H)$. Then H acts on $K=C_L(z)$, and $C_K(H)=C_L(H)$ is 2-isolated. Since $O_2(C_L(H))=1$, it follows that $H\cap L=1$ and $O_2(K)=1$. Thus, z is an outer involutory automorphism with $O_2(C_L(z))=1$. This remark, together with 2.1 and 2.3, implies $L\not\cong SU_3(3)$, ${}^2F_4(2)'$, $G_2(2)$, or ${}^2F_4(2)$. Similarly, if $L\cong A_6$ or $Sp_4(2)$, then z is not contained in the subgroup of A isomorphic to $Sp_4(2)$, and it follows that z interchanges the nontrivial parabolic subgroups of L containing a given H-invariant Sylow 2-subgroup of L.

Assume therefore $L\in\mathrm{Chev}(2)^-$ but $L\not\cong SL_2(2^m)$, $PSU_3(2^m)$, or $Sz(2^{2m-1})$, $m\geqq 2$. If furthermore $L\not\cong PSL_3(2^m)$ or $Sp_4(2^m)$ then, as $O_2(C_L(z))=1$, results in Sections 8 and 19 of [**2**] show that $N=E(K)$, the semisimple part of K, is contained in Chev(2). As $C_N(H)$ is 2-isolated by 1.5, the induction hypothesis implies that N is one of the quasisimple groups mentioned in (1) and (2). Then $L\cong PSL_4(2^m)$, $PSL_5(2^m)$, $PSU_4(2^m)$, or $PSU_5(2^m)$, and $K\cong Sp_4(2^m)$ again by [**2**]. If $L\cong PSL_4(2^m)$ or $PSL_5(2^m)$, then z is a graph automorphism. However, as $\langle z\rangle\neq H$, the structure of Out(L) shows that $H=Z(H)$ contains an involutory field automorphism, which is a contradiction. Therefore, $L\cong PSL_3(2^m)$, $Sp_4(2^m)$, $PSU_4(2^m)$, or $PSU_5(2^m)$.

Let P_i, $i=1,2$, be the nontrivial parabolic subgroups of L containing an H-invariant Sylow 2-subgroup U of L. Suppose (2) does not hold. Then H normalizes both P_1 and P_2 as $P_1\not\cong P_2$ when $L\cong PSU_4(2^m)$ or $PSU_5(2^m)$. So z is a field automorphism and, if $L\cong PSL_3(2^m)$ or $Sp_4(2^m)$, then $K\cong PSL_3(2^{m/2})$ or $Sp_4(2^{m/2})$. We argue that $U\cap K\in\mathrm{Syl}_2(K)$ and that $P_i\cap K$, $i=1,2$, are the nontrivial parabolic subgroups of K containing $U\cap K$. It is clear that, for some $y\in zU\cap z^A$, $C_U(y)\in \mathrm{Syl}_2(C_L(y))$ and the $C_{P_i}(y)$, $i=1,2$, are the nontrivial parabolic subgroups of $C_L(y)$ containing $C_U(y)$. So it suffices to prove $yU\cap y^A=y^U$. This, however, has been proved in the course of calculation in [2] and our paper "Finite groups with a standard subgroup isomorphic to $Sp(4,2^n)$" (*Japan. J. Math.* 4 (1978), 1-76; see 4J and 6K). Now the induction hypothesis implies that some element of H interchanges $P_1\cap K$ and $P_2\cap K$. This is a contradiction as we are assuming that H normalizes P_1 and P_2.

2.5 Theorem. *Suppose G is a group having a normal subgroup $L\in\mathrm{Chev}(2)$. Then if $L\leqq X\leqq G$ and $H\in\mathcal{H}_{0,X}$, one of the following holds:*

(1) *$C_L(H)$ is 2-isolated;*

(2) *$\mathcal{H}_L$ posseses a nonidentity element I such that $N_X^*(H)\leqq N_G(I)$ and $|G:N_G(I)|$ is odd;*

(3) *$L\cong(P)SL_3(2^m)$, $Sp_4(2^m)'$, or $\hat{A}_6$ and $N_L(H\cap L)$ is a nontrivial parabolic subgroup of L with $O_2(N_L(H\cap L))=H\cap L$. Some 2-element of G interchanges, by conjugation, the two nontrivial parabolic subgroups of L containing a Sylow 2-subgroup of L.*

Proof. Let $I=H\cap L$. Clearly $N_G(H)\leqq N_G(I)$, and $I\in\mathcal{H}_L$ by 1.1. If $I=1$, (1) holds by 1.5; so we assume $I\neq 1$. By 2.2, $N_L(I)$ is a parabolic subgroup of L. Let U be an H-invariant Sylow 2-subgroup of $N_L(I)$ and let $UH\leqq T\in\mathrm{Syl}_2(G)$.

We assume $T \nleqq N_G(I)$ as (2) clearly holds in the contrary case. Then $I \neq U$ as $U = T \cap L$, whence $N_L(H)/I$ has even order and the rank of L is at least 2. We assume that the rank of L is greater than 2, for if it is 2 then (3) clearly holds.

Now let $N = \langle N_L(I)^T \rangle$. If $N \neq L$, then $O_2(N) \neq 1$ and 1.4 shows that there is an element $J \neq 1$ of $\mathscr{H}_L$ such that $NT \leqq N_G(J)$. This implies that (2) holds as $N_X^*(H) \leqq N_L(H)H \leqq NT$. We therefore assume $N = L$. In this case, we shall derive a contradiction by showing that L has rank 2. Now since $N_L(I)$ is a nontrivial parabolic subgroup of L, it follows that $O^{2'}(N_L(I)/I)$ is a central product of the subgroups K_i/I, $i = 1, 2, \cdots, r$, each of which is contained in the family

$$\mathrm{Chev}(2)^- \cup \{SL_2(2),\ (P)SU_3(2),\ Sp_4(2)\}.$$

The Krull-Remak-Schmidt theorem applied to the central factor of $K_1K_2\cdots K_r/I$ yields that H permutes the K_i by conjugation. As $C_{N_L(I)/I}(H) = N_L(H)/I$ is 2-isolated, 1.8 shows that H is transitive on the K_i and that $C_{K_1/I}(H_1)$ is 2-isolated, where $H_1 = N_H(K_1)$. So by 2.4, one of the following holds:

(i) $K_1/I \cong SL_2(2^m)$, $(P)SU_3(2^m)$, or $Sz(2^{2m-1})$;

(ii) $K_1/I \cong (P)SL_3(2^m)$ or $Sp_4(2^m)$ and some element of H_1 interchanges, by conjugation, the two nontrivial parabolic subgroups of K_1/I containing $U \cap K_1/I$.

Now T acts, by conjugation, on the set of all minimal nontrivial parabolic subgroups of L containing U, and each T-orbit on the set has length at most 2. If P is a minimal nontrivial parabolic subgroup of L such that $N_L(U) < P \leqq N_L(I)$, then $P \cap K_i/I$ is a minimal nontrivial parabolic subgroup of K_i/I containing $U \cap K_i/I$ for some uniquely determined i. Hence if $r \neq 1$, then P is not H-invariant and so $\langle P^T \rangle = \langle P^H \rangle \leqq N_L(I)$. This is a contradiction as $N_L(I)$ is generated by all of such P and we are assuming that $N_L(I)$ is not T-invariant. Thus $r = 1$, and an analogous argument shows that (ii) does not hold. Therefore, (i) holds, which implies that $N_L(I)$ is a minimal nontrivial parabolic subgroup of L. But then $|N_L(I)^T| = 2$ and, since $L = N$, it follows that L has rank 2, a contradiction.

2.6 Corollary. *Under the hypotheses of* 2.5, G *has a proper subgroup* Y *of odd index containing* $N_X^*(H)$ *such that* $N_L(Y \cap L) = Y \cap L$ *and* $O^2(C_{Y \cap L}(O_2(Y \cap L))) = Z(L)$, *except when one of the following holds*:

(1) $L \cong SL_2(2^m)$, $(P)SU_3(2^m)$, *or* $Sz(2^{2m-1})$, $m \geqq 2$;

(2) $L \cong (P)SL_3(2^m)$, $Sp_4(2^m)'$, *or* $\hat{A}_6$, *and if* $T \in \mathrm{Syl}_2(G)$ *then some element of* T *interchanges, by conjugation, the two nontrivial parabolic subgroups of* L *containing* $T \cap L$.

Proof. If neither (1) nor (2) holds, then 2.4 and 2.5 show that $\mathscr{H}_L$ contains an element $I \neq 1$ such that $N_X^*(H) \leqq N_G(I)$ and $|G : N_G(I)|$ is odd. As $N_L(I)$ is a proper parabolic subgroup of $L \in \mathrm{Chev}(2)$, $N_L(N_L(I)) = N_L(I)$ and $O^2(C_L(I)) = Z(L)$.

So $Y=N_G(I)$ meets all the requirements.

3. The Set $\mathscr{H}_{0,2,G}$ and the Alternating Groups

In this section, G is a group isomorphic to the symmetric group Σ_n on n letters, and L is the subgroup isomorphic to the alternating group A_n. The main result of this section is Theorem 3.11, which is a consequence of the theorems 3.3, 3.5, 3.9, and 3.10.

Throughout the section, we fix a permutation representation of G on the set $\{1, 2, \cdots, n\}$. The symbols $\cong$, $\hookrightarrow$, and $\times$ respectively denote the isomorphism, embedding, and direct product of permutation groups as well as those of abstract groups. The symbol $\int$ stands for the wreath product of permutation groups. Let $V=V(e, 2)$ be the vector space of dimension e over $GF(2)$ and $A(e, 2)$ be the group of all affine transformations α of the form $x^\alpha=x^\lambda+a$ $(x\in V)$, where λ is a nonsingular linear transformation of V and $a\in V$. Then $A(e, 2)$ is a doubly transitive permutation group on V with a regular normal subgroup, and the stabilizer of a point is isomorphic to $GL(e, 2)$. Let $A(e, k, r)=A(e, 2)\int\Sigma_k\times\Sigma_r$ for $e\geqq 1$, $k\geqq 1$, and $r\geqq 0$. Notice that $A(1, 2)\cong\Sigma_2$ and $A(2, 2)\cong\Sigma_4$.

3.1 Lemma. *Maximal 2-local subgroups of G are isomorphic to $A(e, k, r)$ for some e, k, and r with $n=2^e k+r$. Conversely, subgroups of G isomorphic to $A(e, k, r)$, $n=2^e k+r$, are maximal 2-local subgroups of G with the exception of $A(1, 2, r)$, $A(e, k, 2)$ $(2^e k\neq 4)$, $A(e, k, 4)$ $(2^e k\neq 2)$, and $A(2, 1, 4)$.*

Proof. See [5].

3.2 Lemma. *If $S\in\mathrm{Syl}_2(G)$, then $N_G(S)=S$*

Proof. We argue by induction on n. Let M be a maximal 2-local subgroup of G containing $N_G(S)$. If M is intransitive, then $M\hookrightarrow\Sigma_q\times\Sigma_r$, where $n=q+r$ and $qr\neq 0$; so the assertion follows from the induction hypothesis applied to Σ_q and Σ_r. Assume that M is transitive. Then $M\cong A(e, k, 0)$ with $n=2^e k$ by 3.1, and so M has a normal subgroup N such that $M/N\cong\Sigma_k$ and $N\cong A(e, 2)\times\cdots\times A(e, 2)$ (k copies). Let $T=S\cap N$, then as $N/O_2(N)\cong GL(e, 2)\times\cdots\times GL(e, 2)$, $N_N(T)=T$. Hence $N_M(T)/T\cong\Sigma_k$, and the proof is completed by the induction hypothesis applied to Σ_k.

3.3 Theorem. *Suppose n is odd and $H\in\mathscr{H}_{0,G}$. Then either $N_G(H)\hookrightarrow\Sigma_{n-1}\times\Sigma_1$ or $N_G(H)\cong S\times\Sigma_3$, where $S\in\mathrm{Syl}_2(\Sigma_{n-3})$. In the latter case, $N_G(H)\hookrightarrow\Sigma_{2^m+1}\times\Sigma_{n-2^m-1}$, where 2^m is the highest power of 2 dividing $n-1$.*

Proof. Since n is odd and H is a nonidentity 2-subgroup, it follows that $N_G(H)$ is contained in a subgroup K of G isomorphic to $\Sigma_{n-r}\times\Sigma_r$ for some odd integer $r<n$. Choose K so that r is minimal subject to this condition. As $H\in\mathscr{H}_K$, 1.3 shows that $N_G(H)\cong N_{\Sigma_{n-r}}(I)\times N_{\Sigma_r}(J)$, where $I\in\mathscr{H}_{\Sigma_{n-r}}$ and $J\in\mathscr{H}_{\Sigma_r}$.

As r is odd, the minimality of r forces $J=1$, whence $N_G(H)\cong N_{\Sigma_{n-r}}(I)\times\Sigma_r$. Now since $N_G(H)/H$ is 2-isolated, it follows that either $r=1$ or $r=3$ and $I\in\mathrm{Syl}_2(\Sigma_{n-r})$. In the latter case, $N_{\Sigma_{n-r}}(I)=I$ by 3.2. Thus, we have proved the first half of the theorem.

In proving the second half, we may assume that $N_G(H)$ is the direct product of $S\in\mathrm{Syl}_2(G_{1,2,3})$ and $G_{4,5,\dots,n}$. Let $<S,(2,3)>\leqq T\in\mathrm{Syl}_2(G)$ and let $M=<T,N_G(H)>$. Then $M=<T,(1,2)>=T<(1,2)^T>$, whence $M\leqq N_G(<(1,2)^T>)$. As T fixes 1, $<(1,2)^T>=<(1,i)\,;\, i\in 2^T>$. Therefore, $M\hookrightarrow\Sigma_{2^m+1}\times\Sigma_{n-2^m-1}$.

3.4 LEMMA. *If $n\geqq 6$ and $S\in\mathrm{Syl}_2(L)$, then $N_L(S)=S$.*

Proof. We argue by induction on n. When $n=6$, maximal 2-local subgroups of L containing S are isomorphic to Σ_4 by 2.1 and so $N_L(S)=S$ by 3.2. Assume $n>6$ and let M be a maximal 2-local subgroup of G containing $N_G(S)$. Then $M\cong A(e,k,r)$ with $n=2^ek+r$ by 3.1. Let $q=2^ek$. We distinguish three cases:

Case 1: $r>1$. The $N_G(S)$ is contained in a subgroup $K=K_q\times K_r$ with $K_q\cong\Sigma_q$ and $K_r\cong\Sigma_r$. Let $S\leqq T\in\mathrm{Syl}_2(K)$. Then $T\in\mathrm{Syl}_2(G)$ and $T=(SK_r\cap K_q)(SK_q\cap K_r)$ as $q\neq 1\neq r$. Therefore, $N_G(S)\leqq N_G(T)$ and then 3.2 shows that $N_L(S)=S$.

Case 2: $r=1$. We have $N_L(S)\hookrightarrow A_q$, and $q\geqq 6$ as q is even. Therefore, $N_L(S)=S$ by the induction hypothesis applied to A_q.

Case 3: $r=0$. We have $M\cong A(e,k,0)$ with $n=2^ek$, and $e=1$ or 2 as $|G:M|$ is odd. If $e=1$, then $O_2(M)\nleqq L$ and so $T=SO_2(M)$ is a Sylow 2-subgroup of G normalized by $N_G(S)$, whence $N_L(S)=S$ as before. Assume $e=2$. Then M has a normal subgroup N such that $N\cong\Sigma_4\times\cdots\times\Sigma_4$ (k copies) and $M/N\cong\Sigma_k$. Let $U=S\cap N$. Then since $k\geqq 2$, it follows that $N_{N\cap L}(U)=U$. Therefore, $N_{M\cap L}(U)/U\cong\Sigma_k$, and we have $N_L(S)=S$ by 3.2 applied to Σ_k.

3.5 THEOREM. *Suppose n is odd and $H\in\mathcal{H}_{0,L}\cap\mathcal{H}_G$. Then either $N_L^*(H)\hookrightarrow\Sigma_{n-1}\times\Sigma_1$ or $n=7$ and $N_G(H)\hookrightarrow\Sigma_4\times\Sigma_3$.*

Proof. Arguing as in the proof of 3.3, we have that $N_G(H)\cong N_{\Sigma_q}(I)\times\Sigma_r$, where $n=q+r$, q is even $\neq 0$, and $I\in\mathcal{H}_{\Sigma_q}$. As $H\cong I$ under the above isomorphism, we have furthermore that $q\geqq 4$. If $|N_{A_q}(I)/I|$ is even or $r=1$, then $N_L^*(H)\hookrightarrow N_{A_q}(I)$ and so $N_L^*(H)\hookrightarrow\Sigma_{n-1}\times\Sigma_1$. Assume therefore that $|N_{A_q}(I)/I|$ is odd and $r\neq 1$. Then $I\in\mathrm{Syl}_2(A_q)$ and, unless $q=4$, $N_{\Sigma_q}(I)$ is a 2-group greater than I by 3.4. As $I\in\mathcal{H}_{\Sigma_q}$, we must have $q=4$. Now since $N_L(H)/H$ is 2-isolated, it follows that $r=3$ or 5. But if $r=5$, $N_L(H)/H$ is an extension of Z_3 by Σ_5 and so is not 2-isolated. Therefore, $r=3$ and $N_G(H)\hookrightarrow\Sigma_4\times\Sigma_3$.

3.6 LEMMA. *Let U_m be the group of all upper triangular matrices in $GL(m,2)$, $m\geqq 2$, and E an elementary abelian subgroup of U_m of maximal order. Then if m is even, E is the group E_m of all matrices (x_{ij}) in U_m with $x_{ij}=0$ for $i<j\leqq m/2$ or $m/2<i<j$. If m is odd, E is the group E_m^ε, $\varepsilon=\pm 1$, of all matrices (x_{ij}) in U_m with $x_{ij}=0$ for $i<j\leqq(m+\varepsilon)/2$ or $(m+\varepsilon)/2<i<j$.*

Proof. We denote by $r(X)$ the rank of a 2-group X; that is, the rank of an elementary abelian subgroup of maximal order. We present a proof in the case where m is even as the argument for odd m is similar. The assertion is clearly true when $m=2$. So assume $m\geqq 4$ and let P be the group of all matrices (x_{ij}) in U_m with $x_{ij}=0$ for $1<i<j<m$. Then P is a normal extraspecial subgroup of order 2^{2m-3}. Hence $r(E\cap P)\leqq m-1$ and, since $r(E)\geqq r(E_m)$, it follows that $r(EP/P)\geqq\{(m-2)/2\}^2$. Let S be the group of all matrices (x_{ij}) in U_m with $x_{ij}=0$ for $1=i<j$ or $i<j=m$, and let $F=EP\cap S$. Then $U_m=SP$, $S\cap P=1$, and $S\cong U_{m-2}$. As $r(F)=r(EP/P)\geqq\{(m-2)/2\}^2$, we get $F=E_m\cap S$ by the induction argument and, in particular, $r(F)=\{(m-2)/2\}^2$. So $r(E)=(m/2)^2$ and $r(E\cap P)=m-1$. Now, as $EP=FP=E_mP$, $E\cap P/Z(P)\leqq C_{P/Z(P)}(E_m)=E_m\cap P/Z(P)$. Comparing orders, we get $E\cap P=E_m\cap P$, whence $E\leqq C_{E_mP}(E_m\cap P)=E_m$. This completes the proof.

3.7 Lemma. *Let $X\cong A(e,2)$ and $S\in\mathrm{Syl}_2(X)$. Then*

(1) *the rank of S is equal to $e(e+2)/4$ when e is even and to $(e+1)^2/4$ when e is odd*;

(2) *if $e\geqq 3$ and $Y/O_2(X)$ is a minimal nontrivial parabolic subgroup of $X/O_2(X)\cong GL(e,2)$ containing $S/O_2(X)$, then Y is contained either in $N_X(J(S))$ or in $C_X(Z(S))$, where $J(S)$ is the "Thompson subgroup" generated by all elementary abelian subgroups of S of maximal order.*

Proof. We may assume that X consists of all matrices (x_{ij}) in $GL(e+1,2)$ with $x_{i1}=0$ for $i>1$ and that $S=U_{e+1}$ in the notation of 3.6. Therefore, (1) is a consequence of 3.6. For $i=1,2,\cdots,e$, let s_i be the matrix obtained by interchanging the i-th row and the $(i+1)$-th row of the identity matrix, and let $Y_i=\langle S,s_i\rangle$. Then $Y=Y_i$ for some i. Now, $J(S)=E_{e+1}$ or $\langle E_{e+1}^{+1},E_{e+1}^{-1}\rangle$ according as e is odd or even by 3.6 and, since $e+1\geqq 4$, it follows that Y_1 and Y_e are contained in $N_X(J(S))$ and the Y_i, $2\leqq i\leqq e-1$, are contained in $C_X(Z(S))$. This proves (2).

3.8 Lemma. *Let $X\cong A(e,2)\wr\Sigma_k$, $e\geqq 2$, and let Y be the normal subgroup of X such that $Y\cong A(e,2)\times\cdots\times A(e,2)$ (k copies) and $X/Y\cong\Sigma_k$. Let $S\in\mathrm{Syl}_2(Y)$ and $S\leqq T\in\mathrm{Syl}_2(X)$. Then $J(S)=J(T)$.*

Proof. Let r be the rank of a Sylow 2-subgroup of $A(e,2)$. Then $r(S)=rk$ and r is given in 3.7. Let E be an elementary abelian 2-subgroup of T of maximal order, and suppose $E\nleqq S$. Then $\bar{E}=ES/S$ is a nonidentity permutation group on the set D of k factors of Y. For $i=1,2,\cdots$, define the subgroup $\bar{E}_i$ of $\bar{E}$ and a subset D_i of D in the following way. Let $\bar{E}_1=\bar{E}$ and let D_1 be an $\bar{E}_1$-orbit on D of maximal length. For $i=2,3,\cdots$, let $\bar{E}_i$ be the kernel of $\bar{E}_{i-1}$ on D_{i-1} and let D_i be an $\bar{E}_i$-orbit on D of maximal length. Let $l_i=|D_i|$ and let n_i be the number of the $\bar{E}_i$-orbits on D. Choose a positive integer m so that $\bar{E}_m\neq 1$ and $\bar{E}_{m+1}=1$. As E is abelian, $|\bar{E}_i:\bar{E}_{i+1}|=l_i$ for each i and so

$$|\bar{E}|=l_1l_2\cdots l_m. \tag{1}$$

As $n_{i+1} \geqq (n_i - 1) + l_i$ by the definition of the n_i and l_i,

$$n_{i+1} - n_i \geqq l_i - 1, \qquad 1 \leqq i \leqq m.$$

Summing up, we have

$$k - n_1 \geqq \sum_{i=1}^{m} (l_i - 1) \tag{2}$$

as $n_{m+1} = k$. It follows from 1.7 that

$$r(S \cap E) \leqq r(C_S(E)) \leqq rn_1. \tag{3}$$

By (1) and (3),

$$|E| = |\bar{E}|\ |S \cap E| \leqq l_1 l_2 \cdots l_m 2^{rn_1}.$$

As $r(E) \geqq r(S) = rk$, we have

$$2^{r(k-n_1)} \leqq l_1 l_2 \cdots l_m.$$

On the other hand, (2) shows

$$2^{r(k-n_1)} \geqq (2^r)^{l_1 - 1} (2^r)^{l_2 - 1} \cdots (2^r)^{l_m - 1}.$$

Combining these two inequalities, we conclude that $l_i = 2$ for each i and that $r = 1$. However, since $e \geqq 2$, it follows that $r \neq 1$. This contradiction shows $E \leqq S$, proving the lemma.

3.9 THEOREM. *Suppose n is even and $H \in \mathcal{H}_{0,G}$. Then $N_G(H)$ is contained in a subgroup of G isomorphic to $\Sigma_2 \int \Sigma_{n/2}$ or $\Sigma_4 \int \Sigma_{n/4}$ or $\Sigma_4 \int \Sigma_{(n-2)/4} \times \Sigma_2$.*

Proof. We argue by induction on n. Assume first that $N_G(H)$ is intransitive. Then $N_G(H) \hookrightarrow \Sigma_q \times \Sigma_r$, where q and r are even nonzero integers such that $n = q + r$, and so $N_G(H) \cong N_{\Sigma_q}(H_q) \times N_{\Sigma_r}(H_r)$ for some $H_q \in \mathcal{H}_{\Sigma_q}$ and $H_r \in \mathcal{H}_{\Sigma_r}$ by 1.3. As $N_G(H)/H$ is 2-isolated, $H_q \neq 1 \neq H_r$ and we may assume, by symmetry, that $N_{\Sigma_q}(H_q)/H_q$ is 2-isolated and $N_{\Sigma_r}(H_r)/H_r$ is of odd order. Thus, $H_q \in \mathcal{H}_{0,\Sigma_q}$ and the induction hypothesis implies that $N_{\Sigma_q}(H_q) \hookrightarrow \Sigma_2 \int \Sigma_{q/2}$ or $\Sigma_4 \int \Sigma_{q/4}$ or $\Sigma_4 \int \Sigma_{(q-2)/4} \times \Sigma_2$. As $N_{\Sigma_r}(H_r) = H_r \in \mathrm{Syl}_2(\Sigma_r)$ by 3.2, $N_{\Sigma_r}(H_r) \hookrightarrow \Sigma_2 \int \Sigma_{r/2}$ and, if $4 | r$, $N_{\Sigma_r}(H_r) \hookrightarrow \Sigma_4 \int \Sigma_{r/4}$. Therefore, it suffices to consider the case where $N_{\Sigma_q}(H_q) \hookrightarrow \Sigma_4 \int \Sigma_{q/4}$ or $\Sigma_4 \int \Sigma_{(q-2)/4} \times \Sigma_2$, $4 \nmid r$, and $r \neq 2$. Then H_r has an orbit of length 2 and $4 | (r-2)$. Hence, $H_r \hookrightarrow \Sigma_4 \int \Sigma_{(r-2)/4} \times \Sigma_2$, and the theorem is proved.

Assume next that $N_G(H)$ is transitive, and choose a maximal 2-local subgroup M of G containing $N_G(H)$ so that the 2-part of $|M|$ is maximal. Then $M \cong A(e, k, 0)$ with $n = 2^e k$ by 3.1. If $e \leqq 2$, the theorem holds; so we assume $e \geqq 3$. Let $N = N_1 \times \cdots \times N_k$ be the normal subgroup of M such that $N_i \cong A(e, 2)$ for each i and $M/N \cong \Sigma_k$, and let $K = H \cap N$. Then $K \in \mathcal{H}_N$ by 1.1, and so $K = K_1 \times \cdots \times K_k$ with $K_i \in \mathcal{H}_{N_i}$ for each i by 1.3. Since $N_G(H)$ is transitive, it follows that the K_i are all conjugate under $N_G(H)$, and consequently $N_{N_1}(K_1) \cong$

$N_{N_i}(K_i)$ for each i. As $O_2(N_1)\leqq K_1$ by 1.1, $N_{N_1}(K_1)/O_2(N_1)$ is a parabolic subgroup of $N_1/O_2(N_1)\cong GL(e,2)$ by 2.2, and either

(A) $N_{N_1}(K_1)/K_1$ is a nonempty direct product of groups isomorphic to $GL(d,2)$, $2\leqq d\leqq e$, or

(B) $N_{N_1}(K_1)=K_1\in \mathrm{Syl}_2(N_1)$.

Assume that (A) holds. Then $N_N(K)/K=L_1/K\times\cdots\times L_l/K$, $k|l$, with $L_j/K\cong GL(d_j,2)$ for some d_j, $2\leqq d_j\leqq e$, and H permutes the L_j by conjugation. As $C_{N_N(K)/K}(H)=N_N(H)/K$ is 2-isolated, 1.8 shows that the L_j, $1\leqq j\leqq l$, are conjugate under H and so the N_i, $1\leqq i\leqq k$, are conjugate under H. Furthermore, $C_{L_1/K}(H_1)$ is 2-isolated, where $H_1=N_H(L_1)$. Thus, 2.4 shows that $L_1/K\cong GL(2,2)$ or $GL(3,2)$ and, if $L_1/K\cong GL(3,2)$, an element of H_1 induces a graph automorphism on L_1/K. We may suppose $L_1\leqq N_{N_1}(K_1)K$. Then $1\neq O^2(L_1)\leqq N_1$; so $H_1\leqq N_H(N_1)$ and if $L_1{}^h\leqq N_{N_1}(K_1)K$, $h\in H$, then $h\in N_H(N_1)$. As $N_M(N_1)=N_1C_M(N_1)$, we conclude that $N_{N_1}(K_1)/K_1\cong GL(2,2)$ in Case (A). Now let S be an H-invariant Sylow 2-subgroup of $N_N(K)$ and let $SH\leqq T\in\mathrm{Syl}_2(M)$. As $e\geqq 3$ and as the N_i are conjugate under H, 3.7 shows that $N_N(K)\leqq N_N(J(S))$ or $C_N(Z(S))$. Also, $J(S)=J(T)$ by 3.8 and $Z(T)=<z_1\cdots z_k>$, where $<z_i>=Z(S\cap N_i)$. Therefore, $N_M(K)\leqq N_N(K)N_M(S)\leqq N_M(J(T))$ or $C_M(Z(T))$. However, if X is a maximal 2-local subgroup of G containing $N_G(J(T))$ or $N_G(Z(T))$, then $N_G(H)\leqq X$ and the 2-part of $|X|$ is greater than that of $|M|$ as $T\notin\mathrm{Syl}_2(G)$. This is a contradiction completing the proof.

3.10 THEOREM. *Suppose n is even, $H\in\mathcal{H}_{0,L}\cap\mathcal{H}_G$, and $N_G(H)\nleqq L$. Then $N_L^*(H)$ is contained in a subgroup of G isomorphic to $\Sigma_2\wr\Sigma_{n/2}$ or $\Sigma_4\wr\Sigma_{n/4}$ or $\Sigma_4\wr\Sigma_{(n-2)/4}\times\Sigma_2$.*

Proof. We argue by induction on n. Assume that $N_G(H)$ is transitive, and let M be a maximal 2-local subgroup of G containing $N_G(H)$. Then $M\cong A(e,k,0)$ with $n=2^e k$ by 3.1. As $O_2(M)\leqq H\leqq L$ by 1.1, we have $e>1$, and as $M\nleqq L$, we have $e<3$. Therefore, $M\cong\Sigma_4\wr\Sigma_k$ and the theorem holds.

Assume therefore that $N_G(H)$ is intransitive. Then arguing as in the proof of 3.9, we have that $N_G(H)\cong N_{\Sigma_q}(H_q)\times N_{\Sigma_r}(H_r)$ for some $H_q\in\mathcal{H}_{\Sigma_q}$ and $H_r\in\mathcal{H}_{\Sigma_r}$, where q and r are even nonzero integers with $n=q+r$. As $H\leqq L$, $H_q\leqq A_q$ and $H_r\leqq A_r$, and consequently $q\neq 2\neq r$. As $N_L(H)/H$ is 2-isolated, we may assume, by symmetry, that $|N_{A_r}(H_r)/H_r|$ is odd. Then $H_r\in\mathrm{Syl}_2(A_r)$ and $|N_{\Sigma_r}(H_r):N_{A_r}(H_r)|=2$. If $r\neq 4$, then $N_{A_r}(H_r)=H_r$ by 3.4 and so $N_{\Sigma_r}(H_r)$ is a 2-group, a contradiction because $H_r\in\mathcal{H}_{\Sigma_r}$. Therefore, $r=4$. If $|N_{A_q}(H_q)/H_q|$ is odd, then similarly $q=4$ and $N_G(H)\hookrightarrow\Sigma_4\times\Sigma_4\hookrightarrow\Sigma_4\wr\Sigma_2$. So assume that $|N_{A_q}(H_q)/H_q|$ is even. Then $N_{A_q}(H_q)/H_q$ is 2-isolated and, as q is even, $H_q\neq 1$. Thus, $H_q\in\mathcal{H}_{0,A_q}\cap\mathcal{H}_{\Sigma_q}$ and moreover $N_L^*(H)\hookrightarrow N_{A_q}^*(H_q)\times H_r$. If $N_{\Sigma_q}(H_q)\leqq A_q$, then $H_q\in\mathcal{H}_{0,\Sigma_q}$ and so $N_{\Sigma_q}(H_q)\hookrightarrow\Sigma_2\wr\Sigma_{q/2}$ or $\Sigma_4\wr\Sigma_{q/4}$ or $\Sigma_4\wr\Sigma_{(q-2)/4}\times\Sigma_2$ by 3.9. If $N_{\Sigma_q}(H_q)\nleqq A_q$, then $N_{A_q}^*(H_q)\hookrightarrow\Sigma_2\wr\Sigma_{q/2}$ or $\Sigma_4\wr\Sigma_{q/4}$ or $\Sigma_4\wr\Sigma_{(q-2)/4}\times\Sigma_2$ by the induction hypothesis. As $H_r\hookrightarrow\Sigma_2\wr\Sigma_2\hookrightarrow\Sigma_4$, the theorem holds.

3.11 THEOREM. *Suppose* $L \leqq X \leqq G$ *and* $H \in \mathcal{H}_{0,X}$. *Then* $N_X^*(H)$ *is contained in a subgroup of* G *isomorphic to one of the groups on the following list*:

$$\Sigma_2 \int \Sigma_{n/2},\ \Sigma_4 \int \Sigma_{n/4},\ \Sigma_4 \int \Sigma_{(n-2)/4} \times \Sigma_2,$$

$$\Sigma_2 \int \Sigma_{(n-1)/2} \times \Sigma_1,\ \Sigma_4 \int \Sigma_{(n-1)/4} \times \Sigma_1,$$

$$\Sigma_2 \int \Sigma_{(n-3)/2} \times \Sigma_3,\ \Sigma_4 \int \Sigma_{(n-3)/4} \times \Sigma_3.$$

If n *is odd and* 2^m *is the highest power of* 2 *dividing* $n-1$, *then* $N_X^*(H)$ *is contained in a subgroup of* G *isomorphic to* $\Sigma_{n-1} \times \Sigma_1$ *or* $\Sigma_{2^m+1} \times \Sigma_{n-2^m-1}$.

Proof. If $X=G$ and n is even, then by 3.9, $N_G(H)$ is contained in a subgroup isomorphic to one of the first three groups on the above list. Suppose $X=G$ and n is odd. Then by 3.3, either $N_G(H) \cong N_{\Sigma_{n-1}}(K) \times \Sigma_1$ for some $K \in \mathcal{H}_{0,\Sigma_{n-1}}$ or $N_G(H) \cong S \times \Sigma_3$, $S \in \mathrm{Syl}_2(\Sigma_{n-3})$. As $S \hookrightarrow \Sigma_2 \int \Sigma_{(n-3)/2}$, $N_G(H)$ is contained in one of the last four groups on the list. Suppose $X=L$ and n is even. In this case, we may assume $N_G(H) \nleqq L$. Let $K=O_2(N_G(H))$, then $H=K \cap L$ and $N_G(H)=N_G(K)$ by 1.1. If $H \neq K$, then $K \in \mathcal{H}_{0,G}$; so we may assume $H=K$. But then H satisfies the hypotheses of 3.10, and it follows that $N_L^*(H)$ is contained in one of the first three groups on the list. Finally, suppose $X=L$ and n is odd. As before, we may assume $H \in \mathcal{H}_G$ and 3.5 shows that either $N_L^*(H) \hookrightarrow \Sigma_{n-1} \times \Sigma_1$ or $n=7$ and $N_G(H) \hookrightarrow \Sigma_4 \times \Sigma_3$. In the former case, $N_L^*(H) \cong N_{A_{n-1}}^*(K) \times \Sigma_1$ for some $K \in \mathcal{H}_{0,A_{n-1}}$ by 1.6. Thus, we have proved the first part of the theorem. The second part follows from 3.3 and 3.5.

3.12 COROLLARY. *Suppose* $L \leqq X \leqq G$ *and* $H \in \mathcal{H}_{0,X}$. *If* $n \neq 2^m+1$ *for any integer* m, *then* G *has a proper subgroup* Y *of odd index containing* $N_X^*(H)$ *such that* $N_G(Y \cap L)=Y$. *If furthermore* n *is even or* $n \equiv 3$ (*mod* 4), *then such* Y *may be chosen so that, respectively,* $C_{Y \cap L}(O_2(Y \cap L)) \leqq O_2(Y \cap L)$ *or* $O^2(C_{Y \cap L}(O_2(Y \cap L))) \leqq \langle z \rangle$, *where* z *is a* 3-*cycle in* L.

Proof. Assume that n is even. Then $N_X^*(H)$ is contained in a subgroup Y of G isomorphic to one of the first three groups on the list of 3.11, which are of odd index in G and satisfy $C_Y(O_2(Y)) \leqq O_2(Y)$. So it suffices to prove $N_G(Y \cap L)=Y$. Suppose $Y \cong \Sigma_2 \int \Sigma_k$ or $\Sigma_4 \int \Sigma_k$. Then Y is a maximal 2-local subgroup of G by 3.1 as $k \geqq 3$ when $Y \cong \Sigma_2 \int \Sigma_k$. Since $Y \leqq N_G(Y \cap L) \leqq N_G(O_2(Y \cap L))$ and $1 \neq O_2(Y) \cap L \leqq O_2(Y \cap L)$, it follows that $N_G(Y \cap L)=Y$, as desired. If $Y \cong \Sigma_4 \int \Sigma_k \times \Sigma_2$, then $O_2(Y \cap L)$ has k orbits of length 4 and two fixed points; so $N_G(O_2(Y \cap L)) \hookrightarrow \Sigma_4 \int \Sigma_k \times \Sigma_2$ and, comparing orders, we get $N_G(Y \cap L)=Y$.

Assume $n \equiv 3$ (mod 4). Then $N_X^*(H)$ is contained in $Y \cong \Sigma_2 \int \Sigma_k \times \Sigma_1$ ($k \geqq 3$) or $\Sigma_2 \int \Sigma_k \times \Sigma_3$ (k even $\neq 2$) or $\Sigma_4 \int \Sigma_k \times \Sigma_3$, which are of odd index in G and satisfy $O^2(C_Y(O_2(Y))) \cong \Sigma_1$ or A_3. These groups are maximal 2-local subgroups of G by 3.1 and $O_2(Y) \cap L \neq 1$, whence $N_G(Y \cap L)=Y$ as before.

Suppose n is odd and $n \neq 2^m+1$ for any integer m. The $N_X^*(H)$ is contained in $Y \cong \Sigma_{n-1} \times \Sigma_1$ or $\Sigma_{2^m+1} \times \Sigma_{n-2^m-1}$, where 2^m is the highest power of 2 dividing

$n-1$. These groups are proper and of odd index in G. Looking at the lengths of the $Y\cap L$-orbits, we have as before that $N_G(Y\cap L)=Y$.

4. The Main Theorems

In this section, we assume the hypothesis 4.1 below. The main result of this paper is stated in the theorems 4.2 and 4.3.

4.1 HYPOTHESIS. *G is a finite group, N is a normal subgroup of G, G/N is a 2-group, and N is a central product of the quasisimple groups $L=L_1, L_2, \cdots, L_k$, which are all conjugate in G.*

4.2 THEOREM. *Under Hypothesis* 4.1 *with $L\in\mathrm{Chev}(2)$, if $H\in\mathcal{H}_{0,2,G}$, then G has a proper subgroup M of odd index containing $N_G^*(H)$ such that $O^2(C_M(O_2(M)))=Z(N)$, except when one of the following holds*:

(1) $L\cong SL_2(2^m)$, $(P)SU_3(2^m)$, *or* $Sz(2^{2m-1})$, $m\geqq 2$;

(2) *$L\cong(P)SL_3(2^m)$, $Sp_4(2^m)'$, or $\hat{A}_6$, and if $S\in\mathrm{Syl}_2(G)$, then some element of $N_S(L)$ interchanges, by conjugation, the two nontrivial parabolic subgroups of L containing $S\cap L$.*

4.3 THEOREM. *Under Hypothesis* 4.1 *with $L\cong A_n$, $n\geqq 7$, if $H\in\mathcal{H}_{0,2,G}$, then the following holds*:

(1) *if $n\neq 2^m+1$ for any integer m, then G has a proper subgroup of odd index containing $N_G^*(H)$*;

(2) *if n is even, then G has a proper subgroup M of odd index containing $N_G^*(H)$ such that $C_M(O_2(M))\leqq O_2(M)$*;

(3) *if $n\equiv 3$ (mod 4), then G has a proper subgroup M of odd index containing $N_G^*(H)$ such that $O^2(C_M(O_2(M)))=1$ or $\langle z_1, z_2, \cdots, z_k\rangle$, where z_i is a 3-cycle in $L_i\cong A_n$ for each i.*

Proof of 4.2 *and* 4.3. Assume that G satisfies Hypothesis 4.1 with $L\in\mathrm{Chev}(2)$ or $L\cong A_n$, $n\geqq 7$. When $L\in\mathrm{Chev}(2)$, assume that neither (1) nor (2) of 4.2 holds, and when $L\cong A_n$, $n\geqq 7$, assume that $n\neq 2^m+1$ for any integer m. Now let $H\in\mathcal{H}_{0,2,G}$ and define $J=H\cap N$. Then $N_G(H)\leqq N_G(J)$ and $J\in\mathcal{H}_N$ by 1.1, and so $J=J_1J_2\cdots J_k$ by 1.3, where $J_i\in\mathcal{H}_{L_i}$ and $J_i=J\cap L_i$ for each i. Since G/N is a 2-group, it follows that $O^2(C_X(O_2(X)))=O^2(C_{X\cap N}(O_2(X\cap N)))$ for any subgroup X of G. Hence if $J\in\mathrm{Syl}_2(N)$, then $M=N_G(J)$ is a proper subgroup of odd index containing $N_G^*(H)$ such that $O^2(C_M(O_2(M)))=Z(N)$. Therefore, assume $J\notin\mathrm{Syl}_2(N)$ and, changing the numbering of the L_i, let $N_{L_i}(J_i)/J_i$ have even order for $1\leqq i\leqq l$ and odd order for $l<i\leqq k$, where $1\leqq l\leqq k$. Then, as $C_{N_N(J)/J}(H)=N_N(H)/J$ is 2-isolated by 1.5, 1.8 shows that the L_i, $1\leqq i\leqq l$, are conjugate under H. Let

$H_i=N_H(L_i)$ for each i. Then H_i acts by conjugation on $N_{L_i}(J_i)/J_i$, and we can define the set C_i/J_i of all fixed points under this action of H_i. For simplicity, let $I=H_1$, $K=J_1$, and $D=C_1$. Now, D/K is 2-isolated by 1.8. Hence if $K=1$, then $C_L(I)=D$ is 2-isolated. But then $L\not\cong A_n$, $n\geqq 7$, and (1) or (2) of 4.2 holds by 2.4. Therefore, we assume $K\neq 1$. Then, as $N_{LI}(I)/I\cong D/K$, $I\in\mathscr{H}_{0,LI}$ and $O_2(LI)$ is a proper subgroup of I by 1.1. Let $I\leqq T\in\mathrm{Syl}_2(N_G(L))$ and let bars denote images in $LT/O_2(LT)$. As $\bar{I}\in\mathscr{H}_{0,\overline{LT}}$, 2.6 and 3.12 show that $\overline{LT}$ has a proper subgroup $\bar{Y}=Y/O_2(LT)$ of odd index containing $N^*_{\overline{LT}}(\bar{I})$ such that $N_{\bar{L}}(\bar{Y}\cap\bar{L})=\bar{Y}\cap\bar{L}$. Furthermore, if $L\in\mathrm{Chev}(2)$ or $L\cong A_{2m}$, $m\geqq 4$, we may choose Y so that $O^2(C_{\bar{Y}\cap\bar{L}}(O_2(\bar{Y}\cap\bar{L})))=Z(\bar{L})$, while if $L\cong A_n$, $n\equiv 3 \pmod 4$, we may choose Y so that $O^2(C_{\bar{Y}\cap\bar{L}}(O_2(\bar{Y}\cap\bar{L})))\leqq<\bar{z}>$, where $\bar{z}$ is a 3-cycle in $\bar{L}$. As Y contains I and has odd index in LT, we may assume $T\leqq Y$. Let $T\leqq S\in\mathrm{Syl}_2(G)$ and define $P=<(Y\cap L)^S>$. Then, as $N_S(L)=T\leqq N_G(Y\cap L)$ and the L_i, $1\leqq i\leqq k$, are conjugate under S, we see that

$$P=\prod_{i=1}^{k}(Y\cap L)^{s_i},$$

where the s_i, $1\leqq i\leqq k$, are elements of S satisfying $L^{s_i}=L_i$. Similarly,

$$<(Y\cap L)^H>=\prod_{i=1}^{l}(Y\cap L)^{h_i},$$

where $h_i\in H$ and $L^{h_i}=L_i$, $1\leqq i\leqq l$. Now, $N_G(L)=N_{NS}(L)=NT=LN_G(Y\cap L)$, whence $N_G(L_i)=N_G((Y\cap L)^{s_i})L_i$ for each i, $1\leqq i\leqq k$. If $1\leqq i\leqq l$, $(Y\cap L)^{s_i}$ and $(Y\cap L)^{h_i}$ are conjugate in $N_G(L_i)$, and so L_i has an element x_i such that $(Y\cap L)^{h_i}=(Y\cap L)^{s_ix_i}$. Let $\Lambda_1, \Lambda_2, \cdots, \Lambda_m$ be the H-orbits on the set $\{L_{l+1}, \cdots, L_k\}$ and, changing the numbering, assume that $L_{l+1}, \cdots, L_{l+m}$ are representatives of $\Lambda_1, \cdots, \Lambda_m$, respectively. Let $l+1\leqq j\leqq l+m$. As $N_G(Y\cap L)$ contains a Sylow 2-subgroup of $N_G(L)$, L_j has an element x_j such that $H_j\leqq N_G((Y\cap L)^{s_jx_j})$, whence as before

$$<((Y\cap L)^{s_jx_j})^H>=\prod_{h}(Y\cap L)^{s_jx_jh},$$

where h ranges over representatives of the left cosets of H_j in H. If $L_j{}^h=L_i$, $h\in H$, then as before L_i has an element y such that $(Y\cap L)^{s_jx_jh}=(Y\cap L)^{s_iy}$. Thus, we conclude that

$$P^x=<(Y\cap L)^H,\ ((Y\cap L)^{s_jx_jH};\ l+1\leqq j\leqq l+m>$$

for some element $x\in N$. Now define $M=N_G(P^x)$. Then M has odd index $\neq 1$ in G as $S^x\leqq M$. Also, $H\leqq M$ and $C_1^*\leqq Y\cap L\leqq P^x$, where C_i^*/J_i, $1\leqq i\leqq l$, is the unique minimal subnormal subgroup of C_i/J_i of even order, whence $C_1^*\cdots C_l^*H=<C_1^*, H>\leqq M$. As $N_G(H)$ permutes the C_i^* by conjugation, $<C_1^*, H>\cap N_G(H)/H$ is a normal subgroup of $N_G(H)/H$ of even order whence $N_G^*(H)\leqq<C_1^*, H>\cap N_G(H)\leqq M$. We have shown that M is a proper subgroup of odd index containing $N_G^*(H)$, completing the proof of 4.3.1. Now since $N_L(Y\cap L)=Y\cap L$, it follows that $P\cap L=Y\cap L$ and $M\cap N=P^x$. If $L\in\mathrm{Chev}(2)$ or $L\cong A_n$, n even $\geqq 8$, the choice of Y shows $O^2(C_P(O_2(P)))=Z(N)$, while if $L\cong A_n$, $n\equiv 3 \pmod 4$, $O^2(C_P(O_2(P)))=1$ or $<z_1, \cdots, z_k>$, where z_i is a 3-cylce in L_i for $1\leqq i\leqq k$. This completes the proof of 4.2, 4.3.2, and 4.3.3.

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