# On Alperin-Goldschmidt's Fusion Theorem

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In this note, we shall make some remarks on Alperin-Goldschmidt's fusion theorem and its applications.

Let $G$ be a finite group and $p$ be a prime. We denote by $\mathscr{H}$ the set of all non-identity $p$-subgroups $H$ of $G$ such that $N_G(H)/H$ is $p$-isolated. (See §1 for the definition and properties of $p$-isolated group.) For a Sylow $p$-subgroup $P$ of $G$, we denote by $\mathscr{H}(P)$ the set of subgroups $H$ in $\mathscr{H}$ such that $N_P(H)$ is a Sylow $p$-subgroup of $N_G(H)$. Finally, for $H \in \mathscr{H}$, let $X(H)$ be the smallest normal subgroup of $N_G(H)$ subject to the condition that $X(H) \supseteq H$ and $p \,|\, |X(H)/H|$. (See Lemma 1 in §1 for the existence of $X(H)$.)

Now we can slightly sharpen Alperin-Goldschmidt's fusion theorem [1], [5].

THEOREM. *The family* $\{(H, O^p(X(H))) | H \in \mathscr{H}(P)\}$ *is a conjugation family. More precisely, if $A$ and $B$ are subsets of $P$ such that $A^g = B \neq 1$ for some $g \in G$, there exist $H_i \in \mathscr{H}(P)$, $x_i \in O^p(X(H_i))$ $(i=1, 2, \cdots, n)$ and $y \in N(P)$ such that*

(i) $g = x_1 x_2 \cdots x_n y$

(ii) $A \subseteq H_1$ *and* $A^{x_1 x_2 \cdots x_i} \subseteq H_{i+1}$ $(i=1, 2, \cdots, n-1)$.

In §2, we shall prove this theorem by using a method of Gomi [4]. In §3, we note firstly that several conjugation families which are defined in Goldschmidt [5], [6] and Solomon [8] can be obtained as special cases of the theorem, and secondly that a theorem of Glauberman [2], which says that a strongly closed abelian $p$-subgroup controls strong $p$-fusion in $G$, can be proved by an application of Alperin-Goldschmidt's theorem. In an interesting paper [7], by which the theorem was suggested, Puig defined a characteristic functor $L(P)$ and proved "$ZL$-Theorem" which says that, if $G$ is $p$-stable, $ZL(P)$ controls strong $p$-fusion in $G$. Finally we note that $ZL$-Theorem can be also proved by an application of the theorem.

## § 1. Some properties of $p$-isolated groups.

Let $X$ be a finite group and $p$ be a prime. An equivalence relation on $\mathrm{Syl}_p(X)$, the set of all Sylow $p$-subgroups of $X$, is defined as follows: For $P, Q \in \mathrm{Syl}_p(X)$, $P \sim Q$ if and only if there exists a sequence $P=P_0, P_1, \cdots, P_n=Q \in \mathrm{Syl}_p(X)$ such that $P_{i-1} \cap P_i \neq 1$ $(i=1,2,\cdots,n)$. If $X$ has more than one equivalence class, $X$ is called *$p$-isolated*. It is well known that $X$ is $p$-isolated if and only if $X$ possesses a proper subgroup $Y$ such that $p \mid |Y|$ and $p \nmid |Y^x \cap Y|$ for any $x \in X-Y$. Such a subgroup $Y$ is called *strongly $p$-embedded subgroup* of $X$ and has the following properties:

(1.1) $Y$ contains some Sylow $p$-subgroup of $X$.

(1.2) If $R$ is a subgroup of $Y$ with $p \mid |R|$, $N_X(R) \subseteq Y$.

(1.3) If $N \lhd X$ and $p \mid |N|$, $X = YO^p(N)$.

(1.1) and (1.2) follow from the definition of strongly $p$-embedded subgroup, and (1.3) follows from (1.2) and Frattini argument.

The next lemma is proved in L. Puig [7]. For convenience of readers, we include the proof.

LEMMA 1. *Let $X$ be a $p$-isolated group and $N_i$ $(i=1,2)$ be normal subgroups of $X$ such that $p \mid |N_i|$.* Then $p \mid |N_1 \cap N_2|$.

*Proof.* Let $Y$ be a strongly $p$-embedded subgroup of $X$ and $P$ be a Sylow $p$-subgroup of $Y$. Suppose by way of contradiction that $p \nmid |N_1 \cap N_2|$. Then we have $[P \cap N_1, P \cap N_2] \subseteq P \cap N_1 \cap N_2 = 1$ and so $P$ is noncyclic, as $P \cap N_i \neq 1$ $(i=1,2)$ by (1.1). Therefore $N_1 \cap N_2 = \langle C_{N_1 \cap N_2}(x) \mid 1 \neq x \in P \rangle \subseteq Y$ by (1.2) and $Y \supseteq (P \cap N_1)(N_1 \cap N_2)$. Since $N_2$ normalizes $(P \cap N_1)(N_1 \cap N_2)$, we have $Y \supseteq N_2$ by (1.2). Then we get $Y \supseteq N(N_2) = X$ by (1.2), a contradiction.

*Remark.* By lemma 1, a $p$-isolated group $X$ possesses the smallest normal subgroup $X_0$ whose order is divisible by $p$. In [7], it is shown that $X_0/O_{p'}(X_0)$ is an (abelian or non abelian) simple group.

## § 2. Fusion Theorems.

Let $G$ be a finite group and $P$ be a Sylow $p$-subgroup of $G$. Let $\mathcal{H}$, $\mathcal{H}(P)$ and $X(H)$ $(H \in \mathcal{H})$ be as in the introduction.

The following lemma is a slight generalization of K. Gomi [4, Prop. (2.3)].

LEMMA 2. *If $P, Q \in \mathrm{Syl}_p(G)$ and $P \cap Q \neq 1$, there exists a sequence $P_0 = P, P_1, \cdots$ $P_n = Q \in \mathrm{Syl}_p(G)$ such that*

(i) $H_i=P_{i-1}\cap P_i\in\mathcal{H}$ $(i=1, 2, \cdots, n)$

(ii) *$H_i$ is a tame intersection of $P_{i-1}$ and $P_i$*

(iii) *there exist* $x_i\in O^p(X(H_i))$ such that $P_i^{x_i}=P_{i-1}$

(iv) $P\cap Q=H_1\cap H_2\cap\cdots\cap H_n$.

*Proof.* We proceed by induction on $[P: P\cap Q]$. Firstly suppose that $P\cap Q$ is a maximal intersection. Then $P=P_0$ and $Q=P_1$ satisfy the conditions (i), (ii) and (iv). We will show that (iii) holds. Let $H=P_0\cap P_1$ and $M=N(P_1)\cap N(H)$. Then $M/H$ is a strongly $p$-embedded subgroup of $N(H)/H$, as $H=P_0\cap P_1$ is a maximal intersection. Then, by (1.3) applied as $Y=M/H$ and $N=X(H)/H$, we have

$$(2.1)\qquad N(H)=MO^p(X(H)), \quad (M=N(P_1)\cap N(H)).$$

Since $N_{P_0}(H)$, $N_{P_1}(H)\in\mathrm{Syl}_p(N(H))$, there exists $x\in N(H)$ such that $N_{P_1}(H)^x=N_{P_0}(H)$. By (2.1), we can write $x=mx_1$ $(m\in M, x_1\in O^p(X(H)))$. Then we have $N_{P_0}(H)=N_{P_1}(H)^x=N_{P_1}(H)^{mx_1}=N_{P_1}(H)^{x_1}$. Since $P_0\cap P_1^{x_1}\supseteq N_{P_0}(H)\supsetneqq H$ and $P_0\cap P_1$ is a maximal intersection, we must have $P_1^{x_1}=P_0$. Thus $P=P_0$ and $Q=P_1$ satisfy the condition (iii).

Let $H=P\cap Q$. Take $R, S\in\mathrm{Syl}_p(G)$ such that $N_P(H)\subseteq N_R(H)\in\mathrm{Syl}_p(N(H))$, $N_Q(H)\subseteq N_S(H)\in\mathrm{Syl}_p(N(H))$. Then we have $P\cap R\supseteq N_P(H)\supsetneqq H$ and $S\cap Q\supseteq N_Q(H)\supsetneqq H$. If $R\cap S\supsetneqq H$, we can find a sequence of Sylow $p$-subgroups which satisfies the conditions (i)$\sim$(iv) by induction applied to three pairs $(P, R)$, $(R, S)$ and $(S, Q)$. So we may assume $H=R\cap S$. If $N(H)/H$ is not $p$-isolated, we have a sequence $R_0=R, R_1, \cdots, R_m=S\in\mathrm{Syl}_p(G)$ such that $R_{i-1}\cap R_i\supsetneqq H$ $(i=1, 2, \cdots, m)$. Then we can apply induction to $(R_{i-1}, R_i)$ $(1\leq i\leq m)$. Therefore we may assume $N(H)/H$ is $p$-isolated. Let $\mathcal{C}$ be an equivalence class on $\mathrm{Syl}_p(N(H)/H)$ containing $N_S(H)/H$ (cf. §1) and $M=\{x\in N(H)\mid \mathcal{C}^x=\mathcal{C}\}$. Then $M/H$ is a strongly $p$-embedded subgroup of $N(H)/H$ and $\mathcal{C}=\mathrm{Syl}_p(M/H)\ni N_S(H)/H$. Then, by (1.3) applied as $Y=M/H$ and $N=X(H)/H$, we have $N(H)=MO^p(X(H))$. Take $x\in N(H)$ such that $N_S(H)^x=N_R(H)$ (Sylow's Theorem). Let $x=mx_1$ $(m\in M, x_1\in O^p(X(H))$. Then we have $R\cap S^{mx_1}\supseteq N_R(H)\supsetneqq H$ and $S^{mx_1}\cap S^m$ is a tame intersection of $S^{mx_1}$ and $S^m$. Furthermore it follows from $\mathcal{C}=\mathrm{Syl}_p(M/H)\ni N_S(H)/H$, $N_{S^m}(H)/H$ that there exists a sequence $S^m=S_0, S_1, \cdots, S_n=S$ of Sylow $p$-subgroups of $G$ suth that $N_{S_i}(H)\in\mathrm{Syl}_p N(H)$ and $S_{i-1}\cap S_i\supsetneqq H$ $(i=1, 2, \cdots, n)$. Thus we can find a sequence of Sylow $p$-subgroups satisfying (i)$\sim$(iv) by applying induction to $(P, R)$, $(R, S^{mx_1})$, $(S_{i-1}, S_i)$ and $(S, Q)$, q.e.d.

Now we can prove the theorem stated in the introduction.

THEOREM 1. *If $A, B$ are subsets of $P$ such that $A^g=B\neq 1$ for some $g\in G$, there exist $H_i\in\mathcal{H}(P)$, $x_i\in O^p(X(H_i))$ $(i=1, 2, \cdots, n)$ and $y\in N(P)$ such that*

(i) $g=x_1x_2\cdots x_ny$

(ii) $A\subseteq H_1$ and $A^{x_1x_2\cdots x_i}\subseteq H_{i+1}$ $(i=1, 2, \cdots, n-1)$.

*Proof.* Let $K_H=O^p(X(H))$ for $H\in\mathcal{H}$. Then we have clearly

$$K_{Hx}=K_H^x \quad \text{for any} \quad x\in G. \tag{2.2}$$

Let $A, B$ be subsets of $P$ with $A^g=B\neq 1$ for some $g\in G$. Then we have $A\subseteq P\cap P^{g^{-1}}\neq 1$. Therefore, by Lemma 2, there exists a sequence

$$P=P_0, P_1, \cdots, P_n=P^{g^{-1}}$$

of Sylow $p$-subgroups such that

(i) $L_i=P_{i-1}\cap P_i\in \mathcal{H}$

(ii) $L_i$ is a tame intersection of $P_{i-1}$ and $P_i$

(iii) there exists $y_i\in K_{L_i}$ such that $P_i^{y_i}=P_{i-1}$

(iv) $P\cap P^{g^{-1}}=L_1\cap L_2\cap\cdots\cap L_n$.

Set

$$x_1=y_1, \quad x_i=y_i^{y_{i-1}\cdots y_1}, \quad H_i=L_i^{y_iy_{i-1}\cdots y_1} \quad (i=1,2,\cdots,n). \tag{2.3}$$

Then we have

$$x_1x_2\cdots x_{i-1}=y_{i-1}\cdots y_1 \quad \text{and} \quad H_i\in\mathcal{H}(P), \tag{2.4}$$

as $H_i\subseteq P_i^{y_i\cdots y_1}=P$ and $N_P(H_i)=N_{P_i}(L_i)^{y_i\cdots y_1}$. By (2.2) and (2.3), we have

$$x_i=y_i^{y_{i-1}\cdots y_1}\in K_{L_i}^{y_{i-1}\cdots y_1}=K_{H_i}.$$

By (2.4) we have

$$A^{x_1x_2\cdots x_{i-1}}\subseteq L_i^{y_{i-1}\cdots y_1}=H_i.$$

Since $P=P_n^{y_n\cdots y_1}=(P^{g^{-1}})^{x_1\cdots x_n}$, we have $g^{-1}x_1\cdots x_n\in N(P)$ and so $g=x_1x_2\cdots x_ny$ for some $y\in N(P)$. Thus $H_i$, $x_i\in K_{H_i}$ and $y\in N(P)$ have the required properties, q.e.d.

To state the following theorem which is a corollary of Theorem 1, we introduce some notations.

For any $p$-subgroup $H$ of $G$, set

$$\tilde{C}(H)=O_p(N(H) \bmod C(H)).$$

Let

$$\mathcal{H}'(P)=\{H\in\mathcal{H}(P)\,|\,H\in \mathrm{Syl}_p(\tilde{C}(H))\}.$$

Note that $H\in\mathrm{Syl}_p(\tilde{C}(H))$ if and only if $H\in\mathrm{Syl}_p(O_{p'p}(N(H)))$ and $H\in\mathrm{Syl}_p(HC(H))$, and also that, if $H\in\mathcal{H}'(P)$, $N(H)$ is $p$-constrained and $\tilde{C}(H)=HC(H)=O_{p'p}(N(H))=O_{p'}(N(H))\times H$.

THEOREM 2. *If $A, B$ are subsets of $P$ such that $A^g=B\neq 1$ for some $g\in G$, there exist $H_i\in\mathcal{H}'(P)$, $x_i\in O_p(X(H_i))$ $(i=1,2,\cdots,n)$, $c\in C(A)$ and $y\in N(P)$ such that*

(i) $g=cx_1x_2\cdots x_ny$

(ii) $A \subseteq H_1$ *and* $A^{x_1 x_2 \cdots x_{i-1}} \subseteq H_i$.

*Proof*. If $H \in \mathscr{H}(P) - \mathscr{H}'(P)$, we have

$$C(H) \supseteq O^p(X(H)). \tag{2.5}$$

In fact, we have $\tilde{C}(H) \supseteq X(H)$ as $N(H) \triangleright \tilde{C}(H) \supseteq H$ and $p \mid |\tilde{C}(H)/H|$, and so $C(H) \supseteq O^p(\tilde{C}(H)) \supseteq O^p(X(H))$. By Theorem 1, we have $H_i \in \mathscr{H}(P)$, $y_i \in O^p(X(H_i))$ $(i=1, 2, \cdots, n)$ and $y \in N(P)$ such that

(i) $g = y_1 y_2 \cdots y_n y$

(ii) $A \subseteq H_1$ and $A^{y_1 y_2 \cdots y_i} \subseteq H_{i+1}$ $(i=1, 2, \cdots, n-1)$.

If $H_i \notin \mathscr{H}'(P)$, we have $y_i \in C(H_i)$ by (2.5) and so $y_i^{(y_1 y_2 \cdots y_{i-1})^{-1}} \in C(H_i^{(y_1 y_2 \cdots y_{i-1})^{-1}}) \subseteq C(A)$. Thus we have

$$g = y_i^{(y_1 y_2 \cdots y_{i-1})^{-1}} y_1 y_2 \cdots y_{i-1} y_{i+1} \cdots y_n y$$

and

$$y_i^{(y_1 y_2 \cdots y_{i-1})^{-1}} \in C(A).$$

The repeated applications of this fact complete the proof of Theorem 2.

## § 3. Applications.

3.1. Some examples.

*Example* 1 (Goldschmidt [5]). For $H \in \mathscr{H}(P)$, let

$$T_H = \begin{cases} C(H), & \text{if} \quad H \notin \mathrm{Syl}_p(O_{p'p}(N(H))) \quad \text{or} \\ & \phantom{\text{if} \quad} H \notin \mathrm{Syl}_p(HC(H)) \\ N(H), & \text{otherwise} \end{cases}$$

As remarked in § 2, the condition $H \notin \mathrm{Syl}_p(O_{p'p}(N(H)))$ or $H \notin \mathrm{Syl}_p(HC(H))$ is equivalent to $H \notin \mathrm{Syl}_p(\tilde{C}(H))$. Furthermore, if $H \notin \mathrm{Syl}_p(\tilde{C}(H))$, we have $C(H) \supseteq O^p(X(H))$ by (2.5). Thus for $H \in \mathscr{H}(P)$, we have $T_H \supseteq O^p(X(H))$. Then, by Th. 1, the family $\{(H, T_H) | H \in \mathscr{H}(P)\}$ is a conjugation family. This family is the one defined in Goldschmidt [5].

*Example* 2 (Goldschmidt [6, § 9 (9, 1)]). Let $Z$ be a nonidentity subgroup of $Z(P)$. For $H \in \mathscr{H}(P)$, let

$$T_H = \begin{cases} C(H), & \text{if} \quad H \notin \mathrm{Syl}_p(\tilde{C}(H)) \\ C(Z^{N(H)}) \cap N(H), & \text{if} \quad H \in \mathrm{Syl}_p(\tilde{C}(H)) \quad \text{and} \\ & \phantom{\text{if} \quad} H \notin \mathrm{Syl}_p(C(Z^{N(H)}) \cap N(H)) \\ N(H), & \text{otherwise.} \end{cases}$$

If $H\in\mathrm{Syl}_p(\tilde{C}(H))$, we have $Z\subseteq H$ and $[Z,H]=1$. So $H\subseteq C(Z^{N(H)})\cap N(H)\lhd N(H)$. Thus if $H\in\mathrm{Syl}_p(\tilde{C}(H))$ and $H\notin\mathrm{Syl}_p(C(Z^{N(H)})\cap N(H))$, we have $C(Z^{N(H)})\cap N(H)\supseteq X(H)$. Therefore we have $T_H\supseteq O^p(X(H))$ for any $H\in\mathcal{H}(P)$. Then, by Th. 1, the family $\{(H,T_H)|H\in\mathcal{H}(P)\}$ is a conjugation family.

*Example* 3 (Solomon [8]). For $H\in\mathcal{H}(P)$,

$$T_H=\begin{cases} C(H)\,, & \text{if } H\notin\mathrm{Syl}_p(\tilde{C}(H)) \\ C(\Omega_1(Z(H)))\cap N(H)\,, & \text{if } H\in\mathrm{Syl}_p(\tilde{C}(H)) \text{ and} \\ & \quad H\notin\mathrm{Syl}_p(C(\Omega_1(Z(H)))\cap N(H) \\ N(H)\,, & \text{otherwise.}\end{cases}$$

Similarly as example 2, we see $T_H\supseteq O^p(X(H))$ for any $H\in\mathcal{H}(P)$. Therefore, by Th. 1, $\{(H,T_H)|H\in\mathcal{H}(P)\}$ is a conjugation family.

3.2. (A theorem of Glauberman). We shall give an alternate proof of the following theorem which Glauberman proved in [2] and [3].

THEOREM (Glauberman). *If $A$ is a strongly closed abelian $p$-subgroup of $G$, then $N(A)$ controls strong $p$-fusion in $G$.*

*Proof*. Let $P$ be a Sylow $p$-subgroup of $G$ containing $A$. In view of Th. 2, it suffices to show

$$N(H)\subseteq N(A) \quad \text{for any} \quad H\in\mathcal{H}'(P)\,.$$

To prove this, firstly we note that, if $A_0$ is a strongly closed abelian $p$-subgroup of a finite group $X$, then we have

$$X=C(O_p(X))N(A_0)\,. \tag{3.1}$$

In particular, if $X$ is $p$-constrained,

$$X\rhd A_0O_{p'}(X)\,. \tag{3.2}$$

Let $T=O_p(X)$ and $C=C(T\cap A_0)\cap C(T/T\cap A_0)$. As $A_0$ is strongly closed, we have $T\cap A_0\lhd X$ and so $C\lhd X$. Then Frattini argument yields $X=C\cdot N(C\cap P_0)$, where $P_0$ is a Sylow $p$-subgroup of $X$ containing $A_0$. Then we get $X=C(T)N(C\cap P_0)$, since $C/C(T)$ is $p$-group and so $C=C(T)(C\cap P_0)$. But since $A_0$ is abelian, we have $C\cap P_0\supseteq A_0$ and then the strong closure of $A_0$ yields $N(C\cap P_0)\subseteq N(A_0)$. Thus we must have $X=C(T)N(A_0)$, which proves (3.1). (3.2) follows from (3.1) applied to $X/O_{p'}(X)$.

Now let $H\in\mathcal{H}'(P)$. Then $N(H)$ is $p$-constrained and $O_{p'p}(N(H))=O_{p'}(N(H))\times H$. Since $N_A(H)$ is a strongly closed abelian $p$-subgroup of $N(H)$, we get

$$N(H)\rhd N_A(H)O_{p'}(N(H))$$

by applying (3.2) to $X=N(H)$. Then we have $N_A(H)\subseteq O_{p'p}(N(H))=O_{p'}(N(H))\times H$ and so $N_A(H)\subseteq H$. This implies $A\subseteq H$ and then the strong closure of $A$ yields $N(H)\subseteq N(A)$, q.e.d.

3.3. (Puig's $ZL$-Theorem). In [7], L. Puig defined a conjugacy functor $L(P)$ for a $p$-group $P$. $L(P)$ is defined as follows:

Let $L_*(P)$ be a subgroup generated by all abelian normal subgroups of $P$ and $L^*(P)$ be a subgroup generated by all abelian subgroups of $P$ normalized by $L_*(P)$. Then we define

$$L_0{}^*(P)=P, \quad L_i{}^*(P)=L^*(L_{i-1}^*(P)) \quad (i=1, 2, \cdots)$$

and

$$L(P)=\bigcap_{i=0}^{\infty} L_i{}^*(P).$$

The following lemma is proved in [7] by elementary arguments. For the proof, we refer to [7, p. 54-56].

LEMMA 3. *Let $Q$ be a subgroup of a $p$-group $P$. Then we have*

(i) $C_P(L(P))\subseteq L(P)$

(ii) *If $Q\supseteq L(P)$, we have $L(P)=L(Q)$.*

(iii) *If $Q\not\supseteq L(P)$, there exists a subgroup $B$ of $P$ such that $B\subseteq N_P(ZL(Q))$, $B\not\subseteq Q$ and $[ZL(Q), B, B]=1$.*

(iv) *If $Q\supseteq L_*(L(P))$, we have $ZL(Q)\supseteq ZL(P)$*

(v) *If $Q\not\supseteq L_*(L(P))$, there exists a subgroup $B$ of $P$ such that $B\subseteq N_P(L(Q))$, $B\not\subseteq Q$ and $[L(Q), B, B]=1$.*

Now we can prove the following theorem as an application of Theorem 2.

THEOREM (L. Puig). *Let $G$ be a finite group and $p$ be an odd prime. If $G$ is $p$-stable, $ZL(P)$ controls strong fusion in $G$, where $P$ is a Sylow $p$-subgroup of $G$.*

*Proof.* Recall that $G$ is $p$-stable, if $G$ satisfies the following conditions:

Whenever $Q$ is a $p$-subgroup of $G$ and $x\in N(Q)$ with $[N(Q), x, x]=1$, we have $x\in\tilde{C}(Q)$.

In view of Theorem 2, it will be sufficient to show

(3.3) $$O^p(X(H))\subseteq N(ZL(P)) \quad \text{for any} \quad H\in\mathcal{H}'(P).$$

Let $H\in\mathcal{H}'(P)$. If $H\supseteq L(P)$, we have $L(H)=L(P)$ by Lemma 3 (ii), and so $O^p(X(H))$

$\subseteq N(H)\subseteq N(L(H))=N(L(P))\subseteq N(ZL(P))$. Therefore suppose $H\not\supseteq L(P)$. Then, by Lemma 3 (iii), there exists a subgroup $B$ of $P$ such that $B\subseteq N_P(ZL(H))$, $B\nsubseteq H$ and $[ZL(H), B, B]=1$. Then, by $p$-stability of $G$, we have $B\subseteq\tilde{C}(ZL(H))$ and so $H\notin \mathrm{Syl}_p(H\tilde{C}(ZL(H))\cap N(H))$, as $H\tilde{C}(ZL(H))\cap P\supseteq\langle B, H\rangle\supsetneqq H$. This implies $X(H)\subseteq H\tilde{C}(ZL(H))\cap N(H)$. Thus we get

$$O^p(X(H))\subseteq C(ZL(H)), \tag{3.4}$$

since $H\tilde{C}(ZL(H))/C(ZL(H))$ is $p$-group. Now we will show

$$ZL(H)\supseteq ZL(P) \quad \text{for any} \quad H\in\mathcal{H}'(P). \tag{3.5}$$

Then (3.4) and (3.5) will yield $O^p(X(H))\subseteq C(ZL(H))\subseteq C(ZL(P))\subseteq N(ZL(P))$ which will imply (3.3).

So suppose by way of contradiction that $ZL(H)\not\supseteq ZL(P)$ for some $H\in\mathcal{H}'(P)$. Then, by Lemma 3 (iv), (v) we have $H\not\supseteq L_*(L(P))$ and there exists a subgroup $B$ of $P$ such that $B\subseteq N_P(L(H))$, $B\nsubseteq H$ and $[L(H), B, B]=1$. By $p$-stability of $G$, we have $B\subseteq\tilde{C}(L(H))$ and then

$$H\notin\mathrm{Syl}_p(H\tilde{C}(L(H))\cap N(H)) \tag{3.6}$$

as $H\tilde{C}(L(H))\cap P\supseteq\langle B, H\rangle\supsetneqq H$. But we have

$$\tilde{C}(L(H))\cap N(H)\subseteq\tilde{C}(H). \tag{3.7}$$

In fact, if $x$ is a $p'$-element of $\tilde{C}(L(H))\cap N(H)$, then we have $x\in O^p(\tilde{C}(L(H)))\subseteq C(L(H))$ and so $[x, L(H), H]=1$. Then 3-subgroup lemma yields $[x, H]\subseteq C(L(H))$. So $x$ stabilizes a chain $H\supseteq L(H)\supseteq 1$, since $C_H(L(H))\subseteq L(H)$ by Lemma 3 (i). Therefore we get $x\in C(H)$ and $\tilde{C}(L(H))\cap N(H)/C(H)$ is $p$-group, which implies (3.7). As $H\in\mathcal{H}'(P)$, we have $H\in\mathrm{Syl}_p(\tilde{C}(H))$ and so $H\in\mathrm{Syl}_p(H\tilde{C}(L(H))\cap N(H))$ by (3.7), contrary to (3.6). This contradiction implies (3.5), q.e.d.

## References


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