Automorphisms of a Finite Group and Their Fixed Points

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1. Let A be a noncyclic elementary abelian p-group where p is a prime, and X be a p'-group on which A acts. When informations on the nilpotent length of the groups $C_X(a)$ $(a\in A^{\#})$ are given, what can we say about the nilpotent length of the underlying group X? This sort of results have been obtained by Kurzweil [3] and J. N. Ward [4], [5].

The purpose of this note is to prove the following:

Theorem. *Let A and X be as above. Assume that, for any $a\in A^{\#}$, $C_X(a)$ is solvable. Let n be an integer defined by*

$$n=\operatorname*{Max}_{a\in A^{\#}} n(C_X(a))$$

where $n(C_X(a))$ denotes the nilpotent length of $C_X(a)$. If $m(A)\geqq n+2$, X is solvable with $n(X)\leq n$.

Notations. For a solvable group Y, $F(Y)$ denotes the Fitting subgroup (=the largest normal nilpotent subgroup) of Y, and we define $F_m(Y)$ inductively:

$$F_1(Y)=F(Y) \quad \text{and} \quad F_m(Y)=F(Y \bmod F_{m-1}(Y)) \ (m\geqq 2)$$

If $Y=F_n(Y)$ and $Y\supsetneqq F_{n-1}(Y)$, n is called the nilpotent length of a solvable group Y and is denoted by $n(Y)$. For an abelian group B, $m(B)$ denotes the minimal number of generators of B. The other notations used in this note are standard.

Remarks. (1) The assertion in the theorem that X is solvable is an immediate corollary of the solvable signalizer functor theorem of Goldschmidt [1], if $n\geqq 2$ and so $m(A)\geqq 4$. We shall use this fact in the proof of the theorem.

(2) If $m(A)\leq n+1$, we can construct a solvable group X with $n(X)=n+1$ which satisfies the hypothesis of the theorem (cf. § 4). Thus the assumption $m(A)\geqq n+2$ is essential.

2. Lemma. *Let Y be a finite group on which a finite group B acts. Assume that the orders of B and Y are coprime. Then the followings hold:*

(i) *For any prime q dividing $|Y|$, there exists a B-invariant Sylow q-subgroup of Y.*

(ii) *Let Q be a B-invariant Sylow q-subgroup of Y. For any subgroup V of B, $C_Q(V)$ is a Sylow q-subgroup of $C_Y(V)$.*

(iii) *Let Y_0 be a B-invariant normal subgroup of Y. Set $\bar{Y}=Y/Y_0$. Then, for any subgroup V of B, we have $C_{\bar{Y}}(V)=\overline{C_Y(V)}$,*

(iv) *If B is abelian, we have*

$$Y=\langle C_Y(V)|B/V=\text{cyclic}\rangle.$$

(v) *If B is noncyclic abelian and Y is solvable, we have*

$$\bigcap_{b\in B^\#} F(C_Y(b))\subseteq F(Y).$$

Proof. The assertions (i), (ii) and (iii) are consequences of the Schur-Zassenhaus's Theorem [2, p. 224]. For (iv), refer to [2, p. 225]. We shall prove (v). By [1, Lemma 2, 3], we have

$$\bigcap_{b\in B^\#} O_{q'}(C_Y(b))\subseteq O_{q'}(Y)$$

for any prime q. Take a prime r dividing $|Y|$. Then we have

$$\bigcap_{q\neq r} O_{q'}(Y)=O_r(Y)$$

and

$$\bigcap_{q\neq r}\ \bigcap_{b\in B^\#} O_{q'}(C_Y(b)) = \bigcap_{b\in B^\#} O_r(C_X(b)).$$

Thus we get

$$\bigcap_{b\in B^\#} O_r(C_Y(b))\subseteq O_r(Y).$$

This yields (v).

3. The proof of the theorem. We proceed by induction on n, where n is an integer defined in the theorem:

$$n=\operatorname*{Max}_{a\in A^\#} n(C_X(a)).$$

Firstly suppose $n=1$. Then our hypothesis is that $C_X(a)$ is nilpotent for any $a\in A^\#$ and $m(A)\geqq 3$, and we will have to show that X is nilpotent. Let q be a prime such that q divides $|X|$ and Q be an A-invariant Sylow q-subgroup of X. We shall show

(1) $$Q\lhd X.$$

By Lemma (iv), we have

(2) $$X=\langle C_X(V) \mid A/V=\text{cyclic}\rangle.$$

The second application of Lemma (iv) yields

(3) $$Q=\langle C_Q(U) \mid V/U=\text{cyclic}\rangle.$$

where V is any fixed subgroup such that A/V is cyclic. (Note that V is noncyclic, as $m(A)\geqq 3$). Then $C_Q(U)$ is a Sylow q-subgroup of $C_X(U)$ for any U with $V/U=$cyclic (see Lemma (ii)). Noting that $U\neq 1$ as V is noncyclic, $C_X(U)$ is nilpotent and so $C_X(V)$ normalizes $C_Q(U)$, as $C_X(V)\subseteq C_X(U)$. By (3), $C_X(V)$ normalizes Q. Since V is arbitrary, X normalizes Q by (2). This proves (1). Thus X is nilpotent. We have proved the case $n=1$.

Let $n>1$. Then we have $m(A)\geqq n+2\geqq 4$. A theorem of Goldschmidt [1] yields that X is solvable. Set $\bar{X}=X/F(X)$. We shall show that, for any subgroup V of A with $m(V)=2$,

(4) $$n(C_{\bar{X}}(V))\leqslant n-1.$$

By Lemma (v), we have

$$\bigcap_{v\in V^{\#}} F(C_X(v))\subseteq F(X).$$

This implies

(5) $$F(X)\cap C_X(V)=\bigcap_{v\in V^{\#}} C_X(v).$$

Then we have

(6) $$\begin{aligned} C_{\bar{X}}(V)&=\overline{C_X(V)} && \text{(Lemma (iv))}\\ &\cong C_X(V)/\bigcap_{v\in V^{\#}}(F(C_X(v))\cap C_X(V)) && \text{(by (5)).}\end{aligned}$$

But we have

$$C_X(v)/F(C_X(v))\cap C_X(V)\cong C_X(V)F(C_X(v))/F(C_X(v)).$$

Since $n(C_X(v))\leqslant n$ by the assumption of the theorem and $C_X(V)\subseteq C_X(v)$, we have

$$n(C_X(V)F(C_X(v))/F(C_X(v)))\leqslant n-1.$$

Then (6) yields (4), because the nilpotent length of a direct product is the largest of those of direct factors and the nilpotent length of a homomorphic image does not exceed the one of the original group. Next we shall show, for any $a\in A^{\#}$,

(7) $$n(C_{\bar{X}}(a))\leqslant n-1.$$

If we set $Y=C_{\bar{X}}(a)$ and $A=\langle a\rangle\oplus A_1$, A_1 acts on Y. Then (4) yields

$$n(C_Y(b))=n(C_{\bar{X}}(\langle a, b\rangle)\leqslant n-1$$

for any $b\in A_1^\sharp$. Since $m(A_1)\geqq(n-1)+2$, the induction applies to get (7).

Finally (7) and the induction yield $n(\bar{X})\leqslant n-1$, which implies $n(X)\leqslant n$. This completes the proof of the theorem.

4. The construction of examples mentioned in the remark. In this section, we shall construct a pair of groups (A_n, X_n) for each integer $n\geqq 2$ as follows:

A_n: elementary abelian p-group of order p^n.

X_n: solvable p'-group with $n(X_n)=n$,

A_n acts on X_n in such a way that, for any $a\in A^\sharp$, $n(C_{X_n}(a))\leqq n-1$.

We proceed by induction on n. We know that (A_2, X_2) can be constructed (cf. [4, p. 469]). So we shall show how to construct (A_{n+1}, X_{n+1}) from (A_n, X_n). Set

$$Y_n=A_n\cdot X_n \qquad \text{(a semi-direct product)}.$$

Take a prime q such that p divides $q-1$ and $(q, |F(X_n)|)=1$. Let V_n be a vector space of finite dimension over the field F_q of q elements such that Y_n operates faithfully on V_n and $Y_n\cap Z(GL(V_n))=1$. (For example, look at the regular representation of Y_n). Let

$$\phi_n=\begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}=\alpha I\in GL(V_n)$$

where $\alpha^p=1$ and $1\neq\alpha\in F_q$. Regarding $Y_n\subseteq GL(V_n)$, set

$$U_{n+1}=\langle \phi_n, Y_n\rangle$$

and

$$A_{n+1}=\langle \phi_n, A_n\rangle.$$

Constructing a semidirect product $Y_{n+1}=U_{n+1}\cdot V_n$, let

$$X_{n+1}=O_{p'}(Y_{n+1}).$$

Then (A_{n+1}, X_{n+1}) is what we require. In fact, we have $F(X_{n+1})=V_n$ because $(q, |F(X_n)|)=1$ and Y_n is faithful on V_n. We note that $X_{n+1}=X_n\cdot V_n$ and X_n, V_n are invariant under A_{n+1}. Since $C_{X_{n+1}}(\phi_n)=X_n$, we have $n(C_{X_{n+1}}(\phi_n))=n$. If $\psi\in A_n$, we have

$$C_{X_{n+1}}(\psi)=C_{V_n}(\psi)C_{X_n}(\psi)$$

and so $n(C_{X_{n+1}}(\psi))\leqslant n$ since $n(C_{X_n}(\psi))\leqslant n-1$. If $\psi\in A_{n+1}-A_n$ and $\psi=\phi_n^m\psi_1$ $(\psi_1\in A_n$

and $m \neq 0$), we have

$$C_{X_{n+1}}(\psi)=C_{V_n}(\psi)C_{X_n}(\psi)$$
$$=C_{V_n}(\psi)C_{X_n}(\psi_1)$$

since $[\phi_n, X_n]=1$ by construction. Then we have

$$n(C_{X_{n+1}}(\psi))\leqslant n.$$

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