On the Essential Spectrum of Schrödinger Operators with Vector Potentials

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1. A Schrödinger operator with a vector potential has the form

$$Lu(x)=-\sum_{j,k=1}^{n}(\partial_j+ib_j(x))a_{jk}(x)(\partial_k+ib_k(x))u(x)+c(x)u(x).$$

Recently, Ikebe and Saito [1] and Mochizuki [5] investigated the spectral properties of L by the limiting absorption method and proved among other things that the spectrum of L is absolutely continuous in $(0, \infty)$. In deriving this result, it is assumed that c and the rotation of b, but not necessarily b_j themselves, tend to zero as $|x|\to\infty$ with the specified order of decrease. From this assumption it also follows that the essential spectrum $\sigma_{ess}(L)$ of L is contained in $[0, \infty)$. As is mentioned in [5], however, it seems that the relation $\sigma_{ess}(L)=[0, \infty)$ has remained unproved.

In the present note we prove $\sigma_{ess}(L)=[0, \infty)$. The method is simple and based on the fact that a guage transformation in the electromagnetic theory does not change the spectrum. By virtue of this fact it suffices to find a guage transformation which makes new b_j decrease at infinity. In this note we are concerned with selfadjoint problems, but the method may be applied to non-selfadjoint problems as treated by Saito [6]. For previous results for selfadjoint problems, see [2], [7].

In formulation we follow Mochizuki's work [5] on exterior problems. However, it is convenient for us to work with quadratic forms rather than operators. This is because only new b_j, not necessarily their derivatives, decrease at infinity. Assumptions on a_{jk} and c will be slightly weakened.

2. Let Ω be a domain in R^n with smooth boundary $\Gamma=\partial\Omega$ such that the complement of Ω is compact. Let $a_{jk}(x)$, $b_j(x)$, and $c(x)$ be real functions on Ω and σ a real function on Γ. We consider the quadratic form

$$h[u, v]=\sum_{j,k=1}^{n}(a_{jk}(\partial_k+ib_k)u, (\partial_j+ib_j)v)+(cu, v)+(\sigma u, v)_\Gamma \tag{2.1}$$

in $L^2(\Omega)$, where $(\ ,\ )$ and $(\ ,\ )_\Gamma$ denote the inner products in $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. We assume that:

(2.2) $\quad a_{jk}$ is bounded, measurable, and satisfies $a_{jk}(x)=a_{kj}(x)$;

(2.3) $$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2 \quad \text{for all} \quad x\in R^n \quad \text{and} \quad \xi\in R^n \quad (c_0>0);$$

(2.4) $$|a_{jk}(x)-\delta_{jk}|\to 0 \quad \text{as} \quad |x|\to\infty;$$

(2.5) $\quad b_j\in C^1(\bar{\Omega})$ and there exist $c>0$ and $\varepsilon>0$ such that

$$|\partial_j b_k(x)-\partial_k b_j(x)|\leq c(1+|x|)^{-1-\varepsilon}$$

for all $x\in R^n$ and $j, k=1, \cdots, n$;

(2.6) $$\sup_{y\in\Omega}\int_{\substack{|x-y|\leq 1\\ x\in\Omega}} \frac{|c(y)|}{|y-x|^{n-\alpha}}dx<\infty \quad \text{for some } \alpha,\ 0<\alpha<2;$$

(2.7) $$\lim_{|y|\to\infty}\int_{\substack{|y-x|\leq 1\\ x\in\Omega}} |c(x)|dx=0;$$

(2.8) $$\sigma\in C^1(\Gamma).$$

We denote by H^m the Sobolev space of order m.

THEOREM. *Let*

$$\mathfrak{D}_b=\{u\in L^2(\Omega)\cap H^{1,\mathrm{loc}}(\bar{\Omega})\,|\,(\partial_k+ib_k)u\in L^2(\Omega),\ k=1, \cdots, n\}$$

and let h be the quadratic form with domain $\mathfrak{D}(h)=\mathfrak{D}_b$ defined by expression (2.1). *Then*: i) $\mathfrak{D}(h)=\{e^{-i\phi}u\,|\,u\in H^1(\Omega)\}$ *for some* $\phi\in C^1(\bar{\Omega})$; ii) *$h$ is a closed Hermitian symmetric form which is bounded from below*; *and* iii) *the essential spectrum of the selfadjoint operator H associated with the form h coincides with* $[0, \infty)$.

Remark 1. For the sake of brevity we formulated the theorem for the case that the Robin boundary condition is satisfied. However, the theorem remains valid with a suitable change in the case that the Dirichlet condition is imposed on some of the connected components of Γ.

Remark 2. It is seen from the proof that a unitary transformation converts the form h to form (2.1) with an additional condition that

(2.9) $$|b_j(x)|\leq c_1(1+|x|)^{-\min(\varepsilon,1)} \quad \text{if} \quad \varepsilon\neq 1.$$

3. Proof of the theorem.

LEMMA. *Let b_j, $j=1, \cdots, n$, satisfy* (2.5) *and put $b(x)=(b_1(x), \cdots, b_n(x))$. Then,* *according as* $\varepsilon \neq 1$ *or* $\varepsilon = 1$ *there exists* $\phi \in C^1(\bar{\Omega})$ *such that*

$$|b(x) - \operatorname{grad} \phi(x)| \leq \begin{cases} c_2(1+|x|)^{-\min(\varepsilon, 1)}, & \text{if } \varepsilon \neq 1, \\ c_2(1+|x|)^{-1} \log(1+|x|), & \text{if } \varepsilon = 1. \end{cases} \tag{3.1}$$

Proof. For the simplicity of exposition the proof will be given for $n=3$. The proof is essentially the same for general n. It suffices to assume $\Omega = R^3$ and construct $\phi \in C^1(R^3 - \{0\})$ satisfying (3.1) for $|x| \geq 1$. We define ϕ as

$$\phi(x) = \int_0^x b(x) \cdot d\boldsymbol{s} = \int_0^x b_1(x)dx_1 + b_2(x)dx_2 + b_3(x)dx_3,$$

where we integrate along the straight line from 0 to x. Let $e_1 = (1, 0, 0)$ and let h be real. Let γ be a triangular contour passing the vertices $0, x+he_1, x, 0$ in this order and C the triangle encircled by γ. Then, using Stokes's theorem we obtain

$$\begin{aligned} \phi(x+he_1) - \phi(x) &= \int_\gamma b(x) \cdot d\boldsymbol{s} + \int_x^{x+he_1} b(x) \cdot d\boldsymbol{s} \\ &= \int_C \operatorname{rot} b(x) \cdot \boldsymbol{n} dS + \int_{x_1}^{x_1+h} b_1(t, x_2, x_3) dt, \end{aligned} \tag{3.2}$$

where $\boldsymbol{n}$ is the unit vector in the (positive) normal direction of C, dS the area element of C, and $x = (x_1, x_2, x_3)$. Dividing both sides of (3.2) by h and letting $h \to 0$, we obtain

$$\frac{\partial}{\partial x} \phi(x) = b_1(x) + \sin\theta \int_0^{|x|} \operatorname{rot} b(|x|^{-1}sx) \cdot \boldsymbol{n} |x|^{-1} s ds,$$

where θ is the angle between x and e_1. This and the similar formulas for $\partial\phi/\partial x_2$ and $\partial\phi/\partial x_3$ show that $\phi \in C^1(R^n - \{0\})$. Furthermore, it follows from (2.5) that

$$|b(x) - \operatorname{grad} \phi(x)| \leq \frac{3c}{|x|} \int_0^{|x|} \frac{s ds}{(1+s)^{1+\varepsilon}}.$$

(3.1) for $|x| > 1$ follows from this at once.

Remark. If the first order derivatives of $\operatorname{rot} b(x)$ satisfies condition (2.5), we can prove an inequality similar to (3.1) for the derivatives of $b - \operatorname{grad} \phi$.

Proof of the theorem. Let ϕ be as given in the lemma and put $b^{(1)} = b - \operatorname{grad} \phi$. Let W be the unitary operator in $L^2(\Omega)$ defined as $Wu = e^{-i\phi}u$. Then, as is immediately seen, one has $(\partial_k + ib_k)W = W(\partial_k + ib_k^{(1)})$*. Hence $\mathfrak{D}_b = W\mathfrak{D}_b^{(1)}$. Since $b^{(1)}$

* The change from b to $b^{(1)}$ is a (time-independent) "guage transformation." It is well-known in physics that such a transformation induces a change of phase of wave functions as described by W (cf., e.g., Landau and Lifschitz [4; Chapt. XV]). Similar use of guage transformations in the study of Schrödinger operators was made also in a recent work of Simon [8].

is bounded, we have $\mathfrak{D}_b^{(1)}=H^1(\Omega)$. Thus, statement i) is proved.

The quadratic form $h[Wu, Wv]$, $u, v\in W^*\mathfrak{D}_b=\mathfrak{D}_b^{(1)}$, has the expression (2. 1) with b replaced by $b^{(1)}$. Since statements ii) and iii) of the theorem are concerned with unitarily invariant properties, we may replace b by $b^{(1)}$ and assume that b itself satisfies (2. 9). Under this assumption statements ii) and iii) are essentially known and may be proved as follows. (Note, however, that the following proof does not require any information on the behavior of $\partial_j b_k$ at infinity.)

It follows from the boundedness of b, the relation $\mathfrak{D}_b=H^1(\Omega)$, and assumptions (2. 2)—(2. 4) that the positive Hermitian form $h_1[u, v]=\sum(a_{jk}(\partial_k+ib_k)u, (\partial_j+ib_j)v)$ on $\mathfrak{D}_b$ is closed and that $h_1[u]+\|u\|^2$ is equivalent to the norm of $H^1(\Omega)$. Assumptions (2. 6) and (2. 7) imply that the multiplication by $|c|^{1/2}$ is a compact operator from $H^1(\Omega)$ to $L^2(\Omega)$ (see Schechter [7; pp. 105 and 111]). Hence, the form (cu, v) is defined on $\mathfrak{D}_b$ and is h_1-compact. It is also clear that the form $(\sigma u, v)_\Gamma$ is h_1-compact. Since h_1-compact forms are h_1-bounded with relative bound 0, statement ii) follows.

Let $h_0[u, v]=\sum_{k=1}^{n}(\partial_k u, \partial_k v)$ with $\mathfrak{D}(h_0)=H^1(\Omega)$. Then, the form h can be expressed as

$$h[u, v]=h_0[u, v]+\sum_{\nu}(B_\nu u, A_\nu v)+(C_1 u, C_2 v)+(\sigma u, v)_\Gamma, \tag{3. 3}$$

where $\sum_{\nu}$ is a finite sum and A_ν, B_ν, and C's have the following form: A_ν is one of ∂_j and the identity; B_ν is one of $(a_{jk}-\delta_{jk})\partial_k$, $ia_{jk}b_k$, $-ib_j a_{jk}\partial_k$, and $a_{jk}b_j b_k$; $C_1=\operatorname{sgn} c|c|^{1/2}\cdot$ and $C_2=|c|^{1/2}\cdot$. Let $\mathfrak{K}=(\sum_{\nu}\oplus L^2(\Omega))\oplus L^2(\Omega)\oplus L^2(\Gamma)$, where $\sum_{\nu}\oplus L^2(\Omega)$ stands for the direct sum of copies of $L^2(\Omega)$ indexed by ν. We define operators A and B from $L^2(\Omega)$ to $\mathfrak{K}$ as follows:

$$Au=(\sum_{\nu}\oplus A_\nu u)\oplus C_2 u\oplus u|_\Gamma,$$

$$Bu=(\sum_{\nu}\oplus B_\nu u)\oplus C_1 u\oplus \sigma u|_\Gamma.$$

Then (3. 3) can be written as

$$h[u, v]=h_0[u, v]+(Bu, Av)\,.$$

Let H_0 and H be selfadjoint operators associated with h_0 and h and take a negative λ such that $\lambda<H$. Put $R(\lambda)=(H-\lambda)^{-1}$ and $R_0(\lambda)=(H_0-\lambda)^{-1}$. Then we know that

$$R(\lambda)-R_0(\lambda)=[R_0(\lambda)A^*]^a(1+[BR_0(\lambda)A^*]^a)^{-1}BR_0(\lambda),$$

where $[T]^a$ stands for the closure of an operator T (see Kuroda [3; § 2]). It is clear that $B_\nu R_0(\lambda)$ is compact for every ν. Furthermore, the compactness of $|c|^{1/2}$ (or σ) as an operator from $H^1(\Omega)$ to $L^2(\Omega)$ (or $L^2(\Gamma)$) implies the compactness of $C_1R_0(\lambda)$ (or $\sigma R_0(\lambda)$). Thus, $BR_0(\lambda)$ and hence $R(\lambda)-R_0(\lambda)$ are compact. Therefore, $\sigma_{ess}(R(\lambda))=\sigma_{ess}(R_0(\lambda))$ by Weyl's theorem. This gives $\sigma_{ess}(H)=\sigma_{ess}(H_0)=[0, \infty)$ and statement iii) is proved.

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