A Theorem on a Multiplication Operator in Certain Function Spaces

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Finite-dimensional Minkowsky-Farkas lemma [4] was generalized by Hurwicz [5] to infinite dimensional cases. We shall generalize his result further to the case of densely defined linear operators which are not necessarily everywhere defined. Next, we shall apply the result to the case of a multiplication operator between concrete function spaces on [0, 1].

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Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be locally convex linear topological spaces on $\boldsymbol{R}$ and let P_Y be a closed convex cone in $\boldsymbol{Y}$. The cone P_Y induces order relations (cf. e.g. [6]) in $\boldsymbol{Y}$ and $\boldsymbol{Y}^*$.

PROPOSITION 1. *Let A be a linear operator from $\boldsymbol{X}$ into $\boldsymbol{Y}$ whose domain $D(A)$ is dense in $\boldsymbol{X}$. We assume that:*

(α) *any non-negative y^* in $\boldsymbol{Y}^*$ is approximated in the w^*-topology by non-negative elements of $D(A^*)$.*

Let $V_A=\{x^*\in \boldsymbol{X}^* : x^*(x)\geqq 0$ for all $x\in D(A)$ such that $Ax\geqq 0\}$ and let $Z_{A^*}=\{A^*y^* : 0\leqq y^*\in D(A^*)\}$.

In order that $Z_{A^*}=V_A$, it is necessary and sufficient that Z_{A^*} is closed in the $\sigma(\boldsymbol{X}^*, D(A))$-topology, whose definition is found e.g. in [2].

Proof. (*Sufficiency*) An inclusion $V_A\supset Z_{A^*}$ always holds. Let Z_{A^*} be $\sigma(\boldsymbol{X}^*, D(A))$-closed. We take an $x_0^*\notin Z_{A^*}$. Then, for some $x_0\in D(A)$,

$$\inf_{x^*\in Z_{A^*}} x^*(x_0)=\inf_{0\leqq y^*\in D(A^*)} y^*(Ax_0)=0>x_0^*(x_0)\ ,$$

i.e.,

$$y^*(Ax_0)\geqq 0>x_0^*(x_0) \text{ for any } 0\leqq y^*\in D(A^*)\ . \tag{1}$$

We note that the inequality (1) holds also for any non-negative $y^*\in \boldsymbol{Y}^*$ by the assumption (α).

Next, we assert that

$$Ax_0\geqq 0\ . \tag{2}$$

In fact, if (2) is false, then Ax_0 does not belong to the closed convex cone P_Y. Therefore Ax_0 and P_Y are strictly separated by a functional $y_0^* \in Y^*$ such as

$$\inf_{y \in P_Y} y_0^*(y) = 0 > y_0^*(Ax_0) ,$$

i.e.,

$$y_0^*(Ax_0) < 0 , \quad y_0^* \geqq 0 .$$

which contradicts (1). Thus we have the inequality (2). The inequalities (1) and (2) show that $x_0^* \notin V_A$.

Necessity of the condition is evident, because V_A is $\sigma(X^*, D(A))$-closed.

Next, we shall consider cases where X and Y are $C \equiv C$ [0, 1] or $L_p \equiv L_p$ [0, 1] with $1 \leqq p \leqq +\infty$ and P_Y is the set of non-negative functions in Y. Let A be an operator defined by multiplication by a fixed function $a(t)$, i.e.,

$$Ax(t) = a(t)x(t) ,$$

$$D(A) = \{x \in X;\ ax \in Y\} .$$

Our aim is to study conditions to be imposed on the function a so that we can apply Proposition 1. Let a be a measurable function on [0, 1] and let $S = \{t;\ a(t) \neq 0\}$. We define functions a^{-1} and $a^{(-1)}$ as follows:

$$a^{-1}(t) = a(t)^{-1} \text{ for } t \in S \text{ and } a^{-1}(t) = +\infty \quad \text{for } t \notin S ,$$

$$a^{(-1)}(t) = a(t)^{-1} \text{ for } t \in S \text{ and } a^{(-1)}(t) = 0 \quad \text{for } t \notin S .$$

We say that a function h *belongs locally to* L_p *at* t, if $h|_U$ belongs to $L_p(U)$ for some neighborhood U of t. Let Γ_p be the set of such a function h that h^{-1} belongs locally to L_p at almost all t where h does not vanish. We denote by M the set of all Lebesgue-measurable functions on [0, 1].

PROPOSITION 2. *In each combination of X and Y, a necessary and sufficient condition for $D(A)$ to be dense in X is that the function $a(t)$ belongs to a set listed in the table, where $1 \leqq p, p' < +\infty$.*

X \ Y	C	L_∞	$L_{p'}$
C	1) C	2) L_∞	3) $L_{p'}$
L_∞	4) $\{0\}$	5) L_∞	6) $L_{p'}$
L_p	7)*) Γ_p	8) M	9) M

*) The result is due to F. Niiro.

In each case, if $D(A)$ is dense, $D(A^*)$ satisfies the condition (α).

Proof. (I) Sufficiency of the conditions except for 7), 8) and 9) is clear.
Case 1). If a is discontinuous at t_0, the domain $D(A)$ is contained in a set $\{x\in \boldsymbol{C};\ x(t_0)=0\}$ which is not dense in $\boldsymbol{C}$.
Case 5). Let a be not essentially bounded. We put $C_n=\{t;\ n\leqq |a(t)|<n+1\}$. There exists a subsequence $\{n_j\}$ such that the Lebesgue measure $|C_{n_j}|$ of C_{n_j} is positive. For any $x\in D(A)$, we have

$$|x(t)|n_j\leqq \|ax\|_{L_\infty}<+\infty \quad \text{for almost all } t\in C_{n_j}\,,$$

i.e.,

$$|x(t)|\leqq \frac{\|ax\|_{L_\infty}}{n_j} \quad \text{for almost all } t\in C_{n_j}\,.$$

Therefore $\|1-x\|_{L_\infty}\geqq 1-\dfrac{\|ax\|_{L_\infty}}{n_j}$ for all j. Letting $j\to+\infty$, we have $\|1-x\|_{L_\infty}\geqq 1$ which shows that $D(A)$ is not dense in $\boldsymbol{L}_\infty$.
Case 2). Proof goes in the same way as in Case 5).
Case 4). Suppose that a does not vanish almost everywhere. The Lebesgue measure of a set $Q=\{t;\ \gamma\leqq |a(t)|\leqq \delta\}$ is positive for some γ and $\delta>0$. We find a t_0 in Q such that for any $\varepsilon>0$, $|Q\cap(t_0-\varepsilon,\ t_0)|$ and $|Q\cap(t_0,\ t_0+\varepsilon)|$ are both positive. In fact, if every $t\in Q$ has a neighborhood $(t-\varepsilon_t,\ t+\varepsilon_t)$ such that one of $|Q\cap(t-\varepsilon_t,\ t)|$ and $|Q\cap(t,\ t+\varepsilon_t)|$ is zero, then the measure of Q is zero, since Q is covered by at most a countable number of such intervals.

Putting $x_0(t)=\chi_{Q\cap[0,t_0]}(t)$, we have for $x\in D(A)$,

$$\begin{aligned}
&\|a^{-1}(t)x_0(t)-x(t)\|_{L_\infty}\\
&\geqq \|a^{-1}(t)x_0(t)-x(t)\|_{L_\infty(Q)}\\
&=\|a^{-1}(t)\{x_0(t)-a(t)x(t)\}\|_{L_\infty(Q)}\\
&\geqq \delta^{-1}\|x_0(t)-a(t)x(t)\|_{L_\infty(Q)}\,.
\end{aligned}$$

Since $ax\in \boldsymbol{C}$, the right hand side of the inequality is not less than $\delta^{-1}/2$, which shows that $D(A)$ is not dense in $\boldsymbol{L}_\infty$.
Case 3). Suppose $a\notin \boldsymbol{L}_{p'}$. Then $\sum_{n=0}^{\infty} n^{p'}|C_n|=+\infty$, where $C_n=\{t;\ n\leqq |a(t)|<n+1\}$.
For $x\in D(A)$, we have $x_0\equiv\inf_t x(t)=0$, since $x_0\sum n^{p'}|C_n|\leqq \int |a(t)x(t)|^{p'}dt<+\infty$.
Therefore every x in $D(A)$ has zeros, which implies that $D(A)$ is not dense in $\boldsymbol{C}$.
Case 6). We see as above that if $a\notin \boldsymbol{L}_{p'}$, the essential infimum of $x(t)$ is equal to zero for any $x\in D(A)$.
Case 8, 9). (*Sufficiency*) Let a be a Lebesgue-measurable function. We put $B_n=\{t;\ |a(t)|\leqq n\}$. The domain $D(A)$ includes a set $\{x\chi_{B_n};\ x\in \boldsymbol{L}_\infty,\ n\geqq 1\}$, which

is dense in $\boldsymbol{L}_p$.
(*Necessity*) The domain $D(A)$ being dense in $\boldsymbol{L}_p$, we can extract a sequence $\{x_n\}$ from $D(A)$ converging to 1. A subsequence $\{x_{n_j}\}$ converges almost everywhere to 1. The function $a(t)=\lim a(t)x_{n_j}(t)$ is Lebesgue-measurable, since ax_{n_j} is so.
Case 7). Let G be the union of open sets O such that $a^{-1}|_O \in \boldsymbol{L}_p(O)$. We see that $|G\setminus S|=0$. In fact, if not, $|O\setminus S|>0$ for some open set O such that $a^{-1}|_O\in \boldsymbol{L}_p(O)$, which is absurd.

LEMMA 1. *A continuous function y belongs to the range $R(A)$ of A, if the support S_y of y is contained in G.*

Proof. For every t in S_y, there is an open neighborhood U_t of t such that $a^{-1}|_{U_t}\in \boldsymbol{L}_p(U_t)$. The neighborhood U_t is contained in G. The compact set S_y is covered by a finite number of the open sets $U_{t_1}, U_{t_2}, \cdots, U_{t_n}$. Let $U=\bigcup_{j=1}^{n} U_{t_j}$. The function $a^{-1}|_U$ belongs to $\boldsymbol{L}_p(U)$. We define a function x as follows:

$$x(t)=a^{-1}(t)y(t) \quad \text{for } t\in U\,,$$

$$x(t)=0 \qquad\qquad \text{for } t\notin U\,.$$

Vanishing on U^c, the function y is equal to ax. The function $|x|$ is majorized by $\|y\|_C|a^{-1}|$ on U and vanishes on U^c, therefore it belongs to $\boldsymbol{L}_p$.

LEMMA 2. *There is a y in $R(A)$ such that $y(t_0)\neq 0$, if and only if a^{-1} belongs locally to $\boldsymbol{L}_p$ at t_0.*

Proof. Let a^{-1} belong locally to $\boldsymbol{L}_p$ at t_0. For some neighborhood U of t_0, $a^{-1}|_U$ belongs to $\boldsymbol{L}_p(U)$. Take a continuous function y with the support in U such that $y(t_0)=1$. The function x defined as in proof of Lemma 1 belongs to $\boldsymbol{L}_p$. Therefore $y=ax$ belongs to $R(A)$.

Conversely, let $y=ax$ be a function in $R(A)$ such that $y(t_0)\neq 0$. We may assume that $y(t_0)=1$. A set $U=\{t;\ y(t)>1/2\}$ is a neighborhood of t_0. If $t\in U$, then $a(t)\neq 0$, therefore $|x(t)|=|a^{-1}(t)|y(t)>1/2|a^{-1}(t)|$. Since $x\in \boldsymbol{L}_p$, $a^{-1}|_U\in \boldsymbol{L}_p(U)$.

We now prove the sufficiency of the condition in Case 7). If $a\in \Gamma_p$, then $|S\setminus G|=0$, therefore $|S\ominus G|=0$. Without loss of generality, we may assume that $S=G$. It is sufficient to prove that a characteristic function of any measurable set is approximated in $\boldsymbol{L}_p$ by elements of $D(A)$.

Let E be a measurable set. Since $\chi_{S^c\cap E}\in D(A)$ and $\chi_E=\chi_{S\cap E}+\chi_{S^c\cap E}$, we may assume $E\subset S$. Fix an arbitrary positive number ε. There are a positive constant c and a measurable set $E_1\subset E$ such that $|E\setminus E_1|<(\varepsilon/3)^p$ and such that $|a(t)|\geqq c$ on E_1. Let $\delta=\min\{(\varepsilon/3)^p, c\varepsilon/(2^{(1/p)}\cdot 3)\}$. By Egoroff's theorem, there are a compact set $F\subset E_1$ and a continuous function y such that $|E_1\setminus F|<\delta$ and such that $|a(t)-y(t)|<\delta$ on F. Put $k=\sup_{t\in F} y(t)$. There is an open set O with $F\subset \bar{O}\subset G$ such that

$$\int_{O\setminus F}|a^{-1}(t)|^p dt<\frac{\varepsilon^p}{2\cdot 3^p k^p}\ .$$

We find a continuous functions y_1 satisfying relations:

$$S_{y_1}\subset O\ ,$$

$$y_1=y \text{ on } F\ ,$$

$$\|y_1\|_{C}\leqq k\ .$$

By Lemma 1, $y_1\in R(A)$. Hence for some $x\in D(A)$, $y=ax$. We may assume $x(t)=0$ on O^c. We have:

$$\|\chi_E-\chi_{E_1}\|_{L_p}\leqq|E\setminus E_1|^{1/p}<\frac{\varepsilon}{3}\ ,$$

$$\|\chi_{E_1}-\chi_F\|_{L_p}\leqq|E_1\setminus F|^{1/p}<\delta^{1/p}<\frac{\varepsilon}{3}\ ,$$

$$\|\chi_F-\chi\|_{L_p}=\left\{\int_F|1-a^{-1}y_1|^p dt+\int_{O\setminus F}|a^{-1}y_1|^p\,dt\right\}^{1/p}$$

$$\leqq\left\{(c^{-1}\delta)^p+k^p\int_{O\setminus F}|a^{-1}(t)|^p\,dt\right\}^{1/p}$$

$$\leqq\left(\frac{\varepsilon^p}{2\cdot 3^p}+k^p\frac{\varepsilon^p}{2\cdot 3^p\cdot k^p}\right)^{1/p}=\frac{\varepsilon}{3}\ .$$

Therefore $\|\chi_E-x\|_{L_p}<\varepsilon$.
(*Necessity*) Suppose that $|S\setminus G|>0$. Take any $x\in D(A)$. By Lemma 2, $y(t_0)=a(t_0)x(t_0)=0$ for any $t_0\in S\setminus G$. Since $a(t_0)\neq 0$, $x(t_0)=0$. Thus x vanishes on $S\setminus G$. Therefore $\chi_{S\setminus G}$ is not in the closure of $D(A)$.
(II) We prove the last statement. In Cases 1—6), A is everywhere defined and continuous on $\boldsymbol{X}$. Therefore A^* is also everywhere defined. The condition (α) is satisfied. We see that $A^*y^*=ay^*$ for $y^*\in D(A^*)=Y^*$.
Cases 7, 8). Let K be a set $\{y(t);\,0\leqq y\in \boldsymbol{L}_1,\,ay\in \boldsymbol{L}_p{}^*\}$ in $\boldsymbol{Y}^*$. As is easily seen, K is included in $D(A^*)$.

In Case 7), a non-negative measure in $\boldsymbol{C}^*$ is approximated in the w^*-topology by elements of K, which implies validity of the condition (α).

In Case 8), the w^*-closure of $L_+=\{ydt\colon 0\leqq y\in \boldsymbol{L}_1\}$ is equal to a set $\{y^*\in \boldsymbol{L}_\infty{}^*,\ y^*\geqq 0\}$ by Lemma 1 of [6]. Every element of L_+ is w^*-approximated by elements of K, since K is strongly dense in L_+ as is seen in Case 9) of (I).
Case 9). We have $D(A^*)=\{y^*\in \boldsymbol{L}_{p'}{}^*:\,ay^*\in \boldsymbol{L}_p{}^*\}$ and $A^*y^*=ay^*$ for $y^*\in D(A^*)$. As is seen in (I), $D(A^*)$ is strongly dense in $\boldsymbol{L}_{p'}{}^*$. For any non-negative $y^*\in \boldsymbol{L}_{p'}{}^*$, there is a sequence $y_n{}^*$ in $D(A^*)$ strongly converging to y^*. Let $\bar{y}_j{}^*=y_{n_j}{}^*\vee 0$ for a subsequence $y_{n_j}{}^*$ converging to y^* a.e.. Then $0\leqq\bar{y}_j{}^*\in D(A^*)$ and $\bar{y}_j{}^*\to y^*$ strongly, therefore in the w^*-topology.

REMARK. *In each case, a necessary and sufficient condition for A to be defined everywhere on X is that the function $a(t)$ belongs to a set listed in the table, where* $+\infty>p_1>p_2\geqq 1$ and $p_0^{-1}+p_1^{-1}=p_2^{-1}$.

X \ Y	C	L_∞	L_{p_1}	L_{p_2}
C	1) C	2) L_∞	3) L_{p_1}	L_{p_2}
L_∞	4) $\{0\}$	5) L_∞	6) L_{p_1}	L_{p_2}
L_{p_1}	7) $\{0\}$	8) $\{0\}$	9) L_∞	10) L_{p_0}
L_{p_2}	$\{0\}$	$\{0\}$	11) $\{0\}$	L_∞

Proof. The sufficiency of the conditions is obvious. In Cases 1—6), necessity follows from Proposition 2.

We prove necessity in Cases 7—11).

Case 7). The result follows immediately from that in Case 4).

Case 10). Suppose a $\notin L_{p_0}$, i.e., $|a|^{p_2}\notin L_{p_0p_2^{-1}}$. Since $L_{p_0p_2^{-1}}$ is the dual space of $L_{p_1p_2^{-1}}$, there exists a function $x_0\in L_{p_1p_2^{-1}}$ such that $\int|a(t)|^{p_2}|x_0(t)|dt=+\infty$. Putting $x_1=|x_0|^{1/p_2}$, we have $x_1\in L_{p_1}$ and $ax\notin L_{p_2}$. The operator A is not defined for $x_1\in L_{p_1}$.

Case 9). The result follows in a way analogous to that of Case 10).

Case 11). Suppose that $a(t)$ does not vanish almost everywhere. Then, for some $\delta>0$, a set $P=\{t;\ \delta\leqq|a(t)|\}$ has the positive Lebesgue measure. There is a countable number of mutually disjoint sets $\{T_n\}$ with the positive Lebesgue measures in P. That is,

$$P\supset\sum_{n=1}^{\infty}T_n,\ |T_n|>0\ .$$

Given $\gamma>(1-p_2/p_1)^{-1}$, there is a subsequence $\{n_l\}$ such that $|T_{n_l}|\leqq l^{-\gamma}$. Putting $x(t)=\sum_{l=1}^{\infty}|T_{n_l}|^{(-1/p_1)}\chi_{T_{n_l}}(t)$, we have $x\in L_{p_2}$ and $ax\notin L_{p_1}$.

Case 8). The result follows from that in Case 11).

THEOREM. *In each case, in order that $D(A)$ is dense, that $D(A^*)$ satisfies the condition (α) and that Z_{A^*} is $\sigma(X^*, D(A))$-closed, it is necessary and sufficient that the function a satisfies the conditions listed in the table, where* $+\infty>p_1>p_2\geqq 1$ *and* $p_0^{-1}+p_1^{-1}=p_2^{-1}$.

X \ Y	C	L_∞	L_{p_1}	L_{p_2}
C	1) $a, a^{(-1)} \in C$	2) $a, a^{(-1)} \in L_\infty$	$a=0$ *a.e.*	3) $a=0$ *a.e.*
L_∞	4) $a=0$ *a.e.*	5) $a, a^{(-1)} \in L_\infty$	$a=0$ *a.e.*	6) $a=0$ *a.e.*
L_{p_1}	$a \in \Gamma_{p_1}$, $a^{(-1)} \in L_{p_1}$	$a^{(-1)} \in L_{p_1}$	$a^{(-1)} \in L_\infty$	7) $a=0$ *a.e.*
L_{p_2}	8) *) $a \in \Gamma_{p_2}$ $a^{(-1)} \in L_{p_2}$	9) $a^{(-1)} \in L_{p_2}$	10) $a^{(-1)} \in L_{p_0}$	11) $a^{(-1)} \in L_\infty$

Proof. Case 5) (*Necessity*) As was seen before $D(A^*)=L_\infty^*$ and $A^*y^*=ay^*$ for $y^* \in L_\infty^*$. Suppose $a^{(-1)} \notin L_\infty$. Since L_∞ is the dual space of L_1, there is a function $x^* \in L_1$ with $a^{(-1)}(t)x^*(t) \geqq 0$ such that $\int a^{(-1)}(t)x^*(t)dt=+\infty$. As a positive measure $y_n^*(t)dt=\chi_{\{a^{-1}x^* \leqq n\}}(t)a^{(-1)}(t)x^*(t)dt$ belongs to $L_\infty^*=D(A^*)$, a measure $a(t)y_n^*(t)dt$ belongs to Z_{A^*}. We see that $a(t)y_n^*(t)dt \to x^*(t)dt$ in the w^*-topology. But $x^*(t)dt$ does not belong to Z_{A^*}, since $a^{(-1)}(t)x^*(t)dt$ does not belong to L_∞^*. Therefore Z_{A^*} is not w^*-closed.
(*Sufficiency*) Let $x_\lambda^*=A^*y_\lambda^*$ in Z_{A^*} converge to some $x^* \in L_\infty^*$ in $\sigma(X^*, D(A))$-topology. For any $x \in D(A)=L_\infty$, we have

$$\int a(t)x(t)y_\lambda^*(t)dt \to \int x(t)x^*(t)dt\,.$$

Take an $\bar{x} \geqq 0$ from L_∞, then a function $x=a^{(-1)}\bar{x}$ which belongs to L_∞, makes the left-hand side non-negative. We see that $a^{(-1)}(t)x^*(t)dt \geqq 0$, since $\int \bar{x}(t)a^{(-1)}(t)$ $x^*(t)dt \geqq 0$. Therefore $x^*=A^*(a^{(-1)}x^*)$ belongs to Z_{A^*}.
Case 2). Necessity is proved in the same way as above.
(*Sufficiency*)**) The set Z_{A^*} is convex, so that by the Krein-Shmulyan theorem (cf. [3]) it is sufficient to prove that $(Z_{A^*})_N=\{x^* \in Z_{A^*};\ \|x^*\| \leqq N\}$ is w^*-closed in $\{x^*;\ \|x^*\| \leqq N\}$ for any N.

For a moment we assume following

LEMMA 3. *For any $x^* \in Z_{A^*}$, we can find a non-negative $y^* \in L_\infty^*$ with $x^*=A^*y^*$ such that $\|y^*\|\ \|a^{(-1)}\|_{L_\infty}\|x^*\|$.*

Let a directed set $\{x_\lambda^*\}$ in $(Z_{A^*})_N$ w^*-converge to x_0^* with $\|x_0^*\| \leqq N$. By Lemma 3, we find $y_\lambda^* \geqq 0$ with $x_\lambda^*=A^*y_\lambda^*$ such that $\|y_\lambda^*\| \leqq \|a^{(-1)}\|_{L_\infty}\|x_\lambda^*\| \leqq \|a^{(-1)}\|_{L_\infty} \cdot N$. Being bounded in norm, the set $\{y_\lambda^*\}$ is sequentially w^*-compact. A sequence $\{y_n^*\}$ w^*-converges to some y_0^* in L_∞^*. By the resonance theorem, $\|y_0^*\| \leqq \|a^{(-1)}\|_{L_\infty} \cdot N$. The operator A^* is a continuous mapping from (L_∞^*, w^*)

*) The result is due to F. Niiro.
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into $(\boldsymbol{C}^*, w^*)$, so that $x_0^*=A^*y_0^*$. Hence $x_0^*\in(Z_{A^*})_N$.

Proof of Lemma 3. Let $x^*=A^*y^*$ with $y^*\geqq 0$. The signed measure x^* is decomposed into the positive part x_+^* and the negative one x_-^* (Jordan's decomposition). Let $a_\pm=(\pm a)\vee 0$, then $a=a^+-a^-$. We denote by $A_\pm$ operators defined by multiplication by $a_\pm$. Since $A^*=A_+^*-A_-^*$, $x^*=A_+^*y^*-A_-^*y^*$. The operators $A_\pm^*$ being non-negative, the measures $A_\pm^*y^*$ are also non-negative. Minimality of Jordan's decomposition implies that $0\leqq x_\pm^*\leqq A_\pm^*y^*$. For any $0\leqq x\in\boldsymbol{C}$, $0\leqq\int x\,dx_\pm^*\leqq\int x\,a_\pm dy^*$.

In view of Theorem 2 of Ando [1], there are extensions $\bar{x}_\pm^*$ of $x_\pm^*$ to elements of $\boldsymbol{L}_\infty^*$ such that $0\leqq\bar{x}_\pm^*\leqq a_\pm dy^*$ Multiplying each side by $a_\pm^{(-1)}$, we get $0\leqq a_\pm^{(-1)}x_\pm^*\leqq dy^*$. Let $\bar{y}^*=\bar{y}_+^*+\bar{y}_-^*$, where $\bar{y}_\pm^*=a_\pm^{(-1)}x_\pm^*$. We see that

$$A^*\bar{y}^*=A^*\bar{y}_+^*+A^*\bar{y}_-^*=x_+^*-x_-^*=x^* .$$

Hence $\|\bar{y}^*\|=\bar{y}^*(1)=\bar{y}_+^*(1)+\bar{y}_-^*(1)\leqq\|a^{(-1)}\|_{L_\infty}\{x_+^*(1)+(x_-^*(1)\}=\|a^{(-1)}\|_{L_\infty}\|x^*\|$.

Case 4). The result follows immediately from proposition 2.

Case 9). (*Necessity*) Suppose $a^{(-1)}\notin\boldsymbol{L}_{p_2}$ Then for some $x_0\in\boldsymbol{L}_{q_2}$, $\int|a^{(-1)}(t)x_0(t)|dt=+\infty$, where $p_2^{-1}+q_2^{-1}=1$. We can take x_0 such as $a^{(-1)}(t)x_0(t)\geqq 0$. If $x_0\in Z_{A^*}$, there is a non-negative $y_0^*\in D(A^*)$ such that

$$\int x_0(t)x(t)dt=\int a(t)x(t)y_0^*(t)dt .$$

for any $x\in D(A)$. For $x_n(t)=\chi_{\{|a^{-1}|\leqq n\}}(t)a^{(-1)}(t)$ which belongs to $D(A)$, the left-hand side of the above equality diverges to infinity, while the right-hand side is bounded by $\|y^*\|_{L_1}$ as $n\to+\infty$. Thus $x_0\notin Z_{A^*}$. We have

$$D(A^*)\supset K'\equiv\{y^*(t)dt;\ y^*\in\boldsymbol{L}_1,\ ay^*\in\boldsymbol{L}_{q_2}\} ,$$

$$A^*y^*dt=ay^*dt \text{ for } y^*(t)dt\in K' .$$

As is easily seen, $y_n^*(t)dt\equiv\chi_{\{a^{(-1)}x_0\leqq n\}}(t)a^{-1}(t)dt\in K'$ and $A^*y_n^*dt=ay_n^*dt\to x_0(t)dt$ in the w^*-topology. Hence Z_A^* is not $\sigma(\boldsymbol{X}^*, D(A))$-closed.

(*Sufficiency*) Let $x_\lambda^*dt=A^*dy_\lambda^*\in Z_{A^*}$ $\sigma(\boldsymbol{X}^*, D(A))$-converge to some x^*, i.e.,

$$\int axdy_\lambda^*\to\int x(t)x^*(t)dt=\int a(t)x(t)a^{(-1)}(t)x^*(t)dt \quad \text{for any } x\in D(A).$$

Since the left-hand side is non-negative for $x=a^{(-1)}y$ with $0\leqq y\in\boldsymbol{L}_\infty$, so is the right-hand side, i.e., $\int y(t)a^{(-1)}(t)x^*(t)dt\geqq 0$. Thus $a^{(-1)}x^*dt\geqq 0$ and $x^*dt=A^*(a^{(-1)}x^*dt)\in Z_{A^*}$.

Case 11). (*Necessity*) Let $p_2>1$. Suppose $a^{(-1)}\notin\boldsymbol{L}_\infty$, i.e., $|a^{(-1)}|^{q_2}\notin\boldsymbol{L}_\infty$. There is a function $|x^*|^{q_2}\in\boldsymbol{L}_1$ with $a^{(-1)}x^*\geqq 0$ such that $\int|a^{(-1)}(t)|^{q_2}|x^*(t)|^{q_2}dt=+\infty$.

Put $y_n^*=\chi_{\{a^{(-1)}x^*\leqq n\}}a^{(-1)}(t)x^*(t)$, then $A^*y_n^*$ converges in the $\sigma(X^*, D(A))$-topology to x^*, which is not in Z_{A^*} as in Case 5).

In case where $p_2=1$, replacing x^* by $\mathrm{sign}(a^{(-1)}(t))$, we can apply the same argument.

Sufficiency is proved in the same way as in Case 9).

Case 10). (*Necessity*) We see that $D(A^*)=\{y^*\in L_{q_1};\ ay^*\in L_{q_2}\}$ with $p_j^{-1}+q_j^{-1}=1(j=1,2)$ and that $A^*y^*=ay^*$ for $y^*\in D(A^*)$. If $p_2=1$, then $q_2=+\infty$. Suppose $a^{(-1)}\notin L_{p_0}$, i.e., $|a^{(-1)}|^{q_1}\notin L_{p_0q_1^{-1}}$. Noting that $L^*_{p_0q_1^{-1}}=L_{q_2q_1^{-1}}$, we see that for some $|x_0^*|^{q_1}\in L_{q_2q_1^{-1}}$ (i.e., $x_0^*\in L_{q_2}$), $\int |a^{(-1)}|^{q_1}|x_0^*|^{q_1}dt=+\infty$, i.e., $a^{(-1)}x_0^*\notin L_{q_1}$. We can take x_0^* such as $a^{(-1)}x_0^*\geqq 0$. Put $y_n^*=\chi_{\{a^{-1}x_0^*\leqq n\}}(t)a^{(-1)}(t)x_0^*(t)$, then $0\leqq y_n^*\in D(A^*)$. A sequence of functions $x_n^*=A^*y_n^*=\chi_{\{a^{-1}x_0^*\leqq n\}}\ x^*(t)$ which belong to Z_A^*, converges in the $\sigma(X^*, D(A))w^*$-topology to x_0^*. Since $a^{(-1)}x_0^*\notin L_{q_1}$, x_0^* is not in Z_A^*.

Sufficiency is easily proved.

Case 6). As the operator A is everywhere defined and bounded, so is the operator A^*. Suppose that a does not vanish almost everywhere. A set $P=\{t;\ \delta\leqq|a(t)|\}$ has the positive measure for some $\delta>0$. We choose T_n as in proof of Remark to Proposition 2. For a fixed $\gamma>p_2$, we take a subsequence $\{n_l\}$ such that $|T_{n_l}|\leqq l^{-\gamma}$. Putting

$$x_N^*(t)=(\mathrm{sign}\ a(t))\sum_{l=1}^{N}|T_{n_l}|^{-\beta}\chi_{T_{n_l}}(t)\in L_\infty^*\ ,$$

$$x^*(t)=(\mathrm{sign}\ a(t))\sum_{l=1}^{\infty}|T_{n_l}|^{-\beta}\chi_{T_{n_l}}(t)\in L_1\subset L_\infty^*\ ,$$

where $1-1/\gamma>\beta>1-1/p_2$, we have

$$x_N^*=A^*(a^{(-1)}x_N^*)\in Z_A^*\ ,$$

$$w^*\text{-}\lim_{N\to+\infty}x_N^*=x^*\ .$$

On the other hand $x^*\notin Z_A^*$, since $a^{(-1)}x^*\notin L_{q_2}=L_{p_2}^*$.

Case 3). The argument goes in the same way as above.

Case 7). We can apply the same argument as in Case 6) by taking

$$\gamma>(1-q_1q_2^{-1})^{-1}\ ,$$

$$q_1^{-1}(1-1/\gamma)>\beta>q_2^{-1}\ ,$$

$$x_N^*=(\mathrm{sign}\ a(t))\sum_{l=1}^{N}|T_{n_l}|^{-\beta}\chi_{T_{n_l}}(t)\ ,$$

$$x^*=(\mathrm{sign}\ a(t))\sum_{l=1}^{\infty}|T_{n_l}|^{-\beta}\chi_{T_{n_l}}(t)\ ,$$

where $q_1^{-1}+p_1^{-1}=1$ and $q_2^{-1}+p_2^{-1}=1$.
Case 8) We first prove

LEMMA 4. *We assume that $D(A)$ is dense in L_p with $p\geqq 1$. Then $Z_{A^*}=\{x^*dt;$ $x^*\in L_q=L_p^*,\ 0\leqq a^{(-1)}x^*\in L_1,\ x^*=0$ on $S^c\}$.*

Proof. It is evident that the right-hand side is contained in Z_{A^*}.
Let $x^*dt=A^*dy^*$ for some $dy^*\in D(A^*)$. For any $x\in D(A)$,

$$\int xx^*dt=\int axdy^* .$$

Put $x_0(t)=\chi_{S^c}(t)\operatorname{sign}(x^*(t))$. Then $ax_0=0$, hence $x_0\in D(A)$, and $\int_{S^c}|x^*|dt=\int x_0x^*dt=0$. Therefore $x^*=0$ on S^c.

Choose an open set O such that $O\subset\bar{O}\subset G$, where G is the set defined in proof of Proposition 2. Let $dy_1^*=a^{(-1)}x^*\chi_O dt$ and $dy_2^*=\chi_O dy^*$, the former is a well-defined measure, because $a^{(-1)}\in L_p(\bar{O})$ by Proposition 2. If the support S_y of a continuous function y is contained in O, then $y\in R(A)$ by Lemma 1. For such y, $y=\chi_O y$ and $a^{(-1)}y\in D(A)$, so that

$$\int a^{(-1)}yx^*dt=\int ydy^* ,$$

therefore

$$\int ydy_1^*=\int ydy_2^* .$$

As $y_1^*(O^c)=y_2^*(O^c)=0$, $dy_1^*=dy_2^*$. For $z=\operatorname{sign}(a^{(-1)}x^*)$,

$$\int_O|a^{(-1)}x^*|dt=\int_O zdy_1^*=\int_O zdy_2^*\leqq\int_O|dy_2^*|<+\infty .$$

Letting $O\uparrow G=S$, we have $a^{(-1)}x^*\in L_1$. Non-negativity of $a^{(-1)}x^*$ is evident. Thus Lemma 4 is proved.

We now prove necessity of the conditions. The first condition is evidently necessary by Proposition 2. Suppose that $a^{(-1)}\notin L_{p_2}$. There is an $x^*\in L_{q_2}=L_{p_2}^*$ such that $a^{(-1)}x^*\notin L_1$. We may choose x^* so that $a^{(-1)}x^*\geqq 0$. Take a sequence of compact sets $\{F_n\}$ with $\lim_{n\to\infty}F_n=S$. Since $a\in\Gamma_{p_2}$, $a^{(-1)}\chi_{F_n}\in L_{p_2}(F_n)$, so that $a^{(-1)}\chi_{F_n}x^*\in L_1$. Therefore $\chi_{F_n}x^*\in Z_{A^*}$ by Lemma 4. As is easily seen, $\chi_{F_n}x^*\to\chi_S x^*$ in the $\sigma(X^*, D(A))$-topology. But $\chi_S x^*\notin Z_{A^*}$, since $a^{(-1)}\chi_S x^*\notin L_1$.
(*Sufficiency*) Let $x_\lambda^*dt=A^*dy_\lambda^*\in Z_A^*$ converge to $x^*\in L_{p_2}^*=L_{q_2}$ in the $\sigma(X^*, D(A))$-topology. Namely, for any $x\in D(A)$

$$\int axdy_\lambda^*\to\int x^*xdt .$$

As in proof of Lemma 4, we see that $x^*=0$ on S^c. Since $a^{(-1)}\in \boldsymbol{L}_{p_2}$, $a^{(-1)}x^*\in \boldsymbol{L}_1$. For any $0\leqq y\in \boldsymbol{C}$ with its support $S_y\subset S$, which is in $R(A)$ by Lemma 1, $\int ydy_\lambda^*\geqq 0$. Therefore $\int ya^{(-1)}x^*dt=\int x^*xdt\geqq 0$, where $y=Ax=ax$. Thus $a^{(-1)}x^*\geqq 0$. By Lemma 4, we see that $x^*\in Z_{A^*}$.

Let f be a real valued function defined on $\boldsymbol{L}_\infty=\boldsymbol{L}_\infty[0.\ 1]$. We assume that f is Fréchet-differentiable, i.e., there exists a limit $\delta f(x_0;\ x)=\lim_{\varepsilon\to 0}(f(x_0+\varepsilon x)-f(x))/\varepsilon$ for any x_0 and $x\in \boldsymbol{L}_\infty$ and that for some constant $C(x_0)$, $|\delta f(x_0;\ x)|\leqq C(x_0)\|x\|_{L_1}$ for any $x\in \boldsymbol{L}_\infty$, i.e., $\delta f(x_0;\ \cdot\ )$ is extended to an element $\nabla f(x_0)$ of $\boldsymbol{L}_1^*$. A mapping $g(x)=1-x^2$ from $\boldsymbol{L}_\infty$ into itself has the continuous Fréchet differential $\delta g(x_0;x)=-2x_0x$. We put $\Phi(x,\ \lambda^*)=f(x)+\langle g(x),\ \lambda^*\rangle$ for $x\in \boldsymbol{L}_\infty$ and $\lambda^*\in \boldsymbol{L}_\infty^*$.

PROPOSITION 3. *If the maximum of $f(x)$ subject to the constraint $g(x)\geqq 0$ is attained at a, then there exists a non-negative $\lambda_0^*\in \boldsymbol{L}_\infty^*$ satisfying relations*:

$$\delta_x\Phi((a;\ x),\ \lambda_0^*)=\delta f(a;\ x)+\langle\delta g(a;\ x),\ \lambda_0^*\rangle=0 \quad \text{for any } x\in \boldsymbol{L}_\infty\ ,$$

$$\langle g(a),\ \lambda_0^*\rangle=0\ .$$

If $a^{(-1)}\in \boldsymbol{L}_p$, we can choose λ_0^* from $\boldsymbol{L}_p$, where $1<p\leqq+\infty$.

Proof. Let $\boldsymbol{X}=\boldsymbol{L}_\infty\times \boldsymbol{R}$ with the natural order relation. We consider an operator A on $\boldsymbol{X}$ into itself which is defined by:

$$A\boldsymbol{x}=\begin{pmatrix}-2a & 1-a^2\\ 0 & 1\end{pmatrix}\boldsymbol{x}=\begin{pmatrix}-2ax+(1-a^2)\xi\\ \xi\end{pmatrix} \text{ for } \boldsymbol{x}=\begin{pmatrix}x\\ \xi\end{pmatrix}\in \boldsymbol{X}\ .$$

A functional $\boldsymbol{x}_0^*\in \boldsymbol{X}^*$ defined by $\boldsymbol{x}_0^*(\boldsymbol{x})=-\delta f(a;\ x)$ for $\boldsymbol{x}=\begin{pmatrix}x\\ \xi\end{pmatrix}\in \boldsymbol{X}$, belongs to V_A. To see this, we take any $\boldsymbol{x}\in \boldsymbol{X}$ such that $A\boldsymbol{x}\geqq 0$, i.e., $-2ax+(1-a^2)\xi\geqq 0$ and $\xi\geqq 0$. By Lemma 5, $1-(a+\varepsilon x)^2\geqq 0$ for $\varepsilon\in[0,\ 1/(1+\xi+\|x\|_{L_\infty})]$. Since $f(a)$ is the maximum of $f(x)$ under the constraint $g(x)\geqq 0$, $f(a+\varepsilon x)\leqq f(a)$ for small positive ε. Thus $\boldsymbol{x}_0^*(\boldsymbol{x})=-\delta f(a;\ x)\geqq 0$.

A set $U=\{(\boldsymbol{y},\zeta)\in \boldsymbol{X}\times \boldsymbol{R};\ (\boldsymbol{y},\zeta)\leqq(A\boldsymbol{x},\ -\boldsymbol{x}_0^*(\boldsymbol{x}))$ for some $\boldsymbol{x}\in \boldsymbol{X}\}$ contains all non-positive elements in $\boldsymbol{X}\times \boldsymbol{R}$, hence U has interior points. Since $\boldsymbol{x}_0^*\in V_A$, $(\boldsymbol{0},\zeta)\notin U$ for any $\zeta>0$. Therefore the zero $(\boldsymbol{0},0)$ of $\boldsymbol{X}\times \boldsymbol{R}$ is on the boundary of U. By the Hahn-Banach theorem, there exists a non-null functional $(\boldsymbol{y}_0^*,\ \zeta_0^*)\in \boldsymbol{X}^*\times \boldsymbol{R}^*$ such that

$$\langle \boldsymbol{y},\ \boldsymbol{y}_0^*\rangle+\langle\zeta,\ \zeta_0^*\rangle\leqq 0\ ,$$

for any $(\boldsymbol{y},\zeta)\in U$. Since U contains all non-positive elements, $(\boldsymbol{y}_0^*,\zeta_0^*)\geqq 0$.

We shall show that $\zeta_0^*>0$. In fact, if $\zeta_0^*=0$, we have $\langle A\boldsymbol{x},\ \boldsymbol{y}_0^*\rangle\leqq 0$ for any $\boldsymbol{x}\in \boldsymbol{X}$. Replacing $\boldsymbol{x}$ by $-\boldsymbol{x}$, we have $\langle A\boldsymbol{x},\ \boldsymbol{y}_0^*\rangle=0$ for $\boldsymbol{x}\in \boldsymbol{X}$, i.e.,

$$\langle -2ax,\ y_0^*\rangle+\langle 1-a^2,\ y_0^*\rangle\xi+\eta_0^*\xi=0\ ,$$

where $\boldsymbol{y}_0^*=\begin{pmatrix} y_0^* \\ \eta_0^* \end{pmatrix}$ and $\boldsymbol{x}=\begin{pmatrix} x \\ \xi \end{pmatrix}$. The last equality shows that

$$\langle ax, y_0^*\rangle=\langle 1-a^2, y_0^*\rangle=0 .$$

Putting $x=a\in \boldsymbol{L}_\infty$, we obtain $\langle 1, y_0^*\rangle=0$, which is absurd, since y_0^* is a non-negative, non-null functional. Thus $\zeta_0^*>0$.

Since $(A\boldsymbol{x}, -\boldsymbol{x}_0^*(\boldsymbol{x}))\in U$, $\langle A\boldsymbol{x}, \boldsymbol{y}_0^*\rangle+\langle -\boldsymbol{x}_0^*(\boldsymbol{x}), \zeta_0^*\rangle\leqq 0$ for any $\boldsymbol{x}\in \boldsymbol{X}$. Replacing $\boldsymbol{x}$ by $-\boldsymbol{x}$, we have

$$\langle A\boldsymbol{x}, \boldsymbol{y}_0^*\rangle+\langle -\boldsymbol{x}_0^*(\boldsymbol{x}), \zeta_0^*\rangle=0 .$$

Therefore

$$\langle A\boldsymbol{x}, \boldsymbol{\lambda}_0^*\rangle-\boldsymbol{x}_0^*(\boldsymbol{x})=0 ,$$

where $\boldsymbol{\lambda}_0^*=\begin{pmatrix} \lambda_0^* \\ l_0^* \end{pmatrix}=(\zeta_0^*)^{-1}\boldsymbol{y}_0^*\geqq 0$. That is,

$$\langle -2ax, \lambda_0^*\rangle+\langle 1-a^2, \lambda_0^*\rangle+\xi l_0^*=-\delta f(a; x) ,$$

for any $x\in \boldsymbol{L}_\infty$ and $\xi\in \boldsymbol{R}$. Hence

$$\delta_x\Phi((a; x), \lambda_0^*)=\delta f(a; x)+\langle \delta g(a; x), \lambda_0^*\rangle=0 \quad \text{for any } x\in \boldsymbol{L}_\infty ,$$

and

$$\langle g(a), \lambda_0^*\rangle=0 .$$

We now prove the second statement in case where $1<p<+\infty$. Let $\boldsymbol{X}=\boldsymbol{L}_1\times \boldsymbol{R}$ and let $\boldsymbol{Y}=\boldsymbol{L}_p^*\times \boldsymbol{R}$. We consider an operator A from $\boldsymbol{X}$ into $\boldsymbol{Y}$, which is defined as above on $D(A)=\boldsymbol{L}_\infty\times \boldsymbol{R}\subset \boldsymbol{X}$. The domain $D(A)$ is dense in $\boldsymbol{X}$.

For $\boldsymbol{y}^*=\begin{pmatrix} y^* \\ \eta^* \end{pmatrix}\in D(A^*)$ and $\boldsymbol{x}=\begin{pmatrix} x \\ \xi \end{pmatrix}\in D(A)$, we have

$$\begin{aligned}\langle \boldsymbol{x}, A^*\boldsymbol{y}^*\rangle=\langle A\boldsymbol{x}, \boldsymbol{y}^*\rangle&=\left\langle \begin{pmatrix} -2ax+(1-a^2)\xi \\ \xi \end{pmatrix}, \begin{pmatrix} y^* \\ \eta^* \end{pmatrix}\right\rangle \\ &=\left\langle \begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} -2ay^* \\ \langle 1-a^2, y^*\rangle+\eta^* \end{pmatrix}\right\rangle \\ &=\left\langle \boldsymbol{x}, \begin{pmatrix} -2ay^* \\ \langle 1-a^2, y^*\rangle+\eta^* \end{pmatrix}\right\rangle .\end{aligned}$$

We see that $D(A^*)\supset K\times \boldsymbol{R}$, where $K=\{y^*\in \boldsymbol{L}_p;\ ay^*\in \boldsymbol{L}_\infty\}$. The set K is a strongly dense sublattice of $\boldsymbol{L}_p$ as is seen in Case 8) of Proposition 2. Therefore any non-negative element of $\boldsymbol{Y}^*=\boldsymbol{L}_p\times \boldsymbol{R}$ is w^*-approximated by non-negative elements of $K\times \boldsymbol{R}$. Hence the condition (α) is satisfied.

The set Z_{A^*} is $\sigma(X^*, D(A))$-closed as will be seen in Lemma 6. Therefore $V_A = Z_{A^*}$ by Proposition 1.

As is seen above, $x_0^* \in V_A = Z_{A^*}$. Hence $x_0^* = A^*\lambda_0^*$ for some non-negative $\lambda_0^* = \begin{pmatrix} \lambda_0^* \\ l_0^* \end{pmatrix} \in D(A^*) \subset Y^* = L_p \times R$. We have, for any $x = \begin{pmatrix} x \\ \xi \end{pmatrix} \in D(A)$,

$$-\delta f(a : x) = \langle x, x_0^* \rangle = \langle x, A^*\lambda_0^* \rangle$$

$$= \left\langle \begin{pmatrix} x \\ \xi \end{pmatrix}, \begin{pmatrix} -2a\lambda_0^* \\ \langle 1-a^2, \lambda_0^* \rangle + l_0^* \end{pmatrix} \right\rangle$$

$$= \langle x, -2a\lambda_0^* \rangle + (\langle 1-a^2, \lambda_0^* \rangle + l_0^*)\xi .$$

Hence $-\delta f(a : x) = \langle -2ax, \lambda_0^* \rangle$ and $\langle 1-a^2, \lambda_0^* \rangle = 0$.

In case where $p = +\infty$, we can apply the same argument as above by replacing Y by $L_1 \times R$. The domain $D(A^*)$ coincides with Y^*.

LEMMA 5. *Let a, x, ξ and ε be real numbers satisfying the following inequalities*: $1-a^2 \geqq 0$, $-2ax + (1-a^2)\xi \geqq 0$, $\xi \geqq 0$ *and* $0 \leqq \varepsilon \leqq 1/(1+\xi+|x|)$. *Then we have* $1-(a+\varepsilon x)^2 \geqq 0$.

Proof. If $x>0$ and $a \leqq (1+\xi)/(1+\xi+|x|)$, then $1-a-\varepsilon x \geqq 1-(1+\xi)/(1+\xi+|x|) - |x|/(1+\xi+|x|) = 0$, and $1+a+\varepsilon x \geqq 0$, which proves $1-(a+\varepsilon x)^2 \geqq 0$.

If $x>0$ and $a \geqq (1+\xi)/(1+\xi+|x|)$, then $1-a-\varepsilon x = \{1-a^2-\varepsilon x(1+a)\}/(1+a) \geqq \{2ax/(1+\xi) - 2x/(1+\xi+|x|)\}/(1+a) \geqq \{2x/(1+\xi+|x|) - 2x/(1+\xi+|x|)\}/(1+a) = 0$ and $1+a+\varepsilon x \geqq 0$. Case when $x<0$ can be reduced to the above, since $-x>0$ and $-2(-a)(-x) + \{1-(-a)^2\} \geqq 0$.

LEMMA 6. *We denote $L_1 \times R$ by X. Let Y be $L_p^* \times R$ in case $1<p<+\infty$ and $L_1 \times R$ in case $p=+\infty$. We define an operator A as in the proof of the second statement of Proposition 3 with essentially bounded a.*

If $a^{(-1)} \in L_p$, then Z_{A^*} is $\sigma(X^*, D(A))$-closed.

Proof. Let $x_\lambda^* = A^*\begin{pmatrix} y_\lambda^* \\ \eta_\lambda^* \end{pmatrix} \in Z_{A^*}$ converge to $x_\infty^* = \begin{pmatrix} x_\infty^* \\ \xi_\infty^* \end{pmatrix} \in X^*$ in the $\sigma(X^*, D(A))$-topology. In other words,

$$\langle x, -2ay_\lambda^* \rangle \to \langle x, x_\infty^* \rangle \text{ for any } x \in L_\infty ,$$

$$\langle 1-a^2, y_\lambda^* \rangle + \eta_\lambda^* \to \xi_\infty^* .$$

As x_∞^* vanishes on the complement of $S = \{t : a(t) \neq 0\}$, $x_\infty^* = -2ay_\infty^*$ for $y_\infty^* = -2^{-1}a^{(-1)}x_\infty^*$. We can easily see that $0 \leqq y_\infty^* \in L_p$. (Cf. Case 10) of Theorem). Suppose $\xi_\infty^* < \langle 1-a^2, y_\infty^* \rangle$. The right-hand side is equal to $\langle 1, y_\infty^* \rangle - \lim_\lambda \langle a^2, y_\lambda^* \rangle$, since $\langle a^2, y_\lambda^* \langle = \langle -a/2, -2ay_\lambda^* \rangle$.

For $x_N = -(a^{(-1)}/2)\chi_{\{|a^{-1}| \leqq N\}} \in L_\infty$, $\lim_\lambda \langle \chi_{\{|a^{-1}| \leqq N\}}, y_\lambda^* \rangle = \lim_\lambda \langle x_N, x_\lambda^* \rangle = \langle x_N, x_\infty^* \rangle = \langle \chi_{\{|a^{-1}| \leqq N\}}, y_\infty^* \rangle$.

The last term converges to $\langle 1, y_\infty^* \rangle$ as $N \to +\infty$. Hence for large N, we have

$$\begin{aligned}\xi_\infty^* &< \langle \chi_{\{|a^{-1}| \leqq N\}}, y_\infty^* \rangle - \lim_\lambda \langle a^2, y_\lambda^* \rangle \\ &= \lim_\lambda \langle \chi_{\{|a^{-1}| \leqq N\}}, y_\lambda^* \rangle - \lim_\lambda \langle a^2, y_\lambda^* \rangle \\ &\leqq \underline{\lim}_\lambda \langle 1-a^2, y_\lambda^* \rangle \\ &\leqq \lim_\lambda \{\langle 1-a^2, y_\lambda^* \rangle + \eta_\lambda^*\} = \xi_\infty^* ,\end{aligned}$$

which is a contradiction. Thus $\xi_\infty^* \geqq \langle 1-a^2, y_\infty^* \rangle$. Putting $\eta_\infty^* = \xi_\infty^* - \langle 1-a^2, y_\infty^* \rangle \geqq 0$, we have $\boldsymbol{x}_\infty^* = A^* \begin{pmatrix} y_\infty^* \\ \eta_\infty^* \end{pmatrix}$.

REMARK. A direct calculation leads to a finer result. That is, *if* $\nabla f(a) \in \boldsymbol{L}_p$, *then* $\lambda_0^* \in \boldsymbol{L}_p$. To see this, we prove.

LEMMA 7. *Assume that* $\nabla f(a) \in \boldsymbol{L}_p$. *Then*, $\nabla f(a)(t) = 0$ *for almost all* t *where* $|a(t)| \neq 1$. $\nabla f(a)(t) \geqq 0$ *for almost all* t *where* $a(t) = 1$. $\nabla f(a)(t) \leqq 0$ *for almost all* t *where* $a(t) = -1$.

Proof. Let $J_n = \{t : |a(t)| \leqq 1 - 1/n\}$. For any measurable set $\Gamma \subset J_n$ and $0 < \varepsilon < 1/n$, $g(a \pm \varepsilon \chi_\Gamma) \geqq 0$. Therefore $\delta f(a : \pm \chi_\Gamma) = \langle \nabla f(a), \pm \chi_\Gamma \rangle \leqq 0$, which implies that $\nabla f(a) = 0$ a.e. on J_n.

Let $J = \{t : a(t) = 1\}$. For any measurable set $\Gamma \subset J$ and $0 < \varepsilon < 1$, $g(a - \varepsilon \chi_\Gamma) \geqq 0$. Therefore $f(a : -\chi_\Gamma) = \langle \nabla f(a), -\chi_\Gamma \rangle \leqq 0$, which implies that $\nabla f(a) \geqq 0$ a.e. on J.

Put $\lambda_0^*(t) = (\nabla f(a)(t))/(2a(t))$ for t such that $a(t) \neq 0$ and put $\lambda_0^*(t) = 0$ for t where $a(t) = 0$. By above Lemma, λ_0^* is non-negative and belongs to $\boldsymbol{L}_p$. It is easy to see that λ_0^* is a unique function that satisfies the relations in Proposition 3.

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