# Some Counter Examples in the Theory of Positive Operators

By Ikuko SAWASHIMA

Department of Mathematics, Faculty of Science, Ochanomizu University
(Introduced by Prof. Y. MIMURA)

(Received September 17, 1964)

In the case of a finite dimensional vector space with natural order the following fact is well known; the proper value on the circle $|\lambda|=1$ of a positive operator (non negative matrix) with spectral radius 1 is a $k$th root of unity. This result was generalized by T. Ando [1] to the case of completely continuous operators on a $\sigma$-complete Banach lattice and by H. Schaefer [5] to the case of $L_p$ and $C$. However the above fact does not necessarily hold when the positive cone is not minihedral. Indeed, the example given in our previous paper [3] shows also that if a 3-dimensional vector space is ordered by the Lorenz cone then the operator represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

has the proper values $\{1, e^{i\theta}, e^{-i\theta}\}$. Therefore if $\theta/\pi$ is irrational then the proper value $e^{i\theta}$ is not the $k$th root.

Further, applying the method of the above example to the case of a separable Hilbert space ordered by the Lorentz cone, we obtain a counter example for the following problem proposed by H. Schaefer [4]:

Let $E$ be a partially ordered Banach space with a positive cone $K$ and $T$ a positive operator on $E$ with spectral radius $r$. Under what general condition, if any (e.g. $K$ is a normal $B$-cone in $E$), are the following implications true?

a. If $r$ is an isolated singularity of $R(\lambda, T)$[1], every singularity of $R(\lambda, T)$ on $|\lambda|=r$ is isolated.

b. If $r$ is a pole of $R(\lambda, T)$, $R(\lambda, T)$ has no singularities on $|\lambda|=r$ other than poles.

In the foregoing paper [2], F. Niiro solved the problem b. affirmatively, when $E$ is $l_p$ with its natural order, assuming further that $T$ is an indecomposable positive operator which is equivalent to the following condition (A) [2; Theorem 1]:

(A) $T$ and $T^*$ have the spectral radius $r(T)=r(T^*)=r$ and the proper

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1) $R(\lambda, T)$ denotes the resolvent of $T$.

space of each of them corresponding to the proper value $r$ is one dimensional with non-support base.

Our counter example also satisfies this condition (A). The reason for this may be based on the fact that the Lorenz cone is not minihedral.

Let $E$ be the infinite dimensional real Hilbert space and $x_0$ be an element of $E$ with norm 1. Let $E_1$ be the space spanned by the sole element $x_0$, and $E_2$ be the orthogonal complement of $E_1$. We denote the projections on $E_1$ and $E_2$ by $P_1$ and $P_2$ respectively. The Lorenz cone $K=\{x;\, x\in E,\ (x, x_0)\geqq \|P_2x\|\}$ is normal and has non empty interior. Let $U$ be an isometric operator on $E_2$, i.e., a linear operator mapping $E_2$ onto itself such that $\|Ux\|=\|x\|$ whenever $x\in E_2$. Then the bounded linear operator $T$ defined by $Tx=P_1x+UP_2x$ is a positive operator with respect to the cone $K$. If $1\notin\sigma(U)$[2] then the point 1 is a pole of $R(\lambda, T)$ with order 1 and the proper space of $T$ corresponding to 1 is $E_1$ whose base $x_0$ is an interior element of $K$. Since $\sigma(T)=\sigma(U)\cup\{1\}$, $\sigma(T)$ is on the circle $|\lambda|=1$. It is known that for any closed subset of the circle $|\lambda|=1$, symmetric with respect to the real axis, there exists an isometric operator with this subset as its spectrum. Let $U$ be an isometric operator on $E_2$ where $\sigma(U)\not\ni 1$ and $\sigma(U)$ has at least one accumulation point $\lambda_0$ on the circle $|\lambda|=1$. Then we obtain the following result. The positive operator $T=P_1+UP_2$ has the spectral radius 1 and the point $\lambda=1$ is a pole of $R(\lambda, T)$, but $\lambda_0$ is not an isolated singularity of $R(\lambda, T)$. Further, $T$ has the property (A)[3].

*Remark.* By putting $\theta=\pi$ in the first example we see that the dimension of the proper space corresponding to $-1$ is larger than that corresponding to 1. Similarly in the second example we can choose $T$ so that the proper space corresponding to $\mu_0\neq 1$ ($|\mu_0|=1$) may be infinite dimensional, although the proper space corresponding to 1 is one-dimensional.

## References

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2) $\sigma(T)$ denotes the spectrum of $T$.

3) Because $T^*=P_1+U^*P_2$ and $\sigma(U)=\sigma(U^*)$.