An Extension of Tannaka Duality Theorem for Homogeneous Spaces

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1. One of the authors has given in a former paper [1] a formulation of the well-known Tannaka duality theorem as a characterization of the right translation of the ring of Fourier polynomials on a compact group. The purpose of this paper is to extend this theorem to the case of homogeneous spaces which are factor spaces of compact groups.

Let $G \ni a, b, \ldots, x, \ldots$ be a compact group acting transitively on a compact space $M \in P, Q, \ldots$. Let us fix a point P_0 of M and we denote by H the isotropy group of P_0:

$$H = \{a \in G\,;\, aP_0 = P_0\}.$$

Then the mapping $aH \rightarrow aP_0$ is a homeomorphism from the homogenous space G/H onto M.

Now let us denote by $C(G)$, $C(M)$ the algebra of all complex valued continuous functions on G, M respectively. $C(G)$, $C(M)$ are algebras over the field of complex numbers. To $f \in C(M)$, we associate a function $\tilde{f} \in C(G)$ as follows: $\tilde{f}(a) = f(aP_0)$ $(a \in G)$. Then the mapping $f \rightarrow \tilde{f}$ is an injection from $C(M)$ into $C(G)$. We shall regard $C(M)$ as a subalgebra of $C(G)$ under this injection: $C(M) \subset C(G)$. Then it is clear that a function $f \in C(G)$ is in $C(M)$ if and only if f is constant on every coset of H, *i. e.*

$$f(xh) = f(x) \text{ for every } x \in G,\ h \in H.$$

For an element a in G, left (right) translation L_a (R_a) is defined as a linear operator on $C(G)$ as follows:

$$(L_a f)(x) = f(a^{-1}x), \quad (R_a f)(x) = f(xa) \quad (x \in G,\ f \in C(G)).$$

It is obvious that $C(M)$ is a stable subspace under every $L_a (a \in G)$. Now a function $f \in C(G)$ is called *Fourier polynomial* on G if the set $\{L_a f;\ a \in G\}$ contains only a finite number of linearly independent functions. In other words, a function f in $C(G)$ is a Fourier polynomial if and only if the smallest left G-invariant subspace $\mathfrak{M}(f)$ of $C(G)$ containing f is finite-dimensional. Let us denote by $R(G)$ the set of all Fourier polynomials on G. Then $R(G)$

is a subalgebra of $C(G)$. In fact if $f_1, f_2 \in R(G)$, then the product $\mathfrak{M}(f_1)\cdot\mathfrak{M}(f_2)$ is also a finite-dimensional G-invariant subspace of $C(G)$ containing $f_1 f_2$. Hence we have $\mathfrak{M}(f_1 f_2)\subset\mathfrak{M}(f_1)\cdot\mathfrak{M}(f_2)$ and $\dim \mathfrak{M}(f_1 f_2)<\infty$. Thus we have $f_1 f_2 \in R(G)$. Similarly we have $\lambda f_1+\mu f_2 \in R(G)$ for every complex number λ, μ.

We note that a function f in $C(G)$ belongs to $R(G)$ if and only if f is expressible in the form:

$$f(x)=\sum c_{ij}^{(\alpha)} d_{ij}^{(\alpha)}(x) \qquad \text{(finite sum)} \tag{1}$$

where $c_{ij}^{(\alpha)}$'s are complex numbers and $d_{ij}^{(\alpha)}(x)$'s are components of some continuous irreducible representation $x\to D^{(\alpha)}(x)=(d_{ij}^{(\alpha)}(x))$ of G by matrices of a finite degree. In fact, by $L_a d_{ij}^{(\alpha)}=\sum_k d_{ik}^{(\alpha)}(a^{-1})d_{kj}^{(\alpha)}$, we have $d_{ij}^{(\alpha)} \in R(G)$. Hence if f has the form (1), then $f \in R(G)$. Conversely, if $f \in R(G)$, then the smallest left G-invariant subspace $\mathfrak{M}(f)$ of $C(G)$ is finite dimensional. Let $f_1, \dots, f_k$ be a base of $\mathfrak{M}(f)$ such that

$$\int_G f_i(x)\overline{f_j(x)}dx=\delta_{ij} \qquad (dx\text{: a Haar measure of } G). \tag{2}$$

$L_a f_i$ is a linear combination of the f_j's:

$$L_a f_i=\sum_j d_{ji}(a) f_j. \tag{3}$$

Then $a\to D(a)=(d_{ij}(a))$ is a representation of G. Moreover d_{ij} is a continuous function on G as is seen from the following equation:

$$d_{ij}(a)=(L_a f_i, f_j)=\int_G f_i(a^{-}x)\overline{f_j(x)}dx.$$

Then, decomposing the representation $a\to D(a)$ into irreducible components, we see that every d_{ji} is of the form (1). Now by (3) we have

$$f_i(a^{-1})=\sum_j d_{ji}(a) f_j(e) \quad (e\text{: the unit element of } G).$$

Since dx is left-invariant, $D(a)$ is a unitary matrix. Thus we have

$$f_i(a)=\sum d_{ji}(a^{-1})f_j(e)=\sum\overline{d_{ij}(a)}f_j(e). \tag{4}$$

Hence every f_i is of the form (1). Then f is also of the form (1).

Thus our assertion is verified.

Now let us put

$$R(M)=R(G)\cap C(M). \tag{5}$$

Then $R(M)$ is a subalgebra of $R(G)$. A function f in $R(M)$ is called a *spherical function* on M (under the operation of G). In other words, a continuous function f on M is called a spherical function on M if the set $\{L_a f;\ a \in G\}$ contains only a finite number of linearly independent functions, *i.e.* if $\dim \mathfrak{M}(f) < \infty$. (If f is a spherical function and $\mathfrak{M}(f)$ if G-irreducible, then $\mathfrak{M}(f)$ defines an irreducible representation $a \rightarrow D(a)$ as in (3). Then f is called a spherical function of irreducible type D.)

Now let us note that every continuous function on M can be approximated uniformly on M by spherical functions on M. In fact, let f be any function in $C(M)$. Then by Peter-Weyl's theorem, there exists for any $\varepsilon > 0$, a function f_0 in $R(G)$ such that

$$|f(x) - f_0(x)| \leq \varepsilon \qquad \text{(for any } x \in G). \tag{6}$$

Let dh be the Haar measure on H such that $\int_H dh = 1$. Let us denote by f^*, f_0^* the "right means" of f, f_0 on H, *i.e.*

$$f^*(x) = \int_H f(xh)dh, \quad f_0^*(x) = \int_H f_0(xh)dh, \quad (x \in G).$$

Then, since $f \in C(M)$ we have $f(xh) = f(x)$ for every $h \in H$, $x \in G$. Hence $f = f^*$. Moreover we have $f_0^*(xh) = f_0^*(x)$ for every $h \in H$, $x \in G$. Thus f_0^* is in $C(M)$. Now f_0^* is in $R(G)$. In fact, since f_0 has the form

$f_0(x) = \sum_{\alpha, i, j} c_{ij}^{(\alpha)}(x)$ ($d_{ij}^{(\alpha)}(x)$'s are components of representations of G), we have

$$f_0^*(x) = \sum_{\alpha, i, j} c_{ij}^{(\alpha)} \int_H d_{ij}^{(\alpha)}(xh)dh = \sum_{\alpha, i, j} c_{ij}^{(\alpha)} \sum_k d_{ik}^{(\alpha)}(x) \int_H d_{kj}^{(\alpha)}(h)dh.$$

Hence f_0^* has the form (1). Thus we have $f_0^* \in R(M)$.

Now integrating $|f(xh) - f_0(xh)| \leqq \varepsilon$ $(x \in G,\ h \in H)$ over H with respect to h, we have $|f^*(x) - f_0^*(x)| \leqq \varepsilon$, *i.e.* $|f(x) - f_0^*(x)| \leqq \varepsilon$. Thus our assertion is verified.

Now let us denote by $N(H)$ the nomalizer of H in G:

$$N(H) = \{a \in G;\ aHa^{-1} = H\}.$$

Then we have the following

LEMMA. *A right translation R_a leaves $C(M)$ invariant if and only if $a \in N(H)$.*

In fact, let $a \in N(H)$, $f \in C(M)$. Then for any $h \in H$, we can choose an element $h' \in H$ such that $ha=ah'$. Hence we have

$$f(xha)=f(xah')=f(xa) \quad (\text{since } f \in C(M)).$$

This means that $R_h(R_a f)=R_a f$ (for any $h \in H$). Hence we have $R_a f \in C(M)$.

Conversely let $R_a(C(M)) \subset C(M)$. Then, for any $f \in C(M)$ we have $R_a f \in C(M)$, *i. e.* $f(xha)=f(xa)$ $(x \in G, h \in H)$. Now, any two different points of M can be separated by a function in $C(M)$. In other words, if $b_1, b_2 \in G$ satisfies

$$f(b_1)=f(b_2) \quad \text{for any} \quad f \in C(M),$$

then we have $b_1H=b_2H$. Hence we have

$$xhaH=xaH \quad \text{for any} \quad x \in G,\ h \in H.$$

Then we have $a \in N(H)$. Thus our lemma is proved.

COROLLARY. *A right translation R_a leaves $R(M)$ invariant if and only if $a \in N(H)$.*

If $a \in N(H)$, we have $R_a(C(M)) \subset C(M)$, $R_a(R(G)) \subset R(G)$. Hence we have $R_a(R(M)) \subset R(M)$. Conversely, let $R_a(R(M)) \subset R(M)$. Since $R(M)$ is uniformly dense in $C(M)$ and R_a conserves the uniform norm, we have $R_a(C(M)) \subset C(M)$. Thus $a \in N(H)$, *q. e. d.*

2. Now let us denote by $\mathfrak{A}$ the full automorphism group of the algebra $R(M)$. An automorphism S of the algebra $R(M)$ is called *real* if

$$S\bar{f}=\overline{Sf} \quad \text{for any} \quad f \in R(M),$$

where $\bar{f}$ denotes the complex conjugate of f (note that $\overline{R(M)}=R(M)$). Then the set $\mathfrak{S}$ of all real automorphisms in the automorphism group $\mathfrak{A}$ forms a subgroup of $\mathfrak{A}$. Obviously, the restriction of L_a on $R(M)$ (which we denote by L_a') is in the real automorphism group $\mathfrak{S}$, for any $a \in G$. The set $\{L_a' ; a \in G\}$ is a subgroup of $\mathfrak{S}$. This subgroup is denoted by $\tilde{G}$.

Now, let us denote by $Z(\tilde{G})$ the centralizer of $\tilde{G}$ in $\mathfrak{S}$:

$$Z(\tilde{G})=\{S \in \mathfrak{S} ; SL_a'=L_a'S \quad \text{for any} \quad a \in G\}.$$

For $a \in N(H)$, the restriction of R_a on $R(M)$ (which we denote by R_a') is in $Z(\tilde{G})$. In fact, R_a' is in $\mathfrak{S}$ and $L_b' R_a'=R_a' L_b'$ for any $b \in G$.

Thus we obtain a homomorphism $a \to R_a'$ from the group $N(H)$ into the group $Z(\tilde{G})$. Now our extension of Tannaka duality theorem is stated as follows:

THEOREM. *The homomorphism $a \to R_a'$ from $N(H)$ into $Z(\tilde{G})$ is an onto-*

homomorphism with kernel H:

$$N(H)/H \cong Z(\tilde{G}).$$

This contains Tannaka duality theorem as formulated in [1] as the special case $H=\{e\}$. The proof runs analogously to the proof in [1]. But we shall give it here for completeness' sake.

Let S be in $Z(\tilde{G})$. We first establish that

$$\int_G f(x)dx = \int_G (Sf)(x)dx \tag{7}$$

for every f in $R(M)$. Let $\mathfrak{M}(f)$ be the smallest left G-invariant subspace of $C(M)$ containing f. Let

$$\mathfrak{M}(f) = \mathfrak{M}_1 + \cdots + \mathfrak{M}_r \text{ (direct sum)}$$

be a decomposition of $\mathfrak{M}(f)$ into irreducible components. Then, f can be expressed as $f=f_1+\cdots+f_r$, $f_i \in \mathfrak{M}_i$ $(i=1, \ldots, r)$.
Hence we may prove (7) under the assumption that $\mathfrak{M}(f)$ is irreducible. In case where $\mathfrak{M}(f)$ is irreducible, we distinguish the following two cases:
(i) The representation of G on $\mathfrak{M}(f)$ is trivial, *i. e.* the case where $\mathfrak{M}(f)$ is one-dimensional and

$$L_a f = f \quad \text{for every} \quad a \in G.$$

Then f is constant and we have $Sf=f$. Thus we have (7) in this case.
(ii) The representation of G on $\mathfrak{M}(f)$ is non-trivial. Let $f_1, \ldots, f_n$ be a base of $\mathfrak{M}(f)$ which satisfies (2). Then the matrix $(d_{ij}(a))$ defined in (3) is a unitary one. From (3) and $SL_a' = L_a'S$, we have

$$L_a S f_i = \sum_j d_{ji}(a) S f_j,$$

i. e.

$$(Sf_i)(a^{-1}x) = \sum_j d_{ji}(a)(Sf_j)(x) \qquad (a, x \in G).$$

Putting $x=e$, we have

$$(Sf_i)(a^{-1}) = \sum d_{ji}(a)(Sf_i)(e),$$

or

$$(Sf_i)(a) = \sum \overline{d_{ij}(a)}(Sf)(e) \qquad (a \in G).$$

Now, since the representation $a \to (d_{ij}(a))$ is irreducible and non-trivial, we have $\int_G (Sf_i)(a)da = 0$. On the other hand (4) implies that $\int_G f_i(a)da = 0$.

Thus (7) holds for any $f_1, \ldots, f_n$, hence for f also.

From (7) we have

$$(8) \qquad \|Sf\|_{2p}=\|f\|_{2p} \quad (p=1, 2,\ldots)$$

for any $f \in R(M)$, where

$$\|f\|_{2p}=\left\{\int_G |f(x)|^{2p}dx\right\}^{\frac{1}{2p}}=\left\{\int_G (f(x)\overline{f(x)})^p dx\right\}^{\frac{1}{2p}}.$$

In fact, since S is a real automorphism of $R(M)$, we have

$$S(|f|^{2p})=|Sf|^{2p}.$$

Now (8) implies that

$$(9) \qquad \|Sf\|_\infty=\|f\|_\infty$$

for any $f \in R(M)$, where $\|f\|_\infty=\max\limits_{x \in G}|f(x)|$. In fact, the well-known formula

$$\|f\|_\infty=\lim_{p\to\infty}\|f\|_p,$$

and (8) imply (9).

Thus S is continuous with respect to the uniform norm in $R(M)$, S can be uniquely extended to a continuous linear operator $\tilde{S}$ of $C(M)$. $\tilde{S}$ is an automorphism of the algebra $C(M)$ since S^{-1} has also the extension to $C(M)$ which is the inverse of $\tilde{S}$. Obviously $\tilde{S}$ commutes with the restriction L_a'' of L_a on $C(M)$ for any $a \in G$. Let a be any element of G. We denote by $\mathfrak{J}_a$ the maximal ideal of $C(M)$ defined by

$$(10) \qquad \mathfrak{J}_a=\{f \in C(M)\,;f(a)=0\}.$$

Obviously if a_1, a_2 are congruent mod. H, then $\mathfrak{J}_{a_1}=\mathfrak{J}_{a_2}$ and conversely. Every maximal ideal $\mathfrak{J}$ of $C(M)$ is expressed in the form $\mathfrak{J}=\mathfrak{J}_a$ for some $a \in G$. Now since $\tilde{S}$ is an automorphism of the algebra $C(M)$, $\tilde{S}(\mathfrak{J}_e)$ is also a maximal ideal of $C(M)$. Hence there exists an element b in G such that

$$\tilde{S}(\mathfrak{J}_e)=\mathfrak{J}_b.$$

Now, as is seen easily, we have $L_a''(\mathfrak{J}_b)=\mathfrak{J}_{ab}$ for any $a \in G$. Now let f be any function in $C(M)$. Put $g=\tilde{S}f$. Let a be any element in G. Then we have $f-f(a) \in \mathfrak{J}_a$. Hence we have

$$g-f(a) \in \tilde{S}(\mathfrak{J}_a)=\tilde{S}(L_a''\mathfrak{J}_e)=L_a''\tilde{S}(\mathfrak{J}_e)=L_a''\mathfrak{J}_b=\mathfrak{J}_{ab}.$$

Thus we have

$$g(ab)=f(a) \quad \text{for any} \quad a \in G.$$

Writing $ab=x$, we have $g(x)=f(xb^{-1})$, *i.e.*

$$\tilde{S}f=R_{b^{-1}}f.$$

Thus, we see that the right translation $R_{b^{-1}}$ leaves $C(M)$ invariant and the restriction of $R_{b^{-1}}$ on $C(M)$ coincides with $\tilde{S}$. Hence we have $b\in N(H)$ by the lemma, and we see that the restriction $R_{b^{-1}}'$ of $R_{b^{-1}}$ on $R(M)$ coincides with S. Thus, we have established that the homomorphism $a\to R_a'$ from $N(H)$ into $Z(\tilde{G})$ is an onto-homomorphism.

Now the kernel of the homomorphism $a\to R_a'$ is determined easily: $R_a'=1$ implies that

$$f(xa)=f(x) \tag{11}$$

for any $f\in R(M)$ and $x\in G$. Then, since $R(M)$ is dense in $C(M)$, (11) holds for any $f\in C(M)$ and $x\in G$, then $xaH=xH$ for any $x\in G$. Thus we have $a\in H$. Conversely, if a $a\in H$, then we have clearly $R_a'=1$. Thus our theorem is proved.

Remark. Let us introduce in the group $Z(\tilde{G})=\{R_a'; a\in N(H)\}$ the weak topology: for any finite sequence $f_1, \ldots, f_k\in R(M)$ and for any $\varepsilon>0$, put

$$\mathfrak{U}(f_1, \ldots, f_k; \varepsilon)=\{S\in Z(\tilde{G}); \|Sf_1-f_1\|<\varepsilon, \ldots, \|Sf_k-f_k\|<\varepsilon\}$$

Then taking $\{\mathfrak{U}(f_1, \ldots, f_k; \varepsilon)\}$ as the neighborhood system of the unit element in $Z(\tilde{G})$, $Z(\tilde{G})$ becomes a topological group. Moreover the homomorphism $a\to R_a'$ is a continuous mapping from the compact group $N(H)$ onto $Z(\tilde{G})$. Thus, we have $N(H)/H\cong Z(\tilde{G})$ as topological groups.

Example. Let G be a connected, compact Lie group, and H be a maximal torus in G. Then, $N(H)/H$ is isomorphic with the Weyl group of G. Or more precisely, $N(H)/H$ is isomorphic with the Weyl group of the complex form of the Lie algebra of G. Hence we have a characterization of the Weyl group of G as the centralizer of $\tilde{G}=\{L_a'; a\in G\}$ in the real automorphism group $\mathfrak{G}$ of the algebra of spherical functions on $M=G/H$.

Reference

[1] Nobuko Iwahori, A proof of Tannaka duality theorem, *Sci. Pap. Coll. Gen. Educ., Univ. Tokyo*, 8, 1-4 (1958).