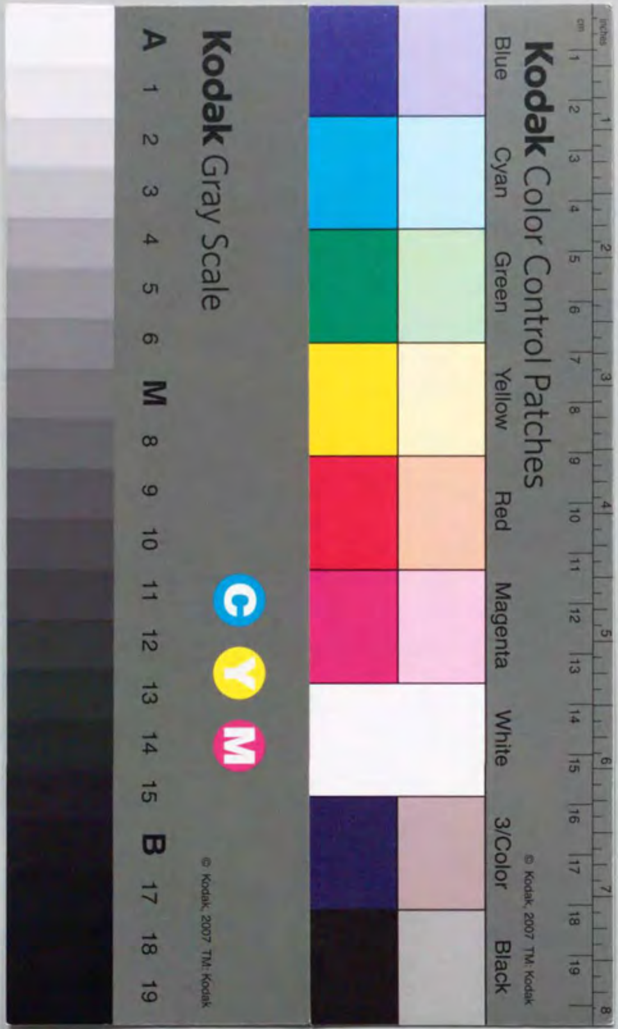

Dynamical Properties
of
Mixed State in Type-II
Superconductors

Masahiko Hayashi

第二種超伝導体混合状態の

動的特性

林　正彦

Thesis

Dynamical Properties of Mixed State in Type-II Superconductors

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1. Introduction

The mixed state of type-II superconductors has been one of the most intensively studied topics in the field of the superconductivity because of its importance not only in the theoretical physics but also in the applied physics. The birth of this field goes back to 1957, when the pioneering work by Abrikosov [1] based on the Ginzburg-Landau (GL) theory appeared. This year is the same year, miraculously, as the year when the dawn of the theoretical superconductivity was proclaimed by the BCS theory. Since then much theoretical progress has been made in this field [2, 3, 4]. While the Abrikosov's mean field theory was being improved taking into account the knowledge obtained from the microscopic theory, (see Fetter and Hohenberg in [2]) the elastic fluctuations of the Abrikosov vortex lattice were studied by several authors [5, 6]. On the other hand, the dissipative nature of the dynamics of the vortices were first clarified by Bardeen and Stephen [7], and then analyzed in detail based on the time-dependent GL equation by several authors [8, 9, 10]. Their theory made a way to the wide field of the flux flow phenomena. The effects of disorder on the flux flow were first studied by Anderson and Kim [11], now known as the flux creep theory, which has been extended to the collective flux creep theory by taking into account the correlation among vortex lines by Larkin and Ovchinnikov [12].

The research field of the mixed state in type-II superconductors, thus, has been making steady progress, until the surprising discovery of the high-T_c cuprates in 1986. As well as the theories of superconducting mechanism are attacked by this new materials, the phenomenology of the mixed state also started to go into the new circumstances where the effects of the thermal fluctuations and the disorder are large enough to make the established understanding based on the conventional materials change. Especially studies on the effects of the enhanced thermal fluctuations challenge the conventional understanding of the mixed state drastically. Theoretical investigations concluded that the $H - T$ phase diagram should be modified in several aspects from the mean field one; the second order phase transition at H_{c2}, present within the mean field treatment, becomes a crossover once the effects of thermal fluctuations are taken into account and, in place of it, a new phase transition appears, i.e. the vortex lattice melting transition [13, 14, 15]. Moreover it is proposed that the vortex liquid phase is further divided into the entangled and the disentangled vortex liquid phase and other new phases. Although these theoretical predictions are under controversy at present, the latest experimental studies [16, 17, 18, 19, 20] seem to confirm the existence of the vortex lattice melting transition in extremely clean samples.

Besides the above mentioned equilibrium properties, dynamical properties have also been studied by many authors. Especially the flux flow and flux creep phenomena are still under intensive investigations [21] because of its importance especially in the experimental and applied physics, although these theories only consider the response of the system to the uniform electric current (zero wave vector) flowing perpendicular to the applied magnetic field. However there are now a wide variety of the experiments, in which the geometry of the samples plays important roles [22, 23]. In order to analyze these experiments theoretically we need to study the response of the system to the external field with finite wave vectors, which is the viewpoint completely disregarded in the treatment of flux flow or flux creep theories. Such an attempt has been done by Huse [24] recently who proposed a phenomenological description of the system with various geometry.

In this thesis we propose a new framework to study the electromagnetic response of the type-II superconductors taking the wave vector and the frequency dependence into account. All electromagnetic properties of the system, within the linear response theory, are contained in the superfluid density. This characterizes the response of the supercurrent against the perturbing vector potential. Once we obtain this we can also evaluate the electric conductivity and magnetic susceptibility. In calculating the superfluid density we paid a special attention to distinguish the external vector potential and the internal one. This is important when we treat systems which show strong screening effects. Technically when we investigate the linear response of the systems we must treat the internal field as the perturbation to the system. Such situations are similar to the case of the system of electrons interacting via the Coulomb interaction, which is closely studied in detail in [25]. We emphasize that this point is completely disregarded in the conventional treatments of flux flow and flux creep phenomena, where the internal

vector potential is integrated out from the beginning and is never discussed in this context. Because of this, we found important difference from the conventional results.

Next we describe the model and method we used to formulate the theory. Since we are interested in the properties of the ideal type-II superconductors we ignored the anisotropy and inhomogeneity, though they are characteristic to the high-T_c superconductors. Therefore our model is for the isotropic and homogeneous type-II superconductors. We consider the case, $H_{c1} \ll H \ll H_{c2}$, which will cover a wide region in the phase diagram including both vortex lattice and vortex liquid states and, therefore, is appropriate to investigate the difference of the two states in their dynamical properties. In this case, we can employ the London theory, namely since the amplitude of the Ginzburg-Landau (GL) order parameter is well developed, we neglect its fluctuation. Therefore the fluctuations of the condensate are described only by the fluctuations of the "phase", which are governed by the fluctuations of the positions of the vortices. Thus the superfluid density, which is given by the autocorrelation function of the supercurrent in the equilibrium, is expressed by the autocorrelation function of the vortex density. This will be calculated both for vortex lattice and vortex liquid state by use of the different theoretical framework.

The vortex lattice state will be examined employing the method of the elastic theory. Our method differs from the one employed by Brandt [6] in the following points. First the elastic moduli obtained from our treatment do not include the screening effects by the internal vector potential fluctuations. Second we use a new method in evaluating the reciprocal lattice summation in caluculating the elastic moduli. This method is developed to calculate the elastic moduli in two dimensional classical Wigner crystal [26]. In our case, it is valid in the limit of vanishing coherence length, ξ_{sc}. As a result our compression and tilt modulus are different from those obtained in the London limit by Brandt [6] although the shear modulus is same. The differences are important when we discuss the magnetic permeability. In introducing the equations of motion of the vortices, we assumed the Bardeen-Stephen theory, namely we assumed that the dissipation is expressed as a form of the friction of the vortex motion. The dynamical superfluid density is calculated on these bases and the electric conductivity and the magnetic permeability as well as the static superfluid density are discussed based on it.

In the vortex liquid state we will employ the hydrodynamic treatment. We tried to extend the well-defined phenomenological Markoffian equation for the scalar density field (e.g. the particle density), which is called *model B* in the classification made by Hohenberg and Halperin [27], to the vector density field (the vortex line density). In introducing the dynamics we employed the Bardeen-Stephen theory again. Our hydrodynamic equations are basically the same as those previously used by Marchetti and Nelson [28] except for the treatment of the screening effects of the internal vector potential as in the lattice state. In our theory the internal properties of the vortex liquid is contained in the kinetic constants and, particularly, the entanglement and cutting of vortices play important roles in the hydrodynamic equations. This framework is applied to the calculation of the dynamical superfluid density in the liquid state.

This thesis is composed as follows. In chapter 2, we first express the free energy of the system in terms of the degree of freedom of the vortices. Then we describe the linear response theory, according to which, we see that the dynamical superfluid density is given by the autocorrelation function of supercurrent and that it is further expressed as the density-density correlation function (or dynamical form factor) of the vortices. Therefore our purpose is reduced into the calculation of the correlation function. In chapter 3, we use the elastic theory to analyze the vortex lattice state. We will discuss the superfluid density, the electric conductivity and the magnetic susceptibility. We also calculate the corrections to these quantities due to the effects of thermal fluctuations. In chapter 4, we use the hydrodynamic equations to analyze the vortex liquid state. Especially we show how the effects of the vortex line entanglement and cutting are taken into account in the hydrodynamic equations. The superfluid density and the electric conductivity are obtained. In chapter 5, we summarize the results of this thesis and give discussions. Especially we clarify the relation between our treatment and the conventional one. The details of the calculations are given in Appendices.

2. Model

2.1 Ginzburg-Landau free energy and vortices

We assume that the condensate free energy can be written in the Ginzburg-Landau form. Moreover, since we are now interested in the region of the magnetic field, H; $H_{c1} \ll H \ll H_{c2}$, we may assume the London limit, i.e. the fluctuations of the condensate order parameter,

$$\psi(\mathbf{r},t) = |\psi_0|\, e^{i\theta(\mathbf{r},t)}, \tag{2.1}$$

are only in the phase angle, $\theta(\mathbf{r},t)$, and the amplitude, $|\psi_0|$, is assumed to be constant except that it drops to zero at the vortex cores. The radius of the core may be taken to be the superconducting coherence length, ξ_{sc}, and $|\psi_0|^2 = \rho_0^s$ is the bare superfluid density.

The Gibbs' free energy, $\mathcal{G}_c$, of the condensate can be written as follows,

$$\mathcal{G}_c = \frac{K}{2}\int \mathrm{d}^3\mathbf{r}\, \left\{ \frac{\nabla\theta(\mathbf{r},t)}{2\pi} - \frac{\mathbf{A}(\mathbf{r},t)}{\phi_0} \right\}^2 \tag{2.2}$$

where $K = m^*\rho_0^s\kappa_c^2$, $\kappa_c = h/m^*$ and $\phi_0 = hc/e^*$ with m^* and e^* being the effective mass and the charge of Cooper pairs, respectively. The magnetic penetration depth is given by $\lambda = \phi_0/\sqrt{4\pi K} = \sqrt{m^* c^2/4\pi\rho_0^s e^{*2}}$. The vector potential, here denoted as $\mathbf{A}(\mathbf{r},t)$, is the internal vector potential, namely the vector potential experienced by the condensate electrons locally. By denoting the constant (apatially and temporary) part of the internal magnetic field as B the vector potential is expressed as,

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2}B\hat{\mathbf{z}} \times \mathbf{r} + \mathbf{a}(\mathbf{r},t), \tag{2.3}$$

where $\mathbf{a}(\mathbf{r},t)$ denotes the time-dependent fluctuation of the vector potential. Here we made an assumption that the spatial variation of the magnetic field in the equilibrium is negligible, which is valid in the field range of interest, $H_{c1} \ll H \ll H_{c2}$, since the typical scale of the magnetic fluctuation, i.e. the penetration depth, λ, is much larger than the typical scale of the vortex system, the mean vortex spacing given by $\sqrt{B/\phi_0}$.

The total transport current density in the system, $\mathbf{j}^{el}(\mathbf{r},t)$, is written as

$$\mathbf{j}^{el}(\mathbf{r},t) = \mathbf{j}^{n}(\mathbf{r},t) + \mathbf{j}^{s}(\mathbf{r},t), \tag{2.4}$$

where $\mathbf{j}^{n}(\mathbf{r},t)$ and $\mathbf{j}^{s}(\mathbf{r},t)$ are the normal current and the supercurrent, respectively. The supercurrent, $\mathbf{j}^{s}(\mathbf{r},t)$, is given by

$$\begin{aligned}\mathbf{j}^{s}(\mathbf{r},t) &= -c\,\frac{\delta \mathcal{G}_c}{\delta \mathbf{A}(\mathbf{r},t)} \\ &= \rho_0^s e^* \kappa_c \left\{ \frac{\nabla\theta(\mathbf{r},t)}{2\pi} - \frac{\mathbf{A}(\mathbf{r},t)}{\phi_0} \right\}.\end{aligned} \tag{2.5}$$

We also assign $\mathbf{H}^{ext}(\mathbf{r},t)$ and $\mathbf{D}^{ext}(\mathbf{r},t)$ to the externally applied magnetic and electric field, respectively. These external electromagnetic fields consist of two parts, the constant magnetic field, $\mathbf{H}^a$, and the electromagnetic perturbation probing the superconducting properties of the mixed state, which can be written in terms of the vector potential as $\mathbf{a}^{ext}(\mathbf{r},t)$. Therefore $\mathbf{H}^{ext}(\mathbf{r},t)$ and $\mathbf{D}^{ext}(\mathbf{r},t)$ are expressed as

$$\begin{aligned}\mathbf{H}^{ext}(\mathbf{r},t) &= H^a\hat{\mathbf{z}} + \nabla\times\mathbf{a}^{ext}(\mathbf{r},t), \\ \mathbf{D}^{ext}(\mathbf{r},t) &= -\frac{1}{c}\frac{\partial\,\mathbf{a}^{ext}(\mathbf{r},t)}{\partial t},\end{aligned} \tag{2.6}$$

where we put $\mathbf{H}^a = H^a\hat{\mathbf{z}}$ with $\hat{\mathbf{z}}$ the unit vector in z-direction.

Above mentioned quantities must be related to each other by the following relations. First B is determined so as to minimize the total free energy under the applied field, H^a, therefore B can be expressed as $B(H^a)$. Second $\mathbf{a}(\mathbf{r},t)$, $\mathbf{a}^{ext}(\mathbf{r},t)$, and $\mathbf{j}^{s}(\mathbf{r},t)$ are related by the Maxwell equation as,

$$-\nabla^2\left\{\mathbf{a}(\mathbf{r},t) - \mathbf{a}^{ext}(\mathbf{r},t)\right\} = \frac{4\pi}{c}\mathbf{j}^{s}(\mathbf{r},t). \tag{2.7}$$

In this thesis we limit ourselves to the trnaverse response only and, therefore, we impose the condition $\nabla\cdot\mathbf{a}(\mathbf{r},t) = 0$, and $\nabla\cdot\mathbf{a}^{ext}(\mathbf{r},t) = 0$ which is consistent with the gauge condition, $\nabla\cdot\mathbf{A}(\mathbf{r},t) = 0$. The scalar potential is taken to be zero.

Here we emphasize that $\mathbf{a}^{ext}(\mathbf{r},t)$ and $\mathbf{a}(\mathbf{r},t)$ should be strictly distinguished [25]; $\mathbf{a}^{ext}(\mathbf{r},t)$ is the externally applied field and $\mathbf{a}(\mathbf{r},t)$ is so called the "local field ", which is determined by treating the effect of screening selfconsistently. Therefore,

in general, $\mathbf{a}^{ext}(\mathbf{r},t) \neq \mathbf{a}(\mathbf{r},t)$. In this thesis we do not integrate out $\mathbf{a}(\mathbf{r},t)$ but keep them as the dynamical variable in the equations of motion. The superfluid density and the electric conductivity are calculated as a response function to $\mathbf{a}(\mathbf{r},t)$. Within our knowledge this point has been completely ignored in the preceding theories treating the vortex dynamics.

Here let us rewrite the free energy in terms of the degrees of freedom of the vortex configurations. The phase of the order parameter can be separated into two parts, the multi-valued part due to the vortices and the single-valued smoothly-varying part due to "spin-wave". In this thesis, we consider the vortices with winding number ± 1 only, i.e. the phase changes $\pm 2\pi$ by going around them once. The vortices with higher winding numbers are neglected since they require higher energy to be created. Accordingly $\nabla\theta(\mathbf{r},t)$ can be decomposed into two parts as follows,

$$\nabla\theta(\mathbf{r},t) = \nabla\theta^s(\mathbf{r},t) + 2\pi\nabla\times\int \mathrm{d}^3\mathbf{r}' G(\mathbf{r}-\mathbf{r}')\,\mathbf{n}(\mathbf{r}',t). \tag{2.8}$$

The single-valued part of $\theta(\mathbf{r},t)$ is denoted as $\theta^s(\mathbf{r},t)$, which is neglected from now on since it does not couple to the vortices and the transverse vector potential. This is because $\nabla\theta^s(\mathbf{r},t)$ gives rise to the longitudinal current while the vortices and vector potential are related to the transverse current. The vortex line density, $\mathbf{n}(\mathbf{r},t)$, is defined by

$$\mathbf{n}(\mathbf{r},t) = \sum_\nu \int \mathrm{d}\,l_\nu\,\frac{\partial\mathbf{r}_\nu(l_\nu,t)}{\partial l_\nu}\delta^{(3)}(\mathbf{r}-\mathbf{r}_\nu(l_\nu,t)), \tag{2.9}$$

where l_ν is the parameter along the ν-th vortex line and $\mathbf{r}_\nu(l_\nu,t)$ is the position of the ν-th vortex line element. In Eq. (2.8) $G(\mathbf{r})$ is defined as

$$G(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{k^2}e^{-i\,\mathbf{k}\cdot\mathbf{r}}. \tag{2.10}$$

Since the vortices are almost aligned with the external magnetic field both in the lattice state and liquid state, we take as l_ν the z-coordinate of the vortex line elements. Therefore $\mathbf{r}_\nu(l_\nu,t)$ is written as $\mathbf{r}_\nu(z,t) = (x_\nu(z,t), y_\nu(z,t), z)$. Note that, from the continuity condition of vortex lines, which is required since each vortex line does not have ends except at the boundary of the sample, $\mathbf{n}(\mathbf{r},t)$ satisfies the relation,

$$\nabla\cdot\mathbf{n}(\mathbf{r},t) = 0. \tag{2.11}$$

Substituting Eq. (2.8) into Eq. (2.2) and Eq. (2.5) and neglecting $\theta^s(\mathbf{r},t)$ we obtain the free energy of the vortex system as follows,

$$
\begin{aligned}
\mathcal{G}_c \equiv& \mathcal{G}_v + \mathcal{G}_{v-em}, \\
\mathcal{G}_v =& \frac{K}{2}\int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{r}' G_v(\mathbf{r}-\mathbf{r}')\, \delta\mathbf{n}(\mathbf{r},t)\cdot\delta\mathbf{n}(\mathbf{r}',t), \\
\mathcal{G}_{v-em} =& \frac{1}{c}\int \mathrm{d}^3\mathbf{r}\; \mathbf{j}^p(\mathbf{r},t)\cdot\mathbf{a}(\mathbf{r},t) + \frac{K}{2{\phi_0}^2}\int \mathrm{d}^3\mathbf{r}\; \{\mathbf{a}(\mathbf{r},t)\}^2,
\end{aligned}
\tag{2.12}
$$

where

$$\delta\mathbf{n}(\mathbf{r},t) \equiv \mathbf{n}(\mathbf{r},t) - n_B\hat{\mathbf{z}}, \tag{2.13}$$

where we defined the average vortex density as $n_B \equiv B/\phi_0$. The current, $\mathbf{j}^p(\mathbf{r},t)$, given by

$$
\begin{aligned}
\mathbf{j}^s(\mathbf{r},t) \equiv& \mathbf{j}^p(\mathbf{r},t) + \mathbf{j}^d(\mathbf{r},t), \\
\mathbf{j}^p(\mathbf{r},t) =& \frac{cK}{\phi_0}\int \mathrm{d}^3\mathbf{r}' G_v(\mathbf{r}-\mathbf{r}')\, \nabla'\times\delta\mathbf{n}(\mathbf{r}',t), \\
\mathbf{j}^d(\mathbf{r},t) =& -\frac{cK}{{\phi_0}^2}\mathbf{a}(\mathbf{r},t).
\end{aligned}
\tag{2.14}
$$

$\mathbf{j}^p(\mathbf{r},t)$ is the paramagnetic current caused by the vortices and $\mathbf{j}^d(\mathbf{r},t)$ is the diamagnetic screening current associated with $\mathbf{a}(\mathbf{r},t)$. The vortex Green function, $G_v(\mathbf{r})$, should be cut off at short distance of the order of the superconducting coherence length, ξ_{sc}, corresponding to the destruction of the GL order parameter amplitude at the vortex cores and we put

$$G_v(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \frac{e^{-\xi_{sc}|\mathbf{k}|}}{k^2} e^{-i\,\mathbf{k}\cdot\mathbf{r}}. \tag{2.15}$$

Here we comment on the gauge invariance of our theory. The gauge transformation changes $\theta^s(\mathbf{r},t)$ and the longitudinal part of vector potential (we do not consider the singular gauge transformation). Therefore the gauge invariance is undertaken by these degrees of freedom, which are neglected since we are not interested in them in this thesis. Therefore the free energy, given in Eq. (2.12), is written by gauge invariant quantities. Especially, we should note that the vortex position is a gauge invariant quantitiy.

2.2 Dynamical superfluid density: linear response theory

The superconducting properties are best described by the superfluid density tensor, $\rho^s(\mathbf{k},\omega)$, defined by

$$j^s_\alpha(\mathbf{k},\omega) = -\frac{e^{*2}}{m^* c}\rho^s_{\alpha\beta}(\mathbf{k},\omega)a_\beta(\mathbf{k},\omega), \tag{2.16}$$

where $\alpha,\beta = \{x,y,z\}$, and $\mathbf{a}(\mathbf{k},\omega)$ and $\mathbf{j}^s(\mathbf{k},\omega)$ are Fourier transforms of $\mathbf{a}(\mathbf{r},t)$ and $\mathbf{j}^s(\mathbf{r},t)$, respectively. Without vortices, $\rho^s_{\alpha\beta}(\mathbf{k},\omega)$ reduces to the "bare" superfluid density, $\rho^s_0\delta_{\alpha\beta}$, since in that case $\mathbf{j}_p(\mathbf{r},t) = 0$ in Eq. (2.14). Here we note that $\rho^s_0\delta_{\alpha\beta}$ can be written as $\rho^s_0\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)$, since we are interested in the transverse vector potential satisfying $\mathbf{k}\cdot\mathbf{a}(\mathbf{k},\omega) = 0$.

The linear response theory tells us that the superfluid density can be expressed in terms of the equilibrium autocorrelation function of the supercurrent caused by vortices (paramagnetic supercurrent) as

$$\rho^s_{\alpha\beta}(\mathbf{k},\omega) = \rho^s_0\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) + \frac{m^*}{e^{*2}}\frac{1}{k_B T}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\frac{\omega'\, Q_{\alpha\beta}(\mathbf{k},\omega')}{\omega'-\omega-i\epsilon}, \tag{2.17}$$

where

$$Q_{\alpha\beta}(\mathbf{k},\omega) = \int_{-\infty}^{\infty}\mathrm{d}t\int \mathrm{d}^3\mathbf{r}\left\{\left\langle j^p_\alpha(\mathbf{r},t)j^p_\beta(0,0)\right\rangle - \left\langle j^p_\alpha(\mathbf{r},t)\right\rangle\left\langle j^p_\beta(0,0)\right\rangle\right\} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}. \tag{2.18}$$

Here $\langle\cdots\rangle$ means the average in the equilibrium and ϵ is an infinitesimally small positive constant introduced to satisfy the Kramers-Kronig relation. The first term of Eq. (2.17) is originated from the diamagnetic current, denoted as $\mathbf{j}^d(\mathbf{r},t)$ in Eq. (2.14).

By taking into account the symmetry of the mixed state (the system has uniaxial anisotropy in z-direction), it is convenient to go to the following new coordinate system. We define the unit vectors, $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}^{(in)}$, and $\hat{\mathbf{k}}^{(out)}$, which are orthogonal each other, by

$$
\begin{aligned}
\hat{\mathbf{k}} \equiv& \frac{\mathbf{k}}{k} \\
\hat{\mathbf{k}}^{(in)} \equiv& \frac{\mathbf{k}_\perp\times\hat{\mathbf{z}}}{k_\perp} \\
\hat{\mathbf{k}}^{(out)} \equiv& \hat{\mathbf{k}}\times\hat{\mathbf{k}}^{(in)} = \frac{k_z}{k}\frac{\mathbf{k}_\perp}{k_\perp} - \frac{k_\perp}{k}\hat{\mathbf{z}},
\end{aligned}
\tag{2.19}
$$

where $\mathbf{k}_\perp \equiv (k_x, k_y, 0)$. The configurations of $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}^{(in)}$ and $\hat{\mathbf{k}}^{(out)}$ are shown in Fig. 1. Here we also introduce the following spherical coordinate,

$$\mathbf{k} = (k_x, k_y, k_z) = (k\cos\vartheta\cos\varphi, k\cos\vartheta\sin\varphi, k\sin\vartheta). \tag{2.20}$$

Since $\nabla\cdot\mathbf{a}(\mathbf{r},t) = 0$ there are only two independent components in $\mathbf{a}(\mathbf{r},t)$, i.e. $\mathbf{a}(\mathbf{r},t)$ can be written as

$$\mathbf{a}(\mathbf{k},\omega) = a^{(in)}(\mathbf{k},\omega)\, i\hat{\mathbf{k}}^{(in)} + a^{(out)}(\mathbf{k},\omega)\, \hat{\mathbf{k}}^{(out)}. \tag{2.21}$$

The imaginery, i, in the first term of r.h.s. is inserted in order to satisfy the condition, $a^{(in)*}(\mathbf{k},\omega) = a^{(in)}(-\mathbf{k},-\omega)$.

In this thesis we disregarded the Magnus force. This is a good approximation for such superconductors with strong dissipation as the high-T_c cuprates. Therefore, in the vortex density autocorreltaion function and superfluid density, the terms corresponding to the Hall effect which behave like $\hat{\mathbf{k}}^{(in)}_\alpha \hat{\mathbf{k}}^{(out)}_\beta$ do not appear. For this reason, the autocorreltaion function of the supercurrent and the superfluid density can be decomposed as

$$\begin{aligned} \rho^s_{\alpha\beta}(\mathbf{k},\omega) =& \rho^{s(in)}(\mathbf{k},\omega)\ P^{(in)}_{\alpha\beta} + \rho^{s(out)}(\mathbf{k},\omega)\ P^{(out)}_{\alpha\beta}, & (2.22)\\ Q_{\alpha\beta}(\mathbf{k},\omega) =& Q^{(in)}(\mathbf{k},\omega)\ P^{(in)}_{\alpha\beta} + Q^{(out)}(\mathbf{k},\omega)\ P^{(out)}_{\alpha\beta}, & (2.23)\end{aligned}$$

where

$$\begin{aligned} P^{(in)}_{\alpha\beta} &= \hat{k}^{(in)}_\alpha \hat{k}^{(in)}_\beta \\ P^{(out)}_{\alpha\beta} &= \hat{k}^{(out)}_\alpha \hat{k}^{(out)}_\beta. \end{aligned} \tag{2.24}$$

The paramagnetic supercurrent is expressed in terms of the vortex density as

$$\begin{aligned} \mathbf{j}^p(\mathbf{k},\omega) =& -\frac{cK}{\phi_0}\frac{i\mathbf{k}}{k^2}\times\delta\mathbf{n}(\mathbf{k},\omega), \\ =& \frac{cK}{\phi_0}\frac{1}{k^2}\left\{\hat{\mathbf{k}}^{(out)}\delta n^{(in)}(\mathbf{k},\omega) + i\ \hat{\mathbf{k}}^{(in)}\delta n^{(out)}(\mathbf{k},\omega)\right\}, \end{aligned} \tag{2.25}$$

where $\delta n^{(in)}(\mathbf{k},\omega)$ and $\delta n^{(out)}(\mathbf{k},\omega)$ are defined by

$$\delta\mathbf{n}(\mathbf{k},\omega) \equiv i\ \hat{\mathbf{k}}^{(in)}\delta n^{(in)}(\mathbf{k},\omega) + \hat{\mathbf{k}}^{(out)}\delta n^{(out)}(\mathbf{k},\omega). \tag{2.26}$$

Therefore the autocorrelation function of paramagnetic current is expressed as

$$\begin{aligned} \left\langle j^p_\alpha(\mathbf{k},\omega) j^p_\beta(-\mathbf{k},-\omega)\right\rangle =& \left(\frac{cK}{\phi_0}\right)^2 \frac{1}{k^2}\left\{\left\langle \delta n^{(in)}(\mathbf{k},\omega)\delta n^{(in)}(-\mathbf{k},-\omega)\right\rangle P^{(out)}_{\alpha\beta}\right. \\ &\left. + \left\langle \delta n^{(out)}(\mathbf{k},\omega)\delta n^{(out)}(-\mathbf{k},-\omega)\right\rangle P^{(in)}_{\alpha\beta}\right\}. \end{aligned} \tag{2.27}$$

By defining the autocorrelation function of vortex density (or the dynamical form factor of the vortices) by

$$\begin{aligned} S_{\alpha\beta}(\mathbf{k},\omega) =& \int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}^3\mathbf{r} \Big\{ \langle \delta n_\alpha(\mathbf{r},t)\delta n_\beta(\mathbf{0},0)\rangle - \langle \delta n_\alpha(\mathbf{r},t)\rangle \langle \delta n_\beta(\mathbf{0},0)\rangle \Big\} e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \\ \equiv& S^{(in)}(\mathbf{k},\omega) P^{(in)}_{\alpha\beta} + S^{(out)}(\mathbf{k},\omega) P^{(out)}_{\alpha\beta}, \end{aligned} \tag{2.28}$$

the correlation functions, $Q^{(in)}(\mathbf{k},\omega)$ and $Q^{(out)}(\mathbf{k},\omega)$, can be described as

$$\begin{aligned} Q^{(in)}(\mathbf{k},\omega) &= \left(\frac{cK}{\phi_0}\right)^2 \frac{1}{k^2} S^{(out)}(\mathbf{k},\omega), \\ Q^{(out)}(\mathbf{k},\omega) &= \left(\frac{cK}{\phi_0}\right)^2 \frac{1}{k^2} S^{(in)}(\mathbf{k},\omega). \end{aligned} \tag{2.29}$$

It should be noted that the correlation functions of vortex density in Eq. (2.29) must not include the screening effects by the internal field fluctuation.

A. Static superfluid density

If we take the limit of $\omega \to 0$ in the dynamical superfluid density we obtain the static superfluid density, $\rho^s_{\alpha\beta}(\mathbf{k},\omega\to 0)$, which describes the response of the system to the static vector potential. Therefore since this quantity is closely related to the properties of the thermodynamic equilibrium, we can characterize the superconducting order from this quantity.

B. Electric conductivity

The electric current, $\mathbf{j}^{el}(\mathbf{k},\omega)$, can be expressed using the dynamical superfluid density, $\rho^s_{\alpha\beta}(\mathbf{k},\omega)$, defined in Eq. (2.16) as [29],

$$\begin{aligned} j^{el}_\alpha(\mathbf{k},\omega) =& \left\{\sigma^n\delta_{\alpha\beta} + \frac{e^{*2}}{m^*}\frac{\rho^s_{\alpha\beta}(\mathbf{k},\omega)}{i\omega+\epsilon}\right\} E_\beta(\mathbf{k},\omega) \\ \equiv& \sigma_{\alpha\beta}(\mathbf{k},\omega)\ E_\beta(\mathbf{k},\omega), \end{aligned} \tag{2.30}$$

where σ^n is the electric conductivity of the normal component, which originates from $\mathbf{j}^n(\mathbf{k},\omega) = \sigma^n \mathbf{E}(\mathbf{k},\omega)$. We can obtain the macroscopic electric conductivity, $\sigma(\omega)$, from Eq. (2.30) taking the limit of $\mathbf{k} \to \mathbf{0}$.

C. Magnetic permeability

Next we comment on the relation between the magnetic susceptibility and the superfluid density. From Eq. (2.7) and Eq. (2.16) we obtain

$$k^2 \left\{a_\alpha - a_\alpha^{ext}(\mathbf{k},\omega)\right\} = -\frac{1}{\lambda^2 \rho_0^s} \rho_{\alpha\beta}^s(\mathbf{k},\omega) a_\beta(\mathbf{k},\omega), \tag{2.31}$$

which can be further expressed into

$$\frac{a^{(in,out)}(\mathbf{k},\omega)}{a^{ext\,(in,out)}(\mathbf{k},\omega)} = \left\{1 + \frac{1}{\lambda^2 \rho_0^s k^2} \rho^{s\,(in,out)}(\mathbf{k},\omega)\right\}^{-1}. \tag{2.32}$$

Here we consider the static limit, $\omega \to 0$. The external magnetic field perturbation and the internal magnetic field fluctuation can be expressed as

$$\begin{aligned} \delta\mathbf{H}^{ext}(\mathbf{k},\omega) =& i\ \hat{\mathbf{k}}^{(in)} \delta H^{ext\,(in)}(\mathbf{k},\omega) + \hat{\mathbf{k}}^{(out)} \delta H^{ext\,(out)}(\mathbf{k},\omega), \\ \delta\mathbf{B}(\mathbf{k},\omega) =& i\ \hat{\mathbf{k}}^{(in)} \delta B^{(in)}(\mathbf{k},\omega) + \hat{\mathbf{k}}^{(out)} \delta B^{(out)}(\mathbf{k},\omega), \end{aligned} \tag{2.33}$$

with

$$\begin{aligned} \delta H^{ext\,(in)}(\mathbf{k},\omega) = ik a^{ext\,(out)}(\mathbf{k},\omega), \qquad & \delta H^{ext\,(out)}(\mathbf{k},\omega) = ik a^{ext\,(in)}(\mathbf{k},\omega), \\ \delta B^{(in)}(\mathbf{k},\omega) = ik a^{(out)}(\mathbf{k},\omega), \qquad & \delta B^{(out)}(\mathbf{k},\omega) = ik a^{(in)}(\mathbf{k},\omega). \end{aligned} \tag{2.34}$$

Here we define the wave vector and frequency dependent "local "magnetic permeabilities, $\mu^{(in)}(\mathbf{k},\omega;\mathbf{H}^a)$ and $\mu^{(out)}(\mathbf{k},\omega;\mathbf{H}^a)$, and "local "magnetic susceptibilities, $\chi^{(in)}(\mathbf{k},\omega;\mathbf{H}^a)$ and $\chi^{(out)}(\mathbf{k},\omega;\mathbf{H}^a)$ as follows,

$$\mu^{(in)}(\mathbf{k},\omega;\mathbf{H}^a) = \frac{\delta B^{(in)}(\mathbf{k},\omega)}{\delta H^{ext\,(in)}(\mathbf{k},\omega)} = \frac{a^{(out)}(\mathbf{k},\omega)}{a^{ext\,(out)}(\mathbf{k},\omega)}, \tag{2.35}$$

$$\mu^{(out)}(\mathbf{k},\omega;\mathbf{H}^a) = \frac{\delta B^{(out)}(\mathbf{k},\omega)}{\delta H^{ext\,(out)}(\mathbf{k},\omega)} = \frac{a^{(in)}(\mathbf{k},\omega)}{a^{ext\,(in)}(\mathbf{k},\omega)},$$

$$\begin{aligned} \mu^{(in)}(\mathbf{k},\omega;\mathbf{H}^a) =& 1 + 4\pi\chi^{(in)}(\mathbf{k},\omega;\mathbf{H}^a) \\ \mu^{(out)}(\mathbf{k},\omega;\mathbf{H}^a) =& 1 + 4\pi\chi^{(out)}(\mathbf{k},\omega;\mathbf{H}^a), \end{aligned} \tag{2.36}$$

where "local "means that these quantities represent the magnetic properties of the system under the applied magnetic field, $\mathbf{H}^a$. In order to compare these with the conventional definitions of the permeabilities and the susceptibilities we also define the frequency independent total permeabilitiy, μ^{tot}, and differential permeability, μ^{dif}, and the susceptibilities corresponding to each as

$$\begin{aligned} \mu^{tot}(H^a) =& \frac{B(H^a)}{H^a}, \qquad & \mu^{dif}(H^a) = \frac{\partial B(H^a)}{\partial H^a}, \\ \mu^{tot}(H^a) =& 1 + 4\pi\chi^{tot}(H^a), \qquad & \mu^{dif}(H^a) = 1 + 4\pi\chi^{dif}(H^a). \end{aligned} \tag{2.37}$$

From the physical considerations, we obtain the following relations in various limitimg cases,

$$\lim_{\mathbf{k}\to\mathbf{0}} \mu^{(in)}(\mathbf{k},\omega=0;\mathbf{H}^a) = \mu^{tot}(H^a), \tag{2.38}$$

$$\lim_{\mathbf{k}\to\mathbf{0}+\epsilon\hat{\mathbf{z}}} \mu^{(out)}(\mathbf{k},\omega=0;\mathbf{H}^a) = \mu^{tot}(H^a), \tag{2.39}$$

$$\lim_{\mathbf{k}\to\mathbf{0}+\epsilon\hat{\mathbf{x}}} \mu^{(out)}(\mathbf{k},\omega=0;\mathbf{H}^a) = \mu^{dif}(H^a), \tag{2.40}$$

where $\mathbf{k} \to \mathbf{0} + \epsilon\hat{\mathbf{x}}$ means that the limit is taken along the x-axis. The reason for these are understood as follows. The first two permeabilities correspond to the perturbation, $\delta\mathbf{H}^{ext}$, perpendicular to $\mathbf{H}^a$, namely the perturbations tilting the external field, $\mathbf{H}^a$, without changing the magnitude of $\mathbf{H}^{ext}$. Therefore they correspond to $\mu^{tot}(H^a)$. The last permeability correspond to the perturbation, $\delta\mathbf{H}^{ext}$, parallel to $\mathbf{H}^a$ which changes the magnitude of $\mathbf{H}^{ext}$. Therefore it gives $\mu^{dif}(H^a)$. In this thesis we discuss the permeability closely especially in the vortex lattice state.

3. Vortex lattice state

3.1 Elastic free energy

In this section we introduce the effective free energy which describes the low energy fluctuations of the vortex lattice state. Our calculations are based on the method of elastic theory [6, 15]. Since we are interested in the electromagnetic response of the vortex lattice in this thesis, we should be careful in treating the vector potential. In order to make the role of the vector potential fluctuations clear, we do not integrate it out in constructing the elastic free energy, but keep it as another independent degree of freedom in the equation of motions.

We decompose the position of the vortex line element of the ν-th vortex line, $\mathbf{r}_\nu(z,t)$, into the equilibrium lattice point, $\mathbf{r}^0_\nu(z)$, and the displacements from them, $\mathbf{s}_\nu(z,t)$, namely,

$$\begin{aligned}\mathbf{r}_\nu(z,t) \equiv& \mathbf{r}^0_\nu(z) + \mathbf{s}_\nu(z,t),\\ \mathbf{r}^0_\nu(z) =& (x^0_\nu, y^0_\nu, z),\\ \mathbf{s}_\nu(z,t) =& (s_{\nu,x}(z,t), s_{\nu,y}(z,t), 0)\,.\end{aligned} \tag{3.1}$$

The vortices are assumed to form a regular triangular lattice in the equilibrium state and x^0_ν and y^0_ν are the x and y component of the triangular lattice points. The elastic free energy is written as

$$\begin{aligned}\mathcal{G}_v =& \mathcal{G}_0 + \mathcal{G}_{el}\\ \mathcal{G}_0 =& V\left[\frac{K}{2}n_B^2\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}G}}{G^2} + \frac{1}{8\pi}(B-H^a)^2\right]\end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{G}_{el} =& \frac{1}{2n_B^2}\int_D\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Big[c_{11}(\mathbf{k})k_\alpha k_\beta + c_{66}(\mathbf{k})(k_\perp^2\delta_{\alpha\beta} - k_\alpha k_\beta)\\ &+ c_{44}(\mathbf{k})k_z^2\delta_{\alpha\beta}\Big]s_\alpha(\mathbf{k},t)s_\beta(\mathbf{k},t),\end{aligned} \tag{3.3}$$

where $\{\alpha,\beta\} = \{x,y\}$, $\mathbf{G}$ is the reciprocal lattice vector of the vortex lattice and D denotes the region where k_x and k_y are limited to the first Brillouin zone of the triangular lattice and k_z is limited to $-k_c < k_z < k_c$ with $k_c \sim \xi_{sc}^{-1}$ because of the cutoff in the vortex-vortex interaction.

The internal magnetic field, B, is determined by the minimization the total free energy. In this thesis we neglecte the corrections to the equilibrium free energy due to thermal fluctuations and determine B by minimizing the mean field free energy, $\mathcal{G}_0$. As a result we obtain,

$$H^a = B + H_{c1}\frac{\ln(H_{c2}/B)}{2\ln\kappa}, \tag{3.4}$$

where $\kappa = \lambda/\xi_{sc}$, $H_{c1} = (\phi_0/4\pi\lambda^2)\ln\kappa$, and $H_{c2} = \phi_0/2\pi\xi_{sc}^2$. The elastic moduli are given by (see Appendix B)

$$\begin{aligned}c_{11}(\mathbf{k}) =& Kn_B^2\left(\frac{1}{k^2} - \frac{1}{16\pi n_B}\right) = \frac{B^2}{4\pi}\frac{1}{k^2\lambda^2} - \frac{BH_{c2}}{4\pi}\frac{1}{8\kappa^2}\\ c_{66}(\mathbf{k}) =& \frac{Kn_B}{16\pi} = \frac{BH_{c2}}{4\pi}\frac{1}{8\kappa^2}\\ c_{44}(\mathbf{k}) =& Kn_B^2\left(\frac{1}{k^2} - \frac{\ln(4\pi n_B\,\xi_{sc}^2)}{8\pi n_B}\right) = \frac{B^2}{4\pi}\frac{1}{k^2\lambda^2} + \frac{B(H^a-B)}{4\pi},\end{aligned} \tag{3.5}$$

and

$$\mathbf{s}(\mathbf{k},t) \equiv \sum_\mu\int\mathrm{d}z\;\mathbf{s}_\nu(z,t)\;e^{i\,\mathbf{k}\cdot\mathbf{r}^0_\mu(z,t)}. \tag{3.6}$$

Note that the compression and tilt modulus, $c_{11}(\mathbf{k})$ and $c_{44}(\mathbf{k})$, diverge in the $\mathbf{k}\to 0$ limit since we have not integrated out the gauge field fluctuation yet and, hence, the vortex-vortex interaction is long range at this stage. We confirm later that if we integrate out the fluctuations of vector potential in terms of the equations of motion, the vortex interaction is screened and we obtain the elastic moduli to be compared with the the conventional treatment [6].

We decompose $\mathbf{s}(\mathbf{k},t)$ into the longitudinal (compressional) part, $s^l(\mathbf{k},t)$, and the transverse (shear) part, $s^t(\mathbf{k},t)$, and rewrite the equations into,

$$\begin{aligned}\mathbf{s}(\mathbf{k},t) =& \mathbf{k}^{(l)}\;s^l(\mathbf{k},t) + \hat{\mathbf{k}}^{(in)}\;s^t(\mathbf{k},t),\\ n_B^2E^l(\mathbf{k}) =& c_{11}(\mathbf{k})\;k_\perp^2 + c_{44}(\mathbf{k})\;k_z^2,\\ n_B^2E^t(\mathbf{k}) =& c_{66}(\mathbf{k})\;k_\perp^2 + c_{44}(\mathbf{k})\;k_z^2,\end{aligned} \tag{3.7}$$

where $\mathbf{k}^{(l)} \equiv \mathbf{k}_\perp/k_\perp$. We can rewrite $\mathcal{G}_{el}$ as

$$\mathcal{G}_{el} = \frac{1}{2}\int_D\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[E^l(\mathbf{k})|s^l(\mathbf{k},t)|^2 + E^t(\mathbf{k})|s^t(\mathbf{k},t)|^2\right], \tag{3.8}$$

Employing the free energy derived in this section, we derive the equation of motion of $\mathbf{s}_\nu(z,t)$ in the next section.

3.2 Equations of motion of vortex lattice

In this section we describe the equations of motion of the elastic field, $\mathbf{s}_\nu(z,t)$. As stated in chapter 1, we concern ourselves with the case of strong dissipation where the Magnus force is negligible. The force on the vortices consists of two parts; the forces caused by the vector potential perturbation and the interaction with the other vortices. These two forces act in the direction opposite to each other. In this section, we disregard the force from the internal vector potential to estimate the density autocorrelation function of vortices without including the screening effect by internal vector potential. The mobility of the vortex per unit length, μ, is given by the inverse of the Bardeen-Stephen viscosity,

$$\mu = \frac{\phi_0^2 \sigma_n}{2\pi \xi_{sc}^2 c^2}, \tag{3.9}$$

Therefore the equations of motion for the lattice state is given by,

$$\frac{\partial \mathbf{s}_\nu(z,t)}{\partial t} = -\mu \frac{\delta \mathcal{G}_{el}}{\delta \mathbf{s}_\nu(z,t)} + \eta_\nu(z,t) \tag{3.10}$$

where $\eta_\nu(z,t)$ is the fluctuating Langevin force necessary to bring $\mathbf{s}_\nu(z,t)$ into thermodynamic equilibrium. The first term in the r.h.s. of Eq. (3.10) come from the Lorentz force caused by the other vortices.

By introducing the Fourier transformation with respect to time, t, Eq. (3.10) can be written as,

$$i\omega s^l(\mathbf{k},\omega) = -\Gamma E^l(\mathbf{k}) s^l(\mathbf{k},\omega) + R^l(\mathbf{k},\omega) \tag{3.11}$$

$$i\omega s^t(\mathbf{k},\omega) = -\Gamma E^t(\mathbf{k}) s^t(\mathbf{k},\omega) + R^t(\mathbf{k},\omega). \tag{3.12}$$

where $\Gamma = \mu n_B$, and $R_l(\mathbf{k},\omega)$ and $R_t(\mathbf{k},\omega)$ are the l- and t-components of the Langevin force, $\eta_\nu(z,t)$, respectively. In order to make the equations of motion, Eq. (3.11) and Eq. (3.12), lead to the correct expectation value in the equilibrium state, the Langevin force must satisfy the following relations,

$$\begin{aligned} \langle R^l(\mathbf{k},\omega)\rangle &= \langle R^t(\mathbf{k},\omega)\rangle = 0, \\ \langle R^{l,t}(\mathbf{k},\omega) R^{l,t}(\mathbf{k}',\omega')\rangle &= 4\pi k_B T \Gamma \, (2\pi)^4 \, \delta^{(3)}(\mathbf{k}+\mathbf{k}') \, \delta(\omega+\omega'). \end{aligned} \tag{3.13}$$

Here we comment on the relation to the well known elastic theory [6]. The difference between the two treatments lies in the treatment of the internal magnetic fluctuations, which, in our framework, are taken to be the perturbation to the vortex system whereas, in [6], they are integrated out as the modes attached to the vortices. This difference is crucial in treating the linear response of the vortex system. Since we are now interested in the elastic fluctuations in equilibrium, we put $\mathbf{a}^{ext}(\mathbf{r},t) = \mathbf{0}$. We can eliminate $a^{(in)}(\mathbf{k},\omega)$ and $a^{(out)}(\mathbf{k},\omega)$ by solving the Maxwell equation, Eq. (2.31), as

$$\begin{aligned} a^{(in)}(\mathbf{k},\omega) &= \frac{\phi_0}{\lambda^2} \frac{1}{k^2+\lambda^{-2}} s^l(\mathbf{k},\omega), \\ a^{(out)}(\mathbf{k},\omega) &= -\frac{\phi_0}{\lambda^2} \frac{1}{k^2+\lambda^{-2}} \frac{k_z}{k} s^l(\mathbf{k},\omega). \end{aligned} \tag{3.14}$$

Substituting the solution into Eq. (3.11) and Eq. (3.12) we obtain the effective equation of motions of the vortex lattice with the screened vortex-vortex interaction. As a result the three elastic moduli of Eq. (3.5) are modified to,

$$\begin{aligned} c_{11}^\lambda(\mathbf{k}) &= K n_B^2 \left(\frac{1}{k^2+\lambda^{-2}} - \frac{1}{16\pi n_B} \right) = \frac{B^2}{4\pi} \frac{1}{\lambda^2} \frac{1}{k^2+\lambda^{-2}} - \frac{B H_{c2}}{4\pi} \frac{1}{8\kappa^2} \\ c_{66}^\lambda(\mathbf{k}) &= \frac{K n_B}{16\pi} = \frac{B H_{c2}}{4\pi} \frac{1}{8\kappa^2} \\ c_{44}^\lambda(\mathbf{k}) &= K n_B^2 \left(\frac{1}{k^2+\lambda^{-2}} - \frac{\ln(4\pi n_B \, \xi_{sc}^2)}{8\pi n_B} \right) = \frac{B^2}{4\pi} \frac{1}{\lambda^2} \frac{1}{k^2+\lambda^{-2}} + \frac{B(H^a - B)}{4\pi}. \end{aligned} \tag{3.15}$$

Here we comment on the similarity and the dissimilarity with the conventional results derived by Brandt [6] in the London limit. In the Brandt's treatment the second terms in $c_{11}(\mathbf{k})$ and $c_{44}(\mathbf{k})$ are ignored, whereas, in our theory, these terms have rather important meaning; first they are related to the diamagnetic susceptibility as shown in the next section, secondly the following sum rule holds for our results,

$$\left\{ E^l(\mathbf{k}) + E^t(\mathbf{k}) \right\} \Big|_{k_z=0} = K, \tag{3.16}$$

which is called the Kohn's sum rule satisfied in the two dimensional Wigner crystal [26]. It is interesting to see that these terms come from the lattice sum over the reciprocal lattice as shown in Appendix B. In the elastic moduli derived by Brandt in the region near H_{c2}, the similar terms appear because of the attractive interactions between the vortex cores. We believe that the origin of them is different since our results are derived for $\xi_{sc} >> k_{BZ}$ whereas the Brandt's results are for the region $H^a \lesssim H_{c2}$ and $\xi_{sc} \sim k_{BZ}$.

We can estimate the melting temperature from these. Let us employ the Lindemann criterion [14, 15]; a simple way to estimate the melting temperature, T_m. Hence T_m is given by the relation, $\langle \mathbf{s}_\nu(z,t)^2 \rangle^{\frac{1}{2}} = c_L\, d$, where d is the lattice constant given (in case of the triangular lattice) by $\{2/(\sqrt{3}n_B)\}^{\frac{1}{2}}$ and c_L is the empirical constant of the order of ~ 0.1. Now $\langle \mathbf{s}_\nu(z,t)^2 \rangle$ can be calculated as

$$\begin{aligned}\langle \mathbf{s}_\nu(z,t)^2 \rangle =& \frac{1}{n_B^2 V} \sum_{\mathbf{k},\mathbf{k}' \in D} \langle \mathbf{s}(\mathbf{k},t) \cdot \mathbf{s}(\mathbf{k}',t) \rangle \; e^{-i(\mathbf{k}+\mathbf{k}')\cdot \mathbf{r}_\nu^0(z)} \\ =& \frac{k_B T_m}{n_B^2} \int_D \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left\{ \frac{1}{\tilde{E}^l(\mathbf{k})} + \frac{1}{\tilde{E}^t(\mathbf{k})} \right\}, \end{aligned} \tag{3.17}$$

where $\tilde{E}^l(\mathbf{k})$ and $\tilde{E}^t(\mathbf{k})$ are obtained by replacing $c_{11}(\mathbf{k})$, $c_{66}(\mathbf{k})$ and $c_{44}(\mathbf{k})$ by $c_{11}^\lambda(\mathbf{k})$, $c_{66}^\lambda(\mathbf{k})$ and $c_{44}^\lambda(\mathbf{k})$ in $E^l(\mathbf{k})$ and $E^t(\mathbf{k})$. We approximate the integration over the first Brillouin zone in Eq. (3.17) by integration over the circle of same area (the radius is $\sqrt{4\pi n_B}$ which is denoted as k_{BZ}). By carrying out the integration numerically we obtain the melting temperature as

$$k_B T_m = f\; c_L^2 \left\{ \frac{\phi_0}{B} \right\}^{\frac{1}{2}} \frac{\phi^2}{4\pi\lambda^2}, \tag{3.18}$$

where f is the numerical constant comes from the integration over the region D. Although f depends on the GL parameter κ and the density of the vortices, n_B, it is a constant of the order of 1: e.g. $f \sim 0.2-0.3$ for $\xi_{sc} = 10$ (Å) and $B = 0.25(T)$. This result is almost same as one obtained in [15].

3.3 Dynamical superfluid density in vortex lattice state

We calculate the superfluid density in the vortex lattice state. From Eq. (2.17) and Eq. (2.29), we see that the dynamical superfluid density is determined by calculating the density autocorrelation function, $S_{\alpha\beta}(\mathbf{k},\omega)$. Therefore we calculate $S_{\alpha\beta}(\mathbf{k},\omega)$ first.

$S^{(in)}(\mathbf{k},\omega)$ and $S^{(out)}(\mathbf{k},\omega)$ defined Eq. (2.28) are given to the second order in $\mathbf{s}(\mathbf{k},\omega)$ as

$$\begin{aligned} S^{(in)}(\mathbf{k},\omega) =& \Omega^{-1} \left\langle \delta n^{(in)}(\mathbf{k},\omega) \delta n^{(in)}(-\mathbf{k},-\omega) \right\rangle \\ =& \Omega^{-1}\; k_z^2 \langle s^t(\mathbf{k},\omega) s^t(-\mathbf{k},-\omega) \rangle, \\ S^{(out)}(\mathbf{k},\omega) =& \Omega^{-1} \left\langle \delta n^{(out)}(\mathbf{k},\omega) \delta n^{(out)}(-\mathbf{k},-\omega) \right\rangle \\ =& \Omega^{-1}\; k^2 \langle s^l(\mathbf{k},\omega) s^l(-\mathbf{k},-\omega) \rangle, \end{aligned} \tag{3.19}$$

as shown in Appendix C, where we defined $\Omega \equiv (2\pi)^4\, \delta^{(4)}(0)$. By Eq. (3.11), Eq. (3.12) and Eq. (3.13) the correlations of $s^l(\mathbf{k},\omega)$ and $s^t(\mathbf{k},\omega)$ are given as

$$\langle s^{l,t}(\mathbf{k},\omega) s^{l,t}(\mathbf{k}',\omega') \rangle = \frac{2\Gamma k_B T (2\pi)^4 \delta^{(3)}(\mathbf{k}+\mathbf{k}')\, \delta(\omega+\omega')}{\omega^2 + \{\Gamma E^{l,t}(\mathbf{k})\}^2}. \tag{3.20}$$

Therefore the superfluid density, $\rho^{s\,(in)}(\mathbf{k},\omega)$ and $\rho^{s\,(out)}(\mathbf{k},\omega)$ defined by Eq. (2.22) is obtained as

$$\begin{aligned} \rho^{s\,(in)}(\mathbf{k},\omega) =& \rho_0^s \left\{ 1 - \frac{\Gamma}{i\omega + \Gamma E^l(\mathbf{k})} \right\}, \\ =& \rho_0^s \left\{ 1 - \frac{1}{i\omega/\Gamma K + 1 - c_1 k_\perp^2 + c_2 k_z^2} \right\}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \rho^{s\,(out)}(\mathbf{k},\omega) =& \rho_0^s \left\{ 1 - \left(\frac{k_z}{k} \right)^2 \frac{\Gamma}{i\omega + \Gamma E^t(\mathbf{k})} \right\}, \\ =& \rho_0^s \left\{ 1 - \frac{(k_z/k)^2}{i\omega/\Gamma K + (k_z/k)^2 + c_1 k_\perp^2 + c_2 k_z^2} \right\}, \end{aligned} \tag{3.22}$$

where

$$c_1 = 1/16\pi n_B, \tag{3.23}$$

$$c_2 = -(1/8\pi n_B) \ln(4\pi n_B \xi_{sc}^2) = (H^a - B)\lambda^2/B. \tag{3.24}$$

Here we should note that whether the system is superconducting or not in the thermodynamic limit is determined by taking the limit of $\omega \to 0$ first and then taking the limit of $\mathbf{k} \to \mathbf{0}$. On the other hand whether the electric conductivity of the system is divergent or not is determined by taking the limit in the opposite way. Let us consider the both cases separately.

A. Static superfluid density

If we take $\omega \to 0$ limit first our result reduces to the static superfluid density,

$$\rho^{s\,(in)}(\mathbf{k},0) = \rho_0^s \left\{1 - \frac{1}{1 - c_1 k_\perp^2 + c_2 k_z^2}\right\}, \tag{3.25}$$

$$\rho^{s\,(out)}(\mathbf{k},0) = \rho_0^s \left\{1 - \frac{(k_z/k)^2}{(k_z/k)^2 + c_1 k_\perp^2 + c_2 k_z^2}\right\}. \tag{3.26}$$

Whether the system is in the superconducting state or not is determined by the small $\mathbf{k}$ behavior of these static superfluid density. In general if $\rho^s(\mathbf{k}) \to 0$ in the limit of $\mathbf{k} \to \mathbf{0}$ the system is nonsuperconducting and no static supercurrent can flow. On the other hand if $\rho^s(\mathbf{k}) \neq 0$ in the limit of $\mathbf{k} \to \mathbf{0}$ the static supercurrent can flow in the system.

Although it can easily be understood that $\rho^{s\,(in)}(\mathbf{k}) \to 0$ in limit $\mathbf{k} \to \mathbf{0}$, as known as the "flux flow "behavior, the behavior of $\rho^{s\,(out)}(\mathbf{k})$ is rather complicated. First, if we take $\mathbf{k} \to \mathbf{0}$ limit with $\mathbf{k}$ keeping parallel to the z-axis (i.e. $\mathbf{k} = (0,0,k_z)$), we obtain $\rho^{s\,(out)}(\mathbf{k}) \to 0$. This is because if $\mathbf{k}$ is parallel to $\hat{\mathbf{z}}$, $\hat{\mathbf{k}}^{(out)}$ is perpendicular to $\hat{\mathbf{z}}$ and $a^{(out)}(\mathbf{k})$ is also perpendicular to $\hat{\mathbf{z}}$. (see Fig. 1) On the other hand if we take $\mathbf{k} \to \mathbf{0}$ limit with $\mathbf{k}$ keeping perpendicular to the z-axis (i.e. $\mathbf{k} = (k_\perp, 0, 0)$ for example), $\rho^{s\,(out)}(\mathbf{k}) \to \rho_0^s$, i.e. the system behaves like a superconductor. In this case $a^{(out)}(\mathbf{k})$ is parallel to $\hat{\mathbf{z}}$. Therefore we conclude that the vortex lattice state is superconducting only in the direction of the externally applied magnetic field. It should be noted that the existence of the shear modulus is essential to the result, which corresponds to the $c_1 k_\perp^2$ term in the denominator of the second term of Eq. (3.26).

B. Electric conductivity

The electric conductivity can be obtained from the dynamical superfluid density by applying the formula Eq. (2.30). The two formulas of superfluid density denoted by *in* and *out* in Eq. (3.21) and Eq. (3.22) are combined into the one formula in $\mathbf{k} \to \mathbf{0}$ limit and the electric conductivity is expressed as,

$$\sigma(\theta,\omega) = \sigma_n + \frac{e^{*2}\rho_0^s}{m^*}\frac{1}{i\omega + \Gamma K \sin^2\theta}, \tag{3.27}$$

where θ is the angle between $\mathbf{k}$ and the xy plane introduced in Eq. (2.20) and shown in Fig. 1. It should be noted that $\sigma(\omega,\vartheta)$ diverges at $\vartheta = 0, \pi$ in the limit

of $\omega \to 0$, which is the characteristic form of the superconductors. Therefore we conclude that the vortex lattice has infinite conductivity in the direction parallel to the magnetic field. In the following section we examine whether this result holds even when the effects of the thermal fluctuation are considered.

C. Magnetic permeability

Now it is interesting to see how the superfluid density is related to the diamagnetic permeability. Here we limit our discussions to the static case, $\omega = 0$. By Eq. (3.21) and Eq. (3.22), we obtain the expression of the "local "permeability, $\mu^{(in)}(\mathbf{k};\mathbf{H}^a)$ and $\mu^{(out)}(\mathbf{k};\mathbf{H}^a)$, defined in Eq. (2.35) as

$$\mu^{(in)}(\mathbf{k};\mathbf{H}^a) = \left\{1 + \frac{1}{\lambda^2}(c_1\frac{k_\perp^2}{k_z^2} + c_2)\right\}^{-1} = \left\{1 + \frac{1}{\lambda^2}(c_1 \cot^2\theta + c_2)\right\}^{-1}, \tag{3.28}$$

$$\mu^{(out)}(\mathbf{k};\mathbf{H}^a) = \left\{1 + \frac{1}{\lambda^2}(-c_1\frac{k_\perp^2}{k^2} + c_2\frac{k_z^2}{k^2})\right\}^{-1} = \left\{1 + \frac{1}{\lambda^2}(-c_1\cos^2\theta + c_2\sin^2\theta)\right\}^{-1}, \tag{3.29}$$

for small $\mathbf{k}$ limit. Here $\mu^{(in)}(\mathbf{k};\mathbf{H}^a)$ has a divergence in the denominator at $\theta = 0$ corresponding to the nonvanishing superfluid density in the direction of the external magnetic field, $\mathbf{H}^a$.

Here we consider the two special cases, i.e. $\mathbf{H}^a \perp \delta\mathbf{H}^{ext}$ and $\mathbf{H}^a // \delta\mathbf{H}^{ext}$.

(1) The case of $\mathbf{H}^a \perp \delta\mathbf{H}^{ext}$.

This case corresponds to Eq. (2.38) and Eq. (2.39). In this configuration $\delta\mathbf{H}^{ext}$ works so as to tilt $\mathbf{H}^a$. From the result of the elastic theory we have the "local "permeability as,

$$\mu^{(in)}(\theta) = \left\{1 + \frac{1}{\lambda^2}(c_1\cot^2\theta + c_2)\right\}^{-1} \neq \mu^{tot}, \tag{3.30}$$

$$\mu^{(out)}\Big|_{\theta=\frac{\pi}{2}} = \left(1 + \frac{c_2}{\lambda^2}\right)^{-1} = \frac{B}{H^a} = \mu^{tot}. \tag{3.31}$$

We used Eq. (3.24) in the last equation. In Eq. (3.31) the relation $\mathbf{H}^{ext} + \delta\mathbf{H}^{ext} // \mathbf{B} + \delta\mathbf{B}$ is naturally satisfied but is not satisfied in Eq. (3.30) except when $\theta = \pi/2$. We think that the discrepancy is attributed to the dependence of the magnetic susceptibility on the penetration process of the magnetic filed in vortex state.

(2) The case of $\mathbf{H}^a // \delta\mathbf{H}^{ext}$.

This case corresponds to the Eq. (2.40). In this configuration the perturbation, $\delta\mathbf{H}^{ext}$, works so as to increase the magnitude of $\mathbf{H}^a$. Therefore the "local "magnetic permeability in $\mathbf{k} \to 0$ limit is directly related to the differential permeability. From the results of the elastic theory we obtain,

$$\mu^{(out)}\Big|_{\theta=0} = \left(1 - \frac{c_1}{\lambda^2}\right)^{-1} = \left(1 - \frac{\phi_0}{16\pi B\lambda^2}\right)^{-1}. \tag{3.32}$$

Therefore we obtain the differential permeability as

$$\mu_{el}^{dif}(H^a) = \left(1 - \frac{\phi_0}{16\pi B\lambda^2}\right)^{-1}, \tag{3.33}$$

where the suffix "el "is added to specify the result by the elastic theory. In contrast to these results, the mean field equation, Eq. (3.4), gives us,

$$\mu_{m.f.}^{def}(H^a) = \left(1 - \frac{\phi_0}{8\pi B\lambda^2}\right)^{-1}. \tag{3.34}$$

Here we recognize the serious disagreement between the two. The calculation by the elastic theory, Eq. (3.33), gives only a half contribution to the first term of the denominator of r.h.s. in contrast to the mean field result, Eq. (3.34).

The reason of the discrepancy is discussed in the following section, where the effects of the thermal fluctuations are taken into account.

3.4 Corrections due to thermal fluctuations to superfluid density

A. Helical deformation of vortices

In this section we examine whether the nonvanishing superfluid density in the direction of the magnetic field is stable or not when the effects of the thermal fluctuation is taken into account. This is nothig but a consideration of the corrections of the higher order term in $\mathbf{s}(\mathbf{k},\omega)$ to the dynamical form factor, especially to $S^{(in)}(\mathbf{k},\omega)$. We denote the thermal correction terms to the dynamical structure

factor as $S_{th.}^{(in)}(\mathbf{k},\omega)$ and $S_{th.}^{(out)}(\mathbf{k},\omega)$. For the purpose of this section we have only to consider the effects of $S_{th.}^{(in)}(\mathbf{k},\omega)$. From Appendix C we have

$$\begin{aligned}
S_{th.}^{(in)}(k_\mu) = \frac{1}{\Omega}\int_D \frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0\mathrm{d}q_0'}{(2\pi)^2}\Bigg[&\frac{-(k_z+2q_z)^2}{4}\\
&\times\left\langle \hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle\\
&\times\left\langle \mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle\\
&-\frac{k_z^2-4q_z^2}{4}\left\langle \hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle\\
&\times\left\langle \hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle\Bigg]
\end{aligned} \tag{3.35}$$

where $k_0 = \omega$, $k_\mu = (\mathbf{k}, k_0)$ and $\Omega = (2\pi)^4\delta^{(4)}(0)$.

The dominant contribution to the superfluid density, which we denote as $\rho_{th.}^{s(out)}(\mathbf{k},\omega)$, is calculated from

$$\rho_{th.}^{s(out)}(\mathbf{k},\omega) = \rho_0^s\frac{K}{k_BT}\frac{1}{k^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}\frac{\omega' S_{th.}^{(in)}(\mathbf{k},\omega)}{\omega'-\omega-i\epsilon}. \tag{3.36}$$

We assume $\mathbf{k}\perp\hat{\mathbf{z}}$ or equivalently $k_z = 0$ since this is the configuration we are interested in. We also assume the static limit $\omega \to 0$. After integrating over ω' we obtain

$$\lim_{\mathbf{k}_\perp\to 0}\rho_{th.}^{s(out)}(\mathbf{k}_\perp, k_z=0, \omega=0) = -\rho_0^s\frac{k_BTK}{n_B^2}\int_D\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{q_z^2}{E^l(\mathbf{q})E^t(\mathbf{q})} \tag{3.37}$$

Estimating the integral over $\mathbf{q}$ numerically we obtain

$$\begin{aligned}
\rho_{zz}^s(0,0) =&\rho_0^s + \lim_{\mathbf{k}_\perp\to 0}\rho_{th.}^{s(out)}(\mathbf{k}_\perp, k_z=0,\omega=0)\\
=&\rho_0^s\times\left(1-4.126\frac{k_BT\sqrt{n}_B}{K}\right).
\end{aligned} \tag{3.38}$$

By comparing with the estimate of the melting temperature, T_m, (see Eq. (3.18)) we conclude that the second term in the parenthesis is not larger than 0.03 even at the melting temperature, which implies that the vortex lattice state is superconducting in the direction of the external magnetic field even when the effects of the thermal fluctuation are taken into account. The effects considered in this section are actually same as the one known as the effects of the helical deformation of vortices in the Lorentz force free configuration. [30] Our results qualitatively agree with the results obtained by Brandt.

B. Correction to magnetic permeability

In the section 3.3 we mentioned about the discrepancy between the results of the susceptibility of the calculations based on the elastic theory, Eq. (3.33), and the mean field theory, Eq. (3.34). In order to investigate its reason we calculate the contributions of the thermal fluctuations to the susceptibility. We can study this by considering the corrections of the higher order terms in $\mathbf{s}(\mathbf{k},\omega)$ to the dynamical form factor just as we carried out in the last section. This time we must consider the corrections to $S^{(out)}(\mathbf{k},\omega)$, denoted as $S^{(out)}_{th.}(\mathbf{k},\omega)$. From Appendix C we have

$$\begin{aligned} S^{(in)}_{th.}(k_\mu) =& \frac{1}{\Omega}\int_D \frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\Bigg[\,\frac{1}{2}\,k^2\,k_\perp^2 \\ &\times\left\langle \mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle \\ &\times\left\langle \mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu,\omega\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle\Bigg]. \end{aligned} \tag{3.39}$$

The dominant contribution to the superfluid density, $\rho^{s\,(in)}(\mathbf{k},\omega)$, in the static limit, $\omega\to 0$, of $S^{(out)}_{th.}(\mathbf{k},\omega)$ can be expressed as

$$\begin{aligned} \lim_{k_z\to 0}\lim_{\omega\to 0}\rho^{s\,(in)}_{th.}(\mathbf{k},\omega) \cong& \rho^s_0\frac{k_BTK}{n_B^2}\frac{k_\perp^2}{2}\int_D\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{q_y^4}{\{(\frac{k_\perp}{2}-q_x)^2+q_t^2\}\{(\frac{k_\perp}{2}+q_x)^2+q_t^2\}} \\ &\times\frac{1}{E^t(\frac{k_\perp}{2}-q_x,q_yq_z)}\,\frac{1}{E^t(\frac{k_\perp}{2}+q_x,q_yq_z)} \\ \sim& -\rho^s_0\frac{3\,k_BT\sqrt{\pi}}{2\,K\sqrt{n}_B}\,k_\perp^2\ln k_\perp, \end{aligned} \tag{3.40}$$

where we put $\mathbf{k}=(k_\perp,0,0)$ in the calculations which is general because of the rotational symmetry in xy-plane in the small $\mathbf{k}$ limit. We neglected the contributions of the less divergent compressional mode, $E^l(\mathbf{k})$. Here we obtain a remarkably large contribution in the coefficient of $k_\perp^2$ term, i.e. the $\ln k_\perp$ divergence. Although this divergence, if the electric conductivity is considered, is negligible because it vanishes in the $\mathbf{k}\to\mathbf{0}$ limit, it has large contribution to the susceptibility because in this case the coefficients of k^2 terms come into question. Actually we have

$$\mu^{(out)}\Big|_{k_z,\omega=0} \cong \left\{1+\frac{1}{\lambda^2}(-c_1+\frac{3\,k_BT\sqrt{\pi}}{2\,K\sqrt{n}_B}\,\ln k_\perp)\right\}^{-1}, \tag{3.41}$$

and the log correction appears as the correction to the constant, c_1, in comparison with Eq. (3.29). We consider that this log-divergence is deeply related to the discrepancy of the susceptibility calculated by the mean field theory and the elastic theory, which we pointed out in the section 3.3.

The reason of the discrepancy and the log-divergence can be considered as follows: There are two ways to increase the density of the lattice. One is to compress the lattice isotropically. This compression corresponds to the calculation of the mean field theory, in which the triangular lattice structure is always kept in increasing the field. (see Fig. 2(a)) The other way is to compress the lattice only in the one direction. In this case the triangular lattice structure is not kept in increasing the field. (see Fig. 2(b)) This compression corresponds to the calculation of the elastic theory. Actually we can confirm this by calculating the change of the condensate free energy, which is the first term of $\mathcal{G}_0$ in Eq. (3.2) corresponding to the two, Fig. 2; in the case of (a) we have

$$\delta\mathcal{G}_0/V = -\frac{K}{2}\frac{1}{8\pi n_B}\times\left(\frac{\delta B}{\phi_0}\right)^2, \tag{3.42}$$

whereas we have

$$\delta\mathcal{G}_0/V = -\frac{K}{2}\frac{1}{16\pi n_B}\times\left(\frac{\delta B}{\phi_0}\right)^2, \tag{3.43}$$

in case of (b). The magnetic susceptibility calculated from Eq. (3.42) and Eq. (3.43) give the result of the mean field theory and the elastic theory, respectively as shown in Appendix D.

The origin of the log-divergence can be understood as follows. When perturbed by the magnetic field, the lattice is compressed in some regions and decompressed in other regions. If the wave vector of the perturbation is sufficiently small, the lattice is more stable if the structure is reconstructed region by region into the regular triangular lattice with different lattice constants. This reconstruction destroys the lattice in the regions inbetween the regular lattice structures. Physically, this regions may become unstable to the formation of the dislocations.

Reflecting on our results, the log-divergence comes from a higher order term in $\mathbf{s}(\mathbf{k},\omega)$, which is originated from the shear fluctuations in the lattice distorted by the magnetic perturbation. We consider that this is consistent with the abovementioned physical picture.

We also note that the log-divergence stated in this section have nothing to do with the superconductivity, since $k_\perp^2 \log k_\perp \to 0$ in the $k_\perp \to 0$ limit.

4. Vortex liquid state

4.1 Effective free energy of vortex liquid

The low energy and long wave length properties of the fluctuations in the vortex liquid state can be well described by the hydrodynamic equations [28], according to which the free energy is expressed by the "coarse grained "vortex density, a slowly varing variable both in time and space. Here we illustrate the basic properties of the expected free energy. The effective free energy is expected to be given by

$$\mathcal{G}_v = \frac{K}{2}\int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{r}' G^{liq}_{\alpha\beta}(\mathbf{r}-\mathbf{r}')\ \delta n_\alpha(\mathbf{r},t)\ \delta n_\beta(\mathbf{r}',t), \tag{4.1}$$

where $\delta\mathbf{n}(\mathbf{r},t) = \mathbf{n}(\mathbf{r},t) - n_B\hat{\mathbf{z}}$. Note that $\mathbf{n}(\mathbf{r},t)$ is the coarse grained variable, which should be distinguished from its "microscopic"definition, Eq. (2.9). The effective vortex-vortex interaction in the liquid state, $G^{liq}_{\alpha\beta}(\mathbf{r})$, is given by

$$G^{liq}_{\alpha\beta}(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left\{ \frac{\delta_{\alpha\beta}}{k^2} + f_{\alpha\beta}(\mathbf{k}) \right\} e^{-i\mathbf{k}\cdot\mathbf{r}}. \tag{4.2}$$

The leading term of this free energy is the isotropic long range force or the $1/k^2$ singularity in small $\mathbf{k}$ limit. There are also the contributions from the effects of vortex line correlations in $f_{\alpha\beta}(\mathbf{k})$, which is expected to be anisotropic in general because of the uniaxial anisotropy of applied field.

In this thesis we consider only the leading term, i.e.,

$$\mathcal{G}_v = \frac{K}{2}\int \mathrm{d}^3\mathbf{r} \int \mathrm{d}^3\mathbf{r}' G^{liq}(\mathbf{r}-\mathbf{r}')\delta\mathbf{n}(\mathbf{r},t)\cdot\delta\mathbf{n}(\mathbf{r}',t), \tag{4.3}$$

where

$$G^{liq}(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \frac{1}{k^2}\, e^{-i\mathbf{k}\cdot\mathbf{r}}, \tag{4.4}$$

and neglect the rather complicated $f_{\alpha\beta}(\mathbf{k})$ term here. Therefore the discussions in the rest of this thesis should be limited to the contributions of the leading term. Because of this limitation, we cannot discuss the magnetic susceptibility in the liquid state but the other quantities, i.e. the static superfluid density and the electric conductivity, can be discussed as in the lattice state.

4.2 Hydrodynamic description of vortex liquid

The basic equations which describe the vortex motion in the vortex liquid state can be obtained as follows. First we describe the conservation of vortex density, $\mathbf{n}(\mathbf{r},t)$. The vortex current is defined by

$$J_{\alpha\beta}(\mathbf{r},t)=\sum_{\nu}\int \mathrm{d}l_\nu \frac{\partial r_{\nu,\alpha}(l_\nu,t)}{\partial t}\frac{\partial r_{\nu,\beta}(l_\nu,t)}{\partial l_\nu}\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_\nu(l_\nu,t)), \tag{4.5}$$

where the dummy indices α,β take x,y,z. The vortex current is a tensor since it has two suffices corresponding to the direction of vortex line element and the direction of motion. The right hand side corresponds to the current density of the vortices oriented in the β-direction moving in the α-direction. This expression satisfies the conservation equation for the vortex density,

$$\frac{\partial n_\alpha(\mathbf{r},t)}{\partial t}+\partial_\beta\left\{J_{\alpha\beta}(\mathbf{r},t)-J_{\beta\alpha}(\mathbf{r},t)\right\}=0, \tag{4.6}$$

which can easily be confirmed by substituting Eq. (2.9) and Eq. (4.5). Note that the diagonal components, $J_{\alpha\alpha}(\mathbf{r},t)$, are unphysical, since they represent the motion of the vortex lines parallel to themselves. Therefore the anti-symmetrized form of $J_{\alpha\beta}(\mathbf{r},t)$ in Eq. (4.6) is quite natural.

The most convincing way to construct the equations of motion of vortex liquid is to extend the Markoffian equations of motion for the conserved scalar field, which is called "*model B*" according to the notation used by Hohenberg and Halperin [27], to the three-dimensional conserved vector field (the vortex density in our problem). First we describe how this is worked out.

To derive the equations of motion of vortex density, $\delta\mathbf{n}(\mathbf{r},t)$, (hereafter we use $\delta\mathbf{n}(\mathbf{r},t)$ in place of $\mathbf{n}(\mathbf{r},t)$) it is simple to modify the Fick's law to the tensor current and to introduce the thermally fluctuating random currents, which we denote as $\zeta_{\alpha\beta}(\mathbf{r},t)$. Employing the free energy defined in Eq. (4.3) we obtain,

$$\begin{aligned}
J_{z\alpha}(\mathbf{r},t)=&-\Gamma\partial_\alpha\frac{\delta\mathcal{G}_v}{\delta\left[\delta n_z(\mathbf{r},t)\right]}+\zeta_{z\alpha}(\mathbf{r},t)\\
=&-\Gamma\partial_\alpha\int \mathrm{d}^3\mathbf{r}' G^{liq}(\mathbf{r}-\mathbf{r}')\delta n_z(\mathbf{r}',t)+\zeta_{z\alpha}(\mathbf{r},t), && (4.7)\\
J_{\alpha z}(\mathbf{r},t)=&-\Gamma\partial_z\frac{\delta\mathcal{G}_v}{\delta\left[\delta n_\alpha(\mathbf{r},t)\right]}+\zeta_{\alpha z}(\mathbf{r},t)
\end{aligned}$$

$$\begin{aligned}
=&-\Gamma\partial_z\int \mathrm{d}^3\mathbf{r}' G^{liq}(\mathbf{r}-\mathbf{r}')\delta n_\alpha(\mathbf{r}',t)+\zeta_{\alpha z}(\mathbf{r},t), && (4.8)\\
J_{\alpha\beta}(\mathbf{r},t)=&-\Gamma_t\partial_\beta\frac{\delta\mathcal{G}_v}{\delta\left[\delta n_\alpha(\mathbf{r},t)\right]}+\zeta_{\alpha\beta}(\mathbf{r},t)\\
=&-\Gamma_t\partial_\beta\int \mathrm{d}^3\mathbf{r}' G^{liq}(\mathbf{r}-\mathbf{r}')\delta n_\alpha(\mathbf{r}',t)+\zeta_{\alpha\beta}(\mathbf{r},t), && (4.9)
\end{aligned}$$

where α and β takes only x and y, $\Gamma=\mu n_B$, $\Gamma_t=\mu_t n_t$ with μ being same as Eq. (3.10) and μ_t being the mobility of the vortices oriented in x-direction and moving in y-direction, whose "microscopic"estimation is given later. Here we neglected the contributions of the viscosity (the internal friction) of the vortex liquid, since they are irrelevant in the long wave behavior considered in this thesis.

We introduce the following notations,

$$\delta\mathbf{n}(\mathbf{k},t)=\ i\ \hat{\mathbf{k}}^{(in)}\delta n^{(in)}(\mathbf{k},t)+\hat{\mathbf{k}}^{(out)}\delta n^{(out)}(\mathbf{k},t). \tag{4.10}$$

The continuity condition of vortex lines, Eq. (2.11), is naturally satisfied by Eq. (4.10) because $\mathbf{k}\cdot\delta\mathbf{n}(\mathbf{k},t)=k\,\hat{\mathbf{k}}\cdot\delta\mathbf{n}(\mathbf{k},t)=0$. After some manipulations, we obtain the following equations of motion of the vortex density,

$$\begin{aligned}
i\omega\delta n^{(in)}(\mathbf{k},\omega)=&-(\Gamma_t k_\perp^2+\Gamma k_z^2)G^{liq}(\mathbf{k})\delta n^{(in)}(\mathbf{k},\omega)+\zeta^{(in)}(\mathbf{k},\omega) && (4.11)\\
i\omega\delta n^{(out)}(\mathbf{k},\omega)=&-\Gamma k^2 G^{liq}(\mathbf{k})\delta n^{(out)}(\mathbf{k},\omega)+\zeta^{(out)}(\mathbf{k},\omega). && (4.12)
\end{aligned}$$

where $\zeta^{(in)}(\mathbf{k},\omega)$ and $\zeta^{(out)}(\mathbf{k},\omega)$ are the random forces, the correlations of which are given by,

$$\begin{aligned}
\left\langle\zeta^{(in)}(\mathbf{k},\omega)\right\rangle=&\left\langle\zeta^{(out)}(\mathbf{k},\omega)\right\rangle=0, && (4.13)\\
\left\langle\zeta^{(in)}(\mathbf{k},\omega)\zeta^{(in)}(\mathbf{k}',\omega')\right\rangle=&2k_BT(\Gamma_t k_\perp^2+\Gamma k_z^2)(2\pi)^4\ \delta^{(3)}(\mathbf{k}+\mathbf{k}')\ \delta(\omega+\omega') && (4.14)\\
\left\langle\zeta^{(out)}(\mathbf{k},\omega)\zeta^{(out)}(\mathbf{k}',\omega')\right\rangle=&2k_BT\Gamma k^2(2\pi)^4\ \delta^{(3)}(\mathbf{k}+\mathbf{k}')\ \delta(\omega+\omega'). && (4.15)
\end{aligned}$$

The correlations of the higher order products of $\zeta^{(in)}$ and $\zeta^{(out)}$ are neglected here.

4.3 Entanglement and cutting of vortices

In this section, we estimate the kinetic constant Γ_t of vortex liquid state, which has been introduced in Eq. (4.9). It is determined by n_t and μ_t, which are the free vortex density and the mobility of the vortices oriented in the direction perpendicular to the applied magnetic field, $\mathbf{H}^a$, and moving in the direction perpendicular to the orientation and $\mathbf{H}^a$.

Before estimating Γ_t we classify the vortex liquid. So far at least three thermodynamic phases has been proposed for the vortex liquid states, which are:

(a) entangled vortex liquid state

(b) disentangled vortex liquid state

(c) topological glass state.

The entangled vortex liquid state is the most common one [14, 31] where the vortices are entangled and the vortex cutting is induced by the thermal fluctuations.

The disentangled vortex liquid state is also proposed by several authors [14, 32] in which the vortices are almost straight and entanglements are absent but the configuration of such straight vortices are not in the triangular lattice and the system has no shear rigidity. This state is usually assumed to appear between the entangled vortex liquid state and the vortex lattice state.

The topological glass state is proposed by Obukhov and Rubinstein [33], in which the energy barrier is too large for the vortex cutting to be thermally induced. Although this proposal has been debated by several authors [34], based on the fact that the cutting barrier is always finite in the type-II superconductors, we consider that this glass state may provide us a qualitative view of the case where cutting barrier is larger than the energy scale of thermal fluctuations.

We in this section investigate how Γ_t changes in these three states.

The free vortices in xy-direction may consist of the xy-component of the field induced vortices. In estimating their density we employ the discussion of Nelson et al. [13, 14, 31]. Especially the estimation of the "diffusion constant"of vortices (the diffusion constant of the vortex positions, z, considered as the fictitious "time "coordinate) made by Marchetti [31] will be a good starting point.

The "diffusion constant", D, of one tagged vortex in the entangled vortex liquid, introduced by Nelson and Seung [14], is defined by

$$\sqrt{\langle|\mathbf{r}(z)-\mathbf{r}(0)|^2\rangle} = \sqrt{2D|z|}, \tag{4.16}$$

where $\mathbf{r}(z)$ is the position of the tagged vortex. The "diffusion constant"without other vortex lines is given by

$$D_0 = \frac{k_B T}{\varepsilon_1}, \tag{4.17}$$

with ε_1 being the line tension given by $(\phi_0/4\pi\lambda^2)^2 \ln\kappa$. Marchetti [31] pointed out that the correlation effects in the vortex liquid enhances the line tension ε_1 to the renormalized one, ε_1^r. According to his estimate, we may put $\varepsilon_1^r \sim 10\times\varepsilon_1$.

With this effective line tension of the vortices we can estimate the characteristic spacing between the entanglements along z-axis, ξ_z, as

$$\xi_z = \frac{\varepsilon_1^r}{2\,k_B T\, b}, \tag{4.18}$$

which is the distance in z-direction necessary for a vortex to wander one mean vortex spacing. Since one vortex segment oriented in x or y direction is expected per area $2\xi_z d$ where $d \sim 1/\sqrt{n_B}$, the vortex density, n_t, in case of (a) will be given by

$$n_t = \frac{1}{2\xi_z d} = \frac{k_B T}{\varepsilon_1^r} n_B^{3/2}. \tag{4.19}$$

If ξ_z exceeds the length of the sample, there are effectively no entanglement. In this case we may put $n_t = 0$ and this corresponds to the disentangled vortex liquid: the case (b).

Next we estimate μ_t, namely the mobility of the vortex segment oriented in the direction perpendicular to $\mathbf{H}^a$ and moving in the direction perpendicular to the orientation and $\mathbf{H}^a$. In the usual entangled vortex liquid, the vortex cutting is expected. Since the vortex cutting barrier, U_c, depends on the actual cutting processes [34], the estimation may be complicated in the realistic entangled vortex liquid. In Fig. 3 (a)$\sim$(c), the diffusion processes of the above mentioned vortex segment about one mean vortex spacing are schematically shown. We put the characteristic length of the vortex segment which diffuses collectively as d_s and consider an approximate equation of motion of the segment. In this case U_c should also depend on d_s since U_c is determined from the collective cutting process as shown in Fig. 3 (d) for example.

Here we give a simple estimation. We assume that the equation of motion of the vortex segment oriented in xy-direction is given by,

$$d_s\,\eta\,\dot{x}_v(t) = -\frac{\phi_0\,d^s}{c}J_T + \frac{\pi U_c}{d}\sin\left\{\frac{2\pi}{d}x_v(t)\right\} + L(t), \tag{4.20}$$

where x_v is the position of the segment, and L is the fluctuating random force, satisfying the relation $\langle L(t)L(0)\rangle = 2\eta d_s k_B T\delta(t)$. The vortex cutting barrier is approximated by the periodic potential for simplicity. Eq. (4.20) has been studied by Ambegaokar and Halperin [35] in the context of the Josephson junction. From their analysis we expect that, if the barrier U_c satisfies $U_c \gtrsim k_B T$, the mobility of the segment, μ_t, is given by

$$\mu_t = \mu\frac{2\pi U_c}{k_B T}e^{-\frac{U_c}{k_B T}}. \tag{4.21}$$

It is clear that if $U_c \gg k_B T$ the mobility of the vortices is reduced extremely as compared to the mobility of the free diffusion, $\mu = 1/\eta$. This result holds for the case (a).

The case (c) corresponds to the situation, $U_c \gg k_B T$, in case (b).

Therefore we obtain the following results for Γ_t.

(a) entangled vortex liquid state (from Eq. (4.19) and Eq. (4.21))

$$\Gamma_t = \mu\frac{2\pi U_c b^{3/2}}{\varepsilon_1^r}e^{-U_c/k_B T} \propto e^{-U_c/k_B T}. \tag{4.22}$$

(b) disentangled vortex liquid state ($n_t = 0$)

$$\Gamma_t = 0. \tag{4.23}$$

(c) topological glass state ($U_c \gg k_B T$)

$$\Gamma_t = 0. \tag{4.24}$$

The similar picture has proposed by Feigelman et.al. [32] which is derived based on the analogy between the statics of three dimensional vortex liquid and the dynamics of two dimensional bose system.

4.4 Dynamical superfluid density in vortex liquid state

By Eq. (4.11) and Eq. (4.12) with Eq. (4.13)-Eq. (4.15) we obtain the dynamical form factor of vortices as follows,

$$S^{(in)}(\mathbf{k},\omega) = \frac{2k_B T(\Gamma_t k_\perp^2 + \Gamma k_z^2)}{\omega^2 + \{(\Gamma_t k_\perp^2 + \Gamma k_z^2)G^{liq}(\mathbf{k})\}^2} \tag{4.25}$$

$$S^{(out)}(\mathbf{k},\omega) = \frac{2k_B T\Gamma k^2}{\omega^2 + \{\Gamma k^2 G^{liq}(\mathbf{k})\}^2}. \tag{4.26}$$

Then employing the linear response theory we obtain the superfluid density as

$$\begin{aligned}\rho^{s\,(in)}(\mathbf{k},\omega) &= \rho_0^s\left\{1 - \frac{\Gamma K}{i\omega + \Gamma k^2 G^{liq}(\mathbf{k})}\right\}\\ &= \rho_0^s\left\{1 - \frac{1}{i\omega/(\Gamma K) + 1}\right\}\end{aligned} \tag{4.27}$$

$$\begin{aligned}\rho^{s\,(out)}(\mathbf{k},\omega) &= \rho_0^s\left\{1 - \frac{1}{k^2}\frac{(\Gamma k_z^2 + \Gamma_t k_\perp^2)K}{i\omega + (\Gamma k_z^2 + \Gamma_t k_\perp^2)G^{liq}(\mathbf{k})}\right\}\\ &= \rho_0^s\left\{1 - \left\{\frac{i\omega}{(\Gamma\sin^2\theta + \Gamma_t\cos^2\theta)K} + 1\right\}^{-1}\right\}.\end{aligned} \tag{4.28}$$

A. Static superfluid density

In the static limit, $\omega \to 0$, the superfluid density becomes

$$\rho^{s\,(in)}(\mathbf{k},\omega) = \rho^{s\,(out)}(\mathbf{k},\omega) = 0. \tag{4.29}$$

The result does not depend on the way of taking limits, $\mathbf{k} \to \mathbf{0}$, $\omega \to 0$. From this expression, we conclude that the vortex liquid is not superconducting in any direction in the thermodynamic sense. This conclusion is in striking contrast with the lattice state which is confirmed to be superconducting in the direction parallel to the applied magnetic field according to Eq. (3.21) and Eq. (3.22) As seen from Eqs. (3.25,3.26) and Eqs. (4.27,4.28), the origin of this difference in the static limit lies in the existence or absence of shear rigidity. This result holds for all cases, (a)-(c), described in the preceding section, since the static results do not depend on the kinetic constant Γ_t.

B. Electric conductivity

The frequency dependent electric conductivity is given by,

$$\sigma(\vartheta,\omega) = \sigma_n + \frac{e^{*2}\rho_0^s}{m^*}\frac{1}{i\omega + K(\Gamma_t\cos^2\vartheta + \Gamma\sin^2\vartheta)}. \tag{4.30}$$

In case of the entangled vortex liquid state (the case (a)), this form has no divergence even if we put $\theta = 0, \pi$ but has a finite value given by $\sigma_n + e^{*2}\rho_0^s/m^* K\Gamma_t$, since $\Gamma_t \neq 0$ in contrast with the vortex lattice state, Eq. (3.27). In the cases of the disentangled vortex liquid state (b) and topological glass state (c), the electric conductivity given by Eq. (4.30) diverges in the direction of the applied field as in the case of vortex lattice state.

As is mentioned before we can not discuss the magnetic susceptibility here. To study this quantity we have to construct a theory which can treat the vortices at more "microscopic "level, i.e., we have to treat the individual vortices. This problem is left to future studies.

5. Summary and discussion

5.1 Resistivity in vortex state

Let us, first, summarize the characteristic features of the resistivity in the mixed state. (Although we calculated the conductivity so far we discuss in the form of the resistivity here.) The behavior of the resistivity in all configurations of the perturbations in the lattice and the liquid states of the vortices can be given by the expression of the electric conductivity in Eq. (3.27) and Eq. (4.30), but first we will discuss two specific cases, i.e., the case of the electric field parallel ($R^{//}$) and perpendicular ($R^{\perp}$) to the applied magnetic field, $\mathbf{H}^a$. As is schematically shown in Fig. 4, the behavior of $R^{\perp}$ does not differ between the lattice and the liquid states and shows the well known "flux flow "behavior in both cases. On the other hand, $R^{//}$, which corresponds to the "Lorentz force free configuration", differs in behavior between two states. There are several possibilities.

1) The system without disentangled vortex liquid state.

There are two thermodynamic states, the vortex lattice state and the entangled vortex liquid state. In this case $R^{//}$ has the discontinuity at the vortex lattice melting transition temperature, T_m, since in general $R^{//} = 0$ in the lattice state and $R^{//} \neq 0$ in the entangled liquid state. $R^{//}$ may show an activation type behavior ($\propto e^{-U_c/k_B T}$) in the low temperature region of the vortex liquid state.

2) The system with the disentangled vortex liquid state.

There are three thermodynamic phase, the vortex lattice state, the disentangled and the entangled vortex liquid states. In this case the discontinuity of $R^{//}$ occurs at the entangled-disentangled phase transition (or crossover) temperature, T_{e-d}.

3) The system with the topological glass state.

The topological glass phase transition, if possible, is only a crossover. (We write the crossover temperature as T_{tg}. The system transforms from lattice state to topological glass state and, then, shows crossover to entangled vortex liquid state as the temperature increases. In this case the vortex cutting barrier is so large just above the melting temperature and, therefore, $R^{//} \sim 0$. Therefore the discontinuity of $R^{//}$ may not be visible. The onset of $R^{//}$ occurs at T_{tg}.

The possible temperature dependence of $R^{\perp}$ and $R^{//}$ are schematically shown in Fig. 4.

Next we comment on the realistic superconductors, especially on the effects of disorder. As is the case in high-T_c materials, every superconductor has inhomogeneities (or pinning centers) more or less, which causes a drastic change in the thermodynamic and transport properties of the mixed state. For this reason it is actually hard to extract the intrinsic behavior of the ideal superconductors from actual experiments. It is however, encouraging to observe that the clean samples have been realized recently where the melting transition is identified from the vortex glass transition [18] and we may expect the above mentioned behavior in the resistivity in the "Lorentz force free configuration".

It is interesting to note that the obtained temperature dependence of $R^{\perp}$ in the vortex liquid region has a similarity to the resistivity of the low temperature region of the one-dimensional (1D) GL model [36, 37]; both have the activation type temperature dependence and the activation energy is the vortex cutting barrier and the phase slip energy for the vortex liquid and the 1D GL model, respectively, i.e. the phase slip in 1D GL model corresponds to the vortex cutting in 3D vortex liquid. This similarity seems to support the concept of the dimensional reduction in the fluctuation regime [38, 39, 40]. Therefore we consider that the obtained activation type behavior in the low temperature region of the vortex liquid state may be smoothly connected to the behavior of the fluctuation regime as in the case of 1D GL model.

5.2 Magnetic properties

It is known that if the superfluid density is not zero in all directions the electric conductivity diverges in all direction and the Meissner effect occurs. Then what happen if the superfluid density is not zero in only one direction just as in the vortex lattice state. This problem is closely related to the magnetization process in the mixed state, which has been studied by several authors [5, 41].

We assume that the external magnetic field is changed from $\mathbf{H}^a$ to $\mathbf{H}^a+\delta\mathbf{H}^{ext}$. The sample of the finite length scale is considered here and, hence, the sample has edges in all directions. If $\delta\mathbf{H}^{ext}$ is parallel to $\mathbf{H}^a$ the screening current due to $\delta\mathbf{H}^{ext}$ flows near the sample edges in the direction perpendicular to $\mathbf{H}^a$. In this case the screening current is perpendicular to the vortices and the flux penetrates into the sample by the usual flux flow processes, i.e. the new vortices flow into or flow out of the sample from the edges. As a result the internal magnetic field changes from $\mathbf{B}$ to $\mathbf{B}+\delta\mathbf{B}$ corresponding to $\delta\mathbf{H}^{ext}$.

If $\delta\mathbf{H}^{ext}$ is in the direction perpendicular to $\mathbf{H}^a$, the screening currents, $\mathbf{j}^{//}_{scr.}$ and $\mathbf{j}^{\perp}_{scr.}$, flow as shown in Fig. 5 (a). In this case the process of the penetration of flux lines is different between in the entangled vortex liquid state and in the vortex lattice state because of the following reason. Since the vortex lattice state is stable to the current parallel to $\mathbf{H}^a$, the penetration occurs only from the upper and lower edges, where the currents are perpendicular to the vortices as shown in Fig. 5 (b). In contrast to this the penetration occurs also from the side of the sample in the entangled vortex liquid state, as shown in Fig. 5 (c), because the vortex liquid state is no longer stable to those currents. From our theory we can estimate the ratio of the two penetration process as $\Gamma : \Gamma_t$.

Therefore we conclude that the nonvanishing superfluid density of vortex lattice state in the "Lorentz force free configuration "do not imply the Meissner effect in the case of the sample with finite dimensions. If we imagine the sample with infinite dimension, e.g. in the direction of $\mathbf{H}^a$, the situations change in case of the vortex lattice state because there are no edges from which the flux can penetrate into the sample. However this is an unphysical situation, so we consider our conclusion above is true in all physical situations. We consider that the origin of the discrepancy found in Eq. (3.30) also lies in the same fact and is resolved by considering the samples with finite dimensions.

5.3 Relation to previous treatments and future extensions

In this thesis we emphasized that the distinction between the internal and the external vector potential is completely disregarded in the conventional treatments of the flux flow or flux creep phenomena. Here we discuss when the conventional treatments are justified. The conventional treatments are valid only when the difference of the internal and the external vector potential is small. By Eq. (2.35) the ratio of the two is expressed by the magnetic permeability. Therefore we should examine the magnetic permeability corresponding to the flux flow or flux creep phenomena. In Fig. 6 we showed a schematic graph of the typical B-H^a curve in the mixed state. The ratio of the internal and the external vector potential in the the flux flow or flux creep state is given by the magnetic permeability μ^{dif}, which is given by the gradient of the curve at a fixed H^a. From Fig. 6 it is clear that $\mu^{dif} \cong 1$ almost everywhere in the mixed state. The only exception is the region near H_{c1}. Therefore if we limit ourselves to the flux flow or flux creep phenomena the conventional treatments apply. That is the validity and at the same time the limitation of the conventional treatments. The conventional treatment fail in other configurations where the screening effect is strong, for example the conductivity parallel to the applied field.

It has been known that the resistivity of the mixed state of high-T_c cuprates strongly depends on the angle between the electric field and the applied magnetic field [42]. At this stage our theory can not be compared with the quantitative explanation of the experimental results unfortunately, since, in our model, the large anisotropy of the high-T_c cuprates, which is expected to be playing important roles, is not taken into account. In order to apply our treatment to such system, we need to extend the present approach in the following points. First the anisotropy should be taken into the form of the free energy, for example as the anisotropy of the mass of Cooper pairs. Second the Bardeen-Stephen viscosity should also be extended to anisotropic case. In addition, the layer structure, especially the intrinsic pinning by the layers, may be affecting the behavior of high-T_c cuprates, which should also be considered. These effects are expected to change the magnitude of the resistivity largely and, therefore, even the qualitative properties can be different from the isotropic superconductors, evaluated in our theory. We leave these extensions for the future studies.

Appendix A Vortex density fluctuations in lattice state

We will derive the expression of the vortex density fluctuations, $\delta\mathbf{n}(\mathbf{r},t)$, in terms of the elastic field, which can be carried out by expanding $\delta\mathbf{n}(\mathbf{r},t)$ in terms of $\mathbf{s}_\nu(z,t)$ as,

$$\begin{aligned}\delta\mathbf{n}(\mathbf{r},t) =&\sum_\nu \int \mathrm{d}\,z' \left\{\hat{\mathbf{z}} + \frac{\partial \mathbf{s}_\nu(z',t)}{\partial z'}\right\} \delta^{(3)}(\mathbf{r}-\mathbf{r}^0_\nu(z') - \mathbf{s}_\nu(z',t)) - n_B\,\hat{\mathbf{z}}\\ =&\sum_\nu \int \mathrm{d}\,z' \left\{\hat{\mathbf{z}} + \frac{\partial \mathbf{s}_\nu(z',t)}{\partial z'}\right\} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}^0_\nu(z'))}\\ &\times\left\{1 - i\mathbf{k}\cdot\mathbf{s}_\nu(z',t) - \frac{1}{2}(\mathbf{k}\cdot\mathbf{s}_\nu(z',t))^2 - \frac{i}{6}\,(\mathbf{k}\cdot\mathbf{s}_\nu(z',t))^3\right\}\\ &- n_B + o(\mathbf{s}^4_\nu)\,\hat{\mathbf{z}}.\end{aligned} \tag{A1}$$

Then going to the Fourier space, we have,

$$\delta\mathbf{n}^{(0)}(\mathbf{k},\mathbf{G},t) = n_B\,\hat{\mathbf{z}}\,\delta^{(3)}(\mathbf{k})\,(1-\delta_{\mathbf{G},\mathbf{0}}), \tag{A2}$$

$$\delta\mathbf{n}^{(1)}(\mathbf{k},\mathbf{G},t) = ik_z\mathbf{s}(\mathbf{k},t) - i\hat{\mathbf{z}}\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{k},t), \tag{A3}$$

$$\begin{aligned}\delta\mathbf{n}^{(2)}(\mathbf{k},\mathbf{G},t) =& -\frac{1}{2}\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 n_B}\,\hat{\mathbf{z}}\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{q},t)\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{k}-\mathbf{q},t)\\ &-\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 n_B}\,k'_z\mathbf{s}(\mathbf{q},t)\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{k}-\mathbf{q},t),\end{aligned} \tag{A4}$$

$$\begin{aligned}\delta\mathbf{n}^{(3)}(\mathbf{k},\mathbf{G},t) =& -\frac{i}{6}\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 n_B}\int_D \frac{\mathrm{d}^3\mathbf{q}'}{(2\pi)^3 n_B}\,\hat{\mathbf{z}}\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{q},t)\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{q}',t)\\ &\times(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{k}-\mathbf{q}-\mathbf{q}',t)\\ &+\frac{n_B i}{2}\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\int_D \frac{\mathrm{d}^3\mathbf{q}'}{(2\pi)^3}\,q_z\,\mathbf{s}(\mathbf{q},t)\,(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{q}',t)\\ &\times(\mathbf{k}+\mathbf{G})\cdot\mathbf{s}(\mathbf{k}-\mathbf{q}-\mathbf{q}',t),\end{aligned} \tag{A5}$$

where

$$\delta\mathbf{n}(\mathbf{k},\mathbf{G},t) = \delta\mathbf{n}^{(0)}(\mathbf{k},\mathbf{G},t) + \delta\mathbf{n}^{(1)}(\mathbf{k},\mathbf{G},t) + \delta\mathbf{n}^{(2)}(\mathbf{k},\mathbf{G},t) + \delta\mathbf{n}^{(3)}(\mathbf{k},\mathbf{G},t) + o(\mathbf{s}^4), \tag{A6}$$

and

$$\delta\mathbf{n}(\mathbf{r},t) = \sum_{\mathbf{G}}\int_D \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\delta\mathbf{n}(\mathbf{k},\mathbf{G},t)\,e^{-i(\mathbf{k}+\mathbf{G})\cdot\mathbf{r}}, \tag{A7}$$

with $\mathbf{G}$ being the reciprocal lattice vector of the triangular vortex lattice.

Appendix B Derivation of elastic free energy

The elastic moduli, which do not include the screening effects due to the vector potential fluctuation, are obtained in the following way.

We start from Eq. (2.12) and expand it to the second order in the elastic field, $\mathbf{s}(\mathbf{k},t)$. In this Appendix we omit the time coordinate, t. Since $\mathcal{G}_v$ is expressed as

$$\mathcal{G}_v = \sum_{\mathbf{G}} \int_D \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, G_v(\mathbf{k}+\mathbf{G})\, \delta\mathbf{n}(\mathbf{k},\mathbf{G}) \cdot \delta\mathbf{n}(-\mathbf{k},-\mathbf{G}), \tag{B1}$$

we obtain

$$\begin{aligned}\mathcal{G}_v = &\sum_{\mathbf{G}} \int_D \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, G_v(\mathbf{k}+\mathbf{G}) \Big[\delta\mathbf{n}^{(0)}(\mathbf{k},\mathbf{G}) \cdot \delta\mathbf{n}^{(0)}(-\mathbf{k},-\mathbf{G}) \\ &+ 2\,\delta\mathbf{n}^{(0)}(\mathbf{k},\mathbf{G}) \cdot \delta\mathbf{n}^{(1)}(-\mathbf{k},-\mathbf{G}) \\ &+ \delta\mathbf{n}^{(1)}(\mathbf{k},\mathbf{G}) \cdot \delta\mathbf{n}^{(1)}(-\mathbf{k},-\mathbf{G}) + 2\,\delta\mathbf{n}^{(0)}(\mathbf{k},\mathbf{G}) \cdot \delta\mathbf{n}^{(2)}(-\mathbf{k},-\mathbf{G})\Big]. \end{aligned} \tag{B2}$$

Since the vortex lattice is assumed to be in the thermodynamically stable lattice structure, the second line of Eq. (B2) must vanish. The other two terms give,

$$\begin{aligned}\mathcal{G}_v = &\frac{K}{2} \sum_{\mathbf{G}\neq\mathbf{0}} G_v(\mathbf{G}) \\ &+ \frac{K}{2} \sum_{\mathbf{k}\in D} \{K_{\alpha\beta}(\mathbf{k}) + L_{\alpha\beta}(\mathbf{k})\}\ \mathbf{s}_\alpha(\mathbf{k},t)\, \mathbf{s}_\beta(-\mathbf{k},t), \\ K_{\alpha\beta}(\mathbf{k}) \equiv &\sum_{\mathbf{G}} \left\{ \frac{(\mathbf{k}+\mathbf{G})_\alpha (\mathbf{k}+\mathbf{G})_\beta}{|\mathbf{k}+\mathbf{G}|^2} e^{-\xi_{sc}|\mathbf{k}+\mathbf{G}|} - \frac{\mathbf{G}_\alpha \mathbf{G}_\beta}{G^2} e^{-\xi_{sc}G} \right\}, \\ L_{\alpha\beta}(\mathbf{k}) \equiv &\sum_{\mathbf{G}} \frac{k_z^2\, \delta_{\alpha\beta}}{|\mathbf{k}+\mathbf{G}|^2} e^{-\xi_{sc}|\mathbf{k}+\mathbf{G}|}. \end{aligned} \tag{B3}$$

The first term in $\mathcal{G}_v$ is the gain of the potential energy of vortices due to the lattice structure, which contributes to $\mathcal{G}_0$ in Eq. (3.3). The summation in $K_{\alpha\beta}(\mathbf{k})$ and $L_{\alpha\beta}(\mathbf{k})$ can be carried out by the following way. First we note the following replacement which are valid under the summation over the reciprocal lattice vector of triangular lattice: [26]

$$G_\beta\, G_\beta \longrightarrow \frac{1}{2} G^2\, \delta_{\alpha\beta} \ , \quad G_\beta\, G_\beta\, (\mathbf{G}\cdot\mathbf{k})^2 \longrightarrow \frac{1}{4} G^4\, k_\alpha\, k_\beta + \frac{1}{8} G^4\, k_\perp^2\, \delta_{\alpha\beta}.$$

Using these we obtain

$$\begin{aligned} K_{\alpha\beta}(\mathbf{k}) = &\frac{k_\alpha\, k_\beta}{k^2} \end{aligned}$$

$$\begin{aligned} &+ \sum_{\mathbf{G}\neq\mathbf{0}} \left\{ k_\alpha\, k_\beta \left(-\frac{3\xi_{sc}}{8G} + \frac{\xi_{sc}^2}{8} \right) e^{-\xi_{sc}G} + k_\perp^2 \delta_{\alpha\beta} \left(\frac{\xi_{sc}^2}{16} + \frac{\xi_{sc}}{16G} \right) e^{-\xi_{sc}G} \right. \\ &\qquad \left. + k_z^2 \delta_{\alpha\beta} \left(-\frac{1}{2G^2} - \frac{\xi_{sc}}{4G} \right) e^{-\xi_{sc}G} \right\}, \\ L_{\alpha\beta}(\mathbf{k}) = &\frac{k_z^2}{k^2} \delta_{\alpha\beta} + \sum_{\mathbf{G}\neq\mathbf{0}} k_z^2 \delta_{\alpha\beta} \frac{1}{G^2} e^{-\xi_{sc}G}. \end{aligned} \tag{B4}$$

The summation over $\mathbf{G}$ can be approximated by the intergation

$$\begin{aligned} \sum_{\mathbf{G}\neq\mathbf{0}} \xi_{sc}^2\, e^{-\xi_{sc}|\mathbf{G}|} &\cong \frac{1}{2\pi k_{BZ}^2} \int_{k_{BZ}}^{\infty} 2\pi G \mathrm{d}G\, \xi_{sc}^2\, e^{-\xi_{sc}G} \\ &\xrightarrow[\xi_{sc}\to 0]{} \frac{1}{2\pi n_B}, \end{aligned} \tag{B5}$$

$$\begin{aligned} \sum_{\mathbf{G}\neq\mathbf{0}} \frac{\xi_{sc}}{|\mathbf{G}|}\, e^{-\xi_{sc}|\mathbf{G}|} &\cong \frac{1}{2\pi k_{BZ}^2} \int_{k_{BZ}}^{\infty} 2\pi G \mathrm{d}G\, \frac{\xi_{sc}}{G}\, e^{-\xi_{sc}G} \\ &\xrightarrow[\xi_{sc}\to 0]{} \frac{1}{2\pi n_B}, \end{aligned} \tag{B6}$$

$$\begin{aligned} \sum_{\mathbf{G}\neq\mathbf{0}} \frac{1}{|\mathbf{G}|^2}\, e^{-\xi_{sc}|\mathbf{G}|} &\cong \frac{1}{2\pi k_{BZ}^2} \int_{k_{BZ}}^{\infty} 2\pi G \mathrm{d}G\, \frac{\xi_{sc}}{G}\, e^{-\xi_{sc}G} \\ &\xrightarrow[\xi_{sc}\to 0]{} -\frac{1}{2\pi n_B} \ln(\sqrt{k_{BZ}}\xi_{sc}), \end{aligned} \tag{B7}$$

where $G \equiv |\mathbf{G}|$, $k_{BZ} = \sqrt{4\pi n_B}$ and after integration we put $\xi_{sc} \to 0$. Employing Eq. (B5)~Eq. (B7) we obtain,

$$\begin{aligned} K_{\alpha\beta}(\mathbf{k}) = &\left(\frac{1}{k^2} - \frac{1}{16\pi n_B} \right) k_\alpha\, k_\beta + \frac{1}{16\pi n_B} \left(k_\perp^2 \delta_{\alpha\beta} - k_\alpha\, k_\beta \right) \\ &- \left(\frac{1}{4\pi n_B} \ln(\sqrt{4\pi n_B}\xi_{sc})^{-1} + \frac{1}{8\pi n_B} \right) k_z^2 \delta_{\alpha\beta} \\ L_{\alpha\beta}(\mathbf{k}) = &\left(\frac{1}{k^2} + \frac{1}{2\pi n_B} \ln(\sqrt{4\pi n_B}\xi_{sc})^{-1} \right) k_z^2 \delta_{\alpha\beta}. \end{aligned} \tag{B8}$$

Comparing Eq. (B8) with the definition of the elastic moduli in Eq. (3.3) we obtain

$$\begin{aligned} c_{11}(\mathbf{k}) = &K n_B^2 \left(\frac{1}{k^2} - \frac{1}{16\pi n_B} \right) \\ c_{66}(\mathbf{k}) = &\frac{K n_B}{16\pi} \\ c_{44}(\mathbf{k}) = &K n_B^2 \left(\frac{1}{k^2} - \frac{\ln(4\pi n_B\, \xi_{sc}^2)}{8\pi n_B} \right). \end{aligned} \tag{B9}$$

The logarithmic term in $c_{44}(\mathbf{k})$ can be rewritten in the form given in Eq. (3.5) using the formula, Eq. (3.4).

Appendix C Dynamical form factor of vortices in vortex lattice state

From Eq. (A2)-Eq. (A5) the vortex density in the long wavelength limit can be obtained by neglecting the terms of higher reciprocal lattice vector, $\mathbf{G}$, as follows,

$$
\begin{aligned}
&\delta n^{(in)}(k_\mu) = k_z\,\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}(k_\mu)\\
&+\frac{1}{2}\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 n_B}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0}{2\pi}\,(k_z+2q_z)\,k_\perp\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\\
&+\frac{i}{6}\int_D\frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0\mathrm{d}q_0'}{(2\pi)^2}\,(k_z+3q_z)\,k_\perp^2\;\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}+q_\mu\right)\\
&\qquad\times\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}+q_\mu'\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}-q_\mu-q_\mu'\right)+o(\mathbf{s}^4),
\end{aligned}
\tag{C1}
$$

$$
\begin{aligned}
&\delta n^{(out)}(k_\mu) = k\,\mathbf{k}^{(l)}\cdot\mathbf{s}(k_\mu)\\
&+\frac{1}{2}\int_D \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3 n_B}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0}{2\pi}\,k\,k_\perp\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\\
&+\frac{i}{6}\int_D\frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0\mathrm{d}q_0'}{(2\pi)^2}\,k\,k_\perp^2\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}+q_\mu'\right)\\
&\qquad\times\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{3}-q_\mu-q_\mu'\right)+o(\mathbf{s}^4),
\end{aligned}
\tag{C2}
$$

where $k_\mu=(\mathbf{k},k_0)$ and $q_\mu=(\mathbf{q},q_0)$. Therefore the dynamical form factors (or the density-density correlation functions) of the vortices are estimated using Eq. (C1) and Eq. (C2) as follows,

$$
\begin{aligned}
S^{(in)}(k_\mu) =&\frac{1}{\Omega}\,k_z^2\langle s^t(k_\mu)s^t(-k_\mu)\rangle\\
&+\frac{1}{\Omega}\int_D\frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0\mathrm{d}q_0'}{(2\pi)^2}\Bigg[\,\frac{1}{4}\,(k_z+2q_z)(-k_z+2q_z')\,k_\perp^2\\
&\quad\times\left\langle\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\right.\\
&\quad\left.\times\,\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu'\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu'\right)\right\rangle_C\\
&+\frac{1}{3}\,k_z\,(-k_z+3q_z)\,k_\perp^2\\
&\quad\times\left\langle\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\,(k_\mu)\;\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(-\frac{k_\mu}{3}+q_\mu\right)\right.\\
&\quad\left.\times\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{3}+q_\mu'\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{3}-q_\mu'-q_\mu\right)\right\rangle_C\Bigg]\\
&+o(\mathbf{s}^6),
\end{aligned}
\tag{C3}
$$

$$
\begin{aligned}
S^{(out)}(k_\mu) =&\frac{1}{\Omega}\,k^2\langle s^l(k_\mu)s^l(-k_\mu)\rangle\\
&+\frac{1}{\Omega}\int_D\frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\Bigg[\,\frac{1}{4}\,k^2\,k_\perp^2\\
&\quad\times\left\langle\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu,\omega\right)\right.\\
&\quad\left.\times\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu'\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu'\right)\right\rangle_C\\
&+\frac{1}{3}\,k^2\,k_\perp^2\\
&\quad\times\left\langle\mathbf{k}^{(l)}\cdot\mathbf{s}\,(k_\mu)\;\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{3}+q_\mu\right)\right.\\
&\quad\left.\times\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{3}+q_\mu'\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{\mathbf{k}}{3}-\mathbf{q}'-\mathbf{q},\omega\right)\right\rangle_C\Bigg]\\
&+o(\mathbf{s}^6),
\end{aligned}
\tag{C4}
$$

where $S^{(in)}(k_\mu)$ and $S^{(out)}(k_\mu)$ are defined as

$$
S^{(in)}(k_\mu)=\frac{1}{\Omega}\left\{\left\langle\delta n^{(in)}(k_\mu)\,\delta n^{(in)}(-k_\mu)\right\rangle-\left\langle\delta n^{(in)}(k_\mu)\right\rangle\left\langle\delta n^{(in)}(-k_\mu)\right\rangle\right\},\tag{C5}
$$

$$
S^{(out)}(k_\mu)=\frac{1}{\Omega}\left\{\left\langle\delta n^{(out)}(k_\mu)\,\delta n^{(out)}(-k_\mu)\right\rangle-\left\langle\delta n^{(out)}(k_\mu)\right\rangle\left\langle\delta n^{(out)}(-k_\mu)\right\rangle\right\}.\tag{C6}
$$

Here $\langle\cdots\rangle_C$ counts only contributions from the "connected diagrams ", which means that only the terms connecting $\delta n^{(in,\,out)}(k_\mu)$ and $\delta n^{(in,\,out)}(-k_\mu)$ should be taken, because of the subtraction of $\langle\delta n^{(in,\,out)}(k_\mu)\rangle\,\langle\delta n^{(in,\,out)}(-k_\mu)\rangle$ in Eq. (C5) and Eq. (C6). Here we define the $o(\mathbf{s})^4$ terms as $\delta S^{(in)}_{th.}(k_\mu)$ and $\delta S^{(out)}_{th.}(k_\mu)$. Then we obtain

$$
\begin{aligned}
\delta S^{(in)}_{th.}(k_\mu)=&\frac{1}{\Omega}\int_D\frac{\mathrm{d}^3\mathbf{q}\mathrm{d}^3\mathbf{q}'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}q_0\mathrm{d}q_0'}{(2\pi)^2}\Bigg[\;\frac{-(k_z+2q_z)^2}{4}\\
&\times\left\langle\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle\\
&\times\left\langle\mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle\\
&-\frac{k_z^2-4q_z^2}{4}\left\langle\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle\\
&\times\left\langle\hat{\mathbf{k}}^{(in)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\,\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle
\end{aligned}
\tag{C7}
$$

and

$$\delta S^{(in)}_{th.}(k_\mu) = \frac{1}{\Omega}\int_D \frac{\mathrm{d}^3q\mathrm{d}^3q'}{\{(2\pi)^3 n_B\}^2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\left[\frac{1}{4}\,k^2\,k_\perp^2\right.$$
$$\times\left\langle \mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}+q_\mu\right)\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}-q_\mu\right)\right\rangle$$
$$\times\left\langle \mathbf{k}^{(l)}\cdot\mathbf{s}\left(\frac{k_\mu}{2}-q_\mu\right)\mathbf{k}^{(l)}\cdot\mathbf{s}\left(-\frac{k_\mu}{2}+q_\mu\right)\right\rangle. \tag{C8}$$

Here the second term of order $\mathbf{s}(k_\mu)^4$ in Eq. (C3) and Eq. (C4) are neglected because they have smaller contributions compared to above terms.

Appendix D Free energy of compressed vortex lattice

The decrease of the mean field free energy $\mathcal{G}_0$ in Eq. (3.2) when the vortex lattice is compressed either isotropically or anisotropically is calculated as follows. If we change the vortex density as $n_B \to (1+\delta)\,n_B$, we have only to deform the reciprocal lattice as follows,

$$\begin{array}{lll} \text{i)} & \mathbf{G} \longrightarrow \sqrt{1+\delta}\;\mathbf{G} & \text{(isotropic compression),} \\ \text{ii)} & G_y \longrightarrow (1+\delta)\;G_y & \text{(anisotropic compression).} \end{array}$$

We then calculate the decrease of $\mathcal{G}_0$ for each case.

i) Isotropic compression

In this case the change of $\mathcal{G}_0$, which we denote as $\delta\mathcal{G}_0^{\mathrm{iso.}}$, is estimated as follows,

$$\begin{aligned}\frac{\delta\mathcal{G}_0^{\mathrm{iso.}}}{V} =& \frac{K\,n_B^2}{2}(1+\delta)^2\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}\sqrt{1+\delta}G}}{(1+\delta)\;G^2} - \frac{K\,n_B^2}{2}\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}\;G}}{G^2}, \\ =& \frac{K\,n_B^2}{2}\frac{1}{\pi k_{BZ}^2}\int_{k_{BZ}}^{\infty}2\,\pi\,G\mathrm{d}G\frac{e^{-\xi_{sc}\;G}}{G^2}\;\left\{\delta\;(1-\frac{1}{2}\xi_{sc}\;G)-\delta^2\xi_{sc}^2\;G^2\right\}, \\ \cong& -\frac{K\;n_B^2}{2}\;\delta^2\;\frac{1}{8\pi n_B}. \end{aligned} \tag{D1}$$

In the last line we used the approximation given in Eq. (B5)~Eq. (B7).

ii) Anisotropic compression

In this case we define $\delta\mathcal{G}_0^{\mathrm{an.}}$ as the change of free energy.

$$\begin{aligned}\frac{\delta\mathcal{G}_0^{\mathrm{an.}}}{V} =& \frac{K\,n_B^2}{2}(1+\delta)^2\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}\sqrt{G_x^2+(1+\delta)^2G_y^2}}}{G_x^2+(1+\delta)^2G_y^2} - \frac{K\,n_B^2}{2}\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}\;G}}{G^2}, \\ =& \frac{K\,n_B^2}{2}\sum_{\mathbf{G}\neq\mathbf{0}}\frac{e^{-\xi_{sc}\;G}}{G^2}\left\{\delta\;(2-\xi_{sc}\frac{G_y^2}{G}-2\frac{G_y^2}{G^2})\right. \\ & \left. -\delta^2\left(1-5\frac{G_y^2}{G^2}+4\frac{G_y^4}{G^4}-\frac{5}{2}\frac{G_x^2G_y^2}{G^4}+\frac{1}{2}\xi_{sc}^2\frac{G_y^4}{G^2}\right)\right\}, \\ =& \frac{K\,n_B^2}{2}\frac{1}{\pi k_{BZ}^2}\int_{k_{BZ}}^{\infty}G\mathrm{d}G\int_0^{2\pi}\mathrm{d}\theta \\ & \times\frac{e^{-\xi_{sc}\;G}}{G^2}\;\delta^2\;(1-5\cos^2\theta+4\cos^4\theta-\frac{5}{2}\xi_{sc}G\cos^2\theta\sin^2\theta+\frac{1}{2}\cos^4\theta) \\ \cong& -\frac{K\;n_B^2}{2}\;\delta^2\;\frac{1}{16\pi n_B}. \end{aligned} \tag{D2}$$

In estimating the last line we used,

$$\int_0^{2\pi}\mathrm{d}\theta\cos^n\theta\sin^m\theta = 2\pi\frac{(m-1)!!\;(n-1)!!}{(m+n)!!}, \tag{D3}$$

which holds for even m and n and $n!!$ represents $n \times (n-2) \cdots 4 \times 2$ for even n and $n \times (n-2) \cdots 3 \times 1$ for odd n.

From these equations we obtain the change of the total free energy, $\delta \mathcal{G}^{\text{iso.}}$ and $\delta \mathcal{G}^{\text{an.}}$ in each case as,

$$
\begin{aligned}
\delta \mathcal{G}^{\text{iso.}} &= -\frac{K}{2}(n_B\delta)^2 \frac{1}{8\pi n_B} + \frac{\phi_0^2}{8\pi}(n_B\delta - \delta h)^2, \\
\delta \mathcal{G}^{\text{an.}} &= -\frac{K}{2}(n_B\delta)^2 \frac{1}{16\pi n_B} + \frac{\phi_0^2}{8\pi}(n_B\delta - \delta h)^2,
\end{aligned} \tag{D4}
$$

where $\delta h \equiv \delta H^{ext}/\phi_0$ is the perturbation of the external field and $n_B \times \delta$, by definition, corresponds to the change of the internal magnetic field caused by the perturbation. From the present result we obtain the magnetic permeability as

$$
\begin{aligned}
\mu^{dif} &= \frac{n_B \times \delta}{\delta h} \\
&= \begin{cases} \left(1 - \dfrac{\phi_0}{8\pi B\lambda^2}\right)^{-1} & \text{(isotopic case)} \\ \left(1 - \dfrac{\phi_0}{16\pi B\lambda^2}\right)^{-1} & \text{(anisotropic case)} \end{cases}
\end{aligned} \tag{D5}
$$

Comparing these results with Eq. (3.33) and Eq. (3.34) it is understood that the discrepancy in the results of the mean field theory and the elastic theory arises from the difference of compressional mode of the vortex lattice which are taken into account.

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Figure Captions

Fig. 1. The configuration of $\hat{\mathbf{k}}^{(in)}$, $\hat{\mathbf{k}}^{(out)}$ and $\hat{\mathbf{k}}$.

Fig. 2. The original lattice and the compressed lattice are shown. (a) original lattice, (b) isotropically compressed lattice, (c) anisotropically compressed lattice.

Fig. 3. A typical diffusion process of the vortex segments contributing to Γ_t. In (a) the vortices in the second raw are tilted to generate the vortex component perpendicular to the magnetic field. In (b) one vortex cutting process takes place (indicated by arrows) to shift the vortex line by about a half of the mean vortex spacing. In (c) one more cutting process takes place to complete the shifting the vortex line by one mean vortex spacing. In the text, the energy barrier of these collective cutting process, shown in (d), is expressed by U_c with the size of the segment d_s.

Fig. 4. The schematic view of the behavior of "flux flow "($R^{\perp}$) and "Lorentz force free "($R^{//}$) resistivity.

Fig. 5. (a) The additional magnetic field, $\delta\mathbf{H}^{ext}$, applied perpendicular to the original field, $\mathbf{H}^a$ and the screening currents caused by $\delta\mathbf{H}^{ext}$. (b) The penetration of vortices in the vortex lattice state : the penetration occurs only from the upper and lower edges of the sample. (c) The penetration of vortices in the entangled vortex liquid state: the penetration occurs also from the side of the sample due to the vortex cutting process.

Fig. 6. The schematic view of the B-H^a curve in the mixed state of type-II superconductors.

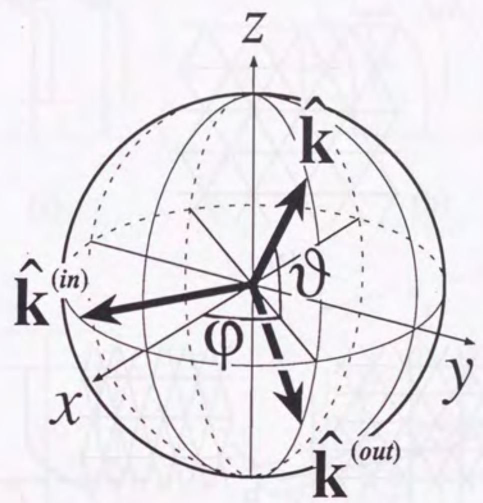


Fig. 1

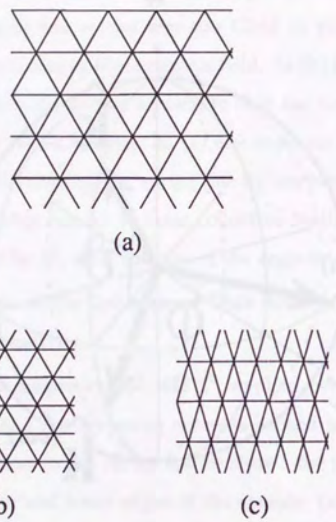


Fig. 2

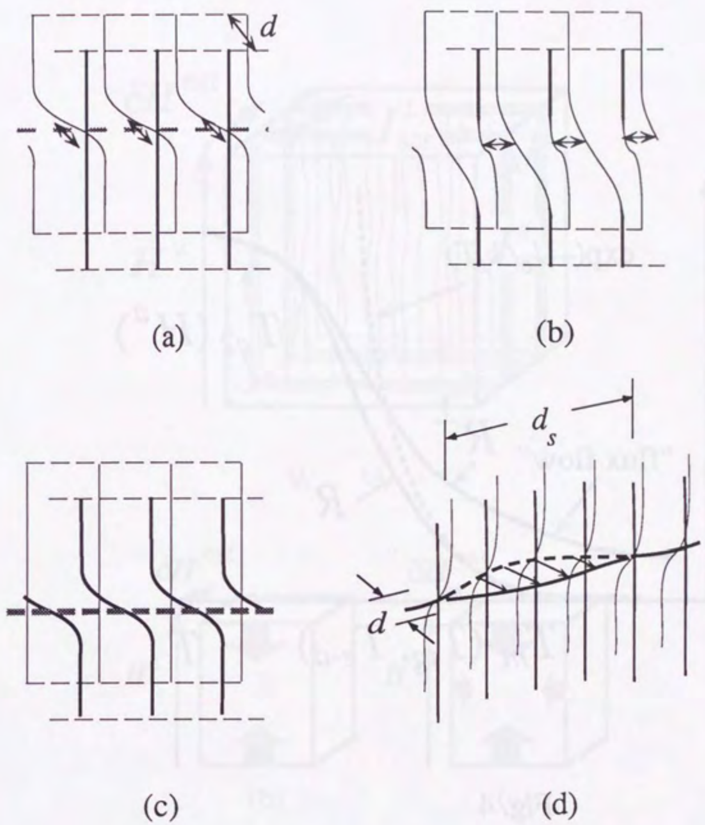


Fig. 3

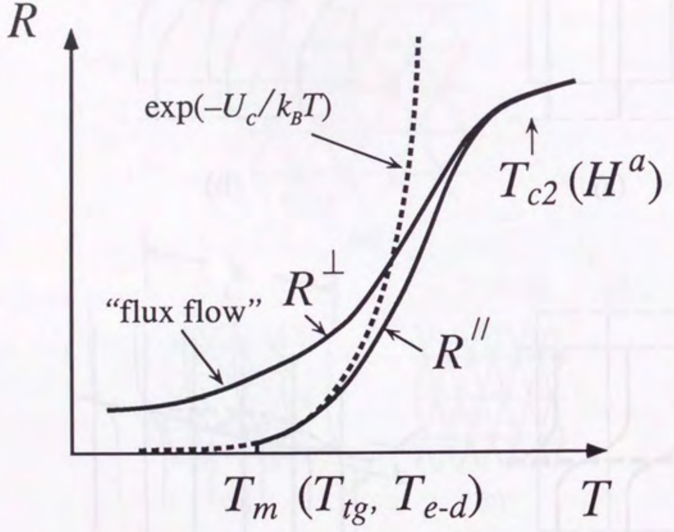
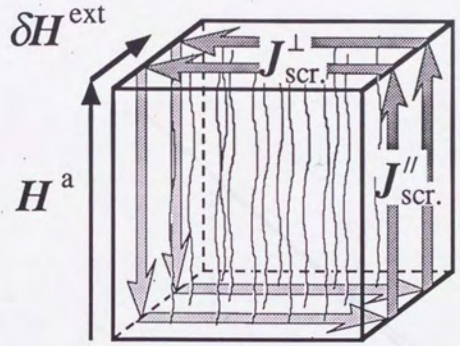
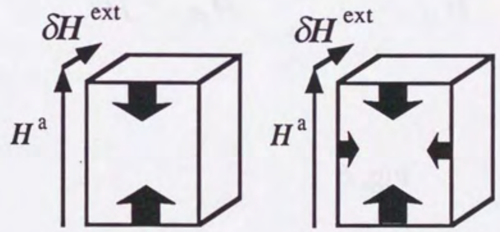


Fig. 4



(a)



(b) (c)

Fig. 5

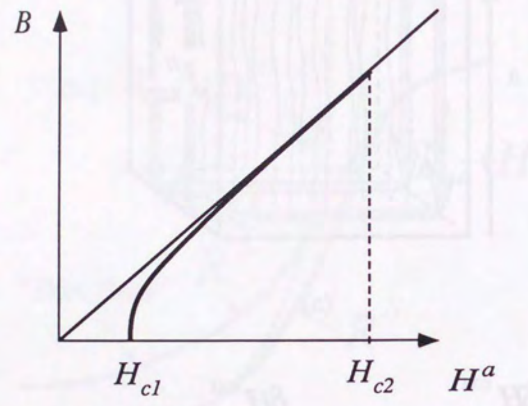


Fig. 6