博士論文

論文題目: Study of the Kähler-Ricci Flow and its Application in Algebraic Geometry

(ケーラー・リッチ流の研究とその代数幾何学における応用)

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Acknowledgments

The author would like to express his deep gratidude to Professor Shigeharu Takayama for continuous encouragement and many suggestions for many years. He would like to thank Takayuki Koike and Genki Hosono for discussions. He is appreciate for Doctor Henri Guenancia for deep hospitality and many discussion while his visit to Stony Brook. He is grateful to Professor Steven Lu for his kindness and advices.

Throughout the doctor couse, he is supported by the Program for Leading Graduate Schools, MEXT, Japan.

Chapter 1

Introduction and Main Results

This thesis is threefold. The fist one is the positivity of the holomorphic sectional curvature of compact Kähler manifolds. The second one is the Schwarz lemma for conical Kähler metrics. The third one is the scalar curvature behavior of the conical Kähler-Ricci flow. In this chapter, we summarize the backgrounds and main results for each topic.

In Chapter 3, we treat compact Kähler manifolds with positive or negative holomorphic sectional curvature. For a compact Kähler manifold (X, ω), we can associate two notions of curvature: the Ricci curvature $\mathrm{Ric}(\omega)$, and the holomorphic sectional curvature $H(\omega)$. The relation between them is not yet clear. For instance, it is still open that if X admits a Kähler form with semi-positive Ricci curvature, then X admits an another Kähler form with semi-positive holomorphic sectional curvature. However, by considering the $\mathrm{Ric}(\omega)$ represents the 1st Chern class of the anti-canonical bundle $-K_X$, we can ask the following weaker queation: What is the relationship between the positivity of K_X and the negativity of holomorphic sectional curvature? This question was originally raised by Yau (see [HLW16, Conjecture 1.2]). Recently, Wu-Yau and Tosatti-Yang obtained the following answer to this question.

Theorem A (=Theorem 3.3.1, [WY16a, Theorem 2], [ToY15, Corollary 1.3]). *If X admits a Kähler form with strictly negative holomorphic sectional curvature, then the canonical bundle K_X is ample. In particular, X is projective.*

Theorem B (=Theorem 3.3.2, [ToY15, Theorem 1.1]). *If X admits a Kähler form with semi-negative holomorphic sectional curvature, then the canonical bundle K_X is nef.*

The original proofs of both theorems are based on the following idea, in [WY16a], constructing a Kähler form $\omega_\varepsilon \in 2\pi c_1(K_X) + \varepsilon[\widehat{\omega}]$ satisfying

$$\mathrm{Ric}(\omega_\varepsilon) = -\omega_\varepsilon + \varepsilon\widehat{\omega},$$

and considering the limiting behavior of ω_ε as $\varepsilon \searrow 0$. Here, $\widehat{\omega}$ is a Kähler form whose holomorphic sectional curvature is (strictly/semi-) negative.

In Chapter 3, we give an another proof of these theorems via Kähler-Ricci flow. Our proof is, in a sense, a parabolic proof of them [Nom16a].

We remark that Diverio and Trapani [DT16] showed that the amplenss of K_X can be obtained under the assumption that the holomorphic sectional curvature is semi-negative everywhere and strictly negative at one point. For the moment, we can only prove the above two theorems.

In Chapter 4, we treat the Schwarz lemma for conical Kähler metrics. The Schwarz–Pick lemma states that any holomorphic map between the unit disks in the complex plane decreases the Poincaré metrics. After that, Ahlfors [Ahl38] generalized it to a holomorphic map from the unit disk to a hyperbolic Riemann surface. For higher dimensions, Yau [Yau78a] showed that any holomorphic map from complete Kähler manifold whose Ricci curvature is bounded from below to a Hermitian manifold whose holomorphic bisectional curvature is bounded by a negative constant decreases the metric up to a multiplicative constant. Also, he showed that, under similar conditions on curvatures, any holomorphic map decreases the volume forms up to a multiplicative constant. Both results essentially based on his maximum principle for complete Riemannian manifolds. Later on, many generalizations obtained in various geometric settigs.

We forcus on the conical Kähler metrics, for short, cone metrics. Let X be a compact Kähler manifold of dimension n, D be a smooth divisor on X, and β be a real number satisfying $0 < \beta < 1$. The cone metric ω with cone angle $2\pi\beta$ along D is a Kähler metric on $X \setminus D$ which is locally quasi-isometric to the standard cone metric

$$\omega_\beta := \frac{\beta^2}{|z|^{2(1-\beta)}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\overline{z^1} + \sum_{i=2}^{n} \frac{\sqrt{-1}}{2} dz^i \wedge d\overline{z^i},$$

and satisfies some regularity conditions. For a precise definition of the cone metric, see Definition 4.2.2. The notion of cone metrics plays an important role in recent advances in Kähler geometries, in particular Kähler -Einstein problems, for instance see [CDS15a, CDS15b, CDS15c], and [Tia15].

To state the theorems, we use the following setups and notations. Let X and Y be compact Kähler manifolds, $D \subset X$, $E \subset Y$ be smooth divisors, and $f\colon X \to Y$ be a surjective holomorphic map satisfying $f^*(E) = kD$ with $k \in \mathbb{Z}_{>0}$. Let ω_X (resp. ω_Y) be a cone metric with cone angle $2\pi\alpha$ (resp. $2\pi\beta$) along D (resp. E) on X (resp. Y). Let $s \in H^0(X, \mathcal{O}_X(D))$ be a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D and h be a smooth Hermitian metric on it satisfying $|s|_h \leq 1$. Let $C > 0$ be an upper bound for the Chern curvature of h i.e. $\sqrt{-1}\, R_h \leq C\omega_X$. For a Kähler form ω, we will denote by $\mathrm{Ric}(\omega)$ the Ricci curvature of ω, $R(\omega)$ the scalar curvature of ω, and $\mathrm{Bisec}(\omega)$ the bisectional curvature of ω.

Schwarz lemma for the cone metrics obtained by Jeffres [Jef00a] is states as follows.

Theorem ([Jef00a, Theorem]). *Assume that* $\dim X = \dim Y = n$*, the cone angles satisfy* $\alpha \leq \beta$ *and there exists non-negative constants* $A, B \geq 0$ *satisfying*

$$R(\omega_X) \geq -A, \ \mathrm{Ric}(\omega_Y) \leq -B\omega_Y < 0.$$

Then, the volume forms satisfy

$$f^*\omega_Y^n \le \left(\frac{A}{B}\right)^n \omega_X^n \quad \textit{on } X \setminus D.$$

Since the cone metric is not complete on $X \setminus D$, we cannot apply the maximum principle argument directly. Jeffers overcame this difficulty by using a barrier function, called "Jeffres' trick". However, his original proof seems to need more assumptions on the regularity of the cone metrics along D as in Definition 4.2.1 (see the proof of Proposition 4.3.6). We will generalize this theorem to a general cone angle and prove a Schwarz lemma for cone metrics [Nom16b].

Theorem C (=Theorem 4.3.3)**.** *Assume that* $\dim X = \dim Y = n$ *and the curvature condition in above theorem holds.*

(a) Suppose $\alpha \le k\beta$*. Then we have*

$$f^*\omega_Y^n \le \left(\frac{A}{nB}\right)^n \omega_X^n \quad \textit{on } X \setminus D.$$

(b) Suppose $\alpha > k\beta$*. Then we have*

$$f^*\omega_Y^n \le \left(\frac{A + (\alpha - k\beta)C}{nB}\right)^n \frac{\omega_X^n}{|s|_h^{2(\alpha - k\beta)}} \quad \textit{on } X \setminus D.$$

Theorem D (=Theorem 4.3.4)**.** *Assume that there exists non-negative constants* $A, B \ge 0$ *such that the curvatures satisfy the following:*

$$\mathrm{Ric}(\omega_X) \ge -A\omega_X, \ \mathrm{Bisec}(\omega_Y) \le -B < 0.$$

(a) Suppose $\alpha \le k\beta$*. Then we have*

$$f^*\omega_Y \le \frac{A}{B}\omega_X \quad \textit{on } X \setminus D.$$

(b) Suppose $\alpha > k\beta$*. Then we have*

$$f^*\omega_Y \le \frac{A + (\alpha - k\beta)C}{B} \frac{\omega_X}{|s|_h^{2(\alpha - k\beta)}} \quad \textit{on } X \setminus D.$$

We remark that the condition $\alpha \le k\beta$ on cone angles in the statement (a) is weaker than assumptions in Jeffres' Schwarz Lemma.

In Chapter 5, we consider the normalized conical Kähler-Ricci flow ω_t on X which is a family of cone metrics with cone angle $2\pi\beta$ along D satisfying the following parabolic equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_t & = -\mathrm{Ric}(\omega_t) - \omega_t + 2\pi(1-\beta)[D], \\ \omega_t|_{t=0} & = \omega^*, \end{cases}$$

where $[D]$ is the current of integration over D, and ω^* is a certain initial cone metric defined later (see (1.0.1)). In the case of $D = 0$, ω_t is called the normalized Kähler-Ricci flow. This case has been studied extensively in the past decades (see [TZ06, ST16b, ST12, CW12, BEG13, CW14a, CT15, GSW15, TZ16] and the references therein).

The maximal existence time T of the normalized conical Kähler-Ricci flow ω_t is characterized by the following cohomological condition:

$$T = \sup\{t > 0 \mid [\omega_t] = e^{-t}[\omega_0] + (1 - e^{-t})2\pi c_1(K_X + (1-\beta)D) \text{ is Kähler } \}.$$

In particular, the limiting class $[\omega_T]$ is nef but not Kähler. This characterization is due to Shen [She14a, She14b]. As t tends to T, the flow ω_t might form singularities. The analysis of the singularities, in particular its curvature behavior, is one of the main objects in the study of the geometric flows. Our purpose here is to investigate the scalar curvature behavior of ω_t with finite time singularites (i.e. $T < \infty$) as t approaches to T.

In the infinite time singularites case (i.e. $T = \infty$), the uniform boundedness of the scalar curvature of the normalized Kähler-Ricci flow (i.e. $D = 0$) was proved by Zhang [Zha09] when K_X is nef and big. This result was extended by Song-Tian [ST16a] when K_X is semi-ample. Furthermore, Edwards [Edw15] generalized these results to the conic setting. In the case of Fano manifolds (i.e. $-K_X$ is ample), Perelman established a uniform boundedness of it (see [SeT08]) and Liu-Zhang [LZ14] extended to the conic case.

On the other hand, in the finite time singularities case, Zhang [Zha10] showed that the scalar curvature $R(\omega_t)$ of the normalized Kähler-Ricci flow ω_t satisfies

$$R(\omega_t) \leq \frac{C}{(T-t)^2}$$

assuming the semi-ampleness of $[\omega_T]$. This condition is natural in terms of the deep relationship between the Kähler-Ricci flow and the minimal model program (see [ST16b, Zha10]). Our main theorem generalizes this to the conic setting.

We assume the following contraction type condition on the cohomology class $[\omega_T]$. Let $f \colon X \to Z$ be a holomophic map between compact Kähler manifolds, whose image is contained in a normal irreducible subvariety Y of Z. Let D_Y be an effective Cartier divisor on Y such that the pullback of D_Y satisfies $D = f^*D_Y$. Let h_Y be a smooth Hermitian metric on the line bundle $\mathcal{O}_Y(D_Y)$ in the sense of [EGZ09, Section 5], and s_Y be a holomorphic section of $\mathcal{O}_Y(D_Y)$ whose zero divisor is D_Y. We define the initial cone metric ω^* by

$$\omega^* := \omega_0 + k\sqrt{-1}\,\partial\bar{\partial}|s|_h^{2\beta}, \tag{1.0.1}$$

where ω_0 is a smooth Kähler form on X, $k \in \mathbb{R}_{>0}$ is a sufficiently small real number, $s := f^*s_Y$ is the holomorphic section of $\mathcal{O}_X(D)$, and $h := f^*h_Y$ is the smooth Hermitian metric on $\mathcal{O}_X(D)$. We remark that if we take k sufficiently small, ω^* is actually a cone metric with cone angle $2\pi\beta$ along D.

Let ω_t be the normalized conical Kähler-Ricci flow with initial cone metric ω^*, and T be the maximal existence time of ω_t. We further assume that T is finite and there exists a smooth Kähler form ω_Z on Z satisfying

$$[f^*\omega_Z] = [\omega_T] \in H^{1,1}(X, \mathbb{R}).$$

Under these assumptions, we have the following theorem [Nom16c].

Theorem E (=Theorem 5.1.5)**.** *The scalar curvature $R(\omega_t)$ of ω_t satisfies*

$$R(\omega_t) \le \frac{C}{(T-t)^2} \quad \text{on } X \setminus D,$$

where $C > 0$ is a constant independent of t.

In contrast with Zhang's result, we need to treat with the singularities of ω_t along D. This is overcame by using the approximation technique used in [CGP13, She14a, LZ14, Edw15].

Chapter 2

Preliminaries

In this chapter, we recall some definitions and properties of Kähler manifolds. In Section 2.1 to 2.4, we recall basic Kähler geometry, especially the notion of curvature. In Section 2.5, we fix some notations. In Section 2.6, we recall the maximum principle argument which will be used in many places. In Section 2.7, we prove the Laplacian estimate called Chern-Lu's inequality.

2.1 Curvature of Hermitian Holomorphic Vector Bundles

Let E be a complex vector bundle on a C^∞-manifold on M. A $\mathbb{C}$-linear map $\nabla\colon C^\infty(M,E) \to C^\infty(M, \bigwedge^1 \otimes E)$ is called a *connection* of E if it satisfies the Leibniz rule i.e for any function $f \in C^\infty(M,\mathbb{C})$ and any section $s \in C^\infty(M,E)$, we have the following:

$$\nabla(fs) = (df) \otimes s + f(\nabla s).$$

For any vector field $X \in C^\infty(M,TM)$, we set $\nabla_X s := (\nabla s)(X) \in C^\infty(M,E)$. We can extend ∇ to a $\mathbb{C}$-linear map $d_\nabla\colon C^\infty(M, \bigwedge^p \otimes E) \to C^\infty(M, \bigwedge^{p+1} \otimes E)$ which is defined by

$$\begin{aligned}(d_\nabla s)(X_1,\ldots,X_{p+1}) := &\sum_{i=1}^{p}(-1)^{i+1}\nabla_{X_i}\Big(s(X_1,\ldots,\widehat{X_i},\ldots,X_{p+1})\Big)\\ &+\sum_{i<j}(-1)^{i+j}s([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_{p+1}),\end{aligned}$$

where $s \in C^\infty(M, \bigwedge^p \otimes E)$ and $X_1,\cdots,X_{p+1} \in C^\infty(M,TM)$. The map d_∇ is called the *covariant differentiation* associated to ∇. Since the definition of d_∇ is similar to that of the exterior derivative d, we have the following generalized Leibniz rule: for any $\eta \in C^\infty(M,\bigwedge^k)$ and $s \in C^\infty(M, \bigwedge^p \otimes E)$, we have

$$d_\nabla(\eta \wedge s) = (d\eta) \wedge s + (-1)^k \eta \wedge (d_\nabla s).$$

It is easily seen that $d_\nabla \circ d_\nabla$ is $C^\infty(X,\mathbb{C})$-linear map. Thus, we can define the *curvature* $R_\nabla \in C^\infty(M, \bigwedge^2 \otimes \mathrm{End}(E))$ of the connection ∇ satisfying

$$(d_\nabla \circ d_\nabla)s = R_\nabla \wedge s,$$

for any $s \in C^\infty(M, \bigwedge^p \otimes E)$. By taking local frame, we can show that $\mathrm{tr}(R_\nabla)$ is a closed 2-form on M.

Let (E, h) be a Hermitian complex vector bundle on M. A connection ∇ of E is said to be *h-compatible* if for any section $s, t \in C^\infty(M, E)$ the following holds:

$$d \langle s, t \rangle_h = \langle \nabla s, t \rangle_h + \langle s, \nabla t \rangle_h . \tag{2.1.1}$$

Here $\langle \cdot, \cdot \rangle_h : C^\infty(M, \bigwedge^p \otimes E) \times C^\infty(M, \bigwedge^q \otimes E) \to C^\infty(M, \bigwedge^{p+q} \otimes E)$ is s sesquilinear form obtained by combining the wedge product and h. It is easily seen that for all $s \in C^\infty(M, \bigwedge^p \otimes E)$ and $t \in C^\infty(M, \bigwedge^q \otimes E)$, the following holds:

$$d \langle s, t \rangle_h = \langle \nabla s, t \rangle_h + (-1)^p \langle s, \nabla t \rangle_h .$$

Therefore, by applying d again, we get the curvature R_∇ of h-compatible connection ∇ is skew-Hermitian i.e. $\langle R_\nabla s, t \rangle_h = - \langle s, R_\nabla t \rangle_h$. In particular $\mathrm{tr}(\sqrt{-1}\, R_h)$ is a closed real 2-form on M.

Let X be a complex manifold of dimension n and $J \in C^\infty(X, \mathrm{End}(TX))$ be the complex structure of X. Since $J^2 = -\mathrm{id}_{TX}$, the complexified tanget bundle $T^{\mathbb{C}}X := TX \otimes \mathbb{C}$ is decomposed into two eigen spaces:

$$\begin{aligned} T^{\mathbb{C}}X &= T^{1,0}X \oplus T^{0,1}X, \\ T^{1,0}X &:= \{v \in T^{\mathbb{C}}X \mid Jv = \sqrt{-1}\, v\}, \\ T^{0,1}X &:= \{v \in T^{\mathbb{C}}X \mid Jv = -\sqrt{-1}\, v\}. \end{aligned}$$

Then, $T^{1,0}X$ is a holomorphic vector bundle, and $T^{0,1}X$ is an anti-holomorphic vector bundle. Dually, the complexified cotanget bundle $T^{*\mathbb{C}}X := T^*X \otimes \mathbb{C}$ also decomposed into $T^{*\mathbb{C}}X = T^{*1,0}X \oplus T^{*0,1}X$. We set $\bigwedge^{p,q} := \bigwedge^p T^{*1,0}X \otimes \bigwedge^q T^{*0,1}X$. For a section $s \in C^\infty(X, \bigwedge^{p,q})$, s is called a (p,q)-form and (p,q) is called the bidegree of s. By using the fact that J is integrable, the exterior covariant derivative $d \colon C^\infty(X, \bigwedge^{p,q}) \to C^\infty(X, \bigwedge^{p+q+1})$ is decomposed into $d = \partial + \overline{\partial}$ where $\partial \colon C^\infty(X, \bigwedge^{p,q}) \to C^\infty(X, \bigwedge^{p+1,q})$, $\overline{\partial} \colon C^\infty(X, \bigwedge^{p,q}) \to C^\infty(X, \bigwedge^{p,q+1})$. If we take a local holomorhic coordinate $(z^1, \cdots, z^n)$ of X, ∂ and $\overline{\partial}$ is expressed as

$$\begin{aligned} \partial \eta &= \sum_{|I|=p, |J|=q} \sum_{i=1}^n \frac{\partial \eta_{I,\overline{J}}}{\partial z^i} dz^i \wedge dz^I \wedge d\overline{z^J}, \\ \overline{\partial} \eta &= \sum_{|I|=p, |J|=q} \sum_{i=1}^n \frac{\partial \eta_{I,\overline{J}}}{\partial \overline{z^i}} d\overline{z^i} \wedge dz^I \wedge d\overline{z^J}, \end{aligned}$$

where $\eta = \sum_{|I|=p,|J|=q} \eta_{I,\overline{J}} dz^I \wedge d\overline{z^J}$ is a (p,q)-form and the sum runs for all $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_q)$ satisfying $1 \leq i_1 < \cdots < i_p \leq n$ and $1 \leq j_1 < \cdots < j_q \leq n$. Since $d \circ d = 0$, by considering the bidegree, we get $\partial \circ \partial = \overline{\partial} \circ \overline{\partial} = 0$ and $\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0$.

For a holomorphic vector bundle E of rank r on X, we can naturally associate a $\mathbb{C}$-linear map $\overline{\partial}_E \colon C^\infty(X, E) \to C^\infty(X, \bigwedge^{0,1} \otimes E)$ called the *$\overline{\partial}$-operator* of E. By using a local holomorphic frame $(e_1, \cdots, e_r)$ of E, $\overline{\partial}_E$ is defined as

$$\overline{\partial}_E s = \overline{\partial}_E \left(\sum_{\lambda=1}^r s^\lambda e_\lambda \right) := \sum_{\lambda=1}^r (\overline{\partial} s^\lambda) \otimes e_\lambda,$$

where $s \in C^\infty(X, E)$ is locally denoted as $s = \sum_{\lambda=1}^r s^\lambda e_\lambda$. Since E is holomorphic, $\overline{\partial}_E$ is a well-defined $\mathbb{C}$-linear map satisfying Leibniz rule i.e. for any function $f \in C^\infty(X, \mathbb{C})$ and for any section $s \in C^\infty(X, E)$, we have the following:

$$\overline{\partial}_E(fs) = (\overline{\partial} f) \otimes s + f(\overline{\partial}_E s).$$

As in the case of the connection, we can extend $\overline{\partial}_E$ to a $\mathbb{C}$-linear map $\overline{\partial}_E \colon C^\infty(X, \bigwedge^{p,q} \otimes E) \to C^\infty(X, \bigwedge^{p,q+1} \otimes E)$ such that for any $\eta \in C^\infty(X, \bigwedge^k)$ and $s \in C^\infty(X, \bigwedge^{p,q} \otimes E)$, we have

$$\overline{\partial}_E(\eta \wedge s) = (\overline{\partial}\eta) \wedge s + (-1)^k \eta \wedge (\overline{\partial}_E s).$$

We remark that the $\overline{\partial}_E$ satisfies $\overline{\partial}_E \circ \overline{\partial}_E = 0$.

For a Hermitian holomorphic vector bundle (E, h) on X, there exists a unique connection ∇ on E, called the *Chern connection* of (E, h), which is h-compatible and $(0,1)$-part of ∇ coincides with $\overline{\partial}_E$. We denote $\nabla = \nabla_h = \partial_h + \overline{\partial}_E$ where ∂_h is the $(1,0)$-part of ∇. The exterior covariant derivative $d_h := d_{\nabla_h}$ of the Chern connection ∇_h decomposed into $d_h = \partial_h + \overline{\partial}_E$ where we extend ∂_h to a $\mathbb{C}$-linear map $C^\infty(X, \bigwedge^{p,q} \otimes E) \to C^\infty(X, \bigwedge^{p+1,q} \otimes E)$. By considering the bidegree in (2.1.1), we get

$$\partial \langle s, t \rangle_h = \langle \partial_h s, t \rangle_h + \left\langle s, \overline{\partial}_E t \right\rangle_h, \ \overline{\partial} \langle s, t \rangle_h = \left\langle \overline{\partial}_E s, t \right\rangle_h + \langle s, \partial_h t \rangle_h . \tag{2.1.2}$$

Let $(e_1, \cdots, e_r)$ be a local holomorphic frame of E, $(e^1, \cdots, e^r)$ be the dual frame of E^* and $(z^1, \cdots, z^n)$ be a holomorphic local coordinate of X. We set $h_{\lambda\overline{\mu}} := h(e_\lambda, e_\nu)$. By using (2.1.2), we can represent the Chern connection as follows:

$$\partial_h e_\lambda = h^{\nu\overline{\mu}} \partial h_{\lambda\overline{\mu}} \otimes e_\nu, \ \overline{\partial}_E e_\lambda = 0.$$

Also, the *Chern curvature* $R_h := R_\nabla$ of the Chern connection is represented as

$$\begin{aligned} R_h =& \overline{\partial}_E \partial_h + \partial_h \overline{\partial}_E = R_{i\bar{j}\lambda}{}^\mu dz^i \wedge d\overline{z^j} \otimes e^\lambda \otimes e_\mu, \\ & R_{i\bar{j}\lambda}{}^\mu = -h^{\mu\overline{\nu}}(\partial_i \partial_{\bar{j}} h_{\lambda\overline{\nu}}) + h^{\mu\overline{\nu}} h^{\kappa\overline{\tau}} (\partial_i h_{\lambda\overline{\tau}})(\partial_{\bar{j}} h_{\kappa\overline{\nu}}). \end{aligned}$$

In particular if (L, h) is a Hermitian holomorphic line bundle i.e. $\mathrm{rank}(L) = 1$ and if we take a holomorphic frame e of L and set $H := h(e, e)$, we have the formula

$$\sqrt{-1}\, R_h = -\sqrt{-1}\, \partial\overline{\partial} \log H.$$

2.2 Kähler Manifolds

Let X be a complex manifold of complex dimension n and $J \in C^\infty(X, \mathrm{End}(TX))$ be the complex structure of X. A Riemannian metric g on X is called *J-invariant* if for any tangent vector $v, w \in TX$, g satisfies

$$g(Jv, Jw) = g(v, w).$$

By extending $\mathbb{C}$-bilinearly, g defines a non-degenerate $\mathbb{C}$-bilinear form on $T^{\mathbb{C}}X$, which we also write g. J-invariant Riemannian metric g defines a Hermitian metric h_g on $T^{\mathbb{C}}X$ by $h_g(v,w) := g(v,\overline{w})$, and 2-form ω_g by $\omega_g := g(J\cdot,\cdot)$. Since g is J-invariant, $T^{1,0}X$ and $T^{0,1}X$ is orthogonal with respect to h_g, and ω_g is a real $(1,1)$-form on X. If we take a local holomorphic chart $(U,(z^1.\cdots,z^n))$, ω_g is locally written as

$$\omega_g = \sqrt{-1}\sum_{i,j=1}^{n} g_{i\bar{j}}dz^i \wedge d\overline{z^j},\ g_{i\bar{j}} := g\left(\frac{\partial}{\partial z^i},\frac{\partial}{\partial \overline{z^j}}\right).$$

In this notation, $\left(g_{i\bar{j}}(x)\right)_{i,j}$ is a positive definite Hermitian matrix for any $x \in U$.

A J-invariant Riemannian metric g on X is called a *Kähler metric* if correspoinding $(1,1)$-form ω_g is d-closd. This condition is equivalent to

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial g_{k\bar{l}}}{\partial z^i}$$

for any $i,j,k = 1,\cdots,n$. We call ω_g a *Kähler form* associated to g, and (X,J,g,ω_g) a Kähler manifold.

A real $(1,1)$-form ω is called *positive* if we write ω as $\omega = \sqrt{-1}\sum_{i,j=1}^n g_{i\bar{j}}dz^i\wedge d\overline{z^j}$ locally, the Hermitian matrix $\left(g_{i\bar{j}}(x)\right)_{i,j}$ is a positive definite. For instance, any Kähler form is positive. We remark that for any positive closed real $(1,1)$-form ω, if we set g by $g := \omega(\cdot, J\cdot)$, then g is a Kähler metric satisfying $\omega_g = \omega$. In the following, a positive closed real $(1,1)$-form ω is also called a Kähler form on X without specifing corresponding Kähler metric.

2.3 Cohomology of Compact Kähler Manifolds

Let E be a holomorphic vector bundle on a compact complex manifold X. Since $\overline{\partial}_E \circ \overline{\partial}_E = 0$, we define the *Dolbeault cohomology* of E by

$$H^{p,q}(X,E) := \frac{\mathrm{Ker}(\overline{\partial}_E\colon C^\infty(X,\bigwedge^{p,q}\otimes E) \to C^\infty(X,\bigwedge^{p,q+1}\otimes E))}{\mathrm{Im}(\overline{\partial}_E\colon C^\infty(X,\bigwedge^{p,q-1}\otimes E) \to C^\infty(X,\bigwedge^{p,q}\otimes E))}.$$

It is well-known that $H^{p,q}(X,E)$ is finite dimensional vector space. By the Dolbeault-Grothendieck lemma, $H^{p,q}(X,E)$ is isomorphic to the q-th cohomology group $H^q(X,\Omega_X^p\otimes E)$ of sheaf of holomophic E-valued p-forms. We set $H^{p,q}(X,\mathbb{C}) := H^{p,q}(X,\mathcal{O}_X)$ and $H^{p,p}(X,\mathbb{R}) := H^{p,p}(X,\mathbb{C}) \cap H^{2p}_{dR}(X,\mathbb{R})$

On a compact Kähler manifold X, the following $\partial\overline{\partial}$-lemma is essential.

Lemma 2.3.1 ($\partial\overline{\partial}$-lemma)**.** *Assume that $p,q \geq 1$ and $u \in C^\infty(X,\bigwedge^{p,q})$ is d-exact i.e. $u = dv$ for some $v \in C^\infty(X,\bigwedge^{p+q-1})$. Then there exists $w \in C^\infty(X,\bigwedge^{p-1,q-1})$ such that $u = \partial\overline{\partial}w$.*

This lemma is a corollary of the Hodge theory. In particular, we have the following identification:

$$H^{1,1}(X,\mathbb{R}) \cong \frac{\{\eta \in C^\infty(X,\bigwedge^{1,1}) \mid d\eta = 0,\ \alpha = \overline{\alpha}\}}{\sqrt{-1}\,\partial\overline{\partial}C^\infty(X,\mathbb{R})}.$$

We define the *Kähler cone* $\mathcal{K}_X$ of X by

$$\begin{aligned}\mathcal{K}_X := \{\alpha \in H^{1,1}(X,\mathbb{R}) \mid\ &\alpha \text{ is represented by a Kähler form on } X\} \\ \subset H^{1,1}(X,\mathbb{R}).\end{aligned}$$

A cohomology class $\alpha \in H^{1,1}(X,\mathbb{R})$ is said to be a *Kähler class* if $\alpha \in \mathcal{K}_X$ and a *nef class* if $\alpha \in \overline{\mathcal{K}_X}$. Here $\overline{\mathcal{K}_X}$ is the closure of $\mathcal{K}_X$ with respect to the Euclidean topology in the finite dimensional vector space $H^{1,1}(X,\mathbb{R})$. It is easily seen that $\mathcal{K}_X$ is an open convex cone in $H^{1,1}(X,\mathbb{R})$.

By the Chern-Weil theory, for any complex vector bundle E on X with connection ∇, the 1st Chern class $2\pi c_1(E)$ of E is represented by $\mathrm{tr}(\sqrt{-1}\,R_\nabla)$ in $H^2_{dR}(X,\mathbb{R})$. In particular, since the bidegree of the Chern curvature is $(1,1)$, for any holomophic vector bundle E, the 1st Chern class $2\pi c_1(E)$ of E belongs to $H^{1,1}(X,\mathbb{R})$. Furthermore, thanks to Kodaira's embedding theorem, the ampleness of a holomophic line bundle L on X is equivalent to $2\pi c_1(L) \in \mathcal{K}_X$. Therefore, the Kähler cone $\mathcal{K}_X$ closely related to the notion of positivity.

2.4 Curvature of Kähler Manifolds

In this section, we define various notions of curvature of Kähler manifolds. Let (X, J, g, ω) be a Kähler manifold of dimension n. The Kähler metric g defines the Levi-Civita connection ∇^{LC} on the tangent bundle TX by

$$\begin{aligned}2g(\nabla^{LC}_u v, w) := &\, ug(v,w) + vg(w,u) - wg(u,v) \\ &+ g([u,v],w) - g([v,w],u) - g([u,w],v)\end{aligned}$$

where $u, v, w \in C^\infty(X, TX)$. This is the unique g-compatible torsion free connection i.e. for any vector fields $u, v, w \in C^\infty(X, TX)$, ∇^{LC} satisfies

$$\begin{aligned}u(g(v,w)) &= g(\nabla^{LC}_u v, w) + g(v, \nabla^{LC}_u w), \\ \nabla^{LC}_u v &= \nabla^{LC}_v u + [u,v].\end{aligned}$$

By extending $\mathbb{C}$-linearly, ∇^{LC} induces a connection on $T^{\mathbb{C}}X$, which we write simply ∇^{LC}. We set D the restriction of ∇^{LC} to $T^{1,0}X$. By the definition of ∇^{LC} and the Kähler condition $d\omega = 0$, D coincides with the Chern connection of the Hermitian holomorphic vector bundle $(T^{1,0}X, h_g)$. We decompose D into $D = \nabla + \overline{\nabla}$ where ∇ is the $(1,0)$-part and $\overline{\nabla}$ is the $(0,1)$-part of D. If we take a local holomorphic cordinate $(z^1.\cdots, z^n)$, the curvature tensor $R_D \in C^\infty(X, \bigwedge^{1,1} \otimes \mathrm{End}(T^{1,0}X))$ of D is locally written as

$$R_D = R_{i\bar{j}k}{}^l dz^i \wedge d\overline{z^j} \otimes dz^k \otimes \frac{\partial}{\partial z^l},\ R_{i\bar{j}k}{}^l = -g^{l\bar{r}}(\partial_i \partial_{\bar{j}} g_{k\bar{r}}) + g^{l\bar{r}} g^{p\bar{q}} (\partial_i g_{l\bar{q}})(\partial_{\bar{j}} g_{p\bar{r}}).$$

We define the *Riemann curvature tensor* $\mathrm{Rm}(\omega)$ by

$$\begin{aligned}\mathrm{Rm}(\omega) = &R_{i\bar{j}k\bar{l}} dz^i \otimes dz^j \otimes dz^k \wedge d\overline{z^l} := g_{p\bar{j}} R_i{}^p{}_{k\bar{l}} dz^i \otimes dz^j \otimes dz^k \wedge d\overline{z^l}, \\ &R_{i\bar{j}k\bar{l}} = -\partial_i \partial_{\bar{j}} g_{k\bar{l}} + g^{p\bar{q}} (\partial_i g_{k\bar{q}})(\partial_{\bar{j}} g_{p\bar{l}}),\end{aligned}$$

the *Ricci curvature* $\mathrm{Ric}(\omega)$ by

$$\mathrm{Ric}(\omega) := \sqrt{-1}\, R_{i\bar{j}} dz^i \wedge d\overline{z^j},\ R_{i\bar{j}} = g^{k\bar{l}} R_{k\bar{l}i\bar{j}} = \partial_i \partial_{\bar{j}} \log\det(g_{p\bar{q}}),$$

and the *scalar curvature* $R(\omega)$ by

$$R(\omega) := g^{i\bar{j}} R_{i\bar{j}} = g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log\det(g_{p\bar{q}}).$$

Since ω^n defines a Hermitian metric on the anti-canonical bundle $-K_X := \bigwedge^n T^{1,0}X$ and its Chern curvature is equal to the Ricci curvature of ω, the Ricci curvature $\mathrm{Ric}(\omega)$ represents the first Chern class $2\pi c_1(X) = -2\pi c_1(K_X) \in H^{1,1}(X,\mathbb{R})$ of X.

For any tangent vector $\xi = \xi^i \partial/\partial z^i, \eta = \eta^i \partial/\partial z^i \in T^{1,0}X$, we define the *holomorphic bisectional curvature* $\mathrm{Bisec}(\omega)(\xi,\eta)$ by

$$\mathrm{Bisec}(\omega)(\xi,\eta) := \frac{\mathrm{Rm}(\omega)(\xi,\bar{\xi},\eta,\bar{\eta})}{|\xi|^2_\omega\, |\eta|^2_\omega} = \frac{R_{i\bar{j}k\bar{l}}\xi^i\overline{\xi^j}\eta^k\overline{\eta^l}}{(g_{i\bar{j}}\xi^i\overline{\xi^j})(g_{k\bar{l}}\eta^k\overline{\eta^l})},$$

the *holomorphic sectional curvature* $H(\omega)(\xi)$ by

$$H(\omega)(\xi) := \mathrm{Bisec}(\omega)(\xi,\xi) = \frac{\mathrm{Rm}(\omega)(\xi,\bar{\xi},\xi,\bar{\xi})}{|\xi|^4_\omega} = \frac{R_{i\bar{j}k\bar{l}}\xi^i\overline{\xi^j}\xi^k\overline{\xi^l}}{(g_{i\bar{j}}\xi^i\overline{\xi^j})^2},$$

Here, abuse of notation, we denote $|\xi|^2_\omega := h_g(\xi,\xi)$.

2.5 Some Notations

Let (X, J, g, ω) be a Kähler manifold X of dimension n. For a real $(1,1)$-form α on X, we set the trace $\mathrm{tr}_\omega(\alpha)$ of α with respect to ω by

$$\mathrm{tr}_\omega(\alpha) := \frac{n\alpha \wedge \omega^{n-1}}{\omega^n} = g^{i\bar{j}}\alpha_{i\bar{j}},$$

where we denote $\omega = \sqrt{-1}\, g_{i\bar{j}} dz^i \wedge d\overline{z^j}$, $\alpha = \sqrt{-1}\, \alpha_{i\bar{j}} dz^i \wedge d\overline{z^j}$. In this notation, (minus sign of) the $\bar{\partial}$-Laplace operator associated to ω acts on the space of smooth functions is represented as

$$\Delta_\omega f = \mathrm{tr}_\omega\left(\sqrt{-1}\, \partial\bar{\partial} f\right) = g^{i\bar{j}}\partial_i\partial_{\bar{j}} f,$$

and the scalar curvature of ω is represented as $R(\omega) = \mathrm{tr}_\omega(\mathrm{Ric}(\omega))$.

For any tensors T, S on X of the same type, we denote $\langle T, S\rangle_\omega$ the Hermitian inner product of T and S, $|T|_\omega$ the norm of T measured by the Hermitian metric on tensor bundles induced by h_g. For instance, if $T = T_{ij}{}^{\bar{k}} dz^i \otimes dz^j \otimes \partial/\partial\overline{z^k}$, then the norm of T is locally written as

$$|T|^2_\omega = g^{i\bar{j}} g^{k\bar{l}} g_{p\bar{q}} T_{ik}{}^{\bar{q}} \overline{T_{jl}{}^{\bar{p}}}.$$

We denote ∇ (resp. $\overline{\nabla}$) the $(1,0)$-part (resp. $(0,1)$-part) of the exterior covariant derivative d_D of the Levi-Civita connection D of ω. In this notation, for a smooth function f on X, we have $\nabla f = \partial f$ and $\overline{\nabla} f = \bar{\partial} f$.

2.6 The Maximum Principle Argument

In this section, we recall the maximum principle argument which is a fundamental technique to obtain estimates for non-linear parabolic equations. Let $f = f(x,t) \in C^\infty(X \times [0,T], \mathbb{R})$ be a smooth function on a compact complex manifold X where $T \in \mathbb{R}_{>0}$. Since $X \times [0,T]$ is compact, f attains its maximum at some point $(x_0, t_0) \in X \times [0,T]$. Then, at this point, we have the following inequalities:

$$\sqrt{-1}\,\partial\bar{\partial} f(x_0,t_0) \le 0, \tag{2.6.1}$$

and

$$\frac{\partial}{\partial t} f(x_0,t_0) \begin{cases} \le 0 & \text{if } \ t_0 = 0 \\ = 0 & \text{if } \ 0 < t_0 < T \\ \ge 0 & \text{if } \ t_0 = T. \end{cases} \tag{2.6.2}$$

These inequalites are elementary but very useful in the subsequent arguments. The direct application is the following estimate.

Proposition 2.6.3. *Let ω_t be a smooth family of Kähler forms. For any function $f = f(x,t) \in C^\infty(X \times [0,T], \mathbb{R})$ satisfying*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) f \le 0,$$

the following estimate holds:

$$f \le \max_{X \times \{0\}} f.$$

Proof. For any constant $\varepsilon > 0$, we set $f_\varepsilon := f - \varepsilon t$. It is obvious that f_ε satisfies

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) f_\varepsilon \le -\varepsilon < 0.$$

Therefore, by applying the inequalities (2.6.1), (2.6.2) to f_ε, any maximum point $(x_0,t_0) \in X \times [0,T]$ of f_ε satisfies $t_0 = 0$. This implies that

$$f_\varepsilon \le \max_{X \times \{0\}} f_\varepsilon = \max_{X \times \{0\}} f.$$

By taking $\varepsilon \searrow 0$, we get the assertion. □

Of course, similar statement for minumum also holds if we replace $\le$ by $\ge$.

2.7 Chern-Lu's Inequality

In this section, we prove well-known Laplacian estimate called Chern-Lu's inequality [Che68, Lu68] which will be used later. We setup notations. Let $f\colon (X,\omega) \to (Y,\omega_Y)$ be a holomorphic map between compact Kähler manifolds. We set $E := \mathrm{Hom}(T^{1,0}X, f^*T^{*1,0}Y) = T^{*1,0}X \otimes f^*T^{*1,0}Y$, denote h the Hermitian metric on E induced by ω and ω_Y, and ∇_E the Chern connection of (E,h). We regard ∂f as a holomorphic section of E. Let $C \in \mathbb{R}$ be an upper bound for the bisectional curvature of ω_Y i.e. $\mathrm{Bisec}(\omega_Y) \le C$.

Proposition 2.7.1. *In the above setting, we get the following estimates.*

(a) Estimate for the Laplacian of $\mathrm{tr}_\omega (f^*\omega_Y)$*:*

$$\begin{aligned}\Delta_\omega \mathrm{tr}_\omega (f^*\omega_Y) &= \langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega - \mathrm{tr}_\omega^{1,2}\mathrm{tr}_\omega^{3,4}(f^*\mathrm{Rm}(\omega_Y)) + |\nabla_E \partial f|^2_{\omega,h} \\ &\geq \langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega - C(\mathrm{tr}_\omega (f^*\omega_Y))^2 + \frac{|\nabla \mathrm{tr}_\omega (f^*\omega_Y)|^2_\omega}{\mathrm{tr}_\omega (f^*\omega_Y)} \\ &\geq \langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega - C(\mathrm{tr}_\omega (f^*\omega_Y))^2\end{aligned}$$

(b) Estimate for the Laplacian of $\log \mathrm{tr}_\omega (f^*\omega_Y)$*:*

$$\begin{aligned}\Delta_\omega \log \mathrm{tr}_\omega (f^*\omega_Y) &= \frac{1}{\mathrm{tr}_\omega (f^*\omega_Y)}\Big(\langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega - \mathrm{tr}_\omega^{1,2}\mathrm{tr}_\omega^{3,4}(f^*\mathrm{Rm}(\omega_Y))\Big) \\ &\quad + \frac{1}{(\mathrm{tr}_\omega (f^*\omega_Y))^2}\Big(\mathrm{tr}_\omega (f^*\omega_Y)|\nabla_E \partial f|^2_{\omega,h} - |\nabla \mathrm{tr}_\omega (f^*\omega_Y)|^2_\omega\Big) \\ &\geq \frac{1}{\mathrm{tr}_\omega (f^*\omega_Y)}\Big(\langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega - \mathrm{tr}_\omega^{1,2}\mathrm{tr}_\omega^{3,4}(f^*\mathrm{Rm}(\omega_Y))\Big) \\ &\geq \frac{\langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega}{\mathrm{tr}_\omega (f^*\omega_Y)} - C\mathrm{tr}_\omega (f^*\omega_Y)\end{aligned}$$

Proof. Let $(z^1, \ldots, z^n)$ and $(w^1, \ldots, w^n)$ be normal coodinates on X and Y respectively. We set

$$\begin{aligned}\omega = \sqrt{-1}\, g_{i\bar{j}} dz^i \wedge d\overline{z^j},\ \omega_Y = \sqrt{-1}\, h_{\alpha\bar{\beta}} dw^\alpha \wedge d\overline{w^\beta}, \\ f^*\omega_Y = \sqrt{-1}\, h^*_{i\bar{j}} dz^i \wedge d\overline{z^j} := \sqrt{-1}(h_{\alpha\bar{\beta}} \circ f)(\partial_i f^\alpha)\overline{(\partial_j f^\beta)} dz^i \wedge d\overline{z^j},\end{aligned}$$

and denote $R_{i\bar{j}k\bar{l}}$ and $S_{\alpha\bar{\beta}\gamma\bar{\delta}}$ by the curvature tensor of ω and ω_Y respectively. Then we have the following inequalities.

$$\begin{aligned}\Delta_\omega \mathrm{tr}_\omega (f^*\omega_Y) &= g^{k\bar{l}}\partial_k\partial_{\bar{l}}\left(g^{i\bar{j}}h^*_{i\bar{j}}\right) = g^{k\bar{l}}\left(\partial_k\partial_l g^{i\bar{j}}\right)h^*_{i\bar{j}} + g^{k\bar{l}}g^{i\bar{j}}\left(\partial_k\partial_l h^*_{i\bar{j}}\right) \\ &= g^{k\bar{l}}R_{k\bar{l}}{}^{i\bar{j}}h^*_{i\bar{j}} + \Big(g^{k\bar{l}}g^{i\bar{j}}(\partial_i\partial_k f^\alpha)\overline{(\partial_j\partial_l f^\beta)}(h_{\alpha\bar{\beta}} \circ f) \\ &\quad - g^{k\bar{l}}g^{i\bar{j}}(\partial_i f^\alpha)\overline{(\partial_j f^\beta)}(\partial_k f^\gamma)\overline{(\partial_l f^\delta)}(S_{\alpha\bar{\beta}\gamma\bar{\delta}} \circ f)\Big) \\ &= \langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega + |\nabla_E \partial f|^2_{\omega,h} - \mathrm{tr}_\omega^{1,2}\mathrm{tr}_\omega^{3,4}(f^*\mathrm{Rm}(\omega_Y)) \\ &\geq \langle \mathrm{Ric}(\omega), f^*\omega_Y\rangle_\omega + \frac{|\nabla \mathrm{tr}_\omega (f^*\omega_Y)|^2_\omega}{\mathrm{tr}_\omega (f^*\omega_Y)} - \mathrm{tr}_\omega^{1,2}\mathrm{tr}_\omega^{3,4}(f^*\mathrm{Rm}(\omega_Y)).\end{aligned}$$

In the last inequality, we used the inequality $|\nabla \mathrm{tr}_\omega (f^*\omega_Y)|^2_\omega \leq |\nabla_E \partial f|^2_{\omega,h}\, \mathrm{tr}_\omega (f^*\omega_Y)$

which can be obtained as follows:

$$\begin{aligned}
|\nabla \mathrm{tr}_\omega\,(f^*\omega_Y)|^2_\omega &= g^{i\bar{j}}(\partial_i g^{k\bar{l}} h^*_{k\bar{l}})(\partial_{\bar{j}} g^{p\bar{q}} h^*_{p\bar{q}}) = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} (\partial_i h^*_{k\bar{l}})(\partial_{\bar{j}} h^*_{p\bar{q}}) \\
&= \sum_{i,k,p,\alpha,\beta} (\partial_i\partial_k f^\alpha)\overline{(\partial_k f^\alpha)(\partial_i\partial_p f^\beta)}(\partial_p f^\beta) \\
&\le \sum_{k,p,\alpha,\beta}\left(|\partial_p f^\beta||\partial_k f^\alpha| \left(\sum_i |\partial_i\partial_k f^\alpha|^2\right)^{1/2} \left(\sum_j |\partial_j\partial_p f^\beta|^2\right)^{1/2}\right) \\
&= \left(\sum_{k,\alpha} |\partial_k f^\beta| \left(\sum_i |(\partial_i\partial_k f^\alpha|^2\right)^{1/2}\right)^2 \\
&\le \left(\sum_{l,\beta} |\partial_l f^\beta|^2\right)\left(\sum_{i,k,\alpha} |\partial_i\partial_k f^\alpha|^2\right) \\
&= \mathrm{tr}_\omega\,(f^*\omega_Y)\, g^{k\bar{l}} g^{i\bar{j}} (\partial_i\partial_k f^\alpha)\overline{(\partial_j\partial_l f^\beta)} \\
&= |\nabla_E \partial f|^2_{\omega,h}\,\mathrm{tr}_\omega\,(f^*\omega_Y)\,.
\end{aligned}$$

Here, we used the Cauchy-Schwarz inequalities. By using the upper bound for the bisectional curvature of ω_Y, the term $\mathrm{tr}^{1,2}_\omega \mathrm{tr}^{3,4}_\omega\,(f^*\mathrm{Rm}(\omega_Y))$ estimated as

$$\mathrm{tr}^{1,2}_\omega \mathrm{tr}^{3,4}_\omega\,(f^*\mathrm{Rm}(\omega_Y)) = C\,(\mathrm{tr}_\omega\,(f^*\omega_Y))^2\,.$$

This follows from the following calculation: if we set ξ_i by $(df)(\partial/\partial z^i)$, their norm is

$$|\xi_i|^2_{\omega_Y} = h_{\alpha\overline{\beta}}(y_0)\frac{\partial f^\alpha}{\partial z^i}(x_0)\overline{\frac{\partial f^\beta}{\partial z^i}(x_0)}.$$

By combining the definition of the bisectional curvature, we get

$$\begin{aligned}
\mathrm{tr}^{1,2}_\omega \mathrm{tr}^{3,4}_\omega\,(f^*\mathrm{Rm}(\omega_Y)) &= g^{i\bar{j}} g^{k\bar{l}} (\partial_i f^\alpha)\overline{(\partial_j f^\beta)}(\partial_k f^\gamma)\overline{(\partial_l f^\delta)}(S_{\alpha\overline{\beta}\gamma\overline{\delta}}\circ f) \\
&= \sum_{i,k} \mathrm{Rm}(\omega_Y)(\xi_i, \overline{\xi_i}, \xi_k, \overline{\xi_k})(y_0) \\
&\le \sum_{i,k} C\cdot |\xi_i|^2_{\omega_Y} |\xi_k|^2_{\omega_Y} \\
&= C\cdot \sum_i \left(h_{\alpha\overline{\beta}}\frac{\partial f^\alpha}{\partial z^i}\overline{\frac{\partial f^\beta}{\partial z^i}}\right)\cdot \sum_k \left(h_{\alpha\overline{\beta}}\frac{\partial f^\alpha}{\partial z^k}\overline{\frac{\partial f^\beta}{\partial z^k}}\right) \\
&= C\cdot \left(g^{i\bar{j}} h_{\alpha\overline{\beta}}\frac{\partial f^\alpha}{\partial z^i}\overline{\frac{\partial f^\beta}{\partial z^j}}\right)\left(g^{k\bar{l}} h_{\alpha\overline{\beta}}\frac{\partial f^\alpha}{\partial z^k}\overline{\frac{\partial f^\beta}{\partial z^l}}\right) \\
&= C\,(\mathrm{tr}_\omega\,(f^*\omega_Y))^2\,.
\end{aligned}$$

These estimates gives the desired inequality (a).

The proof of (b) is straightforward:

$$
\begin{aligned}
\Delta_\omega \log \operatorname{tr}_\omega\left(f^*\omega_Y\right) &= \frac{\Delta_\omega \operatorname{tr}_\omega\left(f^*\omega_Y\right)}{\operatorname{tr}_\omega\left(f^*\omega_Y\right)} - \frac{\left|\nabla \operatorname{tr}_\omega\left(f^*\omega_Y\right)\right|_\omega^2}{\left(\operatorname{tr}_\omega\left(f^*\omega_Y\right)\right)^2} \\
&= \frac{1}{\operatorname{tr}_\omega\left(f^*\omega_Y\right)}\Big(\langle \operatorname{Ric}(\omega), f^*\omega_Y\rangle_\omega - \operatorname{tr}_\omega^{1,2}\operatorname{tr}_\omega^{3,4}\left(f^*\operatorname{Rm}(\omega_Y)\right)\Big) \\
&\quad + \frac{1}{\left(\operatorname{tr}_\omega\left(f^*\omega_Y\right)\right)^2}\Big(\operatorname{tr}_\omega\left(f^*\omega_Y\right)\left|\nabla_E \partial f\right|_{\omega,h}^2 - \left|\nabla \operatorname{tr}_\omega\left(f^*\omega_Y\right)\right|_\omega^2\Big) \\
&\geq \frac{1}{\operatorname{tr}_\omega\left(f^*\omega_Y\right)}\Big(\langle \operatorname{Ric}(\omega), f^*\omega_Y\rangle_\omega - \operatorname{tr}_\omega^{1,2}\operatorname{tr}_\omega^{3,4}\left(f^*\operatorname{Rm}(\omega_Y)\right)\Big) \\
&\geq \frac{\langle \operatorname{Ric}(\omega), f^*\omega_Y\rangle_\omega}{\operatorname{tr}_\omega\left(f^*\omega_Y\right)} - C\operatorname{tr}_\omega\left(f^*\omega_Y\right).
\end{aligned}
$$

$\square$

Chapter 3

Compact Kähler Manifolds with Positive or Negative Holomorphic Sectional Curvature

In this chapter, we consider Kähler manifolds with positive or negative Holomorphic Sectional Curvature. In Section 3.1, we quickly review known results for positive holomorphic sectional curvature case. In Section 3.2, we summarize basic properties of the Kähler-Ricci flow which will be used in the next section. The main part of this chapter is Section 3.3. In this section, by using the Kähler-Ricci flow, we provide a new proof to Wu-Yau's and Tosatti-Yang's theorems which represent the relationship between the negativity of the holomorphic sectional curvature and the positivity of the canonical bundle K_X of a compact Kähler manifold X [Nom16a].

3.1 Kähler Manifolds with Positive Holomorphic Sectional Curvature

In this section, we review related results.

Theorem 3.1.1 ([HW12, Theorem 1.1])**.** *Let X be a smooth projective variety. If X admits a Kähler form ω satisfying*

$$\int_X R(\omega)\omega^n > 0,$$

then X is uniruled. In particular, the Kodaira dimension of X is $-\infty$.

We remark that this condition is cohomological (see (3.1.2)). Since the scalar curvature is determined by the holomorphic sectional curvature by the formula

$$R(\omega)(x) = \frac{n(n+1)}{2} \int_{\xi \in \mathbb{P}(T_x^{1,0}X)} H(\omega)(\xi)\ \omega_{FS}^{n-1},$$

if X admits a Kähler form with strictly positive holomorphic sectional curvature, then X is uniruled.

Proof. If we assume that X is not uniruled, then K_X is pseudoeffective i.e. $2\pi c_1(K_X)$ is represented by some closed positive $(1,1)$-current on X. This deep result is due to Boucksom-Demailly-Păun-Peternell [BDPP13, Corollary 0.3]. Therefore, the intersection number satisfies $(2\pi c_1(K_X)\cdot[\omega]^{n-1}) \geq 0$. On the other hand, by using the assumption on the scalar curvature, we have the following esitimate:

$$(2\pi c_1(K_X)\cdot[\omega]^{n-1}) = \int_X -\mathrm{Ric}(\omega)\wedge\omega^{n-1} = \frac{-1}{n}\int_X R(\omega)\omega^n < 0, \tag{3.1.2}$$

hence we get the contradiction. □

In the Hermitian setting, Yang showed the following theorem.

Theorem 3.1.3 ([Yan16, Theorem 1.2]). *Let (X,ω) be a compact Hermitian manifold. Asuume that the holomorphic sectional curvature $H(\omega) \geq 0$ and not identically zero, then the Kodaira dimension of X is $-\infty$.*

The proof is based on some Bochner type formula for a Hermitian metric which is conformal to the Gauduchon metric.

3.2 The Kähler-Ricci Flow

In this section, we summarize well-known properties of the Kähler-Ricci flow which will be used later. For more detailed exposition, we refere the book [BEG13]. In the following argument, we will denote by X a compact Kähler manifold of dimension n.

Definition 3.2.1. A smooth family of Kähler forms $\{\omega_t\}_{t\geq 0}$ is called the *Kähler-Ricci flow* (resp. the *normalized Kähler-Ricci flow*) if it satisfies the following equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_t & = -\mathrm{Ric}(\omega_t) + \lambda\omega_t, \\ \omega_t\big|_{t=0} & = \omega_0, \end{cases} \tag{3.2.2}$$

where $\lambda = 0$ (resp. $\lambda = -1$).

By considering the cohomology class in $H^{1,1}(X,\mathbb{R})$ of (3.2.2), ω_t belongs to $\alpha_t \in H^{1,1}(X,\mathbb{R})$ which is defined as

$$\alpha_t = \begin{cases} [\omega_0] + 2\pi t c_1(K_X) & \text{if } \lambda = 0, \\ e^{-t}[\omega_0] + (1-e^{-t})2\pi c_1(K_X) & \text{if } \lambda = -1. \end{cases} \tag{3.2.3}$$

The maximal existence theorem for the Kähler-Ricci flow is stated as follows.

Theorem 3.2.4 ([Cao85, Tsu88, TZ06], see also [BEG13, 3.3.1]). *For any Kähler form ω_0, the Kähler-Ricci flow (resp. the normalized Kähler-Ricci flow) ω_t starting* *from ω_0 exists uniquely for $t \in [0, T)$ and cannnot extend beyond T, where T is defined by*

$$T := \sup\{t > 0 \mid \alpha_t \textit{ defined by (3.2.3) is Kähler } \}, \tag{3.2.5}$$

and called the maiximal existence time. In particular, ω_t exists for $t \in [0, \infty)$ if and only if K_X is nef, i.e. $2\pi c_1(K_X)$ belongs to the closure of the Kähler cone of X.

We prove well-known estimates which will be used in the proof of Theorem 3.3.1, and 3.3.2.

The following lower bound for the scalar curvature along the Kähler-Ricci flow due to Hamilton [Ham82] is obtained by a simple maximum principle argument.

Proposition 3.2.6 ([BEG13, 3.2.2])**.** *For the Kähler-Ricci flow ω_t, we have the following lower bound for the scalar curvature:*

$$R(\omega_t) \geq \lambda n - Ce^{\lambda t},$$

where $C := -\min_X (R(\omega_0) - \lambda n)$.

Proof. The time derivative of the Ricci curvature is calculated as follows:

$$\frac{\partial}{\partial t}\operatorname{Ric}(\omega_t) = -\sqrt{-1}\,\partial\overline{\partial}\left(\frac{\partial}{\partial t}\log\omega_t^n\right) = -\sqrt{-1}\,\partial\overline{\partial}\left(\operatorname{tr}_{\omega_t}\left(\frac{\partial}{\partial t}\omega_t\right)\right) = \sqrt{-1}\,\partial\overline{\partial}R(\omega_t).$$

The scalar curvature evolves as

$$\begin{aligned}
\frac{\partial}{\partial t}R(\omega_t) &= \frac{\partial}{\partial t}\operatorname{tr}_{\omega_t}(\operatorname{Ric}(\omega_t)) \\
&= -\left\langle \frac{\partial}{\partial t}\omega_t, \operatorname{Ric}(\omega_t)\right\rangle_{\omega_t} + \operatorname{tr}_{\omega_t}\left(\frac{\partial}{\partial t}\operatorname{Ric}(\omega_t)\right) \\
&= |\operatorname{Ric}(\omega_t)|^2_{\omega_t} - \lambda R(\omega_t) + \Delta_{\omega_t}R(\omega_t) \\
&\geq \frac{1}{n}\left(\operatorname{tr}_{\omega_t}(\operatorname{Ric}(\omega_t))\right)^2 - \lambda R(\omega_t) + \Delta_{\omega_t}R(\omega_t) \\
&= -\lambda R(\omega_t) + \frac{1}{n}R(\omega_t)^2 + \Delta_{\omega_t}R(\omega_t).
\end{aligned}$$

Therefore we get

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right)\left(e^{-\lambda t}(R(\omega_t) - \lambda n)\right) \geq 0,$$

which gives the disired result. □

Since the time derivative of the volume form ω_t^n is computated as

$$\begin{aligned}
\frac{\partial}{\partial t}\log\frac{\omega_t^n}{\omega_0} = \operatorname{tr}_{\omega_t}\left(\frac{\partial}{\partial t}\omega_t\right) &= \operatorname{tr}_{\omega_t}(-\operatorname{Ric}(\omega_t) + \lambda\omega_t + n) \\
&= -R(\omega_t) + \lambda n,
\end{aligned}$$

the lower bound for $R(\omega_t)$ gives the following volume upper bounds.

Proposition 3.2.7 ([BEG13, 3.2.3]). *The following volume bounds hold:*

(a) For any Kähler-Ricci flow ω_t, there exists a constant $C > 0$ such that for all $t \in [0, T)$, $\omega_t^n \leq e^{Ct}\omega_0^n$ holds.

(b) For any normalized Kähler-Ricci flow ω_t, there exists a constant $C > 0$ such that for all $t \in [0, T)$, $\omega_t^n \leq C\omega_0^n$ holds.

To get C^2-estimate, we need the parabolic Schwarz lemma obtained by Song-Tian [ST07] applied to the identity map (see also [BEG13, 3.2.6]). This is a parabolic analogue of the Schwarz lemma due to Yau [Yau78a] (see Section 4.1).

Proposition 3.2.8. *Let ω_t be the Kähler-Ricci flow (resp. the normalized Kähler-Ricci flow), $f\colon X \to Y$ be a holomorphic map between compact Kähler manifolds and ω_Y be a Kähler form on Y. We set $E := \mathrm{Hom}(T^{1,0}X, f^*T^{*1,0}Y) = T^{*1,0}X \otimes f^*T^{*1,0}Y$, denote h_t the Hermitian metric on E induced by ω_t and ω_Y, and ∇_{E_t} Chern connection of (E, h_t). Then we have the following inequality:*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \mathrm{tr}_{\omega_t}(f^*\omega_Y) \\
&= -\lambda + \frac{\mathrm{tr}_{\omega_t}^{1,2}\mathrm{tr}_{\omega_t}^{3,4}(f^*\mathrm{Rm}(\omega_Y))}{\mathrm{tr}_{\omega_t}(f^*\omega_Y)} - \frac{\left(\mathrm{tr}_{\omega_t}(f^*\omega_Y)\,|\nabla_{E_t}\partial f|^2_{\omega_t,h_t} - |\nabla \mathrm{tr}_{\omega_t}(f^*\omega_Y)|^2_{\omega_t}\right)}{(\mathrm{tr}_{\omega_t}(f^*\omega_Y))^2} \\
&\leq -\lambda + \frac{\mathrm{tr}_{\omega_t}^{1,2}\mathrm{tr}_{\omega_t}^{3,4}(f^*\mathrm{Rm}(\omega_Y))}{\mathrm{tr}_{\omega_t}(f^*\omega_Y)}
\end{aligned}$$

In particular, applying $f = \mathrm{id}_X\colon (X, \omega_t) \to (X, \widehat{\omega})$, we get the following:

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \mathrm{tr}_{\omega_t}(\widehat{\omega}) \leq -\lambda + \frac{g^{i\bar{j}}(t)g^{k\bar{l}}(t)\widehat{R}_{i\bar{j}k\bar{l}}}{\mathrm{tr}_{\omega_t}(\widehat{\omega})},$$

where λ is in (3.2.2), $\omega_t = \sqrt{-1}\, g_{i\bar{j}}(t) dz^i \wedge d\overline{z^j}$ and $\widehat{R}_{i\bar{j}k\bar{l}}$ is the curvature tensor of $\widehat{\omega}$.

Proof. The direct computation shows the following equality:

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{tr}_{\omega_t}(f^*\omega_Y) &= -\left\langle \frac{\partial}{\partial t}\omega_t, f^*\omega_Y \right\rangle_{\omega_t} \\
&= -\left\langle -\mathrm{Ric}(\omega_t) + \lambda\omega_t + \eta, f^*\omega_Y \right\rangle_{\omega_t} \\
&= -\lambda \mathrm{tr}_{\omega_t}(f^*\omega_Y) + \left\langle \mathrm{Ric}(\omega_t), f^*\omega_Y \right\rangle_{\omega_t}, \\
\frac{\partial}{\partial t}\log \mathrm{tr}_{\omega_t}(f^*\omega_Y) &= \frac{1}{\mathrm{tr}_{\omega_t}(f^*\omega_Y)}\frac{\partial}{\partial t}\mathrm{tr}_{\omega_t}(f^*\omega_Y) \\
&= \frac{1}{\mathrm{tr}_{\omega_t}(f^*\omega_Y)}\left(-\lambda \mathrm{tr}_{\omega_t}(f^*\omega_Y) + \left\langle \mathrm{Ric}(\omega_t), f^*\omega_Y \right\rangle_{\omega_t}\right) \\
&= -\lambda + \frac{\left\langle \mathrm{Ric}(\omega_t), f^*\omega_Y \right\rangle_{\omega_t}}{\mathrm{tr}_{\omega_t}(f^*\omega_Y)}.
\end{aligned}$$

Recall that in Proposition 2.7.1, we obtained the estimate for $\Delta_{\omega_t} \log \mathrm{tr}_{\omega_t} (f^*\omega_Y)$:

$$\begin{aligned}\Delta_{\omega_t} \log \mathrm{tr}_{\omega_t} (f^*\omega_Y) &= \frac{1}{\mathrm{tr}_{\omega_t} (f^*\omega_Y)} \Big(\langle \mathrm{Ric}(\omega_t), f^*\omega_Y \rangle_\omega - \mathrm{tr}^{1,2}_{\omega_t} \mathrm{tr}^{3,4}_{\omega_t} (f^*\mathrm{Rm}(\omega_Y)) \Big) \\ &\quad + \frac{1}{(\mathrm{tr}_{\omega_t} (f^*\omega_Y))^2} \Big(\mathrm{tr}_{\omega_t} (f^*\omega_Y) |\nabla_{E_t} \partial f|^2_{\omega_t, h_t} - |\nabla \mathrm{tr}_\omega (f^*\omega_Y)|^2_{\omega_t} \Big) \\ &\geq \frac{1}{\mathrm{tr}_{\omega_t} (f^*\omega_Y)} \Big(\langle \mathrm{Ric}(\omega_t), f^*\omega_Y \rangle_\omega - \mathrm{tr}^{1,2}_{\omega_t} \mathrm{tr}^{3,4}_{\omega_t} (f^*\mathrm{Rm}(\omega_Y)) \Big)\end{aligned}$$

Combining these, we get the assertion. □

The next proposition due to Royden [Roy80, Lemma] (see also [WWY12, Lemma 2.1]) will be used to obtain the C^2-estimate under the negativity assumption on the holomorphic sectional curvature. This is essentially based on the symmetry of the curvature tensor of the Kähler forms.

Proposition 3.2.9. *Let $f\colon (X, \omega) \to (Y, \omega_Y)$ be a holomorphic map between Kähler manifolds and r be the maximal rank of Jacobian of f. Assume that there exists a non-negative constant $\kappa \geq 0$ such that for any tangent vector $\xi \in T^{1,0}Y$, we have*

$$H(\omega_Y)(\xi) \leq -\kappa \leq 0.$$

Then the following inequality holds:

$$\mathrm{tr}^{1,2}_\omega \mathrm{tr}^{3,4}_\omega (f^*\mathrm{Rm}(\omega_Y)) \leq -\frac{\kappa(r+1)}{2r} (\mathrm{tr}_\omega (f^*\omega_Y))^2 .$$

In particular, for any Kähler form $\widehat{\omega}$ on X whose holomorphic sectional curvature $\widehat{H}$ satisfying

$$\widehat{H}(\xi) \leq -\kappa \leq 0, \tag{3.2.10}$$

we have

$$g^{i\bar{j}} g^{k\bar{l}} \widehat{R}_{i\bar{j}k\bar{l}} \leq -\kappa \frac{n+1}{2n} (\mathrm{tr}_\omega (\widehat{\omega}))^2 \leq 0,$$

where $\omega = \sqrt{-1}\, g_{i\bar{j}} dz^i \wedge d\overline{z^j}$ and $\widehat{R}_{i\bar{j}k\bar{l}}$ is the curvature tensor of $\widehat{\omega}$.

Proof. Let $S\colon V \times V \times V \times V \to \mathbb{C}$ be a symmetric bi-Hermitian form on a Hermitian vector space (V, h) i.e. for any $\xi, \eta, \zeta, \omega \in V$, the following holds:

$$S(\xi, \overline{\eta}, \zeta, \overline{\omega}) = S(\zeta, \overline{\eta}, \xi, \overline{\omega}) = \overline{S(\eta, \overline{\xi}, \omega, \overline{\zeta})}.$$

Assume that there exists a constant $\kappa \geq 0$ satisfying $S(\xi, \overline{\xi}, \xi, \overline{\xi}) \leq -\kappa |\xi|^4_h$. To prove the proposition, we only need to show the following inequality: for any orthonormal system (not necessarily a basis) $\xi_1, \cdots, \xi_N$, we have

$$\sum_{i,j} S(\xi_i, \overline{\xi_i}, \xi_j, \overline{\xi_j}) \leq -\frac{\kappa}{2} \left(\sum_i |\xi_i|^4_h \right) - \frac{\kappa}{2} \left(\sum_i |\xi_i|^2_h \right)^2 \leq -\frac{\kappa}{2} \left(1 + \frac{1}{N} \right) \sum_i |\xi_i|^2_h .$$

Since the second inequality follows from the Cauchy-Schwarz inequality and $\kappa \geq 0$, we prove the first one.

We set $A := \{a = (a_1, \ldots, a_N) \in \mathbb{C}^N \mid a_i = \pm 1, \pm\sqrt{-1}\}$ and denote $\xi_a \in V$ by

$$\xi_a := \sum_{i=1}^{N} a_i \xi_i$$

for any $a \in A$. Since S is bi-Hermitian and symmetric, we have

$$\begin{aligned}
\frac{1}{4^N} \sum_{a \in A} S(\xi_a, \overline{\xi_a}, \xi_a, \overline{\xi_a}) &= \frac{1}{4^N} \sum_{a \in A} \sum_{i,j,k,l} a_i \overline{a_j} a_k \overline{a_l}\, S(\xi_i, \overline{\xi_j}, \xi_k, \overline{\xi_l}) \\
&= \sum_i S(\xi_i, \overline{\xi_i}, \xi_i, \overline{\xi_i}) + \sum_{i \neq j} \Big(S(\xi_i, \overline{\xi_i}, \xi_j, \overline{\xi_j}) + S(\xi_i, \overline{\xi_j}, \xi_j, \overline{\xi_i}) \Big) \\
&= \sum_i S(\xi_i, \overline{\xi_i}, \xi_i, \overline{\xi_i}) + 2 \sum_{i \neq j} S(\xi_i, \overline{\xi_i}, \xi_j, \overline{\xi_j}).
\end{aligned}$$

By using the assumption, we get

$$\begin{aligned}
2 \sum_{i,j} S(\xi_i, \overline{\xi_i}, \xi_j, \overline{\xi_j}) &= \left(\sum_i S(\xi_i, \overline{\xi_i}, \xi_i, \overline{\xi_i}) \right) \\
&\quad + \left(\sum_i S(\xi_i, \overline{\xi_i}, \xi_i, \overline{\xi_i}) + 2 \sum_{i \neq j} S(\xi_i, \overline{\xi_i}, \xi_j, \overline{\xi_j}) \right) \\
&\leq -\kappa \left(\sum_i |\xi_i|_h^4 \right) - \kappa \left(\sum_i |\xi_i|_h^2 \right)^2.
\end{aligned}$$

□

3.3 Kähler Manifolds with Negative Holomophic Sectional Curvature

In this section, we prove the following two theorems by a method of the Kähler-Ricci flow which are main results in this chapter [Nom16a].

Theorem 3.3.1 ([WY16a, Theorem 2], [ToY15, Corollary 1.3])**.** *If X admits a Kähler form with strictly negative holomorphic sectional curvature, then the canonical bundle K_X is ample. In particular, X is projective.*

Theorem 3.3.2 ([ToY15, Theorem 1.1])**.** *If X admits a Kähler form with semi-negative holomorphic sectional curvature, then the canonical bundle K_X is nef.*

Proof of Theorem 3.3.2. By the assumption in Theorem 3.3.2, there exists a Kähler form $\widehat{\omega}$ whose holomorphic sectional curvature is semi-negative i.e. $\kappa = 0$ in (3.2.10). Let ω_t be the Kähler-Ricci flow stating from arbitrary Kähler form ω_0 on X. By Theorem 3.2.4, the nefness of K_X is equivalent to the long time existence of ω_t. By definition of the maximal existence time (3.2.5) and Theorem 3.2.4, it is enough to show that if ω_t exists for $[0, T_0)$ with $T_0 < \infty$, then α_{T_0} is a Kähler class.

By Proposition 3.2.8 and ,Proposition 3.2.9 we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \operatorname{tr}_{\omega_t}(\widehat{\omega}) \le \frac{g^{i\bar{j}}(t) g^{k\bar{l}}(t) \widehat{R}_{i\bar{j}k\bar{l}}}{\operatorname{tr}_{\omega_t}(\widehat{\omega})} \le 0.$$

Applying the maximum principle, we have $\operatorname{tr}_{\omega_t}(\widehat{\omega}) \le \max_X \operatorname{tr}_{\omega_0}(\widehat{\omega}) =: C$ and hence for all $t \in [0, T_0)$ we get

$$\frac{1}{C}\widehat{\omega} \le \omega_t. \tag{3.3.3}$$

Therefore, for any irreducible subvariety $V \subset X$ of positive dimension, the intersection number can be estimated as follows:

$$\int_V [\alpha_{T_0}]^{\dim V} = \lim_{t \nearrow T_0} \int_V \omega_t^{\dim V} \ge \frac{1}{C^{\dim V}} \int_V \widehat{\omega}^{\dim V} > 0.$$

By Demailly-Păun's characterization of the Kähler cone [DP04, Main Theorem 0.1], the limiting class α_{T_0} is Kähler. $\square$

The idea of avoiding higher order estimates by using the Demailly-Păun's theorem can be found in the proof of [Zha10, Theorem 1.1].

Remark 3.3.4. We can also prove that ω_t converges to a smooth Kähler form as $t \nearrow T_0$, in particular α_{T_0} is Kähler. In fact, by using (3.3.3) and Proposition 3.2.7 (a), we get the uniform C^2-estimate for ω_t:

$$\frac{1}{C}\widehat{\omega} \le \omega_t \le C'\widehat{\omega}. \tag{3.3.5}$$

Thefore we obtain the higher order estimates (see for example [BEG13, 3.2.16]), which guarantees the convergence.

Proof of Theorem 3.3.1. By the assumption in Theorem 3.3.1, there exists a Kähler form $\widehat{\omega}$ whose holomorphic sectional curvature is strictly negative i.e. $\kappa > 0$ in (3.2.10). Let ω_t be the normalized Kähler-Ricci flow stating from arbitrary Kähler form ω_0 on X. By Theorem 3.3.2, K_X is nef, and therefore ω_t exists for $t \in [0, \infty)$.

By Proposition 3.2.8 and Proposition 3.2.9, we get

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) \log \operatorname{tr}_{\omega_t}(\widehat{\omega}) \le 1 + \frac{g^{i\bar{j}}(t) g^{k\bar{l}}(t) \widehat{R}_{i\bar{j}k\bar{l}}}{\operatorname{tr}_{\omega_t}(\widehat{\omega})} \le 1 - \kappa \frac{n+1}{2n} \operatorname{tr}_{\omega_t}(\widehat{\omega}).$$

Applying the maximum principle, we have $\operatorname{tr}_{\omega_t}(\widehat{\omega}) \le C$ where

$$C := \max\left\{\frac{2n}{\kappa(n+1)}, \max_X \operatorname{tr}_{\omega_t}(\widehat{\omega})\right\} > 0.$$

This gives, for any $t \in [0, \infty)$,

$$\frac{1}{C}\widehat{\omega} \le \omega_t. \tag{3.3.6}$$

Since $\alpha_t = [\omega_t]$ converges to $2\pi c_1(K_X)$ as $t \to \infty$, the same argument as in the proof of Theorem 3.3.2 shows the ampleness of K_X. $\square$

Remark 3.3.7. As in Remark 3.3.4, combining Proposition 3.2.7 (b) and (3.3.6), we get the C^2-estimate for ω_t, and hence the higher order estimates for ω_t. By Arzelà-Ascoli argument, there exists a subsequence t_i satisfying $t_i \nearrow \infty$ such that ω_{t_i} converges smoothly to a Kähler form ω_∞. Since this Kähler form ω_∞ represents $2\pi c_1(K_X)$ by (3.2.3), we get the ampleness of K_X.

Remark 3.3.8. A classical result due to Cao [Cao85] show that under the assumption on the ampleness of K_X, any normalized Kähler-Ricci flow ω_t converges to the Kähler-Einstein metric with negative Ricci curvature.

Chapter 4

Schwarz Lemma for Conical Kähler Metrics

The Schwarz–Pick lemma is a fundamental result in complex analysis. It is well-known that Yau generalized it to the higher dimensional manifolds by applying his maximum principle for complete Riemannian manifolds. Jeffres obtained Schwarz lemma for volume forms of conical Kähler metrics, based on a barrier function and the maximum principle argument. The objective of this chapter is to generalize Jeffres' result to general cone angles including the case when the pullback of the metric would blows up along the divisors.

In Section 4.1, we review Yau's Schwarz lemma. In Section 4.2, we recall the definition of conical Kähler metrics according to Donaldson [Don12]. In Section 4.3, we prove Schwarz lemma for conical Kähler metrics which is the main result in this chapter.

4.1 Yau's Schwarz Lemma

In this section, we review Yau's original Schwarz lemma [Yau78a]. Let $f\colon (X,\omega_X) \to (Y,\omega_Y)$ be a holomoprhic map from a complete Kähler manifold of dimension n to a Hermitian manifold of dimension m. Then Yau's Schwarz lemma is stated as follows.

Theorem 4.1.1 ([Yau78a, Theorem 2]). *Assume that there exists non-negative constants $A, B \geq 0$ such that the curvatures satisfy the following:*

$$\mathrm{Ric}(\omega_X) \geq -A\omega_X,\ \mathrm{Bisec}(\omega_Y) \leq -B < 0. \tag{4.1.2}$$

Then we have the following:

$$f^*\omega_Y \leq \frac{A}{B}\omega_X.$$

Theorem 4.1.3 ([Yau78a, Theorem 3]). *Assume that* $\dim X = \dim Y = n$ *and the following curvature condition holds:*

$$R(\omega_X) \geq -A,\ \mathrm{Ric}(\omega_Y) \leq -B\omega_Y < 0, \tag{4.1.4}$$

and $\mathrm{Ric}(\omega_X)$ *is bounded from below. Then the volume form is estimated as follows:*

$$f^*\omega_Y^n \le \left(\frac{A}{nB}\right)^n \omega_X^n.$$

Remark 4.1.5. If we assume that (Y, ω_Y) is Kähler, by applying Royden's Lemma (Proposition 3.2.9), the assumption on bisectional curvature can be replaced to that on holomorphic sectional curvature (see [Roy80, Theorem 1]).

The proof of both Theorems is based on the maximum principle for comlete Riemannian manifolds due to Omori [Omo67] and Yau [Yau75].

Theorem 4.1.6. *Let* (M, g) *be a complete Riemannian manifold with Ricci curvature is bounded from below. Then for any function* $f \in C^2(M, \mathbb{R})$ *bounded from below and for any* $\varepsilon > 0$*, there exists a point* $x_\varepsilon \in M$ *such that*

$$|\nabla f|_g^2(x_\varepsilon) < \varepsilon,\ \Delta_g f(x_\varepsilon) > -\varepsilon,\ f(x_\varepsilon) < \inf_M f + \varepsilon. \tag{4.1.7}$$

Since M is not necessarily compact, the function f does not have minimum in general. This theorem states that there exists a point $x_\varepsilon \in M$ which is very close to minumum point under the completeness and curvature bound.

Corollary 4.1.8. *Let* (X, ω_X) *be a complete Kähler manifold whose Ricci curvature is bounded from below. If* $u \in C^2(M, \mathbb{R})$ *is non-negative and satisfies*

$$\Delta_{\omega_X} u \ge u(Bu^p - A) \tag{4.1.9}$$

for some $p, B > 0$*. Then* $A \ge 0$ *and the following holds:*

$$u \le \left(\frac{A}{B}\right)^{1/p}.$$

Proof. We denote $f := (u + c)^{-p/2}$ with fixed constant $c > 0$. The direct computation shows that

$$\Delta_{\omega_X} u = -\frac{2}{p}(u + c)^{1+p/2}\Delta_{\omega_X} f + \frac{2(p+2)}{p^2}(u + c)^{1+p}|\nabla f|_{\omega_X}^2. \tag{4.1.10}$$

Applying Theorem 4.1.6 to $f \ge 0$, we get a sequence $x_\varepsilon \in X$ satisfying (4.1.7). Combining with the assumption (4.1.9) and (4.1.10), we have the following inequality which holds at x_ε:

$$Bu^{1+p} - Au \le \varepsilon\left(\frac{2}{p}(u + c)^{1+p/2} + \frac{2(p+2)}{p^2}(u + c)^{1+p}\right). \tag{4.1.11}$$

We remark that as ε goes to 0, $f(x_\varepsilon)$ converges to $\inf_X f$ and $u(x_\varepsilon)$ to $\sup_X u$. Combining (4.1.11) and $B > 0$, $\sup_X u$ is finite. Therefore, by using (4.1.11) again, we have the desired estimate. $\square$

Thanks to this corollary, we only need to prove that $v := f^*\omega_Y^n/\omega_X^n$ and $u := \mathrm{tr}_{\omega_X}(f^*\omega_Y)$ satisfies the Laplacian estimate (4.1.9). The estimate for v is due to [MY83, Section 1], and that for u is due to Chern-Lu inequality (Proposition 2.7.1). We remark that both inequalities holds even if Y is a Hermitian manifold. For simplicity, we only prove when Y is Kähler.

Proposition 4.1.12. *Let X, Y be (not necessarily compact) Kähler manifolds, and $f\colon X \to Y$ be a holomorphic map. Let ω_X (resp. ω_Y) be a smooth Kähler metric on X (resp. Y). We set $v := f^*\omega_Y^n/\omega_X^n$, and $u := \mathrm{tr}_{\omega_X}(f^*\omega_Y)$.*

(a) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $R(\omega_X) \geq -A$, $\mathrm{Ric}(\omega_Y) \leq -B\omega_Y$, and $\dim X = \dim Y = n$. Then we have

$$\Delta_{\omega_X} \log v \geq nBv^{1/n} - A,$$
$$\Delta_{\omega_X} v \geq v(nBv^{1/n} - A).$$

(b) Suppose that there exists non-negative constants $A, B \geq 0$ satisfying $\mathrm{Ric}(\omega_X) \geq -A\omega_X$, $\mathrm{Bisec}(\omega_Y) \leq -B\omega_Y$. Then we have

$$\Delta_{\omega_X} \log u \geq Bu - A,$$
$$\Delta_{\omega_X} u \geq u(Bu - A).$$

Proof. (b) follows from Proposition 2.7.1. We prove (a). Let $(z^1, \ldots, z^n)$ and $(w^1, \ldots, w^n)$ be normal coodinates on X and Y respectively. We set

$$\omega_X = \sqrt{-1}\, g_{i\bar{j}} dz^i \wedge d\overline{z^j}, \ \omega_Y = \sqrt{-1}\, h_{\alpha\bar{\beta}} dw^\alpha \wedge d\overline{w^\beta}.$$

v is locally denoted as

$$v = \frac{f^*\omega_Y^n}{\omega_X^n} = \frac{\det(h_{\alpha\bar{\beta}} \circ f)\,|\det J(f)|^2}{\det(g_{i\bar{j}})} \tag{4.1.13}$$

where $J(f)$ is the Jacobian of f. Therefore, on $\Omega := \{x \in X \mid \det J(f)(x) \neq 0\}$, we obtain

$$\begin{aligned} \sqrt{-1}\,\partial\bar{\partial} \log v &= f^* \sqrt{-1}\,\partial\bar{\partial} \log \det(h_{\alpha\bar{\beta}}) + \sqrt{-1}\,\partial\bar{\partial} \log \det(g_{i\bar{j}}) \\ &\quad - \sqrt{-1}\,\partial\bar{\partial} \log |\det J(f)|^2 \\ &= f^*(-\mathrm{Ric}(\omega_Y)) + \mathrm{Ric}(\omega_X). \end{aligned}$$

By the assumption on curvatures and the inequality of arithmetic and geometric means, we have the following estimates on Ω:

$$\begin{aligned} \Delta_{\omega_X} \log v &= \mathrm{tr}_{\omega_X}\left(\sqrt{-1}\,\partial\bar{\partial} \log v\right) = \mathrm{tr}_{\omega_X}(f^*(-\mathrm{Ric}(\omega_Y))) + R(\omega_X) \\ &\geq B\mathrm{tr}_{\omega_X}(f^*\omega_Y) - A \\ &\geq nBv^{1/n} - A, \\ \Delta_{\omega_X} v &= \Delta_{\omega_X} e^{\log v} = e^{\log v}\left(|\nabla \log v|^2_{\omega_X} + \Delta_{\omega_X} \log v\right) \\ &\geq v\Delta_{\omega_X} \log v \\ &\geq v(nBv^{1/n} - A). \end{aligned}$$

By continuity, the last inequality holds on the whole X. □

4.2 Conical Kähler Metrics

In this section, we recall the definition of conical Kähler metrics, for short cone metrics, following [Don12, Section 4]. Let X be a compact Kähler manifold of dimension n, D be a smooth divisor on X, and β be a real number satisfying $0 < \beta < 1$. If we take a local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, the standard cone metric ω_β is defined as

$$\omega_\beta := \frac{\beta^2}{|z|^{2(1-\beta)}} \frac{\sqrt{-1}}{2} dz^1 \wedge d\overline{z^1} + \sum_{i=2}^{n} \frac{\sqrt{-1}}{2} dz^i \wedge d\overline{z^i}.$$

We remark that ω_β induces a distance function d_β on U which is expressed as

$$d_\beta(z, w) = \left(\left| (z^1)^\beta - (w^1)^\beta \right|^2 + |z^2 - w^2|^2 + \cdots + |z^n - w^n|^2 \right)^{1/2},$$

where $z = (z^1, \dots, z^n), w = (w^1, \dots, w^n)$. Here, we take a suitable branch of z^β.

Definition 4.2.1 ($C^{2,\alpha,\beta}$-functions)**.** Let α be a constant satiftying $0 < \alpha < \min\{1/\beta - 1, 1\}$. We define the regularites of functions along D as follows.

1. A function f on X is said to be *of class* $C^{,\alpha,\beta}$ if for any local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$, f is an α-Hölder continuous function on U with respect to the distance function d_β.

   This definition is equivalent to the following statement which is the original definition in [Don12]. We set $\widetilde{f}$ by $\widetilde{f}(\xi, z^2, \dots, z^n) := f(|\xi|^{1/\beta - 1}\, \xi, z^2, \dots, z^n)$. Then $\widetilde{f}$ is an α-Hölder continuous function with respect to $\xi, z^2, \dots, z^n$ with respect to the Euclidean distance.

2. A $(1,0)$-form τ is said to be *of class* $C^{,\alpha,\beta}$ if

$$\begin{aligned} \left| z^1 \right|^{1-\beta} \tau \left(\frac{\partial}{\partial z^1} \right) &\in C^{,\alpha,\beta}, \\ \tau \left(\frac{\partial}{\partial z^i} \right) &\in C^{,\alpha,\beta} \quad \text{for} \quad i = 2, \dots, n \end{aligned}$$

3. A $(1,1)$-form σ is said to be *of class* $C^{,\alpha,\beta}$ if

$$\begin{aligned} \left| z^1 \right|^{2(1-\beta)} \sigma \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \overline{z^1}} \right) &\in C^{,\alpha,\beta}, \\ \left| z^1 \right|^{1-\beta} \sigma \left(\frac{\partial}{\partial z^1}, \frac{\partial}{\partial \overline{z^i}} \right) &\in C^{,\alpha,\beta} \quad \text{for} \quad i = 2, \dots, n, \\ \left| z^1 \right|^{1-\beta} \sigma \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z^1}} \right) &\in C^{,\alpha,\beta} \quad \text{for} \quad i = 2, \dots, n, \\ \sigma \left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z^j}} \right) &\in C^{,\alpha,\beta} \quad \text{for} \quad i, j = 2, \dots, n. \end{aligned}$$

4. A function f is said to be *of class* $C^{2,\alpha,\beta}$ if f, ∂f, $\overline{\partial} f$, $\sqrt{-1}\,\partial\overline{\partial} f$ are of class $C^{,\alpha,\beta}$.

Definition 4.2.2 (Cone metrics)**.** A closed positive $(1,1)$-current ω on X is called a *cone metric with cone angle* $2\pi\beta$ *along* D if it satisfies the following three conditions:

(i) ω is a Kähler metric on $X \setminus D$

(ii) For each point $x \in D$, there exists a local holomorphic chart $(U, (z^1, \dots, z^n))$ satisfying $D \cap U = \{z^1 = 0\}$ such that ω is quasi-isometric to the standard cone metric ω_β on $U \setminus D$, that is, there exists a constant $C = C_U > 0$ such that

$$\frac{1}{C}\omega_\beta \le \omega \le C\omega_\beta \quad \text{on } U \setminus D.$$

(iii) There exists a smooth Kähler form ω_0 on X, and a $C^{2,\alpha,\beta}$-function φ such that

$$\omega = \omega_0 + \sqrt{-1}\,\partial\overline{\partial}\varphi.$$

In [Jef00a], the regularity condition (iii) does not assumed. However, we assume here.

A typical example of the cone metric is $\omega := \omega_0 + \delta\sqrt{-1}\,\partial\overline{\partial}|s|_h^\beta$, where ω_0 is a smooth Kähler metric on X, δ is a sufficiently small constant, $s \in H^0(X, \mathcal{O}_X(D))$ is a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D, and h is a smooth Hermitian metric.

4.3 Schwarz Lemma for Conical Kähler Metrics

In this section, we prove Schwarz lemma for conical Kähler metrics, which is the main result in this chapter [Nom16b].

To state the theorems, we use the following setups and notations. Let X and Y be compact Kähler manifolds, $D \subset X$, $E \subset Y$ be smooth divisors, and $f\colon X \to Y$ be a surjective holomorphic map satisfying $f^*(E) = kD$ with $k \in \mathbb{Z}_{>0}$. Let ω_X (resp. ω_Y) be a cone metric with cone angle $2\pi\alpha$ (resp. $2\pi\beta$) along D (resp. E) on X (resp. Y). Let $s \in H^0(X, \mathcal{O}_X(D))$ be a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D and h be a smooth Hermitian metric on it satisfying $|s|_h \le 1$. Let $C > 0$ be an upper bound for the Chern curvature of h i.e. $\sqrt{-1}\,R_h \le C\omega_X$. For a Kähler form ω, we will denote by $\mathrm{Ric}(\omega)$ the Ricci curvature of ω, $R(\omega)$ the scalar curvature of ω, and $\mathrm{Bisec}(\omega)$ the bisectional curvature of ω.

Schwarz lemma for the cone metrics obtained by Jeffres [Jef00a] is states as follows.

Theorem 4.3.1 ([Jef00a, Theorem])**.** *Assume that* $\dim X = \dim Y = n$*, the cone angles satisfy* $\alpha \leq \beta$ *and there exists non-negative constants* $A, B \geq 0$ *satisfying*

$$R(\omega_X) \geq -A, \ \mathrm{Ric}(\omega_Y) \leq -B\omega_Y < 0. \tag{4.3.2}$$

Then, the volume forms satisfy

$$f^*\omega_Y^n \leq \left(\frac{A}{B}\right)^n \omega_X^n \quad \text{on } X \setminus D.$$

Since the cone metric is not complete on $X \setminus D$, we cannot apply the maximum principle argument directly. Jeffers overcame this difficulty by using a barrier function, called "Jeffres' trick". However, his original proof seems to need more assumptions on the regularity of the cone metrics along D as in Definition 4.2.1 (see the proof of Proposition 4.3.6).

We will generalize this theorem to a general cone angle and prove a Schwarz lemma for cone metrics.

Theorem 4.3.3 (Volume forms)**.** *Assume that* $\dim X = \dim Y = n$ *and the curvature condition (4.3.2) holds.*

(a) Suppose $\alpha \leq k\beta$*. Then we have*

$$f^*\omega_Y^n \leq \left(\frac{A}{nB}\right)^n \omega_X^n \quad \text{on } X \setminus D.$$

(b) Suppose $\alpha > k\beta$*. Then we have*

$$f^*\omega_Y^n \leq \left(\frac{A + (\alpha - k\beta)C}{nB}\right)^n \frac{\omega_X^n}{|s|_h^{2(\alpha - k\beta)}} \quad \text{on } X \setminus D.$$

We remark that the condition $\alpha \leq k\beta$ on cone angles in the statement (a) is weaker than assumptions in Theorem 4.3.1.

Theorem 4.3.4 (Metrics)**.** *Assume that there exists non-negative constants* $A, B \geq 0$ *such that the curvatures satisfy the following:*

$$\mathrm{Ric}(\omega_X) \geq -A\omega_X, \ \mathrm{Bisec}(\omega_Y) \leq -B < 0. \tag{4.3.5}$$

(a) Suppose $\alpha \leq k\beta$*. Then we have*

$$f^*\omega_Y \leq \frac{A}{B}\omega_X \quad \text{on } X \setminus D.$$

(b) Suppose $\alpha > k\beta$*. Then we have*

$$f^*\omega_Y \leq \frac{A + (\alpha - k\beta)C}{B} \frac{\omega_X}{|s|_h^{2(\alpha - k\beta)}} \quad \text{on } X \setminus D.$$

If the cone angle satisfies $\alpha > k\beta$, the pullback $f^*\omega_Y$ has singularites along D. In fact, even in a one-dimensional case, the pullback of the standard cone metric $\omega_\beta = (\beta^2/|w|^{2(1-\beta)})\sqrt{-1}\,dw \wedge d\overline{w}/2$ by $f : z \mapsto w = z^k$ is given by

$$f^*\omega_\beta = \beta^2 k^2 |z|^{2(k\beta-1)} \frac{\sqrt{-1}}{2} dz \wedge d\overline{z},$$

therefore we have

$$\frac{f^*\omega_\beta}{\omega_\alpha} = \frac{\beta^2 k^2}{\alpha^2} |z|^{2(k\beta-\alpha)},$$

which is singular if $\alpha > k\beta$.

The next proposition is the so-called "Jeffres' trick" and needs regularity on the definition of cone metrics.

Proposition 4.3.6 ([Jef00a, Section 4])**.** *Let X be a compact Kähler manifold, D be a smooth divisor, and β be a real number satisfying $0 < \beta < 1$. Let $s \in H^0(X, \mathcal{O}_X(D))$ be a holomorphic section of the line bundle $\mathcal{O}_X(D)$ whose zero divisor is D, and h is a smooth Hermitian metric. Then, for any function $u \in C^{,\alpha,\beta}$ and $\varepsilon > 0$, every maximum point of the function*

$$u_\varepsilon := u + \varepsilon |s|_h^{2\gamma}$$

on X belongs to $X \setminus D$ if $0 < 2\gamma < \alpha\beta$.

Proof. We assume that u_δ takes maximum at $x_0 \in D$. Let $(U, (z^1, \dots, z^n))$ be a holomorphic chart centered at x_0 satisfying $D \cap U = \{z^1 = 0\}$. By the definition of x_0, for any $x = (z, 0, \dots, 0) \in U$, we have

$$\frac{|u(x) - u(x_0)|}{d_\beta(x, x_0)^\alpha} = \frac{|u(x) - u(x_0)|}{|z|^{\alpha\beta}} \geq \frac{\varepsilon |s|_h^{2\gamma}(x)}{|z|^{\alpha\beta}} \geq \frac{\varepsilon}{C} \frac{|z|^{2\gamma}}{|z|^{\alpha\beta}}.$$

Since $0 < 2\gamma < \alpha\beta$, the right hand side goes to ∞ as $z \to 0$. This contradicts with the definition of $C^{,\alpha,\beta}$. $\square$

Theorem 4.3.3 and Theorem 4.3.4 can be shown in a smilar manner. We only prove Theorem 4.3.3 here.

Proof of Theorem 4.3.3 (a). Since f can be represented as $(w^1, \dots, w^n) = ((z^1)^k, f_2(z), \dots, f_n(z))$ such that $D = \{z^1 = 0\}$ and $E = \{w^1 = 0\}$, the direct computation gives that f is locally Hölder continuous with respect to d_α and d_β if $\alpha \leq k\beta$. Combining with (4.1.13) and the definition of the cone metrics, $v := f^*\omega_Y^n/\omega_X^n$ is a $C^{,\sigma,\beta}$ function for some $0 < \sigma < 1$. By Proposition 4.3.6, all maximum points of $v_\delta := v + \varepsilon |s|_h^{2\gamma}$ belong to $X \setminus D$ where γ is sufficiently small. Since v_ε is smooth on $X \setminus D$, we can apply the maximum principle argument to v_ε. The direct computation show that

$$\begin{aligned}\sqrt{-1}\,\partial\overline{\partial}|s|_h^{2\gamma} = \sqrt{-1}\,\partial\overline{\partial} e^{\gamma \log |s|_h^2} &= |s|_h^{2\gamma}(\gamma\sqrt{-1}\,\partial\overline{\partial}\log|s|_h^2 + \gamma^2\sqrt{-1}\,\partial\log|s|_h^2 \wedge \overline{\partial}\log|s|_h^2) \\ &\geq -\gamma |s|_h^{2\gamma}\sqrt{-1}\,R_h.\end{aligned}$$

Therefore, there exists a constant $C > 0$ (which is independent of ε) satisfying

$$\Delta_{\omega_X} |s|_h^{2\gamma} \geq -C.$$

Let $x_0 \in X \setminus D$ be a maximum point of v_ε. At this point, by Proposition 4.1.12 (a), we have

$$0 \geq \Delta_{\omega_X} v_\varepsilon = \Delta_{\omega_X} v + \varepsilon \Delta_{\omega_X} |s|_h^{2\gamma} \geq v(nBv^{1/n} - A) - \varepsilon C.$$

Simple calculus show that the function $t \mapsto t^n(nBt - A) - \varepsilon C$ takes non-positive values exactly on some bounded interval $[0, T_\varepsilon]$ and $T_\varepsilon \to A/(nB)$ as $\varepsilon \to 0$. It follows that

$$v_\varepsilon(x_0) = v(x_0) + \varepsilon |s|_h^{2\gamma}(x_0) \leq T_\varepsilon^n + \varepsilon \sup_X |s|_h^{2\gamma}.$$

Since the right hand side does not depend on x_0 and x_0 is any maximum point of v_ε, this inequality holds on whole X. Therefore, we have the following inequality

$$v = v_\varepsilon - \varepsilon |s|_h^{2\gamma} \leq v_\varepsilon \leq T_\varepsilon^n + \varepsilon \sup_X |s|_h^{2\gamma}$$

on X. By taking $\varepsilon \to 0$, we obtain $v \leq (A/(nB))^n$. □

Proof of Theorem 4.3.3 (b). By definition of the cone metric, we can easily see that for any $\varepsilon > 0$,

$$v_\varepsilon := |s|_h^{2(\ell+\varepsilon)} v = |s|_h^{2(\ell+\varepsilon)} \frac{f^*\omega_Y^n}{\omega_X^n}$$

tends to 0 as x approaches to D, where $\ell := \alpha - k\beta > 0$. Then, combining the Laplacian estimate in Proposition 4.1.12 (a), we have

$$\begin{aligned} \Delta_{\omega_X} \log v_\varepsilon &= -(\ell + \varepsilon) \mathrm{tr}_{\omega_X} \left(\sqrt{-1}\, R_h \right) + \Delta_{\omega_X} \log v \\ &\geq -(\ell + \varepsilon) C - A + nBv^{1/n}, \\ \Delta_{\omega_X} v_\varepsilon &\geq v_\varepsilon(-(\ell + \varepsilon) C - A + nBv^{1/n}). \end{aligned}$$

If $x_0 \in X$ is a maximum of v_ε, we can assume that $x_0 \in X \setminus D$. At this point, by applying the maximum principle, we have

$$v(x_0) \leq \left(\frac{A + (\ell + \varepsilon) C}{nB} \right)^n.$$

Therefore, we get

$$v_\varepsilon(x_0) \leq |s|_h^{\ell+\varepsilon}(x_0) \left(\frac{A + (\ell + \varepsilon) C}{nB} \right)^n \leq \left(\frac{A + (l + \varepsilon) C}{nB} \right)^n.$$

Since the right hand side does not depend on x_0, this inequality holds on X. Taking $\varepsilon \to 0$, we obtain

$$|s|_h^{2\ell} \frac{f^*\omega_Y^n}{\omega_X^n} \leq \left(\frac{A + \ell C}{nB} \right)^n.$$

□

Chapter 5

Blowup Behavior of the Conical Kähler-Ricci Flow

In this chapter, we investigate the scalar curvature behavior along the normalized conical Kähler-Ricci flow ω_t, which is the conic version of the normalized Kähler-Ricci flow, with finite maximal existence time $T < \infty$. In Theorem 5.1.5, we prove that the scalar curvature of ω_t is bounded from above by $C/(T-t)^2$ under the existence of a contraction associated to the limiting cohomology class $[\omega_T]$ [Nom16c]. This generalizes Zhang's work [Zha10] to the conic case.

5.1 Statement of the Result

In this section, we recall the definitions and properties for the conical Kähler-Ricci flow and state the main result Theorem 5.1.5 in this chapter.

Let X be a compact Kähler manifold of dimension n, D be a smooth divisor on X, and β be a positive real number satisfying $0 < \beta < 1$.

Definition 5.1.1. A family of cone metrics ω_t with cone angle $2\pi\beta$ along D called the *conical Kähler-Ricci flow* if it satisfies the following parabolic equation:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_t &= -\mathrm{Ric}(\omega_t) - \omega_t + 2\pi(1-\beta)[D], \\ \omega_t|_{t=0} &= \omega^*, \end{cases} \tag{5.1.2}$$

where $[D]$ is the current of integration over D, and ω^* is a initial cone metric.

The maximal existence theorem for conical Kähler-Ricci flow holds for certain inital metrics which is similar to the Kähler-Ricci flow case Theorem 3.2.4.

Theorem 5.1.3 ([She14a, She14b])**.** *Let $s \in H(X, \mathcal{O}_X(D))$ be a section whose zero divisor is D, h be a smooth Hermitian metric on $\mathcal{O}_X(D)$ and ω_0 be a smooth Kähler form on X. We set $\omega^* := \omega_0 + k\sqrt{-1}\,\partial\overline{\partial}|s|_h^{2\beta}$ with sufficiently small $k \in \mathbb{R}_{>0}$ such that ω^* is a cone metric with cone angle $2\pi\beta$ along D. Then the conical Kähler-Ricci flow starting from ω^* uniquely exists for $t \in [0, T)$ where*

$$T = \sup\{t > 0 \mid [\omega_t] = e^{-t}[\omega_0] + (1-e^{-t})2\pi c_1(K_X + (1-\beta)D) \ \ \textit{is Kähler}\ \}.$$

We call T the *maximal existence time* for the conical Kähler-Ricci flow. By Theorem 3.2.4, the Kähler-Ricci flowis closely related to the canonical bundle K_X, on the other hand, by Theorem 5.1.3, the conical Kähler-Ricci flowis to the log-canonical bundle $K_X + (1-\beta)D$.

In this chapter, we treat the singularity behavior of the scalar curvature along the conical Kähler-Ricci flow as t approaches to T. We assume the following contraction type condition on the cohomology class $[\omega_T]$. Let $f\colon X \to Z$ be a holomophic map between compact Kähler manifolds, whose image is contained in a normal irreducible subvariety Y of Z. Let D_Y be an effective Cartier divisor on Y such that the pullback of D_Y satisfies $D = f^*D_Y$. Let h_Y be a smooth Hermitian metric on the line bundle $\mathcal{O}_Y(D_Y)$ in the sense of [EGZ09, Section 5], and s_Y be a holomorphic section of $\mathcal{O}_Y(D_Y)$ whose zero divisor is D_Y. We define the initial cone metric ω^* by

$$\omega^* := \omega_0 + k\sqrt{-1}\,\partial\overline{\partial}|s|_h^{2\beta}, \tag{5.1.4}$$

where ω_0 is a smooth Kähler form on X, $k \in \mathbb{R}_{>0}$ is a sufficiently small real number, $s := f^*s_Y$ is the holomorphic section of $\mathcal{O}_X(D)$, and $h := f^*h_Y$ is the smooth Hermitian metric on $\mathcal{O}_X(D)$. We remark that if we take k sufficiently small, ω^* is actually a cone metric with cone angle $2\pi\beta$ along D.

Let ω_t be the normalized conical Kähler-Ricci flow with initial cone metric ω^*, and T be the maximal existence time of ω_t. We further assume that T is finite and there exists a smooth Kähler form ω_Z on Z satisfying

$$[f^*\omega_Z] = [\omega_T] \in H^{1,1}(X, \mathbb{R}).$$

Under these assumptions, we have the following theorem.

Theorem 5.1.5. *The scalar curvature $R(\omega_t)$ of ω_t satisfies*

$$R(\omega_t) \le \frac{C}{(T-t)^2} \quad \text{on } X \setminus D,$$

where $C > 0$ is a constant independent of t.

This Theorem is a cone metric analogue to the following Zhang's result.

Theorem 5.1.6 ([Zha10])**.** *Let ω_0 be a Kähler form such that the maximal existence time T_{KRF} for the normalized Kähler-Ricci flow (3.2.5) is finite. Assume that there exists a holomorphic map $f\colon X \to (Z, \omega_Z)$ between compact Kähler manifolds such that $[f^*\omega_Z] = [\omega_{T_{KRF}}]$. Then there exists a constant $C > 0$ such that the scalar curvature of the normalized Kähler-Ricci flowsatisfies*

$$R(\omega_t) \le \frac{C}{(T_{KRF}-t)^2}.$$

In contrast with Zhang's result, we need to treat with the singularities of ω_t along D. This is overcame by using the approximation technique used in [CGP13, She14a, LZ14, Edw15].

Remark 5.1.7. If we replace $(1-\beta)D$ by $\sum_{i\in I}(1-\beta_i)D_i$ where D_i are smooth divisors intersecting transversely, the same argument below gives the same conclusion. But for simplicity, we only treat one smooth divisor case.

5.2 Approximation by the Twisted Normalized Kähler-Ricci Flow

In the following argument, we assume that the conditions in Theorem 5.1.5 are always satisfied. We first define a family of reference smooth Kähler forms $\widehat{\omega}_t$ whose cohomology classes are equal to $[\omega_t]$. We set $\widehat{\omega}_\infty$ by

$$\begin{aligned}\widehat{\omega}_\infty &:= -\frac{e^{-T}}{1-e^{-T}}\omega_0 + \frac{1}{1-e^{-T}} f^*\omega_Z \\ &\in -\frac{e^{-T}}{1-e^{-T}}[\omega_0] + \frac{1}{1-e^{-T}}[\omega_T] = 2\pi c_1(K_X + (1-\beta)D),\end{aligned}$$

and $\widehat{\omega}_t$ by

$$\widehat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\widehat{\omega}_\infty = a_t\omega_0 + (1-a_t)\widehat{\omega}_T, \tag{5.2.1}$$

where $a_t := (e^{-t} - e^{-T})/(1-e^{-T})$. In this setting, $\widehat{\omega}_T = f^*\omega_Z \geq 0$ is semi-positive, hence $\widehat{\omega}_t$ are smooth Kähler forms for any $t \in [0, T)$. The cohomology class of $\widehat{\omega}_t$ coincide with $[\omega_t]$.

We next define a family of reference smooth Kähler forms $\widetilde{\omega}_{\varepsilon,t}$ whose cohomology classes are equal to $[\omega_t]$. We use the approximation method as in [She14a, LZ14, Edw15] originated from [CGP13]. We denote $\rho_\varepsilon := \chi(|s|_h^2, \varepsilon^2)$, where

$$\chi(u, \varepsilon^2) := \beta \int_0^u \frac{(r+\varepsilon^2)^\beta - \varepsilon^{2\beta}}{r} dr.$$

Then, ρ_ε are smooth functions on X and converge to $|s|_h^{2\beta}$ in $C^\infty_{\mathrm{loc}}(X \setminus D)$ as $\varepsilon \to 0$. In this notation, we define reference smooth Kähler forms $\widetilde{\omega}_{\varepsilon,t}$ by

$$\widetilde{\omega}_{\varepsilon,t} := \widehat{\omega}_t + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon = a_t\widetilde{\omega}_{\varepsilon,0} + (1-a_t)\widetilde{\omega}_{\varepsilon,T}. \tag{5.2.2}$$

These forms converge to $\widetilde{\omega_t}$ in $C^\infty_{\mathrm{loc}}(X \setminus D)$ and as current on X when ε tends to 0.

We prove that if we take k sufficiently small, $\widetilde{\omega}_{\varepsilon,t}$ is positive for all $t \in [0, T)$. Let $C_1 > 0$ be a constant satisfying

$$-C_1\omega_Z \leq \sqrt{-1}\,R_{h_Y} \leq C_1\omega_Z \quad \text{on } Y, \tag{5.2.3}$$

where R_{h_Y} is the Chern curvature of h_Y. Since $h = f^*h_Y$ and $\widehat{\omega}_T = f^*\omega_Z$, we have

$$-C_1\widehat{\omega}_T \leq \sqrt{-1}\,R_h \leq C_1\widehat{\omega}_T \quad \text{on } X. \tag{5.2.4}$$

Let $C_2 > 0$ and $C_3 > 1$ be constants such that

$$\sup_Y |s_Y|_{h_Y} \leq C_2, \tag{5.2.5}$$

$$\widehat{\omega}_T = f^*\omega_Z \leq C_3\omega_0 \quad \text{on } X. \tag{5.2.6}$$

By (5.2.5), there exists a constant $C_4 > 0$ independent of ε such that

$$0 \leq \rho_\varepsilon \leq C_4 \quad \text{on } X. \tag{5.2.7}$$

By the computation in [CGP13, Section 3], we have

$$\begin{aligned}\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon &= \beta^2\frac{\sqrt{-1}\langle\nabla s,\nabla s\rangle_h}{(|s|_h^2+\varepsilon^2)^{1-\beta}} - \beta\Big((|s|_h^2+\varepsilon^2)^\beta - \varepsilon^{2\beta}\Big)\sqrt{-1}\,R_h \\ &\geq -\beta C_1 C_2^{2\beta}\widehat{\omega}_T,\end{aligned} \tag{5.2.8}$$

where ∇ is the Chern connection of the line bundle $(\mathcal{O}_X(D),h)$, R_h is its Chern curvature, and $\sqrt{-1}\langle\nabla s\wedge\nabla s\rangle_h$ is a semi-positive closed real $(1,1)$-form combining the wedge product of differential forms with the Hermitian metric h on $\mathcal{O}_X(D)$. By (5.2.2), (5.2.8), and (5.2.6), we obtain the following inequalities:

$$\widetilde{\omega}_{\varepsilon,T} = \widehat{\omega}_T + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon \geq (1-k\beta C_1C_2^{2\beta})\widehat{\omega}_T \geq (1-k\beta C_1C_2^{2\beta}C_3)\widehat{\omega}_T, \tag{5.2.9}$$

$$\widetilde{\omega}_{\varepsilon,0} = \omega_0 + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon \geq \omega_0 - k\beta C_1C_2^{2\beta}\widehat{\omega}_T \geq (1-k\beta C_1C_2^{2\beta}C_3)\omega_0. \tag{5.2.10}$$

Finally, these inequalities give the positivity of $\widetilde{\omega}_{\varepsilon,t}$ for any $t\in[0,T)$:

$$\widetilde{\omega}_{\varepsilon,t} = \widehat{\omega}_t + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon = a_t\widetilde{\omega}_{\varepsilon,0} + (1-a_t)\widetilde{\omega}_{\varepsilon,T} \geq (1-k\beta C_1C_2^{2\beta}C_3)\widehat{\omega}_t > 0.$$

By using these reference smooth Kähler forms, we consider the following approximate flow:

$$\begin{cases}\dfrac{\partial}{\partial t}\varphi_{\varepsilon,t} &= \log\dfrac{(\widetilde{\omega}_{\varepsilon,t}+\sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t})^n}{\Omega} - \varphi_{\varepsilon,t} + (1-\beta)\log(|s|_h^2+\varepsilon^2) - k\rho_\varepsilon, \\ \varphi_{\varepsilon,t}|_{t=0} &= 0,\end{cases} \tag{5.2.11}$$

where Ω is a smooth volume form on X satisfying

$$-\mathrm{Ric}(\Omega) + (1-\beta)\sqrt{-1}\,R_h = \widehat{\omega}_\infty\ \in 2\pi c_1(K_X+(1-\beta)D).$$

We set $\omega_{\varepsilon,t}$ by

$$\omega_{\varepsilon,t} := \widetilde{\omega}_{\varepsilon,t} + \sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t}. \tag{5.2.12}$$

Then, $\omega_{\varepsilon,t}$ satisfies the following twisted Kähler-Ricci flow:

$$\begin{cases}\dfrac{\partial}{\partial t}\omega_{\varepsilon,t} &= -\mathrm{Ric}(\omega_{\varepsilon,t}) - \omega_{\varepsilon,t} + \eta_\varepsilon, \\ \omega_t|_{t=0} &= \widetilde{\omega}_{\varepsilon,0}(:=\omega_0 + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon),\end{cases} \tag{5.2.13}$$

where η_ε is a closed real $(1,1)$-form defined by $\eta_\varepsilon := (1-\beta)\sqrt{-1}\,\partial\overline{\partial}\log(|s|_h^2+\varepsilon^2) + (1-\beta)\sqrt{-1}\,R_h$. η_ε converges to $2\pi(1-\beta)[D]$ in $C^\infty_{\mathrm{loc}}(X\setminus D)$ and as current on X when ε goes to 0.

The validity of these approximations (5.2.11), (5.2.13) is justified by the following theorem due to Shen [She14a].

Theorem 5.2.14 ([She14a, Section 2]). *There exists a subsequence ε_i converging to 0 as $i\to\infty$ such that $\varphi_{\varepsilon_i,t}$ converges to φ_t in $C^\infty_{\mathrm{loc}}(X\setminus D)$ and $\omega_{\varepsilon_i,t}$ converges to ω_t in $C^\infty_{\mathrm{loc}}(X\setminus D)$ and as current on X.*

Thanks to this theorem, we only need to estimate $\varphi_{\varepsilon,t}$ and $\omega_{\varepsilon,t}$.

5.3 Overview of the Proof of Theorem 5.1.5

In this section, we outline the proof of Theorem A. First, we need the following formulas.

Proposition 5.3.1. *The Ricci curvature* $\mathrm{Ric}(\omega_{\varepsilon,t})$ *and the scalar curvature* $R(\omega_{\varepsilon,t})$ *satisfy the following formulas:*

(a)
$$(1-e^{t-T})\,(\mathrm{Ric}(\omega_{\varepsilon,t})-\eta_\varepsilon) = -\sqrt{-1}\,\partial\overline{\partial}v_{\varepsilon,t} + e^{t-T}\omega_{\varepsilon,t} - \widehat{\omega}_T,$$

(b)
$$(1-e^{t-T})(R(\omega_{\varepsilon,t})-\mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon)) = -\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} + ne^{t-T} - \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T),$$

where

$$v_{\varepsilon,t} := (1-e^{t-T})\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon.$$

Proof. (b) follows from (a) by taking traces. We prove (a). By (5.2.13), (5.2.2), and (5.2.12), we have

$$\begin{aligned}
\mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon &= -\frac{\partial}{\partial t}\,\omega_{\varepsilon,t} - \omega_{\varepsilon,t}\\
&= -\left(\frac{\partial}{\partial t}\,\widehat{\omega}_t + \frac{\partial}{\partial t}\,\sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t}\right) - \left(\widehat{\omega}_t + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon + \sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t}\right)\\
&= -\sqrt{-1}\,\partial\overline{\partial}(\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon) - \left(\widehat{\omega}_t + \frac{\partial}{\partial t}\,\widehat{\omega}_t\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
-e^{t-T}(\mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon) &= -e^{t-T}\left(-\frac{\partial}{\partial t}\,\omega_{\varepsilon,t} - \omega_{\varepsilon,t}\right)\\
&= e^{t-T}\left(\frac{\partial}{\partial t}\,\widehat{\omega}_t + \frac{\partial}{\partial t}\,\sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t}\right) + e^{t-T}\omega_{\varepsilon,t}\\
&= \sqrt{-1}\,\partial\overline{\partial}\left(e^{t-T}\dot{\varphi}_{\varepsilon,t}\right) + e^{t-T}\omega_{\varepsilon,t} + e^{t-T}\frac{\partial}{\partial t}\,\widehat{\omega}_t.
\end{aligned}$$

Combining these equalities and (5.2.1), we have (a). □

By this proposition, to obtain the upper bound for the scalar curvature $R(\omega_{\varepsilon,t})$, we only need to estimate $u_{\varepsilon,t} := \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)$ and $\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}$. We devide our argument into the following 5 steps:

Step 1. The C^0-estimate for $v_{\varepsilon,t}$ (Section 5.4).

Step 2. The C^0-estimate for $u_{\varepsilon,t} := \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)$ using Step 1 and the parabolic Schwarz lemma (Section 5.5).

Step 3. The gradient estimate for $v_{\varepsilon,t}$ (Section 5.6).

Step 4. The Laplacian estimate for $v_{\varepsilon,t}$ (Section 5.7):

$$\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t} \geq -\frac{C}{T-t}.$$

Step 5. Proof of Theorem A (Section 5.7).

5.4 The C^0-estimate for $v_{\varepsilon,t}$

In this section, we prove the C^0-estimates for $v_{\varepsilon,t}$. More precisely, we prove the following proposition.

Proposition 5.4.1. *There exists a constant $C_5 > 0$ independent of ε and t such that*

$$\|v_{\varepsilon,t}\|_{C^0} \leq C_5$$

holds.

To apply the maximum principle, we need the following lemma.

Lemma 5.4.2. *$v_{\varepsilon,t}$ satisfies the following evolution equation*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) v_{\varepsilon,t} = -n + u_{\varepsilon,t},$$

where $u_{\varepsilon,t} := \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)$.

Proof. Differentiating (5.2.11) with respect to t, we have

$$\frac{\partial}{\partial t}\,\dot{\varphi}_{\varepsilon,t} = \mathrm{tr}_{\omega_{\varepsilon,t}}\left(\frac{\partial}{\partial t}\left(\widetilde{\omega}_{\varepsilon,t} + \sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t}\right)\right) - \dot{\varphi}_{\varepsilon,t}$$

$$\text{i.e. } \frac{\partial}{\partial t}\,(\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t}) = \mathrm{tr}_{\omega_{\varepsilon,t}}\left(\frac{\partial}{\partial t}\,\widehat{\omega}_t\right) + \Delta_{\omega_{\varepsilon,t}}\dot{\varphi}_{\varepsilon,t}. \tag{5.4.3}$$

On the other hand, by (5.2.12) and (5.2.2), we have

$$\Delta_{\omega_{\varepsilon,t}}\varphi_{\varepsilon,t} = \mathrm{tr}_{\omega_{\varepsilon,t}}(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\widetilde{\omega}_{\varepsilon,t}) = n - \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_t) - \Delta_{\omega_{\varepsilon,t}}(k\rho_\varepsilon).$$

Combing these, we obtain

$$\frac{\partial}{\partial t}\left(\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon\right) = \Delta_{\omega_{\varepsilon,t}}(\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon) - n + \mathrm{tr}_{\omega_{\varepsilon,t}}\left(\widehat{\omega}_t + \frac{\partial}{\partial t}\,\widehat{\omega}_t\right). \tag{5.4.4}$$

Next, by using (5.4.3), we have

$$\frac{\partial}{\partial t}\left(-e^{t-T}\dot{\varphi}_{\varepsilon,t}\right) = -e^{t-T}\frac{\partial}{\partial t}\,(\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t}) = -\mathrm{tr}_{\omega_{\varepsilon,t}}\left(e^{t-T}\frac{\partial}{\partial t}\,\widehat{\omega}_t\right) - \Delta_{\omega_{\varepsilon,t}}(e^{t-T}\dot{\varphi}_{\varepsilon,t}). \tag{5.4.5}$$

By (5.4.4), (5.4.5), and (5.2.1), we get the assertion. $\square$

Next, we prove the uniform volume estimate of the reference metrics $\widetilde{\omega}_{\varepsilon,t}$.

Lemma 5.4.6. *There exists a constant $C_6 > 0$ independent of ε ant t satisfying the following inequalites:*

(a)

$$\frac{1}{C_6}\frac{\Omega}{(|s|_h^2+\varepsilon^2)^{1-\beta}} \leq \widetilde{\omega}_{\varepsilon,0}^n \leq C_6\frac{\Omega}{(|s|_h^2+\varepsilon^2)^{1-\beta}}.$$

(b)

$$\widetilde{\omega}_{\varepsilon,t}^n \leq C_3^n C_6 \frac{\Omega}{(|s|_h^2+\varepsilon^2)^{1-\beta}}.$$

Proof. The first inequality follows from (5.2.8). We prove the second one. For $0 < k < C_3$, by (5.2.4) and (5.2.2), we have

$$\widetilde{\omega}_{\varepsilon,T} = \widehat{\omega}_T + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon \leq C_3\omega_0 + k\sqrt{-1}\,\partial\overline{\partial}\rho_\varepsilon \leq C_3\widetilde{\omega}_{\varepsilon,0}.$$

Since $C_3 > 1$, we have

$$\widetilde{\omega}_{\varepsilon,t} = a_t\widetilde{\omega}_{\varepsilon,0} + (1-a_t)\widetilde{\omega}_{\varepsilon,T} \leq a_t\widetilde{\omega}_{\varepsilon,0} + C_3(1-a_t)\widetilde{\omega}_{\varepsilon,0} \leq C_3\widetilde{\omega}_{\varepsilon,0}.$$

Therefore, we get the assertion. □

Using these lemmas, we can prove the uniform lower boundedness of $v_{\varepsilon,t}$.

Proposition 5.4.7. *$v_{\varepsilon,t}$ is uniformly lower bounded. More precisely, there exists a constant $C_7 > 0$ independent of ε and t such that*

$$v_{\varepsilon,t} \geq -C_7.$$

Proof. By Lemma 5.4.2 and the semi-positivity of $\widehat{\omega}_T$, we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)(v_{\varepsilon,t} + nt) = u_{\varepsilon,t} = \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T) \geq 0.$$

Thus, the maximum principle for $v_{\varepsilon,t} + nt$ gives the following:

$$v_{\varepsilon,t} + nt \geq \min_{X\times\{0\}}(v_{\varepsilon,t} + nt) = (1-e^{-T})\dot{\varphi}_{\varepsilon,0} + k\rho_\varepsilon \geq (1-e^{-T})\dot{\varphi}_{\varepsilon,0}.$$

Lemma 5.4.6 (a) and (5.2.7) give the lower boundedness of right hand side as follows:

$$\dot{\varphi}_{\varepsilon,0} = \log\frac{\widetilde{\omega}_{\varepsilon,0}^n}{\Omega/(|s|_h^2+\varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,0} - k\rho_\varepsilon \geq -\log C_6 - kC_4.$$

Therefore we get the assertion. □

To prove the uniform upper boundedness of $v_{\varepsilon,t} = (1-e^{t-T})\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon$, it is enough to show that $\varphi_{\varepsilon,t}$ and $\dot{\varphi}_{\varepsilon,t}$ are unifomly upper bounded.

Proposition 5.4.8. *We have the following inequalities:*

(a) $\varphi_{\varepsilon,t} \leq C_8$,

(b) $\dot{\varphi}_{\varepsilon,t} \leq C_9$,

where $C_8 > 0$, $C_9 > 0$ independent of ε and t.

Proof. (a) Since $\varphi_{\varepsilon,0} = 0$, we may assume that $\varphi_{\varepsilon,t}$ takes maximum at $(x_0, t_0) \in X \times (0, T)$. By Lemma 5.4.6 (b), we have the following inequatliy which holds at (x_0, t_0):

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t} \varphi_{\varepsilon,t} &= \log \frac{(\widetilde{\omega}_{\varepsilon,t} + \sqrt{-1}\,\partial\overline{\partial}\varphi_{\varepsilon,t})^n}{\Omega/(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,t} - k\rho_\varepsilon \\ &\leq \log \frac{\widetilde{\omega}_{\varepsilon,t}^n}{\Omega/(|s|_h^2 + \varepsilon^2)^{1-\beta}} - \varphi_{\varepsilon,t} - k\rho_\varepsilon \\ &\leq \log(C_3^n C_6) - \varphi_{\varepsilon,t}. \end{aligned}$$

Therefore, we obtain

$$\varphi_{\varepsilon,t}(x_0, t_0) \leq \log(C_3^n C_6) \ =: \ C_8.$$

Since (x_0, t_0) is arbitrary, $\varphi_{\varepsilon,t} \leq C_8$ holds on $X \times [0, T)$.

(b) We set $H_{\varepsilon,t} := (1 - e^t)\dot{\varphi}_{\varepsilon,t} + \varphi_{\varepsilon,t} + k\rho_\varepsilon + nt$. The same computation in Lemma 5.4.2 gives

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) H_{\varepsilon,t} = \mathrm{tr}_{\omega_{\varepsilon,t}}(\omega_0) > 0.$$

By the maximum principle for $H_{\varepsilon,t}$, we have

$$H_{\varepsilon,t} \geq \min_{X \times \{0\}} H_{\varepsilon,t} = k\rho_\varepsilon \geq 0.$$

Therefore, combining with (a) and (5.2.7), we get the upper bound for $\dot{\varphi}_{\varepsilon,t}$:

$$\dot{\varphi}_{\varepsilon,t} \leq \frac{\varphi_{\varepsilon,t} + k\rho_\varepsilon + nt}{e^t - 1} \leq \frac{C_8 + kC_4 + nT}{e^t - 1}.$$

Combining with the uniform local estimate for the parabolic equation, we get the assertion. $\square$

5.5 The C^0-estimate for $u_{\varepsilon,t}$

In this section, we prove the following proposition.

Proposition 5.5.1. *There exists a constant $C_{10} > 0$ independent of ε and t such that*

$$0 \leq u_{\varepsilon,t} := \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T) \leq C_{10}.$$

To prove this proposition, we need to estimate on $\eta_\varepsilon := (1-\beta)\sqrt{-1}\,\partial\overline{\partial}\log(|s|_h^2 + \varepsilon^2) + (1-\beta)\sqrt{-1}\,R_h$ and the parabolic Schwartz lemma. A direct computation gives the following.

Lemma 5.5.2. *We have the following inequalities of* η_ε*.*

(a) Lower boundedness of η_ε*:*

$$\eta_\varepsilon = (1-\beta)\frac{\varepsilon^2}{|s|_h^2+\varepsilon^2}\left(\frac{\sqrt{-1}\langle\nabla s, \nabla s\rangle_h}{|s|_h^2+\varepsilon^2} + \sqrt{-1}\,R_h\right) \geq -(1-\beta)C_1\widehat{\omega}_T.$$

(b) For any Kähler form ω*, we have*

$$-\langle\eta_\varepsilon, \widehat{\omega}_T\rangle_\omega \leq (1-\beta)C_1|\widehat{\omega}_T|_\omega^2 \leq (1-\beta)C_1(\mathrm{tr}_\omega(\widehat{\omega}_T))^2.$$

By the fact that $\widehat{\omega}_T$ is the pullback of ω_Z by f, we can use the parabolic Schwarz lemma which is obtained by Song-Tian [ST07]. This is the parabolic version of [Yau78a]. This lemma follows from similar computation as in Proposition 3.2.8 (see also Proposition 2.7.1).

Lemma 5.5.3 (Parabolic Schwarz lemma)**.** $u_{\varepsilon,t}$ *and* $\log u_{\varepsilon,t}$ *satisfy the following inequalites.*

(a)

$$\begin{aligned}\Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} &\geq -C_Z u_{\varepsilon,t}^2 + \langle\mathrm{Ric}(\omega_{\varepsilon,t}), \widehat{\omega}_T\rangle_{\omega_{\varepsilon,t}} \\ &\geq -C_{11}u_{\varepsilon,t}^2 + \langle\mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \widehat{\omega}_T\rangle_{\omega_{\varepsilon,t}}.\end{aligned}$$

(b)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)u_{\varepsilon,t} &\leq u_{\varepsilon,t} + C_Z u_{\varepsilon,t}^2 - \langle\eta_\varepsilon, \widehat{\omega}_T\rangle_{\omega_{\varepsilon,t}} - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}} \\ &\leq u_{\varepsilon,t} + C_{11}u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}}^2}{u_{\varepsilon,t}}.\end{aligned}$$

(c)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\log u_{\varepsilon,t} &\leq C_Z u_{\varepsilon,t} + 1 - \frac{\langle\eta_\varepsilon, \widehat{\omega}_T\rangle_{\omega_{\varepsilon,t}}}{u_{\varepsilon,t}} \\ &\leq C_{11}u_{\varepsilon,t} + 1.\end{aligned}$$

Here, ∇ *is* $(1,0)$*-part of the Levi-Civita connection of* $\omega_{\varepsilon,t}$*,* $C_Z > 0$ *is an upper bound for the bisectional curvature of* ω_Z*, and* $C_{11} := C_Z + (1-\beta)C_1 > 0$*.*

Proof of Proposition 5.5.1 We set $G_{\varepsilon,t} := \log u_{\varepsilon,t} - C_{12}v_{\varepsilon,t}$ where $C_{12} := C_{11}+1 > 0$ is a uniform constant. The uniform upper boundedness of $G_{\varepsilon,0}$ follows from (5.2.6), (5.2.10) and Proposition 5.4.1. If we suppose that $G_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0,T)$, we have $u_{\varepsilon,t}(x_0,t_0) > 0$ and

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)G_{\varepsilon,t} &= \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\log u_{\varepsilon,t} - C_{12}\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)v_{\varepsilon,t} \\ &\leq (C_{11}u_{\varepsilon,t}+1) - C_{12}(u_{\varepsilon,t} - n) \\ &= -u_{\varepsilon,t} + (C_{12}n+1) \quad \text{at } (x_0,t_0).\end{aligned}$$

By using the uniform boundedness of $v_{\varepsilon,t}$ (Proposition 5.4.1), we obtain

$$G_{\varepsilon,t} \leq \log(C_{12}n+1) - C_{12}v_{\varepsilon,t} \leq \log(C_{12}n+1) + C_{12}C_5 \quad \text{at } (x_0,t_0).$$

Since (x_0,t_0) is arbitrary, we have $G_{\varepsilon,t} \leq C_{13}$ on $X \times [0,T)$. Hence, using the definition of $G_{\varepsilon,t}$ and Proposition 5.4.1, we obtain

$$\log u_{\varepsilon,t} \leq C_{12}v_{\varepsilon,t} + C_{13} \leq C_{12}C_5 + C_{13},$$

which prove the assertion. □

5.6 The Gradient Estimate for $v_{\varepsilon,t}$

In this section, we prove the following gradient estimate.

Proposition 5.6.1. *There exists a uniform constant $C_{14} > 0$ which is independent of ε and t such that*

$$|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \leq C_{14}.$$

To prove this proposition, as in [Zha10], we set $\Psi_{\varepsilon,t} := \dfrac{|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{A - v_{\varepsilon,t}}$, where $A > C_5 + 1$ is a fixed constant (see Proposition 5.4.1). We will use the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. The direct computation gives the following formulas.

Lemma 5.6.2. *We have the following formulas.*

(a)

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} = |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \eta_\varepsilon(\nabla v_{\varepsilon,t}, \overline{\nabla} v_{\varepsilon,t}) + 2\,\mathrm{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} - |\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}$$

(b)

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} = \Delta_{\omega_{\varepsilon,t}}(v_{\varepsilon,t} + u_{\varepsilon,t}) + \langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}$$

(c)

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Psi_{\varepsilon,t} \\
&= \frac{1}{A - v_{\varepsilon,t}}\Big(|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \eta_\varepsilon(\nabla v_{\varepsilon,t}, \overline{\nabla} v_{\varepsilon,t}) \\
&\quad + 2\,\mathrm{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}\Big) \\
&\quad + \frac{1}{(A - v_{\varepsilon,t})^2}\left((u_{\varepsilon,t} - n)|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - 2\,\mathrm{Re}\langle \nabla|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}\right) \\
&\quad - \frac{2}{(A - v_{\varepsilon,t})^3}|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}
\end{aligned}$$

Proof of Proposition 5.6.1 We will apply the maximum principle to $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$. First, we estimate $\left(\dfrac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Psi_{\varepsilon,t}$. By Lemma 5.5.2 (a), we have

$$-\eta_\varepsilon(\nabla v_{\varepsilon,t}, \overline{\nabla} v_{\varepsilon,t}) \le (1-\beta)C_1|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.$$

For sufficiently small constant $\delta > 0$ which will be determined later, we have

$$2\operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \le 2|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}} \le \frac{1}{\delta}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.$$

Since

$$\nabla\Psi_{\varepsilon,t} = \frac{\nabla|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{A - v_{\varepsilon,t}} + \frac{|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{(A-v_{\varepsilon,t})^2}\nabla v_{\varepsilon,t}, \tag{5.6.3}$$

we have

$$\begin{aligned}
&-\frac{2-\delta}{(A-v_{\varepsilon,t})^2}\operatorname{Re}\langle \nabla|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \\
&= -\frac{2-\delta}{A-v_{\varepsilon,t}}\operatorname{Re}\langle \nabla\Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} + (2-\delta)\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A-v_{\varepsilon,t})^3}.
\end{aligned}$$

On the other hand, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
|\langle \nabla|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}| &= \left| g^{i\bar{j}}g^{k\bar{l}}\Big((\partial_k\partial_i v_{\varepsilon,t})(\partial_{\bar{j}}v_{\varepsilon,t})(\partial_{\bar{l}}v_{\varepsilon,t}) + (\partial_i v_{\varepsilon,t})(\partial_k\partial_{\bar{j}}v_{\varepsilon,t})(\partial_{\bar{l}}v_{\varepsilon,t})\Big)\right| \\
&\le |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}(|\nabla\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}} + |\nabla\overline{\nabla} v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}) \\
&\le \sqrt{2}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}(|\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}})^{1/2}.
\end{aligned}$$

Therefore, we obtain the following:

$$\begin{aligned}
&\frac{-\delta}{(A-v_{\varepsilon,t})^2}\operatorname{Re}\langle \nabla|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \\
&\le \frac{\delta}{(A-v_{\varepsilon,t})^2}\left(\sqrt{2}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}(|\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}})^{1/2}\right) \\
&= \sqrt{2}\delta\frac{|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{(A-v_{\varepsilon,t})^{3/2}}\frac{(|\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}})^{1/2}}{(A-v_{\varepsilon,t})^{1/2}} \\
&\le \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A-v_{\varepsilon,t})^3} + \delta\frac{|\nabla\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{A-v_{\varepsilon,t}}.
\end{aligned}$$

Combining these inequalities with Proposition 5.5.1, Lemma 5.6.2 (c), and $A - v_{\varepsilon,t} > 1$, we obtain the following inequality:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Psi_{\varepsilon,t} \\
&\le C_{15}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{2-\delta}{A-v_{\varepsilon,t}}\operatorname{Re}\langle \nabla\Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A-v_{\varepsilon,t})^3}
\end{aligned}$$

where $C_{15} := 1 + (1-\beta)C_1 + (1/\delta) + C_{10} > 0$.

On the other hand, by Lemma 5.5.3 (a), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) u_{\varepsilon,t} &\le u_{\varepsilon,t} + C_{11}u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{u_{\varepsilon,t}}\\ &\le C_{10} + C_{11}C_{10}^2 - 2\delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}\\ &=: C_{16} - 2\delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}\end{aligned}$$

Here, we take $0 < \delta < 1/(2C_{10})$. Finally, we obtain the following:

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)(\Psi_{\varepsilon,t} + u_{\varepsilon,t})\\ &\le \left(C_{15}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{2-\delta}{A - v_{\varepsilon,t}}\operatorname{Re}\langle\nabla\Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^3}\right)\\ &\quad + \left(C_{16} - 2\delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}\right)\\ &= C_{16} + C_{15}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{2-\delta}{A - v_{\varepsilon,t}}\operatorname{Re}\langle\nabla\Psi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^3}\\ &\le C_{16} + \left(C_{15} + \frac{1}{\delta}\right)|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{2-\delta}{A - v_{\varepsilon,t}}\operatorname{Re}\langle\nabla(\Psi_{\varepsilon,t} + u_{\varepsilon,t}), \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^3}.\end{aligned} \tag{5.6.4}$$

Here, we used the following inequality:

$$\frac{2-\delta}{A - v_{\varepsilon,t}}\operatorname{Re}\langle\nabla u_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \le 2|\nabla v_{\varepsilon,t}|_{\omega_{\varepsilon,t}}|\nabla u_{\varepsilon,t}|_{\omega_{\varepsilon,t}} \le \frac{1}{\delta}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \delta|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.$$

The uniform boundedness of $\Psi_{\varepsilon,0}+u_{\varepsilon,0}$ follows from [CGP13, Section 4], Proposition 5.4.1 and Proposition 5.5.1. If $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$, by (5.6.4), we have the following:

$$\begin{aligned}0 &\le C_{16} + \left(C_{15} + \frac{1}{\delta}\right)|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}}}{(A - v_{\varepsilon,t})^3}\\ &\le C_{16} + \left(C_{15} + \frac{1}{\delta}\right)|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} - \frac{\delta}{2}\frac{1}{(A + C_5)^3}|\nabla v_{\varepsilon,t}|^4_{\omega_{\varepsilon,t}} \quad \text{at } (x_0, t_0).\end{aligned}$$

Therefore there exists a constant $C_{17} > 0$ satisfying

$$|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \le C_{17} \quad \text{at } (x_0, t_0),$$

which does not depend on ε and t. By using the definition of $\Psi_{\varepsilon,t}$, $A - v_{\varepsilon,t} > 1$, and Proposition 5.5.1, we have the uniform upper bound of $\Psi_{\varepsilon,t} + u_{\varepsilon,t}$ on $X \times [0, T)$, and therefore we obtain the uniform upper bound of $|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}$. □

5.7 The Laplacian Estimate for $v_{\varepsilon,t}$

In this section, we estimate $\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}$. In order to prove the uniform upper boundedness of $\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}$, we need the lower boundedness of the scalar curvature. It is obtained by [Edw15, Corollary 4.3], which is proved by the maximum principle argument as in the case of normalized Kähler-Ricci flow (see Proposition 3.2.6).

Proposition 5.7.1 ([Edw15, Corollary 4.3])**.** *The scalar curvature $R(\omega_{\varepsilon,t})$ evolves as*

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Big(R(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon)\Big) = |\mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon|^2_{\omega_{\varepsilon,t}} + \Big(R(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon)\Big),$$

and is uniformly bounded from below by

$$R(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon) \geq -C_{18},$$

where $C_{18} > 0$ is a constant independent of ε and t.

Using this estimate, we can easily obtain the following upper bound.

Proposition 5.7.2. *There exists a uniform constant $C_{19} > 0$ which is independent of ε and t such that*

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \leq C_{19}.$$

Proof. By Proposition 5.3.1, 5.7.1, and $u_{\varepsilon,t} \geq 0$, we have

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} = ne^{t-T} - u_{\varepsilon,t} - (1 - e^{t-T})(R(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon)) \leq n + C_{18} =: C_{19},$$

which proves the assertion. □

Proposition 5.7.3. *There exists a constant $C_{20} > 0$ independent of ε and t such that*

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \geq -\frac{C_{20}}{T-t}.$$

Proof. As in [Zha10, Section 3.3], we set

$$\Phi_{\varepsilon,t} := \frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}},$$

where $B > 0$ is a sufficiently large uniform constant satisfying $B - C_{19} > 0$, and $B - C_5 > 1$ so that the numerator and the denominator of $\Phi_{\varepsilon,t}$ are positive. Straightforward calculations show that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\Phi_{\varepsilon,t} \\
&\quad = \frac{-1}{B - v_{\varepsilon,t}}\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^2}(u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) && (5.7.4)\\
&\qquad - \frac{1}{B - v_{\varepsilon,t}}\Big(\langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} + \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t}\Big) && (5.7.5)\\
&\qquad + \frac{2}{(B - v_{\varepsilon,t})^2}\,\mathrm{Re}\langle \nabla v_{\varepsilon,t}, \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} && (5.7.6)\\
&\qquad - \frac{2}{(B - v_{\varepsilon,t})^3}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t})|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.
\end{aligned}$$

By using $B - v_{\varepsilon,t} > 1$, $B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} > 0$, and Proposition 5.5.1, (5.7.4) is estimated as follows:

$$
\begin{aligned}
&\frac{-1}{B - v_{\varepsilon,t}} \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + \frac{1}{(B - v_{\varepsilon,t})^2}(u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&= \left(\frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}} + \frac{-B}{B - v_{\varepsilon,t}} \right) + \frac{1}{(B - v_{\varepsilon,t})^2}(u_{\varepsilon,t} - n)(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&\leq \frac{B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}}{B - v_{\varepsilon,t}} + \frac{C_{10}}{(B - v_{\varepsilon,t})^2}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&\leq C_{21}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}),
\end{aligned}
$$

where $C_{21} := 1 + C_{10} > 0$.

We next estimate (5.7.5). By using Lemma 5.5.3 (a) and Proposition 5.3.1 (a), we obtain

$$
\begin{aligned}
&- \langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} \\
&\quad \leq -\langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + (C_{22} - \langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}}) \\
&\quad = -\langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} + \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} + C_{22} \\
&\quad = \frac{1}{1 - e^{t-T}} |\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} + \widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} - \frac{e^{t-T}}{1 - e^{t-T}}(\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + u_{\varepsilon,t}) + C_{22},
\end{aligned}
$$

where $C_{22} := C_Z C_{10}^2 + (1 - \beta) C_1 C_{10}^2$. The first term is estimated as follows:

$$
\begin{aligned}
|\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} + \widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} &= |\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} + 2\,\mathrm{Re}\langle \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}, \widehat{\omega}_T \rangle_{\omega_{\varepsilon,t}} \\
&\leq |\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + |\widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} + \delta |\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \frac{1}{\delta} |\widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} \\
&= (1 + \delta)|\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + (1 + 1/\delta)|\widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} \\
&\leq (1 + \delta)|\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + (1 + 1/\delta) C_{10}^2,
\end{aligned}
$$

where $\delta > 0$ is a uniform constant determined later. Here, we used $|\sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} = |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}$, $|\widehat{\omega}_T|^2_{\omega_{\varepsilon,t}} \leq \mathrm{tr}_{\omega_{\varepsilon,t}}(\widehat{\omega}_T)^2 = u_{\varepsilon,t}^2 \leq C_{10}^2$. For the second term, we have

$$
\begin{aligned}
-\frac{e^{t-T}}{1 - e^{t-T}}(\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + u_{\varepsilon,t}) &= \frac{e^{t-T}}{1 - e^{t-T}}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) - \frac{B e^{t-T}}{1 - e^{t-T}} \\
&\leq \frac{1}{1 - e^{t-T}}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&- \frac{1}{B - v_{\varepsilon,t}} \left(\langle \mathrm{Ric}(\omega_{\varepsilon,t}) - \eta_\varepsilon, \sqrt{-1}\,\partial\overline{\partial} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + \Delta_{\omega_{\varepsilon,t}} u_{\varepsilon,t} \right) \\
&\leq \frac{C_T}{T - t} \left(\frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla\overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \left(1 + \frac{1}{\delta}\right) C_{10}^2 \right) + \frac{C_T}{T - t}(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_{22},
\end{aligned}
$$

where $C_T > 0$ is a uniform constant satisfying

$$
\frac{1}{1 - e^{t-T}} \leq \frac{C_T}{T - t}
$$

for $0 \le t < T$. Since, we have

$$\nabla \Phi_{\varepsilon,t} = \frac{(B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t})}{(B - v_{\varepsilon,t})^2} \nabla v_{\varepsilon,t} - \frac{1}{B - v_{\varepsilon,t}} \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t},$$

(5.7.6) can be computated as follows:

$$\begin{aligned}
&\frac{2}{(B - v_{\varepsilon,t})^2} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\
&= -\frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} + \frac{2}{(B - v_{\varepsilon,t})^3} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) |\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.
\end{aligned}$$

Combining these estimates, we get

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\
&\le C_{21} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) \\
&\quad + \frac{C_T}{T - t} \left(\frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \left(1 + \frac{1}{\delta} \right) C_{10}^2 \right) + \frac{C_T}{T - t} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_{22} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} \\
&\le \frac{C_{23}}{T - t} + \frac{C_{23}}{T - t} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + \frac{C_T}{T - t} \frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) (T - t) \Phi_{\varepsilon,t} = -\Phi_{\varepsilon,t} + (T - t) \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\
&\le (T - t) \left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \Phi_{\varepsilon,t} \\
&\le C_{23} + C_{23} (B - \Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t}) + C_T \frac{1 + \delta}{B - v_{\varepsilon,t}} |\nabla \overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla (T - t) \Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}}.
\end{aligned}$$

We set $\widetilde{\Psi}_{\varepsilon,t} := \dfrac{|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{B - v_{\varepsilon,t}}$. Combining with Lemma 5.6.2 (c) and (5.6.3), we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}} \right) \widetilde{\Psi}_{\varepsilon,t} \\
&\le C_{24} - \frac{|\nabla \overline{\nabla} v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{B - v_{\varepsilon,t}} + \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}} - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla \widetilde{\Psi}_{\varepsilon,t}, \nabla v_{\varepsilon,t} \rangle_{\omega_{\varepsilon,t}},
\end{aligned}$$

where $C_{24} := C_{14} + (1-\beta)C_1C_{14} + C_{14}C_{10} > 0$. On the other hand, we have

$$\frac{4}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \leq \delta |\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} + \frac{4}{\delta}|\nabla v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}.$$

By using Lemma 5.5.3 (b), Proposition 5.5.1, and Proposition 5.6.1, we get

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right) u_{\varepsilon,t} &\leq u_{\varepsilon,t} + C_{11}u_{\varepsilon,t}^2 - \frac{|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{u_{\varepsilon,t}} \\
&\leq C_{10} + C_{11}C_{10}^2 - \frac{1}{C_{10}}|\nabla u_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\leq -\frac{4}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} + C_{25},
\end{aligned}$$

where we take $0 < \delta < 1/C_{10}$, and $C_{25} := C_{10} + C_{11}C_{10}^2 + 4C_{14}/\delta > 0$.

Combining these inequalities, we have

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{\omega_{\varepsilon,t}}\right)\left((T-t)\Phi_{\varepsilon,t} + 2C_T\widetilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}\right) \\
&\leq C_{23} + C_{23}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) + C_T\frac{1+\delta}{B - v_{\varepsilon,t}}|\nabla\overline{\nabla}v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla(T-t)\Phi_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \\
&\quad + 2C_T\biggl(C_{24} - \frac{|\nabla\overline{\nabla}v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}}}{B - v_{\varepsilon,t}} + \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla\widetilde{\Psi}_{\varepsilon,t}, \nabla v_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}}\biggr) \\
&\quad + 2C_T\left(-\frac{4}{B - v_{\varepsilon,t}} \operatorname{Re}\langle \nabla v_{\varepsilon,t}, \nabla u_{\varepsilon,t}\rangle_{\omega_{\varepsilon,t}} + C_{25}\right) \\
&\leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T\frac{1-\delta}{B - v_{\varepsilon,t}}|\nabla\overline{\nabla}v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\quad - \frac{2}{B - v_{\varepsilon,t}} \operatorname{Re}\left\langle \nabla\left((T-t)\Phi_{\varepsilon,t} + 2C_T\widetilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}\right), \nabla v_{\varepsilon,t}\right\rangle_{\omega_{\varepsilon,t}}.
\end{aligned}$$

The uniform boundedness of $(T-t)\Phi_{\varepsilon,t} + 2C_T\widetilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}$ at $t = 0$ follows from [CGP13, Section 4], Proposition 5.4.1, Proposition 5.5.1 and Proposition 5.6.1. If $(T-t)\Phi_{\varepsilon,t} + 2C_T\widetilde{\Psi}_{\varepsilon,t} + 2C_Tu_{\varepsilon,t}$ achieves maximum at $(x_0, t_0) \in X \times (0, T)$, we have the following at this point:

$$\begin{aligned}
0 &\leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T\frac{1-\delta}{B - v_{\varepsilon,t}}|\nabla\overline{\nabla}v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} \\
&\leq C_{26} + C_{26}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t}) - C_T\frac{1-\delta}{B + C_5}\left(\frac{1}{n}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t})^2 - \frac{B^2}{n}\right).
\end{aligned}$$

Here, we used Proposition 5.4.1, and

$$|\nabla\overline{\nabla}v_{\varepsilon,t}|^2_{\omega_{\varepsilon,t}} = g^{i\bar{j}}g^{k\bar{l}}(\partial_i\partial_{\bar{l}}v_{\varepsilon,t})(\partial_{\bar{j}}\partial_kv_{\varepsilon,t}) \geq \frac{1}{n}(\Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t})^2 \geq \frac{1}{n}(B - \Delta_{\omega_{\varepsilon,t}}v_{\varepsilon,t})^2 - \frac{B^2}{n}.$$

Therefore, at this point, there exists a constant C_{27} satisfying

$$-\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \leq C_{27} \quad \text{at } (x_0, t_0)$$

which is independent of ε, t, and (x_0, t_0). Combining Proposition 5.4.1, Proposition 5.6.1, and Proposition 5.5.1, we obtain the uniform upper boundedness of $(T-t)\Phi_{\varepsilon,t} + 2C_T \widetilde{\Psi}_{\varepsilon,t} + 2C_T u_{\varepsilon,t}$ on $X \times [0, T)$. Finally, we conclude that

$$\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} \geq -\frac{C_{20}}{T-t}$$

□

Proof of Theorem A By Proposition 5.3.1, and Proposition 5.7.3, we have

$$\begin{aligned} R(\omega_{\varepsilon,t}) - \mathrm{tr}_{\omega_{\varepsilon,t}}(\eta_\varepsilon) &= \frac{1}{1-e^{t-T}} \left(-\Delta_{\omega_{\varepsilon,t}} v_{\varepsilon,t} + ne^{t-T} - u_{\varepsilon,t} \right) \\ &\leq \frac{C_T}{T-t} \left(\frac{C_{20}}{T-t} + n \right) \leq \frac{C}{(T-t)^2}, \end{aligned}$$

where $C > 0$ does not depend on ε and t. Therefore, by taking $\varepsilon_i \to 0$, we get the asssertion. □

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