博士論文

論文題目　　The Infinite Regress Problem in Choice of a Collective Decision Procedure: An Axiomatic Study

（集団的意思決定における手続き正当化を巡る無限後退の解消に関する研究）

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1 Introduction

1.1 Infinite Regress in Procedural Choice

Imagine a group of individuals faces a collective choice problem from the set of alternatives X without an ex ante agreement on the procedure that will be used to aggregate their preferences. As Nurmi (1992) points out, different voting procedures can result in different outcomes even if we fix each individual's preferences over X. Nurmi provides an example that shows plurality, runoff, amendment, Borda count, and approval voting each result in different outcomes for a given preference profile. Even when we restrict our attention to scoring rules only, Saari (1992) shows that if there are ten alternatives, millions of different rankings of X can be achieved by the choice of scoring rule. In fact, many researchers have verified such possibilities based on real election data such as those from the 1968 (Roderick, 1979) and the 1992 (Brams & Merrill, 1994) U.S. presidential elections. These observations demonstrate that the choice of procedure is no less important than the choice of X.

In social choice theory, there are many axiomatic studies of voting rules, such as May's (1952) characterization of majority rule, based on the premise that a *good* rule is one that satisfies normative and/or intuitive criteria such as Condorcet's criterion, unanimity, etc. However, many negative results, the best known of which are Arrow's and Gibbard and Satterthwaite's, suggest that there is no *perfect* voting rule. On the other hand, there is another point of view that a *good* rule is one that is favored by the group of individuals themselves, even though such a procedure might fail to satisfy the normative axioms that social choice researchers esteem. Dietrich (2005) formally defines this view as Procedural Autonomy (PA). It demands that the procedure by which the society aggregates voters' procedural judgments should be entirely determined by the procedural judgments, i.e. their (true) preferences over the set of possible procedures, within the group. However, taking PA literally we could face an infinite regress problem as follows. When a society faces a decision-making problem, X, PA demands that the rule to aggregate the society members' opinions over X must be determined by their opinions over such rules. This means the society faces a new decision-making problem: how to choose the rules to choose X. Using PA again, it follows that the society needs to aggregate its members' opinions over the rules to choose the rule to choose X. This process can go on ad infinitum unless there is an ex ante agreement at some meta level, because no procedure is legitimate before it is selected by the meta rule to choose such procedures—this is the infinite regress problem in procedural choice[1].

The objective of this research is to find a rational way to stop and solve this infinite

[1] Similar regress problems have appeared in many academic disciplines. For instance, the epistemic regress problem, i.e., a belief B1 must by justified by belief B2 but B2 must be also justified by belief B3, and so on, is a classic problem in epistemology (See, Steup, 2006).

regress problem; I propose a new concept—weak/strong convergence—as the solution. The objective is stated in detail in section 1.2, with reference to relevant literature. Section 1.2 also introduces the basic concept of convergence. Section 1.3 provides a more formal introduction with some preliminary results demonstrating the basic difficulties with procedural choice. Chapter 2 gives a rigorous definition of the weak/strong convergence concept and shows some initial results. In Chapter 3, I discuss the design possibilities for menus of voting rules following a convergence approach.

In the convergence model (and most of the related literature referred to in section 1.2), the society—the set of individuals who have the right to vote—is supposed to be fixed a priori. There are, however, some cases where this implicit assumption is not appropriate. In Section 1.3, I introduce the classic *boundary problem*, or how to determine the "society" itself and I briefly sketch its expression as an aggregation problem. Chapter 4 includes the arguments related to the strategic aspect of such aggregation procedures. Concluding remarks are given in Chapter 5.

1.2 Related Literature

The infinite regress problem of procedural choice is a classic problem that Buchanan and Tullock (1962) referred to, arguing the importance of unanimity of consent at the constitutional level. Rae (1969) also studies individuals' procedural judgments in terms of minimizing the expected frequency of losing in the future. Lagunoff (1992) argues for a possible solution to the infinite regress problem, showing that a society can reach a Pareto-optimal outcome by repeatedly dropping the unsuitable mechanisms that fail to satisfy his "Free Choice" condition, which rules out such mechanisms that make some agent locked in to an equilibrium outcome.

Recently, a sequence of studies examined the procedural choice problem based on so-called fixed point approach (Barbera & Jackson, 2004; Koray, 2000; Koray & Slinko, 2006; Kultti & Paavo, 2009). Intuitively, a social choice function (SCF) is called self-selective if it chooses itself from among other rival SCFs (Koray, 2000). If procedural choice is to be made using the existing procedure (e.g., the amendment procedure of the Constitution of Japan) self-selectivity is a powerful tool for detecting stable states. Barbera and Jackson (2004) considered the process of constitutional design, where one alternative is the status quo, and studied the class of voting rules that choose themselves (i.e., self-stable voting rules). Kultti and Paavo (2009) extended the notion of stability so that the model incorporates higher-level meta procedures. There are, however, some impossibilities on the design of self-selective procedure. Koray (2000) shows that for unanimous and neutral SCFs, the (universal) self-selectivity is logically equivalent to dictatorship, in the proof of which Koray shows the logical relationship with Arrow's impossibility result. Subsequently, Koray and Slinko (2006) characterized the class of dictatorship and anti-dictatorship using a weaker requirement of self-selectivity.

While these researchers considered single voting rules, the notion of stability was later extended to apply to menus of voting rules. Houy (2004) states that a menu of social choice rules (SCRs) satisfies the condition of first-level stability if, for all preference profiles over the voting rules, the menu includes one and only one SCR that chooses itself. Houy then shows the negative result that no menu of SCRs can satisfy first-level stability and two more intuitive conditions (this result is discussed further in Chapter 3). On the other hand, Diss, Louichi, Merlin, and Smaoui (2012) and Diss and Merlin (2010) studied the actual probability that a menu of SCRs is stable (i.e., there is at least one SCR that chooses itself) under the Impartial Culture (IC) and Impartial Anonymous Culture (IAC) models, respectively. Their results show that when the population is (infinitely) large, the probability that the set of {plurality (P), Borda (B), anti-plurality (A)} is stable is 84.49% in the IC model and 84.10% in the IAC model.

These studies show the difficulty of determining the most legitimate procedure based on voters' own procedural judgments. The difficulty exists even when a society chooses the procedure from among three popular voting rules (e.g., P, B, and A). The objective of this study is to eliminate these difficulties, specifically, 1) to provide a procedural choice method that can determine the unique legitimate outcome without failure and 2) to enable choice from a set of familiar voting procedures, such as the set: {P, B, A}. To achieve this objective, I propose a new approach based on the concept of weak/strong convergence. Using two examples below, I outline the idea of convergence. The examples also indicate why we need the concept of convergence instead of the previous concept of stability (e.g., of a menu of voting rules). In the following examples, let $X = \{a, b, c\}$ be a set of mutually exclusive alternatives (social states) and $F = \{P, B, A\}$ be a set of admissible SCRs.

Example 1: The Menu is Stable but the Outcome is not Uniquely Determined

Suppose $n = 42$ with the following preferences:

- 9 voters' preferences are a, b, c and P, B, A, (i.e., among the alternatives they prefer a to b and b to c, while among voting rules, they prefer P to B and B to A)
- 11 voters' preferences are a, c, b and P, A, B,
- 17 voters' preferences are b, c, a and B, A, P,
- 1 voter's preference is c, a, b and A, P, B, and
- 4 voters' preference is c, b, a and A, B, P.

(The reader might wonder about the plausibility of such preferences with regard to X and F, however, I will formally show in Chapter 2 that this profile is possible with the assumption of consequentialism.)

Let L^0 and L^1 be the combination of such preferences (the preference profile) regarding X and F, respectively. Once L^0 and L^1 are given, the reader can easily check that each procedure chooses itself among F, i.e. $P(L^1) = P$, $B(L^1) = B$, and $A(L^1) = A$. The menu is clearly stable at L^1 in Diss and Merlin's (2010) sense. However, there arises a new problem: which of the self-selecting procedures should be used when each of them results in a different outcome?

Example 2: The Menu is not Stable but the Outcome is Uniquely Determined

Suppose $n = 14$ and the individuals have the following preferences:

- 4 voters' preferences are a, b, c and P, B, A,
- 6 voters' preferences are a, b, c and B, P, A, and
- 4 voters' preferences are b, c, a and A, P, B.

In this case, no voting rule chooses itself. So, the menu F is not stable at this preference profile regarding F. However, each P, B, A—when used as a rule to choose the rule to choose from X—results in the same outcome (see Figure 1). In such a case, the failure of stability seems less problematic—the ultimate outcome is the same no matter which of the rules to choose the rule is selected.

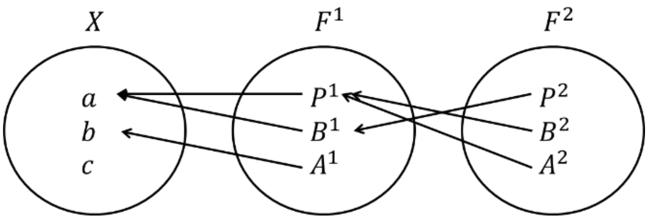

Figure 1. Example of a regress convergence

F^1 denotes the set of voting rules for the choice of alternatives and F^2 denotes the set of voting rules for the choice of F^1.

These examples show that the stability of a menu does not indicate its ability to determine a unique outcome. Indeed, Example 2 shows the possibility that each procedure may ultimately reach the same outcome at some meta level even though the judgments of the procedures do not coincide and no procedure chooses itself. The phenomenon of every procedure ultimately reaching the same outcome is what I name convergence. The formal definition and technical results are presented in Chapters 2 and 3.

I conclude this section with some technical clarifications based on my literature review. In the formulation of procedural choice, the timing of the procedural choice is a key assumption. In the literature, there are two types of assumption with respect to this timing: type 0, when there is no specific agenda such as Rawls' veil of ignorance (Barbera & Jackson, 2004; Houy, 2004; Rae, 1969), and type 1, when a society is facing a specific agenda (Koray, 2000; Koray & Slinko, 2006; Lagunoff, 1992). This dissertation makes the latter assumption, because it seems more suited to the assumption of PA that voters' procedural judgments can and may differ for different agendas.

Another important assumption in the model is what type of procedural judgments are allowed. Roughly speaking, in studies assuming a type 0 situation, voters' procedural judgments are evaluated in terms of the expected payoff or probability of being in the losing side in the future events. In contrast, in most of the studies assuming a type 1 situation, each individual is assumed to hold a consequential procedural preference: they are assumed to evaluate meta-level procedures according to their outcomes (consequentialism). Although the consequentialism assumption is easy to deal with, other types of meta-preferences are also considered in related literature. For example, Nurmi (2015) argues the preference over the criteria of voting rules such as Condorcet winner criterion, monotonicity, etc. From the deliberating point of view, List (2007) argues the possibility that votes agree on the conceptualization of the decision problem. In the next section, I give a formal mathematical introduction to the procedural choice problem and show the basic impossibility arising when we consider any type of meta-preference. This negative result motivates the analysis in Chapters 2 and 3, which gives a solution concept under a consequential society.

1.3 Preliminary Formal Discussion[2]

In this section, I provide a formal introduction to the procedural choice problem and show the basic difficulty with procedural choice that motivates the analysis in the next chapters. I formulate a procedural choice rule (PCR), which is the rule for aggregating voters' procedural judgments and is technically close to the decision rule studied in Dietrich (2005). I introduce several new axioms, which I argue are necessary for the process of choosing voting procedures without an ex ante agreement, and show an impossibility.

1.3.1 Notation

Let $N = \{1,2,\dots,n\}$ denote a society with at least two individuals, $n \geq 2$, that is to make a collective decision. Let X denote the set of decision alternatives, whose cardinality is $2 \leq |X| < \infty$. The society is supposed to make an endogenous decision over X without an agreement on the procedure to aggregate their preferences.

[2] Main results of this subsection are originally published in the following: Takahiro Suzuki and Masahide Horita, "How to Order the Alternatives, Rules, and the Rules to Choose Rules: When the Endogenous Procedural Choice Regresses". Outlooks and Insights on Group Decision and Negotiation. Springer International Publishing, 2015. p.47-59.

A binary relation R over a non-empty set A is defined as a subset of $A \times A$. As usual, for $a, b \in A$, I often write aRb instead of $(a, b) \in R$. For a binary relation R on A,

- R is reflexive if aRa for all $a \in A$.
- R is transitive if, for all $a, b, c \in A$, aRb and bRc implies aRc.
- R is complete if, for all $a, b \in A$, aRb or bRa holds.
- R is anti-symmetric if, for all $a, b \in A$, $[aRb \ \& \ bRa]$ implies $a = b$.
- R is a weak order if it is reflexive, transitive, and complete.
- R is a linear order if it is an anti-symmetric weak order.

Let $\mathcal{W}(A)$ and $\mathcal{L}(A)$ be the set of all weak orders and linear orders over A, respectively. Let $P(R)$ and $I(R)$ respectively denote the asymmetric and symmetric parts of binary relation R, i.e.:

$$P(R) := \{(a, b) \in A \times A \,|\, (a, b) \in R \text{ and } (b, a) \notin R\}.$$
$$I(R) := \{(a, b) \in A \times A \,|\, (a, b) \in R \text{ and } (b, a) \in R\}.$$

Given a binary relation R over A and a nonempty subset $B \subseteq A$, I denote by $G(R, B)$ the greatest elements of B relative to R, i.e., $G(R, B) := \{x \in B \,|\, xRy \text{ for all } y \in X\}$.

A preference profile over a nonempty set A is an n-tuple of weak orders $R = (R_1, R_2, \ldots, R_n) \in \mathcal{W}(A)^n$, where the i^{th} element R_i represents individual i's preference. A social choice function (SCF) f over A is a function that assigns an alternative to each preference profile over A, such that $f : \mathcal{W}(A)^n \to A$. Let $\mathbb{N} = \{1,2,3, \ldots\}$ denote the set of positive integers. I define a sequence $F^0, F^1, F^2, \ldots$ of sets of SCFs as compatible if:

(1) $F^0 = X$, and

(2) $\forall k \in \mathbb{N}$, F^k is a set of SCFs over F^{k-1}.

Given such a compatible sequence, I define an element of F^k as a level-k SCF (or level-k procedure, interchangeably). In words, a level-k SCF is a rule [to choose the rule] ($(k-1)$ times) to choose an alternative. Note that there are many compatible sequences.

Example 3: Universal Domain

For all $k \in \mathbb{N}$, let F^k be the set of *all* SCFs over F^{k-1}. It is easy to see that, defined in this way, the sequence $F^0, F^1, F^2, \ldots$ is compatible.

According to Dietrich (2005), Universal Domain (UD) is an assumption that the society considers all level-1 procedures. The sequence in Example 3 is a straight extension of UD because it also considers any meta-level procedures. In the following argument, I call such $F^0, F^1, F^2, \ldots$ the UD-sequence.

Given a compatible sequence $F^0, F^1, F^2, \ldots$, I assume that each individual $i \in N$ has a preference order $R_i^k \in \mathcal{W}(F^k)$ for all $k \in \mathbb{N}$. A level-k preference profile $R^k =$

$\left(R_1^k, R_2^k, \dots, R_n^k\right)$ is a preference profile over F^k. Integrating the level-k $(k = 0,1,\dots,L)$ preference profile $R^0, R^1, \dots, R^L$, I call $R = (R^0, R^1, \dots, R^L)$ a level-L meta-profile.

Definition 1. Procedural Choice Rule

Let $L \in \mathbb{N}$ and $D \subseteq \mathcal{W}(X)^n \times \mathcal{W}(F^1)^n \times \cdots \times \mathcal{W}(F^L)^n$. A level-$L$ PCR E of domain D is defined as a function assigning a level-L social meta preference $E = (E^0, E^1, E^2, \dots, E^L)$ to each level-L meta-profile, i.e., $E: D \to \mathcal{W}(X)^n \times \mathcal{W}(F^1)^n \cdots \times \mathcal{W}(F^L)^n$.

A PCR expresses a way of determining a society's collective judgments over the meta-level procedures. Given a meta-profile, i.e., each individual's procedural judgment, the PCR returns the collective procedural judgment of the society. Unlike usual social welfare functions, a PCR considers people's procedural judgments. Note that the PCR is an extension of the concept of the decision rule in Dietrich (2005), which is a correspondence assigning a subset of X for each level-1 meta-profile given the individuals' procedural judgments[3]. I define the Universal Preference Domain (UPD), $\bar{D}$, as:

$$\bar{D} \coloneqq \mathcal{W}(X)^n \times \mathcal{W}(F^1) \times \cdots \times \mathcal{W}(F^L)^n.$$

Example 4: Dictatorial PCR

Take an individual, called a dictator, $i^* \in N$ and define a dictatorial PCR E_d as for all level-L meta-profile R,

$$\forall k \in \{0,1,\dots,L\},\ \forall f,g \in F^k,\ f E_d^k(R) g \Leftrightarrow f R_{i^*}^k g.$$

This is a PCR that judges each element in F^k according to the will of the dictator. A possible problem concerning the dictatorial PCR is that if the dictator i^*'s meta-preference is not *consistent*, the PCR itself must also fail to be consistent.

Definition 2: Inter-Level Consistency (ILC)

A level-L PCR of domain D satisfies the axiom of Inter-Level Consistency (ILC) if, and only if, for all $R \in D$, $k \in \{1,\dots,L\}$ and $f,g \in F^k$, $\left[f E^k g \Leftrightarrow f\left(R^{k-1}\right) E^{k-1} g\left(R^{k-1}\right)\right]$.

This consistency property rules out inconsistent social meta-preferences such as those that evaluate SCF f as being at least as good as SCF g, even though f's outcome is not as good as g's. Behind the axiom of ILC lies the idea that procedural judgments must be made for the very decision-making problem that the society faces. It is not that the society determines a universally

[3] Note that, technically speaking, the decision rule is more than a level-1 PCR, because Dietrich (2005) does not restrict his attention to weak order preferences. Dietrich's argument is made without specifying the messages on X.

desirable procedure that can be applied to any possible agenda or decision-making process. The model allows for an individual who esteems a supermajority rule for amendments to the nation's constitution while that same individual supports the simple majority rule for ordinary legislation.

The next axiom demands that a better rule must result in a better outcome, and a better outcome must be supported by a better rule.

Definition 3: Arbitrary Focus (AF)

A level-L PCR E of domain D satisfies the axiom of Arbitrary Focus (AF) if, for all $j = 0,1,\ldots,L-1$, $R = (R^0, R^1, \ldots, R^L)$, and $R' = \left(R'^0, R'^1, \ldots, R'^L\right) \in D$, if $R^\mu = R'^\mu$ for all $\mu \geq j$, then $E^j(R) = E^j(R')$.

AF states that the level-j social meta-preference is entirely determined by the level-j or higher level meta-profile. For all $f, g \in F^j$, AF demands that the collective decision over f and g is determined by the rule to evaluate them, not by their outcomes. To put it differently, AF assumes that the choice from F^k can be treated independently of the original choice problem X.

1.3.2 Results and Discussion

Before stating the impossibility theorem concerning the design of PCR, I will introduce the technical condition for a connected sequence.

Definition 4: Connected Sequence of Sets of Procedures

$F^0, F^1, F^2, \ldots$ is called a connected sequence (CON-sequence, hereafter) if it satisfies the condition that for all $k \in \mathbb{N}$ and $f, g \in F^k$ there exist $p \in \mathbb{N}$ and $h_0, h_1, \ldots, h_p \in F^k$ such that for some $R_1, R_2, \ldots, R_p \in D$,

$$h_{q-1}\left(R_q\right) = h_q\left(R_q\right) \text{ for all } q \in \{1,2,\ldots,p\}.$$

Although this definition looks technical, it is not very demanding. Assuming UPD (i.e., allowing all logically possible meta-profiles) and supposing also that each F^k is the set of unanimous SCFs where for all level-$(k-1)$ SCF $f \in F^{k-1}$, if everyone ranks f as superior to any other level-$(k-1)$ SCF, then the level-k SCF chooses f. We can verify that the sequence is a CON-sequence when $p = 1$ because all the level-k procedures yield the same outcome at this profile. It is also verified that the UD-sequence is an example of a CON-sequence under UPD.

Now, I state the impossibility result. Let E_I be the indifferent PCR such that for all $k \in$

$\{0,1,2,\ldots,L\}$, $f,g \in F^k$, and $R \in D$, it follows that $fI\left(E^k(R)\right)g$. In other words, this is a PCR that judges any two elements of any level as indifferent.

Theorem 1

Under any CON-sequence $F^0, F^1, F^2, \ldots$ and UPD, a PCR satisfies ILC and AF if and only if it is E_I.

Proof of Theorem 1.

The 'if' part is straightforward. Let me show the 'only if' part. Let E be a level-L PCR. Take any $k \in \{1,2,\ldots,L\}$ and $f,g \in F^k$. Take also $R = (R^0, R^1, R^2, \ldots, R^L) \in D$.

Because $F^0, F^1, F^2, \ldots$ is assumed to be connected, there exist level-$(k-1)$ profiles $R_1, R_2, \ldots, R_p \in D^k$ and $h_0(=f), h_1, \ldots, h_p(=g) \in F^k$ such that $h_{q-1}(R_q) = h_q(R_q)$ for all $q = 1,2,\ldots,p$. Suppose to the contrary that $fP(E^k)g$. Then, it follows that $(h_{q-1}, h_q) \notin I(E^k)$ for some $q = 1,2,\ldots,p$ [otherwise, that is if $(h_0,h_1),(h_1,h_2),\ldots,(h_{p-1},h_p) \in I(E^k)$, the transitivity of E^k requires $(h_0,h_p) \in I(E^k)$, and this is contradictory with regard to $fP(E^k)g$]. Now, let $R' \in D$ be a meta-profile obtained from R by substituting R_q for R^k. Because $f(R_q) = g(R_q)$, ILC demands that $fI(E^k)g$. This contradicts $fP(E^k)g$. ■

The theorem states that there is no PCR (the way a society ranks each alternative, rule, the rule to choose rules, and so on) that satisfies ILC and AF without being degenerate. In addition to ILC and AF, I imposed two assumptions, UPD and CON-sequence. For interpretation of the theorem, let me add some notes on these assumptions.

The first comment is on CON-sequence. I already noted that the UD-sequence is a CON-sequence. So, it follows that Theorem 1 holds even if we substitute UD-sequence for CON-sequence. Recall that the UD-sequence represents a situation where a given society does not have any agreement about the (meta-level) rules. Once the procedural choice is entirely entrusted to a society, it is irrational to drop several SCFs beforehand: even the notorious dictatorship might be selected, for example, in the situation where all the group members favor it. In other cases, however, the procedural choice is made among a restricted number of SCFs, say {plurality, Borda count, anti-plurality}.

The second comment is on UPD. The next result says what happens if the allowed meta-profile is restricted.

Definition 5

Let $k \in \{1,2,\ldots,L\}$, $f \in F^k$, and $R = (R^0, R^1, \ldots, R^L) \in D$. I define the class of f at R, denoted $C_f[R]$, as follows:

- For $k=1$, $C_f[R]=f(R^0)$.
- For $k\geq 2$, $C_f[R]=C_{f(R^{k-1})}[R]$.

Definition 6

Let $R\in D$. I call i's level-L meta-preference $R_i=(R_i^0,R_i^1,\ldots,R_i^L)$ extremely consequential if, for all $k\in\mathbb{N}$ and $f,g\in F^k$, $C_f[R]R_i^0C_g[R]\Leftrightarrow fR_i^kg$.

In general, ILC is such a strong condition that $E^1,E^2,\ldots$ is unique with respect to E^0. I say a PCR E is derived from a social welfare function (SWF) h on X if

1) for all $f,g\in F^1$, $fE^1g\Leftrightarrow f(R^0)h(R^0)g(R^0)$, and

2) for all $k\geq 2$ and $f,g\in F^k$, $fE^kg\Leftrightarrow C_f[R]h(R^0)C_g[R]$.

Then, it is easy to verify the following (I omit the proof).

Lemma 1

A level-L PCR E of domain $D\subseteq\overline{D}$ satisfies ILC if and only if it is derived from a SWF h on X.

Let $D_{C^*}\subseteq\mathcal{W}(X)\times\mathcal{W}(F^1)\times\cdots\times\mathcal{W}(F^L)$ be the set of all extremely consequential meta-profiles. We have the following:

Theorem 2

Take UD-sequence $F^0,F^1,F^2,\ldots$ and $L\in\mathbb{N}$. Let $D=D_0\times D_1\times\ldots\times D_k$, where $\phi\neq D_k\subseteq\mathcal{W}(F^k)$ for all $k=0,1,2,\ldots,L$.

(1) If $D=D_{C^*}$, there exists a PCR that satisfies ILC and AF.

(2) If $D\cap(\overline{D}\setminus D_{C^*})\neq\phi$, there is no PCR that satisfies ILC and AF.

Proof of Theorem 2

Proof of (1). It is straightforward to check that a PCR derived from some SWF h on X satisfies the two axioms.

Proof of (2). Suppose R_i with respect to $R=(R_1,\ldots,R_n)\in D\cap(\overline{D}\setminus D_{C^*})$ is not extremely consequential. Let $k\in\{1,2,\ldots,L\}$ be the smallest level at which the condition collapses. Then, we have $f,g\in F^k$ such that $C_f[R]R_i^0C_g[R]$ but not fR_i^kg. R_i^k is assumed to be complete, and so we have $gP(R_i^k)f$.

(a) If $R'^{k-1}\in\mathcal{W}(F^{k-1})^n$ exists such that $f(R'^{k-1})=g(R'^{k-1})$, let $R'\in D$ be a meta-profile obtained from R by substituting ${R'}^{k-1}$ for R^{k-1}. AF demands that $E^k = {E'}^k$. So, we have $gP({E'}^k)f$. However, ILC demands $gI({E'}^k)f$. Contradiction.

(b) Consider the other case, i.e., there is no ${R'}^{k-1} \in \mathcal{W}(F^{k-1})^n$ such that $f({R'}^{k-1}) = g({R'}^{k-1})$. Because we consider a UD-sequence, there exist $h \in F^k$ such that $h(R^{k-1}) = f(R^{k-1})$ and $h({R'}^{k-1}) = g({R'}^{k-1})$ for all ${R'}^{k-1} \in D_{k-1}$. With the argument in (a), we have that $fI(E^k)h$ and $hI(E^k)g$. With the transitivity of E^k, we have $fI(E^k)h$. ■

Under UD-sequence, Theorem 2 states the necessary and sufficient condition for a PCL satisfying ILC and AF to exist. It says that it matters whether there exists an individual that is not extremely consequential.

To conclude, the present section outlines a preliminary model of procedural choice. While Theorem 1 states the basic impossibility faced when any type of meta-preference is considered (as well as some axioms of PCRs), Theorem 2 shows that the impossibility disappears when a society made up of consequential individuals is considered. Based on these observations, Chapters 2 and 3 address the situation where a consequential society has a restricted number of voting procedures.

1.4 Determination of the Society

To begin the last part of this introduction, I consider a pre-step of the procedural choice considered above. Although most of the research referred to in Section 1.1 and the analysis in Section 1.3 assume that the society has been defined prior to the voting step, there are some cases where there is ambiguity in the definition of which individuals have the right to vote. Indeed, the boundary problem—who should be eligible to take part in which decision-making processes (Arrhenius, 2005; Dahl, 1991)—is a classical problem in political science. While a number of solutions have been proposed, Schumpeter (1942) argues that it is the people involved who should determine who is entitled to participate in the democratic process:

> Observe: it is not relevant whether we, the observers, admit the validity of those reasons or of the practical rules by which they are made to exclude portions of the population; all that matters is that the society in question admits it. (Schumpeter, 1942, p. 244).

In Chapter 4, I consider the boundary problem as an aggregation problem, in other words, to determine or define a given society based on individuals' views on who is (or should be) included in the society and who is not (or should not be). Formally speaking, let $\bar{N}$ be the set of potential individuals where each $i \in \bar{N}$ is assumed to have an opinion $N_i \subseteq \bar{N}$ about who he or she thinks should be included in the society. Seen as an aggregation problem, one can describe the boundary problem as the need to determine the aggregator, hereafter called the nomination rule, φ, that maps each profile $(N_1, N_2, \ldots, N_n)$ into $\varphi(N_1, N_2, \ldots, N_n) \subseteq \bar{N}$.

Holzman and Moulin (2010, 2013) made axiomatic studies of such nomination rules from a technical perspective, using the determination of prize winners as an example. This model is different from an ordinary social choice problem in that each individual $i \in \bar{N}$ is a candidate as well as a voter. Therefore, if they are selfish in the sense that they care greatly or only about whether they are themselves selected, certain types of strategic voting can occur: a rational voter $i \in \bar{N}$ might present a misrepresentation of his or her opinion as $\widetilde{N_i}$ instead of presenting his or her true opinion N_i so that i can win. Indeed, approval voting (AV), often noted for its strategy-proofness (Endriss 2013), is nonetheless shown to be fragile to this kind of manipulation. Holzman and Moulin (2013) proposed an axiom of impartiality (IMP), which demands that the nomination rule be robust against such manipulations.

At the same time, however, Holzman and Moulin (2013) show that the constant rule, which selects the same individual no matter what the ballots are, is the unique nomination function that satisfies the both IMP and the Anonymous Ballots axiom (AB), which corresponds with the usual anonymity condition. Their result is based on the assumption that each N_i is a singleton—each person is supposed to submit another person's name (the person who they think deserves the prize), and that the prize winner $\varphi(N_1, N_2, \ldots, N_n)$ is only one person. In other words, they think of the nomination rule φ as a function from the domain $\{(N_1, N_2, \ldots, N_n) \mid N_i \in \bar{N} \setminus \{i\} \text{ for all } i \in \bar{N}\}$ to codomain $\bar{N}$. In their subsequent work, Tamura (2015) and Tamura and Ohseto (2014) considered nomination correspondence (i.e., allowing multiple winners) and show that the impossibility shown in Holzman and Moulin (2013) can be relieved. Other domains and codomains have also been studied: the domain of approval ballots $N_i \subseteq \bar{N}$ (Alon et al. 2011), the codomain of $\bar{N} \cup \{\phi\}$ (Mackenzie 2015)[4], etc.

Although a variety of studies have considered the design possibility of impartial nomination rules under each domain-codomain pair, there seems no systematic study of the comparisons between popular pairs. In Chapter 4, I aim to answer the question of which domain-codomain pairs perform well in terms of design possibility by the comparative study of each domain-codomain pair. As Dietrich (2005) argues, some axioms (anonymity, neutrality, and monotonicity) are considered to be essential under PA. Chapter 4 is also designed to find impartial nomination rules satisfying these axioms. In Chapter 4, I first show the common structure that an impartial and anonymous nomination rule has under various domain-codomain pairs (Lemma 7). Later, the design possibility under each domain-codomain setting will be discussed.

Finally, I introduce literature that relates to the nomination rules. The framework of

[4] While some of them (Alon et al. 2011; Holzman and Moulin 2013) also consider nondeterministic rules (i.e., the codomain is the set of probability distributions over $\bar{N}$), I restrict my attention to deterministic rules only throughout the chapter.

nomination rules is very similar to the endogenous choice of representative committees (Brams, Kilgour, & Sanver, 2007; Kilgour, Brams, & Sanver, 2006). Indeed, Brams et al. (2007) studied the aggregation of approval ballots, where each of the ballots is the set of individuals who the voter thinks should be on the committee. They proposed the Minimax procedure based on the minimization of the Humming distance from the voters' ballots. For strategic aspects of endogenous choice, Amorós (2009, 2011) considered a strategy-proof mechanism in the slightly different context that there exists a unique person who everyone thinks is the *best* person to be chosen.

In the nomination rule, each individual $i \in \overline{N}$ is assumed to submit $N_i \subseteq \overline{N}$. Technically speaking, such N_i can be regarded as i's (presented) dichotomous preference over $\overline{N}$. In general, a preference relation on a certain set of alternatives is called dichotomous if it has at most two indifferent classes, usually interpreted as the acceptable class and the unacceptable class. The study of approval voting (AV) in this preference domain has resulted in a lot of concern (Vorsatz, 2008; Vorsatz, 2007; Sato, 2014). As I noted above, however, AV is not impartial. Therefore, a natural question is how can the mechanism of AV be modified to satisfy impartiality without losing its preferable properties such as anonymity and neutrality? My comparative study of various rules described in Chapter 4 provides an answer to this question.

2 Regress Convergence[5]

2.1 Intuition of Regress Convergence

In this chapter, I formulate a phenomenon—weak/strong convergence (of a preference profile)—where the regress argument is supposed to naturally disappear within finite steps. Intuitively speaking, convergence is a phenomenon where every voting rule in the menu ultimately provides the same outcome[6]. The aim of this chapter is to show how and how often this phenomenon occurs.

I will first explain the basic idea using an example. Suppose a society of 14 individuals must choose one of three candidates—$a, b,$ and c—and there is an ex ante agreement on the set, F, of potential voting rules, where $F = \{\text{plurality } (P), \text{Borda } (B), \text{anti} - \text{plurality } (A)\}$. When the preference profile on the set of candidates X is given as $L^0_{1-10}{:}\,abc$, and $L^0_{11-14}{:}\,bca$ (i.e., individuals $1,2,\dots,10$ prefer a to b and b to c; individuals $11, 12, 13,$ and 14 prefer b to c and c to a), the three voting rules $P, B,$ and A yield $\{a\}, \{a\},$ and $\{b\}$, respectively. Suppose now that the same society votes on which rule in F to use. If everyone is consequential (i.e., preferring those rules that yield their own preferred results) and is required to submit a linear order, it is understood that the first 10 individuals submit either "PBA" or "BPA," and the remaining four individuals submit "APB" or "ABP". If they submit as: $L^1_{1-4}{:}\,PBA$, $L^1_{5-10}{:}\,BPA$, and $L^1_{11-14}{:}\,APB$, then applying the same three voting rules to this profile $(L^1_1, L^1_2 \dots, L^1_{14})$, P yields $\{B\}$ while B and A yield $\{P\}$ (see Figure 12).

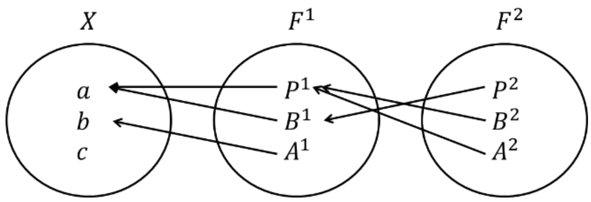

Figure 2. Example of a regress convergence

F^1 denotes the set of voting rules for the choice of candidates and F^2 denotes the set of voting rules for the choice of F^1.

Note that each $P^2, B^2,$ and A^2 (the rule to choose the rule) ultimately reaches the same outcome $\{a\}$.

[5] Main results of this section are originally published in the following: Takahiro Suzuki and Masahide Horita (2017), "Plurality, Borda count, or anti-plurality: regress convergence phenomenon in the procedural choice". Bajwa, D., Koeszegi, S. T., and Vetschera, R. (eds) Group Decision and Negotiation. Theory, Empirical Evidence, and Application: 16th International Conference, GDN 2016, Bellingham, WA, USA, June 20-24, 2016, Revised Selected Papers, LNBIP Vol.274, 43-56.

[6] Saari & Tataru (1999) argue in their introduction that "Except in extreme cases such as where the voters are in total agreement, or where all procedures give a common outcome, it is debatable how to determine the 'true wishes' of the voters." Clearly, the intuition of regress convergence lies in these latter "extreme cases" where all procedures (rules) produce the same (ultimate) outcome, although our results show that the phenomenon can occur relatively frequently in some familiar menus.

Thus, no matter which rule in $F^2, F^3, \ldots$ is selected, the ultimate outcome is the same. Thus, further regress has no meaning for the determination of the ultimate outcome. In general, a profile L^0 is said to weakly converge to $C \subseteq X$ if such a (sequence of) consequential profile(s) exists and any higher-level meta rules ultimately result in the same C.

The current chapter is organized as follows. Section 2.2 shows basic notation. In Section 2.3, I show the formal definition of convergence. Section 2.4 states the probability model and the basic technique of the probability calculation. In Section 2.5, I show theoretical results.

2.2 Basic Notation

Let $N = \{1,2,\ldots,n\}$ be a society of n individuals, where $2 \leq n < +\infty$. For any nonempty and finite set A, $\mathcal{L}(A)$ denotes the set of all linear orders over A. A preference profile over A is an n-tuple of linear orders $(L_1, L_2, \ldots, L_n) \in \mathcal{L}(A)^n$, where the i^{th} element L_i is interpreted as individual i's preference. For any nonempty and finite set of alternatives A, a social choice rule (SCR) f maps the preference profile $L = (L_1, \ldots, L_n) \in \mathcal{L}(A)^n$ into a nonempty subset of A, i.e. $\phi \neq f(L; A) \subseteq A$. A SCR f is called a social choice function (SCF) if it is always singleton-valued. When f is a SCF, with a slight abuse of notation, I write $f(L) = x$ instead of $f(L) = \{x\}$.

Let A and B be any nonempty and finite sets with the same cardinalities, $0 < |A| = |B| < \infty$ (A and B can be identical). For any preference profile $L = (L_1, L_2, \ldots, L_n) \in \mathcal{L}(A)^n$ and a bijection $\sigma: B \to A$, I define a (permuted) preference profile $L^\sigma = (L_1^\sigma, L_2^\sigma, \ldots, L_n^\sigma) \in \mathcal{L}(B)^n$ on B as follows: for all $a, b \in B$ and $i \in N$,

$$aL_i^\sigma b \Leftrightarrow \sigma(a) L_i \sigma(b).$$

I say a SCR is neutral if, for any finite nonempty sets A and B with $|A| = |B|$, alternative $b \in B$, bijection $\sigma: B \to A$, and profile $L \in \mathcal{L}(A)^n$,

$$\sigma(b) \in f(L; A) \Leftrightarrow a \in f(L^\sigma; B).$$

This axiom demands that the outcome of the SCR must not depend on the names of the alternatives. Following are brief descriptions of several SCRs[7] that are well-known in social choice theory. Note that all of them are neutral.

(1) Scoring Rules (Positional Rules)

A scoring rule f is characterized with the combination of vectors $[s_1^m, s_2^m, \ldots, s_m^m]_{m \geq 3}$. For a given set A with $|A| = m \geq 2$ and a preference profile $L \in \mathcal{L}(A)^n$, f assigns to each alternative s_j^m points ($j = 1,2,\ldots,m$) if it is ranked at the j^{th} position in one's preference, where we assume that $1 = s_1^m \geq s_2^m \geq \cdots \geq s_m^m = 0$ for each $m \geq 2$[8]. The choice set $f(\cdot)$ is defined as the set of options

[7] Nurmi (2002) gives a more detailed description of these voting rules, including their axiomatic properties and the related paradox.

[8] As Saari (2012) and many other authors point out, without loss of generality we can standardize arbitrary scoring rules into this form.

with the highest scores. For example, plurality, denoted f_P, has the score assignment $[1,0,0]_{m=3}$ and $[1,0,0,0]_{m=4}$, Borda count, denoted f_B, has $[1, 1/2\,, 0]_{m=3}$ and $[1, 2/3\,, 1/3\,, 0]_{m=4}$, and anti-plurality, denoted f_A, has $[1,1,0]_{m=3}$ and $[1,1,1,0]_{m=4}$. In general, Borda count[9] assigns $s_j^m = \frac{m-j}{m-1}$ for each $j = 1,2,\dots,m$. For $k \in \mathbb{N}$, a SCR f is called a k-approval voting E_k if it is a scoring rule with the assignment $[s_1^m, s_2^m, \dots, s_m^m]_{m\geq 3}$, where if $m > k$, then

$$s_j^m = \begin{cases} 1 & \text{if } j \leq k \\ 0 & \text{otherwise.} \end{cases}$$

In words, a k-approval voting rule E_k assigns 1 point to the $1^{\text{st}}, 2^{\text{nd}}, \dots, k^{\text{th}}$ ranked alternatives and zero points to the others. Note that we do not specify how E_k works if there are equal to or less than k alternatives. Therefore, technically speaking, E_k just specifies the class of scoring rules.

(2) Sequential Positional Rules

The sequential positional rules have multiple rounds to determine the winners. From the first to the $(m-2)$ round, the score of each remaining alternative is calculated and the alternative with the lowest score is eliminated. In the $(m-1)$ round (note that exactly two alternatives remain now), the winning alternative is determined by the majority rule. For the score calculation in each round, Hare's system f_H uses the plurality rule and Coomb's procedure f_C uses the anti-plurality rule. Nanson's procedure is defined in a similar way. In each round, it eliminates all the candidates whose Borda score (the scores of candidates evaluated by Borda count) do not surpass the average Borda score.

(3) Maximin Rule (f_M)

The Maximin score of alternative $x \in X$ is defined as $\min_{y\in X}\left|\{i \in N \mid xL_i y\}\right|$. Then, f_M chooses the alternative(s) with the highest scores.

(4) Black's Rule (f_{Bl})

Black's rule chooses the Condorcet winner, if it exists. Otherwise, it chooses the Borda winner.

2.3 Definition of Convergence

To help the reader understand the formal definition of convergence that follows, I will first outline the hypothetical situation. Assume that a society faces a decision requiring a choice from a set of

[9] Note that my model normalizes the score assignment. In common use, Borda count assigns $(m-j)$ points for the alternative ranked at the j^{th} position (when there are m options). Dividing the assignments by the constant $(m-1)$, my $s_j^m = \frac{m-j}{m-1}$ is obtained.

alternatives $X = \{x_1, x_2, \dots, x_M\}$, where $3 \leq M < +\infty$ and that they have in mind a menu $F = \{f_1, \dots, f_m\}$ $(2 \leq m < +\infty)$ of possible SCRs (throughout Chapters 2 and 3, I use the letter F as a menu of SCRs only). For instance, they agree on the use of either the plurality, Borda count, or anti-plurality rule but there is no agreement on which of them should be used for the current agenda. At the first level, the society applies each SCR in F^1 (upper script expresses the argument level) to the collected preference profile L^0 over X. If every SCR gives the same outcome (*convergence*), the regress stops. Otherwise, the society tries to vote on F^1. Then, the society applies each SCR in F^2 to the collected preference profile L^1 over F^1. If every SCR gives essentially the same outcome (*convergence*), the regress stops. Otherwise, the society tries to vote on F^2. The process can go on ad infinitum unless the society finds a *convergence*.

Definition 7: Level[10]

The level-1 issue is the choice of X using each $f_j \in F$. In this context, each f_j $(j = 1,2,\dots,m)$ is called a level-1 SCR and denoted f_j^1 and the level-1 menu is denoted $F^1 = \{f_1^1, \dots, f_m^1\}$. For any integer $k \geq 2$, the level-k issue is the choice of F^{k-1} using $f_1, f_2, \dots, f_m$. In this context, each f_j $(j = 1,2,\dots,m)$ is called a level-k SCR and denoted f_j^k and the level-k menu is denoted $F^k = \{f_1^k, f_2^k, \dots, f_m^k\}$.

Definition 8: Class

- For any level-1 SCR $f^1 \in F^1$, its class at a level-0 preference profile $L^0 \in \mathcal{L}(X)^n$, denoted $C_{f^1}[L^0]$, is defined as $C_{f^1}[L^0] = f^1(L^0)$.
- For any level-$k(\geq 2)$ SCR $f^k \in \mathcal{F}^k$, its class at a level-$0,1,2,\dots,(k-1)$ preference profile $L^0, L^1, \dots, L^{k-1}$, denoted $C_{f^k}[L^0, L^1, \dots, L^{k-1}]$, is defined as

$$C_{f^k}[L^0, L^1, \dots, L^{k-1}] := \bigcup_{g^{k-1} \in f^k(L^{k-1})} C_{g^{k-1}}[L^0, L^1, \dots, L^{k-2}].$$

Remark. Let $\sim$ be a binary relation over F^k such that for all $f^k, g^k \in F^k$,

$$f^k \sim g^k \Leftrightarrow C_{f^k} = C_{g^k}.$$

Then it is clear that $\sim$ makes an equivalence relation and each equivalence class is made up of the rules with the same class. It is in this sense that I use the term "class" here.

Intuitively, the class of $f^k \in \mathcal{F}^k$ represents the ultimate outcome that f^k derives into X. When the sequence $L^0, L^1, \dots, L^{k-1}$ is obvious in the context, I write simply as C_{f^k} instead of $C_{f^k}[L^0, L^1, \dots, L^{k-1}]$.

[10] In this article, I suppose that the society uses the fixed set of SCRs, $f_1, \dots, f_m$ for any level. The distinction between f_j^1 and f_j^2 by the superscripts is made based on the supposed agenda.

Example 5

Let $f_1, f_2 \in F$. Let L^0 and L^1 be profiles over X and F^1, respectively. Suppose $f_1^1(L^0) = \{x\} \subseteq X$, $f_2^1(L^0) = \{x, y\} \subseteq X$, and $f_1^2(L^1) = \{f_1^1, f_2^1\}$. Then, the class of f_1^1 at L^0 is $\{x\}$ while the class of f_1^2 at (L^0, L^1) is $\{x, y\}$. These are denoted as follows.

$$C_{f_1^1}[L^0] = \{x\}$$

$$C_{f_1^2}[L^0, L^1] = \{x, y\}.$$

Definition 9: Preference Extension System

For each $i \in N$, I define $e_i: \mathcal{L}(X) \to \mathcal{L}(\mathfrak{P}(X) \setminus \{\phi\})$ as a preference extension system if it satisfies the following:

1) for each $a, b \in X$ and $L_i^0 \in \mathcal{L}(X)$, if $(a, b) \in L_i^0$, then $\{a\}e_i(L_i^0)\{b\}$.

2) for any set $A \subseteq X$ and $b \in X \setminus A$ such that bL_i^0a for all $a \in A$, $A \cup \{b\}e_i(L_i^0)A$.

In words, e_i maps each $L_i \in \mathcal{L}(X)$ to a linear order preference over the power set of X (without the empty set). Condition 1 is known in the literature as the Extension Rule (e.g. Barbera, Bossert, & Pattanaik, 2004). Almost all the well-known preference extension systems satisfy this condition. Condition 2 says that if better alternative b is added to A, the new set $A \cup \{b\}$ is evaluated as better than A. This condition is also often referred to in the literature (see, e.g., Gardenfors, 1976; Kannai & Peleg, 1984). Note that there are many preference extension systems that satisfy these two conditions. Throughout this dissertation, I do not specify what kind of e_i each individual has, except when I give a specific example. This guarantees the generality of the following argument.

Definition 10: Consequentially Induced Preference/Profile

For any $i \in N$, $k \in \mathbb{N}$, and $L^0 \in \mathcal{L}(X)^n, L^1 \in \mathcal{L}(F^1)^n, \dots, L^{k-1} \in \mathcal{L}(F^{k-1})^n$, I define $R_i^k \in \mathcal{W}(F^k)$ as the i's level-k consequentially-induced weak order preference if, for each $f^k, g^k \in F^k$,

$$(f^k, g^k) \in L_i^k \Leftrightarrow (C[f^k: L^0, L^1, \dots, L^{k-1}], C[g^k: L^0, L^1, \dots, L^{k-1}]) \in e_i(L_i^0).$$

A linear order $L_i^k \in \mathcal{L}(F^k)$ is called an i's level-k linear order preference or consequentially induced preference (hereafter, level-k CI preference) if it is compatible with the i's level-k consequentially-induced weak order preference. I say $L^0 \in \mathcal{L}(X)^n$, $L^1 \in \mathcal{L}(F^1)^n$, …, $L^k \in \mathcal{L}(F^k)^n$ as a sequence of CI profiles till level-k if L^j $(j = 1,2, \dots, k)$ is a CI profile with respect to the previous-level profiles $L^0, L^1, \dots, L^{j-1}$. I denote by $\mathcal{L}^k[L^0, \dots, L^{k-1}]$ the set of all level-k CI profiles with respect to a given sequence $L^0, L^1, \dots, L^{k-1}$ of CI profiles till level $(k-1)$.

When $k = 1$ and F is made up of SCFs only, the CI preference is nothing but the "induced preference" used in the study of self-selective SCRs (Koray, 2000). In this sense, the CI preference is a generalization of the induced preference so that we can deal with higher levels and SCRs, i.e. correspondences instead of functions.

Definition 11: Weak Convergence

A level-0 preference profile $L^0 \in \mathcal{L}(X)^n$ is said to weakly converge to $C \subseteq X$ if and only if $k \in \mathbb{N}$ and a sequence of CI profiles till level $(k-1)$ $L^0, L^1, \dots, L^{k-1}$ exist such that each $f^k \in F^k$ has the same class, i.e., $C[f^k: L^0, L^1, \dots, L^{k-1}] = C$ for all $f^k \in F^k$.

Remark. Whether a profile L^0 weakly converges or not depends on what kind of menu F the society considers, and so it is more precise to say "L^0 weakly converges with respect to the menu F." In the subsequent argument, however, the menu F is explicit from the context. So, we simply say it as "L^0 weakly converges to $C \subseteq X$".

Remark. In the definition of weak convergence, I do not specify individuals' preference extension systems $\{e_i\}_{i \in N}$. Strictly speaking, a profile L^0 is defined as weakly converging to $C \subseteq X$ if and only if, for combinations of all preference extension systems $\{e_i\}_{i \in N}$, the required sequence of CI profiles exists. This point will be exemplified later (see Example 11).

Note that once a profile L^0 weakly converges to C at level k, the class of any rule of level $k' > k$ is also C. Thus, further regress is thought to be meaningless. Following are some examples of the notions introduced in this section.

Example 6: Weak Convergence (The Example Introduced in Section 2.1)

Let $n = 14$, $X = \{a, b, c\}$, and $F = \{f_P, f_B, f_A\}$. Suppose the preference profile $L^0 = (L^0_1, L^0_2, \dots, L^0_{14})$ is $L^0_{1-10}: abc$, and $L^0_{11-14}: bca$. Then, $f_P(L^0) = f_B(L^0) = a$ and $f_A(L^0) = b$, and so the CI preference is

$$\mathcal{L}_i[L^0] = \{f_P f_B f_A, f_B f_P f_A\} \text{ for all } i = 1,2,\dots,10$$

$$\mathcal{L}_i[L^0] = \{f_A f_P f_B, f_A f_B f_A\} \text{ for all } i = 11,12,13,14.$$

Let $L^1_{1-4}: PBA$, $L^1_{5-10}: BPA$, and $L^1_{11-14}: APB$. Now, $L^1 = (L^1_1, L^1_2, \dots, L^1_{14})$ defined in this way is actually in $\mathcal{L}[L^0]$. It follows that $C[f^2_P; L^0, L^1] = C[f^2_B; L^0, L^1] = C[f^2_A; L^0, L^1] = \{a\}$. This means L^0 weakly converges to $\{a\}$.

Example 7: Singleton Menu

I assumed $|F| \geq 2$ in the beginning of this section. This is because if $|F| = 1$, then the society has no other options but to choose the unique procedure, and hence there is no need of procedural choice. The above sequence of definitions, however, applies even for $|F| = 1$. So, only in several examples throughout the chapter, I refer to such singleton menus. Suppose $F = \{f\}$, where f is an arbitrary SCR. It is clear that for any set of alternatives X and for any preference profile $L \in \mathcal{L}(X)^n$, L weakly converges to $f(L) \subseteq X$.

Example 8: Menu of two SCRs

Let f, g be any (distinct) neutral SCFs and let $F = \{f, g\}$. For all $L \in \mathcal{L}(X)^n$, if $a = f(L) = g(L)$, it follows that L strongly converges to a. If $a = f(L) \neq g(L) = b$, then the level-1 CI profile is uniquely determined because $aL_i^0 b \Leftrightarrow fL_i^1 g$ for all $i \in N$ by the extension rule. If $f^2(L^1) = g^2(L^1)$, it follows that L^0 weakly converges. Otherwise, L^0 never (weakly) converges, because it is easy to see that for all $k \geq 2$, level-k CI profile L^k is unique and $f^k(L^{k-1}) = f^{k-1}$ and $g^k(L^{k-1}) = g^{k-1}$ (see Proposition 1).

As a generalization of Example 8, I define a class of profiles called trivial deadlock, where convergence never occurs.

Definition 12: Trivial Deadlock

Let $F = \{f_1, f_2, \dots, f_m\}$ be the menu of SCRs. A preference profile L^0 is said to be in a *trivial deadlock* if:

1) each $f_1^1(L^0), f_2^1(L^0), \dots, f_m^1(L^0)$ is a distinct singleton, and
2) each $f_1^2(L^1), f_2^2(L^1), \dots, f_m^2(L^1)$ is also a distinct singleton for all $L^1 \in \mathcal{L}^1[L^0]$.

From a technical perspective, the intuition behind this trivial deadlock comes from the following:

Proposition 1: Trivial Deadlock Fails to Converge[11]

Let F be a menu of neutral SCRs. If L^0 is in a trivial deadlock, then L^0 does not (weakly) converge.

(From now on, all proofs of lemmas, propositions, and theorems are shown in the Appendix, unless otherwise noted.)

Remark. When $|X| = |F|$, the condition 2) in the definition of trivial deadlock is unnecessary because 1) directly implies 2). This can be easily shown, as can the proof of Proposition 1.

Example 9: Trivial Deadlock

Suppose $n = 42$, $X = \{x_1, x_2, x_3\}$, and $F = \{f_P, f_B, f_A\}$. Let $n_1, n_2, \dots, n_6$ be the number of voters

[11] Note that this proposition does not hold if $|F| = 1$, for in such a case it follows that trivial deadlock implies weak convergence.

whose preferences are $x_1x_2x_3, x_1x_3x_2, x_2x_1x_3, x_2x_3x_1, x_3x_1x_2, x_3x_2x_1$, respectively. If $(n_1, n_2, \dots, n_6) = (9,11,0,17,1,4)$, such profile is in a trivial deadlock. Because of Proposition 1, we know that this profile never weakly converges. The figure below shows the regress structure.

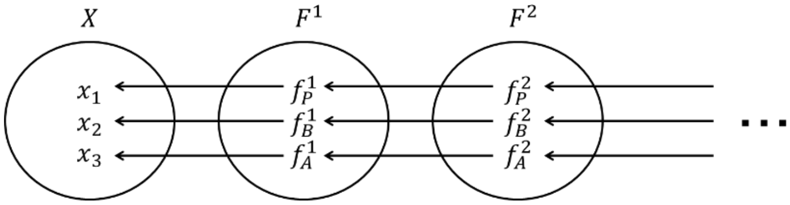

Figure 3. A graph image of trivial deadlock

The proposition tells us that (weak) convergence and trivial deadlock are mutually exclusive as long as we consider neutral SCRs only. Also, as we saw in Example 9, once a profile turns out to be in trivial deadlock, the 'structure' (i.e., which higher-level rule chooses which rule) does not change at all, no matter how high a level is considered. Hence, considering further regress under trivial deadlock has little effective meaning (although, of course, it does not yield weak convergence). Finally, I note that trivial deadlock is, in this sense, the polar opposite of weak convergence. There are, of course, some profiles that are not in trivial deadlock, but do not weakly converge either.

Example 10
Let $n = 17$, $X = \{a, b, c\}$, and $F = \{f_P, f_A, f_D\}$, where f_D: [1,0.75,0]. Suppose that two individuals have level-0 preference abc, three have acb, five have bac, two have bca, three have cab, and two have cba. Then, it is easy to inductively verify that for all $k \geq 2$ and for all CI sequence $L^0, L^1, \dots, L^{k-1}$, the class of f_P^k is $\{a\}$ while the classes of f_D^k and f_A^k are both $\{b\}$.

Example 11
Let $n = 1700$, $X = \{a, b, c\}$, and $F = \{f_P, f_B, f_A\}$. Assume the profile $L^1 \in \mathcal{L}(X)^n$ is as follows:

$$L_i^0: abc \text{ if } 1 \leq i \leq 400 \quad (\mathrm{I})$$
$$L_i^0: acb \text{ if } 401 \leq \mathrm{i} \leq 500 \quad (\mathrm{II})$$
$$L_i^0: bac \text{ if } 501 \leq i \leq 800 \quad (\mathrm{III})$$
$$L_i^0: bca \text{ if } 801 \leq i \leq 1000 \quad (\mathrm{IV})$$
$$L_i^0: cab \text{ if } 1001 \leq i \leq 1400 \quad (\mathrm{V})$$
$$L_i^0: cba \text{ if } 1401 \leq i \leq 1700 \quad (\mathrm{VI}).$$

At this profile L^0, we have $f_P^1(L^0) = \{c\}$, $f_B^1(L^0) = \{a, b, c\}$, and $f_A^1(L^0) = \{a, b\}$. As denoted, we designate the voters whose level-0 preference is $abc, acb, bac, bca, cab, cba$ as type Ⅰ, Ⅱ, Ⅲ, Ⅳ, Ⅴ, Ⅵ, respectively. By the definition of CI preference, type Ⅰ and type Ⅲ voters' level-1 preference must be f_A^1, f_B^1, f_P^1 and type Ⅴ and type Ⅵ voters' level-1 preference must be f_P^1, f_B^1, f_A^1. On the other hand, type Ⅱ and type Ⅳ voters' level-1 preference is indeterminate. I now show that the possibility of weak convergence depends on what kind of preference extension systems $\{e_i\}_{i\in N}$ the voters have. Suppose:

— All the type Ⅱ voters and 35 voters of type Ⅳ have

$$\{a,b\}e_i(L_i^0)\{c\}e_i(L_i^0)\{a,b,c\}.$$

This implies that their level-1 CI preference is f_A^1, f_B^1, f_P^1.

— 105 voters of type Ⅳ have

$$\{a,b,c\}e_i(L_i^0)\{c\}e_i(L_i^0)\{a,b\}.$$

This implies that their level-1 CI preference is f_B^1, f_P^1, f_A^1.

— The other 60 voters of type Ⅳ have

$$\{c\}e_i(L_i^0)\{a,b,c\}e_i(L_i^0)\{a,b\}.$$

This implies that their level-1 CI preference is f_P^1, f_B^1, f_A^1.

The reader can easily check that there exists a preference extension system that is compatible with these preferences. At this level-1 CI preference profile L^1, we have that $f_P^2(L^1) = \{f_A^1\}$, $f_B^2(L^2) = \{f_P^1\}$, and $f_A^2(L^1) = \{f_B^1\}$. The proof that we cannot find weak convergence[12] in the subsequent levels for this profile is similar to the proof of Proposition 1. The profile L^0 is not in trivial deadlock, and so this example shows the existence of a profile that is neither weakly convergent nor in trivial deadlock.

2.4 Probability Model

The examples in the previous section show that the possibility of weak/strong convergence largely depends on the menu F. To state this formally, we need to determine the probability model. In social choice theory, there are two major probability models—the Impartial Culture (IC) model and the Impartial Anonymous Culture (IAC) model. Because of its simplicity, I assume IAC unless otherwise noted. I briefly introduce them here for the reader's convenience.

IC assumes that each voter independently chooses, with equal likelihood, one of the linear orders over X. Therefore, each profile $L^0 \in \mathcal{L}(X)^n$ occurs with the equal probability $1/(|X|!)^n$. On the other hand, IAC assumes that every voting situation, a combination of the numbers of individuals who each have a specific linear order, occurs with equal likelihood. Hence, each $(n_1, \dots, n_{m!})$, where n_j represents the number of individuals who have the j^{th} linear order, occurs with the equal probability $1/{}_{n+m!-1}C_n$. In either model, it is well known that the probability that a certain scoring rule yields a

[12] On the other hand, if all the type Ⅱ and Ⅳ voters have a level-1 preference of f_A^1, f_B^1, f_P^1, it is also easy to verify that L^0 weakly converges to $\{a,b\}$. Thus, whether the profile can weakly converge depends on the preference extension systems $\{e_i\}_{i\in N}$.

tied outcome is negligible as $n \to \infty$[13] (Marchant, 2001, Pritchard and Wilson, 2007, Pritchard and Wilson, 2009, and Diss and Merlin, 2010). This fact allows us to restrict our attention to those profiles where each rule in the menu yields only a singleton, when we restrict our attention to a large society. For convenience in the subsequent argument, I give a similar statement and its elementary proof as a lemma.

Lemma 2

Take any distinct $x, y \in X$. Let $P(\alpha)$ be the probability that exactly $n\alpha$ voters prefer x to y. Under either IC or IAC models and for all $\alpha \in [0,1]$, $P(\alpha) \to 0$ as $n \to \infty$.

2.5 Menu of Three Scoring Rules

From this point on, I show my theoretical results. In this section, I show the fundamental result concerning the weak convergence of a menu of three scoring rules.

Lemma 3

Let $n \geq m$ and $F = \{g_1, g_2, \dots, g_p, h_1, h_2, \dots, h_q\}$ $(p \geq q \geq 0)$ be the menu of scoring SCRs, where $m = p + q \geq 3$. For any sequence $L^0, L^1, \dots, L^{k-1}$ of CI profiles to level $(k-1)$ and alternatives $x, y \in X$, suppose the class of each level-k SCR is:

$$C_{g_j^k}[L^0, L^1, \dots, L^{k-1}] = \{x\} \text{ for all } j = 1,2,\dots,p.$$

$$C_{h_j^k}[L^0, L^1, \dots, L^{k-1}] = \{y\} \text{ for all } j = 1,2,\dots,q.$$

If $\left|\{i \in N \mid xL_i^0 y\}\right| > \left|\{i \in N \mid yL_i^0 x\}\right|$, then L^0 weakly converges to $\{x\}$.

The lemma considers the case where every level-k rule results in either $\{x\}$ or $\{y\}$. It says that if at least half of the rules result in $\{x\}$ and more than half of the people prefer x to y, then the original profile weakly converges to $\{x\}$. Hence, the lemma indicates a specific case of weak convergence. Note that the lemma tells only about the possibility of weak convergence and it is still possible that the same profile weakly converges to $\{y\}$ at the same time. The uniqueness of the convergent outcome will be argued later in relation with the notion of strong convergence in section 2.6.

Lemma 4

Let $n \geq m$, $m = 3$ or 4, and $x, y \in X$ such that $\left|\{i \in N \mid xL_i^0 y\}\right| \neq \frac{n}{2}$. If the menu of SCRs is $F = \{E_1, E_2, \dots, E_{m-1}, f_B\}$ and the class of each level-k SCR is either $\{x\}$ or $\{y\}$ for a given sequence of CI profiles $L^0, L^1, \dots, L^{k-1}$, then L^0 weakly converges.

[13] For a relatively small n, the probabilities of tied outcomes when using well-known scoring rules, such as plurality and Borda count, are studied by Gillet (1977; 1980) and Marchant (2001).

These lemmas both give sufficient conditions for weak convergence. In Lemma 3, no condition is placed on the menu F, but the condition that more than half the people prefer x to y is imposed on the preference profile. Lemma 4, on the other hand, considers the specific menu F that is made up of k-approval voting and Borda count only and placed little condition on the preference profile. As noted in section 2.4, an event such as:

$$\left|\{i \in N \mid xL_i^0 y\}\right| = \frac{n}{2}$$

is unlikely as $n \to \infty$. Therefore, Lemma 4 almost always holds in the case of a large society with such a menu.

Lemma 5 Let $m = 3$ and $F^k = \{g_1^k, g_2^k, g_3^k\}$, where $g_j^k\colon [1, s_j, 0]$. Assume $C_{g_1^k} = C_{g_2^k} = \{x\}$ and $C_{g_3^k} = \{y\}$. Then, there exists $L^k \in \mathcal{L}^k[L^0, L^1, \dots, L^{k-1}]$ such that $\left|s_j\left(g_1^k : L^k\right) - s_j\left(g_2^k : L^k\right)\right| \le 1$ for all $j = 1,2,3$, where $s_j(\quad)$ denotes the score evaluated by g_j^{k+1}.

Theorem 3

Let $F = \{f_1, f_2, f_3\}$ be a menu of SCRs containing three scoring rules, where $f_j\colon [1, s_j, 0]$ $(j = 1,2,3)$, $1 \ge s_1 > s_2 > s_3 \ge 0$, and

$$s_3 \ge 1/2 \quad \text{or} \quad \left[s_3 < 1/2 \quad \text{and} \quad s_2 \le \frac{1 + s_3}{2 - s_3}\right]. \tag{1}$$

Under either IC or IAC, we have

$$p_{WC} + p_D \to 1 \quad \text{as} \quad n \to \infty.$$

Here, p_{WC} denotes the probability of occurrence of those profiles that weakly converge and p_D denotes the probability of occurrence of those profiles that are in trivial deadlock.

It is worth noting that if F contains $\{f_P, f_B\}$ or $\{f_B, f_A\}$, equation (1) automatically holds irrespective of the third scoring rule. For instance, if a large consequential society admits the menu $F = \{f_P, f_B, f_A\}$, the theorem states that there are asymptotically only two possibilities: they face a trivial deadlock or they are endowed with the ability to find weak convergence. In either case, my argument in section 2.3 indicates that further regress has little or no meaning. However, before declaring that the infinite regress is *solved,* the probabilities p_{WC} and p_D must be estimated. This is because trivial deadlock is simply a case where the regress structure does not change at all and, thus, trivial deadlock does not provide a specific answer. The actual calculation of p_{WC} and p_D can be done using the technique presented by Diss et al. (2012) and Diss and Merlin (2010).

Corollary 1

Let $|X| = 3$, $n \to \infty$, and $F = \{f_P, f_B, f_A\}$. Under IC, p_{WC} is 98.2%. Under IAC, p_{WC} is 98.8%.

This result shows for the menu $F = \{f_P, f_B, f_A\}$ that the probability of weak convergence is much larger than that of stability. This fact implies that such a society can solve the infinite regress with quite high probability (98.2% under IC and 98.8% under IAC).

2.6 Strong Convergence

Recall that $L^0 \in \mathcal{L}(X)$ is, by definition, said to weakly converge if at least one consequential sequence of (subsequent) profiles $L^1, L^2, \ldots$ exists[14] that adjusts the rules' ultimate judgments at a certain level. The existence of such $L^1, L^2, \ldots$ guarantees that we can stop the apparent infinite regress arguments through finite steps of regress. One might, however, be concerned that the same L^0 might weakly converge to a distinct C and C' by the choice of sequence. Indeed, the following example shows that such multiplicity can actually occur.

Example 12

Let $X = \{a, b, c\}$ and $F = \{f_1, f_2, f_3(= f_P)\}$, where $f_1\colon \left[1, \frac{137}{589}, 0\right], f_2\colon \left[1, \frac{68}{2945}, 0\right], f_3\colon [1,0,0]$.

Consider $L^0 \in \mathcal{L}(X)^n$ such that 1 voter: abc, 87 voters: acb, 88 voters: bac, 1 voter: bca, 22 voters: cab, and 1 voter: cba $(n = 200)$. Now, we have that $f_1(L^0) = f_2(L^0) = \{a\}$ and $f_3(L^0) = \{b\}$. It is easy to check that there exist $L^1, \tilde{L}^1 \in \mathcal{L}[L^0]$ such that

$$f_1^2(L^1), f_2^2(L^1), f_3^2(L^1) \subseteq \{f_1^1, f_2^1\} \text{ but } f_1^2(\tilde{L}^1), f_2^2(\tilde{L}^1), f_3^2(\tilde{L}^1) \subseteq \{f_3^1\}.$$

To avoid this issue, I define the notion of strong convergence, which completely avoids multiplicity.

Definition 13: Strong Convergence

A level-0 preference profile $L^0 \in \mathcal{L}(X)^n$ is said to strongly converge to $C \subseteq X$ if and only if it weakly converges to $C \subseteq X$ and it does not weakly converge to any other set $C' \neq C$.

It is clear from the definition that strong convergence is logically stronger than weak convergence and that multiplicity entirely disappears once a profile is shown to strongly converge. The next result shows the frequency of strong convergence for the menu $\{f_P, f_B, f_A\}$.

---

[14] Technically speaking, we can find the similar use of a compatible linear order in Koray (2000) and Koray and Slinko (2006). They define a social choice function (SCF) f as self-selective at L^0 relative to the menu of SCFs F^1 if and only if there is a consequentially induced $L^1 \in \mathcal{L}(F^1)^n$ such that $f^2(L^1) = f^1$. If we impose that the rule chooses itself for all compatible linear orders, as Koray and Slinko (2006; p.9) state, "it leads to a vacuous concept." The same applies to regress convergence.

Theorem 4

Let $F = \{f_P, f_B, f_A\}$. Under either IC or IAC, we have

$$p_{SC} + p_D \to 1 \text{ as } n \to \infty.$$

As a direct corollary of Theorem 3, we already know that a large society with the menu $\{f_P, f_B, f_A\}$ has asymptotically only two cases—weak convergence or trivial deadlock. Theorem 4, however, states that the two cases are actually strong convergence or trivial deadlock. To this point, I have mainly restricted attention to large societies, i.e., $n \to \infty$. But strong convergence can also be found in relatively small societies, as follows.

Example 13

The profile I gave in Example 6 strongly converges to $\{a\}$ as can be demonstrated with a slight modification to the proof of Theorem 4.

Example 14

Let $F = \{f, g\}$, where f and g are any (distinct) SCRs. If $f(L^0) = g(L^0) = C$, then L^0 strongly converges to C.

Finally, I deal with the choice of SCFs (i.e., not a correspondence but a function) and provide SCRs with neutral tie-breaking systems. Specifically, for any SCR f_Y, I denote by f_{Y^*} the SCF that breaks ties in favor of $i_Y \in N$, named the tie breaker of f_Y. Note that different SCRs can have different tie breakers (for example, the plurality tie breaker $i_P = 1$ and the Borda count tie breaker $i_B = 2$). Then, Theorem 4 can be revised for a relatively small n.

Theorem 5

Assume $n \geq 3$ is odd, $|X| = 3$, and the menu of SCFs is $\{f_{P^*}, f_{X^*}, f_{A^*}\}$, where f_X is either Borda count, Black's rule, Copeland's method, or the Hare system. Then, any level-0 profile L^0 strongly converges unless it is in a trivial deadlock.

3 Convergent Menus of SCRs

In Chapter 2, we saw basic results for the concepts of weak and strong convergence. Specifically, I showed that a large society with the menu $\{f_P, f_B, f_A\}$ has asymptotically only two possibilities: strong convergence or trivial deadlock. While the probability of the former is quite high when $|X| = 3$, we cannot deny that trivial deadlock can also occur with small but positive probability, with which the society's attempt to determine the appropriate rule ends in vain. In this chapter, I search for menus with which a society can avoid such failure. I assume the IAC model throughout this chapter.

3.1 Convergent Property of a Menu

First, I formally state the axiom of menus that demands that the society can *always* find a convergence.

Definition 14: Convergent Property

(1) If every $L^0 \in \mathcal{L}(X)^n$ weakly/strongly converges, I say that the menu F satisfies the weak/strong convergent property.

(2) Let p_{WC} (p_{SC}) be the probability that those level-0 preference profiles occur that weakly (strongly) converge. If $p_{WC} \to 1$ $(p_{SC} \to 1)$ as $n \to \infty$, I say that F satisfies the asymptotic weak (strong) convergent property.

Clearly, the strong convergent property is logically the strongest of the four axioms and the asymptotic weak convergent property is the weakest. The logical relationship between them is shown below.

$$\begin{array}{ccc}
\text{strong convergent property} & \Rightarrow & \text{asymptotic strong convergent property} \\
\Downarrow & \searrow & \Downarrow \\
\text{weak convergent property} & \Rightarrow & \text{asymptotic weak convergent property}
\end{array}$$

Let us see some examples concerning these axioms.

Example 15: Singleton Menu

If $F = \{f\}$ (a singleton menu), any profile $L^0 \in \mathcal{L}(X)^n$ strongly converges to $f(L^0)$. Hence, any singleton menu satisfies the strong convergent property.

Example 16: Menus of two Neutral SCFs

Let $F = \{f, g\}$ be a menu of two neutral SCFs and $|X| = 2$. If $L^0 \in \mathcal{L}(X)^n$ exists such that $f(L^0) \neq g(L^0)$, then F fails to satisfy the weak convergent property because we can verify that such a profile L^0 causes a trivial deadlock and hence, it never weakly converges.

I previously introduced eight familiar SCRs—plurality, Borda, anti-plurality, Hare, Nanson, Coomb, Maximin, and Black—in section 2.2. If we construct a menu of three SCRs from these eight SCRs, there are

$$\binom{8}{3} = 56$$

different menus. Our next result shows that these 56 menus also have the properties shown in Theorem 3 for a triplet of scoring rules. For the convenience of the proof, let $\mathcal{F}$ be the set of these eight SCRs:

$$\mathcal{F} = \{f_P, f_B, f_A, f_H, f_N, f_C, f_M, f_{Bl}\}.$$

Lemma 6

Suppose n is sufficiently large. Let $x, y \in X$ and $|F| = 3$, where $F \subseteq \mathcal{F}$. Let $L^0, L^1, \dots, L^{k-1}$ be a sequence of CI profiles to level $(k-1)$, where $k \in \mathbb{N}$. Suppose

$$\{C_f[L^0, L^1, \dots, L^{k-1}] \mid f \in F^k\} = \{\{x\}, \{y\}\}.$$

If

$$\#\{i \in N \mid xL_i^0 y\} > \#\{i \in N \mid yL_i^0 x\},$$

then L^0 weakly converges to x.

Theorem 6

Of the 56 menus of SCRs, the following ten menus of SCRs satisfy the asymptotic weak convergent property, i.e., $p_{WC} \to 1$ as $n \to \infty$,

$$\{f_P, f_N, f_M\}, \{f_A, f_N, f_M\}, \{f_B, f_H, f_{Bl}\}, \{f_B, f_N, f_M\}, \{f_B, f_N, f_{Bl}\}$$
$$\{f_B, f_C, f_{Bl}\}, \{f_B, f_M, f_{Bl}\}, \{f_H, f_N, f_M\}, \{f_N, f_C, f_M\}, \{f_N, f_M, f_{Bl}\}$$

The theorem shows a basic possibility concerning the asymptotically weak convergent property. It can be confirmed using familiar SCRs. Note that the menus not cited in the theorem do not have the asymptotically weak convergent property. Using the **barvinok** computer software implemented by Verdoolaege et al. (2004), we can calculate the asymptotic probability of trivial deadlock (see Appendix, 0).

3.2 Strongly Convergent Menus

In the previous section, we saw that many menus of familiar SCRs provide weak convergence with high probability. The next result shows that we can even construct a menu that satisfies the strong convergent property.

Definition 15: Difference

A menu F of SCRs satisfies the criterion of difference if, for all $f, g \in F$ and for any set X of alternatives with $|X| \geq 3$, there exists a profile $L \in \mathcal{L}(X)^n$ such that $f(L) \neq g(L)$.

This axiom is introduced by Houy (2004) and is quite a weak condition: it demands only that F should not include more than two identical SCRs. Houy (2004) shows that there is no set F that satisfies neutrality (i.e., each $f \in F$ is neutral), difference, and strong first-level stability (i.e., exactly one self-selective SCR exists at every $L \in \mathcal{L}(X)^n$). Our first result shows that the impossibility disappears if we substitute strong first-level stability for strong convergent property.

Theorem 7: Strong Convergent Menu

A set F of neutral SCRs exists that satisfies both the condition of difference and the strong convergent property.

3.3 Convergent Expansion

In sections 3.1 and 3.2, I showed that several menus, such as $\{\text{Borda}, \text{Hare}, \text{Black}\}$ or the menu used in the proof of Theorem 7, satisfy the convergent property. A straightforward conclusion from these results is that a (large) society can solve infinite regress once they accept those menus. However, if a society has already accepted a menu, such as $\{\text{plurality}, \text{Borda}, \text{anti} - \text{plurality}\}$, which fails to satisfy the convergent property, it may be, for some reason, difficult to replace this with the technical menu introduced in Theorem 7. This section considers how the convergent property may be given to such menus.

Let us define chair rule φ. Take an individual $i^* \in N$ designated as the chair:

$$\varphi(L) := \begin{cases} f_B(L) & \text{if } f_P(L) \neq f_{Bo}(L) \text{ and } \big(f_P(L), f_B(L)\big) \in e_{i^*}(L_{i^*}) \\ f_P(L) & \text{otherwise.} \end{cases}$$

In words, φ is a SCR in which the chair i^* chooses the outcome among $f_P(\cdot)$ and $f_{Bo}(\cdot)$ according to his or her own preference. Surprisingly, we have the following:

Theorem 8

The menu $\{f_P, f_{Bo}, f_A, \varphi\}$ satisfies the asymptotically weak convergent property.

Definition 16

Let F be a menu of SCRs. $G \supseteq F$ is defined as an asymptotically-convergent expansion (AC expansion) of F if the menu G has the asymptotically weak convergent property.

With this, we can write Theorem 8 as:

$G = \{f_P, f_{Bo}, f_A, \varphi\}$ is an AC expansion of $F = \{f_P, f_{Bo}, f_A\}$.

Definition 17

For a scoring rule $f: [s_1^m, s_2^m, \dots, s_m^m]_{m\geq3}$, its assignment function $\bar{f}^m: [1, m] \to [0,1]$ (on m alternatives) is defined as follows: for all $j \in \{1,2, \dots, m-1\}$,

$$\bar{f}^m \big|_{[j,j+1]}(x) \coloneqq \left(s_{j+1}^m - s_j^m\right)(x - j) + s_j^m.$$

In other words, an assignment function is obtained in two steps. First, plot m points $(1, s_1^m), (2, s_2^m), \dots, (m, s_m^m)$ and second, connect each point to the ones next to it. If the number of alternatives are obvious in the context, we often write the assignment function without its upper script letters, for example, $\bar{f}$ instead of $\bar{f}^m$.

Definition 18: Concave Function

Let $I \subseteq \mathbb{R}$ be an interval. A function $f: I \to \mathbb{R}$ is said to be concave if, for all $x, y \in I$, and $t \in [0,1]$,

$$tf(x) + (1-t)f(y) \leq f(tx + (1-t)y). \qquad (2)$$

Definition 19: Concave Scoring Rule

A scoring rule f is said to be concave if its assignment functions $\{\bar{f}^m\}_{m\geq3}$ are all concave. I denote by $\mathcal{C}$ the set of all concave scoring rules and by $\mathcal{C}_m$ the set of all score assignments $[s_1^m, s_2^m, \dots, s_m^m]$ whose assignment function is concave.

Example 17

Borda count f_B and anti-plurality f_A are both concave, while plurality is not.

Proposition 2

Let $f: [s_1^m, s_2^m, \dots, s_m^m]_{m\geq3}$ be a concave scoring rule. For all $m \geq 3$ and $1 < a < m$, we have $\bar{f_B}^m(x) \leq \bar{f}^m(x) \leq 1$.

Proof of Proposition 2

It is sufficient to show that $\bar{f_B}^m(x) \leq \bar{f}^m(x)$ for all $x = 1,2, \dots, m$. If $x = 1$ or $x = m$, the statement holds trivially because the definition of a scoring rule demands that $\bar{f}^m(1) = \bar{f_B}^m(1) = 1$ and $\bar{f}^m(m) = \bar{f_B}^m(m) = 0$. Let $x = 2,3, \dots, m-1$. Substituting $x = 1$, $y = m$, and $t = (m-x)/(m-1)$ in the equation of Definition 18, we get the proposition. ■

The following two propositions can similarly be obtained from Definition 18.

Proposition 3

Let $f: [s_1^m, s_2^m, \dots, s_m^m]_{m\geq 3}$ be a concave scoring rule. For all $m \geq 3$ and $1 < a < b < m$, we have

(1) $L_{1,b}(a) \leq \bar{f}^m(a) \leq \min\{1, L_{b,m}(a)\}$, and

(2) $L_{a,m}(b) \leq \bar{f}^m(b) \leq L_{1,a}(b)$.

where $L_{p,q}$ (for distinct $p, q \in \{1,2,\dots,m\}$) expresses the equation of the straight line passing through (p, s_p^m) and (q, s_q^m), i.e.:

$$L_{p,q}(x) := \frac{s_q^m - s_p^m}{q - p}(x - p) + s_p^m.$$

Proposition 4

Let $f: [s_1^m, s_2^m, \dots, s_m^m]_{m\geq 3}$ be a concave scoring rule. For all $m \geq 3$ and $1 < a < b < m$, we have

(1) If $1 < c < a$, $L_{1,a}(c) \leq \bar{f}^m(c) \leq \min\{1, L_{a,b}(c)\}$,

(2) If $a < c < b$, $L_{ab}(c) \leq \bar{f}^m(c) \leq \min\{L_{1,a}(c), L_{b,m}(c)\}$, and

(3) If $b < c < m$, $L_{bm}(c) \leq \bar{f}^m(c) \leq L_{ab}(c)$.

where $L_{p,q}$ (for distinct $p, q \in \{1,2,\dots,m\}$) expresses the equation of straight line passing through (p, s_p^m) and (q, s_q^m).

Theorem 9

Assume IAC. Let $F = \{f_1, f_2, \dots, f_M\}$ be a menu of $M \geq 3$ concave scoring rules. Then, there exists $G \supseteq F$ that has the asymptotically weak convergent property.

Theorem 9 says that for any menu F of any finite size, if F is made up of concave scoring rules only, we can expand it to $G \supseteq F$ so that this G has the weak convergent property. Thus, a large society can avoid the risk of trivial deadlock without abandoning the concave scoring rules in the status quo. As a straightforward result from the theorem, I will introduce two specific classes of concave scoring rules.

Corollary 2: **Polynomial Concave Scoring Rule**

The polynomial concave scoring rule p_α (with parameter $\alpha \geq 1$) is defined as a scoring rule such that, for all $m \geq 3$,

$$s_x^m = 1 - \left(\frac{x-1}{m-1}\right)^\alpha.$$

Let $F = \{p_{\alpha_1}, p_{\alpha_2}, \dots, p_{\alpha_\xi}\}$, where $\alpha_1, \alpha_2, \dots, \alpha_\xi \in [1, +\infty)$ are distinct real numbers. Then there exists $G \supseteq F$ that has the asymptotically weak convergent property.

Just as in the proof of **Corollary 2**, we have also the following.

Corollary 3: **Exponential Concave Scoring Rule**

The exponential concave scoring rule e_α (with parameter $0 < \alpha < 1$) is defined as a scoring rule such that, for all $m \geq 3$,

$$s_x^m = \frac{\alpha^x - \alpha^m}{\alpha - \alpha^m}.$$

Let $F = \left\{e_{\alpha_1}, e_{\alpha_2}, \dots, e_{\alpha_\xi}\right\}$, where $\alpha_1, \alpha_2, \dots, \alpha_\xi \in [1, +\infty)$ are distinct real numbers. Then there exists $G \supseteq F$ that has the asymptotically weak convergent property.

Example 18: **Polynomial Concave Scoring Rule**

Let $F = \{f_P, p_1, p_2, f_A\}$ be the menu of concave scoring rules. Note that p_1 is identical to Borda count. Suppose $X = \{a, b, c, d\}$. Let $n_1, n_2, n_3, \dots, n_{24}$ be the number of voters whose level-0 preference is $abcd, abdc, acbd, \dots, dcba$ (lexicographic order), respectively. Suppose the level-0 preference profile L^0 satisfies

$$\begin{aligned}(n_1, n_2, \dots, n_{24}) = (&119,60,61,61,83,61,61,95,61,67,61,61,\\ &65,130,61,61,61,61,61,61,61,147,61,61).\end{aligned}$$

Then it is easily verified that L^0 is in trivial deadlock. However, when we expand $\tilde{F} = \{p_1, p_2, f_A\}$ in the way shown in the proof of Theorem 9 ($\mu = 10$ and $r = 10$) and suppose 44 rules choose $\{a\}$, 44 rules choose $\{b\}$, 44 rules choose $\{c\}$, and 55 rules choose $\{d\}$ at L^0, then, if we construct level-1 CI profile L^1 in the way shown in the same proof, it follows that every level-2 has class $\{d\}$. This means a weak convergence to $\{d\}$. Although such resolution of trivial deadlock is not what Theorem 9 says, this indicates that the expansion of a given menu can be used to solve the trivial deadlock in some cases.

3.4 A Historical Example

Abraham Lincoln (1809–1865), the 16th President of the United States, was elected in 1860. The election, historically known as the impetus for the outbreak of the Civil War, is quite interesting from the perspective of social choice theory. There were four candidates running: Abraham Lincoln (Republican Party), John C. Breckinridge (Southern Democratic Party), John Bell (Constitutional Union Party), and Stephen A. Douglas (Northern Democratic Party). Each of them received a significant number of ballots. Indeed, some researchers argue that if the citizens' preference profiles had been aggregated using other voting procedures, the result might have been different (Riker, 1982; Tabarrok & Spector, 1999). In this section, we use this example to illustrate the notion of convergence.

Although we cannot know the complete preference profile of the citizens at that time, Riker

(1982) and Tabarrok and Spector (1999) give estimations. Riker (1982) estimates the full preference ranking for each state himself (Riker's profile) while Tabarrok and Spector (1999, p.274) "carried out a survey among a number of historians, all of whom had written on the election of 1860 or more generally on the politics of the pre-civil war era." Their estimation, the Mean Historian Profile, is made by taking the average of the 13 entire profiles estimated by the historians. For the convenience of the reader, I cite their results in the tables below.

Table 1. Riker's Profile (Ballots)

LDRB	0	RLDB	0
LDBR	450000	RLBD	0
LRDB	0	RDLB	104000
LRBD	0	RDBL	329000
LBDR	1414000	RBLD	0
LBRD	0	RBDL	413000
DLRB	83000	BLDR	270000
DLBR	318000	BLRD	0
DRLB	173000	BDLR	114000
DRBL	489000	BDRL	28000
DBLR	319000	BRLD	31000
DBRL	0	BRDL	146000

Table 2. Mean Historian Profile (%)

LDRB	0	RLDB	0
LDBR	21.17	RLBD	0
LRDB	0	RDLB	0.13
LRBD	0	RDBL	6.87
LBDR	18.61	RBLD	0
LBRD	0	RBDL	11.19
DLRB	0.11	BLDR	1.7
DLBR	8.04	BLRD	0
DRLB	0.22	BDLR	4.48
DRBL	4.87	BDRL	3.81
DBLR	8.59	BRLD	0.04
DBRL	7.53	BRDL	2.56

Based on these estimated profiles, they showed that different procedures (e.g., plurality and anti-plurality) yield different outputs. I now demonstrate how these discrepancies can be resolved through the notion of weak/strong convergence. Let L_R^0 and L_M^0 be Riker's profile and the Mean Historian Profile over $X = \{\text{Lincoln } (L), \text{Douglas } (D), \text{Bell } (B), \text{Breckinridge } (R)\}$, respectively.

(1) Strong Convergence in Riker's Profile.

Let $F = \{f_P, f_B, f_A\}$. For this profile, it follows that $f_P^1(L_R^0) = \{L\}$ while $f_B^1(L_R^0) = f_A^1(L_R^0) = \{D\}$ (see Figure 4). Just as in the proof of Theorem 8, we can see that L^0 strongly converges to $\{D\}$.

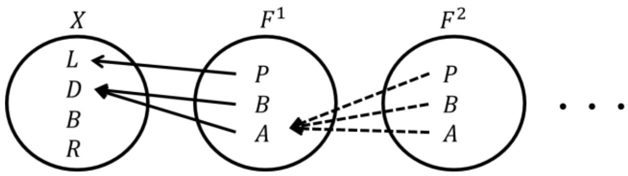

Figure 4. Strong convergence in Riker's profile

(2) Trivial Deadlock in Riker's Profile.

Let $F = \{f_P, f_X, f_A\}$, where f_X is a slight change of f_B such that

$$f_X \colon \left[1, \frac{1}{2}, 0\right], \left[1, 1, \frac{1}{3}, 0\right].$$

It is easy to check that L^0 is in trivial deadlock (see Figure 5).

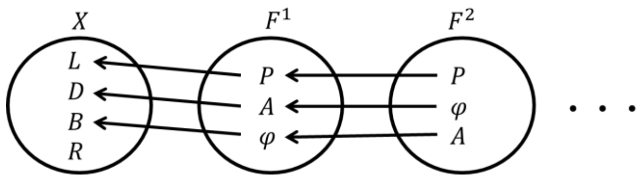

Figure 5. Trivial deadlock in Riker's profile

(3) Strong convergence in the Mean Historian Profile.
Let $F = \{f_P, f_B, f_A\}$ again. Just as in (1), we can verify that L_M^0 strongly converges to $\{D\}$.

(1), (2), and (3) provide a good example of how procedural choice can be made using the notion of convergence. In (1) and (3), L_R^0 and L_M^0 both strongly converges not to Lincoln (L) but to Douglas (D). Part of the reason for this result is that Douglas wins over Lincoln using the simple majority rule under the both profiles. I do not claim that Douglas *should* have been the winner. In terms of convergence theory, whether a specific candidate (e.g., Douglas) should be elected depends on what kind of menu the society accepts. For example, if the U.S. citizens at that time thought that f_P was the unique appropriate procedure, i.e., $F = \{f_P\}$, the convergence clearly shows that the winner should have been Lincoln, because both profiles strongly converges to $\{L\}$. The procedural choice based on convergence depends on which procedures are on the menu.

3.5 Discussion

In Chapters 2 and 3, I investigated the notion of weak/strong convergence. A preference profile L^0 over the set of alternatives X is said to (weakly/strongly) converge if every rule to choose the rule to … to choose the rule to choose from X derives the same subset C of X. In Chapter 2, the results showed that a large society with three "familiar" SCRs can find convergence with relatively high probability when there are three alternatives. Specifically, the probability of weak convergence marks 100% (as $n \to \infty$) for ten menus (Theorem 6). In Chapter 3, we focused on the question of under which menus of SCRs a (large) society can *always* find convergence. When little or no condition is placed on menus, Theorem 7 shows the existence of a menu satisfying the strongest property, i.e., the strong convergent property. On the other hand, I also showed that if F is made up of concave scoring rules, there is an AC-expansion $G \supseteq F$ (Theorem 9). This result enables a large society to acquire the asymptotic convergent property without abandoning the SCRs that they have already accepted. To conclude the chapter, I will add several comments on these results.

The first comment is on the calculation of probability. Throughout Chapters 2 and 3, the argument originates from the set of alternatives X, and I assume that the probability model of voters' preferences over X follows either the IC or IAC model. Therefore, the probability of convergence or of trivial deadlock can differ once the number of alternatives changes. There is, however, a slightly different use of the IC or IAC model: to assume that people's preferences over the menu F follow one of these models. In this alternative interpretation[15], the choice from menu F should correspond with "the set of alternatives X". If we do not use the original set of alternatives explicitly, in this way, we can determine the probability of convergence without depending on the number of original alternatives, because we can regard Theorem 6 as a purely general case.

The second comment is on the definition of weak convergence. I defined a profile as weakly convergent if *there exists a sequence of CI profiles* that satisfy the required condition. Theoretically, CI preference is a generalization of the "induced preference" used by Koray (2000). Koray defines a voting rule to be self-selective if *there exists an induced preference profile* that satisfies a couple of conditions. As Koray and Slinko (2006; p.6) argued, if we substitute the italic part in the last sentence into "for all induced preference profiles", the notion of self-selectivity turns out to be degenerate, and so too does the notion of convergence. In this sense, our notion of convergence is theoretically close to the notion of self-selectivity or self-stability: I note, however, that they are independent of each other. More specifically, Diss et al. (2012) and Diss and Merlin (2010) define a menu of SCRs as stable if there exists at least one self-selecting SCR for all preference profiles. What I claim here is that the two statements "a menu is stable" and "a menu weakly converges" are independent. To show this, it is sufficient to give two examples. The first is a profile that is stable and not weakly convergent; such a profile was shown in the introduction of trivial deadlock (see Example 9). The second example is a profile that is not stable and is weakly convergent, as shown in Figure 1 (page 6). The reader can see that no (level-1 and level-2) SCR chooses itself in the figure.

The last and concluding comment is on the meta-level profiles. The notion of convergence is, by definition, based on the implicit assumption that voters' meta-preferences are consequential. If everyone is (supposed to be) consequential, the convergence notion performs relatively well to resolve the infinite regress of procedural choice, just by manipulating the indifferent class in consequentially induced weak preference profiles. However, the notion does not work well if there exist some voters whose meta-level preferences are not consequential. For example, suppose a voter, Mr. Z, prefers Douglas to Lincoln and prefers plurality to anti-plurality. Such a voter is not consequential because plurality chooses Lincoln while anti-plurality chooses Douglas (section 3.4). But the point is a little more demanding. To reject the theory of convergence, there must be a voter whose meta-preference is not consequential, regardless of how high a level is considered. If Mr. Z prefers any level-$k \geq 2$ rule (i.e., a rule in F^k) that ultimately chooses level-1 plurality, then we can regard him as consequential

[15] Indeed, Diss et al. (2012) accepts this interpretation. They assume that the probability distribution over of the preference profile over the rules should follow IAC model. Nevertheless, I assumed that voters' preference profile over X follows IC or IAC. The reason is my personal idea that the procedural choice should be made for the very agenda that the society faces. It could be that a man does not care whether his wife decides dictatorially which restaurant to go for lunch but at the same time the man hates the use of dictatorship to determine whether the Diet abolishes a national law. One's procedural judgment can vary according to the agenda. Therefore, our theory treats X explicitly.

from level-2. If every $i \in N$ is consequential at some finite level $k_i \in \mathbb{N}$, then the convergence phenomenon will work, for we can say that everyone is consequential from the level $k^* \coloneqq \max\{k_1, k_2, \dots, k_n\}$ and hence the theory works once we regard X as F^{k^*}. Once this translation is done, my series of theorems works to provide the convergence for a society. Thus, whether the notion of convergence can work depends on whether some individuals are not consequential at any level. For instance, whether there is an individual who prefers plurality at any level, even though level-10 plurality might choose level-9 anti-plurality. To determine what kind of meta-preference (for infinite number of levels) can be an interesting future topic.

4 Determine the Society

In the study of convergence in the previous chapters, the society N is assumed to be given. In other words, an ex ante agreement is assumed to exist as to who has voting rights and who does not. While this implicit assumption is commonly used in the literature referred to in the previous chapter, the determination of N can sometimes be quite controversial, especially when it is not clear who belongs to the set of individuals affected by the decision. In such a case, a voting rule is needed to aggregate individuals' opinions on who should be included in N. This chapter focuses on the strategic aspect of such an aggregation procedure.

Consider that a set of individuals $\bar{N} = \{1,2,\ldots,n\}$, where $n \geq 2$, assigns some (honorable) *position*s among them based on their mutual evaluations. I assume that everyone is selfish, in the sense that they want to win the honor for themselves. These situations differ from an ordinary social choice problem because each individual is a candidate as well as a voter, and therefore, specific kinds of strategic voting may occur. A basic interpretation of the word *positions* in the context of this dissertation is (a person who has) the right to vote. However, the subsequent argument is not specific to this context. Indeed, there are many other decision-making situations that have a similar structure, such as the awarding of prizes at an academic conference, a leadership contest within a political party, and the selection of representatives within a group.

For these situations, Holzman and Moulin (2013) proposed an axiomatic framework of nomination rules and the axiom of Impartiality (IMP). A nomination rule is a rule for choosing the set of winners through the aggregation of individuals' ballots that state who should receive the honorable positions. Under many familiar nomination rules such as approval voting (AV), a rational voter might manipulate his or her ballot in order to improve their own chance of winning. Consider, for instance, a society of four individuals: $1, 2, 3,$ and 4. They choose the prize winner(s) from among themselves by AV, where everyone is obliged to approve others and is not allowed to self-approve or abstain. Suppose individual 1 approves 2, 2 approves 3, 3 approves 4, and 4 approves 3. In this case, AV declares victory for individual 3, because he or she receives the highest score (two points). However, if individual 4 approved 1 instead, AV would declare victory for the entire set of individuals (i.e., 1, 2, 3, and 4) because everyone's score would be the same. Thus, individual 4 would be better off by manipulating his or her ballot[16]. A nomination rule is called impartial if everyone can approve of anyone without fearing that the vote might spoil his or her own chance of winning. The example shows that AV is not impartial, despite its widely accepted robustness against strategic manipulation.

[16] This possibility of manipulation still exists even if the method of AV uses a deterministic tie-breaking rule to restrict the set of winners to singletons only. Suppose individual A wins when A approves B, B approves C, C approves D, and D approves A. In this case, individual A has an opportunity to manipulate at the following ballot profile: A approves C, B approves C, C approves D, and D approves A. Individual A would be better off approving B instead.

Described technically, Holzman and Moulin (2013) study the nomination function—where exactly one person wins—and propose several impossibility theorems. The combination of IMP and Anonymous Ballots (AB), an axiom that demands that each individual be treated equally as a voter, show one of the most striking impossibilities: the constant rule is the only nomination rule that satisfies both of these two axioms. Among subsequent axiomatic studies, Tamura and Ohseto's (2014) is the closest to my study. They showed that by considering nomination correspondences (i.e., allowing multiple winners), the impossibilities can be relieved. However, they faced another impossibility concerning IMP, AB, and Positive Unanimity (PU).

In general, the framework of a nomination rule is determined by two sets: (1) the domain (i.e., the admissible set of ballot profiles), and (2) the codomain (i.e., the admissible set of sets of winners). I refer to this pair as the setting of the nomination rule. As Tamura and Ohseto (2014) show, the extent of design possibility for normative nomination rules can differ among these settings. This strand of research motivates me to consider other popular settings and to find further escape routes from the impossibilities. The comparative study of different settings is also motivated from an empirical point of view. Consider, for example, nominations for the best paper award in some academic societies, or for the position of president of a country. In such cases, the number of winners is supposed to be restricted to one or at least bounded from above. By considering various domains with the number of winners fixed (to one), we can provide an escape route from the Holzman and Moulin impossibility.

In a technical sense, I consider four types of ballot domain:

1) All voters can approve as many other individuals as they like and neither self-approval nor abstention is allowed.
2) All voters can approve as many individuals as they like and abstention is not allowed.
3) All voters can approve as many individuals as they like and self-approval is not allowed.
4) All voters must approve a fixed number of others (self-approval is not allowed).

In Holzman and Moulin (2013), Tamura and Ohseto (2014), and Tamura (2015), each individual was allowed to approve another individual. Their framework is a special case of my ballot profile domain 4). This ballot profile domain was also studied in relation to AV (e.g., Peters, Roy, & Storcken, 2012).

As for codomains, I consider three types:

a) The number of winners is fixed.
b) The number of winners is bounded by a maximum of some fixed number.
c) The number of winners is bounded by a minimum of some fixed number.

By considering every combination of the domains (i.e., 1–4) and codomains (i.e., a–d), we can evaluate the nomination rules for a large number of settings. For example, for $n = 10$ individuals, the number of possible settings is as high as 275. I have investigated, for each of the possible settings, whether a nomination rule exists that will satisfy IMP, AB, Pairwise Candidate Neutrality (2CN), Weak PU (WPU), and Negative Unanimity (NU). 2CN is a new axiom that I have formulated to express the idea of neutrality for endogenous nominating settings. My comparison analysis is possible because I succeed in describing the general structure that the impartial nomination rules commonly have under various settings (Lemma 7, Lemma 8, Lemma 9, Lemma 10, and Lemma 11 in subsection 4.3.2). Comparative results will be described in subsection 4.3.3.

Roughly speaking, the result shows that the threshold rule performs well in many settings. The threshold rule is defined as a rule that chooses all individuals whose scores (i.e., the number of approvals received by the individual) reach the fixed threshold. For example, if the threshold is fixed to two, every individual wins if and only if he or she is approved by at least two individuals. Indeed, the threshold rule, if it is well-defined for the setting, satisfies IMP and AB for almost all settings except those where self-approvals are allowed. The intuitive reason for this is that, when self-approval is not allowed, individuals cannot change their own score. This implies that individuals' own ballots cannot affect whether their score reaches the fixed threshold. I will show that the threshold rule is characterized using IMP and some of the axioms well-known in relation to AV: anonymity, neutrality, positive/negative unanimity, and weak monotonicity.

The current chapter is organized as follows. Section 4.1 denotes the basic notation and section 4.2 describes the axioms of nomination rules in detail. I show the technical results of my comparative study in section 4.3. Further comments and discussion on the results are given in section 4.4. All proofs are in the Appendix.

4.1 Notation

Let $\bar{N} = \{1,2,\dots,n\}$ be a society consisting of n $(3 \leq n < \infty)$ individuals. Each individual $i \in \bar{N}$ casts a ballot $N_i \subset \bar{N}$, where the ballot N_i is interpreted as the set of candidates approved by i. I refer to several circumstances that differ in the kinds of ballots that are admitted and winners that can be chosen. These pieces of information are formally expressed as the domains and codomains of nomination rules.

Definition 20: Ballot Profile Domains—The Domain of the Nomination Rule

Let $k \in \{1,2,\dots,n-2\}$. For any $i \in \bar{N}$ I define four types of admissible ballot domains $\mathfrak{N}_i, \mathfrak{N}_i^{self}, \mathfrak{N}_i^{AB}, \mathfrak{N}_i^{k} \subseteq \mathfrak{P}(\bar{N})$ as follows:

$$\mathfrak{N}_i = \left\{ N_i \in \mathfrak{P}(\bar{N}) \,\middle|\, \phi \neq N_i \subseteq \bar{N} \setminus \{i\} \right\}$$

$$\mathfrak{N}_i^{self} = \left\{N_i \in \mathfrak{P}(\bar{N}) \,\middle|\, \phi \neq N_i \subseteq \bar{N} \setminus \{i\}\right\}$$

$$\mathfrak{N}_i^{AB} = \left\{N_i \in \mathfrak{P}(\bar{N}) \,\middle|\, N_i \subseteq \bar{N} \setminus \{i\}\right\}$$

$$\mathfrak{N}_i^{k} = \left\{N_i \in \mathfrak{N}(\bar{N}) \,\middle|\, N_i \subseteq \bar{N} \setminus \{\phi\} \ \ and \ \ |N_i| = k\right\}$$

I refer to each $\mathfrak{N}_{\mathrm{i}}$, $\mathfrak{N}_i^{self}$, $\mathfrak{N}_i^{AB}$, and $\mathfrak{N}_i^{k}$ as i's ballot domain, generally denoted by $\mathfrak{D}_i$. A combination of all individuals' ballots is called a ballot profile. I denote by $\mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^{k}$ the corresponding set of admissible ballot profiles for each of $\mathfrak{D}_i = \mathfrak{N}_i, \mathfrak{N}_i^{AB}, \mathfrak{N}_i^{self}, \mathfrak{N}_i^{k}$:

$$\mathfrak{N} = \mathfrak{N}_1 \times \ldots \times \mathfrak{N}_n$$
$$\mathfrak{N}^{self} = \mathfrak{N}_1^{self} \times \ldots \times \mathfrak{N}_n^{self}$$
$$\mathfrak{N}^{AB} = (\mathfrak{N}_1^{AB} \times \ldots \times \mathfrak{N}_n^{AB}) \setminus \{(\phi, \phi, \ldots, \phi)\}$$
$$\mathfrak{N}^{k} = \mathfrak{N}_1^{k} \times \ldots \times \mathfrak{N}_n^{k}$$

I refer to each $\mathfrak{N}$, $\mathfrak{N}^{self}$, $\mathfrak{N}^{AB}$, and $\mathfrak{N}^{k}$ as a (ballot profile) domain, generally denoted by $\mathfrak{D}$.

A ballot profile domain $\mathfrak{D}_i$ expresses what kind of ballot i can cast. As we can see from the definition, $\mathfrak{N}_{\mathrm{i}}$ allows individual i to approve as many individuals as he or she likes, if i approves at least one individual (i.e., no abstention) and does not approve him or herself (i.e., no self-approval). The next two ballot profile domains $\mathfrak{N}^{self}$ and $\mathfrak{N}^{AB}$ are situations where self-approval or abstention, respectively, are permitted. Finally, $\mathfrak{N}_i^{k}$ is a ballot domain where i must approve a fixed number of k individuals from among the others. Note that the condition of $k \leq n-2$ is not restricting because it only rules out $\mathfrak{N}_i^{n-1}$ where i has no choice but to approve all others. Considering all possible combinations of individuals' ballots from the corresponding ballot profile domains $\mathfrak{D}_i$, I define the profile domains $\mathfrak{D}$. I note that $\mathfrak{N}^{AB}$ excludes the empty profile, where no one approves anyone.[17] I denote the ballots using capital letters with a subscript representing the individual $N_i, M_i, K_i (\in \mathfrak{D}_i)$. Ballot profiles are denoted by scripted styles $\mathcal{N}, \mathcal{M}, \mathcal{K} (\in \mathfrak{D})$, and ballot profile domains are denoted by fraktur letters $\mathfrak{N}, \mathfrak{D}$, etc.

Definition 21: Possible Winners—The Codomain of the Nomination Rule

Let $l \in \{1,2,\ldots,n-1\}$. We consider several types of the codomain $\mathfrak{X}$ of the nomination rule.

$$\bar{\mathfrak{X}} = \mathfrak{P}(\bar{N}) \setminus \{\phi\}$$
$$\mathfrak{X}^l = \left\{W \in \mathfrak{P}(\bar{N}) \,\middle|\, |W| = l\right\}$$
$$\bar{\mathfrak{X}}^l = \left\{W \in \mathfrak{P}(\bar{N}) \,\middle|\, |W| \geq l\right\}$$
$$\underline{\mathfrak{X}}^l = \left\{W \in \mathfrak{P}(\bar{N}) \,\middle|\, |W| \leq l\right\}$$

I refer to each $\bar{\mathfrak{X}}$, $\underline{\mathfrak{X}}$, $\bar{\mathfrak{X}}^l$, and $\underline{\mathfrak{X}}^l$ as a codomain, generally denoted by $\mathfrak{X}$.

[17] As we describe later, this condition follows the model of Alon et al. (2009).

The codomain $\mathfrak{X} = \overline{\mathfrak{X}}, \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$ contains the information on the possible number of winners. The codomain $\overline{\mathfrak{X}}$ admits any number of winners except zero, while $\mathfrak{X}^l$ admits only the fixed number of winners, $\overline{\mathfrak{X}}^l$ admits l or more winners, and $\underline{\mathfrak{X}}^l$ admits l or less winners. It follows from the definition that $\overline{\mathfrak{X}}$ is a special case of $\overline{\mathfrak{X}}^l$, or $\overline{\mathfrak{X}} = \overline{\mathfrak{X}}^1$. Though I do not consider the case of $l = n$, it is clear that $\mathfrak{X}^n = \overline{\mathfrak{X}}^n$ has little importance, and $\underline{\mathfrak{X}}^n (= \overline{\mathfrak{X}}^1 = \overline{\mathfrak{X}})$ is included in the other types. Thus, the restriction of $l < n$ eliminates the trivial cases only. Hereafter, I call the pair of the domain and codomain a setting. A nomination rule is formally defined for each combination of domain and codomain.

Definition 22: Nomination Rule

The nomination rule φ of setting $(\mathfrak{D}, \mathfrak{X})$ is a function $\varphi: \mathfrak{D} \to \mathfrak{X}$, which assigns to each ballot profile $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{D}$ the set of winners $\varphi(\mathcal{N}) \in \mathfrak{X}$.

My definition deals with many variations in the settings. Holzman and Moulin (2010) study the setting of $(\mathfrak{D}, \mathfrak{X}) = (\mathfrak{N}^1, \mathfrak{X}^1)$, Tamura and Ohseto (2014) study the setting of $(\mathfrak{D}, \mathfrak{X}) = (\mathfrak{N}^1, \overline{\mathfrak{X}})$, and Alon et al. (2009) study the setting of $(\mathfrak{N}^{AB}, \mathfrak{X}^l)$.

Here I will provide a few more notations. For any $\mathcal{N} \in \mathfrak{D}$, I will denote by $s_i(\mathcal{N})$ the i's score at ballot profile $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{D}$, which is calculated as follows:

$$s_i(\mathcal{N}) \coloneqq \left|\{j \in \overline{N} \mid i \in N_j\}\right|.$$

This counts the number of ballots that include i. I denote by $s(\mathcal{N}) = \left(s_1(\mathcal{N}), s_2(\mathcal{N}), \dots, s_n(\mathcal{N})\right)$ the profile of scores at a ballot profile $\mathcal{N}$. To distinguish this from ballot profiles, I denote by $s(\mathcal{N})$ the score profile (with respect to $\mathcal{N}$). I also denote by $s_j^{-i}(\mathcal{N})$ the individual j's score coming from the individuals in $\overline{N} \setminus \{i\}$ as follows:

$$s_j^{-i}(\mathcal{N}) = \left|\{\mu \in \overline{N} \setminus \{i\} \mid j \in N_\mu\}\right|.$$

Finally, I define a special type of ballot profile that is useful for the proof. For all $j \in \mathbb{N}$, I define $\bar{j} \in \overline{N}$ as $\bar{j} = n$ if $j \equiv 0 \pmod{n}$ and $\bar{j} \equiv l$ if $j \equiv l \pmod{n}$ for some $1 \leq l \leq n - 1$. For example, $\bar{\imath} = i$ for all $i \in \overline{N}$, and $\bar{0} = n, \overline{-1} = n - 1, \overline{-2} = n - 2, \dots$ and $\overline{n + 1} = 1, \overline{n + 2} = 2, \dots$. Let me define cyclic ballot profiles $\mathcal{C}^1, \mathcal{C}^2, \dots, \mathcal{C}^{n-1} \in \mathfrak{N}$. For any $1 \leq m \leq n - 1$, I define the m-cyclic ballot profile $\mathcal{C}^m = (C_1^m, C_2^m, \dots, C_n^m) \in \mathfrak{N}$ as $C_i^m \coloneqq \{\overline{\imath + 1}, \overline{\imath + 2}, \dots, \overline{\imath + m}\}$. I further define a reversed m-cyclic ballot profile $\mathcal{R}^m = (R_1^m, \dots, R_n^m)$ such that $R_i^m \coloneqq \{\overline{\imath - 1}, \overline{\imath - 2}, \dots, \overline{\imath - m}\}$ for all $i \in \overline{N}$.

4.2 Axioms for Nomination Rules

4.2.1 Axioms

I introduce some normative axioms for nomination rules. After the definitions, I will show some well-known nomination rules and argue their axiomatic performance in 4.2.2.

Let $\varphi : \mathfrak{D} \to \mathfrak{X}$ be a nomination rule.

(1) φ satisfies IMP if

for any $\mathcal{N} = (N_i, N_{-i}) \in \mathfrak{D}, i \in \bar{N}$, and $N_i' \in \mathfrak{D}_i$, we have $i \in \varphi(N_i, N_{-i}) \Leftrightarrow i \in \varphi(N_i', N_{-i})$.

(2) φ satisfies (strong) PU if

for all $\mathcal{N} \in \mathfrak{D}$, if $i \in \bar{N}$ exists such that $i \in \bigcap_{j \neq i} N_j$, then

$$\varphi(\mathcal{N}) = \{ i \in \bar{N} \mid i \in N_j \text{ for all } j \in \bar{N} \}.$$

(3) φ satisfies WPU if

for any $\mathcal{N} = (N_i, N_{-i}) \in \mathfrak{D}$ and $i \in \bar{N}$, if $i \in N_j$ for all $j \in \bar{N} \setminus \{i\}$, then $i \in \varphi(\mathcal{N})$.

(4) φ satisfies NU if

for all $\mathcal{N} \in \mathfrak{D}$ and $i \in \bar{N}$, if $i \notin N_j$ for all $j \in \bar{N} \setminus \{i\}$, then $i \notin \varphi(\mathcal{N})$.

(5) φ satisfies AB if

for all $\mathcal{N}, \mathcal{N}' \in \mathfrak{D}$, if $s(\mathcal{N}) = s(\mathcal{N}')$, then $\varphi(\mathcal{N}) = \varphi(\mathcal{N}')$.

(6) φ satisfies No Dummy (ND) if

for all $i \in \bar{N}$, $\mathcal{N} = (N_i, N_{-i})$ and $N_i' \in \mathfrak{D}_i$ exist such that $\varphi(\mathcal{N}) \neq \varphi(\mathcal{N}')$.

Note that these axioms, except WPU, coincide with those used in both Holzman and Moulin (2013) and Tamura and Ohseto (2014), if we consider the settings studied in those papers. WPU is my own axiom. To make this dissertation self-contained, I will briefly explain these axioms.

Axiom IMP demands that each voter's ballot has no influence over whether that voter wins or loses. In other words, everyone can approve anyone without fearing that the approval of one's potential rivals decreases one's own chance of winning.

The axioms of PU, WPU, and NU relate to the idea of unanimity. PU and WPU demand that one must win if one earns unanimous approval from all others, and PU furthermore demands that those who fail to obtain unanimous approval from the others cannot win if someone else obtains unanimous approval. Note that PU and WPU are logically equivalent in some settings (e.g., in $(\mathfrak{N}^1, \mathfrak{X}^1)$), which Holzman and Moulin (2013) studied. On the other hand, NU demands that if one cannot obtain approval from any of the others, he or she must not win.

The fifth axiom, AB, states that all individuals should be treated equally as voters. If a rule satisfies AB, then it does not see who approves who, but only the scores of each individual. Note that this condition does not necessarily require individual equality as a candidate. For example, a nomination rule that chooses some fixed individual $i \in \bar{N}$ satisfies AB, although this rule is clearly discriminative over other candidates.

The sixth axiom, ND, states that all voters have at least one situation (i.e., one ballot profile) where they can change the winners $\varphi(\cdot)$. Thus, the constant rule does not satisfy ND because anyone in the society is a dummy voter.

Note also that all the above axioms are satisfied by AV, except for IMP. Formal discussion on the properties of each rule will be given in section 4.2.2.

To state the next axiom, I need to introduce further notation. Let $\sigma = (i, j)$ be any transposition[18] over $\bar{N}$ that swaps i and j. For any ballot profile $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{D}$ such that $i \notin N_j$ and $j \notin N_i$, let $\mathcal{N}^\sigma = (N_1^\sigma, \dots, N_n^\sigma)$ be the transposed ballot profile defined for any $k \in \bar{N}$ and $\mu = i, j$, $\mu \in N_k \Leftrightarrow \sigma(\mu) \in N_k^\sigma$.

In words, $\mathcal{N}^\sigma$ is a ballot profile where the approvers of i and j are swapped from the original ballot profile $\mathcal{N}$: those who approved i at $\mathcal{N}$ will newly approve j at $\mathcal{N}^\sigma$, and those who approved j at $\mathcal{N}$ will approve i at $\mathcal{N}^\sigma$. This means that the individuals' judgments over i and j are swapped with each other. Note that for any transposition $\sigma = (i, j)$ and a ballot profile $\mathcal{N} \in \mathfrak{D}\big(= \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k\big)$ such that $i \notin N_j$ and $j \notin N_i$, it follows that $\mathcal{N}^\sigma \in \mathfrak{D}$. Therefore, we can freely consider the transposed ballot profile as long as $i \notin N_j$ and $j \notin N_i$. Note also that if it were not for the condition of $i \notin N_j$ and $j \notin N_i$, it could be that $\mathcal{N} \in \mathfrak{D}$ but $\mathcal{N}^\sigma \notin \mathfrak{D}$. This is because self-approval is not permitted in the domains $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{\mathrm{AB}}, \mathfrak{N}^k$. Therefore, if i approves j at the original ballot profile $\mathcal{N}$, we cannot define $\mathcal{N}^\sigma$ for $\sigma = (i, j)$ in a direct manner because N_i^σ should include i instead of j, which constitutes self-approval. Using this notation I introduce the next axiom.

(7) φ satisfies Pairwise Candidate Neutrality (2CN) if

for all $\mathcal{N} \in \mathfrak{D}$, $i, j \in \bar{N}$, and transposition $\sigma = (i, j)$, if $i \notin N_j$ and $j \notin N_i$, then we have $i \in \varphi(\mathcal{N}) \Leftrightarrow \sigma(i) \in \varphi(\mathcal{N}^\sigma)$.

(8) φ satisfies Cancellation (C) if

for all $\mathcal{N} \in \mathfrak{D}$, if $s_i(\mathcal{N}) = s_j(\mathcal{N})$ for all $i.j \in \bar{N}$, then $\varphi(\mathcal{N}) = \bar{N}$.

(9) φ satisfies Weak Monotonicity (WM) if

for all $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{D}$, $i \in \varphi(\mathcal{N}) \subseteq \bar{N}$, and $j, k \in \bar{N} \setminus \{i\}$ such that $i \notin N_j$ and $k \in N_j$,

$$i \in \varphi(\mathcal{N}') \text{ for } \mathcal{N}' = \big(N_j', N_{-j}\big) \in \mathfrak{D}, \text{ where } N_j' = \big(N_j \cup \{i\}\big) \setminus \{k\}.$$

2CN reflects the idea of the neutrality axiom in the nomination environment. Roughly speaking, 2CN demands that the swap of i and j in the ballot profile causes the swap of i and j in the

[18] We say that $\sigma = (i, j)$ is a transposition over $\bar{N}$ between i and j if $\sigma: \bar{N} \to \bar{N}$ is a bijection and $\sigma(i) = j$, $\sigma(j) = i$, and $\sigma(k) = k$ for all $k \in \bar{N} \setminus \{i, j\}$.

result of φ. However, 2CN says nothing if there is an internal approval between i and j. Under 2CN, we can say that each individual is treated almost equally. Cancellation states that if the scores of all the individuals are the same, then all the individuals win. As I show in Lemma 10, this axiom is logically connected to the others[19]. The last axiom, WM, states that the original winner $i \in \bar{N}$ is still one of the winners after some j newly approves i instead of some k.

Remark. 2CN is a weaker axiom than Candidate Neutrality (CN), which is used in Mackenzie (2015). To see this, let me give a slight paraphrasing of AC. If we consider $\bar{\mathfrak{N}}$ or $\mathfrak{N}^k$ as the domain, the condition of $[i \notin X_j$ and $j \notin X_i]$ is equivalent to saying $X^\sigma \in D$[20]. Furthermore, $[i \in \varphi(X) \Leftrightarrow j \in \varphi(X^\sigma)]$ is equivalent to saying $[\varphi_{\sigma(i)}(X^\sigma) = \varphi_i(X)]$ if we consider deterministic rules. Therefore, in these domains, 2CN and CN can be expressed as follows:

CN: For each profile $X = (X_1, X_2, \dots, X_n) \in D$, each $i \in N$, and each permutation $\sigma \in S_N$,

$$X^\sigma \in D \text{ implies } \varphi_{\sigma(i)}(X^\sigma) = \varphi_i(X).$$

2CN: For each profile $X = (X_1, X_2, \dots, X_n) \in B$, and each $i \in N$ and each transposition $\sigma = (i, j)$,

$$X^\sigma \in D \text{ implies } \varphi_{\sigma(i)}(X^\sigma) = \varphi_i(X).$$

The difference is whether they consider any permutation or any transposition. Clearly, CN implies 2CN because a transposition is a permutation. It is well known that any permutation can be written as a product of transpositions, and so the reader might think that 2CN and CN are logically equivalent, but they are not. 2CN is strictly logically weaker than CN. The following example shows this fact.

Let $\varphi: \mathfrak{N} \to \mathfrak{P}(N) \setminus \{\phi\}$ be defined as follows: for each $\mathcal{N} = (N_1, N_2, \dots, N_n) \in \bar{\mathcal{X}}$,

$$\varphi(X) := \begin{cases} \{1\} & \text{if } X \in \{C^{n-2}, R^{n-2}\} \\ N_1 & \text{otherwise.} \end{cases}$$

In words, this rule chooses individual 1 only if the ballot profile is either $(n-2)$-cyclic profile $\mathcal{C}^{n-2}$ or its reverse $\mathcal{R}^{n-2}$. Otherwise, it chooses those who individual 1 approves. Let me show two statements:

[1] the rule φ satisfies 2CN, and

[2] the rule φ fails to satisfy CN.

<u>Proof of [1]</u>

I show [1] with four steps. Let $\mathbf{1}: \bar{N} \to \bar{N}$ be the identity function.

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[19] We found some works that use this axiom in the characterization of AV: Fishburn (1978), Laffont (1979), and Alós-Ferrer (2006).

[20] In this remark, I use the following notation so that I can compare the definitions to those of Mackenzie (2015).

$$X_i^\tau = \tau(X_i) = \{\tau(y) \mid y \in X_i\}.$$

Step 1: To show that (a) for each $\sigma \in S_{\overline{N}}$, $[\sigma(\mathcal{C}^{n-2}) \in \mathfrak{D} \Leftrightarrow \sigma$ is either $(1,2,\dots,n)$ or $\mathbf{1}]$, and (b) for each $\sigma \in S_{\overline{N}}$, $[\sigma(\mathcal{R}^{n-2}) \in \mathfrak{D} \Leftrightarrow \sigma$ is either $(n, n-1, \dots, 1)$ or $\mathbf{1}]$.

The proofs are similar, and so let me show (a) only. $[\Leftarrow]$ is straightforward, therefore I show $[\Rightarrow]$. Take any $\sigma \in S_{\overline{N}}$ such that $\sigma(\mathcal{C}^{n-2}) \in \mathfrak{D}$ and $\sigma \neq \mathbf{1}$. Take any $i \in \overline{N}$. Suppose $i \neq \sigma(i)$ and let $j = \sigma(i)$. $\sigma(\mathcal{C}^{n-2}) \in \mathfrak{D}$, and so j does not nominate himself or herself at the profile $\sigma(\mathcal{C}^{n-2})$. Formally,

$$j \notin \left\{\mu \in \overline{N} \,\middle|\, j \in (\mathcal{C}^{n-2})^{\sigma}_{\mu}\right\} = \left\{\mu \in \overline{N} \,\middle|\, i \in \mathcal{C}^{n-2}_{\mu}\right\} = \overline{N} \setminus \{i, \overline{\imath + 1}\}.$$

So, j is either i or $\overline{\imath + 1}$. We assumed $j \neq i$, and so we have $j = \sigma(i) = \overline{\imath + 1}$.

$\sigma \neq \mathbf{1}$, and so there exists at least one individual $i_0 \in \overline{N}$ such that $\sigma(i_0) \neq i_0$. With the argument from the previous paragraph, we have $\sigma(i_0) = \overline{\imath_0 + 1}$. σ is a permutation, and so we have $\sigma(\overline{\imath_0 + 1}) \neq \overline{\imath_0 + 1}$, because otherwise $\sigma(i_0)$ and $\sigma(\overline{\imath_0 + 1})$ would become the same. Inductively, we have $\sigma(i) \neq i$ for all $i \in \overline{N}$. With the previous paragraph, this means $\sigma(i) = \overline{\imath + 1}$ for all $i \in \overline{N}$. As an extra notation, let me denote as $\Sigma = (1,2,\dots,n)$ (and $\Sigma^{-1} = (n, n-1, \dots, 1)$).

Step 2: To confirm that $\Sigma(\mathcal{C}^{n-2}) = \mathcal{R}^{n-2}$ and $\Sigma^{-1}(\mathcal{R}^{n-2}) = \mathcal{C}^{n-2}$.

The confirmation is straightforward. Note that Step 1 and Step 2 together imply that if $\sigma(\mathcal{C}^{n-2}) \in \mathfrak{D}$ for some $\sigma \in S_{\overline{N}}$, it follows that $\sigma(\mathcal{C}^{n-2})$ is either $\mathcal{C}^{n-2}$ or $\mathcal{R}^{n-2}$.

Step 3: To show that there is no $\tau \in S_{\overline{N}}$ and $\mathcal{N} \in \mathfrak{D} \setminus \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$ such that $\tau(\mathcal{N}) \in \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$.

Suppose to the contrary that $\tau(\mathcal{N}) = \mathcal{C}^{n-2}$ for $\tau \in S_{\overline{N}}$ and $\mathcal{N} \in \mathfrak{D} \setminus \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$. Then,

$$\tau^{-1}(\mathcal{C}^{n-2}) = \tau^{-1}\big(\tau(\mathcal{N})\big) = \mathcal{N}.$$

This contradicts Step 2.

Step 4: To show that φ satisfies 2CN.

Take any $i, j \in \overline{N}$ and $\mathcal{N} \in \mathfrak{D}$. Let $\sigma = (i, j)$. If $\mathcal{N} \in \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$, Step 1 tells us that $\sigma(\mathcal{N}) \notin \mathfrak{D}$ (if $n \geq 3$). So, the statement of 2CN automatically holds. If $\mathcal{N} \in \mathfrak{D} \setminus \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$, step 3 shows that $\sigma(\mathcal{N}) \notin \mathfrak{D}$ or $\sigma(\mathcal{N}) \in \mathfrak{D} \setminus \{\mathcal{C}^{n-2}, \mathcal{R}^{n-2}\}$. In the former case, the statement of 2CN automatically holds. In the latter case, 2CN also holds because $\varphi(\mathcal{N}) = N_1$ and $\varphi(\mathcal{N}^{\sigma}) = N_1^{\sigma}$. ■

__Proof of [2]__

To check that φ does not satisfy CN, let us consider profiles $\mathcal{C}^{n-2}$ and $\mathcal{R}^{n-2}$ and permutation Σ defined in the proof of [1]. By definition, $\varphi(\mathcal{C}^{n-2}) = \varphi(\mathcal{R}^{n-2}) = \{1\}$. However, CN demands that $\varphi(\mathcal{R}^{n-2}) = \Sigma(\mathcal{C}^{n-2}) = \{2\}$. Contradiction. ■

4.2.2 Independence of the Axioms

Here I show some basic examples of nomination rules and discuss whether they satisfy the main axioms: IMP, AB, WPU, PU, 2CN, and NU, which will be used frequently later on. Then I show the logical relation of the main axioms, mainly on the setting $(\mathfrak{N}, \overline{\mathfrak{X}})$[21].

(1) Approval Voting $\varphi_{AV}: \mathfrak{D} \to \overline{\mathfrak{X}}$

AV φ_{AV} is the nomination rule that chooses as the winners those with the highest scores. For any ballot profile $\mathcal{N} \in \mathfrak{D}$,

$$\varphi_{AV}(\mathcal{N}) \coloneqq \underset{i \in \overline{N}}{\operatorname{argmax}}\, s_i(\mathcal{N}) = \{i \in \overline{N} | s_i(\mathcal{N}) \geq s_j(\mathcal{N}) \text{ for all } j \in \overline{N}\}.$$

As I noted in the introduction, this rule is not impartial on $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^{k}$. I will briefly show this through two counterexamples.

For the domains $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{AB}, \mathfrak{N}^{self}, \mathfrak{N}^{1}$, let $\overline{N} = \{1,2,3\}$ and consider a ballot profile $\mathcal{N} = (N_1, N_2, N_3) \in \mathfrak{N}$, where $N_1 = \{2\}, N_2 = \{1\}, N_3 = \{2\}$. The score profile is given by $s(\mathcal{N}) = (s_1(\mathcal{N}), s_2(\mathcal{N}), s_3(\mathcal{N})) = (1,2,0)$. Thus, $\varphi_{AV}(\mathcal{N}) = \{2\}$. However, individual 1 can be better off by changing his or her ballot N_1 to $N_1' \coloneqq \{3\}$. AV will choose $\varphi_{AV}(N_1', N_{-1}) = \{1,2,3\}$ at this new ballot profile, for $s_1(\mathcal{N}) = s_2(\mathcal{N}) = s_3(\mathcal{N}) = 1$, thus contradicting IMP.

For the domain $\mathfrak{D} = \mathfrak{N}^{k}$, let $\overline{N} = \{1,2,\dots,n\} (n \geq 3)$ and consider the k-cyclic ballot profile $\mathcal{C}^k \in \mathfrak{N}^k$. Then $\varphi(\mathcal{C}^k) = \overline{N}$. Next, consider individual 1's manipulation as $\mathcal{D} = (D_1, D_{-1}) \in \mathfrak{N}^k$, where $D_1 = (C_1^k \cup \{n\}) \setminus \{2\}$ and $D_{-i} = C_{-i}^k$. Note that $k \leq n-2$ implies $n \notin C_1^k$. Therefore, we can see $\mathcal{D}$ as a ballot profile such that individual 1 approves n instead of 2. Then, we have $s_n(\mathcal{D}) = k+1 > s_1(\mathcal{D}) = k$, which implies $1 \notin \varphi_{AV}(\mathcal{D})$. Therefore, this rule does not satisfy IMP. These examples show a basic gap between the concept of AV and the axiom of IMP.

(2) Constant-C Rule $con_{C}: \mathfrak{D} \to \mathfrak{X}$

Let $C \in \mathfrak{X}$ be a subset of $\overline{N}$. The constant-C rule, con_C, is the nomination rule that always nominates C regardless of the ballots:

$$\varphi_i(\mathcal{N}) = C \text{ for all } \mathcal{N} \in \mathfrak{X}$$

Two illustrative cases are when $C = \{i\}$ for some $i \in \overline{N}$ and $C = \overline{N}$ (i.e., $con_{\{i\}}$ and $con_{\overline{N}}$). The former is shown to be the unique nomination rule on the setting $(\mathfrak{N}^1, \mathfrak{X}^1)$ that satisfies both IMP and AB (Holzman & Moulin, 2013). In fact, it is easy to see that the constant-C rule, where $C \in \overline{\mathfrak{X}}$

[21] For the logical relationship of the main axioms in other typical settings, see Holzman and Moulin (2013) for $(\mathfrak{N}^1, \mathfrak{X}^1)$ and Tamura and Ohseto (2014) for $(\mathfrak{N}^1, \overline{\mathfrak{X}})$.

is any admissible set, satisfies IMP and AB. Consider the latter case $con_{\overline{N}}$ on the setting $(\mathfrak{N}, \overline{\mathfrak{X}})$. In this case, it is clear that $con_{\overline{N}}: \mathfrak{N} \to \overline{\mathfrak{X}}$ satisfies WPU and 2CN, however, it fails to satisfy PU and NU.

(3) Plurality With Runners-up Rule $\boldsymbol{\varphi_P}: \mathfrak{N} \to \overline{\mathfrak{X}}$

The plurality with runners-up rule, φ_P, defined below, is an extension of the original definition proposed in Tamura and Ohseto (2014). Although it was invented on the setting $(\mathfrak{N}^1, \overline{\mathfrak{X}})$, I show that a similar idea works in $(\mathfrak{N}, \overline{\mathfrak{X}})$, in the sense that φ_P satisfies IMP and some other axioms. For any given ballot profile $\mathcal{N} \in \mathfrak{N}$, I define $F_{\mathcal{N}}, S_{\mathcal{N}} \subseteq \overline{N}$ as follows:

$$F_{\mathcal{N}} \coloneqq \{ i \in \overline{N} \mid s_i(\mathcal{N}) \geq s_j(\mathcal{N}) \text{ for all } j \in \overline{N} \}$$

$$S_{\mathcal{N}} \coloneqq \{ i \in \overline{N} \mid s_i(\mathcal{N}) = s_{F_{\mathcal{N}}}(\mathcal{N}) - 1 \}$$

where $s_{F_{\mathcal{N}}}(\mathcal{N}) = s_i(\mathcal{N})$ for some $i \in F_{\mathcal{N}}$. Note that $F_{\mathcal{N}}$ is the set of individuals with the largest scores. Therefore, it is nonempty for all ballot profiles $\mathcal{N} \in \mathfrak{N}$. On the other hand, $S_{\mathcal{N}}$ is the set of individuals whose score is just one point smaller than the largest. Thus, $S_{\mathcal{N}}$ can be empty for some ballot profiles. Let me define the plurality with runners-up rule, φ_P, on $(\mathfrak{N}, \overline{\mathfrak{X}})$ as follows:

$$\text{for all } \mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{N},$$

$$\varphi_P(\mathcal{N}) = F_{\mathcal{N}} \cup \{ i \in S_{\mathcal{N}} \mid F_{\mathcal{N}} \subseteq N_i \}$$

This rule unconditionally chooses all individuals in $F_{\mathcal{N}}$. For individuals in $S_{\mathcal{N}}$, on the other hand, the rule chooses them if and only if they approve all of the individuals in $F_{\mathcal{N}}$ at the given ballot profile $\mathcal{N}$. Note that if we swap the domain $\mathfrak{N}$ in the above definition to domain $\mathfrak{N}^1$, the result is identical to what is proposed in Tamura and Ohseto (2014) under the setting $(\mathfrak{N}^1, \overline{\mathfrak{X}})$. Proposition 5 will show that this rule satisfies WPU, NU, and IMP, but not PU[22] nor 2CN. Furthermore, it fails to satisfy AB if $n \geq 4$.

(4) Threshold-$\boldsymbol{t}$ Rule $\boldsymbol{\varphi^t}$

For all $t \in \{0,1,2,\dots,n\}$, I define the threshold-t rule, $\varphi^t(\mathcal{N})$, for all $\mathcal{N} \in \mathfrak{D}$ as follows:

$$\varphi^t(\mathcal{N}) = \{ i \in \overline{N} \mid s_i(\mathcal{N}) \geq t \}.$$

In words, this rule chooses all of the individuals whose scores reach t. Note that for the threshold rule to be well-defined, the codomain must be rich enough. Consider a society of four individuals $\overline{N} = \{1,2,3,4\}$ and the setting $(\mathfrak{N}, \mathfrak{X}^1)$. At the 1-cyclic ballot profile $\mathcal{C}^1$, we have

$$s_1(\mathcal{C}^1) = s_2(\mathcal{C}^1) = s_3(\mathcal{C}^1) = s_4(\mathcal{C}^1) = 1.$$

Therefore, the threshold-1 rule should choose $\varphi^1(\mathcal{C}^1) = \overline{N}$. However, because $\overline{N} \notin \mathfrak{X}^1$, we can see that the rule is not well-defined on this setting. For the same reason, we cannot provide φ^1 on the setting $(\mathfrak{N}, \mathfrak{X}^l)$ or $(\mathfrak{N}, \underline{\mathfrak{X}}^l)$.

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[22] In their Theorem 1, Tamura and Ohseto (2014) show that their $\varphi_P: \mathfrak{N}^1 \to \overline{\mathfrak{X}}$ also satisfies PU. However, according to my expanded definition, $\varphi_P: \mathfrak{N} \to \overline{\mathfrak{X}}$ does not satisfy PU. Proposition 10 will demonstrate that this is because of the intrinsic impossibility in this setting rather than my failure to properly redefine the rule.

Although unorthodox, I introduce three other nomination rules. These are introduced to show the logical independence of the main axioms.

(5) Pseudo Threshold Rule $\xi^i: \mathfrak{N} \to \bar{\mathfrak{X}}$ for some $i \in \bar{N}$

For all $\mathcal{N} \in \mathfrak{N}$,

$$\xi^i(\mathcal{N}) = \begin{cases} \varphi^1(\mathcal{N}) & \text{if } s_i(\mathcal{N}) \neq n-1 \\ \varphi^1(\mathcal{N}) \setminus \{i\} & \text{if } s_i(\mathcal{N}) = n-1 \end{cases}$$

This rule is very similar to the threshold-1 rule, φ^1, and only differs when individual i receives unanimous approval from the others. It is easy to see that the pseudo threshold rule satisfies IMP, AB, 2CN, and NU, and does not satisfy WPU.

(6) Pseudo Threshold Rule' $\xi'^i: \mathfrak{N} \to \bar{\mathfrak{X}}$ for some $i \in \bar{N}$

For all $\mathcal{N} \in \mathfrak{N}$,

$$\xi'^i(\mathcal{N}) = \begin{cases} \varphi^1(\mathcal{N}) & \text{if } s_i(\mathcal{N}) \neq 1 \\ \varphi^1(\mathcal{N}) \setminus \{i\} & \text{if } s_i(\mathcal{N}) = 1 \end{cases}$$

It is clear that this rule satisfies IMP, AB, WPU, and NU, but not 2CN.

(7) Pseudo-Dictatorial Rule $d^i: \mathfrak{N} \to \bar{\mathfrak{X}}$

$$d^i(\mathcal{N}) = \begin{cases} N_i & \text{if } s_i(\mathcal{N}) < n-1 \\ N_i \cup \{i\} & \text{if } s_i(\mathcal{N}) = n-1 \end{cases}$$

Under this rule, $j \neq i$ wins if and only if j is approved by i while i wins only if i receives unanimous approval from the others. We can verify that the pseudo-dictatorial rule $d^i: \mathfrak{N} \to \bar{\mathfrak{X}}$ satisfies WPU, NU, IMP, and 2CN, but not AB.

The following presents the conclusions from this section.

Proposition 5.

Let $n \geq 3$ and $i \in \bar{N}$. The axioms IMP, AB, 2CN, WPU, and NU are all logically independent under the setting $(\mathfrak{N}, \bar{\mathfrak{X}})$. In fact, we have the following:

(1) φ_{AV} satisfies AB, 2CN, WPU, and NU, but not IMP.

(2) $con_{\bar{N}}$ satisfies IMP, AB, 2CN, and WPU, but not NU.

(3)[23] φ_P satisfies IMP, WPU, and NU, but not 2CN. φ_P satisfies AB if $n = 3$, but fails if $n \geq 4$.

(4) φ^1 satisfies IMP, AB, 2CN, WPU, and NU.

(5) ξ^i satisfies IMP, AB, 2CN, and NU, but not WPU.

[23] Part of the proof that φ_P satisfies I, specifically, can be obtained by modifying the proof from Tamura and Ohseto's (2014) Theorem 1.

(6) ξ'^i satisfies IMP, AB, NU, and WPU, but not 2CN.

(7) d^i satisfies IMP, 2CN, WPU, and NU, but not AB.

These results are described in Table 3. For each entry in the table, 0 means that the rule does not satisfy the axiom, and 1 means that the rule satisfies the axiom. We can infer from the table that all the five axioms are logically independent of each other on the setting $(\mathfrak{N}, \overline{\mathfrak{X}})$.

Table 3. Axiomatic Performances of Each Nomination Rule

	IMP	AB	2CN	WPU	NU
φ_{AV}	0	1	1	1	1
$con_{\overline{N}}$	1	1	1	1	0
φ_P	1	0	0	1	1
φ^1	1	1	1	1	1
ξ^1	1	1	1	0	1
ξ'^{i}	1	1	0	1	1
d^i	1	0	1	1	1

4.3 Results

4.3.1 Known Impossibilities

Before my own contributions in 4.3.2, I will state some other related results.

Proposition 6 (Alon et al., 2011)[24]

Let $l \in \{1,2,\dots,n-1\}$. There is no nomination rule $\varphi: \mathfrak{N}^{AB} \to \mathfrak{X}^l$ that satisfies IMP and NU.

Proposition 7 (Holzman & Moulin, 2013)

Let $\varphi: \mathfrak{N}^1 \to \mathfrak{X}^1$ be a nomination rule.

(1) φ satisfies AB and IMP if and only if it is the constant rule φ_i.

(2) There is no nomination rule that satisfies IMP, PU, and NU.

Proposition 8 (Tamura & Ohseto, 2014)

Let $\varphi: \mathfrak{N}^1 \to \overline{\mathfrak{X}}$ be a nomination rule.

(1) The plurality with runners-up rule satisfies IMP, PU, and NU.

(2) If $n \geq 4$, there is no nomination rule φ that satisfies IMP, AB, and PU.

[24] Indeed, their result is based on the concept of finite approximate ratio, and they do not explicitly refer to NU. However, one can easily derive this from their Theorem 3.1.

Each of these propositions shows a basic impossibility or difficulty regarding IMP and some of the well-known axioms, though most of their results differ both in the domain and the codomain, which makes it difficult to directly compare the extent of the possibilities. Roughly speaking, we can infer from these results that it seems difficult to design a nomination rule that satisfies IMP, AB, and the axioms related to unanimity. This motivates me to investigate the extent of the possibilities for other typical settings.

4.3.2 Basic Results

Clearly, the axiomatic possibility of designing impartial nomination rules $\varphi: \mathfrak{D} \to \mathfrak{X}$ depends largely on the setting $(\mathfrak{D}, \mathfrak{X})$. However, I will first show the structure that impartial nomination rules have in common under various settings, especially as it pertains to AB. For simplicity of description, I introduce another term.

Definition 23

Let $\varphi: \mathfrak{D} \to \mathfrak{X}$ be the nomination rule. For any ballot profiles $\mathcal{N}, \mathcal{N}' \in \mathfrak{D}$ and an individual $i \in \overline{N}$, we say that two ballot profiles $\mathcal{N}$ and $\mathcal{N}'$ are i-equivalent (under the nomination rule φ), or $\mathcal{N} \sim_i \mathcal{N}'$ if and only if $[i \in \varphi(\mathcal{N}) \Leftrightarrow i \in \varphi(\mathcal{N}')]$ holds.

The i-equivalence relationship $\sim_i$ defined in this way makes an equivalence relation over the domain $\mathfrak{D}$ (i.e., it satisfies reflexivity, symmetry, and transitivity). With this terminology, we can rephrase the axiom of IMP as: a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP if and only if for any $i \in \overline{N}$ and for any ballot profiles $\mathcal{N} = (N_i, N_{-i}) \in \mathfrak{D}$, $\mathcal{N}' = (N_i', N_{-i}) \in \mathfrak{D}$, $\mathcal{N}$ and $\mathcal{N}'$ are i-equivalent.

Lemma 7: Table Lemma, the Common Structure Stipulated by IMP and AB[25]

Let $k \in \{1,2,\dots,n-2\}$ and $l \in \{1,2,\dots,n-1\}$. Let $\mathfrak{D}$ be either $\mathfrak{N}, \mathfrak{N}^{\mathrm{AB}}, \mathfrak{N}^{self}, \mathfrak{N}^k$ and let $\phi \neq \mathfrak{X} \subseteq \mathfrak{P}(\overline{N}) \setminus \{\phi\}$. Suppose a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP and AB. For any ballot profiles $\mathcal{N}, \mathcal{N}' \in \mathfrak{D}$ and for any individual $i \in \overline{N}$, if $s_i(\mathcal{N}) = s_i(\mathcal{N}')$, then $\mathcal{N}$ and $\mathcal{N}'$ are i-equivalent.

This lemma states that, under IMP and AB, the i-equivalence class grows much larger than under IMP only. It also states that, for any individual $i \in \overline{N}$, any two ballot profiles $\mathcal{N}, \mathcal{N}' \in \mathfrak{D}$ with individual i's score being the same, or $s_i(\mathcal{N}) = s_i(\mathcal{N}')$, must yield the same result on i. This property is widely observed in all settings that are introduced in section 4.1. Indeed, this lemma applies for all combinations of the domain $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ (i.e., as many as $(n+1)$ domains)

[25] The case of $\mathfrak{D} = \mathfrak{N}^1$ and $\mathfrak{X} = \mathfrak{X}^1$ is implicitly shown in the proof of Holzman and Moulin's (2013) Theorem 3. Thus, this lemma can be interpreted as a generalization result for any setting $(\mathfrak{D}, \mathfrak{X})$ that is introduced in section 4.1.

and the codomain $\mathfrak{X} = \overline{\mathfrak{X}}, \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$ (i.e., as many as $(3n-5)$ codomains).[26] This lemma holds even for other codomains if they are nonempty and do not allow an empty set as a winner set.

The technical implications of this lemma will be shown in the proofs of the following results. Here I provide an intuitive explanation of this lemma. Consider a society with four individuals, $\overline{N} = \{1,2,3,4\}$, and a nomination rule $\varphi: \mathfrak{N} \to \overline{\mathfrak{X}}$. Because the number of possible ballots by any $i \in \overline{N}$ is $2^3 - 1 = 7$, the cardinality of the ballot profile domain $\mathfrak{N}$ is $7^4 = 2401$. The number of possible sets of winners is $|\overline{\mathfrak{X}}| = |\mathfrak{P}(\overline{N}) \setminus \{\phi\}| = 2^4 - 1 = 15$. Therefore, the number of possible nomination rules is as many as $15^{2401} > 10^{2800}$. However, according to Lemma 7, the nomination rule that satisfies IMP and AB is fully expressed by the table below.

Table 4. A Table Expressing a Nomination Rule

$s_i(\mathcal{N}) \setminus i$	1	2	3	4
0	Win	Lose	Win	Win
1	Win	Win	Lose	Win
2	Win	Win	Lose	Lose
3	Win	Lose	Lose	Lose

The columns in table 2 are labeled with the individuals and the rows express the score. For example, the information in row "2" and column "1" states whether individual 1 wins or loses when individual 1's score is two. Because any two ballot profiles with 1's score being two are 1-equivalent, we can say that a nomination rule corresponds with a way to fill in the table. Thus, we know that the number of possible nomination rules that satisfy IMP and AB for four individuals is at most $2^{16} = 65536$.[27]

For a given ballot profile domain $\mathfrak{D}$, we define the score profile domain $\mathbb{S}[\mathfrak{D}]$ as follows:

$$\mathbb{S}[\mathfrak{D}] = \left\{ s = (s_1, \dots, s_n) \in \{0,1,\dots,n\}^n \,\middle|\, \exists \mathcal{N} \in \mathfrak{D} \text{ s.t. } \; s_i(\mathcal{N}) = s_i \text{ for all } i \in \overline{N} \right\}$$

Thus, $\mathbb{S}[\mathfrak{D}]$ is the set of all score profiles that can appear under the ballot profile domain $\mathfrak{D}$. Under AB, any two ballot profiles $\mathcal{N}, \mathcal{N}' \in \mathfrak{D}$ such that $s(\mathcal{N}) = s(\mathcal{N}')$ yield the same result. Thus, we can interpret a given nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ as a function of φ: $\mathbb{S}[\mathfrak{D}] \to \mathfrak{X}$ with a natural manner that for all $s \in \mathbb{S}[\mathfrak{D}]$, $\varphi(s) \coloneqq \varphi(\mathcal{N})$, where $\mathcal{N}$ is a ballot profile such that $s(\mathcal{N}) = s$. The axiom of AB guarantees that $\varphi: \mathbb{S}[\mathfrak{D}] \to \mathfrak{X}$ defined in this manner is well-defined. Lemma 8 shows the structure of $\mathbb{S}[\mathfrak{D}]$ for any $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$.

Lemma 8: **The Relationship Between $\mathfrak{D}$ and $\mathbb{S}[\mathfrak{D}]$**

Let $k \in \{1,2,\dots,n-1\}$.

[26] If $n = 10$, the number of the combinations equals 275.

[27] There are not as many as 65536 different nomination rules. This is because we cannot fill in all the entries in column 2 with 'lose'. Considering $\varphi(\mathcal{C}^2) \neq \phi$, we know that there is at least one individual who wins when he or she receives a score of two.

$$\mathbb{S}[\mathfrak{N}^k] = \left\{(s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n \mid \sum_{i=1}^{n} s_i = nk\right\}$$
$$\mathbb{S}[\mathfrak{N}] = \left\{(s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n \mid \sum_{i=1}^{n} s_i \geq n\right\}$$
$$\mathbb{S}[\mathfrak{N}^{self}] = \left\{(s_1, \dots, s_n) \in \{0,1,\dots,n\}^n \mid \sum_{i=1}^{n} s_i \geq n\right\}$$
$$\mathbb{S}[\mathfrak{N}^{AB}] = \left\{(s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n \mid \sum_{i=1}^{n} s_i \geq 1\right\}.$$

This lemma shows that all the score profile domains related to $\mathfrak{N}^k, \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$ can be captured through a simple arithmetic formula on the sum of the individual scores. The next lemma shows that by imposing 2CN as well, each individual should be treated almost equally in terms of their score. In terms of the table, this implies that the entries in almost every row should be filled in with the same results.

Let $M_{\mathfrak{D}} \in \mathbb{Z}$ be the maximum score possible at the domain $\mathfrak{D}$, viz.

$$M_{\mathfrak{D}} = \begin{cases} n-1 & \text{if } \mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{AB}, \mathfrak{N}^k \\ n & \text{if } \mathfrak{D} = \mathfrak{N}^{self}. \end{cases}$$

Lemma 9

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ be the domain and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. Suppose a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB and 2CN. For any ballot profile $\mathcal{N} \in \mathfrak{D}$ and for any individual $i \in \overline{N}$, suppose $i \in \varphi(\mathcal{N})$ and $0 \leq d = s_i(\mathcal{N}) \leq M_{\mathfrak{D}} - 1$. Then, for any individual $j \in \overline{N}$ and $\mathcal{N}' \in \mathfrak{D}$, if we have $s_j(\mathcal{N}) = d$, then $j \in \varphi(\mathcal{N}')$.

As a direct consequence of these lemmas (8 and 9), the relationship between C and other main axioms can be found, which fact will also be used in the proofs of the results in section 4.3.3.

Lemma 10: Derivation of Cancellation

Let $n \geq 4$, $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^k$, and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. If a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB, WPU, and 2CN, $\varphi(\mathcal{C}^m) = \overline{N}$ holds for any $m \in \{1,2, \dots. n-1\}$.

Proof of Lemma 10

Take any m-cyclic ballot profile $\mathcal{C}^m \in \mathfrak{D}$, where $m \in \{1,2,\dots,n-1\}$.[28] The case of $m = n-1$ is easily verified by WPU. Assume $1 \leq m \leq n-2$. Then Lemma 9 implies that $i \in \varphi(\mathcal{C}^m) \Leftrightarrow j \in \varphi(\mathcal{C}^m)$ for all $i, j \in \overline{N}$. Because $\varphi(\mathcal{C}^m) \neq \phi$, this implies that $\varphi(\mathcal{C}^m) = \overline{N}$. ■

[28] Note that $\mathcal{C}^1, \dots, \mathcal{C}^{n-1}$ are all in $\mathfrak{D} = \mathfrak{N}$ and $\mathcal{C}^k$ is also in $\mathfrak{N}^k$.

Although the statement of the lemma specifies only $\varphi(\mathcal{C}^k) = \overline{N}$, using the AB condition, any ballot profile $\mathcal{N} \in \mathfrak{N}$ such that $s(\mathcal{N}) = s(\mathcal{C}^k)$, in other words $s_1(\mathcal{N}) = s_2(\mathcal{N}) = \cdots = s_n(\mathcal{N}) = k$, yields the same choice as $\mathcal{C}^k$: $\varphi(\mathcal{N}) = \varphi(\mathcal{C}^k)$. Therefore, we can say that the combination of IMP, AB, 2CN, and WPU implies C.

Lemma 11: **Threshold Lemma**

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{AB}, \mathfrak{N}^{self}, \mathfrak{N}^k (1 \leq k \leq n-2)$ and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. Let $\varphi: \mathfrak{D} \to \mathfrak{X}$ be a nomination rule that satisfies IMP, AB, 2CN, and WM. Suppose $i \in \varphi(\mathcal{N})$ for some individual $i \in \overline{N}$ and ballot profile $\mathcal{N} \in \mathfrak{D}$ such that $s_i(\mathcal{N}) \leq M_{\mathfrak{D}} - 1$. Then, for any individual $j \in \overline{N}$ and $\mathcal{N}' \in \mathfrak{D}$, if we have $s_j(\mathcal{N}') \geq s_i(\mathcal{N})$, then $j \in \varphi(\mathcal{N}')$.

This lemma states that under IMP, AB, 2CN, and WM, a possible nomination rule, if it exists, would be the threshold rule. However, I do not intend to claim that the reverse holds. As I have noted in section 4.2.2, there are many settings where the threshold rule is not well-defined, and therefore this lemma fails to characterize the threshold rule. The details pertaining to this will be shown in the next section. Note also that as a corollary of these lemmas, we have the following:

Corollary 4

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{AB}, \mathfrak{N}^{self}, \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l, \underline{\mathfrak{X}}^l$. There is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP, AB, and 2CN.

Proof of Corollary 4

Take any $k \in \{1,2, \dots, n-2\}$. Note that $\mathcal{C}^k \in \mathfrak{N}^k \subseteq \mathfrak{N} = \mathfrak{N}^{AB} \cap \mathfrak{N}^{self}$. Therefore, we have $\mathcal{C}^k \in \mathfrak{D}$. Assume that a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies IMP, AB, and 2CN. Because $\phi \notin \mathfrak{X}$, there is a winner $i \in \varphi(\mathcal{C}^k)$. Based on Lemma 9, the entire society $\overline{N}$ should be the winner set, which contradicts $\phi \notin \mathfrak{X}$. ■

Corollary 5

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^k$ $(1 \leq k \leq n-2)$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$. Suppose a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ fails to satisfy NU. In this case, φ satisfies IMP, AB, 2CN, and WM, if and only if it is $con_{\overline{N}}$.

Proof of Corollary 5

It is clear that $con_{\overline{N}}: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ satisfies IMP, AB, 2CN, and WM, but not NU. Suppose a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ fails to satisfy NU. In this case, there is an individual $i \in \overline{N}$ and a ballot profile $\mathcal{N} \in \mathfrak{D}$ such that $i \in \varphi(\mathcal{N})$ and $s_i(\mathcal{N}) = 0$. Based on Lemma 11, it follows that $j \in \varphi(\mathcal{N}')$ for all $j \in \overline{N}$ and $\mathcal{N}' \in \mathfrak{D}$. This means that φ is identical to $con_{\overline{N}}$. ■

Corollary 6: **IMP, AB, 2CN, WM⇒WPU**

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. If a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB, 2CN, and WM, then it also satisfies WPU.

Proof of Corollary 6

Note that $\mathcal{C}^k \in \mathfrak{D}$ and $s_i(\mathcal{C}^k) = k \leq n-2$ for all $i \in \overline{N}$. Because $\phi \notin \mathfrak{X}$, there is a winner $j \in \varphi(\mathcal{C}^k)$. Therefore, according to Lemma 5, we have for all $i \in \overline{N}$ and for all $\mathcal{N} \in \mathfrak{D}$, if $s_j(\mathcal{N}) \geq k$, then $j \in \varphi(\mathcal{N})$. This means that one can win whenever one obtains a score $M_{\mathfrak{D}}$. ■

Let me discuss these results from the viewpoint of designing a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP and AB. Corollary 4 shows that the codomain of $\mathfrak{X}^l, \underline{\mathfrak{X}}^l$, both of which bound the number of winners from above, are not suited to further impose 2CN; the number of winners must be unbound for consideration of 2CN. Indeed, Corollary 5 shows that the codomain $\overline{\mathfrak{X}}^l$ enables us to impose 2CN as well. Another lesson from Corollary 5 is the importance of NU. Once NU is broken, the possibility of designing an impartial nomination rule is limited by the four axioms of IMP, AB, 2CN, and WM. These results motivate me to consider the class of nomination rules that satisfy IMP, AB, and NU in each of the possible settings, and we will answer this in the next subsection. On the other hand, Corollary 6 can be seen as a relationship between WPU and WM under the axioms of IMP, AB, and 2CN. This result will also be used to compare the possibility results in 4.2.2.

4.3.3 Comparative Results for Various Settings

I first show a basic impossibility result that motivates us to compare various settings.

Proposition 9: **Universal Impossibility**

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{\mathrm{self}}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}, \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. There is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP, AB, and PU.

Note that Proposition 5 is a generalization of Proposition 8 (Tamura & Ohseto, 2014).

Proposition 10: Complementary Results on Proposition 9

[1] Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. A nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies IMP and AB.

[2] Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$. If $\mathfrak{X} = \mathfrak{X}^l, \underline{\mathfrak{X}}^l$, there is no nomination rule that satisfies PU. If $\mathfrak{X} = \overline{\mathfrak{X}}^l$, there is no nomination rule that satisfies IMP and PU.

Let $\mathfrak{D} = \mathfrak{N}^k$.

Let $\mathfrak{X} = \mathfrak{X}^l$. If $k = 1$ and $l = 1$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}^l$ exists that satisfies IMP and PU if and only if $n \geq 4$. Otherwise ($k \geq 2$ or $l \geq 2$), there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP and PU.

If $\mathfrak{X} = \overline{\mathfrak{X}}^l$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies IMP and PU if and only if $k \leq n-3$ and $l = 1$.

If $\mathfrak{X} = \underline{\mathfrak{X}}^l$ and $k > l$, there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies PU.

If $\mathfrak{X} = \underline{\mathfrak{X}}^l$ and $k = n-2$, there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP and PU.

If $\mathfrak{X} = \underline{\mathfrak{X}}^l$ and $k = 1$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies IMP and PU.

[3] Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$. There is a nomination rule that satisfies AB and PU.

Let $\mathfrak{D} = \mathfrak{N}^k$.

If $\mathfrak{X} = \mathfrak{X}^l$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies AB and PU if and only if $k = 1$ and $l = 1$.

If $\mathfrak{X} = \overline{\mathfrak{X}}^l$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies AB and PU if and only if $l = 1$.

If $\mathfrak{X} = \underline{\mathfrak{X}}^l$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies AB and PU if and only if $k \leq l$.

Proposition 9 shows a simple limitation to the design of impartial nomination rules, saying that a nomination rule cannot be designed to satisfy IMP, AB, and PU under any setting we have defined. The essence of the proof is very simple and worth noting. Once we admit IMP and AB, Lemma 7 tells us that the winners are determined by individual scores rather than the structure of the ballot profile itself. If PU is then imposed, the existence of ballot profiles of the form $(\dots, s_i = M_{\mathfrak{D}}, \dots, s_j = x, \dots)$ will inevitably demand that individual j will lose the election whenever j gets score x. Because this argument holds as long as $x < M_{\mathfrak{D}}$, it is very difficult to determines the winner when no one obtains score $M_{\mathfrak{D}}$.

For more detail, Proposition 10 shows the necessity of each axiom to derive the impossibility. We can see that it is generally difficult to satisfy PU. While the axiom pair IMP and AB does not itself yield an impossibility (see [1] in Proposition 10), PU itself or the combination of PU and one of IMP or AB often leads to a negative result. Let me describe the difficulties concerning PU. The first problem comes from the unconditional acceptance of those with a maximum score $M_{\mathfrak{D}}$ (let me call them $M_{\mathfrak{D}}$-holders). Under the domain $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$, for example, the number of $M_{\mathfrak{D}}$-holders can vary from zero to $n-1$. However, if the codomain $\mathfrak{X}$ does not allow that many winners, there is no way to design a nomination rule with PU. This problem occurs when the codomain is $\mathfrak{X} = \overline{\mathfrak{X}}^l, \mathfrak{X}^l$. Note that we cannot escape this problem even if we substitute WPU for PU. The second problem is the exclusiveness of PU. Recall that PU chooses only $M_{\mathfrak{D}}$-holders if they exist. Thus, PU directly yields impossibility under the following circumstances:

$\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l$ and if there is a ballot profile $\mathcal{N} \in \mathfrak{D}$ where less than l (but at least one) individuals are $M_{\mathfrak{D}}$-holders, or

$\mathfrak{X} = \mathfrak{X}^l, \underline{\mathfrak{X}}^l$ and if there is a ballot profile $\mathcal{N} \in \mathfrak{D}$ where more than l individuals are $M_{\mathfrak{D}}$-holders.

In this case, the situation would be expected to improve once we substitute WPU for PU. Furthermore, the exclusiveness of PU is also harmful when we consider its combination with IMP or AB. As shown in the intuitive proof of Proposition 9 in the previous paragraph, this exclusiveness essentially works to derive the impossibility.

Therefore, from this point forward I will mainly consider WPU or other normative axioms as along with the pair of IMP and AB. The following results are attempts to determine the escape routes from the impossibility of Proposition 9 by substituting WPU for PU. In some settings, in fact, I find a very positive result. Assume $1 \leq k \leq n-2$ and $1 \leq l \leq n-1$.

Proposition 11: The Codomain $\mathfrak{X}^l$

Let $\mathfrak{X} = \mathfrak{X}^l$. If $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP and AB if and only if it is the constant rule con_C for any $C \in \mathfrak{X}^l$.

Proposition 11 states that if the possible number of winners is fixed and we consider the domains introduced, the constant rule is the only (class of) nomination rule that satisfies IMP and AB. This is a generalization of Theorem 3 from Holzman and Moulin (2013) in the sense that Proposition 11 shows that their result holds under any setting I tested.

Proposition 12: The Codomain $\overline{\mathfrak{X}}^l$

Let $\mathfrak{X} = \overline{\mathfrak{X}}^l$ $(1 \leq l \leq n-1)$.

If $\mathfrak{D} = \mathfrak{N}^{self}$, a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ satisfies IMP and AB if and only if it is the constant rule con_X for some $X \in \overline{\mathfrak{X}}^l$.

If $\mathfrak{D} = \mathfrak{N}^{AB}$ and $l \geq 2$, there is no nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ that satisfies NU.

If $\mathfrak{D} = \mathfrak{N}^{AB}$ and $l = 1$, a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ satisfies IMP, AB, and NU if and only if it is φ^1.

If $\mathfrak{D} = \mathfrak{N}$ and $l \geq 3$, there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies NU.

If $\mathfrak{D} = \mathfrak{N}$ and $l = 2$, a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ satisfies IMP, AB, and NU if and only if it is φ^1.

If $\mathfrak{D} = \mathfrak{N}$ and $l = 1$, a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ satisfies IMP, AB, 2CN, WPU, and NU if and only if it is φ^1.

If $\mathfrak{D} = \mathfrak{N}^k$ and $l \geq k+2$, there is no nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}^l$ that satisfies NU.

A nomination rule $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}^l$ where $l \leq k+1$ satisfies IMP, AB, 2CN, NU, and WM if and only if it is the threshold-m rule, where x is an integer such that

$$1 \leq x \leq \left\lceil \frac{nk-(l-1)(n-1)}{n-l+1} \right\rceil \left(= \left\lceil \frac{n(k+1-l)+l-1}{n-l+1} \right\rceil \right)$$

Remark on $(\mathfrak{N}^k, \overline{\mathfrak{X}}^l)$, $l \leq k+1$. Let us briefly evaluate the right-hand side of the inequality we obtain for $(\mathfrak{N}^k, \overline{\mathfrak{X}}^l)$, $l \leq k+1$. To see the numerator, we have $k+1-l \geq 0$ and $l-1 \geq 0$ by $1 \leq l \leq k+1$. The equalities hold if $l = k+1$ and $l = 1$, respectively. Because $k \geq 1$, these conditions do not hold at the same time, which means that at least one is strictly positive. Thus, we have

$$\frac{n(k+1-l)+l-1}{n-l+1} > 0.$$

This implies for all $1 \leq k \leq n-2$ and $1 \leq l \leq k+1$,

$$\left\lceil \frac{n(k+1-l)+l-1}{n-l+1} \right\rceil \geq 1.$$

Thus, we know that φ^1 is the nomination rule that satisfies IMP, AB, 2CN, WM, and NU for all $(\mathfrak{N}^k, \overline{\mathfrak{X}}^l)$, $1 \leq k \leq n-2$ and $1 \leq l \leq k+1$. Furthermore, if $l < k$, then it is also easy to see that

$$\left\lceil \frac{n(k+1-l)+l-1}{n-l+1} \right\rceil \geq 2.$$

This means that φ^2 is also well-defined and satisfies the five axioms.

Proposition 13: The Codomain $\underline{\mathfrak{X}}^l$

Let $\mathfrak{X} = \underline{\mathfrak{X}}^l (1 \leq l \leq n-1)$.

If $\mathfrak{D} = \mathfrak{N}^{self}$, a nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ satisfies IMP and AB if and only if it is the constant rule con_X for some $X \in \underline{\mathfrak{X}}^l$.

If $\mathfrak{D} = \mathfrak{N}^{AB}$, there is no nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ that satisfies WPU. Furthermore, there is no nomination rule that satisfies IMP, AB, and NU.

If $\mathfrak{D} = \mathfrak{N}$, there is no nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ that satisfies WPU. If $l \leq n-2$, there is no nomination rule that satisfies IMP, AB, and NU. If $l = n-1$, a nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ satisfies IMP, AB, and NU if and only if it is $\varphi^1_{-i}: \mathfrak{N} \to \underline{\mathfrak{X}}^l$ (for some $i \in \overline{N}$) defined for any ballot profile $\mathcal{N} \in \mathfrak{N}$, $\varphi^1_{-i}(\mathcal{N}) = \varphi^1(\mathcal{N}) \setminus \{i\}$.

If $\mathfrak{D} = \mathfrak{N}^k$ and $l < k$, there is no nomination rule that satisfies WPU.

Let $\mathfrak{D} = \mathfrak{N}^k$ and $l = k$. If $n = 3$, there is no nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ that satisfies IMP and WPU. If $n \geq 4$, there is a nomination rule that satisfies IMP and WPU. However, there is no nomination rule $\varphi: \mathfrak{D} \to \underline{\mathfrak{X}}^l$ that satisfies IMP, AB, and WPU.

If $\mathfrak{D} = \mathfrak{N}^k$ and $l > k$, there is a nomination rule that satisfies IMP, AB, and WPU (but we cannot further impose 2CN).

Table 5. (Im)Possibilities of Nomination Rules in Various Settings

Dom\Codomain	$\mathfrak{X}^l$	$\overline{\mathfrak{X}}^l$	$\underline{\mathfrak{X}}^l$	$\overline{\mathfrak{X}}(=\overline{\mathfrak{X}}^1)$
$\mathfrak{N}$	IMP and AB⇔ Const.	$l \geq 3$: NU⇒ None. $l = 2$: IMP, AB, and NU⇔φ^1. $l = 1$: IMP, AB, 2CN, NU, and WPU⇔φ^1.	WPU⇒None. $l \leq n-2$: IMP, AB, and NU⇒ None. $l = n-1$: IMP, AB, and NU⇔ φ^1_{-i}.	IMP, AB, 2CN, NU, and WPU⇔ φ^1
$\mathfrak{N}^{self}$	IMP and AB⇔ Const.	IMP and AB⇔ Const.	IMP and AB⇔ Const.	IMP and AB⇔ Const.
$\mathfrak{N}^{AB}$	IMP and AB⇔ Const.	$l \geq 2$: NU⇒ None. $l = 1$: IMP, AB, and NU⇔φ^1.	WPU⇒None. IMP, AB, and NU⇒None.	IMP, AB, and NU⇔φ^1.
$\mathfrak{N}^k$	IMP and AB⇔ Const.	$l \geq k+2$: NU⇒ None. $l \leq k+1$: IMP, AB, 2CN, NU, and WM⇔some threshold.	$l < k$: WPU⇒ None. $l = k$ and $n = 3$: IMP and WPU ⇒None. $l = k$ and $n \geq 4$: IMP and WPU. IMP, AB, and WPU⇒None. $l > k$: IMP, AB, and WPU.	IMP, AB, 2CN, NU, and WM⇔ $\varphi^1, \varphi^2, \ldots, \varphi^k$.

Proposition 12 and Proposition 13 are the results from the cases where the possible number of winners is bounded from below or above, respectively. Although these look complicated, we can see that the impossibility shown in Proposition 9 is relieved by substituting WPU for PU. Indeed, in some of the settings, such as $(\mathfrak{N}^k, \overline{\mathfrak{X}}^l)$ $(l \leq k+1)$, we can see that the threshold rule satisfies many normative axioms. Because $\overline{\mathfrak{X}} = \overline{\mathfrak{X}}^1$ by definition, we have the following situation.

Corollary 7: The Codomain $\mathfrak{X} = \overline{\mathfrak{X}}$

Let $\mathfrak{X} = \overline{\mathfrak{X}}(= \overline{\mathfrak{X}}^1)$.

If $\mathfrak{D} = \mathfrak{N}$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB, 2CN, NU, and WPU if and only if it is φ^1.

If $\mathfrak{D} = \mathfrak{N}^{\mathrm{self}}$, there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP and NU. A nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies I and AB if and only if it is constant.

If $\mathfrak{D} = \mathfrak{N}^{AB}$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB, and NU if and only if it is φ^1

If $\mathfrak{D} = \mathfrak{N}^k$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP, AB, 2CN, NU, and WM if and only if it is the threshold-x rule for some $x \in \{1,2,\dots,k\}$.

Proof of Corollary 7

For $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$, the corollary is obvious from Proposition 12.

For $\mathfrak{D} = \mathfrak{N}^k$, the case of $l = 1 \leq k+2$ in Proposition 12 can be applied. We need only check the upper bound of the threshold. When $l = 1$, we have the following:

$$\left\lceil \frac{nk-(l-1)(n-1)}{n-l+1} \right\rceil = \left\lceil \frac{nk}{n+1} \right\rceil = \left\lceil k - \frac{k}{n+1} \right\rceil = k.$$

The final equality is given by $0 < \frac{k}{n+1} < 1$. ■

Table 5 is an aggregation of the preceding results.

Some comments can be made on the comparative results. First, let us examine the table row-by-row. This comparison is expected to provide a lesson on the choice of domain when a society is given a fixed codomain $\mathfrak{X}$. Take, for example, the domains of $\mathfrak{N}$ and $\mathfrak{N}^{self}$ and recall that they differ only in whether they allow self-approval on the ballots. Let us compare rows $\mathfrak{N}$ and $\mathfrak{N}^{self}$, first comparing $(\mathfrak{N}, \mathfrak{X}^l)$ and $(\mathfrak{N}^{self}, \mathfrak{X}^l)$, then $(\mathfrak{N}, \overline{\mathfrak{X}}^l)$ and $(\mathfrak{N}^{self}, \overline{\mathfrak{X}}^l)$, and finally $(\mathfrak{N}, \underline{\mathfrak{X}}^l)$ and $(\mathfrak{N}^{self}, \underline{\mathfrak{X}}^l)$. Then, although we cannot find a difference in the first comparison, we find that $\mathfrak{N}$ works better than $\mathfrak{N}^{self}$ for the given pair of normative axioms. In the second comparison, if the value of l is sufficiently large, both domains fail to generate nomination rules that satisfy NU. When l is small, however, we can find for $\mathfrak{D} = \mathfrak{N}$ many nomination rules that satisfy IMP and NU as well as the other axioms, while we cannot for $\mathfrak{D} = \mathfrak{N}^{self}$. A similar comparison with the codomain $\mathfrak{X} = \overline{\mathfrak{X}}^l$ also suggests that the performance of $\mathfrak{N}$ is at least as good as $\mathfrak{N}^{self}$ for all l (and indeed the former seems better in some l, i.e., $l = n-1$). Note that I do not intend to imply that the results in the table fully describe the advantages and disadvantages of each domain, nor do I say that the comparison entirely determines the relative normative ranking of each domain. However, I conclude that the comparison has some importance for when we face a domain choice problem in terms of the axiomatic possibility. If the domains are evaluated with respect to the proposed axioms, this table shows that the acceptance of self-inclusion $(\mathfrak{N}^{self})$ or the acceptance of abstention $(\mathfrak{N}^{AB})$ will not improve the situation over the normal domain $\mathfrak{N}$.

Next, let us review the table column-by-column. This corresponds with the situation where a society is, for some reason, given the domain and is seeking a good codomain. Thus, this view is close to Tamura and Ohseto (2014), who study the escape routes from the impossibility by expanding the codomain $\mathfrak{X}^1$ to $\overline{\mathfrak{X}}$. The result shows that the codomain of $\overline{\mathfrak{X}}^l$ when $l = 1$ (in the right column $\overline{\mathfrak{X}}$) works as well as any other codomain. This is very intuitive because $\overline{\mathfrak{X}}$ is the largest codomain of all.

The second note concerns the axiomatic property of the threshold rule. The threshold rule, if properly defined on a certain setting $(\mathfrak{D}, \mathfrak{X})$ where $\mathfrak{D} \neq \mathfrak{N}^{self}$, surely satisfies IMP and AB because one cannot change one's own score and thus one cannot change the possibility of winning oneself (IMP), and the winners are determined through scores (AB). It is also clear that it satisfies 2CN, WPU, and NU[29] if the value of the threshold is between one and the maximum score $M_{\mathfrak{D}}$. It also satisfies the axioms of ND and NE (no exclusion). Recall that the well-known AV method defined on $(\mathfrak{N}, \overline{\mathfrak{X}})$, $(\mathfrak{N}^{self}, \overline{\mathfrak{X}})$, and $(\mathfrak{N}^{AB}, \overline{\mathfrak{X}})$ also satisfies all these axioms. In this sense, the basic structure of the threshold rule has many things in common with AV. The difference between them is the axiom of IMP—the robustness against manipulation.

4.4 Discussion

My main contribution in the previous section is shown in Table 5 (Proposition 11, Proposition 12, and Proposition 13), which systematically shows the extent of the possibilities in a variety of settings, the domain and codomain pairs, and the possible strategies to weaken the impossibility results. The most positive result among these is the characterization of the threshold rule for those settings as $(\mathfrak{N}, \overline{\mathfrak{X}}^1), (\mathfrak{N}, \overline{\mathfrak{X}}^2), (\mathfrak{N}^k, \overline{\mathfrak{X}}^1), \ldots, (\mathfrak{N}^k, \overline{\mathfrak{X}}^{k+1})$ by the combination of IMP, AB, 2CN, NU, and either WPU or WM. Indeed, I show that the threshold rule satisfies other normative axioms referred to in previous studies, such as ND or NE. To conclude the chapter, I give here several comments as well as some extra theorems related mainly to the threshold rule.

(1) Manipulability by More Than one Person

Let us consider the codomain $\mathfrak{X} = \overline{\mathfrak{X}}$, which allows as many winners as possible except the empty set. The setting $(\mathfrak{N}^{self}, \overline{\mathfrak{X}})$ is quite often studied in relation to AV, although in endogenous nominating environments I showed that $(\mathfrak{N}^{self}, \overline{\mathfrak{X}})$ and $(\mathfrak{N}^{AB}, \overline{\mathfrak{X}})$ are less promising than $(\mathfrak{N}, \overline{\mathfrak{X}})$. Tamura and Ohseto (2014) show that the use of $(\mathfrak{N}^1, \overline{\mathfrak{X}})$ is effective in relieving the impossibility. In all these

[29] Recall that my definition of $\mathfrak{N}^{AB}$ ensures that there is at least one individual who has a positive score.

settings except $(\mathfrak{N}^{self}, \overline{\mathfrak{X}})$, we have seen that the threshold rule can be characterized with some of IMP, AB, 2CN, NU, WPU, and WM. A strong concern, however, regarding the threshold-1 rule, φ^1, would be the (extreme) simplicity of its winning condition. Because any voter can win simply by obtaining one approval from the others, this rule might be weak against collusion. For example, suppose two individuals, i and j, promise in advance to approve each other. Then φ^1 will choose both i and j, even if they fail to get any support from the individuals in $\overline{N} \setminus \{i, j\}$. Because IMP guarantees that such mutual approvals between two individuals do not cause them to lose their chance of winning, this type of collusion could be understood as a weakly dominant strategy for all voters. My purpose here is to impose a measure of robustness against this type of collusion; I will first define this robustness against collusion.

Definition 24

A nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies weak 2CP (2-person collusion-proof) if and only if for any distinct individuals $i, j \in \overline{N}$ and for any ballot profile $\mathcal{N} = (N_i, N_j, N_{-i,j}) \in \mathfrak{D}$ and $\mathcal{N}' = (N_i', N_j', N_{-i,j}) \in \mathfrak{D}$, if $i \notin \varphi(\mathcal{N})$ and $j \notin \varphi(\mathcal{N}')$, then $i \notin \varphi(\mathcal{N}')$ or $j \notin \varphi(\mathcal{N}')$.

In other words, two individuals, i and j, will not be better off by forming a two-person coalition. This axiom is weak in the sense that it only excludes the possibility of rules under which two persons can be strictly better off at some profile. The following result provides a basic limit for the design of weak 2CP and Impartial nomination rules.

Proposition 14: Collusion Proof

Let $\mathfrak{X} = \overline{\mathfrak{X}}$.

[1] Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ $(2 \leq k \leq n-3)$ and $n \geq 4$. There is no nomination rule of the setting $(\mathfrak{D}, \mathfrak{X})$ that satisfies IMP, AB, ND, and weak 2CP.

[2] Let $\mathfrak{D} = \mathfrak{N}^k$ $(k = 1)$ and $n \geq 4$. Then, a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}$ exists that satisfies IMP, AB, ND, and weak 2CP. However, there is no nomination rule that satisfies IMP, AB, NU, and weak 2CP.

[3] Let $\mathfrak{D} = \mathfrak{N}^k$ $(k = n-2)$. In this case, the threshold-1 rule φ^1 satisfies all of IMP, AB, 2CN, NU, WPU, ND, and weak 2CP.

Under the axioms IMP and AB, [1] says that we cannot expect ND and weak 2CP at the same time, and [2] shows that the domain of $\mathfrak{N}^1$ is promising, but has the limitation that we cannot have NU and weak 2CP as well as IMP and AB. Interestingly, there is a strong possibility in the threshold-1 rule, φ^1, in [3]: $\mathfrak{D} = \mathfrak{N}^{n-2}$. The reason φ^1 satisfies weak 2CP in this setting can be described as follows. According to the definition, a nomination rule can fail to satisfy weak 2CP only if there is a ballot profile where two individuals, $i, j \in \overline{N}$ for example, have a score of zero. However, $\mathfrak{N}^{n-2}$ does not allow that kind of situation. Consider a score profile $s = (s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n$, where $s_1 = \cdots = s_{n-2} = n-1$ (and $s_{n-1} = k$, and $s_n = 0$, for example). Because $s_1, \dots, s_{n-2}$ is maximal at this score profile, we cannot assign more to the first $n-2$ individuals. This means that the sum of the scores of any two individuals must be at least k. Therefore, if k is almost as large as n, we cannot have two individuals with a score of zero.[30]

(2) Relative Ranking Among the Candidates

My characterization results show the high performance of the threshold rule in terms of IMP and other classical axioms such as anonymity, neutrality, or unanimity. The threshold rule, by definition, determines the winner not by the relative score, but by the absolute score of each individual. As a result, the threshold rule can often yield a much larger number of winners compared with other familiar nomination rules like AV. And it also follows that the rule can choose someone who has the lowest number of approvals from the others. Let me discuss this using an example. Consider a society of 10 individuals $\overline{N} = \{1,2,\dots,10\}$ and a ballot profile $\mathcal{N} = (N_1, \dots, N_{10}) \in \mathfrak{N}$ as

$$N_1 = \{2,3,\dots,10\}$$
$$N_2 = \{3,4,\dots,10\}$$
$$N_3 = \{4,5 \dots,10\}$$
$$\dots$$
$$N_9 = \{10\}$$
$$N_{10} = \{1\}$$

At this ballot profile, each individual gets a score of at least 1, and so $\varphi^1(\mathcal{N}) = \{1,2,3,4,5,6,7,8,9,10\}$, while $\varphi_{AV} = \{10\}$ and $\varphi_P = \{9,10\}$. Furthermore, calculation of the scores in the above ballot profile $\mathcal{N}$ shows that individual 10 earns the maximum score 9, individual 9 earns 8, and so on. Thus, the relative ranking of scores is as follows:

$$9 = s_{10}(\mathcal{N}) > s_9(\mathcal{N}) > \cdots > s_2(\mathcal{N}) = s_1(\mathcal{N}) = 1.$$

Although the scores differ greatly, each individual is not distinguished in the eyes of the threshold-1 rule. The threshold rule is, in this sense, does not discern the relative ranking of the scores. Note that these properties can be problematic for certain contexts, such as the determination of prize-winners. Based on this observation, I consider the existence of nomination rules such that (1) the number of the winners is restricted, and (2) the rule excludes those who have *bad* score rankings. To state the latter formally, let me define a term.

[30] This argument does not fully succeed if $k \leq 1 \Leftrightarrow n = 3$. For complete proof, see the Appendix.

Definition 25

For any individual $i \in \bar{N}$ and ballot profile $\mathcal{N} \in \mathfrak{D}$, I define the score ranking of i at $\mathcal{N}$ as $r_i(\mathcal{N})$, where

$$r_i(\mathcal{N}) := \left|\{j \in \bar{N} \mid s_j(\mathcal{N}) > s_i(\mathcal{N})\}\right| + 1$$

Thus, if $s_i(\mathcal{N})$ is the largest among $s_1(\mathcal{N}), s_2(\mathcal{N}), \dots, s_n(\mathcal{N})$, then $r_i(\mathcal{N}) = 1$. If there is just one individual $j \neq i$ such that $s_j(\mathcal{N}) > s_i(\mathcal{N})$, then we have $r_i(\mathcal{N}) = 2$, and so on.

Definition 26[31]

Let $\varphi: \mathfrak{N} \to \mathfrak{X}^l$ be a nomination rule and let $r \in \{1,2,\dots,n\}$. I say φ has rank r if and only if

$$r \leq r_i(\mathcal{N}) \text{ for all } \mathcal{N} \in \mathfrak{D} \text{ and } i \in \varphi(\mathcal{N})$$

In words, a nomination rule φ is said to have rank r if its winners $\varphi(\cdot)$ are always in the top r^{th} ranking of scores. Let us calculate the rank in the previous example $\mathcal{N}$. According to the definition of $r_i(\mathcal{N})$, we can see that

$$r_{10}(\mathcal{N}) = 1$$
$$r_9(\mathcal{N}) = 2$$
$$r_8(\mathcal{N}) = 3$$
$$\dots$$
$$r_3(\mathcal{N}) = 8$$
$$r_2(\mathcal{N}) = r_1(\mathcal{N}) = 9$$

Thus, a nomination rule of rank $r = 3$, for example, must choose the winner from $\{8,9,10\}$, whose ranking is equal to or less than 3, or $r_i(\cdot) \leq 3$. $1,2 \in \varphi^1(\mathcal{N})$, and so we can say that the rank of φ^1 is 9 or larger in the society of 10 individuals. On the other hand, it is clear that $\varphi_{AV}: \mathfrak{N} \to \mathfrak{X}^1$ (with some tie-breaking rule) has rank 1, though it is not impartial. A natural question arises: is there a rank-based impartial nomination rule? However, the following proposition gives a negative result on the setting $(\mathfrak{N}, \mathfrak{X}^l)$.

Proposition 15

Let $n \geq 3$, $1 \leq l \leq n-1$, and $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$. There is no impartial nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}^l$ that has rank $n-1$.

Proposition 15 says that under the setting $(\mathfrak{N}, \mathfrak{X}^l)$, the concepts of rank and impartiality are entirely incompatible. Although the formal proof is a little complicated, we can easily see that no impartial

[31] The reader might wonder about the looseness of this definition, for if a nomination rule φ has rank $r \in \{1,2,\dots,n-1\}$, then it must be that the rule also has rank $r+1, r+2, \dots, n$. This ambiguity can be omitted by adding the extra condition of "$r = r_i(\mathcal{N})$ for some $\mathcal{N} \in \mathfrak{D}$ and $i \in \varphi(\mathcal{N})$" for the definition of rank. For the sake of simplicity I omit the uniqueness because it is unnecessary for stating the result of Proposition 15.

nomination rule on the setting $(\mathfrak{N}, \mathfrak{X}^l)$ has rank less than $n-2$. To demonstrate this, consider a ballot profile $\mathcal{C}^1 = (C_1^1, \dots, C_n^1) \in \mathfrak{N}$. $\phi \notin \mathfrak{X}^l$, and so there is a winner $i \in \varphi(\mathcal{C}^1)$. Now consider a ballot profile $\mathcal{N} \in \mathfrak{N}$ as

$$N_i = \overline{N} \setminus \{i\}$$
$$N_j = C_j^1 \text{ for all } j \in \overline{N} \setminus \{i\}.$$

IMP demands $i \in \varphi(\mathcal{N})$. However, because $s_i(\mathcal{N}) = s_{\overline{\iota+1}}(\mathcal{N}) = 1 < s_\mu(\mathcal{N}) = 2$ for all $\mu \in \overline{N} \setminus \{i, \overline{\iota+1}\}$, the score ranking of individual i at this ballot profile $\mathcal{N}$ is such that $r_i(\mathcal{N}) = n-1$. Thus, we cannot avoid choosing a winner at $n-1$ (or more). The proposition says that there exists a ballot profile for which the impartial nomination rule chooses the individual with the worst score ranking.

5 Conclusion

In this dissertation, I study the infinite regress problem in collective decision making. I give here a brief summary of each chapter and some additional comments.

In Chapter 2, I introduced the notion of (weak/strong) convergence, which I regard as a basic solution concept for the infinite regress of procedural choice, and studied its basic performance on a menu of three scoring rules. Specifically, Theorem 4 states that a large society with the menu {plurality (P), Borda (B), anti-plurality (A)} can almost always find a strong convergence unless it is in a trivial deadlock. Further regress has no effective meaning in each case, and so an interpretation of the theorems in the chapter is as follows: the infinite regress problem degenerates in such a society and, moreover, for the menu $\{\mathrm{P}, \mathrm{B}, \mathrm{A}\}$ and for a set of three alternatives X, its probability of convergence (98.2% under IC and 98.8% under IAC) is shown to be much higher than that of stability (84.49% under IC and 84.10% under IAC). Although trivial deadlock gives no specific answers to the problem of infinite regress, the probability calculation shows the positive effect of considering my convergence notion.

The results in Chapter 2 show that trivial deadlock *can* happen, with a small but positive probability for a large society with the menu $\{\mathrm{P}, \mathrm{B}, \mathrm{A}\}$. This problem motivates my analysis in Chapter 3, which focuses on finding a menu of voting rules with which a society can always find convergence: in my words, menus with the weak/strong convergent property and asymptotically weak/strong convergent property. In the first part of Chapter 3, I investigate the possibility of each property. Specifically, I question if there exists a menu of voting rules that have the strong convergent property, the strongest of the four, and find the answer to be yes (Theorem 7). Such a menu completely releases any society (of any finite size) from the troublesome infinite regress problem.

One problem concerning Theorem 7 might be that the proposed menu is made up of somewhat technical (and not intuitive) voting rules. In the latter part of Chapter 3, therefore, I consider how the convergent property can be obtained for a given menu that does not already have this property such as $\{\mathrm{P}, \mathrm{B}, \mathrm{A}\}$. My Theorem 8 shows that the expanded menu $\{\mathrm{P}, \mathrm{B}, \mathrm{A}, \varphi\}$ has the asymptotically weak convergent property. This means that a society with the menu $\{\mathrm{P}, \mathrm{B}, \mathrm{A}\}$ can acquire the convergent property without abandoning any of P, B, or A. The society has only to add an extra voting rule as an alternative rule. Indeed, such expansion is shown to be possible for many cases (Theorem 9).

To sum up, I find two answers to the question of how to find a convergence. One answer is to equip the society with the menu proposed in Theorem 7. The other, oriented toward a large society, is to expand the menu in the way shown in Theorem 9.

Having discussed the frequency and the mechanism of the convergence phenomenon, I will add some comments about it. My first comment is on how convergence works in a real situation using the example introduced in Chapter 2. Let me show it again here:

- The society is $N = \{1,2,\dots,14\}$
- The set of alternatives is $X = \{a,b,c\}$.
- The level-0 preference profile L^0 is $L^0_{1-10}: abc$ and $L^0_{11-14}: bca$.

Although the outcomes of the first-level SCRs are not the same ($f_P(L^0) = f_B(L^0) = a$ and $f_A(L^0) = b$), the profile L^0 strongly converges to $\{a\}$. This means that

- If the level-1 preference profile L^1 is, for instance, $L^1_{1-4}: PBA$, $L^1_{5-10}: BPA$, and $L^1_{11-14}: APB$, then any level-2 SCR in the menu ultimately results in $\{a\}$.
- No matter what other CI profiles are examined, they do not weakly converge to $\{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}$, or $\{a,b,c\}$.

In the process of finding the convergence, we need to take an appropriate sequence of CI profiles. The fact that L^0 strongly converges to $\{a\}$ does not claim that if each $i \in N$ reports their consequential meta preferences independently, then L^0 *necessarily* converges at some level. It is of course possible that such non-systematic reports never reach convergence. Rather, the fact that L^0 strongly converges to $\{a\}$ means that if L^0 converges, the outcome couldn't be other than $\{a\}$. So, in one sense, the society faces two options: to accept the outcome of strong convergence or to get into the entangled infinite regress without finding an answer. In other words, if everyone agreed to the process in which the authority picked up appropriate voters' meta-level preferences from the submitted level-0 preference profile (i.e. if the authority were admitted to manipulate only the indifference part of the consequentially-induced *weak* preference profile), then the convergence could be found. In this sense, once accepted, the notion of convergence tells us the possible outcome that could be reached from the submitted level-0 preference profile. Thus, each individual has only to submit their preferences over the original set X just as required in an ordinary voting procedure.

As my second comment, I would like to state the mechanism of convergence with respect to a formal description of the infinite regress problem in procedural choice. Gratton (2009) formally states that an infinite regress argument is made up of two propositions: the regress formula, a universal proposition that can be endlessly instantiated, and the triggering statement.[32] Borrowing from Gratton, an example of two such propositions is: "every intelligent act is preceded by an intelligent act" (regress formula) and "act 1 is intelligent" (triggering statement). With the repeated use of the regress formula, we have that "act 1 is preceded by act 2", "act 2 is preceded by act 3", and so on. Using his words, the infinite regress of procedural choice can be described by two propositions "for all level $k \in \mathbb{N}$, a level-k voting procedure is shown to be legitimate if it is selected by a level-$(k+1)$ legitimate voting procedure" (regress formula) and

[32] Technically speaking, Gratton provides some hypothesis for the condition that such argument is truly an infinite regress argument in the subsequent argument.

“we (hope to) show that a level-k^* voting rule f is legitimate”. At first glance, this pair of propositions demonstrates an infinite regress. However, according to Chapters 2 and 3, the regress get degenerated within finite levels, if we restrict our attention to a set of a few voting rules, say $\{\mathrm{P}, \mathrm{B}, \mathrm{A}\}$, and if we have good reasons to regard the voters as consequentialist. Indeed, Corollary 1 and Theorem 4 state that the probability of convergence is quite high (when there are three alternatives). We humans cannot verify the infinitely long process of justification, but convergence says that no matter which rule in a higher level is selected, its ultimate outcome is uniquely determined within finite steps, the proof of which I have shown. Indeed, in the profile L^0 upon which I based my argument in the previous paragraph, I find such a phenomenon at level 1. The convergence is thus a phenomenon that solves the infinite regress within finite levels of arguments.

Subsequently, in Chapter 4, I discuss the axiomatic design of nomination rules:

$$\varphi: (N_1, N_2, \dots, N_n) \mapsto \varphi(N_1, N_2, \dots, N_n) \subseteq \bar{N}.$$

When each individual is a candidate as well as a voter and they want to be chosen themselves, then voters may be inclined to cast ballots that can make themselves better off. Impartiality (IMP), invented by Holzman and Moulin (2013), is an axiom of nomination rules that demands that each individual cannot change his or her own result even by manipulating his or her ballot. My analysis in Chapter 4 aims to find some escape routes from Holzman and Moulin’s impossibility by considering various typical domains and codomains of the nomination rules. I first specify the common structure of nomination rules under various settings (Lemma 7), and then I investigate the design possibility of nomination rules for each setting (Proposition 11, Proposition 12, Proposition 13, and Table 5). The results indicate that the threshold rule works well in many settings in terms of IMP, anonymity, neutrality, and unanimity. In other words, we can acquire impartiality and satisfy other popular axioms by using the objective score of each individual (i.e., whether they reach a set threshold) instead of the relative score (i.e., who gets the highest score, as in AV).

As I suggest at the beginning of Chapter 4, the axiomatic study is oriented to the determination of the society N (i.e., who should have the right to vote) before the procedural choice is made. In environmental issues, for instance, the boundary of the effects of a decision is sometimes vague, and hence there may sometimes be no ex ante answer for the question of whose opinions should be reflected in the decision-making. The framework of nomination rules can be applied to such cases if we interpret N_i as the set of individuals who $i \in \bar{N}$ thinks should have the right to vote and φ as the aggregation rule for people’s ballot profiles $(N_1, N_2, \dots, N_n)$. From his premise of Procedural Autonomy, Dietrich (2005) derives anonymity (precisely speaking, anonymous procedural submission), neutrality, and monotonicity as the basic axioms that should be satisfied by the manner of procedural choice. Indeed, if the determination of the society is made before the procedural choice is made (and therefore, there are no persons or alternatives that have some kind of dominance over others) consideration of anonymity and neutrality is largely noncontroversial because their very demand is that each individual and alternative must be treated equally. Monotonicity (or unanimity as a weaker axiom) is also natural to impose because the determination of a society is supposed to reflect individuals' opinions properly. IMP is a rather empirical, but also rational, axiom that demands that each individual can record his or her true opinion without fearing ruling him- or herself out from the determined society. Therefore, the threshold rule, which I show satisfies these axioms in various domain-codomain settings, can be regarded as the most appropriate way of determining a society.

To conclude, the dissertation consists of two main parts: Chapters 2 and 3 (the first part) and Chapter 4 (the second part). The first part studies the question of "how to determine *how* to choose based on people's preferences" while the second part studies the question of "how to determine *who* should form the society." In each part, I provide answers for each of the "how" and "who" problems with some underlying assumptions. Although these two problems comprise the essential parts of procedural choice, there are additional components of the issue of procedural choice that are of interest, such as the choice of the decision problem itself (Kesting & Lindstädt, 2004). For example, considering the whole process of choosing a constitution could be an interesting future study.

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Appendix

Probability of Trivial Deadlock Under the 56 Menus

When $n \to \infty$, $|X| = 3$, IAC, the probability of trivial deadlock can be calculated in the way described by Diss et al. (2012). Here I show the probability of trivial deadlock under the 56 menus cited in Theorem 6. Figure 6 shows the result. The horizontal axis shows the 56 menus, numbered from 1 to 56 as described in Table 6, and the vertical axis shows the probability of deadlock. Note that each of the 56 menus yields at most a probability of $0.035 = 3.5\%$ of trivial deadlock (the highest is actually 3.35648% in the menu $\{f_P, f_A, f_H\}$; Data number 2). In other words, the probability of weak convergence is at least $100\% - 3.5\% = 96.5\%$ in all menus.

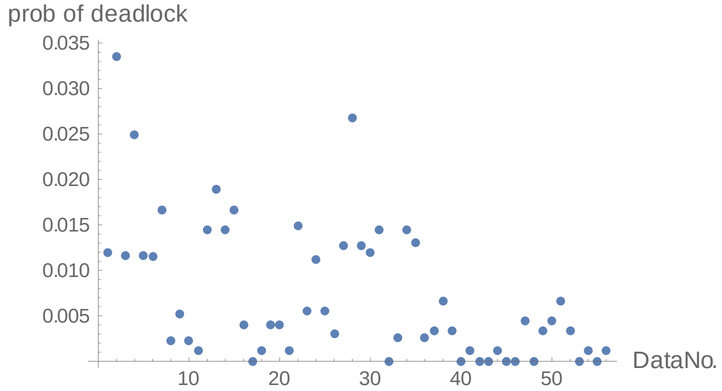

Figure 6. Probability of trivial deadlock under the 56 menus when $|X| = 3$.

The specific ingredients in and probability of each menu is shown in Table 6 below.

Table 6. Specific Values for Each Menu; Data Numbers 1–28 (continued on next page)

Data number	Menu			Prob of trivial deadlock
1	plurality	anti-plurality	Borda	1/84
2	plurality	anti-plurality	Hare	29/864
3	plurality	anti-plurality	Nanson	5/432
4	plurality	anti-plurality	Coomb	43/1728
5	plurality	anti-plurality	Maximin	5/432
6	plurality	anti-plurality	Black	199/17280
7	plurality	Borda	Hare	115/6912
8	plurality	Borda	Nanson	1/432
9	plurality	Borda	Coomb	1/192
10	plurality	Borda	Maximin	1/432
11	plurality	Borda	Black	1/864
12	plurality	Hare	Nanson	25/1728
13	plurality	Hare	Coomb	131/6912
14	plurality	Hare	Maximin	25/1728
15	plurality	Hare	Black	115/6912
16	plurality	Nanson	Coomb	7/1728
17	plurality	Nanson	Maximin	0
18	plurality	Nanson	Black	1/864
19	plurality	Coomb	Maximin	7/1728
20	plurality	Coomb	Black	7/1728
21	plurality	Maximin	Black	1/864
22	anti-plurality	Borda	Hare	241/16128
23	anti-plurality	Borda	Nanson	67/12096
24	anti-plurality	Borda	Coomb	17/1512
25	anti-plurality	Borda	Maximin	67/12096
26	anti-plurality	Borda	Black	181/60480
27	anti-plurality	Hare	Nanson	11/864
28	anti-plurality	Hare	Coomb	185/6912

Table 6: Specific Values for Each Menu; Data Numbers 29–56 (continued from previous page)

29	anti-plurality	Hare	Maximin	11/864
30	anti-plurality	Hare	Black	413/34560
31	anti-plurality	Nanson	Coomb	25/1728
32	anti-plurality	Nanson	Maximin	0
33	anti-plurality	Nanson	Black	11/4320
34	anti-plurality	Coomb	Maximin	25/1728
35	anti-plurality	Coomb	Black	113/8640
36	anti-plurality	Maximin	Black	11/4320
37	Borda	Hare	Nanson	23/6912
38	Borda	Hare	Coomb	23/3456
39	Borda	Hare	Maximin	23/6912
40	Borda	Hare	Black	0
41	Borda	Nanson	Coomb	1/864
42	Borda	Nanson	Maximin	0
43	Borda	Nanson	Black	0
44	Borda	Coomb	Maximin	1/864
45	Borda	Coomb	Black	0
46	Borda	Maximin	Black	0
47	Hare	Nanson	Coomb	31/6912
48	Hare	Nanson	Maximin	0
49	Hare	Nanson	Black	23/6912
50	Hare	Coomb	Maximin	31/6912
51	Hare	Coomb	Black	23/3456
52	Hare	Maximin	Black	23/6912
53	Nanson	Coomb	Maximin	0
54	Nanson	Coomb	Black	1/864
55	Nanson	Maximin	Black	0
56	Coomb	Maximin	Black	1/864

Proof of Proposition 1

Suppose L^0 is in a trivial deadlock under the menu $F = \{f_1, f_2, \dots, f_m\}$. I show the following statement to be true:

> For all $k \geq 2$ and for all sequence $L^0, L^1, \dots, L^{k-1}$ of CI profiles to level k, the class of each $f_1^k, f_2^k, \dots, f_m^k$ with respect to $L^0, L^1, \dots, L^{k-1}$ is a distinct singleton.

(Note that the proposition is straightforward once this statement is proven.) I prove the statement by an induction on $k \geq 2$. For $k = 2$, the statement is straightforward from the definition of trivial deadlock. Let $k_0 \geq 2$ and suppose that the statement holds until $k \leq k_0$. Take any sequence of CI profiles $L^0, L^1, \dots, L^{k_0-1}$. By the assumption of the induction, each $f_1^{k_0}, f_2^{k_0}, \dots, f_m^{k_0}$ is assumed to result in a distinct singleton. This can be denoted as

$$f_p^{k_0}(L^{k_0-1}) = \left\{f_{\sigma(p)}^{k_0-1}\right\} \text{ for all } p \in \{1,2,\dots,m\}.$$

Now, $\sigma: \{1,2,\dots,m\} \to \{1,2,\dots,m\}$ defined in this way is clearly a bijection. By the assumption of the induction again, each SCR in $F^1, F^2, \dots, F^{k_0}$ yields a singleton at the given profile. Hence, their classes are also singleton subsets of X. Take any $f_p^{k_0}, f_q^{k_0} \in F^{k_0}$ and denote their classes as $\{x\}$ and $\{y\}$, respectively $(x, y \in X)$. By the definition of a CI profile, we have that for all $i \in N$, and for all $L^{k_0} \in \mathcal{L}[L^0, L^1, \dots, L^{k_0-1}]$,

$$\begin{aligned} & f_p^{k_0} L_i^{k_0} f_q^{k_0} \\ & \Leftrightarrow \{x\} e_i\left(L_i^0\right) \{y\} \\ & \Leftrightarrow x L_i^0 y \ \ (\because \text{Extension rule}). \end{aligned}$$

Therefore, the level-k_0 CI profile L^{k_0} is uniquely determined. Moreover, it is clear that the classes of $f_{\sigma(p)}^{k_0-1}$ and $f_{\sigma(q)}^{k_0-1}$ are also $\{x\}$ and $\{y\}$, respectively. Similarly, we have that

$$f_{\sigma(p)}^{k_0-1} L_i^{k_0-1} f_{\sigma(q)}^{k_0-1} \Leftrightarrow x L_i^0 y.$$

In summary, we have that $f_p^{k_0} L_i^{k_0} f_q^{k_0} \Leftrightarrow f_{\sigma(p)}^{k_0-1} L_i^{k_0-1} f_{\sigma(q)}^{k_0-1}$. $f_p^{k_0}$ and $f_q^{k_0}$ are arbitrary elements in F^{k_0}, and so this logical equivalence implies that L^{k_0} is a permutated profile from L^{k_0-1} by σ.

Because of the neutrality of the menu, we have that $f_r^{k_0+1}(L^{k_0}) = \left\{f_{\sigma(r)}^{k_0}\right\}$ for all $r \in \{1,2,\dots,m\}$.

σ is a bijection, and so this guarantees the statement when $k = k_0 + 1$. ■

Proof of Lemma 2

Suppose $n\alpha \in \mathbb{N}$.

[Under IC] All the alternatives are treated symmetrically in IC, and so each voter prefers x to y with probability $\frac{1}{2}$ (and y to x with probability $\frac{1}{2}$). Therefore, we have:

$$P(\alpha) = \binom{n}{n\alpha}\left(\frac{1}{2}\right)^{n\alpha}\left(\frac{1}{2}\right)^{n(1-\alpha)} = \binom{n}{n\alpha}\left(\frac{1}{2}\right)^n.$$

Because the proofs are similar, I show the proof only for even n. Let $n = 2p$ $(p \in \mathbb{N})$. Because of the property of combination, we can evaluate this as follows:

$$P(\alpha) \leq P\left(\frac{1}{2}\right) = \binom{2p}{p}\left(\frac{1}{2}\right)^{2p} = \frac{(2p)!}{p!\,p!}\left(\frac{1}{2}\right)^{2p}.$$

Using Stirling's approximation, we can evaluate the right-hand side as

$$\lim_{\substack{p\to\infty \\ (\Leftrightarrow n\to\infty)}} \frac{(2p)!}{p!\,p!}\left(\frac{1}{2}\right)^{2p} = \lim_{p\to\infty} \frac{\sqrt{2\pi\cdot 2k}\left(\frac{2p}{e}\right)^{2p}}{\left(\sqrt{2\pi p}\left(\frac{p}{e}\right)^{p}\right)^{2}}\left(\frac{1}{2}\right)^{2p} = \lim_{p\to\infty}\frac{1}{\sqrt{\pi n}} = 0.$$

[Under IAC] Let $a = \#\{i \in N \mid xL_i^0 y\} = n\alpha$ and $b = n - a$. The probability is described as:

$$P(\alpha) = \binom{a + \frac{|X|!}{2} - 1}{a} \cdot \binom{b + \frac{|X|!}{2} - 1}{b} \Big/ \binom{a + b + |X|! - 1}{a + b}.$$

With a simple calculation, this is shown to converge to zero as $n = a + b \to \infty$. ■

Proof of Lemma 3

Assume that $F = \{g_1, \dots, g_p, h_1, \dots, h_q\}$ and $L^0, L^1, \dots, L^{k-1}$ satisfy the given condition. Let $A = \{1,2,\dots,a\} = \{i \in N \mid xL_i^0 y\}$. If $q = 0$, the lemma is obvious. So, we assume $p \geq q > 0$. It follows that $0 < |A| = a < n$ (if $a = 0$, e.g., no level-1 SCR chooses $\{x\}$, which contradicts $p > 0$). $n \geq m$, and so we have $a \geq (n/2) \geq (m/2) \geq q$. Let $L^k \in \mathcal{L}(F^k)^n$ be defined as follows:

$$L_i^k: g_1^k, g_2^k, \dots, g_p^k, h_1^k, h_2^k, \dots, h_{i-1}^k, h_{i+1}^k, \dots, h_q^k, h_i^k \text{ for all } 1 \leq i \leq q.$$
$$L_i^k: g_1^k, g_2^k, \dots, g_p^k, h_1^k, h_2^k, \dots, h_q^k \text{ for all } q + 1 \leq i \leq a.$$
$$L_i^k: h_1^k, h_2^k, \dots, h_q^k, g_1^k, g_2^k, \dots, g_p^k \text{ for all } i \in N \setminus A.$$

In words, this is a level-k profile where everyone (except the first q individuals) orders $\{g_1^k, \dots, g_p^k\}$ and $\{h_1^k, \dots, h_q^k\}$ lexicographically. Clearly, we have $L^k \in \mathcal{L}^k[L^0, \dots, L^{k-1}]$. Take any $f^{k+1}: [1 = s_1, s_2, \dots, s_m = 0] \in F^{k+1}$ and consider the scores evaluated by this f^{k+1}. Note that h_1^k has the largest score among $h_1^k, \dots, h_q^k$. We have:

$$\begin{aligned} s(g_1^k) - s(h_1^k) &= \{a + (n-a)s_{q+1}\} - \{n - a + (a-1)s_{p+1}\} \\ &\geq 2a - n + (n-a)s_{q+1} - (a-1)s_{q+1} \ \ (\because p \geq q \Rightarrow s_{q+1} \geq s_{p+1}) \\ &= (2a - n)(1 - s_{q+1}) + s_{q+1} > 0 \ \ (\because 2a > n \text{ and } 0 \leq s_{q+1} \leq 1). \end{aligned}$$

This holds for any $f^{k+1} \in F^{k+1}$, and so the profile weak convergences to $\{x\}$. ■

Proof of Lemma 4

Let $A = \{1,2,\dots,a\} = \{i \in N \mid xL_i^0 y\}$, $G \coloneqq \{g \mid C_g = \{x\}\} = \{g_1^k, \dots, g_p^k\}$ $(p = |G|)$ and $H \coloneqq \{h \mid C_h = \{y\}\} = \{h_1^k, \dots, h_q^k\}$ $(q = |H|)$. With Lemma 3, we have only to consider $0 < a < n - a$ and $p > q > 0$ (i.e., $(p,q) = (2,1)$ if $m = 3$ or $(p,q) = (3,1)$ if $m = 4$). Because the proofs are similar, I show only the proof for the latter, $m = 4$. We can check that for all $L^k \in \mathcal{L}^k[L^0, \dots, L^{k-1}]$,

$f_{E_1}(L^k) \subseteq H$ and the scores (at L^k) satisfy:

$$S \coloneqq s_B(g_1^k) + s_B(g_2^k) + s_B(g_3^k) = a(s_1 + s_2 + s_3) + (n-a)(s_2 + s_3 + s_4) = n + a.$$

Let $p_1, \ldots, p_6$ be preferences over G such that $p_1: g_1^k g_2^k g_3^k$, $p_2: g_3^k g_2^k g_1^k$, $p_3: g_3^k g_1^k g_2^k$, $p_4: g_2^k g_1^k g_3^k$, $p_5: g_1^k g_3^k g_2^k$, and $p_6: g_2^k g_3^k g_1^k$. We construct $L^k \in \mathcal{L}^k[L^0, \ldots, L^{k-1}]$ as follows: if $i \equiv j \pmod 6$ then $L_i^k \big|_G = p_j$ $(j = 1,2,\ldots,6)$, and $g_\mu^k L_i^k h_1^k$ $(\mu = 1,2,3) \Leftrightarrow i \leq a$. Because of the symmetry, we obtain that $s_B(g_j^k : L^k) - S/3 \in \{-1/3, 0, 1/3\}$ $(j = 1,2,3)$. Hence:

$$D(L^k) \coloneqq s_B(h_1^k : L^k) - \max\{s_B(g_1^k : L^k), s_B(g_2^k : L^k), s_B(g_3^k : L^k)\} \geq \frac{2}{3}(n - 2a) - \frac{1}{3}.$$

$n - 2a \geq 1$, and so we have $D(L^k) > 0$.

(1) In the case of $n - 2a \geq 2$, we have $D(L^k) \geq 1$. Suppose $\{g, h_1^k\} \in f_{E_{j'}}(L^k)$ for some $g \in G$ and $j' = 2,3$. Let j be the smallest such j'. $s_j(h_1^k) = n - a < n$, and so there exists $i_g \in N$ whose $L_{i_g}^k$ assign zero points to g and one point to $g' \in G \setminus \{g\}$. Now let L'^k be a profile where i_g swaps g and g'. Then we have $s_j(g : L'^k) > s_j(g : L^k) = s_j(h_1^k : L^k) > s_j(h_1^k : L'^k)$. Therefore, $f_{E_j}(L'^k) = \{g\} \subseteq G$. The change in Borda score of g_1^k, g_2^k, g_3^k is at most $2/3$, and so we still have $D(L'^k) \geq 1 - (2/3) > 0$.

(2) In the case of $n - 2a = 1$, because n is odd, we can write $n = 6\mu + \nu$, where $\mu \in \mathbb{N} \cup \{0\}$ and $\nu = 1,3,5$. Note that the swap of $L_i^k \big|_G$ and $L_j^k \big|_G$ for any $i, j \in N$ does not affect $s_1(\cdot)$ and $s_B(\cdot)$. If $n = 6\mu + 1$ ($\mu \geq 1$ because $n \geq m = 4$), let $\left(\mathcal{L}^{(1)}\right)^n \in \mathcal{L}^k[L^0, \ldots, L^{k-1}]$ be defined as: $1 \leq i \leq \mu \Rightarrow {L^{(1)}}_i^k : p_3$, $\mu + 1 \leq i \leq 2\mu \Rightarrow {L^{(1)}}_i^k : p_4$, $2\mu + 1 \leq i \leq 3\mu \Rightarrow {L^{(1)}}_i^k : p_5$, $3\mu + 1 \leq i \leq 4\mu \Rightarrow {L^{(1)}}_i^k : p_1$, $4\mu + 1 \leq i \leq 5\mu \Rightarrow {L^{(1)}}_i^k : p_2$, $5\mu + 1 \leq i \leq 6\mu \Rightarrow {L^{(1)}}_i^k : p_6$, and $i = 6\mu + 1 \Rightarrow {L^{(1)}}_i^k : p_1$. Then we have $s_3\left(g_1^k : {L^{(1)}}^k\right) \geq s_2\left(g_1^k : {L^{(1)}}^k\right) = 3\mu + 2 > 3\mu + 1 = s_2\left(h_1^k : {L^{(1)}}^k\right) = s_3\left(h_1^k : {L^{(1)}}^k\right)$. It follows that $f_{E_2}^{k+1}\left({L^{(1)}}^k\right) \subseteq G$ and $f_{E_3}^{k+1}\left({L^{(1)}}^k\right) \subseteq G$. For the other cases of $n = 6\mu + 3$ and $n = 6\mu + 5$, the following ${L^{(2)}}^k$ (g_3^k wins) and ${L^{(3)}}^k$ (g_1^k wins), respectively, give the corresponding inequalities.

${L^{(2)}}^k$ is defined as: $1 \leq i \leq \mu \Rightarrow p_4$, $\mu + 1 \leq i \leq 2\mu \Rightarrow p_5$, $2\mu + 1 \leq i \leq 3\mu \Rightarrow p_6$, $i = 3\mu + 1 \Rightarrow p_1$, $3\mu + 2 \leq i \leq 4\mu + 1 \Rightarrow p_1$, $4\mu + 2 \leq i \leq 5\mu + 1 \Rightarrow p_2$, $5\mu + 2 \leq i \leq 6\mu + 1 \Rightarrow p_3$, $i = 6\mu + 2 \Rightarrow p_2$, and $i = 6\mu + 3 \Rightarrow p_3$.

${L^{(3)}}^k$ is defined as: $1 \leq i \leq \mu \Rightarrow p_2$, $\mu + 1 \leq i \leq 2\mu \Rightarrow p_3$, $2\mu + 1 \leq i \leq 3\mu \Rightarrow p_4$, $i = 3\mu + 1 \Rightarrow p_3$, $i = 3\mu + 2 \Rightarrow p_4$, $3\mu + 3 \leq i \leq 4\mu + 2 \Rightarrow p_1$, $4\mu + 3 \leq i \leq 5\mu + 2 \Rightarrow p_5$, $5\mu + 3 \leq i \leq 6\mu + 2 \Rightarrow p_6$, $i = 6\mu + 3 \Rightarrow p_1$, and $i = 6\mu + 4 \Rightarrow p_2$, and $i = 6\mu + 5 \Rightarrow p_5$.

In either case above, at least 2 level-$(k+1)$ SCRs have class $\{x\}$ and the other two have either $\{x\}$ or $\{y\}$. So, we can apply Lemma 3 to get the weak convergence. ■

Proof of Lemma 5

Let $A = \{1,2,\ldots,a\} = \{i \in N \mid xL_i^0 y\}$. Assume that both a and $n-a$ are odd. (The cases where at least one of them is even can be similarly, and more simply, proven.) Note that the fact that $C_{g_1^k} = \{x\}$ and $C_{g_3^k} = \{y\}$ guarantees that $a > 0$ and $n - a > 0$.

Let $L^k \in \mathcal{L}(F^k)^n$ be such that $g_1^k L_i^k g_2^k L_i^k g_3^k$ for all i: $1 \le i \le \frac{a}{2} + \frac{1}{2}$, $g_2^k L_i^k g_1^k L_i^k g_3^k$ for all i: $1 \le i \le \frac{a}{2} - \frac{1}{2}$, $g_3^k L_i^k g_2^k L_i^k g_1^k$ for all i: $1 \le i \le \frac{n-a}{2} - \frac{1}{2}$, and $g_3^k L_i^k g_1^k L_i^k g_2^k$ for all i: $1 \le i \le \frac{n-a}{2} + \frac{1}{2}$. Clearly, $L^k \in \mathcal{L}^k[L^0, L^1, \ldots, L^{k-1}]$. We have also that:

$$\left|s(g_1^k) - s(g_2^k)\right| = |(1-s) - s| = |1 - 2s|.$$

The assumption of $0 \le s \le 1$ indicates that this absolute value is at most one. ■

Proof of Theorem 3

As I stated in section 2.4, the probability of a tied outcome is negligible as $n \to \infty$. So, we can consider the case where every level-1 SCR chooses a singleton subset of X. If $f_1^1(L^0) = f_2^1(L^0) = f_3(L^0)$, weak convergence is straightforward. If each $f_1^1(L^0), f_2^1(L^0), f_3^1(L^0)$ is a distinct singleton, L^0 is in trivial deadlock. Therefore, the only nontrivial case is that in which two level-1 SCRs choose $\{x\}$ and the other one chooses $\{y\}$, where $x, y \in X$.

Let $A = \{i \in N \mid xL_i y\}$ and $\alpha := |A|/n$. Let us label them as $g_1^1(L^0) = g_2^1(L^0) = \{x\}$ and $g_3^1(L^0) = \{y\}$, where $F^1 = \{g_1^1, g_2^1, g_3^1\}$. Due to Lemma 3, we need only consider $\alpha < 1/2$. Take any $f\colon [1, s, 0] \in F^2$. With Lemma 5, we have the following:

$$s(g_3^1 \colon L^1) = n - |A|, \max_{L^1 \in \mathcal{L}[L^0]} s(g_1^1 \colon L^1) = |A| + s(n - |A|)$$

$$\min_{L^1 \in \mathcal{L}[L^0]} \max\{\{s(g_1^1 \colon L^1), s(g_2^2 \colon L^1)\}\} \le \frac{1}{2}\{|A|(1+s) + (n - |A|)s\} + \frac{1}{2}.$$

Therefore, f can choose $\{g_1^1\}$ (or $\{g_2^1\}$) if and only if:

$$|A| + s(n - |A|) > n - |A| \Leftrightarrow s > \frac{n - 2|A|}{n - |A|} = \frac{1 - 2\alpha}{1 - \alpha} = \varphi(\alpha).$$

Also, f can choose $\{g_3^1\}$ if:

$$\frac{1}{2}\{|A|(1+s) + (n - |A|)s\} + \frac{1}{2} < n - |A|$$

$$\Leftrightarrow s < 2 - \frac{3|A|}{n} - \frac{1}{n} = 2 - 3\alpha - \frac{1}{n} (\to 2 - 3\alpha = \psi(\alpha) \text{ as } n \to \infty).$$

If $\alpha < 1/3$, we have $\psi(\alpha) > 1$. Thus, any scoring SCR $f\colon [1, s, 0]$ can choose $\{g_3^1\}$. If $1/3 < \alpha < 1/2$, we have three cases. (Note that events such as $\alpha = 1/3$ or $\psi(\alpha) - 1/n < s < \psi(\alpha)$ are negligible because of Lemma 2.)

1) The case of $s_3 \ge \varphi(1/3) = 1/2$. In this case, each f_1^2, f_2^2, f_3^3 can exclude g_3^1 for any $\alpha \in (1/3, 1/2)$.

2) The case of $s_3 < \varphi(1/3)$ and $s_2 \leq \psi(\varphi^{-1}(s_3))$. Note that the event $\alpha = \varphi^{-1}(s_3)$ is negligible because of Lemma 2. In this case, if $1/3 < \alpha < \varphi^{-1}(s_3)$, we have $\psi(\alpha) > s_2$, which implies that $L^1 \in \mathcal{L}^1[L^0]$ exists such that $f_2^2(L^1) = f_3^2(L^1) = \{g_3^1\}$ and $f_1^2(L^0)$ is either $\{g_1^1\}$ or $\{g_3^1\}$. In either case, L^0 is shown to weakly converge to $\{y\}$. If $\varphi^{-1}(s_3) < \alpha < 1/2$, $L^1 \in \mathcal{L}[L^0]$ exists such that $f_1^2(L^1) = f_2^2(L^1) = f_3^2(L^1) = \{g_1^1\}$.

3) The case of $s_3 < \varphi(1/3)$ and $s_2 > \psi(\varphi^{-1}(s_3))$. In this case, an interval of α (with a positive Lebesgue measure) exists where f_1^1 and f_2^1 necessarily choose $\{g_1^1\}$ or $\{g_2^1\}$ and f_3^2 necessarily chooses $\{g_3^1\}$. If α is in this interval, we cannot solve the regress, because inductively we can show for all $k \geq 3$ that $f_1^k(L^{k-1})$ and $f_2^k(L^{k-1})$ are either $\{f_1^{k-1}\}$ or $\{f_2^{k-1}\}$ and $f_3^k(L^{k-1}) = \{f_3^{k-1}\}$. ■

Proof of Corollary 1

Under IC, trivial deadlock corresponds with cases 1, 2, 9, 10, 11, and 27 in Diss and Merlin (2010). Their Table 7 (p. 302) shows that each probability is 0.00299346. Therefore, $p_D = 0.00299346 \times 6 \fallingdotseq 1.8\%$. Under IAC, on the other hand, trivial deadlock corresponds with the cases 1, 2, 9, 10, 11, and 27 in Diss et al. (2012). Their Table 9 (p. 62) shows that each probability is $1/504$. Therefore, $p_D = (1/504) \times 6 \fallingdotseq 1.2\%$. ■

Proof of Theorem 4

The only nontrivial case is $f_1^1(L^0) = f_2^1(L^0) = \{x\}$ and $f_3^1(L^0) = \{y\}$, where $F^1 = \{f_1^1, f_2^1, f_3^1\}$ for distinct $x, y \in X$. Let $A = \{i \in N \mid xL_i^0 y\} = \{1,2,\dots,a\}$. I show that L^0 strongly converges unless α takes several specific values. The case of $\alpha > 2/3$ or $\alpha < 1/3$ is straightforward. Because the proofs are similar, I show only the proof for $1/3 < \alpha < 1/2$. To prove the uniqueness of convergence to $\{y\}$, I inductively show that for any level $k \geq 2$, $f^k \in F^k$ exists whose class is $\{y\}$. For $k = 2$, it follows that $f_P^2(L^1) = \{f_3^1\}$. Assume that the statement holds until $k - 1 (\geq 2)$ and $C_{g_1^{k-1}} = \{y\}$. For the other two rules g_2^k and g_3^k, the class is either $\{x\}, \{x, y\}$, or $\{y\}$. Because g_2^{k-1} and g_3^{k-1} are symmetric, there are six possible cases on the combination of $\left(C_{g_1^{k-1}}, C_{g_2^{k-1}}, C_{g_3^{k-1}}\right)$: Case 1: $(\{y\}, \{x\}, \{x\})$, Case 2: $(\{y\}, \{x\}, \{x, y\})$, Case 3: $(\{y\}, \{x\}, \{y\})$, Case 4: $(\{y\}, \{x, y\}, \{x, y\})$, Case 5: $(\{y\}, \{x, y\}, \{y\})$, and Case 6: $(\{y\}, \{y\}, \{y\})$. For each case, I show that at least one of f_P^k, f_B^k, f_A^k has class $\{y\}$. For cases 1, 3, and 6, this is obvious. For case 2, $\mathcal{L}^{k-1}[L^0, \dots, L^{k-2}]$ is a singleton: L_i^{k-1}: $f_3^{k-1} f_2^{k-1} f_1^{k-1}$ for all $i \in A$ and L_i^{k-1}: $f_1^{k-1} f_3^{k-1} f_2^{k-1}$ for all $i \notin A$. Because $a < n/2$, we have $f_P^k(L^{k-1}) = \{f_1^{k-1}\}$, which means $C_{f_P^k} = \{y\}$. Case 4 is similarly shown. For case 5, we have $f_A^k(L^{k-1}) \subseteq \{f_1^{k-1}, f_3^{k-1}\}$ for all $L^{k-1} \in \mathcal{L}^{k-1}[L^0, \dots, L^{k-1}]$. ■

Proof of Theorem 5

Take a profile $L^0 \in \mathcal{L}(X)^n$ that is not in trivial deadlock. Because of the remark just after Proposition 1, $f_{P^*}(L^0), f_{X^*}(L^0), f_{A^*}(L^0)$ cannot be three distinct singletons. If all three coincide with each other, strong convergence is straightforward. Otherwise, we have that:

$$\{f_{P^*}(L^0), f_{X^*}(L^0), f_{A^*}(L^0)\} = \{\{x\}, \{y\}\}$$

for some distinct $x, y \in X$. Without loss of generality, $f, g \in F^1$ choose $\{x\}$ and h chooses $\{y\}$. We assume n is odd, and so we have $a := \#\{i \in N \mid xL_i^0 y\} \neq \#\{i \in N \mid yL_i^0 x\}$. The rest of the proof, i.e., to check Weak Convergence and Uniqueness, can be done in the same way as in the proof of Theorem 4. ■

Proof of Lemma 6

Given a menu F and a sequence of CI profiles $L^0, \dots, L^{k-1}$ which satisfy the stated conditions, let

$$F_x := \left\{f \in F \,\middle|\, C_f[L^0, L^1, \dots, L^{k-1}] = \{x\}\right\},$$

$$F_y := \left\{f \in F \,\middle|\, C[f\colon L^0, L^1, \dots, L^{k-1}] = \{y\}\right\},$$

and let $\alpha := |F_x|$ and $\beta := |F_y|$. We label the elements as $F_x = \{g_1, g_2, \dots, g_\alpha\}$ and $F_y = \{h_1, h_2, \dots, h_\beta\}$. Also $N_x = \{i \in N \mid xL_i^0 y\}$, $N_y = \{i \in N \mid yL_i^0 x\}$, $n_x = |N_x|$, and $n_y = |N_y|$. Since $\alpha + \beta = 3$, we have two possible cases: (a) $(\alpha, \beta) = (2,1)$ and (b) $(\alpha, \beta) = (1,2)$.

(a) The case of $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbf{2}, \mathbf{1})$.

Define $L^k \in \mathcal{L}^k[L^0, \dots, L^{k-1}]$ as follows.

$$L_i^k\colon \begin{cases} g_1, g_2, h_1 & if \;\; i \in N_x \\ h_1, g_1, g_2 & if \;\; i \in N_y. \end{cases}$$

It is easy to see that every $f \in \mathcal{F}$ chooses a subset of $\{g_1, g_2\}$. So, L^0 weakly converges to $\{x\}$.

(b) The case of $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbf{1}, \mathbf{2})$.

Define $L^k \in \mathcal{L}^k[L^0, L^1, \dots, L^{k-1}]$ as follows.

$$L_i^k\colon \begin{cases} g_1, h_1, h_2 \;\; for \;\; (n_y + 1) \text{ individuals in } N_x \\ g_1, h_2, h_1 \;\; for \;\; n_x - (n_y + 1) \text{ individuals in } N_x \\ h_1, h_2, g_1 \;\; for \;\; \left\lfloor \frac{n}{2} \right\rfloor - (n_y + 1) \text{ individuals in } N_y \\ h_2, h_1, g_1 \;\; for \;\; the \;\; other \;\; individuals. \end{cases}$$

Intuitively, L^k is a profile such that the score of g_1 is the largest and the scores of h_1 and h_2 are the smallest. First, it is easy to see that each level-$(k+1)$ $f_P, f_H, f_C, f_{Bl}, f_M$ chooses $\{g_1\}$.

Consider f_B. For simplicity, we denote by $s(f)$ the score of $f \in F^k$ evaluated by f_B. Since each $i \in N_x$ ranks g_1 at the first position and each $i \in N_y$ ranks it at the third, we have $s(g_1) = n_x$. Since $\left\lfloor \frac{n}{2} \right\rfloor$ individuals rank h_1 above h_2 and $n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$ individuals rank h_2 above h_1, we have $s(h_2) \geq s(h_1)$. Furthermore,

$$\begin{aligned} s(h_2) &= \frac{1}{2}\left[\{n_x - (n_y + 1)\} + \left\{\left\lfloor \frac{n}{2} \right\rfloor - (n_y + 1)\right\}\right] + \left(2n_y + 1 - \left\lfloor \frac{n}{2} \right\rfloor\right) \\ &= \frac{1}{2}n_x + n_y - \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor \\ &\leq \frac{1}{2}n_x + n_y - \frac{1}{2} \cdot n_y \quad \left(\because \left\lfloor \frac{n}{2} \right\rfloor \geq n_y\right) \\ &< n_x = s(g_1) \quad (\because n_x > n_y). \end{aligned}$$

Therefore, $f_B^{k+1}(L^k) = \{g_1\}$. It is straightforward to check that $f_N^{k+1}(L^k) = \{g_1\}$.

Finally, consider f_A. Since n_y individuals rank g_1 at the third position and $(n_y + 1)$ individuals rank h_2 at the third, it follows that $f_A^{k+1}(L^k) \subseteq \{g_1, h_1\}$. Recall that each $f \in \mathcal{F} \setminus \{f_A\}$ chooses $\{g_1\}$ at L^k. If $f_A^{k+1}(L^k) = \{g_1\}$, this implies that L^0 weakly converges to $\{x\}$. If $f_A^{k+1}(L^k) = \{h_1\}$, we can apply the case (a) to the CI sequence $L^0, L^1, \dots, L^k$ (instead of the sequence $L^0, L^1, \dots, L^{k-1}$) to find the convergence. Suppose $f_A^{k+1}(L^k) = \{g_1, h_1\}$. Then, it follows that

$$n_x - (n_y + 1) = n_x.$$

This implies that $n_x = 2n_y + 1$. Then, let $M^k \in \mathcal{L}^k[L^0, L^1, \dots, L^{k-1}]$ as

$$M_i^k: \begin{cases} g_1, h_1, h_2 \ \ for \ \ (n_y + 2) \ \text{individuals in} \ N_x \\ g_1, h_2, h_1 \ \ for \ \ (n_y - 1) \ \text{individuals in} \ N_x \\ h_1, h_2, g_1 \ \ for \ \ \left\lfloor \frac{n}{2} \right\rfloor - (n_y + 2) \ \text{individuals in} \ N_y \\ h_2, h_1, g_1 \ \ for \ the \ other \ individuals. \end{cases}$$

In a similar way, we can check that $f_P, f_H, f_C, f_{Bl}, f_M, f_B, f_N$ chooses $\{g_1\}$ at M^k. Also, we have $f_A^{k+1}(M^k) = \{h_1\}$. So, we can apply the case (a) to the CI sequence $L^0, L^1, \dots, M^k$ to find the convergence. ■

Proof of Theorem 6

Let f_1, f_2, f_3 be distinct SCRs among $f_P, f_B, f_A, f_H, f_N, f_C, f_M, f_{Bl}$. When $n \to \infty$ under IAC, it is easy to see that the probability of tied outcomes by some of f_1^1, f_2^2, f_3^1 is negligible. So, we can discuss only $L^0 \in \mathcal{L}(X)^n$ such that each $f_1^1(L^0), f_2^1(L^0), f_3^1(L^0)$ is a singleton. Let $\mathcal{F} = \{f_P, f_{Bo}, f_A, f_H, f_N, f_C, f_M, f_{Bl}\}$.

(I) The Case of $|\{f_1(L^0), f_2(L^0), f_3(L^0)\}| = 2$

Let $\{f_1(L^0), f_2(L^0), f_3(L^0)\} = \{x, y\}$. When $n \to \infty$, the probability of the event

$$\#\{i \in N \mid xL_i^0 y\} \leq \#\{i \in N \mid yL_i^0 x\} + 2m$$

is negligible. Hence, we can apply Lemma 6 to derive weak convergence.

(2) The Case of $|\{\boldsymbol{f_1(L^0), f_2(L^0), f_3(L^0)}\}| = \mathbf{3}$

In this case, level-1 CI profile L^1 is uniquely determined. It is also straightforward that the probability of tied outcomes by some of the level-2 SCRs is negligible. If $|\{f_1^2(L^1), f_2^2(L^1), f_3^2(L^1)\}| = 2$, we can apply Lemma 6 again to derive weak convergence. I next show that $|\{f_1^2(L^1), f_2^2(L^1), f_3^2(L^1)\}|$ cannot be 3 if the menu F is one of the stated menus in the proposition. Suppose to the contrary that it is 3. Note that

$$\mathcal{L}(F^1) = \{f_1f_2f_3, f_1f_3f_2, f_2f_1f_3, f_2f_3f_1, f_3f_1f_2, f_3f_2f_1\}.$$

Let n_j be the number of individuals who have j^{th} preference. For example, n_1 and n_4 are the numbers of individuals whose level-1 CI preferences are $f_1f_2f_3$ and $f_3f_1f_2$, respectively.

From now on, the proof is similar for the ten menus in the proposition. Let us prove the case of $F = \{f_B, f_H, f_{Bl}\}$. Without loss of generality, we can assume $f_B^2(L^1) = f_1$, $f_H^2(L^1) = f_2$, and $f_{Bl}^2(L^1) = f_3$. With $n_1, \ldots, n_6$, we can rephrase these conditions as follows:

$$f_{Bo}^2(L^1) = f_1^1 : \begin{cases} n_1 + 2n_2 + n_5 > n_3 + 2n_4 + n_6 \\ 2n_1 + n_2 + n_3 > n_4 + n_5 + 2n_6 \end{cases}$$

$$f_H^2(L^1) = f_2^1 : \begin{cases} \begin{cases} n_3 + n_4 > n_1 + n_2 \\ n_5 + n_6 > n_1 + n_2 \\ n_1 + n_3 + n_4 > n_2 + n_5 + n_6 \end{cases} \\ \text{or} \\ \begin{cases} n_1 + n_2 > n_5 + n_6 \\ n_3 + n_4 > n_5 + n_6 \\ n_1 + n_2 + n_5 < n_3 + n_4 + n_6 \end{cases} \end{cases}$$

$f_{Bl}^2(L^1) = f_3^1$:

$$\begin{cases} (n_4 + n_5 + n_6 > n_1 + n_2 + n_3 \text{ and } n_2 + n_5 + n_6 > n_1 + n_3 + n_4) \\ \text{or} \\ \left[\begin{array}{c} (n_3 + n_4 + n_6 > n_1 + n_2 + n_5 \text{ or } n_4 + n_5 + n_6 > n_1 + n_2 + n_3) \\ \text{and} \\ (n_1 + n_2 + n_5 > n_3 + n_4 + n_6 \text{ or } n_2 + n_5 + n_6 > n_1 + n_3 + n_4) \\ \text{and} \\ (n_1 + n_2 + n_3 > n_4 + n_5 + n_6 \text{ or } n_1 + n_3 + n_4 > n_2 + n_5 + n_6) \\ \text{and} \\ n_4 + n_5 + 2n_6 > 2n_1 + n_2 + n_3 \\ \text{and} \\ n_2 + 2n_5 + n_6 > n_1 + 2n_3 + n_4 \end{array} \right] \end{cases}$$

With elementary verification[33], we can see that there is no non-negative integer solution $(n_1, n_2, \ldots, n_6)$ for this system of inequalities. ■

[33] For actual verification, I used the function "FindInstance" in the software Mathematica.

Proof of Theorem 7

I provide extra notation in (0), and then prove for three distinct cases (1), (2), and (3).

(0) Extra Notation

Let $F = \{f_1, f_2, f_3\}$ be a menu of SCFs such that:

f_1: plurality rule where ties are broken in favor of individual 1,

f_2: anti-plurality rule where ties are broken in favor of individual 1, and

f_3: for all set A and $L = (L_1, L_2, \dots, L_n) \in \mathcal{L}(A)^n$, $f_3(L)$ is

- the greatest element for L_1 among $f_P(L)$ if $|f_P(L)| \geq 2$,
- $f_1(L)$ if plurality score of some alternative is greater than $n/2$, and
- $f_2(L)$ otherwise.

I show that this set F satisfies the required conditions (to confirm neutrality and difference is straightforward) and has the strong convergent property.

Take any $L^0 \in \mathcal{L}(X)^n$. Note that $f_3(L^0)$ is either $f_1(L^0)$ or $f_2(L^0)$. If $f_1(L^0) = f_2(L^0)$, strong convergence is straightforward. Assume $f_1(L^0) \neq f_2(L^0)$. We can label $F^1 = \{g_1, g_2, g_3\}$, where $g_1(L^0) = g_2(L^0) = x$ and $g_3(L^0) = y$. Let $N_x \coloneqq \{i \in N \mid xL_i^0 y\}$ and $N_y \coloneqq \{i \in N \mid yL_i^0 x\}$. I denote their cardinalities as $n_x = |N_x|$ and $n_y = |N_y|$. Note that we have $N = N_x \cup N_y$ and $n = n_x + n_y$.

(1) The Case of $n_x > n_y$

I show that L^0 strongly converges to $\{x\}$. To prove this, we need to show two things:

Weak convergence: L^0 weakly converges to $\{x\}$, and

Uniqueness: L^0 does not weakly converge to other $C' \neq \{x\}$.

We prove them one by one.

Weak convergence: Take $L^1 \in \mathcal{L}[L^0]$ such that:

$$L_i^1: g_1, g_2, g_3 \text{ for all } i \in N_x.$$
$$L_i^1: g_3, g_1, g_2 \text{ for all } i \in N_y.$$

$n_x > n_y$, and so it follows that $f_1^2(L^1) = f_2^2(L^1) = f_3^2(L^2) \subseteq \{g_1, g_2\}$. This means that L^0 weakly converges to $\{x\}$.

Uniqueness: I inductively prove the following proposition, which implies that L^0 does not weakly converge to $C' \neq \{x\}$.

Proposition: Assume the conditions in (0) and $n_x > n_y$. For all $k \in \mathbb{N}$, and for all sequences of CI profiles to level $(k-1)$ $L^0, L^1, \dots, L^{k-1}$, there exists $f \in F^k$ such that $C[f: L^0, L^1, \dots, L^{k-1}] = \{x\}$.

Proof of the Proposition:

If $k = 1$, the proposition is straightforward from the assumption. Suppose the statement holds for $k = 1,2, \ldots, k_0 - 1$ ($k_0 \in \mathbb{N}$). Take any sequence of CI profiles $L^0, L^1, \ldots, L^{k_0-1}$.

(a) If every $f_1^{k_0}, f_2^{k_0}, f_3^{k_0}$ has class $\{x\}$, the proposition is straightforward.

(b) Suppose two of $f_1^{k_0}, f_2^{k_0}, f_3^{k_0}$, denoted $h_1^{k_0}$ and $h_2^{k_0}$, have class $\{x\}$ and the other one, denoted $h_3^{k_0}$, has class $\{y\}$. For all $L^{k_0} \in \mathcal{L}[L^0, L^1, \ldots, L^{k_0-1}]$, individuals in N_y rank $h_1^{k_0}$ or $h_2^{k_0}$ at the worst, while individuals in N_x rank $h_3^{k_0}$ at the worst. $n_x > n_y$, and so it follows that $f_2^{k_0+1}(L^{k_0}) \subseteq \{h_1^{k_0}, h_2^{k_0}\}$.

(c) Suppose one of $f_1^{k_0}, f_2^{k_0}, f_3^{k_0}$, denoted $h_1^{k_0}$, has class $\{x\}$ and the other two, denoted $h_2^{k_0}$ and $h_3^{k_0}$, have class $\{y\}$. For any $L^{k_0} \in \mathcal{L}[L^0, L^1, \ldots, L^{k_0-1}]$, individuals in N_y rank $h_2^{k_0}$ or $h_3^{k_0}$ at the top while individuals in N_x rank $h_1^{k_0}$ at the top. $n_x > n_y$, and so it follows that $f_1^{k_0+1}(L^{k_0}) = h_1^{k_0}$.

(2) The Case of $\boldsymbol{n_x < n_y}$

In this case, L^0 strongly converges to $\{y\}$.

Weak convergence: Consider the same $L^1 \in \mathcal{L}[L^0]$ defined in (1). $n_y > n_x$, and so we have $f_1^2(L^1) = g_3$ and $f_3^2(L^1) = g_3$. If $f_2^2(L^1) = g_3$, weak convergence is straightforward. Otherwise, without loss of generality, we can assume $f_2^2(L^1) = g_1$. Let us take $L^2 \in \mathcal{L}[L^0, L^1]$ as follows:

$$L_i^2: f_2, f_1, f_3 \text{ for all } i \in N_x.$$
$$L_i^2: f_1, f_3, f_2 \text{ for all } i \in N_y.$$

It follows that $f_1^3(L^2) = f_2^3(L^2) = f_3^3(L^2) = f_1$.

Uniqueness: The proof can be made in the same way as in (1).

(3) The Case of $\boldsymbol{n_x = n_y}$

Uniqueness proof can be shown as in (1), and so I show only the proof of weak convergence.

(a) Suppose xL_1^0y. In this case, L^0 strongly converges to $\{x\}$. Consider the same $L^1 \in \mathcal{L}[L^0]$ as in (1). $f_P(L^1) = \{g_1, g_3\}$ and $g_1 L_i^1 g_3$, and so we have $f_1(L^1) = g_1$. It also follows that $f_2(L^1) = f_3(L^1) = g_1$.

(b) Suppose yL_1^0x. In this case, L^0 strongly converges to $\{y\}$. Consider the same $L^1 \in \mathcal{L}[L^0]$ as in (1). It follows that $f_1(L^0) = f_3(L^0) = g_3$ and $f_2(L^0) = g_1$. Consider $L^2 \in \mathcal{L}[L^0, L^1]$ that we used in (2). Now, we have $f_1^3(L^2) = f_2^3(L^2) = f_3^3(L^2) = f_1^2$. This completes the proof. ■

Proof of Theorem 8

The probability of tied outcomes at level-1 SCRs can be negligible, and so we can expect that each $f_P^1(L^0), f_B^1(L^0), f_A^1(L^0), \varphi^1(L^0)$ is a singleton. If $|F^1(L^0)| \leq 2$, we can apply Lemma 6 to guarantee the weak convergence. Because of the definition of φ, we know that $\varphi(L^0) \subseteq f_P^1(L^0) \cup f_B^1(L^0) \cup f_A^1(L^0)$. So, we can assume $|F^1(L^0)| = 3$. Let $F^1 = \{g_1^1, g_2^1, g_3^1, g_4^1\}$. Without loss of generality, we can assume $g_1^1(L^0) = g_2^1(L^0) = \{x_1\}$, $g_3^1(L^0) = \{x_2\}$, and $g_4^1(L^0) = \{x_3\}$.

Let $L^1 \in \mathcal{L}^1[L^0]$ be such that everyone ranks g_1^1 above g_2^1. Note that the probability of tied outcomes by some of $f_P^2, f_B^2, f_A^2, \varphi^2$ can be also negligible. So, we can expect that $f_P^2(L^1), f_B^2(L^1), f_A^2(L^1), \varphi^2(L^1)$ are also singletons. It is simple to see that there are six types of preference in L^1. Let $n_1, \dots, n_6$ be the number of individuals who have each specific type of preference as follows:

n_1 individuals: $g_1^1, g_2^1, g_3^1, g_4^1$
n_2 individuals: $g_1^1, g_2^1, g_3^1, g_4^1$
n_3 individuals: $g_3^1, g_1^1, g_2^1, g_4^1$
n_4 individuals: $g_3^1, g_4^1, g_1^1, g_2^1$
n_5 individuals: $g_4^1, g_1^1, g_2^1, g_3^1$
n_6 individuals: $g_4^1, g_3^1, g_1^1, g_2^1$,

where $n = n_1 + n_2 + n_3 + n_4 + n_5 + n_6$. Note also that if $|F^2(L^1)| \leq 2$, then Lemma 6 again guarantees the weak convergence. So, we assume that $|F^2(L^1)| = 3$. At this time, $\varphi^2(L^1)$ is either $f_P^2(L^1)$ or $f_B^2(L^1)$. We can also expect $n_i > 0$ for $i = 1,2,3,4,5,6$ when $n \to \infty$, and so we have that $f_A^2(L^1) = \{g_1^1\}$. Now, we have only two possibilities:

(1) $f_B^2(L^1) = \varphi^2(L^1) = \{g_3^1\}$, $f_P^2(L^1) = \{g_4^1\}$, and $f_A^2(L^1) = \{g_1^1\}$, or
(2) $f_B^2(L^1) = \{g_3^1\}$, $f_B^2(L^1) = \varphi^2(L^1) = \{g_4^1\}$, and $f_A^2(L^1) = \{g_1^1\}$.

(1) The case of $f_B^2(L^1) = \varphi^2(L^1) = \{g_3^1\}$, $f_P^2(L^1) = \{g_4^1\}$, and $f_A^2(L^1) = \{g_1^1\}$.
Let $L^2 \in \left(\mathcal{L}(F^2)\right)^n$ be as follows:

n_1 individuals: $f_A^2, f_B^2, \varphi^2, f_P^2$.
n_2 individuals: $f_A^2, f_P^2, f_B^2, \varphi^2$.
n_3 individuals: $f_B^2, \varphi^2, f_A^2, f_P^2$.
n_4 individuals: $f_B^2, \varphi^2, f_P^2, f_A^2$.
n_5 individuals: $f_P^2, f_A^2, f_B^2, \varphi^2$.
n_6 individuals: $f_P^2, f_B^2, \varphi^2, f_A^2$.

Clearly, we have $L^2 \in \mathcal{L}^2[L^0, L^1]$.
$n_1, \dots, n_6$ are positive, and so we obtain that $f_A^3(L^2) = \{f_B^2\}$. $f_P^2(L^1) = \{g_4^1\}$, and so the plurality score of g_4^1 is greater than those of g_1^1 and g_3^1:

$$\begin{cases} n_5 + n_6 > n_1 + n_{2.} \\ n_5 + n_6 > n_3 + n_4. \end{cases}$$

This also shows that the plurality score of f_P^2 is greater than those of f_A^2 and f_B^2 at L^2. Hence, we have that $f_P^3(L^2) = \{f_P^2\}$. Next, we show that $f_B^3(L^2) = \{f_B^2\}$.

Because $f_B^2(L^1) = \{g_3^1\}$, the Borda scores at L^1 are as follows:

$$s_B(g_3^1) > s_B(g_1^1) \Leftrightarrow n_3 + n_4 + \frac{2}{3}n_6 + \frac{1}{3}n_1 > n_1 + n_2 + \frac{2}{3}(n_3 + n_5) + \frac{1}{3}(n_4 + n_6).$$

$$s_B(g_3^1) > s_B(g_4^1) \Leftrightarrow n_3 + n_4 + \frac{2}{3}n_6 + \frac{1}{3}n_1 > n_5 + n_6 + \frac{2}{3}n_4 + \frac{1}{3}n_2.$$

At L^2, we have:

$$s_B(f_P^2) = n_5 + n_6 + \frac{2}{3}n_2 + \frac{1}{3}n_4.$$
$$s_B(f_B^2) = n_3 + n_4 + \frac{2}{3}(n_1 + n_6) + \frac{1}{3}(n_2 + n_5).$$
$$s_B(f_A^2) = n_1 + n_2 + \frac{2}{3}n_5 + \frac{1}{3}n_3.$$
$$s_B(\varphi^2) < s_B(f_B^2).$$

These equations show that $s_B(f_B^2) > \max\{s_B(f_P^2), s_B(f_A^2), s_B(\varphi^2)\}$.

(2) The case of $f_B^2(L^1) = \{g_3^1\}$, $f_P^2(L^1) = \varphi(L^1) = \{g_4^1\}$, and $f_A^2(L^1) = \{g_1^1\}$.
$f_B^2(L^1) = \{g_3^1\}$, and so the score of g_3^1 at L^1 is strictly greater than those of g_1^1 and g_4^1. Formally, we have that:

$$\begin{aligned} &n_3 + n_4 + \frac{2}{3}n_6 + \frac{1}{3}n_1 > n_1 + n_2 + \frac{2}{3}(n_3 + n_5) + \frac{1}{3}(n_4 + n_6). \\ &n_3 + n_4 + \frac{2}{3}n_6 + \frac{1}{3}n_1 > n_5 + n_6 + \frac{2}{3}n_4 + \frac{1}{3}n_2. \end{aligned} \tag{3}$$

Let $L^2 \in \mathcal{L}^2[L^0, L^1]$ be such that:

n_1 individuals: $f_A^2, f_B^2, f_P^2, \varphi^2$.
n_2 individuals: $f_A^2, f_P^2, \varphi^2, f_B^2$.
n_3 individuals: $f_B^2, f_A^2, f_P^2, \varphi^2$.
n_4 individuals: $f_B^2, f_P^2, \varphi^2, f_A^2$.
n_5 individuals: $f_P^2, \varphi^2, f_A^2, f_B^2$.
n_6 individuals: $f_P^2, \varphi^2, f_B^2, f_A^2$.

In words, this is a consequentially induced preference where everyone ranks f_P^2 above φ^2. Similar to (1), we can check that $f_P^3(L^2) = f_A^3(L^2) = \{f_P^2\}$. Furthermore, the scores of $f_A^2, f_P^2, f_B^2, \varphi^2$ evaluated by f_B^3 are as follows:

$$s_B(f_A^2) = n_1 + n_2 + \frac{2}{3}n_3 + \frac{1}{3}n_5.$$

$$s_B(f_B^2) = n_3 + n_4 + \frac{2}{3}n_1 + \frac{1}{3}n_6.$$

$$s_B(f_P^2) = n_5 + n_6 + \frac{2}{3}(n_2 + n_4) + \frac{1}{3}(n_1 + n_3).$$

With (1) we have that $s_B(f_B^2) > s_B(f_A^2)$. Note that ties between f_B^2, f_P^2 occur only if

$$n_3 + n_4 + \frac{2}{3}n_1 + \frac{1}{3}n_6 = n_5 + n_6 + \frac{2}{3}(n_2 + n_4) + \frac{1}{3}(n_1 + n_3).$$

This event can be negligible as $n \to \infty$. ■

Proof of Theorem 9

Let $F = \{f_1, f_2, \dots, f_M\}$ be a menu of $M \geq 3$ concave scoring rules. Define $E = \{E_1, E_2, \dots, E_\mu\}$ and $C = \{c_{f,j} \mid f \in E \cup C \text{ and } j = 1,2,\dots,r\}$, where $c_{f,j}$ is a voting rule such that

$$c_{f,j}(L: X) \coloneqq \begin{cases} G(L_1, X) & \text{if } |\{i \in N \mid L_i = L_1\}| = j, and \\ f(L: X) & \text{otherwise.} \end{cases}$$

The probability of the event $|\{i \in N \mid L_i = L_1\}| = j$ is negligible as $n \to \infty$, and so each $c_{f,1}, c_{f,2}, \dots, c_{f,r}$ is asymptotically the same as f. In this sense, C is a set of pseudo-copies of the elements in $F \cup E$. We show that $G \coloneqq F \cup E \cup C$ has the asymptotically weak convergent property for μ and r such that $r \geq \mu \gg M$. The proof is made up of several steps.

Step 1: To Prove the Following Statement

Let $k \geq 1$. Let $L^0, L^1, \dots, L^{k-1}$ be a sequence of CI profiles to level $(k-1)$. Suppose:

(1) $\{C_g[L^0, L^1, \dots, L^{k-1}] \mid g \in G^k\} = \{\{y_1\}, \{y_2\}, \dots, \{y_m\}\}$,

(2) $c_{f,j}(L^{k-1}) = f(L^{k-1})$ for all $f \in F \cup E$ and $j = 1,2,\dots,r$, and

(3) $\{j^*\} = \underset{j \in \{1,2,\dots,m\}}{\operatorname{argmax}} |U_j|$, where for all $j = 1,2,\dots,m$,

$$U_j \coloneqq \left\{i \in N \mid y_j L_i^0 y \text{ for all } y \in \{y_1, y_2, \dots, y_m\}\right\}.$$

Then there exists $L^k \in \mathcal{L}^k[L^0, L^1, \dots, L^{k-1}]$ such that

$$C_{E_e^{k+1}}[L^0, \dots, L^{k-1}, L^k] = \{y_{j^*}\} \text{ for all } E_e^{k+1} \in E^{k+1}.$$

Proof of the Statement

For each $j = 1,2,\dots,m$, let us define some extra notation: let $u_j \coloneqq |U_j|$ be the cardinality of each U_j. Let $G_j \coloneqq \{g \in G^k \mid C_g[L^0, \dots, L^{k-1}] = \{y_j\}\}$ be the set of level-k voting rules whose class is y_j and let $b_j \coloneqq |G_j|$ be the cardinality of G_j. We label each element of G_j as

$$G_j = \{g_{j,1}, g_{j,2}, \dots, g_{j,b_j}\}.$$

Without loss of generality, we can assume $j^* = 1$. Next, we define a profile $L^k = (L_1^k, L_2^k, \dots, L_n^k) \in (\mathcal{L}(G^k))^n$ through five steps:

(a) Preferences on $G_{j^*}(= G_1)$.

For all $i \in N$, let

$$L_i^k \mid G_{j^*}: g_{j^*,1}, g_{j^*,2}, \dots, g_{j^*,b_{j^*}}.$$

(b) Re-Label Individuals

There are $m!$ possible preferences over $\{y_1, y_2, \dots, y_m\}$. Suppose we array them in lexicographic order (in terms of the subscripts) and denote them as the $1^{\text{st}}, 2^{\text{nd}}, \dots, (m!)^{\text{th}}$ preferences. For example, the 1^{st} preference is $y_1, y_2, \dots, y_{m-1}, y_m$, the second is $y_1, y_2, \dots, y_m, y_{m-1}$, and the $(m!)^{\text{th}}$ (last) one is $y_m, y_{m-1}, \dots, y_1$. For each $j = 1,2, \dots, m$, let N_j denote the set of individuals who have the j^{th} preference over $\{y_1, y_2, \dots, y_m\}$. Let $n_j = |N_j|$ be its cardinality. Let us re-label individuals as follows: $N_1 = \{i_1, i_2, \dots, i_{n_1}\}$. For each $j \in \{2,3, \dots, m\}$, let

$$N_j = \left\{i_{n_1+n_2+\cdots+n_{j-1}+1}, i_{n_1+n_2+\cdots+n_{j-1}+2}, \cdots, i_{n_1+n_2+\cdots+n_{j-1}+n_j}\right\}.$$

Note that $N_1, N_2, \dots, N_{m!}$ gives a partition of N.

(c) Define Permutations on G_j.

For each $j = 2,3, \dots, m$, let $\sigma_j: G_j \to G_j$ be a permutation on G_j such that

$$\sigma_j(g_{j,p}) = \begin{cases} g_{j,p+1} & \text{if } 1 \le p < b_j, and \\ g_{j,1} & \text{if } p = b_j. \end{cases}$$

As usual, we denote $\sigma_j^q = \sigma_j \circ \sigma_j \circ \cdots \circ \sigma_j$ (q times) for each positive integer q. We interpret σ_j^0 as the identity.

(d) Preferences on $G_j \neq G_{j^*}$.

For each $p = 1,2, \dots, n$ and $j = 2,3, \dots, m$, let

$$L_{i_p}^k \mid_{G_j}: \sigma_j^{p-1}(g_{j,1}), \sigma_j^{p-1}(g_{j,2}), \dots, \sigma_j^{p-1}\left(g_{j,b_j}\right).$$

(e) Preferences Between G_j and $G_{j'}$.

For all distinct $j, j' \in \{1,2, \dots, m\}$, $g \in G_j$, and $g' \in G_{j'}$, let

$$g L_i^k g' \Leftrightarrow y_j L_i^0 y_{j'}.$$

Now, let us confirm that $E_1^{k+1}(L^k), E_2^{k+1}(L^k), \cdots, E_\mu^{k+1}(L^k) \subseteq G_1^k$. Condition (2) demands that each $b_1, b_2, \dots, b_r$ is at least as large as $1 + r$, because at least one element in $F \cup E$ has class y_j and r copies yield the same outcome as theirs. Recall that we assumed $\mu \le r$. E_e $(1 \le e \le \mu)$ looks only at the first, second, …, e^{th} position in the preference profile, and so it follows that voters in U_j assign positive scores only to elements in G_j^k at L^k. Therefore, we have that for all $j = 1,2, \dots, m$

and $g_j \in G_j$,

$$s(g_j) \leq u_j.$$

It is straightforward from (a) that $s(g_{1,1}) = u_1$. u_1 is, by definition, the largest among $u_1, u_2, \dots, u_m$, and so we have

$$s(g_{1,1}) = u_1 > \max_{j \neq 1, g_j \in G_j} s(g_j).$$

This completes the proof of step 1.

In general, the proof process above determines the way we design L^k in the face of $L^0, L^1, \dots, L^{k-1}$ under conditions (1), (2), and (3). From now on, I denote by L^{*k} such a profile L^k.

Step 2: Asymptotic Consequence of step 1

Let us define the effective number (of alternatives) $\{m_k\}_{k=0,1,\dots,|G|}$ inductively. Intuitively, m_k represents the number of classes at level k.

First, let $m_0 := |X|$. When $n \to \infty$, the probability of a tied outcome by some SCR in G is negligible. Hence, we accept that condition (1) holds when $k = 1$. In a similar way, we can check that conditions (2) and (3) also hold when $k = 1$ as $n \to \infty$. Hence, we can infer that L^{*1} is well-defined. Suppose L^{*k} is well-defined for $k = 0,1, \dots, \bar{k}$. Again, we can infer that the probability that any of the conditions (1), (2), or (3) breaks is negligible as $n \to \infty$. Therefore, inductively we can say that if $\bar{k}$ is finite, we can design $L^0, L^1, \dots, L^{\bar{k}}$ so that each condition (1), (2), and (3) holds till level $\bar{k}$.
Now, let m_k be the value of m determined at level $k = 1,2, \dots, \bar{k}$. Clearly, the sequence $m_0, m_1, \dots, m_{|G|}$ is decreasing, i.e., $M = m_0 \geq m_1 \geq m_2 \geq \cdots \geq m_{|G|} \geq 1$ and $m_1 \leq |G|$. If $m_{|G|} = 1$, it directly means convergence. Suppose $m_{|G|} \geq 2$. It follows that $|G|$ integers $m_1, \dots, m_{|G|}$ are between 2 and $|G|$. Therefore, there exists $k_1 \in \{1,2, \dots, |G|\}$ such that $m_{k_1 - 1} = m_{k_1}$.

Step 3: Classes of Level-k_1 Voting Rules

For simplicity, let $k = k_1$ and $m = m_k$ throughout this step. Based on the discussion in step 2, we have that:
(1') $\{C_g[L^0, L^{*1}, \dots, L^{*k-1}] \mid g \in G^k\} = \{\{z_1\}, \{z_2\}, \dots, \{z_m\}\}$,
(2') $c_{f,j}(L^{*k-1}) = f(L^{*k-1})$ for all $f \in F \cup E$ and $j = 1,2, \dots, r$, and
(3') $\{|T_{j^*}|\} = \text{argmax}\{|T_1|, |T_2|, \dots, |T_m|\}$, where for all $j = 1,2, \dots, m$,

$$T_j := \left\{i \in N \mid z_j L_i^0 z \text{ for all } z \in \{z_1, z_2, \dots, z_m\}\right\}.$$

For each $j = 1,2, \dots, m$, let $G_j^k := \left\{g \in G^k \mid C_g[L^0, L^{*1}, \dots, L^{*k-1}] = \{z_j\}\right\}$ and $a_j := |G_j|$. We also denote by t_j the cardinality of T_j. Without loss of generality, we can assume again that $|T_1| > |T_2|, \dots, |T_m|$. Furthermore, step 1 shows that the elements in E and its copies are all in G_1^k. Therefore, we have $a_1 \geq \mu(1+r)$ and $a_2, \dots, a_m \leq M(1+r)$.

Take arbitrary concave score assignments $\left[s_1^{|G|}, s_2^{|G|}, \dots, s_{|G|}^{|G|}\right]$ and calculate the score of each $g \in G^k$ evaluated by this f at L^{*k}. For simplicity, we write s_j instead of s_j^m $(j = 1,2,\dots,m)$ throughout this step.

First, we consider the score of $g_{1,1}^k$. Note that voters in $|T_1|$ rank $g_{1,1}^k$ at the first position and the other voters rank $g_{1,1}^k$ at least at the $(a_2 + a_3 + \cdots + a_m + 1)^{\text{th}}$ position. Hence, we have that

$$s(g_{1,1}^k) \geq s_1 \cdot |T_1| + \sum_{j=2}^{m} s_{a_2+a_3+\cdots+a_m+1} \cdot |T_j|.$$

Take any $g_\lambda \in G_\lambda^k$ $(2 \leq \lambda \leq m)$ and let us evaluate its score $s(g_\lambda^k)$. Because of the recipe for L^{*k}, we can infer that the scores of the elements in G_λ^k do not vary much from each other. Indeed, we have the following proposition.

Proposition. Take any $j \in \{2,3,\dots,m\}$ and $g, g' \in G_j^k$. Then we have $|s(g) - s(g')| \leq |G|!$ at L^{*k}.

Proof. Let us introduce some extra notation. For any $A \subseteq N$ and $h \in G^k$, let $s(g{:}A)$ be the score of h from A. Formally,

$$s(h{:}A) = \sum_{x=1}^{|G|} s_x \cdot R_x(h{:}A)$$

Here, we denote by $R_x(h{:}A)$ the number of individuals in A who rank h at the x^{th} position. With this notion, we can develop the score of g as follows:

$$s(h) = \sum_{j=1}^{m!} s(h{:}N_j).$$

Because of this equation (and $m \leq |G|$), we have only to prove the following: that for each $p = 1,2,\dots,m!$, $|s(g{:}N_p) - s(g'{:}N_p)| \leq |G|$. Take any $N_p \in \{N_1, N_2, N_3, \dots, N_{m!}\}$. Dividing n_p by $|G_j|$, we have

$$n_p = \alpha|G_j| + \beta$$

where $\alpha \in \mathbb{N} \cup \{0\}$ and $0 \leq \beta < |G_j|$. For simplicity, let $v_j := n_1 + n_2 + \cdots + n_{j-1}$. First, look at individuals $I := \left\{i_{v_j+1}, i_{v_j+2}, \cdots, i_{v_j+\alpha|G_j|}\right\}$. Recall that their level-k CI preferences are the same over $G^k \setminus G_j^k$ and

$$L^k_{i_{v_j+1}}\mid_{G^k_j}: \sigma_j^{(i_{v_j}+1)-1}(g_{j,1}), \sigma_j^{(i_{v_j}+1)-1}(g_{j,2}), \ldots, \sigma_j^{(i_{v_j}+1)-1}(g_{j,a_j})$$

$$L^k_{i_{v_j+2}}\mid_{G^k_j}: \sigma_j^{(i_{v_j}+2)-1}(g_{j,1}), \sigma_j^{(i_{v_j}+2)-1}(g_{j,2}), \ldots, \sigma_j^{(i_{v_j}+2)-1}(g_{j,a_j})$$

$$\vdots$$

$$L^k_{i_{v_j+\alpha|G_j|}}\mid_{G^k_j}: \sigma_j^{(i_{v_j}+\alpha|G_j|)-1}(g_{j,1}), \sigma_j^{(i_{v_j}+\alpha|G_j|)-1}(g_{j,2}), \ldots, \sigma_j^{(i_{v_j}+\alpha|G_j|)-1}(g_{j,a_j}).$$

Because $\sigma_j^{|G|} = \sigma_j^0$ (identity), we can see the symmetry between elements, i.e., each element $g \in G_j$ appears at each rank with exactly the same frequency. Hence, we have

$$s(g{:}\,I) = s(g'{:}\,I).$$

Therefore,

$$\begin{aligned}\left|s(g{:}\,N_p) - s(g'{:}\,N_p)\right| &= \left|\{s(g{:}\,I) + s(g{:}\,N_p \setminus I)\} - \{s(g'{:}\,I) + s(g'{:}\,N_p \setminus I)\}\right| \\ &= \left|s(g{:}\,N_p \setminus I) - s(g'{:}\,N_p \setminus I)\right|.\end{aligned}$$

For any individual $i \in N$ and any $h, h' \in G_j$, we have (by the definition of a scoring rule):

$$|s(h,\{i\}) - s(h',\{i\})| \leq 1.$$

Using this, we have

$$\begin{aligned}\left|s(g{:}\,N_p \setminus I) - s(g'{:}\,N_p \setminus I)\right| &\leq \sum_{i \in N_p \setminus I} |s(g{:}\,\{i\}) - s(g'{:}\,\{i\})| \\ &\leq \left|N_p \setminus I\right| \\ &= \beta < \left|G_j\right|.\end{aligned}$$

This completes the proof of the proposition. ■

For each $j \neq \lambda$, voters in T_j rank g_λ at the $a_j + 1^{\text{th}}$ or lower position, because such voters rank the elements of G^k_j at the $1^{\text{st}}, 2^{\text{nd}}, \ldots, a_j^{\text{th}}$ positions. On the other hand, voters in T_λ rank g^k_λ at the 1^{st} or lower position. Therefore, we can evaluate $s(g^k_\lambda)$ as follows:

$$s(g^k_\lambda) < \frac{1}{a_\lambda}\left\{(s_1 + s_2 + \cdots + s_{a_\lambda}) \cdot |T_\lambda| + \sum_{j \neq \lambda}^{m} \left(s_{a_j+1} + s_{a_j+2} + \cdots + s_{a_j+a_\lambda}\right) \cdot \left|T_j\right|\right\} + |G|!.$$

For each $w_1, w_2 \in \mathbb{N}$ such that $w_1 < w_2$, we write for simplicity

$$a_{p\sim q} = a_p + a_{p+1} + \cdots + a_q.$$

$$s_{p\sim q} = s_p + s_{p+1} + \cdots + s_{p+q}.$$

Now the difference between $s(g^k_{1,1})$ and $s(g^k_\lambda)$ can be evaluated as follows:

$$s(g_{1,1}^k) - s(g_\lambda^k) > \left(s_1 - \frac{s_{(a_1+1)\sim(a_1+a_\lambda)}}{a_\lambda}\right) \cdot |T_1| + \left(s_{a_{2\sim m}+1} - \frac{s_{1\sim a_\lambda}}{a_\lambda}\right) \cdot |T_\lambda|$$
$$+ \left\{ \sum_{j \neq 1,\lambda}^{m} \left(s_{a_{2\sim m}+1} - \frac{s_{(a_j+1)\sim(a_j+a_\lambda)}}{a_\lambda}\right) \cdot |T_j| \right\} - |G|!.$$

Recall that $|T_1|$ is assumed to be the largest among $|T_1|, |T_2|, \dots, |T_m|$. Note also that the coefficients of $|T_\lambda|$ and $|T_j|$ for $j \neq 1, \lambda$ are non-positive, because $s_1, s_2, \dots$ is, by definition, a decreasing sequence. Therefore, we have that

$$s(g_{1,1}^k) - s(g_\lambda^k) > \left(s_1 - \frac{s_{(a_1+1)\sim(a_1+a_\lambda)}}{a_\lambda}\right) \cdot |T_1| + \left(s_{a_{2\sim m}+1} - \frac{s_{1\sim a_\lambda}}{a_\lambda}\right) \cdot |T_1|$$
$$+ \left\{ \sum_{j \neq 1,\lambda}^{m} \left(s_{a_{2\sim m}+1} - \frac{s_{(a_j+1)\sim(a_j+a_\lambda)}}{a_\lambda}\right) \cdot |T_1| \right\} - |G|!$$
$$\geq \left(s_1 - \frac{s_{(a_1+1)\sim(a_1+a_\lambda)}}{a_\lambda} + s_{a_{2\sim m}+1} - \frac{s_{1\sim a_\lambda}}{a_\lambda} + \sum_{j \neq 1,\lambda}^{m} \left(s_{a_{2\sim m}+1} - \frac{s_{(a_j+1)\sim(a_j+a_\lambda)}}{a_\lambda}\right) - \frac{|G|!}{|T_1|} \right) \cdot |T_1|.$$

In the next step, I show that the right-hand side is non-negative for any concave score assignment $[s_1, s_2, \dots, s_{|G|}]$. Once that is shown, the proof of Theorem 9 is complete, because $s(g_{1,1}^k) - s(g_\lambda^k) > 0$ (for any concave score assignment) implies that any concave voting rule in G^{k+1} would choose a subset of G_1^k. This means convergence to $\{x_1\}$ at level $(k+1)$.

Step 4: Prove the Inequality

Let $H(s_1, s_2, \dots, s_{|G|})$ be the coefficient part of $|T_1|$ in the last inequality, i.e.,

$$H(s_1, s_2, \dots, s_{|G|})$$
$$:= s_1 - \frac{s_{(a_1+1)\sim(a_1+a_\lambda)}}{a_\lambda} + s_{a_{2\sim m}+1} - \frac{s_{1\sim a_\lambda}}{a_\lambda} + \sum_{j \neq 1,\lambda}^{m} \left(s_{a_{2\sim m}+1} - \frac{s_{(a_j+1)\sim(a_j+a_\lambda)}}{a_\lambda}\right)$$
$$- \frac{|G|!}{|T_1|}.$$

I will show that $H(s_1, s_2, \dots, s_{|G|}) \geq 0$ for all $[s_1, s_2, \dots, s_{|G|}] \in \mathcal{C}_{|G|}$. To show this, let $t := s_{a_{2\sim m}+1}$. Let $\mathcal{D}_t \subseteq \mathcal{C}_{|G|}$ be the set of score assignments in $\mathcal{C}_{|G|}$ such that $s_{a_{2\sim m+1}} = t$. Because of Proposition 2, we have

$$\mathcal{C}_{|G|} = \bigcup_{s_{a_{2\sim m}+1}^B \leq t \leq 1} \mathcal{D}_t.$$

We also define:

$$
\begin{aligned}
\widetilde{H}_t\big(s_1, s_2, \dots, s_{a_{2\sim m}}, s_{a_{2\sim m}+2}, \dots, s_{|G|}\big) \\
&\coloneqq H\big(s_1, s_2, \dots, s_{a_{2\sim m}}, t, s_{a_{2\sim m}+2}, \dots, s_{|G|}\big) \\
= 1 + (m-1)t - \frac{s_{a_1+1\sim a_1+a_\lambda}}{a_\lambda} - \frac{s_{1\sim a_\lambda}}{a_\lambda} - \sum_{j\neq 1,\lambda}^{m} \left(\frac{s_{a_j+1\sim a_j+a_\lambda}}{a_\lambda}\right) \\
&- \frac{|G|}{|T_j|}.
\end{aligned} \tag{4}
$$

In words, $\widetilde{H}_t$ is a $(|G|-1)$-variable function that is generated from H by regarding $s_{a_{2\sim m}+1} = t$ as a fixed parameter. For simplicity, we write this combination of $|G|-1$ elements as $\left[s_j\right]_{j\neq a_{2\sim m}+1}$. Let

$$
l_1(x) = -\frac{1-t}{a_{2\sim m}}(x-1) + 1
$$

be the equation of the straight line passing through $(1,1)$ and $(a_{2\sim m}+1, t)$. Let $e_1 \coloneqq s_{a_{2\sim m}} - l_1(a_{2\sim m}) (> 0)$, and define

$$
l_2(x) \coloneqq -\big(s_{a_{2\sim m}} - t\big)(x - a_{2\sim m} - 1) + t
$$

as the equation of the straight line passing through $\big(a_{2\sim m}, s_{a_{2\sim m}+1}\big)$ and $(a_{2\sim m}+1, t)$. Because of the concavity of $\big(s_1, s_2, \dots, s_{|G|}\big)$, we have that $s_x \leq l_2(x)$ for all $1 \leq x \leq a_{2\sim m}$.
Let

$$
e_2 \coloneqq \max_{x\in[1, a_{2\sim m}]} \{\min\{l_2(x), 1\} - l_1(x)\}.
$$

Because of concavity, we have that $s_x \leq l_2(x)$ for all $a_1 + 1 \leq x \leq a_1 + a_\lambda$.

Next, consider $\left[s_j^*\right]_{j\neq a_{2\sim m}+1} \in \mathcal{D}_t$ such that

$$
s_x^* \coloneqq \begin{cases} l_1(x) & \text{if } x \neq |G|. \\ 0 & \text{if } x = |G|. \end{cases}
$$

It is straightforward to check that this $\left[s_j^*\right]_{j\neq a_{2\sim m}+1}$ is in $\mathcal{D}_t$. Let

$$
e_3 \coloneqq s_{a_1+\xi}^* - \ l_2\big(a_\xi + 1\big) \ \ (\xi = 1,2, \dots, a_\lambda).
$$

Then we have

$$
e_3 \geq \frac{a_1 + 1 - (a_{2\sim m}+1)}{a_{2\sim m} + 1 - x_e} \cdot e_2 \geq \frac{a_1 - a_{2\sim m}}{a_{2\sim m}} \cdot e_2.
$$

When $a_1 \gg a_{2\sim m}$, we have $e_3 \geq (m-1)\cdot e_2$. This means that the middle three terms in equation (4) :

$$
-\frac{s_{a_1+1\sim a_1+a_\lambda}}{a_\lambda} - \frac{s_{1\sim a_\lambda}}{a_\lambda} - \sum_{j\neq 1,\lambda}^{m} \left(\frac{s_{a_j+1\sim a_j+a_\lambda}}{a_\lambda}\right)
$$

have their minimum value at $\left[s_j^*\right]_{j\neq a_{2\sim m}+1}$ and therefore, $\widetilde{H}_t$ has its minimum value when all the

$s_1, s_2, \dots, s_{|G|}$ are on $l_1(x)$. Let $s^*_{|G|} = 0$. $s^*_1, s^*_2, \dots, s^*_{|G|-1}$ is a sequence of numbers with common difference, and so we have:

$$H(s^*_1, \dots, s^*_{|G|}) = 1 + (m-1)t - \frac{s^*_{a_1+1\sim a_1+a_\lambda}}{a_\lambda} - \frac{s^*_{1\sim a_\lambda}}{a_\lambda} - \sum_{j\neq 1,\lambda}^{m} \left(\frac{s^*_{a_j+1\sim a_j+a_\lambda}}{a_\lambda}\right) - \frac{|G|!}{|T_1|}$$
$$= 1 + (m-1)t - h\left(a_1 + \frac{1+a_\lambda}{2}\right) - h\left(\frac{1+a_\lambda}{2}\right) - \sum_{j\neq 1,\lambda}^{m} h\left(a_j + \frac{1+a_\lambda}{2}\right) - \frac{|G|!}{|T_1|}$$
$$= (1-t)\left[1 - m + \frac{a_1 + \left(\sum_{j\neq 1,\lambda}^{m} a_j\right) + \frac{m}{2}(a_\lambda - 1)}{a_{2\sim m}}\right] - \frac{|G|!}{|T_1|}.$$

If $t = 1$, the proof of $H(s^*_1, \dots, s^*_{|G|}) > 0$ is trivial. Suppose $t < 1$. Then, if a_1 is sufficiently large, the equation denoted by $[\cdots]$ can be large enough to make $H(s^*_1, \dots, s^*_{|G|})$ positive. ∎

Proof of Corollary 2

Let us first confirm that p_α defined above is actually a scoring rule. To see this, we need to check that $s^m_1 = 1$, $s^m_m = 0$, and that $s^m_1, s^m_2, \dots, s^m_m$ is decreasing.

$$s^m_1 = 1 - \left(\frac{1-1}{m-1}\right)^\alpha = 1, \text{and}$$

$$s^m_m = 1 - \left(\frac{m-1}{m-1}\right)^\alpha = 0.$$

Let

$$h(x) := 1 - \left(\frac{x-1}{m-1}\right)^\alpha.$$

This function is clearly decreasing, and so we have $s^m_1 = h(1), s^m_2 = h(2), \dots, s^m_m = h(m)$ is also decreasing.

Next, we show that p_α is concave. By taking the second derivative of $h(x)$, we have

$$h''(x) = -\frac{\alpha(\alpha-1)}{(m-1)^\alpha}(x-1)^{\alpha-2} \leq 0.$$

This shows that is $h(x)$ concave. Therefore, p_α is concave. ∎

Proof of Proposition 5

It will be sufficient to show a proof for (3) φ_P, because the other cases are straightforward.

WPU and NU: It is obvious that φ_P satisfies WPU; I will show that φ_P also satisfies NU. Take any $i \in \overline{N}$ and $\mathcal{N} \in \mathfrak{N}$ such that $s_i(\mathcal{N}) = 0$. Because abstention is not allowed, there must be an individual $\mu \in \overline{N}$ that has a score of at least two. Hence, we can say that $i \notin F_{\mathcal{N}} \cup S_{\mathcal{N}}$, which implies $i \notin \varphi_P(\mathcal{N})$.

IMP: Take any individual $i \in \overline{N}$ and ballot profiles $\mathcal{N} = (N_i, N_{-i}), \mathcal{N}' = (N'_i, N_{-i}) \in \mathfrak{N}$.

1) The Case of $i \in F_{\mathcal{N}}$. It follows that $i \in \varphi_P(\mathcal{N})$, and we will show $i \in \varphi_P(\mathcal{N}')$. Note that for all $j \in \overline{N}$, $s_j(\mathcal{N}') \leq s_j(\mathcal{N}) + 1$, where the equality holds only if $j \in N_i'$. So, we have either $i \in F_{\mathcal{N}'}$ or $[i \in S_{\mathcal{N}'}$ and $F_{\mathcal{N}'} \subseteq N_i']$. Thus, $i \in \varphi_P(\mathcal{N}')$.

2) The Case of $i \notin F_{\mathcal{N}}$. Let us first consider the case of $i \in \varphi_P(\mathcal{N})$. It follows that $i \in S_{\mathcal{N}}$ and $F_{\mathcal{N}} \subseteq N_i$. These statements show that for any $j \in \overline{N}$, $s_j(\mathcal{N}') \leq s_{F_{\mathcal{N}}}(\mathcal{N}) = s_i(\mathcal{N}) + 1 = s_i(\mathcal{N}') + 1$. Indeed, $s_j(\mathcal{N}') = s_{F_{\mathcal{N}}}(\mathcal{N})$ holds only if $j \in N_i'$. Thus, we have again either $i \in F_{\mathcal{N}'}$ or $[i \in S_{\mathcal{N}'}$ and $F_{\mathcal{N}'} \subseteq N_i']$, either of which implies $i \in \varphi_P(\mathcal{N}')$.

Next, consider the case of $i \notin \varphi_P(\mathcal{N})$. I will show that $i \notin \varphi_P(\mathcal{N}')$. Because $i \notin \varphi_P(\mathcal{N})$, there is an individual $j \in \overline{N} \setminus \{i\}$ such that either (a) $s_j(\mathcal{N}) \geq s_i(\mathcal{N}) + 2$ or (b) $s_j(\mathcal{N}) = s_i(\mathcal{N}) + 1$ and $j \notin N_i'$. In the case of (a), it is clear that $s_j(\mathcal{N}') \geq s_j(\mathcal{N}) - 1 \geq s_i(\mathcal{N}) + 1 = s_i(\mathcal{N}') + 1$, where $s_j(\mathcal{N}') = s_j(\mathcal{N}) - 1$ holds only if $j \notin N_i'$. So, we have $i \notin \varphi_P(\mathcal{N}')$. In the case of (b), we have $s_j(\mathcal{N}') \geq s_j(\mathcal{N}) = s_i(\mathcal{N}') + 1 = s_i(\mathcal{N}) + 1$, where the equality in the first inequality holds only if $j \notin N_i'$. Again, we can say that $i \notin \varphi_P(\mathcal{N}')$.

Non-2CN: Here I will use a counterexample. Consider a ballot profile $\mathcal{N} = (N_1, \ldots, N_n) \in \mathfrak{N}^1 \subseteq \mathfrak{N}$ (the following proof holds for both settings $(\mathfrak{N}, \overline{\mathfrak{X}})$ and $(\mathfrak{N}^1, \overline{\mathfrak{X}})$) as follows:

$$N_1 = \{3\}, N_2 = \{1\}, \text{and}$$
$$N_i = \{\overline{i+1}\} \text{ for all } i \in \overline{N} \setminus \{1,2\}.$$

Then we have $s_1(\mathcal{N}) = 2$, $s_2(\mathcal{N}) = 0$, $s_3(\mathcal{N}) = \cdots = s_n(\mathcal{N}) = 1$, $F_{\mathcal{N}} = \{1\}$ and $S_{\mathcal{N}} = \overline{N} \setminus \{1,3\}$. Furthermore, we have $\varphi_P(\mathcal{N}) = \{1,2,n\}$. Now consider a transposed ballot profile $\mathcal{N}^\sigma$, where $\sigma = (2,3)$. In this case we have $F_{\mathcal{N}^\sigma} = \{1\}$ and $S_{\mathcal{N}^\sigma} = \overline{N} \setminus \{1,2\}$. Because $1 \notin N_3$, we have $3 \notin \varphi_P(\mathcal{N}^\sigma)$, whereas 2CN demands that $3 \in \varphi_P(\mathcal{N}^\sigma)$.

AB: It follows from Theorems 1 and 2 of Tamura and Ohseto (2014) that φ_P does not satisfy AB under $(\mathfrak{N}^1, \overline{\mathfrak{X}})$ if $n \geq 4$. In fact, we can generalize their result as follows:

1) φ_P satisfies AB if $n = 3$ both under $(\mathfrak{N}, \overline{\mathfrak{X}})$ and $(\mathfrak{N}^1, \overline{\mathfrak{X}})$.
2) φ_P satisfies AB if $n \geq 4$ both under $(\mathfrak{N}, \overline{\mathfrak{X}})$ and $(\mathfrak{N}^1, \overline{\mathfrak{X}})$.

Note that the following proof applies for both settings.

1) If $n = 3$, φ_P satisfies AB. Take any two ballot profiles $\mathcal{N}, \mathcal{M} \in \mathfrak{N}$ such that $s(\mathcal{N}) = s(\mathcal{M})$. I will show that $\varphi(\mathcal{N}) = \varphi(\mathcal{M})$. Let $t := \#\{i \in \overline{N} \mid s_i(\mathcal{N}) = s_i(\mathcal{M}) = 2\}$.

a) The case of $t = 3$. We have $\varphi_P(\mathcal{N}) = \varphi_P(\mathcal{M}) = \overline{N}$, because $F_{\mathcal{N}} = F_{\mathcal{M}} = \overline{N}$.

b) The case of $t = 2$. Suppose $s_i(\mathcal{N}) = s_j(\mathcal{N}) = 2 > s_k(\mathcal{N})$. Because $F_{\mathcal{N}} = F_{\mathcal{M}} = \{i, j\}$, we have $\{i, j\} \subseteq F_{\mathcal{N}} \cap F_{\mathcal{M}}$. Furthermore, $s_i(\mathcal{N}) = s_j(\mathcal{N}) = 2$ implies $(F_{\mathcal{N}} = F_{\mathcal{M}} =)\{i, j\} \subseteq N_k, M_k$, so whether k is in $\varphi_P(\mathcal{N})$ is determined thoroughly by k belonging to $S_{\mathcal{N}}$ and $S_{\mathcal{M}}$, which is also entirely determined by the score profile. Thus, we have $\varphi_P(\mathcal{N}) = \varphi_P(\mathcal{M})$.

c) The case of $t = 1$. Suppose $2 = s_i(\mathcal{N}) > s_j(\mathcal{N}) \geq s_k(\mathcal{N})$. Then $s_i(\mathcal{N}) = s_i(\mathcal{M}) = 2$ implies $i \in N_j \cap N_k \cap M_j \cap M_k$. Recall that in $\sum_{l=1}^{n} s_l \geq n$, there are only two possibilities for the value of the score profile: $(s_i, s_j, s_k) = (2,1,1)$ and $(s_i, s_j, s_k) = (2,1,0)$. If $(s_i, s_j, s_k) = (2,1,1)$, it follows that $\{j,k\} \subseteq S_{\mathcal{N}} \cap S_{\mathcal{M}}$ and we have $\{j,k\} \subseteq \varphi_P(\mathcal{N}) \cap \varphi_P(\mathcal{M})$. On the other hand, if $(s_i, s_j, s_k) = (2,1,0)$, we have $S_{\mathcal{N}} = S_{\mathcal{M}} = \{j\}$ and therefore $j \in \varphi_P(\mathcal{N}) \cap \varphi_P(\mathcal{M})$. In either case, $k \notin \varphi_P(\mathcal{N}) \cup \varphi_P(\mathcal{M})$.

d) The case of $t = 0$**.** Because $\sum_{l=1}^{n} s_l \geq n$, the only possible score profile is $s(\mathcal{N}) = s(\mathcal{M}) = (1,1,1)$. Because $F_{\mathcal{N}} = F_{\mathcal{M}} = \bar{N}$, we have $\varphi_P(\mathcal{N}) = \varphi_P(\mathcal{M}) = \bar{N}$.

In all four of these cases, I have shown that $\varphi_P(\mathcal{N}) = \varphi_P(\mathcal{M})$. Because $\mathcal{N}, \mathcal{M}$ are arbitrary ballot profiles with the same score profile, this means that φ_P satisfies AB if $n = 3$.

2) If $n \geq 4$, φ_P does not satisfy AB. I will provide a counterexample. Consider two ballot profiles $\mathcal{N} = (N_1, \dots, N_n), \mathcal{M} = (M_1, \dots, M_n) \in \mathfrak{N}^1 \subseteq \mathfrak{N}$ as follows:

$$N_1 = \{3\}, N_2 = \{1\}$$
$$N_i = \{\overline{\imath + 1}\} \text{ for all } i \in \bar{N} \setminus \{1,2\}$$

and

$$M_1 = \{3\}, M_2 = \{4\}, M_3 = \{1\}$$
$$M_i = \{\overline{\imath + 1}\} \text{ for all } i \in \bar{N}$$

Because we have $s(\mathcal{N}) = s(\mathcal{M}) = (2,0,1,1,\dots)$, AB demands that $\varphi_P(\mathcal{N}) = \varphi_P(\mathcal{M})$. However, we can check that $3 \notin \varphi_P(\mathcal{N})$ and $3 \in \varphi_P(\mathcal{M})$. This shows that φ_P does not satisfy AB if $n \geq 4$. ■

Proof of Lemma 7

Case 1: $\mathfrak{D} = \mathfrak{N}^k$. To prove case 1, we provide two lemmas.

Lemma 12. Let $\phi \neq \mathfrak{X} \subseteq \mathfrak{P}(\bar{N}) \setminus \{\phi\}$ and $\varphi: \mathfrak{N}^k \to \mathfrak{X}$ be a nomination rule that satisfies IMP and AB. For any distinct individuals $i, \alpha, \beta \in \bar{N}$ and for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$ such that $i \in N_\beta$, and $i \notin N_\alpha$, there is a ballot profile $\mathcal{N}' \in \mathfrak{N}^k$ such that $\mathcal{N} \sim_i \mathcal{N}'$, $i \in N'_\alpha$, $i \notin N'_\beta$, and $i \in N_\gamma \Leftrightarrow i \in N'_\gamma$ for all $\gamma \in \bar{N} \setminus \{i, \alpha, \beta\}$.

Proof of Lemma 12. If there is an individual $\mu \in \bar{N} \setminus \{i, \alpha, \beta\}$ such that $\mu \in N_\alpha$ and $\mu \notin N_\beta$, then let $\mathcal{N}' = (N'_1, \dots, N'_n) \in \mathfrak{N}^k$ be such that:

$$N'_\alpha = (N_\alpha \cup \{i\}) \setminus \{\mu\},$$
$$N'_\beta = (N_\beta \cup \{\mu\}) \setminus \{i\}, and$$
$$N'_\gamma = N_\gamma \text{ for all } \gamma \in \bar{N} \setminus \{\alpha, \beta\}.$$

Because $s(\mathcal{N}) = s(\mathcal{N}')$, AB demands $\mathcal{N} \sim_i \mathcal{N}'$. Therefore, because we also have $i \in N'_\alpha$, $i \notin N'_\beta$, and $N'_\gamma = N_\gamma$ for all $\gamma \in \bar{N} \setminus \{\alpha, \beta\}$, the lemma holds.

Suppose there is no such individual μ. Then for any $\mu \in \bar{N} \setminus \{i, \alpha, \beta\}$, $\mu \in N_\alpha$ implies $\mu \in N_\beta$. It follows that $N_\alpha \setminus \{i, \alpha, \beta\} \subseteq N_\beta \setminus \{i, \alpha, \beta\}$. Because we also have $|N_\alpha| = |N_\beta|$, it follows that $|N_\alpha \cap \{i, \alpha, \beta\}| \geq |N_\beta \cap \{i, \alpha, \beta\}|$. Recall that we have $N_\alpha \cap \{i, \alpha\} = \phi$ and $i \in N_\beta$ by the assumptions. Therefore, we can say that $N_\alpha \cap \{i, \alpha, \beta\} = \{\beta\}$ and $N_\beta \cap \{i, \alpha, \beta\} = \{i\}$.

Because $1 \leq k \leq n - 2$, i's ballot $M_i \in \mathfrak{N}_i^k$ exists such that $\alpha \in M_i$ and $b \notin M_i$. Let us define $\mathcal{M} = (M_i, M_{-i}) \in \mathfrak{N}^k$ as $M_{-i} = N_{-i}$. IMP demands $\mathcal{N} \sim_i \mathcal{M}$. Note that we have $M_\alpha \cap$

$\{i,\alpha,\beta\} = \{\beta\}$, $M_\beta \cap \{i,\alpha,\beta\} = \{i\}$, and $M_i \cap \{i,\alpha,\beta\} = \{\alpha\}$. Therefore, we can construct $\mathcal{M}' = (M_1', \dots, M_n') \in \mathfrak{N}^k$ as follows:

$$M_i' = (M_i \cup \{\beta\}) \setminus \{\alpha\},$$
$$M_\alpha' = (M_\alpha \cup \{i\}) \setminus \{\beta\},$$
$$M_\beta' = (M_\beta \cup \{\alpha\}) \setminus \{i\}, and$$
$$M_\gamma' = M_\gamma = N_\gamma \text{ for all } \gamma \in \bar{N} \setminus \{i,\alpha,\beta\}.$$

By AB, we have $\mathcal{M} \sim_{\mathrm{i}} \mathcal{M}'$. It is clear that $\mathcal{M}'$ has the necessary properties. ∎

Lemma 13.

Let $\phi \neq \mathfrak{X} \subseteq \mathfrak{P}(\bar{N}) \setminus \{\phi\}$ and $\varphi: \mathfrak{N}^k \to \mathfrak{X}$ be a nomination rule that satisfies IMP and AB. Take any individual $i \in \bar{N}$ and a ballot profile $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{N}^k$. For any $\mu, \lambda, \nu \in \{1,2,\dots,n-1\}$, if $\overline{\imath+\lambda} \notin N_{\overline{\imath+\mu}}$ and $\overline{\imath+\nu} \in N_{\overline{\imath+\mu}}$, then we have $\mathcal{N} \sim_i \mathcal{N}' = \left(N_{\overline{\imath+\mu}}', N_{-\overline{\imath+\mu}}'\right)$, where $N_{\overline{\imath+\mu}}' = \left(N_{\overline{\imath+\mu}} \cup \left\{\overline{\imath+\lambda}\right\}\right) \setminus \{\overline{\imath+\nu}\}$.

Proof of Lemma 13. Take any $i \in \bar{N}$, $\mathcal{N} \in \mathfrak{N}^k$ and integers $\mu, \lambda, \nu \in \{1,2,\dots,n-1\}$ such that $\overline{\imath+\lambda} \notin N_{\overline{\imath+\mu}}$ and $\overline{\imath+\nu} \in N_{\overline{\imath+\mu}}$. Because $1 \le k \le n-2$, i's ballot $M_i \in \mathfrak{N}_i^k$ exists such that $\overline{\imath+\lambda} \in M_i$ and $\overline{\imath+\nu} \notin M_i$. Let $\mathcal{M} = (M_i, M_{-i}) \in \mathfrak{N}^k$ be such that $M_{-i} = N_{-i}$. IMP demands $\mathcal{N} \sim_i \mathcal{M}$. Define $\mathcal{M}' = (M_1', \dots, M_n') \in \mathfrak{N}^k$ as follows:

$$M_i' = (M_i \cup \{\overline{\imath+\nu}\}) \setminus \left\{\overline{\imath+\lambda}\right\}.$$
$$M_{\overline{\imath+\mu}}' = \left(M_{\overline{\imath+\mu}} \cup \left\{\overline{\imath+\lambda}\right\}\right) \setminus \{\overline{\imath+\nu}\}.$$
$$M_x' = M_x \text{ for all } x \in \bar{N} \setminus \{i, \overline{\imath+\mu}\}.$$

By AB, we have $\mathcal{M} \sim_i \mathcal{M}'$. Finally, let us define $\mathcal{M}'' = (M_i'', M_{-i}'') \in \mathfrak{N}^k$ as $M_i'' = N_i''$ and $M_{-i}'' = N_{-i}''$. IMP demands $\mathcal{M}' \sim_{\mathrm{i}} \mathcal{M}''$. Clearly, $\mathcal{M}''$ has all the properties required for $\mathcal{N}'$. ∎

***Proof of Case 1**:* $\mathfrak{D} = \mathfrak{N}^k$ (Lemma 7). Suppose a nomination rule $\varphi: \mathfrak{N}^k \to \mathfrak{X}$ satisfies IMP and AB. Fix any $i \in \bar{N}$ through the proof. Let us partition the ballot profile domains $\mathfrak{N}^k$ to $\mathfrak{M}^0, \mathfrak{M}^1, \dots, \mathfrak{M}^{n-1}$, where:

$$\mathfrak{N}^k = \mathfrak{M}^0 \cup \mathfrak{M}^1 \cup \dots \cup \mathfrak{M}^{n-1}, and$$
$$\mathfrak{M}^d = \left\{\mathcal{N} \in \mathfrak{N}^k \,\middle|\, s_i(\mathcal{N}) = d\right\}.$$

In words, $\mathfrak{M}^d$ is the set of ballot profiles where individual i gets score d. For any $d \in \{0,1,\dots,n-1\}$, we define a ballot profile $\mathcal{M}^d = \left(M_1^d, \dots, M_n^d\right) \in \mathfrak{M}^d$ as follows:

$$M_i^d = \left\{\overline{\imath+1}, \dots, \overline{\imath+k}\right\}.$$
$$M_{\overline{\imath+\mu}}^d = \left\{i, \overline{\imath+1}, \dots, \overline{\imath+k}\right\} \setminus \{\overline{\imath+\mu}\} \text{ if } 1 \le \mu \le \min\{d, k+1\}.$$
$$M_{\overline{\imath+\mu}}^d = \left\{\overline{\imath+1}, \dots, \overline{\imath+k+1}\right\} \setminus \{\overline{\imath+\mu}\} \text{ if } d < \mu \le k+1.$$
$$M_{\overline{\imath+\mu}}^d = \left\{i, \overline{\imath+1}, \dots, \overline{\imath+k-1}\right\} \text{ if } k+1 < \mu \le d.$$

$$M^d_{\overline{\iota+\mu}} = \{\overline{\iota+1}, \dots, \overline{\iota+k}\} \text{ if } \max\{d, k+1\} < \mu \leq n-1.$$

In words, this is a ballot profile where individuals' approvals are shifted toward $i, \overline{\iota+1}, \dots, \overline{\iota+k}, \overline{\iota+k+1}$ without changing i's score. Note that for any $d \in \{0,1,\dots,n-1\}$, $\mathcal{M}^d$ is uniquely determined and $\mathcal{M}^d \in \mathfrak{M}^d$. I show that for any d and $\mathcal{N} \in \mathfrak{M}^d$, we have $\mathcal{N} \sim_i \mathcal{M}^d$. Showing this completes the proof because $\mathcal{N}, \mathcal{N}' \in \mathfrak{M}^d$ implies $\mathcal{N} \sim_i \mathcal{M}^d$ and $\mathcal{N}' \sim_i \mathcal{M}^d$, thus $\mathcal{N} \sim_i \mathcal{N}'$.

Take any $d \in \{0,1,\dots,n-1\}$ and $\mathcal{N} \in \mathfrak{M}^d$. With the repetition of the procedure in Lemma 12, we obtain $\mathcal{N}^1 \in \mathfrak{N}^k$ such that $\mathcal{N} \sim_i \mathcal{N}^1$ and only the individuals in $\{\overline{\iota+x} \mid 1 \leq x \leq d\}$ approve i. Next, $\mathcal{N}^2 \in \mathfrak{M}^d$ is defined as $N^2_i = \{\overline{\iota+1}, \dots, \overline{\iota+k}\}$ and $N^2_{-i} = N^1_{-i}$. IMP demands $\mathcal{N}^1 \sim_i \mathcal{N}^2$. Starting from $\mathcal{N}^2$, we sequentially transform $N^2_{\overline{\iota+\mu}}$ to $M^d_{\overline{\iota+\mu}}$ for each $\mu = 1,2,\dots,n-1$ as follows:

$$\mathcal{N}^2 = \left(M^d_i, N^2_{\overline{\iota+1}}, N^2_{\overline{\iota+2}}, N^2_{\overline{\iota+3}} \dots, N^2_{\overline{\iota+n-1}}\right).$$
$$\text{Next, } \left(M^d_i, M^d_{\overline{\iota+1}}, N^2_{\overline{\iota+2}}, N^2_{\overline{\iota+3}} \dots, N^2_{\overline{\iota+n-1}}\right).$$
$$\text{Next, } \left(M^d_i, M^d_{\overline{\iota+1}}, M^d_{\overline{\iota+2}}, N^2_{\overline{\iota+3}} \dots, N^2_{\overline{\iota+n-1}}\right).$$
$$\vdots$$
$$\text{Finally, } \left(M^d_i, M^d_{\overline{\iota+1}}, M^d_{\overline{\iota+2}}, M^d_{\overline{\iota+3}} \dots, M^d_{\overline{\iota+n-1}}\right) = \mathcal{M}^d.$$

Because we have $i \in N^2_x \Leftrightarrow i \in M^d_x$ for all $x \in \overline{N} \setminus \{i\}$, the only difference between N^2_x and M^2_x is on individuals other than i. Recall that Lemma 13 states that the substitution of the approval toward $\overline{\iota+\nu} \in \overline{N} \setminus \{i\}$ with that toward $\overline{\iota+\lambda}$ in someone's ballot retains i-equivalence. Therefore, the above procedure from N^2_x to M^d_x maintains i-equivalence. Hence, we have $\mathcal{N}^2 \sim_i \mathcal{M}^d$. ■

Case 2: $\mathfrak{D} = \mathfrak{N}$. For the proof of case 2, we provide another lemma.

Lemma 14. Let $\phi \neq \mathfrak{X} \subseteq \mathfrak{P}(\overline{N}) \setminus \{\phi\}$ and $\varphi: \mathfrak{N} \to \mathfrak{X}$ be a nomination rule that satisfies IMP and AB. Fix any individual $i \in \overline{N}$. For any ballot profile $\mathcal{N} \in \mathfrak{N}$, if there is an individual $j \in \overline{N}$ such that $|N_j| \geq 2$, then, for any $a \in N_j \setminus \{i\}$, we have that $\mathcal{N}' = (N'_j, N'_{-j}) \in \mathfrak{N}$ and $\mathcal{N}$ are i-equivalent, where $N'_j = N_j \setminus \{a\}$.

Proof of Lemma 14. If $j = i$, the lemma is obvious by IMP. Suppose $j \neq i$ and $|N_j| \geq 2$ at $\mathcal{N} \in \mathfrak{N}$. Take $a \in N_j \setminus \{i\}$. Let $\mathcal{M} \in \mathfrak{N}$ as $M_i = \{j\}$ and $M_{-i} = N_{-i}$. Then we have $a \notin N_i$ and $a \in N_j$ and can define $\mathcal{M}' \in \mathfrak{N}$ as $M'_i = M_i \cup \{\mu\}$, $M'_j = M_j \setminus \{\mu\}$, and $M'_x = M_x$ for all $x \in \overline{N} \setminus \{i,j\}$. Because $s(\mathcal{M}) = s(\mathcal{M}')$, we have $\mathcal{M} \sim_i \mathcal{M}'$. Furthermore, let $\mathcal{M}'' \in \mathfrak{N}$ as $M''_i = N_i$ and $M''_{-i} = M'_{-i}$. IMP demands $\mathcal{M}' \sim_i \mathcal{M}''$. Therefore, $\mathcal{M}''$ satisfies the required property. ■

Proof of Case 2: $\mathfrak{D} = \mathfrak{N}$. Take any $\mathcal{N}^1, \mathcal{N}^2 \in \mathfrak{N}$ such that $s_i(\mathcal{N}^1) = s_i(\mathcal{N}^2)$. For each of these, we can iterate the procedure in Lemma 14 until everyone's ballot becomes a singleton. Let $\mathcal{L}^1, \mathcal{L}^2$ be the final outputs of $\mathcal{N}^1$ and $\mathcal{N}^2$, respectively. Then, we have $\mathcal{N}^1 \sim_i \mathcal{L}^1$, $\mathcal{N}^2 \sim_i \mathcal{L}^2$, and

$\mathcal{L}^1, \mathcal{L}^2 \in \mathfrak{N}^1 \subseteq \mathfrak{N}$. Because $s(\mathcal{L}^1) = s(\mathcal{L}^2)$, case 1 shows that $\mathcal{L}^1 \sim_i \mathcal{L}^2$. Hence, we determine that $\mathcal{N}^1 \sim_i \mathcal{N}^2$. ■

Case 3: $\mathfrak{N} = \mathfrak{N}^{self}$.

Lemma 15. Let $\phi \neq \mathfrak{X} \subseteq \mathfrak{P}(\bar{N}) \setminus \{\phi\}$ and $\varphi: \mathfrak{N}^{self} \to \mathfrak{X}$ be a nomination rule that satisfies IMP and AB. For all $i \in \bar{N}$ and $\mathcal{N} \in \mathfrak{N}^{self}$, if $j \in \bar{N} \setminus \{i\}$ exists such that $j \in N_j$, then there is a ballot profile $\mathcal{N}' \in \mathfrak{N}^{self}$ such that $\mathcal{N} \sim_i \mathcal{N}'$, $s_i(\mathcal{N}) = s_i(\mathcal{N}')$, $j \notin N_j'$, and $N_x = N_x'$ for all $x \in \bar{N} \setminus \{j\}$.

Proof of Lemma 15. Assume $j \in N_j$ for some $j \in \bar{N} \setminus \{i\}$. Take any $\mu \in \bar{N} \setminus \{i, j\}$. Let $\mathcal{M} = (M_i, M_{-i}) \in \mathfrak{N}^{self}$, where $M_i = \{\mu\}$ and $M_{-i} = N_{-i}$. Then IMP demands $\mathcal{N} \sim_i \mathcal{M}$. Let us consider two cases, (a) and (b).

a) The case of $\mu \in M_j$. Let $\mathcal{M}' = (M_1', \dots, M_n') \in \mathfrak{N}^{self}$ such that $M_i' = M_i \cup \{j\}$, $M_j' = M_j \setminus \{j\}$, and $M_\nu' = M_\nu$ for all $\nu \in \bar{N} \setminus \{i, j\}$. Note that the assumption of $\mu \in M_j$ guarantees that $M_j' \neq \phi$. By AB, we have $\mathcal{M} \sim_i \mathcal{M}'$. Let $\mathcal{M}'' \in \mathfrak{N}^{self}$ because $M_i'' = N_i$ and $M_{-i}'' = M_{-i}'$. IMP demands $\mathcal{M}' \sim_i \mathcal{M}''$. Clearly, this ballot profile $\mathcal{M}''$ satisfies the required properties.

b) The case of $\mu \notin M_j$**.** Let $\mathcal{M}''' = (M_1''', \dots, M_n''') \in \mathfrak{N}^{self}$ such that $M_i''' = (M_i \cup \{j\}) \setminus \{\mu\}$, $M_j'' = (M_j \cup \{\mu\}) \setminus \{j\}$, and $M_\nu'' = M_\nu$ for all $\nu \in \bar{N} \setminus \{i, j\}$. AB implies $\mathcal{M} \sim_i \mathcal{M}'''$. Let $\mathcal{M}'''' = (M_i'''', M_{-i}'''') \in \mathfrak{N}^{self}$ because $M_i'''' = N_i$ and $M_{-i}'''' = M_{-i}'''$. IMP demands $\mathcal{M}''' \sim_i \mathcal{M}''''$. Clearly, this ballot profile $\mathcal{M}''''$ satisfies the required properties. ■

Proof of Case 3: $\mathfrak{D} = \mathfrak{N}^{self}$. Take any individual $i \in \bar{N}$ and ballot profiles $\mathcal{N}^1, \mathcal{N}^2 \in \mathfrak{N}^{self}$ such that $s_i(\mathcal{N}^1) = s_i(\mathcal{N}^2)$. The iteration of the procedure in Lemma 15 for each of $\mathcal{N}^1, \mathcal{N}^2$ will give $\mathcal{M}^1, \mathcal{M}^2 \in \mathfrak{N}^{self}$ such that:

$$j \notin M_j^1 \text{ and } j \notin M_j^2 \text{ for all } j \neq i,$$
$$s_i(\mathcal{M}^1) = s_i(\mathcal{M}^2), \text{ and}$$
$$\mathcal{N}^1 \sim_i \mathcal{M}^1 \text{ and } \mathcal{N}^2 \sim_i \mathcal{M}^2.\cdot$$

If $i \notin M_i^1$ and $i \notin M_i^2$, we have $\mathcal{M}^1, \mathcal{M}^2 \in \mathfrak{N}$. Then it follows from case 2 that $\mathcal{M}^1 \sim_i \mathcal{M}^2$ and the proof is done. Therefore, without loss of generality we can focus on two cases: 1) $i \in M_i^1$ and $i \notin M_i^2$, and 2) $i \in M_i^1 \cap M_i^2$. Note that in either case it is enough to prove the i-equivalence of $\mathcal{M}^1$ and $\mathcal{M}^2$.

1) The case of $i \in M_i^1$ *and* $i \notin M_i^2$. Because $s_i(\mathcal{M}^1) = s_i(\mathcal{M}^2)$, $j \in \bar{N} \setminus \{i\}$ exists such that $i \notin M_j^1$. Take an individual $\mu \in M_j^1$. Note that $l \neq j$. Let us define $\mathcal{M}^3 = (M_i^3, M_{-i}^3) \in \mathfrak{N}^{self}$ as $M_i^3 = \{i\}$ and $M_{-i}^3 = M_{-i}^1$. IMP demands $\mathcal{M}^1 \sim_i \mathcal{M}^3$. Note that we have $i \in M_i^3$, $\mu \notin M_i^3$, $i \notin M_j^3$, and $\mu \in M_j^3$. We define $\mathcal{M}^4 \in \mathfrak{N}^{self}$ as follows:

$$M_i^4 = (M_i^3 \cup \{\mu\}) \setminus \{i\},$$
$$M_j^4 = (M_j^3 \cup \{i\}) \setminus \{\mu\}, \text{and}$$

$$M_x^4 = M_x^3 \text{ for all } x \in \overline{N} \setminus \{i, j\}.$$

AB demands $\mathcal{M}^3 \sim_i \mathcal{M}^4$. Note that no one makes a self-approval at $\mathcal{M}^4$, so we have $\mathcal{M}^2, \mathcal{M}^4 \in \mathfrak{N}$. Furthermore, we have $s_i(\mathcal{M}^4) = (= s_i(\mathcal{M}^3) = s_i(\mathcal{M}^1) =) s_i(\mathcal{M}^2)$. Case 2 shows that $\mathcal{M}^4 \sim_i \mathcal{M}^2$, which means that $\mathcal{M}^1 \sim_i \mathcal{M}^2$.

2) The case of $i \in M_i^1 \cap M_i^2$. Let us define ballot profiles $\mathcal{K}^1, \mathcal{K}^2 \in \mathfrak{N}^{self}$ as follows:

$$K_i^1 = (M_i^1 \cup \{\overline{\imath + 1}\}) \setminus \{i\}, K_{-i}^1 = M_{-i}^1, \text{and}$$
$$K_i^2 = \left(M_i^2 \cup \{\overline{\imath + 1}\}\right) \setminus \{i\}, K_{-i}^2 = M_{-i}^2.$$

IMP demands $\mathcal{M}^1 \sim_i \mathcal{K}^1$ and $\mathcal{M}^2 \sim_i \mathcal{K}^2$. Note that $\mathcal{K}^1, \mathcal{K}^2 \in \mathfrak{N}$. Because $s_i(\mathcal{K}^1) = s_i(\mathcal{M}^1) - 1 = s_i(\mathcal{M}^2) - 1 = s_i(\mathcal{K}^2)$, case 2 shows $\mathcal{K}^1 \sim_i \mathcal{K}^2$, which yields $\mathcal{M}^1 \sim_i \mathcal{M}^2$. ■

Case 4: $\mathfrak{N} = \mathfrak{N}^{AB}$. I will first provide a lemma.

Lemma 16. Let $\varphi: \mathfrak{N}^{AB} \to \mathfrak{X}$ be a nomination rule that satisfies IMP and AB. For all $i \in \overline{N}$ and $\mathcal{N} \in \mathfrak{N}^{self}$, if $j \in \overline{N}$ exists such that $N_j = \phi$, then there is a ballot profile $\mathcal{N}' \in \mathfrak{N}^{AB}$ such that $\mathcal{N} \sim_i \mathcal{N}'$, $s_i(\mathcal{N}) = s_i(\mathcal{N}')$, $N_j' \neq \phi$, and $N_x = N_x'$ for all $x \in \overline{N} \setminus \{j\}$.

Proof of Lemma 16. The case of $j = i$ is straightforward from IMP. Assume $j \neq i$ and take any $\mathcal{N} \in \mathfrak{N}^{AB}$. Because $n \geq 3$, $\mu \in \overline{N} \setminus \{i, j\}$ exists. Let us consider $\mathcal{M} \in \mathfrak{N}^{AB}$, where $M_i = \{\mu, j\}$ and $M_{-i} = N_{-i}$. IMP implies $\mathcal{N} \sim_i \mathcal{M}$. Then consider a ballot profile $\mathcal{M}' = (M_1', \ldots, M_n') \in \mathfrak{N}^{AB}$ as follows:

$$M_i' = M_i \setminus \{\mu\},$$
$$M_j' = \{\mu\}, \text{and}$$
$$M_\nu' = M_\nu \text{ for all } \nu \in \overline{N} \setminus \{i, j\}.$$

Because $s(\mathcal{M}) = s(\mathcal{M}')$, AB implies $\mathcal{M} \sim_i \mathcal{M}'$. Finally, let us define $\mathcal{M}'' = (M_i'', M_{-i}'') \in \mathfrak{N}^{AB}$, where $M_i'' = N_i$ and $M_{-i}'' = M_{-i}'$. IMP implies $\mathcal{M}' \sim_i \mathcal{M}''$. Clearly, $\mathcal{M}''$ has the required properties. ■

Proof of Case 4: $\mathfrak{N} = \mathfrak{N}^{AB}$. Take any individual $i \in \overline{N}$ and ballot profiles $\mathcal{N}^1, \mathcal{N}^2 \in \mathfrak{N}^{AB}$ such that $d \coloneqq s_i(\mathcal{N}^1) = s_i(\mathcal{N}^2)$. By iterating the procedure in Lemma 16 from $\mathcal{N}^1, \mathcal{N}^2$ until there is no abstention, we obtain $\mathcal{K}^1, \mathcal{K}^2 \subseteq \mathfrak{N}^{AB}$ such that $\mathcal{N}^1 \sim_i \mathcal{K}^1$, $\mathcal{N}^2 \sim_i \mathcal{K}^2$, and $\mathcal{K}^1, \mathcal{K}^2 \in \mathfrak{N}$. Because the procedure does not change i's score, we have $s_i(\mathcal{K}^1) = s_i(\mathcal{K}^2) = d$. By case 2, we have $\mathcal{K}^1 \sim_i \mathcal{K}^2$, which implies $\mathcal{N}^1 \sim_i \mathcal{N}^2$. ■

Proof of Lemma 8

Case 1: $\mathfrak{D} = \mathfrak{N}^k$. Let $\mathbb{T}$ be the right-hand side of the equality in the lemma. Because $\mathbb{S}[\mathfrak{N}^k] \subseteq \mathbb{T}$ is obvious, I will show $\mathbb{T} \subseteq \mathbb{S}[\mathfrak{N}^k]$. Take any $s = (s_1, \ldots, s_n) \in \mathbb{T}$. I show that $\mathcal{N} \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}) = s$.

I define a class of sets of sets, called assignments, as follows. For any $i \in \bar{N}$, a set of sets $\mathcal{N}^i = (N_1^i, \ldots, N_n^i)$ is called an assignment (from 1) until i (with respect to $s = (s_1, \ldots, s_n)$) if and only if:

$$N_j^i \subseteq \bar{N} \setminus \{i\} \text{ for all } j \in \bar{N},$$
$$|N_j^i| \leq k \text{ for all } j \in \bar{N},$$
$$s_j(\mathcal{N}^i) = s_i \text{ for all } j \leq i, \text{and}$$
$$s_j(\mathcal{N}^i) = 0 \text{ for all } j > i.$$

With a slight abuse of notation, I define $s_j(\mathcal{N}^i) := |\{\mu \in \bar{N} \mid j \in N_\mu^i\}|$ and also $s(\mathcal{N}^i) = (s_1(\mathcal{N}^i), \ldots, s_n(\mathcal{N}^i))$. I denote by $\mathfrak{I}^i$ the set of all assignments until i. Note that an assignment until i, $\mathcal{N}^i \in \mathfrak{I}^i$, expresses a way to take s_1 individuals from $\bar{N} \setminus \{1\}$ so that they approve individual 1 in their ballots, s_2 individuals from $\bar{N} \setminus \{2\}$ so that they can approve individual 2 in their ballots, …, and s_i individuals so that they can approve individual i in their ballots. Note also that if there is an assignment until n, and $\mathcal{N}^n = (N_1^n, \ldots, N_n^n) \in \mathfrak{I}^n$, then $\mathcal{N}^n$ is an element in $\mathfrak{N}^k$ such that $s(\mathcal{N}^n) = s$. This can be easily shown as we can see by definition the following:

$$\sum_{j=1}^{n} |N_j^n| = \sum_{\mu=1}^{n} s_\mu(\mathcal{N}^n) = \sum_{\mu=1}^{n} s_\mu = nk.$$

The last equation is given by $s \in \mathbb{T}$. Because $|N_j^n| \leq k$ for all $j \in \bar{N}$, we have $|N_1^n| = |N_2^n| = \cdots = |N_n^n| = k$. Furthermore, $N_j^n \subseteq \bar{N} \setminus \{j\}$ is also guaranteed by the definition. Therefore, we have $\mathcal{N}^n \in \mathfrak{N}^k$. It is also clear that $s(\mathcal{N}^j) = s$. Thus, the proof is completed if we show $\mathfrak{I}^n \neq \phi$. Indeed, I show that $\mathfrak{I}^i \neq \phi$ for all $i = 1, 2, \ldots, n$ by an induction on i. For any $i \in \bar{N}$ and $\mathcal{N}^i \in \mathfrak{I}^i$, let us denote by $F(\mathcal{N}^i)$ the set of individuals who have already been fully assigned—i.e., $F(\mathcal{N}^i) = \{j \in \bar{N} \mid |N_j^i| = k\}$. For any $\mu \in \bar{N}$, I denote by $F^{-\mu}(\mathcal{N}^i)$ the set of individuals among $\bar{N} \setminus \{\mu\}$, that is, $F^{-\mu}(\mathcal{N}^i) = \{j \in \bar{N} \setminus \{\mu\} \mid |N_j^i| = k\}$.

Let us begin the induction. We first check that $\mathfrak{I}^1 \neq \phi$. Because $0 \leq s_1 \leq n - 1$, we have $2, 3, \ldots, s_1 + 1 \in \bar{N} \setminus \{1\}$. Therefore, let $\mathcal{N}^1 = (N_1^1, \ldots, N_n^1)$ be such that:

$$N_j^1 = \{1\} \text{ if } 2 \leq j \leq s_i + 1, \text{and}$$
$$N_j^1 = \phi \text{ if } j = 1 \text{ or } j > s_i + 1.$$

Clearly, this makes an assignment until 1. So, we have $\mathfrak{I}^1 \neq \phi$.

Now let $i \in \{1, 2, \ldots, n-1\}$ and suppose none of $\mathfrak{I}^1, \mathfrak{I}^2, \ldots, \mathfrak{I}^i$ is empty. I will show $\mathfrak{I}^{i+1} \neq \phi$ with several steps. Suppose to the contrary that $\mathfrak{I}^{i+1} = \phi$. Because $\mathfrak{I}^i \neq \phi$, we can take an assignment $\mathcal{N}^i = (N_1^i, \ldots, N_n^i) \in \mathfrak{I}^i$ that has the minimal $|F^{-(i+1)}(\quad)|$ among $\mathfrak{I}^i$, as follows:

$$\mathcal{N}^i \in \operatorname*{argmin}_{\mathcal{M}^i \in \mathfrak{I}^i} |F^{-(i+1)}(\mathcal{M}^i)|.$$

(1) To show that $\boldsymbol{|F^{-(i+1)}(\mathcal{N}^i)| \geq n - s_{i+1}}$. Because we assumed that $\mathfrak{I}^{i+1} = \phi$, we cannot construct an assignment until $i + 1$ on the basis of $\mathcal{N}^i$. This means that we cannot take

s_{i+1} distinct sets among $N_1^i, \dots, N_i^i, N_{i+2}^i, \dots, N_n^i$ that have cardinality less than k. Therefore, we have the following:

$$\left|(\bar{N} \setminus \{i+1\}) \setminus F^{-(i+1)}(\mathcal{N}^i)\right| < s_{i+1}.$$

This is equivalent to

$$\left|F^{-(i+1)}(\mathcal{N}^i)\right| \geq n - 1 - s_{i+1} + 1 = n - s_{i+1}.$$

Because $n - s_{i+1} > 0$, the above inequality shows $F^{-(i+1)}(\mathcal{N}^i) \neq \phi$.

(2) To show that $\boldsymbol{|N_\mu^i| \in \{k-1, k\}}$ ***for all*** $\boldsymbol{\mu \in \bar{N}}$**.** Suppose to the contrary that $\nu \in \bar{N}$ exists such that $|N_\nu^i| \leq k - 2$. Because $F^{-(i+1)}(\mathcal{N}^i) \neq \phi$, $\lambda \in F^{-(i+1)}(\mathcal{N})$ exists (i.e., $\lambda \in \bar{N} \setminus \{i\}$) such that $|N_\lambda^i| = k$. Because $|N_\nu^i| \leq k - 2$ and $|N_\lambda^i| = k$, $\alpha \in \bar{N} \setminus \{\nu, \lambda\}$ exists such that $\alpha \in N_\lambda^i$ and $\alpha \notin N_\nu^i$. Now, we define another assignment until i, denoted by $\mathcal{M}^i = (M_1^i, \dots, M_n^i) \in \mathfrak{I}^i$, as $M_\lambda^i = N_\lambda^i \setminus \{\alpha\}$, $M_\nu^i = N_\nu^i \cup \{\alpha\}$, and $M_\gamma^i = N_\gamma^i$ for all $\gamma \in \bar{N} \setminus \{\lambda, \nu\}$. It is clear that $\mathcal{M}^i \in \mathfrak{I}^i$. Furthermore, we have the following:

$$\begin{aligned}
&F^{-(i+1)}(\mathcal{N}^i) \setminus \{\lambda, \nu\} = F^{-(i+1)}(\mathcal{M}^i) \setminus \{\lambda, \nu\},\\
&\lambda \in F^{-(i+1)}(\mathcal{N}^i) \text{ and } \lambda \notin F^{-(i+1)}(\mathcal{M}^i), \text{and}\\
&\nu \notin F^{-(i+1)}(\mathcal{N}^i) \text{ and } \nu \notin F^{-(i+1)}(\mathcal{M}^i).
\end{aligned}$$

Therefore, we have

$$\left|F^{-(i+1)}(\mathcal{N}^i)\right| > \left|F^{-(i+1)}(\mathcal{M}^i)\right|.$$

This is in contradiction to the way $\mathcal{N}^i$ is defined.

(3) To show that $\boldsymbol{|N_{i+1}^i| = k}$**.** Assume to the contrary that $|N_{i+1}^i| \leq k - 1$. By (2), this is equivalent to assuming $|N_{i+1}^i| = k - 1$. By (1), $\lambda \in F^{-(i+1)}(\mathcal{N}^i)$ exists. Because we have $i + 1 \notin N_\lambda^i$ (recall that the supporters of $i + 1$ are not yet assigned at the assignment until i), it follows that $\beta \in \bar{N} \setminus \{i, \lambda\}$ exists such that $\beta \in N_\lambda^i$ and $\beta \notin N_{i+1}^i$. Again, we define another assignment until i, $\mathcal{K}^i = (K_1^i, \dots, K_n^i) \in \mathfrak{I}^i$ as $K_\lambda^i = N_\lambda^i \setminus \{\beta\}$, $K_{i+1}^i = N_{i+1}^i \cup \{\beta\}$, and $K_\gamma^i = N_\gamma^i$ for all $\gamma \in \bar{N} \setminus \{\lambda, i+1\}$. Clearly, $\mathcal{K}^i$ is also an assignment until i. Furthermore, because β gets out of $F^{-(i+1)}(\mathcal{K}^i)$, we have the following:

$$\left|F^{-(i+1)}(\mathcal{N}^i)\right| > \left|F^{-(i+1)}(\mathcal{K}^i)\right|.$$

This is in contradiction to the way $\mathcal{N}^i$ is defined.

(4) To complete the induction. Now, recall that we have assumed $\mathfrak{I}^{i+1} = \phi$ and achieved (1), (2) and (3). If we derive a contradiction from these expressions, it follows that $\mathfrak{I}^{i+1} \neq \phi$, which completes the induction. By (1), (2), and (3), we know that $|N_{i+1}^i| = k$, at least $(n - s_{i+1})$ sets among $N_1^i, \dots, N_i^i, N_{i+2}^i, \dots, N_n^i$ have cardinality k, and the rest of the sets have cardinality of at least $k - 1$. Therefore, the sum of the cardinalities satisfies the following:

$$\sum_{j=1}^{n} |N_j^i| \geq k + (n - s_{i+1})k + (s_{i+1} - 1)(k - 1) = nk - s_{i+1} + 1.$$

The definition of the assignments demands that the left-hand side is

$$\sum_{j=1}^{n} \left|N_j^i\right| = \sum_{j=1}^{i} s_j.$$

Thus, we have

$$\sum_{j=1}^{i} s_j = nk - s_{i+1} + 1.$$

This implies

$$\begin{aligned}\sum_{j=1}^{n} s_j &= \sum_{j=1}^{i} s_j + s_{i+1} + \sum_{j=i+2}^{n} s_j \\ &= nk - s_{i+1} + 1 + s_{i+1} + \sum_{j=i+2}^{n} s_j \\ &\geq nk + 1.\end{aligned}$$

On the other hand, because $s \in \mathbb{T}$, the sum must be exactly nk. This contradiction completes the proof. ■

Case 2: $\mathfrak{D} = \mathfrak{N}$. Let $\mathbb{T}$ be the right-hand side of the equality in Lemma 8. Because $\mathbb{S}[\mathfrak{N}] \subseteq \mathbb{T}$ is obvious, I show $\mathbb{T} \subseteq \mathbb{S}[\mathfrak{N}]$. First, I provide another lemma.

Lemma 17.

Let $\mathbb{T} = \left\{(s_1, \dots, s_n) \in \{0,1, \dots, n-1\}^n \,\middle|\, \sum_{i=1}^{n} s_i \geq n\right\}$. For any $s' = (s_1', \dots, s_n') \in \mathbb{S}[\mathfrak{N}]$ and an individual $i \in \bar{N}$, if $s = (s_1, \dots, s_n)$ defined as $s_i = s_i' - 1$ and $s_j = s_j'$ for all $j \in \bar{N} \setminus \{i\}$ is in $\mathbb{T}$, then $s \in \mathbb{S}[\mathfrak{N}]$.

Proof of Lemma 17. Take any $s' \in \mathbb{S}[\mathfrak{N}]$ and $s \in \mathbb{T}$ that satisfy the given conditions. Note that $s \in \mathbb{T}$ implies that $s_i' \geq 1$ and $\sum_{\mu=1}^{n} s_\mu' \geq n + 1$. Because $s' \in \mathbb{S}[\mathfrak{N}]$, we can take a ballot profile $\mathcal{N}' = (N_1', \dots, N_n') \in \mathfrak{N}$ such that $s(\mathcal{N}') = s'$. Because $s_i' \geq 1$, there is an individual $j \in \bar{N} \setminus \{i\}$ such that $i \in N_j'$.

If $\left|N_j'\right| \geq 2$, then it follows that $N_j' \setminus \{i\} \in \mathfrak{N}_j$. Clearly, the n-tuple $\left(N_1', \dots, N_{j-1}', N_j' \setminus \{i\}, N_{j+1}' \dots, N_n'\right)$ makes a ballot profile in $\mathfrak{N}$ whose score profile is s.

Assume $\left|N_j'\right| = 1$. Because $\sum_{\mu=1}^{n} s_\mu' \geq n + 1$ and $\left|N_\mu'\right| \geq 1$ for all $\mu \in \bar{N}$, it follows that $\lambda \in \bar{N} \setminus \{j\}$ exists such that $|N_\lambda'| \geq 2$. If $i \in N_\lambda'$, the n-tuple of $(N_1', \dots, N_{\lambda-1}', N_\lambda' \setminus \{i\}, N_{\lambda+1}', \dots, N_n')$ makes the ballot profile in $\mathfrak{N}$ whose score profile is s. Otherwise (i.e., if $i \notin N_\lambda'$) there is an individual $\nu \in \bar{N} \setminus \{i, j, \lambda\}$ such that $\nu \in N_\lambda'$. Now we can construct a ballot profile $\mathcal{N}'' \in \mathfrak{N}$ as $N_j'' = N_j' \cup \{\nu\}$, $N_\lambda'' = N_\lambda' \setminus \{\nu\}$, and $N_\gamma'' = N_\gamma'$ for all $\gamma \in \bar{N} \setminus \{j, \lambda\}$. We have $\left|N_j''\right| \geq 2$. So, the

n-tuple $\left(N_1'', \dots, N_{j-1}'', N_j'' \setminus \{i\}, N_{j+1}'', \dots, N_n''\right)$ makes a ballot profile in $\mathfrak{N}$ whose score profile is s. ■

Let me begin the proof of case 2: $\mathfrak{D} = \mathfrak{N}$. Take any $s \in \mathbb{T}$. It follows from the definition of $\mathbb{T}$ that for any n-tuple $t \in \{0,1,\dots,n-1\}^n$ such that $s_i \leq t_i$ for all $i \in \bar{N}$, we have $t \in \mathbb{T}$. Therefore, there is a sequence $t^0, t^1, \dots, t^m \in \mathbb{T}$ such that:

1. $t_i^0 = n - 1$ for all $i \in \bar{N}$,
2. $t^m = s$, and
3. for any $p \in \{0,1,\dots,m-1\}$, there is one and only one individual $i \in \bar{N}$ such that $s_i^p = s_i^{p+1} + 1$ and $s_j^{p+1} = s_j^p$ for all $j \in \bar{N} \setminus \{i\}$.

Lemma 17 enables us to prove that $t^m \in \mathbb{S}[\mathfrak{N}]$ inductively along with this sequence. For t^0, we know $C^{n-1} \in \mathfrak{N}$ and $s(C^{n-1}) = t^0$. If $t^p \in \mathbb{S}[\mathfrak{N}]$, we can apply Lemma 17 to obtain $t^{p+1} \in \mathbb{S}[\mathfrak{N}]$, because $t^{p+1} \in \mathbb{T}$. ■

Case 3: $\mathfrak{D} = \mathfrak{N}^{self}$. Let $\mathbb{T}$ be the right-hand side of the equality in the lemma. Because $\mathbb{S}[\mathfrak{N}^{self}] \subseteq \mathbb{T}$ is obvious, I will show $\mathbb{T} \subseteq \mathbb{S}[\mathfrak{N}^{self}]$. For any $s \in \mathbb{T}$, let $M(s) = \#\{i \in \bar{N} \mid s_i = n\}$ be the number of individuals whose score is n at s. Take any $s \in \mathbb{T}$. I will show that $s \in \mathbb{S}[\mathfrak{N}^{self}]$.

If $M(s) = \phi$, from case 2 we have $s \in \mathbb{S}[\mathfrak{N}]$. Because $\mathbb{S}[\mathfrak{N}] \subseteq \mathbb{S}[\mathfrak{N}^{self}]$, it follows that $s \in \mathbb{S}[\mathfrak{N}^{self}]$.

If $s = (0, \dots, 0, n, 0, \dots, 0)$, (i.e., only one individual i gets score n and all the others get scores of zero, then a ballot profile $\mathcal{N} = (\{i\}, \{i\}, \dots, \{i\})$ corresponds with s. Otherwise, (i.e., if $M(s) \neq \phi$ and at least two individuals get positive scores) let $s' = (s_1', \dots, s_n')$ such that:

$$s_i' = n - 1 \text{ for all } i \in M(s), \text{and}$$
$$s_i' = s_i \text{ for all } i \in \bar{N} \setminus M(s).$$

Because the sum of $s_1', \dots, s_n'$ is at least n and each $s_1', \dots, s_n'$ is in $\{0,1,\dots,n-1\}$, case 2 shows $s' \in \mathbb{S}[\mathfrak{N}]$. So, there is a ballot profile $\mathcal{N} \in \mathfrak{N}$ such that $s(\mathcal{N}) = s'$. Let $\mathcal{N}' \in \mathfrak{N}^{self}$ such that $N_i' = N_i \cup M(s)$ for all $i \in \bar{N}$. Clearly, we have $s(\mathcal{N}') = s$ and therefore, this makes a required ballot profile. ■

Case 4: $\mathfrak{D} = \mathfrak{N}^{AB}$. Let $\mathbb{T}$ be the right-hand side of the equality in Lemma 8. Because $\mathbb{S}[\mathfrak{N}^{AB}] \subseteq \mathbb{T}$ is obvious from the definition of the domain $\mathfrak{N}^{AB}$, I will show $\mathbb{T} \subseteq \mathbb{S}[\mathfrak{N}^{AB}]$.

Take any element $s = (s_1, \dots, s_n) \in \mathbb{T}$. Let us directly construct a ballot profile $\mathcal{N} = (N_1, \dots, N_n) \in \mathfrak{N}^{AB}$ as for all $i \in \bar{N}$ and $a \in \{1,2,\dots,n-1\}$, $\overline{i+a} \in N_i \Leftrightarrow a \leq s_{\overline{i+a}}$. In words, this is a ballot profile where any $j \in \bar{N}$ is approved by the preceding s_j individuals. Then, it is easy to see that $\mathcal{N}$ defined in this way is actually a ballot profile in $\mathcal{N}^{AB}$ and its score profile is $s(\mathcal{N}) = s$. ■

Proof of Lemma 9

As the first step toward a proof of Lemma 9, I provide the following lemma.

Lemma 18. Let $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$. Let $\varphi: \mathfrak{D} \to \mathfrak{X}$ be a nomination rule that satisfies IMP, AB and 2CN. For any ballot profile $\mathcal{N} \in \mathfrak{D}$ and for any individuals $i, j \in \overline{N}$, if $0 \leq s_i(\mathcal{N}) = s_j(\mathcal{N}) \leq M_{\mathfrak{D}} - 1$, then $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$.

Proof of Lemma 18. First, let us consider the case of $s_i(\mathcal{N}) = s_j(\mathcal{N}) = 0$. In this case it is clear that $i \notin N_j$ and $j \notin N_i$. Thus, it follows that $\mathcal{N}^\sigma \in \mathfrak{D}$, where $\sigma = (i, j)$ is the transposition over i, j. By 2CN, we have $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N}^\sigma)$. Because $s_i(\mathcal{N}) = s_i(\mathcal{N}^\sigma) = 0$ and $s_j(\mathcal{N}) = s_j(\mathcal{N}^\sigma) = 0$, Lemma 7 demands that $i \in \varphi(\mathcal{N}) \Leftrightarrow i \in \varphi(\mathcal{N}^\sigma)$ and $j \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N}^\sigma)$. Therefore, we have $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$. From here I focus only on the case of $0 \leq s_i(\mathcal{N}) = s_j(\mathcal{N}) \leq M_{\mathfrak{D}} - 1$.

Case 1: $\boldsymbol{\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{AB}, \mathfrak{N}^{self}}$**.** Take any $\mathcal{N} \in \mathfrak{D}$ and two distinct individuals $i, j \in \overline{N}$ such that $1 \leq m := s_i(\mathcal{N}) = s_j(\mathcal{N}) \leq M_{\mathfrak{D}} - 1$. Because $1 \leq m \leq M_{\mathfrak{D}} - 1$, it is easy to find a ballot profile $\mathcal{N}' \in \mathfrak{D}$ such that $j \notin N_i'$, $i \notin N_j'$, and $s_i(\mathcal{N}') = s_j(\mathcal{N}') = m$. Lemma 7 demands that $\mathcal{N} \sim_i \mathcal{N}'$ and $\mathcal{N} \sim_j \mathcal{N}'$. Because $i \notin N_j'$ and $j \notin N_i'$, there is a transposed ballot profile $\mathcal{N}'' = (\mathcal{N}')^\sigma$, where $\sigma = (i, j)$. Because $s(\mathcal{N}') = s(\mathcal{N}'')$, AB demands $\varphi(\mathcal{N}') = \varphi(\mathcal{N}'')$, while 2CN demands $i \in \varphi(\mathcal{N}') \Leftrightarrow j \in \varphi(\mathcal{N}'')$. Therefore, we obtain $i \in \varphi(\mathcal{N}') \Leftrightarrow j \in \varphi(\mathcal{N}')$, and thus, $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$.

Case 2: $\boldsymbol{\mathfrak{D} = \mathfrak{N}^k}$**.** Take any $i, j \in \overline{N}$ and $\mathcal{N} \in \mathfrak{N}^k$ such that $1 \leq m := s_i(\mathcal{N}) = s_j(\mathcal{N}) \leq n - 2$.

1) The case of $i \notin N_j$ *and* $j \notin N_i$. This proof is essentially the same as for case 1.

2) The case of $i \notin N_j$ *and* $j \in N_i$**.** Because $|N_i| = k \leq n - 2$ and $j \in N_i$, there is an individual $\eta \in \overline{N} \setminus \{i, j\}$ such that $\eta \notin N_i$. Consider a ballot profile $\mathcal{M} = (M_i, M_{-i}) \in \mathfrak{N}^k$, where $M_i = (N_i \cup \{\eta\}) \setminus \{j\}$ and $M_{-i} = N_{-i}$. Note that $s_i(\mathcal{N}) = s_j(\mathcal{N}) = s_i(\mathcal{M}) = m$, but $s_j(\mathcal{M}) = m - 1$. Then, IMP demands that $i \in \varphi(\mathcal{N}) \Leftrightarrow i \in \varphi(\mathcal{M})$. Because $i \notin \mathcal{M}_j$ and $j \notin M_i$, we can consider the transposed ballot profile $\mathcal{M}^\sigma$, where $\sigma = (i, j)$. 2CN demands that $i \in \varphi(\mathcal{M}) \Leftrightarrow j \in \varphi(\mathcal{M}^\sigma)$. Because $s_j(\mathcal{M}^\sigma) = s_j(\mathcal{N}) = m$, Lemma 7 implies that $j \in \varphi(\mathcal{M}^\sigma) \Leftrightarrow j \in \varphi(\mathcal{N})$. Therefore, we have $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$.

3) The case of $i \in N_j$ *and* $j \notin N_i$. This case is essentially the same as *2*), above.

4) The case of $i \in N_j$ *and* $j \in N_i$ *where a)* $k \geq 2$ *and b)* $k = 1$.

a) $k \geq 2$**.** Note that $|N_\epsilon| = k \geq 2$. Because $1 \leq s_i(\mathcal{N}) \leq n - 2$ and $i \in N_j$, $\epsilon \in \overline{N} \setminus \{i, j\}$ exists such that $i \notin N_\epsilon$. Therefore, $\lambda \in \overline{N} \setminus \{i, j, \epsilon\}$ exists such that $\lambda \in N_\epsilon$. By the assumption that $k \leq n - 2$, there are at least k individuals other than i, j. This and $i \in N_j$ imply that $\mu \in \overline{N} \setminus \{i, j\}$ exists such that $\mu \notin N_j$. So, the following $\mathcal{N}^1$ makes a ballot profile in $\mathfrak{N}^k$:

$$N_j^1 = \left(N_j \cup \{\mu\}\right) \setminus \{i\},$$
$$N_\epsilon^1 = (N_\epsilon \cup \{i\}) \setminus \{\lambda\}, \text{and}$$

$$N_\nu^1 = N_\nu \ \text{ for all } \ \nu \in \bar{N} \setminus \{j, \epsilon\}.$$

Note that $s_i(\mathcal{N}) = s_i(\mathcal{N}^1) = s_j(\mathcal{N}) = s_j(\mathcal{N}^1) = m$. So, Lemma 7 shows $\mathcal{N} \sim_i \mathcal{N}^1$ and $\mathcal{N} \sim_j \mathcal{N}^1$. Furthermore, because $1 \leq m = s_i(\mathcal{N}^1) = s_j(\mathcal{N}^1) \leq n-2$ and $i \notin N_j^1$, we can apply the approach from *2)*, above, to $\mathcal{N}^1$ to obtain $i \in \varphi(\mathcal{N}^1) \Leftrightarrow j \in \varphi(\mathcal{N}^1)$. Therefore, we have $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$.

b) $k = 1$. If $x \in \bar{N} \setminus \{i, j\}$ exists such that $N_x \cap \{i, j\} = \phi$, let $N_x = \{y\}$ and consider a ballot profile $\mathcal{N}^2 = (N_1^2, \dots, N_n^2) \in \mathfrak{N}^1$ as follows:

$$N_j^2 = \{y\}, N_x^2 = \{i\}, \text{and}$$
$$N_z^2 = N_z \ \text{ for all } \ z \in \bar{N} \setminus \{j, x\}.$$

Then, we have $s_i(\mathcal{N}) = s_i(\mathcal{N}^1) = s_j(\mathcal{N}) = s_j(\mathcal{N}^1) = m$. So, Lemma 7 shows $\mathcal{N} \sim_i \mathcal{N}^2$ and $\mathcal{N} \sim_j \mathcal{N}^2$. Again, we can apply the approach from *2)* to obtain $i \in \varphi(\mathcal{N}^2) \Leftrightarrow j \in \varphi(\mathcal{N}^2)$. Hence, $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$.

Suppose there is no $x \in \bar{N} \setminus \{i, j\}$ such that $N_x \cap \{i, j\} = \phi$. Then it follows that every $\epsilon \in \bar{N} \setminus \{i, j\}$ casts a ballot of either $N_\epsilon = \{i\}$ or $N_\epsilon = \{j\}$. Let $I = \left\{\epsilon \in \bar{N} \setminus \{i, j\} \,\middle|\, N_\epsilon = \{i\}\right\}$ and $J = \left\{\epsilon \in \bar{N} \setminus \{i, j\} \,\middle|\, N_\epsilon = \{j\}\right\}$. Note that the triplet of $\{i, j\}, I, J$ gives a partition of $\bar{N}$. Because $s_i(\mathcal{N}) = s_j(\mathcal{N})$ and $n \geq 3$, we have $|I| = |J| \geq 1$. Take $b \in J$ and $c \in I$. Let $\mathcal{N}^3 \in \mathfrak{N}^1$ be such that:

$$N_i^3 = \{c\}, N_j^3 = \{b\}, N_b^3 = \{i\}, \text{and}$$
$$N_\mu^3 = N_\mu \ \text{ for all } \ \mu \in \bar{N} \setminus \{i, j, b\}.$$

Because $s_i(\mathcal{N}) = s_i(\mathcal{N}^3) = m$, we have $\mathcal{N} \sim_i \mathcal{N}^3$ according to Lemma 7. Furthermore, because $i \notin N_j^3$ and $j \notin N_i^3$, we can construct a transposed ballot profile $\mathcal{N}^4 = (\mathcal{N}^3)^\sigma \in \mathfrak{N}^1$, where $\sigma = (i, j)$. Then, 2CN demands that $i \in \varphi(\mathcal{N}^3) \Leftrightarrow j \in \varphi(\mathcal{N}^4)$. Because $s_j(\mathcal{N}) = s_j(\mathcal{N}^4) = m$, Lemma 7 gives $\mathcal{N} \sim_j \mathcal{N}^4$. Therefore, we obtain $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N})$. ■

Take any $i \in \bar{N}$ and $\mathcal{N} \in \mathfrak{D}$. Suppose $i \in \varphi(\mathcal{N})$ and $0 \leq s_i(\mathcal{N}) = d \leq M_{\mathfrak{D}} - 1$. We will show that for all $j \in \bar{N}$ and $\mathcal{N}' \in \mathfrak{D}$, if $s_j(\mathcal{N}') = d$, then $j \in \varphi(\mathcal{N}')$. Note that the case of $j = i$ is straightforward from Lemma 7, therefore, in this proof I assume $j \neq i$. Note also that, drawing from Lemma 7, we can prove this statement simply by finding a ballot profile $\mathcal{N}^j \in \mathfrak{D}$ such that $s_j(\mathcal{N}^j) = d$ and $j \in \varphi(\mathcal{N}^j)$.

1) The Case of $\boldsymbol{\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}}$***, or*** $\boldsymbol{[\mathfrak{D} = \mathfrak{N}^k \text{ and } 0 \leq nk - 2d \leq (n-2)(n-1)]}$**.** In this case, there uniquely exists a pair of integers $(p, q) \in \mathbb{Z}^2$ such that

$$nk - 2d = p(n-1) + q,$$
$$0 \leq p \leq n-2, \text{and}$$
$$0 \leq q \leq n-2.$$

Note that $q > 0$ holds only if $p < n-2$. Labelling the individuals other than i, j as $\overline{N} \setminus \{i, j\} = \{a_1, \ldots, a_{n-2}\}(\neq \phi)$, I consider an n-tuple of integers $s^1 = (s_1^1, \ldots, s_n^1) \in \{0,1, \ldots, n-1\}^n$ as follows:

$$s_i^1 = s_j^1 = d. s_{a_\mu}^1 = n-1 \text{ for all } 1 \leq \mu \leq p.$$
$$s_{a_{p+1}}^1 = q.$$
$$s_{a_\mu}^1 = 0 \text{ for all } \mu \geq p+2.$$

By the definition of p and q, the sum of these integers is exactly nk. Therefore, according to Lemma 8, $\mathcal{N}^1 \in \mathfrak{D}(= \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k)$ exists such that $s(\mathcal{N}^1) = s^1$. Note that we have $s_i(\mathcal{N}^1) = s_j(\mathcal{N}^2) = d$. Therefore, we can apply Lemma 18 to obtain $i \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N}^1)$. Lemma 7 and $s_i(\mathcal{N}) = s_i(\mathcal{N}^1) = d$ imply $\mathcal{N} \sim_i \mathcal{N}^1$. Because we have assumed $i \in \varphi(\mathcal{N})$, we obtain $j \in \varphi(\mathcal{N}^1)$, where $s_j(\mathcal{N}^1) = d_0$.

2) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $nk - 2d < 0 \Leftrightarrow nk < 2d$. Because we have assumed $n > d$, this case occurs only if $k < 2$, which means $k = 1$. Let us label the individuals as $\overline{N} \setminus \{i, j\} = \{a_1, \ldots, a_{n-2}\}$. Consider an n-tuple of integers $s^3 = (s_1^3, \ldots, s_n^3)$ as follows:

$$s_i^3 = d$$
$$s_j^3 = n - d - 2$$
$$s_{a_1}^3 = 2$$
$$s_a^3 = 0 \text{ for all } a \in \overline{N} \setminus \{i, j, a_1\}$$

Note that the assumption of $d \leq n-2$ yields $n - d - 2 \geq 0$. So, we can say $s^3 \in \{0,1, \ldots, n-1\}^n$. Now, let us consider a ballot profile $\mathcal{N}^3 \in \mathfrak{N}^k$ as follows:

$$N_i^3 = N_j^3 = \{a_1\}.$$
$$N_{a_\mu}^3 = \{i\} \text{ for all } \mu \in \{a_1, a_2, \ldots, a_{d_0}\}.$$
$$N_\mu^3 = \{j\} \text{ for all } \mu \in \overline{N} \setminus \{i, j, a_1, \ldots, a_{d_0}\}.$$

It is clear that $s(\mathcal{N}^3) = s^3$. Because we have $s_i(\mathcal{N}^3) = d = s_i(\mathcal{N}^3)$ and $i \in \varphi(\mathcal{N}^3)$, Lemma 7 implies $i \in \varphi(\mathcal{N}^3)$. Furthermore, because $i \notin N_j^3$ and $j \notin N_i^3$, we can consider a transposed ballot profile $\mathcal{N}^4 = (\mathcal{N}^3)^\sigma$, where $\sigma = (i, j)$. Then, 2CN yields $j \in \varphi(\mathcal{N}^4)$, because we already have $i \in \varphi(\mathcal{N}^3)$. Because $s_j(\mathcal{N}^4) = s_i(\mathcal{N}^3) = d$, this completes the proof of case 2).

3) The Case of $(n-2)(n-1) < nk - 2d$. By focusing on the constraint, we have the following:

$$2d < nk - (n-2)(n-1) = -n\{n - (k+3)\} - 2.$$

In order for the right-hand side to be positive, it is necessary that $n - (k+3) < 0$, or equivalently $n - 2 \leq k$. Because we assumed $k \leq n-2$, we obtain $k = n-2$. We label the individuals as $\overline{N} \setminus \{i, j\} = \{a_1, \ldots, a_{n-2}\}$ and consider an n-tuple of integers $s^5 = (s_1^5, \ldots, s_n^5) \in \{0,1, \ldots, n-1\}^n$ as follows:

$$s_{a_1}^5 = \ldots = s_{a_{n-2}}^5 = n-1.$$
$$s_i^5 = d, s_j^5 = n - 2 - d.$$

Note that the sum of these integers is nk. Therefore, according to Lemma 8, $\mathcal{N}^5 \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}^5) = s^5$. Furthermore, because $|N_j^5| = |N_i^5| = k = n-2$ and $s_{a_1}(\mathcal{N}^5) = \cdots = s_{a_{n-2}}(\mathcal{N}^5) = n-1$, we have $N_j^5 = N_i^5 = \{a_1, a_2, \dots, a_{n-2}\}$. This implies $i \notin N_j^5$ and $j \notin N_i^5$. Thus, there is a transposed ballot profile $\mathcal{N}^6 = (\mathcal{N}^5)^\sigma$, where $\sigma = (i,j)$. 2CN demands that $i \in \varphi(\mathcal{N}^5) \Leftrightarrow j \in \varphi(\mathcal{N}^6)$. Because $s_i(\mathcal{N}^5) = s_i(\mathcal{N}^0) = d$, Lemma 7 demands $i \in \varphi(\mathcal{N}^5)$. Therefore, we have $j \in \varphi(\mathcal{N}^6)$, where $s_j(\mathcal{N}^6) = d$.

Proof of Lemma 11

Take any $i \in \overline{N}$ and $\mathcal{N}^0 \in \mathfrak{D}$. Assume $i \in \varphi(\mathcal{N}^0)$ and let $d_0 \coloneqq s_i(\mathcal{N}^0)$. Let $\Phi(d)$ be a proposition saying that for all $j \in \overline{N}$ and $\mathcal{N} \in \mathfrak{D}$, $[s_j(\mathcal{N}) = d \Rightarrow j \in \varphi(\mathcal{N})]$. We will show the propositions $\Phi(d_0), \Phi(d_0+1), \dots, \Phi(d), \dots, \Phi(M_{\mathfrak{D}})$ with an induction on d. Note that $\Phi(d_0)$ is already shown in Lemma 9. Note also that, drawing from Lemma 7, we can prove $\Phi(d)$ simply by finding for each $j \in \overline{N}$ a ballot profile $\mathcal{N}^j \in \mathfrak{D}$ such that $s_j(\mathcal{N}) = d$ and $j \in \varphi(\mathcal{N}^j)$.

Assume $\Phi(d_0), \Phi(d_0+1), \dots, \Phi(d)$ holds, where $d_0 \le d \le M_{\mathfrak{D}} - 1$. Take any $j \in \overline{N}$ and we will find $\mathcal{N} \in \mathfrak{D}$ such that $j \in \varphi(\mathcal{N})$ and $s_j(\mathcal{N}) = d+1$. Let $t \in \{0,1,\dots,n-1\}^n$ be such that:

$$
\begin{aligned}
&t_j = d,\\
&t_\mu = n-1 \ \text{ for all } \ \mu = \overline{j+\nu}, 1 \le \nu \le k-1,\\
&t_{\overline{j+k}} = n-1-d,\\
&t_{\overline{j+k+1}} = k, \text{and}\\
&t_\mu = 0 \ \text{ for all } \ \mu = \overline{j+\nu}, k+2 \le \nu \le n-1.
\end{aligned}
$$

Then, it is clear that $\sum_i t_i = nk$. So, according to Lemma 8, $\mathcal{M} \in \mathfrak{D}$ exists such that $s(\mathcal{M}) = t$. Because $s_j(\mathcal{M}) = d \le M_{\mathfrak{D}} - 1$, $\lambda \in \overline{N}$ exists such that $j \notin M_\lambda$, $j \in \overline{N}$ if $\mathfrak{D} = \mathfrak{N}^{self}$, and $j \in \overline{N} \setminus \{j\}$ otherwise. Take $\eta \in M_\lambda$ and consider $\mathcal{M}' = (M_1', \dots, M_n') \in \mathfrak{D}$, where:

$$
\begin{aligned}
&M_\lambda' = (M_\lambda \cup \{j\}) \setminus \{\eta\}, \text{and}\\
&M_{-\lambda}' = M_{-\lambda}.
\end{aligned}
$$

Then WM shows that $j \in \varphi(\mathcal{M}')$, where $s_j(\mathcal{M}') = d+1$. Because $j \in \overline{N}$ was arbitrary, this argument shows $\Phi(d+1)$. So, the induction shows $\Phi(d_0), \Phi(d_0+1), \dots, \Phi(M_{\mathfrak{D}})$. ■

Proof of Proposition 9

Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{\text{self}}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l, \overline{\mathfrak{X}}^l, \underline{\mathfrak{X}}^l$, where $1 \le k \le n-2$ and $1 \le l \le n-1$. Suppose $\varphi: \mathfrak{D} \to \mathfrak{X}$ is a nomination rule that satisfies IMP, AB, and PU. Take two distinct individuals $i, j \in \overline{N}$ and label the others as $\overline{N} \setminus \{i,j\} = \{a_1, \dots, a_{n-2}\}$. Let us define an n-tuple of integers $s = (s_1, \dots, s_n)$ as follows:

$$
\begin{aligned}
&s_i = n-1,\\
&s_j = k,
\end{aligned}
$$

$$s_{a_1} = \cdots = s_{a_{n-1-k}} = k-1, \text{and}$$
$$s_\mu = k \text{ for all } \mu \in \overline{N} \setminus \{i, j, a_1, \dots, a_{n-1-k}\}.$$

Note that $n-1-k \geq 0$ is derived from $k \leq n-2$. Clearly, the sum of these integers is exactly nk. So, by Lemma 8, there is a ballot profile $\mathcal{N} \in \mathfrak{D}$ such that $s(\mathcal{N}) = s$. Because $s_i(\mathcal{N}) = n-1 > s_j(\mathcal{N})$, PU demands $j \notin \varphi(\mathcal{N})$. According to Lemma 7, we get $j \notin \varphi(\mathcal{C}^k)$, while $\mathcal{C}^k \in \mathfrak{D}$. Because j was an arbitrary individual, this implies $\varphi(\mathcal{C}^k) = \phi$. This contradicts the condition of $\phi \notin \mathfrak{X}$. ∎

Proof of Proposition 10

Proof of Proposition 10-[1]. It is clear that the constant rule, $con_{\overline{N}}: \mathfrak{D} \to \mathfrak{X}$, satisfies IMP and AB. ∎

Proof of Proposition 10-[2].

1) The Case of $\mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$. Let $\mathfrak{X} = \mathfrak{X}^1, \overline{\mathfrak{X}}^1$. Note that $\mathcal{C}^{n-1} \in \mathfrak{D}$. Although $|\varphi(\mathcal{C}^{n-1})| \leq 1$, PU (or alternatively WPU) demands $\varphi(\mathcal{C}^{n-1}) = \overline{N}$. This contradiction proves the impossibility.

Next, consider the case of $\mathfrak{X} = \overline{\mathfrak{X}}^l$. Suppose to the contrary that $\varphi: \mathfrak{D} \to \mathfrak{X}$ satisfies IMP and PU. Because $\mathcal{C}^{n-2} \in \mathfrak{D}$ and $\phi \notin \mathfrak{X}$, $i \in \overline{N}$ exists such that $i \in \varphi(\mathcal{C}^{n-2})$. Note that $\overline{\imath - 1} \notin C_i^{n-2}$ and $\overline{\imath + 1} \in C_i^{n-2}$. Let us consider $\mathcal{N} \in \mathfrak{D}$ as $N_i = \left(C_i^{n-2} \cup \{\overline{\imath - 1}\}\right) \setminus \{\overline{\imath + 1}\}$ and $N_{-i} = C_{-i}^{n-2}$. IMP demands $i \in \varphi(\mathcal{N})$, but PU demands $\varphi(\mathcal{N}) = \{\overline{\imath - 1}\}$. This is a contradiction. ∎

2) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l$, $k = 1$ and $l = 1$. Suppose first that $n = 3$. I will show that there is no nomination rule $\varphi: \mathfrak{N}^1 \to \mathfrak{X}^1$ that satisfies IMP and PU. Consider the 1-cyclic ballot profile $\mathcal{C}^1 \in \mathfrak{N}^1$ and take individual $i \in \varphi(\mathcal{C}^1)$. Note that $\overline{\imath + 1} \in C_i^1$ and $\overline{\imath - 1} \notin C_i^1$. Consider a ballot profile $\mathcal{N} \in \mathfrak{N}^1$ as $N_i^1 = \{\overline{\imath - 1}\}$ and $N_{-i} = C_{-i}^1$. IMP demands $i \in \varphi(\mathcal{N})$, while PU demands $\varphi(\mathcal{N}) = \{\overline{\imath - 1}\}$. This is a contradiction.

Next, assume that $n \geq 4$. I will show that there is a nomination rule $\varphi: \mathfrak{N}^1 \to \mathfrak{X}^1$ that satisfies IMP and PU. Take the pivotal individual i and denote the other individuals as $\overline{N} \setminus \{i\} = \{a_1, \dots, a_{n-1}\}$. Let us denote the following:

$$B_{\mathcal{N}} = \left\{\mu \in \overline{N} \,\middle|\, s_\mu^{-i}(\mathcal{N}) \geq n-2\right\}.$$

For any ballot profile $\mathcal{N} \in \mathfrak{N}^1$, let

$$\varphi(\mathcal{N}) = \begin{cases} B_{\mathcal{N}} & \text{if } B_{\mathcal{N}} \neq \phi, \text{and} \\ \{i\} & \text{otherwise.} \end{cases}$$

I will show a) φ is a nomination rule on $(\mathfrak{N}^1, \mathfrak{X}^1)$, b) φ satisfies IMP, and c) φ satisfies PU.

Proof of a). Take any $\mathcal{N} \in \mathfrak{N}^1$. If $B_{\mathcal{N}} = \phi$, it is clear that $\varphi(\mathcal{N}) = \{i\} \in \mathfrak{X}^1$. Suppose $B_{\mathcal{N}} \neq \phi$. I will show that $B_{\mathcal{N}}$ is a singleton. Suppose to the contrary that $|B_{\mathcal{N}}| \geq 2$. Then, we have

$$\begin{aligned} \sum_{\mu=1}^{n} s_\mu(\mathcal{N}) &= |N_i| + \sum_{\mu=1}^{n} s_\mu^{-i}(\mathcal{N}) \\ &\geq 1 + 2(n-2) \\ &= n + (n-3) > n. \end{aligned}$$

The last inequality is given by $n \geq 4$. This contradicts Lemma 8.

Proof of b). It is clear that i is a dummy voter, i.e., his or her ballot has no effects on the nomination, under the rule. Take any individual $j \in \bar{N} \setminus \{i\}$ and a ballot profile $\mathcal{N} = (N_j, N_{-j}) \in \mathfrak{D}$. Because $s_j(\quad)$ is determined without j's ballot, we have $j \in B_{\mathcal{N}} \Leftrightarrow j \in B_{\mathcal{N}'}$ for all $\mathcal{N}' = (N_j', N_{-j}) \in \mathfrak{D}$. This implies $j \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N}')$.

Proof of c). For any individual $j \in \bar{N}$ and supposing $s_j(\mathcal{N}) = n - 1$. It follows that $s_j^{-i}(\mathcal{N}) \geq n - 2$, which implies $j \in B_{\mathcal{N}}$. Therefore, we can derive $j \in \varphi(\mathcal{N})$. ■

3) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \mathfrak{X}^l$, $k \geq 2$ or $l \geq 2$. Suppose first that $k \geq 2$ (l is arbitrary). Consider two n-tuples of integers $s = (s_1, \dots, s_n), s' = (s_1', \dots, s_n') \in \{0,1, \dots, n-1\}^n$ as follows:

$$\begin{aligned}
&s_1 = s_2 = \cdots = s_k = n - 1,\\
&s_{k+1} = k,\\
&s_\mu = 0 \text{ for all } \mu \in \{k+2, k+3, \dots, n\},\\
&s_1' = \cdots = s_{k-1}' = n - 1,\\
&s_k' = n - 2,\\
&s_{k+1}' = k(< n - 1),\\
&s_{k+2}' = 1, \text{and}\\
&s_\mu' = 0 \text{ for all } \mu \in \bar{N} \setminus \{1,2, \dots, k+2\}.
\end{aligned}$$

According to Lemma 8, $\mathcal{N}, \mathcal{N}' \in \mathfrak{N}^k$ exist such that $s(\mathcal{N}) = s$ and $s(\mathcal{N}') = s'$. The codomain $\mathfrak{X}^l$ demands $|\varphi(\mathcal{N})| = |\varphi(\mathcal{N}')| = l$. However, PU demands $\varphi(\mathcal{N}) = \{1, \dots, k\}$ and $\varphi(\mathcal{N}') = \{1, \dots, k-1\}$. Whether $k = l$ or $k \neq l$, this ends with a contradiction.

Finally, consider the case of $k = 1$ and $l \geq 2$. Then the score profile s defined above is also well-defined. PU demands $\varphi(\mathcal{N}) = \{1\}$ and $l \geq 2$ implies $\{1\} \notin \mathfrak{X}^l$. This is also a contradiction. ■

4) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$. I will show two things here.

a) If $k = n - 2$ or $l \geq 2$, there is no nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies IMP and PU.

b) If $k \leq n - 3$ and $l = 1$, a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ exists that satisfies IMP and PU.

Proof of a). The impossibility for the case of $k = n - 2$ is shown as a). Let us assume that $l \geq 2$. I will first show the following inequality:

$$0 < nk - (n-1) \leq (n-1)(n-2) \cdots (\star).$$

For the left-hand side, we have

$$nk - (n-1) = n(k-1) + 1 > 0.$$

This inequality is given by $k \geq 1$. For the right-hand side, we have

$$\begin{aligned}
&(n-1)(n-2) - \{nk - (n-1)\}\\
&= n(n-2-k) - n + 2 + n - 1 > 0.
\end{aligned}$$

The inequality is given by $n \geq k + 2$. Thus, we know that $(\star)$ holds. Therefore, there exist a pair of integers $(p, q) \in \mathbb{Z}^2$ such that:

$$nk-(n-1)=p(n-2)+q, \text{ and}$$
$$0 \le p \le n-1 \text{ and } 0 \le q < n-2.$$

Note that $q>0$ can hold only if $p<n-1$. With these integers, we can consider $\mathcal{N} \in \mathfrak{N}^k$ as follows:

$$s_1 = n-1,$$
$$s_\mu = n-2 \text{ for all } 2 \le \mu \le p+1,$$
$$s_{p+2} = q, \text{ and}$$
$$s_\mu = 0 \text{ for all } p+3 \le \mu \le n.$$

Then, it follows that the sum of $s_1, \dots, s_n$ is $(n-1)+p(n-2)+q = nk$ (this holds whether $q>0$ or $q=0$). So, according to Lemma 8, $\mathcal{N} \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}) = s$. PU demands $\varphi(\mathcal{N}) = \{1\}$ and $l \ge 2$ implies $\{1\} \notin \mathfrak{X}^l$. This contradiction proves the proposition. ■

Proof of b). Suppose $k \ge n-3$ and $l=1$. We construct a nomination rule $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}(=\overline{\mathfrak{X}}^1)$ that satisfies IMP and PU. Let us define $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}$ as follows:

For all $\mathcal{N} \in \overline{\mathfrak{X}}$,

$$\varphi(\mathcal{N}) = \begin{cases} F_{\mathcal{N}} \text{ if } s_{F_{\mathcal{N}}} = n-1, \\ \{i \in \overline{N} \mid (F_{\mathcal{N}} \setminus \{i\}) \subseteq N_i\} \text{ if } s_{F_{\mathcal{N}}} = n-2, \text{and} \\ \overline{N}. \end{cases}$$

Recall that $F_{\mathcal{N}}$ is the set of individuals who have the largest scores at $\mathcal{N}$, and $s_{F_{\mathcal{N}}}$ denotes the score of the individuals in $F_{\mathcal{N}}$. I will show the following:

1. φ is a nomination rule on the setting $(\mathfrak{N}^k, \overline{\mathfrak{X}})$, and
2. φ satisfies PU and IMP.

1. φ is a nomination rule on the setting $(\mathfrak{N}^k, \overline{\mathfrak{X}})$. To show this, we need only to prove that $\varphi(\mathcal{N}) \neq \phi$ for all $\mathcal{N} \in \mathfrak{N}^k$. Take $\mathcal{N} \in \mathfrak{N}^k$. If $s_{F_{\mathcal{N}}} \neq n-2$, it is obvious that $\varphi(\mathcal{N}) \neq \phi$. Suppose $s_{F_{\mathcal{N}}} = n-2$. Let $W_{\mathcal{N}} := \{i \in \overline{N} \mid (F_{\mathcal{N}} \setminus \{i\}) \subseteq N_i\}$. If $|F_{\mathcal{N}}| = 1$, its element clearly also belongs to $W_{\mathcal{N}}$. So, we can focus on the case of $|F_{\mathcal{N}}| \ge 2$. Indeed, we can also say that $|F_{\mathcal{N}}| \le n-1$ as follows. If $|F_{\mathcal{N}}| = n$, the sum of individuals' scores would be $n(n-2) > n(n-3) \ge nk$, which contradicts Lemma 8. So, we can conclude that $2 \le |F_{\mathcal{N}}| \le n-1$. Let us label them as $F_{\mathcal{N}} = \{i_1, \dots, i_p\}$, where $2 \le p \le n-1$. Take an individual $j \in \overline{N} \setminus F_{\mathcal{N}}$. Now, assume that $W_{\mathcal{N}} \cap F_{\mathcal{N}} = \phi$. I will show that $j \in W_{\mathcal{N}}$. Because $W_{\mathcal{N}} \cap F_{\mathcal{N}} = \phi$, for any individual $i \in F_{\mathcal{N}}$, there is an individual $a_i \in F_{\mathcal{N}} \setminus \{i\}$ such that $a_i \notin N_i$. If $a_i = a_{i'}$ for some distinct $i, i' \in \{i_1, \dots, i_p\}$, it follows that $s_{a_i}(\mathcal{N}) \Big(= s_{a_{i'}}(\mathcal{N})\Big) \le \overline{N} \setminus \{a_i, i, i'\} \le n-3$, which contradicts $a_i (= a_{i'}) \in F_{\mathcal{N}}$. Thus, we have that $\{i_1, \dots, i_p\} = \{a_1, \dots, a_p\}$. In other words, for any individual $a_i \in F_{\mathcal{N}}$, there is an individual $i \in F_{\mathcal{N}}$ such that $a_i \notin N_i$. Because $s_{F_{\mathcal{N}}} = n-2$, it follows that everyone in $\overline{N} \setminus F_{\mathcal{N}}$ approves the whole $F_{\mathcal{N}}$. Therefore, we have $F_{\mathcal{N}} \subseteq N_j$, which means $j \in W_{\mathcal{N}}$.

2. *φ satisfies PU and IMP.* This rule clearly satisfies PU, and I show that it also satisfies IMP. The basic idea of the proof is similar to that of φ_P in Proposition 5. Note that one's ballot does not impact one's own score.

Take any $i \in \overline{N}$, $\mathcal{N} \in \mathfrak{N}^k$ and $\mathcal{N}' = (N_i', N_{-i}) \in \mathfrak{N}^k$. If $s_i(\mathcal{N}) = n-1$, then it is clear that $i \in \varphi(\mathcal{N})$ and $s_i(\mathcal{N}') = n-1$. Thus, we have $i \in \varphi(\mathcal{N}')$. If $s_i(\mathcal{N}) = n-2$ and $i \in \varphi(\mathcal{N})$, then it follows that $F_{\mathcal{N}} \setminus \{i\} \subseteq N_i$. Therefore, we have $s_j(\mathcal{N}') \leq n-2$ for all $j \in \overline{N} \setminus \{i\}$, where the equality holds only if $j \in N_i$. Because i's score does not change, $F_{\mathcal{N}'}$ is the set of individuals who obtain the score of $n-2$. Thus, we have $F_{\mathcal{N}'} \setminus \{i\} \subseteq N_i'$. This implies $i \in \varphi(\mathcal{N}')$.

If $s_i(\mathcal{N}) = n-2$ and $i \notin \varphi(\mathcal{N})$, it follows that $F_{\mathcal{N}} \setminus N_i \neq \phi$. If $N_i' \cap (F_{\mathcal{N}} \setminus N_i) \neq \phi$, then there is an individual who obtains a score of $n-1$ at the new ballot profile $\mathcal{N}'$, which implies $i \notin \varphi(\mathcal{N}')$. Suppose $N_i' \cap (F_{\mathcal{N}} \setminus N_i) = \phi$. Then it is clear that no one can obtain a score of $n-1$ at the new ballot profile $\mathcal{N}'$. Furthermore, for some individual $j \in F_{\mathcal{N}} \setminus N_i$, we have $s_j(\mathcal{N}') = n-2$. These facts show that $j \in F_{\mathcal{N}'} \setminus N_i'$. Therefore, we obtain $i \notin \varphi(\mathcal{N}')$.

If $s_i(\mathcal{N}) \leq n-3$ and $i \in \varphi(\mathcal{N})$, then it follows that $s_{F_{\mathcal{N}}} \leq n-2$. Furthermore, we have $\{\mu \in \overline{N} \mid s_\mu(\mathcal{N}) = n-2\} \subseteq N_i$ (whether or not the left-hand side is empty, this expression and the following proof hold). Therefore, we have $s_j(\mathcal{N}') \leq n-2$ for all $j \in \overline{N} \setminus \{i\}$, where the equality can hold only if $j \in N_i'$. This implies that we have one of the following:

$$\{\mu \in \overline{N} \mid s_\mu(\mathcal{N}') \geq n-2\} = \phi, \text{or}$$
$$\phi \neq \{\mu \in \overline{N} \mid s_\mu(\mathcal{N}') \geq n-2\} = \{\mu \in \overline{N} \mid s_\mu(\mathcal{N}') = n-2\} \subseteq N_i'.$$

In either case, we have $i \in \varphi(\mathcal{N}')$.

If $s_i(\mathcal{N}) \leq n-3$ and $i \notin \varphi(\mathcal{N}')$, we have one of the following:

$j \in \overline{N} \setminus \{i\}$ exists such that $s_j(\mathcal{N}) = n-1$, or

$j \in \overline{N} \setminus \{i\}$ exists such that $s_j(\mathcal{N}) = n-2$ and $j \notin N_i$.

In either case, we also have one of the following:

$p \in \overline{N} \setminus \{i\}$ exists such that $s_p(\mathcal{N}) = n-1$, or

$p \in \overline{N} \setminus \{i\}$ exists $= n-2$ and $j \notin N_i$.

In either case, we have $i \notin \varphi(\mathcal{N}')$. ■

5) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, $k > l$. It is sufficient to find a ballot profile where k individuals have score $n-1$. Let $s = (s_1, \dots, s_n)$ be such that $s_1 = \cdots = s_k = n-1$, $s_{k+1} = k$, and $s_{k+2} = \cdots = s_n = 0$. Because the sum of them is clearly nk, Lemma 8 says that $s \in \mathbb{S}[\mathfrak{N}^k]$. This completes the proof. ■

6) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, $k = n-2$. This proof is the same as for case 1). ■

7) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, $k = 1$. The rule we referred to in case 2) can be regarded as the nomination rule on the setting $(\mathfrak{N}^1, \underline{\mathfrak{X}}^l)$ for any $l \in \{1,2,\dots,n-1\}^n$. ■

Proof of Proposition 10-[3].

1) The Case of $\mathfrak{X} = \mathfrak{X}^l$. The reason that each of the conditions $k \geq 2$ or $l \geq 2$ yields the impossibility is the same as in 3) of [2]. I will show the existence of the required nomination rule when $k = l = 1$. Let $\underline{F_{\mathcal{N}}}$ be the individual in $F_{\mathcal{N}}$ with the minimum index and define $\psi_{AV}: \mathfrak{N}^1 \to \mathfrak{X}^1$ as follows:

$$\psi_{AV}(\mathcal{N}) = \begin{cases} F_{\mathcal{N}} & \text{if it is a singleton, and} \\ \{\underline{F_{\mathcal{N}}}\} & \text{otherwise.} \end{cases}$$

It is clear that ψ_{AV} satisfies AB. To see PU, we only need to check that the set $\{i \in \bar{N} \,|\, s_i(\mathcal{N}) = n - 1\}$ is a singleton or an empty set for all $\mathcal{N} \in \mathfrak{N}^1$. This is shown in Lemma 8. ∎

2) The Case of $\mathfrak{X} = \bar{\mathfrak{X}}^l$. Recall that we have constructed a score profile $s = (s_1, \dots, s_n) \in \mathbb{S}[\mathfrak{N}^k]$ such that $s_1 = n - 1 > s_j$ for all $j \in \bar{N} \setminus \{1\}$ in 4) of [2]. At the correspondent ballot profile $\mathcal{N}$, PU demands that $\varphi(\mathcal{N})$ be $\{1\}$, which contradicts $\bar{\mathfrak{X}}^l$, where $l \geq 2$. So, the rest of the proof is to show the possibility when $l = 1$. However, the Approval Voting $\varphi_{AV}: \mathfrak{N}^1 \to \bar{\mathfrak{X}}^1$ surely satisfies both axioms. ∎

3) The Case of $\mathfrak{X} = \underline{\mathfrak{X}}^l$. Note that the n-tuple of integers $s = (s_1, \dots, s_n)$, where $s_1 = \cdots = s_k = n - 1$, $s_{k+1} = k$, and $s_{k+1} = \cdots = s_n = 0$ makes a score profile in $\mathfrak{N}^k$. Because a nomination rule that satisfies PU must choose $\{1,2, \dots, k\}$ at the correspondent ballot profile, it is necessary that $k \leq l$. Indeed, if $k \leq l$ holds, we can design a nomination rule that satisfies AB and PU simply by modifying the rule in 1). For any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, I define $\varphi(\mathcal{N})$ as follows:

$$\varphi(\mathcal{N}) = \begin{cases} F_{\mathcal{N}} \;\; \text{if } s_{F_{\mathcal{N}}} = n - 1, \text{and} \\ \quad \{\underline{F_{\mathcal{N}}}\} \;\; \text{otherwise.} \end{cases}$$

It is obvious that this rule satisfies PU and AB. ∎

Proof of Proposition 11

1) The Case of $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}$. It is clear that $con_C: \mathfrak{D} \to \mathfrak{X}^l$ for any $C \in \mathfrak{X}^l$ satisfies IMP and AB. Let $\varphi: \mathfrak{D} \to \mathfrak{X}^l$ be a nomination rule that satisfies IMP and AB. We show that this rule φ is the constant rule. Let us label the individuals as $\varphi(\mathcal{C}^{n-1}) = \{a_1, \dots, a_l\} = A$ and $\bar{N} \setminus \{\mathcal{C}^{n-1}\} = \{b_1, \dots, b_{n-l}\} = B$. Because we assumed $1 \leq l \leq n - 1$, each of A and B is not empty. Suppose to the contrary that φ is not constant. Then, $\mathcal{N} \in \mathfrak{D}$ exists such that $\varphi(\mathcal{N}) \neq A$. Because $|\varphi(\mathcal{N})| = l$, this implies that $b \in B \cap \varphi(\mathcal{N})$ exists. Let $d \coloneqq s_b(\mathcal{N})$. Let us consider an n-tuple of integers $s = (s_1, \dots, s_n)$ as $s_b = d$ and $s_c = n - 1$ for all $c \in \bar{N} \setminus \{b\}$. It follows that

$$\sum_{i=1}^{n} s_i = d + (n-1)(n-1)$$

$$\geq (n-1)^2 = \left(n - \frac{3}{2}\right)^2 - \frac{5}{4} + n$$

$$\geq 1 + n.$$

The final inequality is given by $n \geq 3$. Therefore, according to Lemma 8, $\mathcal{N}' \in \mathfrak{D}$ exists such that $s(\mathcal{N}') = s$. Because $s_b(\mathcal{N}') = s_b(\mathcal{N})$ and $s_a(\mathcal{N}') = s_a(\mathcal{C}^{n-1})$ for all $a \in A$, Lemma 7 shows

$\{b\} \cup A \subseteq \varphi(\mathcal{N}')$, which means $1 + l \leq |\varphi(\mathcal{N}')|$. This is in contradiction with the codomain $\mathfrak{X}^l$. ∎

2) The Case of $\mathfrak{D} = \mathfrak{N}^k$. Let $\varphi: \mathfrak{N}^k \to \mathfrak{X}^l$ be a nomination rule that satisfies IMP and AB—I will prove this by dividing it into two distinct cases. The proof for the first case, if $l = 1$, is a direct extension of that of Holzman and Moulin's (2013) Theorem 3. However, it does not apply generally if $l \geq 2$. So, I will tackle it in a different way. Because we have $\mathcal{C}^k \in \mathfrak{N}^k$, we can label the individuals as $A := \varphi(\mathcal{C}^k) = \{a_1, \dots, a_l\}$ and $B := \bar{N} \setminus \varphi(\mathcal{C}^k) = \{b_1, \dots, b_{n-l}\}$. Note that $A \neq \phi$ and $B \neq \phi$ by the assumption of $1 \leq l \leq n - 1$.

a) $l = 1$. Suppose to the contrary that φ is not constant. Then there is an individual $b \in B$ and a ballot profile $\mathcal{N}^1 \in \mathfrak{D}$ such that $b \in \varphi(\mathcal{N}^1)$. Let $d := s_b(\mathcal{N})$. Consider a special class of score profiles $S \subseteq \mathbb{S}[\mathfrak{N}^k]$ as follows:

$$\tilde{S} = \{(s_1, \dots, s_n) \in \mathbb{S}[\mathfrak{N}^k] \,|\, s_1 = \dots = s_l = k \text{ and } s_b = d\}.$$

Considering how we can assign the scores for the rest of the individual (i.e., $B \setminus \{b\}$), it follows that $\tilde{S}$ is not empty if and only if the following inequality holds:

$$0 \leq nk - (lk + d) \leq (n - l - 1)(n - 1).$$

This is equivalent to

$$(n - l)k - (n - l - 1)(n - 1) \leq d \leq (n - l)k \cdots (\star).$$

Suppose $(\star)$ holds. Then, according to Lemma 8, we can find a ballot profile $\mathcal{N}^2 \in \mathfrak{N}^k$ such that

$$s_1(\mathcal{N}^2) = \dots = s_l(\mathcal{N}^2) = k \left(= s_1(\mathcal{C}^k) = \dots = s_l(\mathcal{C}^k)\right)$$

$$s_b(\mathcal{N}^2) = d = s_b(\mathcal{N}^1).$$

According to Lemma 7 and given that $A \subseteq \varphi(\mathcal{C}^k)$ and $b \in \varphi(\mathcal{N}^1)$, we have $A \cup \{b\} \subseteq \varphi(\mathcal{N}^2)$. However, this contradicts $|\varphi(\mathcal{N}^2)| = l$.

Therefore, we complete the proof if the parameters l and k satisfy $(\star)$ for any value of $d \in \{0,1, \dots, n - 1\}$. The reader can easily check that if $l = 1$, $(\star)$ holds for any $n \geq 3$, $k \in \{1, \dots, n - 1\}$, and $d \in \{0,1, \dots, n - 1\}$. However, this argument does not always succeed. For example, if $n = 10$, $l = 9$, $k = 1$, and $d = 2$, it follows that $d = 2 > (n - l)k = (10 - 9) \cdot 1 = 1$.

b) $l \geq 2$. Let us introduce several notations. For any integers $n_1, n_2 \in \mathbb{Z}$ such that $n_1 \leq n_2$, we write $[\![n_1, n_2]\!] = \{n_1, n_1 + 1, \dots, n_2\}$. With a slight abuse of terms, I write $[\![n_1, n_2]\!]$ to represent an interval (from n_1 to n_2). Let $I = [\![n_1, n_2]\!] \subseteq [\![0, n - 1]\!]$ be an interval, and define two propositions $\Phi_A(I)$ and $\Phi_B(I)$ as follows:

$\Phi_A(I) \Leftrightarrow$ For any $a \in A$ and for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, $[s_a(\mathcal{N}) \in I \Rightarrow a \in \varphi(\mathcal{N})]$

$\Phi_B(I) \Leftrightarrow$ For any $b \in B$ and for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, $[s_b(\mathcal{N}) \in I \Rightarrow b \notin \varphi(\mathcal{N})]$

With this notation, I can state that my goal is to show $\Phi_A([\![0, n - 1]\!])$ and $\Phi_B([\![0, n - 1]\!])$. Note that $\varphi(\mathcal{C}^k) = A$ implies $\Phi_A([\![k, k]\!])$ and $\Phi_B([\![k, k]\!])$. I will first show $\Phi_A([\![k - 1, k + 1]\!])$ and $\Phi_B([\![k - 1, k + 1]\!])$. I will subsequently prove it for the other intervals.

Step 1: To show $\Phi_A(⟦k-1,k+1⟧)$ *and* $\Phi_B(⟦k-1,k+1⟧)$. Because $l \geq 2$, we can take any two distinct individuals $i,j \in A$. I will show that there is a ballot profile $\mathcal{N} \in \mathfrak{N}^k$ such that $\{s_i(\mathcal{N}), s_j(\mathcal{N})\} = \{k-1, k+1\}$ and $\{i,j\} \subseteq \varphi(\mathcal{N})$. Because i and j are arbitrary elements in A, to find such an $\mathcal{N}$ is enough to show $\Phi_A(⟦k-1,k+1⟧)$.

b-1) If $i \in C_j^k$ *and* $j \in C_i^k$**.** If $k=1$, $i \in C_j^k$ and $j \in C_i^k$ imply $i = \overline{j+1}$ and $j = \overline{i+1}$. This contradicts $n \geq 3$, and so we can assume $k \geq 2$. Now, I will show $i \neq \overline{j+1}$. Suppose to the contrary that $i = \overline{j+1}$. This is equivalent to $i = \overline{j-(n-1)}$. However, $j \in C_i^k$ implies $i \in \{\overline{j-1}, \overline{j-2}, \dots, \overline{j-k}\}$. These expressions indicate that $n-1 \leq k$, which contradicts the assumption of $k \leq n-2$. Therefore, we can conclude that $i \neq \overline{j+1}$. Let us now focus on the individual $\overline{j+1}$. According to the definition, $C_{\overline{j+1}}^k = \{\overline{j+2}, \overline{j+3}, \dots, \overline{j+1+k}\}$. Because $1+k \leq n-1$, we have $j \notin C_{\overline{j+1}}^k$. On the other hand, $i \in C_j^k$ and $i \neq \overline{j+1}$ imply $i \in C_j^k \setminus \{\overline{j+1}\} = \{\overline{j+2}, \dots, \overline{j+k}\} = C_{\overline{j+1}}^k \setminus \{\overline{j+1+k}\}$. Therefore, we can conclude that $i \in C_{\overline{j+1}}^k$. In sum, we have $i \in C_{\overline{j+1}}^k$ and $j \notin C_{\overline{j+1}}^k$.

Let $\mu = \overline{j+1}$ and consider a ballot profile $\mathcal{N}^3 = (N_1^3, \dots, N_n^3) \in \mathfrak{N}^k$ such that:

$$N_\mu^3 = (C_\mu^k \cup \{j\}) \setminus \{i\}, \text{and}$$
$$N_{-\mu}^3 = C_{-\mu}^k.$$

Clearly, $s_i(\mathcal{N}^3) = k-1$, $s_j(\mathcal{N}^3) = k+1$, and $s_\nu(\mathcal{N}^3) = s_\nu(\mathcal{C}^k)$ for all $\nu \in \overline{N} \setminus \{i,j\}$. Therefore, by Lemma 7, we have $\varphi(\mathcal{C}^3) \cap (\overline{N} \setminus \{i,j\}) = \varphi(\mathcal{N}^3) \cap (\overline{N} \setminus \{i,j\})$. So, we have $|\varphi(\mathcal{N}^3) \cap \{i,j\}| = |\varphi(\mathcal{C}^1) \cap \{i,j\}| = 2$, which implies $\{i,j\} \subseteq \varphi(\mathcal{N}^3)$.

b-2) If $i \notin C_j^k$ *and* $j \in C_i^k$**.** Because $s_i(\mathcal{C}^k) = s_j(\mathcal{C}^k)$, $\mu \in \overline{N} \setminus \{i,j\}$ exists such that $i \in C_\mu^k$ and $j \notin C_\mu^k$. Therefore, we can consider a ballot profile $\mathcal{N}^4 = (N_1^4, \dots, N_n^4) \in \mathfrak{N}^k$ such that:

$$N_\mu^4 = (C_\mu^k \cup \{j\}) \setminus \{i\}, \text{and}$$
$$N_{-\mu}^4 = C_{-\mu}^k.$$

Clearly, $s_i(\mathcal{N}^4) = k-1$, $s_j(\mathcal{N}^4) = k+1$, and $s_\nu(\mathcal{N}^4) = s_\nu(\mathcal{C}^k)$ for all $\nu \in \overline{N} \setminus \{i,j\}$. Therefore, by Lemma 7, we have $\varphi(\mathcal{C}^4) \cap (\overline{N} \setminus \{i,j\}) = \varphi(\mathcal{N}^4) \cap (\overline{N} \setminus \{i,j\})$. So, we have $|\varphi(\mathcal{N}^3) \cap \{i,j\}| = |\varphi(\mathcal{C}^1) \cap \{i,j\}| = 2$, which implies $\{i,j\} \subseteq \varphi(\mathcal{N}^4)$.

b-3) If $i \in C_j^k$ *and* $j \notin C_i^k$. In this case, for the reversed k-cyclic ballot profile $\mathcal{R}^k = (R_1^k, \dots, R_n^k) \in \mathfrak{N}^k$, we have $i \notin R_j^k$ and $j \in R_i^k$. Therefore, an argument similar to that in b-2) ensures that there is a ballot profile $\mathcal{N}^5 \in \mathfrak{N}^k$ such that $s_i(\mathcal{N}^5) = k+1$, $s_j(\mathcal{N}^5) = k-1$, and $\{i,j\} \subseteq \varphi(\mathcal{N}^5)$.

b-4) If $i \notin C_j^k$ *and* $j \notin C_i^k$**.** Let us consider the individual $\overline{i-1}$. Note that $i \in C_{\overline{i-1}}^k$ and $i \notin C_j^k$ imply $\overline{i-1} \notin \{i,j\}$. I will show $j \notin C_{\overline{i-1}}^k$. Because $j \notin C_i^k = \{\overline{i+1}, \dots, \overline{i+k}\}$, we can say $j \notin C_{\overline{i-1}}^k \subseteq (C_i^k \cup \{i\})$. Therefore, we can say that $j \notin C_{\overline{i-1}}^k$ and $i \in C_{\overline{i-1}}^k$.

Let $\mu = \overline{i-1}$. Then, just as in b-1), we can get a ballot profile $\mathcal{N}^6 \in \mathfrak{N}^k$ such that $s_i(\mathcal{N}^6) = k-1$, $s_j(\mathcal{N}^6) = k+1$, and $\{i,j\} \subseteq \varphi(\mathcal{N}^6)$.

The arguments in b-1) to b-4) show that we have the required ballot profile for any possible case. Because i and j were arbitrary elements in A, we have $\Phi_A(\llbracket k-1,k+1\rrbracket)$ by Lemma 7. Then, it is easy to show $\Phi_B(\llbracket k-1,k+1\rrbracket)$. Take any individual $b \in B$. By Lemma 7, it is sufficient to find two ballot profiles $\mathcal{N},\mathcal{N}' \in \mathfrak{N}^k$ where $s_b(\mathcal{N}) = k-1$, $s_b(\mathcal{N}') = k+1$, and $b \notin \varphi(\mathcal{N})$ and $b \notin \varphi(\mathcal{N}')$. These ballot profiles are constructed in similar ways, and so we will construct only $\mathcal{N}$. Take an element $a \in A$ and consider an n-tuple of integers $s = (s_1,\dots,s_n) \in \{0,1,\dots,n-1\}^n$ as follows:

$$s_a = k+1,$$
$$s_b = k-1, \text{and}$$
$$s_c = k \text{ for all } c \in \overline{N} \setminus \{a,b\}.$$

Then, with Lemma 7, we obtain $\varphi(\mathcal{N}) \cap (\overline{N} \setminus \{b\}) = \varphi(\mathcal{C}^k)$. Because $|\varphi(\mathcal{N})| = |\varphi(\mathcal{C}^k)|$, it follows that $b \notin \varphi(\mathcal{N})$. With a similar argument on $\mathcal{N}'$, we can conclude $\Phi_B(\llbracket k-1,k+1\rrbracket)$.

Step 2: $\Phi_A(\llbracket 0,n-1\rrbracket)$ and $\Phi_B(\llbracket 0,n-1\rrbracket)$. Take any $a \in A$ and $d \in \llbracket 0,n-1\rrbracket \setminus \llbracket k-1,k+1\rrbracket$, and label the other individuals as $C \coloneqq \overline{N} \setminus \{a\} = \{c_1,c_2,\dots,c_{n-1}\}$. Consider an n-tuple of integers $s^7 = (s_1^7,\dots,s_n^7) \in \{0,1,\dots,n-1\}^n$ as follows:

$$s_a^7 = d,$$
$$s_\mu^7 = k - \operatorname{sgn}(d-k) \text{ for all } \mu \in \{c_1,c_2,\dots,c_{|d-k|}\}^{34}, \text{and}$$
$$s_\mu^7 = k \text{ for all } \mu \in \{c_{|d-k|+1},c_{|d-k|+2},\dots,c_{n-1}\}.$$

Then, we can calculate the sum as follows:

$$\sum_{\mu=1}^{n} s_\mu = d + |d-k|\cdot\{k-\operatorname{sgn}(d-k)\} + (n-1-|d-k|)k = nk$$

Therefore, by Lemma 8, $\mathcal{N}^7 \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}^7) = s^7$. With Lemma 7 and step 1, we have $C \cap A \subseteq \varphi(\mathcal{N}^7)$ and $(C \cap B) \cap \varphi(\mathcal{N}^7) = \phi$. By $|\varphi(\mathcal{N}^7)| = l = |C \cap A| + 1$, we have $a \in \varphi(\mathcal{N}^7)$. Because $a \in A$ and $d \in \llbracket 0,n-1\rrbracket \setminus \llbracket k-1,k+1\rrbracket$ were arbitrary, we have $\Phi_A(\llbracket 0,n-1\rrbracket)$. A similar argument for $b \in B$ instead of $a \in A$ derives $\Phi_B(\llbracket 0,n-1\rrbracket)$. ■

Notes on Proposition 11. I will show the necessity of each axiom in the statement.

1) Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$. A nomination rule $\varphi\colon \mathfrak{D} \to \mathfrak{X}^l$ exists that satisfies IMP but is not the constant rule.

The proof. Let us define a nomination rule $\varphi\colon \mathfrak{D} \to \mathfrak{X}^l$ as follows:

For all $\mathcal{N} \in \mathfrak{D}$,

$$\varphi(\mathcal{N}) = \begin{cases} \{1,2,\dots,l\} \text{ if } l \in N_1, \text{and} \\ \{1,2,\dots,l-1,l+1\} \text{ otherwise} \end{cases}$$

[34] For any integer $z \in \mathbb{Z}$, we write the following:

$$\operatorname{sgn}(z) = \begin{cases} +1 \text{ if } z > 0 \\ 0 \text{ if } z = 0 \\ -1 \text{ if } z < 0 \end{cases}$$

Clearly, this rule is impartial but not constant. ■

2) Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^{k}$. A nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}^{l}$ exists that satisfies AB but is not constant.

The proof. Let us define $\varphi: \mathfrak{D} \to \mathfrak{X}^{l}$ as follows:

For all $\mathcal{N} \in \mathfrak{D}$,

$$\varphi(\mathcal{N}) = \begin{cases} \{1,2,\dots,l\} \text{ if } s_i(\mathcal{N}) \text{ is the largest among } \{s_j(\mathcal{N})\}_{j\in\bar{N}}, \text{and} \\ \qquad\qquad \{n, n-1, \dots, n-l+1\} \text{ otherwise} \end{cases}$$

Note that $l \leq n-1$ implies $n-l+1 > 1$. So, this rule satisfies AB but is not constant. ■

Proof of Proposition 12

1) The Case of $\mathfrak{D} = \mathfrak{N}^{self}$. *IMP and AB⇔constant.* For any $X \in \bar{\mathfrak{X}}^{l}$, it is clear that $con_X: \mathfrak{N}^{self} \to \bar{\mathfrak{X}}^{l}$ satisfies IMP and AB. Let $\varphi: \mathfrak{N}^{self} \to \bar{\mathfrak{X}}^{l}$ be a nomination rule that satisfies IMP and AB. Take any individual $i \in \bar{N}$. I will show the following:

For any $d \in \{0,1,\dots,n-1\}$, there is a pair of ballot profiles $\mathcal{N}, \mathcal{N}' \in \mathfrak{N}^{self}$ such that $s_i(\mathcal{N}) = d$, $s_i(\mathcal{N}') = d+1$, and $\mathcal{N} \sim_i \mathcal{N}'$.

If this statement is shown, Lemma 7 guarantees that φ is nothing but the constant rule, because $i \in \bar{N}$ was arbitrary. I first show the statement for the case of $d \in \{1,2,\dots,n-1\}$. In this case, we have $\mathcal{C}^d \in \mathfrak{N}^{self}$. Let us consider a ballot profile $\mathcal{N}^1 \in \mathfrak{N}^{self}$ such that:

$$N_i^1 = C_i^d \cup \{i\}, \text{and}$$
$$N_{-i}^1 = C_{-i}^d.$$

Then, IMP demands $\mathcal{C}^d \sim_i \mathcal{N}^1$, where $s_i(\mathcal{C}^d) = d$ and $s_i(\mathcal{N}^1) = d+1$.

Next, we consider the case of $d = 0$. Consider a ballot profile $\mathcal{N}^2 \in \mathfrak{N}^{self}$ as $N_\mu^2 = \{\mu\}$ for all $\mu \in \bar{N}$. Then, let $\mathcal{N}^3 \in \mathfrak{N}^3$ be such that $N_i^3 = \{\overline{\iota+1}\}$ and $N_{-i}^3 = N_{-i}^2$. IMP demands $\mathcal{N}^3 \sim_i \mathcal{N}^2$, while $s_i(\mathcal{N}^3) = d$ and $s_i(\mathcal{N}^2) = d+1$.

2) The Case of $\mathfrak{D} = \mathfrak{N}^{AB}$.

a) Let $l \geq 2$. NU⇒Impossible. Let us consider a ballot profile $\mathcal{N}^4 \in \mathfrak{N}^{AB}$ as $N_1^4 = \{2\}$ and $N_{-1}^4 = \phi$. Then, $l \geq 2$ implies that there are at least two winners. However, only individual 1 obtains a positive score. This contradicts NU. ■

b) Let $l = 1$. IMP, AB, and NU⇔φ^1. It is clear that $\varphi^1: \mathfrak{N}^{AB} \to \bar{\mathfrak{X}}^1$ satisfies all the three axioms.[35] Take a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}$ that satisfies the three axioms and take any $i \in \bar{N}$. For NU, it is enough to show the following: for all $\mathcal{N} \in \mathfrak{N}^{AB}$, if $s_i(\mathcal{N}) > 0$, then $i \in \varphi(\mathcal{N})$. By Lemma 7, we must only find, for each integer $d \in \{1,2,\dots,n-1\}$, a ballot profile $\mathcal{N} \in \mathfrak{N}^{AB}$ such that $s_i(\mathcal{N}) = d$ and $i \in \varphi(\mathcal{N})$.

For $d \leq n-2$, let us consider $\mathcal{C}^d \in \mathfrak{N}^{AB}$. Because $\phi \notin \bar{\mathfrak{X}}^1$, we have $\mu \in \varphi(\mathcal{C}^d)$, while $s_\mu(\mathcal{C}^d) = s_i(\mathcal{C}^d)$. With Lemma 9, we have $i \in \varphi(\mathcal{C}^d)$. Finally, consider the case of $d = n-1$. Let

[35] It is also easy to see that φ^1 is surely a well-defined nomination rule on this setting.

$\mathcal{N}^5 \in \mathfrak{N}$ as follows:

$$N_i^5 = \phi, \text{and}$$
$$N_j^5 = \{i\} \text{ for all } j \in \bar{N} \setminus \{i\}.$$

Because everyone except i obtains a score of zero, NU implies $(\bar{N} \setminus \{i\}) \cap \varphi(\mathcal{N}^5) = \phi$. So, we have $i \in \varphi(\mathcal{N}^5)$, where $s_i(\mathcal{N}) = n - 1$. ■

3) The Case of $\mathfrak{D} = \mathfrak{N}$ and $l \geq 3$. ***NU⇒Impossible.*** Take any $i \in \bar{N}$ and let $\mathcal{N}^7 \in \mathfrak{N}$ as $N^7_{\overline{\iota+1}} = \{\overline{\iota+2}\}$ and $N^7_\nu = \{\overline{\iota+1}\}$ for all $\nu \in \bar{N} \setminus \{\overline{\iota+1}\}$. Because $|\varphi(\mathcal{N}^7)| \geq 3$, $\mu \in \varphi(\mathcal{N}^7)$ exists such that $s_\mu(\mathcal{N}^7) = 0$, contradicting NU. ■

4) The case of $\mathfrak{D} = \mathfrak{N}$ and $l = 2$: IMP, AB, NU⇔φ^1. This proof can be carried out in the same way as b) in 2). ■

5) The Case of $\mathfrak{D} = \mathfrak{N}$ and $l = 1$. It is clear that φ^1 satisfies the five axioms. Let $\varphi: \mathfrak{N} \to \bar{\mathfrak{X}}^1$ be a nomination rule that satisfies the five axioms. By Lemma 10, we have $\varphi(\mathcal{C}^1) = \cdots = \varphi(\mathcal{C}^{n-1}) = \bar{N}$. With Lemma 7, this shows that any individual wins the election once they obtain positive scores. ■

6) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $l \geq k + 2$.

Let $s = (s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n$ be such that:

$$s_1 = \cdots = s_k = n - 1,$$
$$s_{k+1} = nk - (n-1)k = k, \text{and}$$
$$s_\mu = 0 \text{ for all } \mu \in \bar{N} \setminus \{1, \dots, k+1\}.$$

Because the sum of each set of s_i's is clearly nk, we have from Lemma 8 that $s \in \mathbb{S}[\mathfrak{N}^k]$. Because $|\varphi(s)| \geq l \geq k + 2$, $\mu \in \bar{N} \setminus \{1, \dots, k+1\}$ exists such that $\mu \in \varphi(s)$. This contradicts NU. ■

7) The Case of $\mathfrak{D} = \mathfrak{N}^k$ and $l \leq k + 1$. From Lemma 11, it is obvious that a nomination rule satisfies the five axioms only if it is a threshold rule. Thus, to show the following is enough to prove the proposition.

a) For any integer $x \in \left\{1,2,\dots,\left\lceil\frac{nk-(l-1)(n-1)}{n-l+1}\right\rceil\right\}$, the threshold-$x$ rule φ^x is well defined as a function from $\mathfrak{N}^k$ to $\bar{\mathfrak{X}}^l$, and it satisfies the five axioms.

b) For any individual $i \in \bar{N}$ and $\mathcal{N} \in \mathfrak{N}^k$, if $s_i(\mathcal{N}) \geq \left\lceil\frac{nk-(l-1)(n-1)}{n-l+1}\right\rceil$ then $i \in \varphi(\mathcal{N})$.

Let

$$d := \left\lceil\frac{nk-(l-1)(n-1)}{n-l+1}\right\rceil, \text{and}$$

$$\epsilon := d - \frac{nk-(l-1)(n-1)}{n-l+1}.$$

Note that we have $0 \leq \epsilon < 1$.

a) Take any integer $x \in \{1,2,\dots,d\}$ and take any ballot profile $\mathcal{N} \in \mathfrak{N}^k$. Suppose to the contrary that $\{i \in \bar{N} \,|\, s_i(\mathcal{N}) \geq x\} \leq l-1$. Then, it follows that at least $n-(l-1)$ individuals have a score strictly less than x. Note that $x \leq d$ implies the following:

$$x \leq \left\lceil \frac{nk-(l-1)(n-1)}{n-l+1} \right\rceil$$

$$\Leftrightarrow x-1 \leq \left\lceil \frac{nk-(l-1)(n-1)}{n-l+1} \right\rceil - 1.$$

Because $x-1 \in \mathbb{N} \cup \{0\}$, it follows that

$$0 \leq x-1 < \frac{nk-(l-1)(n-1)}{n-l+1}.$$

Therefore, the sum of individuals' scores can be bounded above as follows:

$$\sum_{i=1}^{n} s_i(\mathcal{N}) \leq (n-l+1)(x-1) + (l-1)(n-1)$$

$$< (n-l+1)\frac{nk-(l-1)(n-1)}{n-l+1} + (l-1)(n-1) = nk.$$

This clearly contradicts Lemma 8. Thus, we determine φ^x is actually a function from $\mathfrak{N}^k$ to $\bar{\mathfrak{X}}^l$.

b) I will find a ballot profile $\mathcal{N} \in \mathfrak{N}^k$ and an individual $i \in \bar{N}$ such that $s_i(\mathcal{N}) = d$ and $i \in \varphi(\mathcal{N})$. According to Lemma 11, this is enough to prove the proposition.

If $l = 1$, it follows that:

$$d = \left\lceil \frac{nk-(1-1)(n-1)}{n-1+1} \right\rceil = \lceil k \rceil = k.$$

Therefore, $\mathcal{C}^k \in \mathfrak{N}^k$ has the required property. Assume $l \geq 2$. Let us consider $s = (s_1, \dots, s_n)$ as follows:

$$s_\mu = n-1 \text{ for all } \mu \in \bar{N}, 1 \leq \mu \leq l-2,$$
$$s_{l-1} = n-1-\epsilon(n-l+1), \text{and}$$
$$s_\mu = d \text{ for all } \mu \in \{l, l+1, \dots, n\}.$$

Indeed, I am going to check that $s = (s_1, \dots, s_n) \in \mathbb{S}[\mathfrak{N}^k]$. To check that $s = (s_1, \dots, s_n) \in \{0,1,\dots,n-1\}^n$, it is sufficient to check that $s_{l-1} \in \{0,1,\dots,n-1\}$ because the others are clearly in the interval.

$$s_{l-1} = n-1-\epsilon(n-l+1) = n-1-\left(d - \frac{nk-(l-1)(n-1)}{n-l+1}\right)(n-l+1)$$

$$= n-1-d(n-l+1)+nk-(l-1)(n-1).$$

So, we have $s_{l-1} \in \mathbb{Z}$. Because $0 \leq \epsilon < 1$ and $n-l+1 > 0$, we also have the following:

$$n-1 \geq s_{l-1} = n-1-\epsilon(n-l+1) \geq n-1-(n-l+1) = l-2 \geq 0.$$

Therefore, we have $s_{l-1} \in \{0,1,\dots,n-1\}$.

In addition, the sum of their scores can be calculated as follows:

$$\sum_{\mu=1}^{n} s_\mu = (n-1)(l-2) + \{n-1-\epsilon(n-l+1)\} + d(n-l+1)$$
$$= nl - n - l + 1 - k + (d-\epsilon)(n-l+1) = nk.$$

The last equation is given by $\epsilon = d - \frac{nk-(l-1)(n-1)}{n-l+1}$. Therefore, Lemma 8 guarantees $s = (s_1, \dots, s_n) \in \mathbb{S}[\mathfrak{N}^k]$. Because $|\varphi(s)| \geq l$, $i \in \{l, l+1, \dots, n\}$ exists such that $i \in \varphi(s)$, while $s_i(\mathcal{N}) = s_i = d$. ■

Notes on Proposition 12

I will show the logical independence of each axiom in each proposition by giving examples. Because in many cases it is quite easy to determine that the proposed nomination rule satisfies the axioms, I introduce most of the examples without proof, while I give some comments for complicated ones.

1) Let $\mathfrak{D} = \mathfrak{N}^{self}$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$. IMP but not AB. Take a pivotal individual $i \in \bar{N}$ and let $\varphi: \mathfrak{N}^{self} \to \bar{\mathfrak{X}}^l$ be a nomination rule such that:

$$\varphi(\mathcal{N}) = \{\bar{\imath}, \overline{\imath+1}, \dots, \overline{\imath+l-1}\} \cup \left(N_i \cap \{\overline{\imath+l}, \dots, \overline{\imath+n-1}\}\right).$$

2) Let $\mathfrak{D} = \mathfrak{N}^{self}$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$. AB but not IMP. Let $\varphi: \mathfrak{N}^{self} \to \bar{\mathfrak{X}}^l$ be a nomination rule such that:

$$\varphi(\mathcal{N}) = \{1,2, \dots, l-1\} \cup \left\{i \in \{l, l+1, \dots, n\} \,\middle|\, s_i(\mathcal{N}) \geq s_j(\mathcal{N}) \text{ for all } j \in \{l, l+1, \dots, n-1\}\right\}$$

Note that because $l \leq n-1$, we have $|\{l, l+1, \dots, n\}| \geq 2$. Hence, this nomination rule is not the constant rule.

3) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$, $l = 1$. IMP, AB, but not NU. $con_{\bar{N}}: \mathfrak{N}^{AB} \to \bar{\mathfrak{X}}^1$ clearly satisfies IMP and AB, but not NU. ■

4) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$, $l = 1$. IMP, NU, but not AB. Let $\varphi_P^1(\mathcal{N}): \mathfrak{N}^{\text{AB}} \to \bar{\mathfrak{X}}^l$ be defined as

$$\varphi(\mathcal{N}) = \begin{cases} \varphi_P(\mathcal{N}) \text{ if } s_{F_\mathcal{N}} \geq 2, \text{and} \\ \quad \varphi^1(\mathcal{N}) \text{ otherwise} \end{cases}$$

The fact that the rule satisfies NU but not AB is obvious. I will show that it also satisfies IMP. Take any $i \in \bar{N}$ and a ballot profile $\mathcal{N} \in \mathfrak{N}^{AB}$. If there is an individual $j \in \bar{N}$ such that $s_j^{-i}(\mathcal{N}) \geq 2$, then we can show that $\varphi(N_i', N_{-i}) = \varphi_P(N_i', N_{-i})$ for all $N_i' \in \mathfrak{N}_i^{AB}$. Thus, $i \in \varphi(\mathcal{N}) \Leftrightarrow i \in \varphi(N_i', N_{-i})$. Suppose there no such j. Then we have either $s_i(\mathcal{N}) = s_i^{-i}(\mathcal{N}) = \{1,0\}$. If $s_i(\mathcal{N}) = 0$, there is no way for i to win. Suppose $s_i(\mathcal{N}) = 1$. Then, i wins, no matter what ballot he or she casts. ■

5) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \bar{\mathfrak{X}}^l$, $l = 1$. AB, NU, but not IMP. The Approval Voting rule φ_{AV} satisfies AB and NU but not IMP. ■

6) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq 2$. NU. The threshold-1 rule $\varphi^1: \mathfrak{N} \to \overline{\mathfrak{X}}^l$ is well-defined and satisfies NU. ■

7) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l = 2$. IMP, AB, but not NU: The constant rule $con_{\overline{N}}$.

8) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l = 2$. IMP, NU, but not AB: The rule in 10) has these properties.

9) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l = 2$. AB, NU, but not IMP.

$$\varphi(\mathcal{N}) = F_{\mathcal{N}} \cup \{i \in \overline{N} \mid \exists j \in F_{\mathcal{N}} \text{ s.t. } \ i \in N_j\}$$

10) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l = 1$: See 4.2.2.

11) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq k + 1$. IMP, AB, 2CN, NU but not WM. Take $i \in \overline{N}$. If $l \leq k$, the nomination rule $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}^l$ defined as follows has the required properties:

$$\varphi(\mathcal{N}) = \begin{cases} \varphi^1(\mathcal{N}) \setminus \{i\} \text{ if } s_i(\mathcal{N}) = n - 1, \text{and} \\ \qquad \varphi^1(\mathcal{N}) \text{ otherwise} \end{cases}$$

If $l = k + 1$, the existence of a required nomination rule depends on n and k. It is obvious that every individual wins once they obtain score of at least k (by considering a score profile where the first $k - 1$ individuals obtain $n - 1$, the k^{th} obtains $m \in \{k, k+1, \ldots, n-1\}$, the $(k+1)^{th}$ obtains score $n - 1 - m$, and the others obtain a score of zero). Thus, if $k = 1$, a nomination rule $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}^l$ satisfies IMP, AB, 2CN, and NU if and only if it is the threshold-1 rule φ^1. On the other hand, if $n = 10$ and $k = 8$, for example, it is easy to see that the following nomination rule has the required properties:

$$\varphi(\mathcal{N}) = \{i \in \overline{N} \mid s_i(\mathcal{N}) \neq 2{,}0\}.$$

12) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq k + 1$. IMP, AB, 2CN, WM, but not NU: The constant rule $\mathrm{con}_{\overline{\mathrm{N}}}$.

13) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq k + 1$. IMP, AB, NU, WM, but not 2CN. It depends. If $k \geq 2$, then let $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}^l$ be such that

$$\varphi(\mathcal{N}) = \begin{cases} \varphi^1(\mathcal{N}) \setminus \{i\} \text{ if } s_i(\mathcal{N}) = 1, \text{and} \\ \qquad \varphi^1(\mathcal{N}) \text{ otherwise} \end{cases}$$

14) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq k + 1$. IMP, 2CN, NU, WM, but not AB. Let $\varphi: \mathfrak{N}^k \to \overline{\mathfrak{X}}^l$ be such that

$$\varphi(\mathcal{N}) = N_i \cup \left\{ j \in \overline{N} \,\middle|\, j \in \bigcup_{\mu \in N_i} N_\mu \right\}.$$

Clearly, this has the necessary properties.

15) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \overline{\mathfrak{X}}^l$, $l \leq k + 1$. AB, 2CN, NU, WM, but not IMP: Let us denote by $W(\mathcal{N})$ the set of individuals who have the smallest score at a ballot profile $\mathcal{N}$ (i.e., $W(\mathcal{N}) = \{i \in \overline{N} \mid s_i(\mathcal{N}) \leq s_j(\mathcal{N}) \text{ for all } j \in \overline{N}\}$). Let us define a nomination rule $\varphi: \mathfrak{N}^{AB} \to \overline{\mathfrak{X}}^l$ as follows: For any $\mathcal{N} \in \mathfrak{N}^{AB}$,

$$\varphi(\mathcal{N}) = \begin{cases} \bar{N} \setminus W(\mathcal{N}) & \text{if } |W(\mathcal{N})| = 1, \text{and} \\ \bar{N} & \text{otherwise.} \end{cases}$$

Proof of Proposition 13

1) The Case of $\mathfrak{D} = \mathfrak{N}^{self}$. *IMP and AB⇔Constant.* This proof is the same as that of $\mathfrak{X} = \bar{\mathfrak{X}}^l$.

2) The Case of $\mathfrak{D} = \mathfrak{N}^{AB}$.

a) WPU ⇒ Impossibility. WPU demands $\varphi(\mathcal{C}^{n-1}) = \bar{N}$, while $\bar{N} \notin \underline{\mathfrak{X}}^l$. Thus, no nomination rule satisfies WPU.

b) IMP, AB, and NU⇒Impossibility. Suppose a nomination rule $\varphi: \mathfrak{N}^{AB} \to \underline{\mathfrak{X}}^l$ exists that satisfies IMP, AB, and NU. Take any $i \in \bar{N}$ and consider a ballot profile $\mathcal{N} \in \mathfrak{N}^{AB}$ as $N_{i-1} = \{i\}$ and $N_j = \phi$ for all $j \in \bar{N}$. Then, by NU, we have $(\bar{N} \setminus \{i\}) \cap \varphi(\mathcal{N}) = \phi$. Because $\phi \notin \underline{\mathfrak{X}}^l$, we have $i \in \varphi(\mathcal{N})$, while $s_i(\mathcal{N}) = 1$. Note that i was an arbitrary individual in $\bar{N}$. Therefore, with Lemma 7, we have $\varphi(\mathcal{N}) = \bar{N}$. This contradicts the codomain of $\underline{\mathfrak{X}}^l$ $(l \leq n-1)$. ■

3) The Case of $\mathfrak{D} = \mathfrak{N}$.

a) WPU⇒Impossibility. Because $\mathcal{C}^{n-1} \in \mathfrak{N}$, WPU demands $\varphi(\mathcal{C}^{n-1}) = \bar{N}$ and $\bar{N} \notin \underline{\mathfrak{X}}^l$. ■

b) Let $l \leq n-2$. IMP, AB, and NU⇒Impossibility. Suppose a nomination rule $\varphi: \mathfrak{N} \to \underline{\mathfrak{X}}^l$ exists that satisfies IMP, AB, and NU. Because $l \leq n-2$, there are two distinct individuals $i, j \in \bar{N}$ such that $\{i,j\} \cap \varphi(\mathcal{C}^{n-1}) = \phi$. However, Lemma 8 says that $s = (s_1, \ldots, s_n) \in \mathbb{S}[\mathfrak{N}]$ such that $s_i = s_j = n-1$ and $s_\mu = 0$ for all $\mu \in \bar{N} \setminus \{i,j\}$. With Lemma 7, we have $i \notin \varphi(s)$ and $j \notin \varphi(s)$. With NU, we determine that $(\bar{N} \setminus \{i,j\}) \cap \varphi(s) = \phi$. Therefore, it follows that $\varphi(s) = \phi$, which contradicts $\phi \notin \underline{\mathfrak{X}}^l$. ■

c) Let $l = n-1$. IMP, AB, and NU⇔φ^1_{-i}. It is clear that φ^1_{-i} satisfies the three axioms. Let $\varphi: \mathfrak{N} \to \underline{\mathfrak{X}}^l$ $(l = n-1)$ be a nomination rule that satisfies IMP, AB, and NU. I will show that φ is identical to φ^1_{-i} for some $i \in \bar{N}$.

Step 1: To show that $\varphi(\mathcal{C}^{n-1}) = \bar{N} \setminus \{i\}$ for some $i \in \bar{N}$. I now show that $|\varphi(\mathcal{C}^{n-1})| \geq n-1$. Suppose to the contrary that there are two distinct individuals $i, j \in \bar{N}$ such that $\{i,j\} \cap \varphi(\mathcal{C}^{n-1}) = \phi$. Then, the same argument as in b) leads to a contradiction. Therefore, we have that $|\varphi(\mathcal{C}^{n-1})| \geq n-1$. Because the codomain is $\underline{\mathfrak{X}}^l$, it follows that $|\varphi(\mathcal{C}^{n-1})| = n-1$. Let us denote as $\{i\} = \bar{N} \setminus \varphi(\mathcal{C}^{n-1})$. This completes step 1.

Next I am going to show that φ is identical to φ^1_{-i} for the individual i. To show this, I need to show the following: (e1) is shown in step 2 and (e2) will be shown in step 3.

(e1). For any individual $j \in \bar{N} \setminus \{i\}$ and for any ballot profile $\mathcal{N} \in \mathfrak{N}$, $s_j(\mathcal{N}) \geq 1 \Leftrightarrow j \in \varphi(\mathcal{N})$.

(e2). For any ballot profile $\mathcal{N} \in \mathfrak{N}$, $i \notin \varphi(\mathcal{N})$.

Step 2: To show that $\varphi^1(\mathcal{N}) \setminus \{i\} \subseteq \varphi(\mathcal{N})$. The "if" part of (e1) is obvious from NU.

Therefore, I will show the "only if" part. According to Lemma 7, I need only find, for each $j \in \bar{N} \setminus \{i\}$ and $d \in \{1,2,\dots,n-1\}$, a ballot profile $\mathcal{N} \in \mathfrak{N}$ such that $s_j(\mathcal{N}) = d$ and $j \in \varphi(\mathcal{N})$. Take any individual $j \in \bar{N} \setminus \{i\}$ and $d \in \{1,2,\dots,n-1\}$. Consider a score profile $s = (s_1,\dots,s_n) \in \{0,1,\dots,n-1\}^n$ as $s_i = n-1$, $s_j = d$, and $s_\mu = 0$ for all $\mu \in \bar{N} \setminus \{i,j\}$. Because $s_i + s_j = d + n - 1 \geq n$, it is certain that $s \in \mathbb{S}[\mathfrak{N}]$ by Lemma 8. Based on step 1 and Lemma 7, we obtain $i \notin \varphi(s)$. NU demands that $(\bar{N} \setminus \{i,j\}) \cap \varphi(s) = \phi$. Because $\phi \notin \underline{\mathfrak{X}}^l$, we have $\varphi(s) = \{j\}$, while $s_j = d$.

Step 3: To show that $i \notin \varphi(\mathcal{N})$ for all $\mathcal{N} \in \mathfrak{N}$. I show (e2) here. By Lemma 7, it is sufficient to find, for each $d = \{0,1,\dots,n-1\}$, a ballot profile $\mathcal{N} \in \mathfrak{N}$ such that $s_i(\mathcal{N}) = d$ and $i \notin \varphi(\mathcal{N})$. The case of $d = 0$ is straightforward from NU. Suppose $d \in \{1,2,\dots,n-1\}$—we then have $\mathcal{C}^d \in \mathfrak{N}$. Furthermore, step 2 and Lemma 7 together show that $(\bar{N} \setminus \{i\}) \subseteq \varphi(\mathcal{C}^d)$. Because the codomain is $\underline{\mathfrak{X}}^l$ and $l \leq n-1$, we have $i \notin \varphi(\mathcal{C}^d)$. ■

4) The Case of $\mathfrak{D} = \mathfrak{N}^k$.

a) Let $l < k$. WPU⇒Impossibility. Suppose a nomination rule $\varphi: \mathfrak{N} \to \underline{\mathfrak{X}}^l$ exists that satisfies WPU. Let $s = (s_1,\dots,s_n)$ be such that $s_\mu = n-1$ for all $\mu \in \{1,2,\dots,k\}$, $s_k = k$, and $s_\mu = 0$ for all $\mu \in \{k+1, k+2, \dots, n\}$. Because the sum of these is exactly nk, Lemma 8 guarantees that $\mathcal{N} \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}) = s$. WPU demands $\{1,2,\dots,k\} \subseteq \varphi(\mathcal{N})$. However, $l < k$ implies $\{1,2,\dots,k\} \notin \underline{\mathfrak{X}}^l$. This is a contradiction. ■

b) Let $l = k$ and $n = 3$. IMP and WPU⇒Impossibility. This case happens only if $k = l = 1$ (because we assumed $1 \leq k \leq n-2$). Suppose $\varphi(\mathcal{C}^k) = \varphi(\mathcal{C}^1) = \{i\}$. Consider $\mathcal{N} \in \mathfrak{N}^k$ as $N_i = \{\overline{\imath + 2}\}$ and $N_\mu = C_\mu^1$ for all $\mu \in \bar{N} \setminus \{i\}$. IMP demands $\mathcal{C}^1 \sim_i \mathcal{N}$, so we have $i \in \varphi(\mathcal{N})$. WPU demands $\overline{\imath + 2} \in \varphi(\mathcal{N})$. Therefore, it follows $\{i, \overline{\imath + 2}\} \subseteq \varphi(\mathcal{N})$, which contradicts the codomain of $\underline{\mathfrak{X}}^l = \underline{\mathfrak{X}}^1$. ■

c) Let $l = k$ and $n \geq 4$. IMP and WPU. I introduce two examples. The former is for $k = l \leq n-3$, while the latter is for $k = l = n-2$.

c1) The case of $k = l \leq n-3$. Take a pivotal individual $i \in \bar{N}$. For any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, we define $U_\mathcal{N} = \left\{ j \in \bar{N} \setminus \{i\} \,\middle|\, j \in N_\mu \text{ for all } \mu \in \bar{N} \setminus \{i,j\} \right\}$ as the set of individuals who obtain the maximum approvals from the individuals in $\bar{N} \setminus \{i\}$. Therefore, we can also express $U_\mathcal{N}$ as follows:

$$U_\mathcal{N} = \left\{ j \in \bar{N} \setminus \{i\} \,\middle|\, s_j^{-i}(\mathcal{N}) = n-2 \right\}.$$

Now, let us define a nomination rule $\varphi: \mathfrak{N}^k \to \underline{\mathfrak{X}}^l$ as follows:

For any ballot profile $\mathcal{N} \in \mathfrak{N}^k$,

$$\varphi(\mathcal{N}) = \begin{cases} U_\mathcal{N} \cup \{i\} & \text{if } |U_\mathcal{N}| < l, \text{and} \\ U_\mathcal{N} & \text{otherwise.} \end{cases}$$

Note that this rule is shown to satisfy WPU but it does not take into account all of the pivotal individual

i's ballots.

To show that for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, $\varphi(\mathcal{N}) \in \underline{\mathfrak{X}}^l$. Take any ballot profile $\mathcal{N} \in \mathfrak{N}^k$. It is clear that $\varphi(\mathcal{N}) \neq \phi$. It is also clear that $|\varphi(\mathcal{N})| = |U_{\mathcal{N}}| + 1 \leq l$ holds whenever $|U_{\mathcal{N}}| < l$. Thus, we show that $|\varphi(\mathcal{N})| \leq l$ holds even if $|U_{\mathcal{N}}| \geq l$. This is equivalent to showing that $|U_{\mathcal{N}}| \leq l$ for all $\mathcal{N} \in \mathfrak{N}^k$. Suppose to the contrary that $|U_{\mathcal{N}}| \geq l + 1$. Then, we have the following:

$$\sum_{j \in \bar{N} \setminus \{i\}} s_j^{-i}(\mathcal{N}) \geq (n-2)(l+1).$$

At the same time, the left-hand side is bounded above as follows:

$$\sum_{j \in \bar{N} \setminus \{i\}} s_j^{-i}(\mathcal{N}) = \sum_{j \in \bar{N} \setminus \{i\}} \left|N_j \setminus \{i\}\right| \leq \sum_{j \in \bar{N} \setminus \{i\}} k = (n-1)k.$$

Therefore, it must be that:

$$(n-2)(l+1) \leq \sum_{j \in \bar{N} \setminus \{i\}} s_j^{-i}(\mathcal{N}) \leq (n-1)k.$$

However, the comparison of the right-hand side and the left-hand side gives the following:

$$(n-1)k - (n-2)(l+1) = -(n-l-2) < 0.$$

The last inequality is given by $l \leq n-3 \Leftrightarrow n-l-3 \geq 0$. This contradiction proves that $|U_{\mathcal{N}}| \leq l$.

To show that φ *satisfies IMP.* Take any individual $j \in \bar{N}$ and $\mathcal{N} = (N_j, N_{-j}) \in \mathfrak{N}^k$. I show that $j \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(N_j', N_{-j})$ for all $N_j' \in \mathfrak{N}^k$. If $j = i$, this is obvious. So, I assume that $j \in \bar{N} \setminus \{i\}$. Take any of j's ballots, $N_j' \in \mathfrak{N}_j$, and consider a ballot profile $\mathcal{N}' = (N_j', N_{-j}) \in \mathfrak{N}^k$. Note that $s_j^{-i}(\mathcal{N}) = s_j^{-i}(\mathcal{N}')$. So, we have $j \in U_{\mathcal{N}} \Leftrightarrow j \in U_{\mathcal{N}'}$. According to the definition, whether $j \in \bar{N} \setminus \{i\}$ wins under the rule or not is entirely determined by whether j belongs to $U_{\mathcal{N}}$. Hence, we determine that $j \in \varphi(\mathcal{N}) \Leftrightarrow j \in \varphi(\mathcal{N}')$.

To show that φ *satisfies WPU.* Take any non-pivotal individual $j \in \bar{N} \setminus \{i\}$ and any ballot profile $\mathcal{N} \in \mathfrak{N}^k$. If $s_j(\mathcal{N}) = n-1$, then we have $s_j^{-i}(\mathcal{N}) = n-2$, which implies $j \in U_{\mathcal{N}}$. So, we can say that $j \in \varphi(\mathcal{N})$.

Let me show WPU on the pivotal individual i. Take any $\mathcal{N} \in \mathfrak{N}^k$ such that $s_i(\mathcal{N}) = n - 1$. Suppose to the contrary that $i \notin \varphi(\mathcal{N})$. It follows that $|U_{\mathcal{N}}| = l$ (recall that $|U_{\mathcal{N}}| \leq l-1$ implies $i \in \varphi(\mathcal{N})$). Therefore, we can calculate the sum of individuals' scores as follows:

$$\begin{aligned}
\sum_{\mu=1}^{n} s_\mu(\mathcal{N}) &= s_i(\mathcal{N}) + \sum_{j \in \bar{N} \setminus \{i\}} s_j(\mathcal{N}) \\
&= s_i(\mathcal{N}) + |N_i| + \sum_{j \in \bar{N} \setminus \{i\}} s_j^{-i}(\mathcal{N}) \\
&\geq n - 1 + k + (n-2)l \\
&= nk + n - k - 1 > nk.
\end{aligned}$$

This contradicts Lemma 8. Therefore, we obtain $|U_{\mathcal{N}}| \leq l-1$ whenever $s_i(\mathcal{N}) = n-1$, which means $i \in \varphi(\mathcal{N})$. ■

c2) The case of $k = l = n-2$. I take a pivotal individual $i \in \bar{N}$ through this proof. I say that a ballot profile $\mathcal{N} \in \mathfrak{N}^k$ satisfies condition $(\star)$ if and only if:

$s_i(\mathcal{N}) \left(= s_i^{-i}(\mathcal{N})\right) = n-2$, and there is one (and only one) individual $j \in \bar{N} \setminus \{i\}$ such that $s_j^{-i}(\mathcal{N}) = n-2$ and $s_l^{-i}(\mathcal{N}) = n-3$ for all $l \in \bar{N} \setminus \{i,j\}$.

With this notation, let us define a nomination rule $\varphi: \mathfrak{N}^k \to \underline{\mathfrak{X}}^l$ as for all $\mathcal{N} \in \mathfrak{N}^k$ as follows:

$$\varphi(\mathcal{N}) = \begin{cases} \{\mu \in \bar{N} \,|\, s_\mu(\mathcal{N}) = n-1\} \cup \{i\} \text{ if } \mathcal{N} \text{ satisfies condition } (\star) \\ \{\mu \in \bar{N} \,|\, s_\mu(\mathcal{N}) = n-1\}. \end{cases}$$

I will show that this rule is well-defined and that it satisfies both IMP and WPU.

To show that for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$*,* $\varphi(\mathcal{N}) \neq \phi$. Take any $\mathcal{N} \in \mathfrak{N}^k$ and suppose $\varphi(\mathcal{N}) = \phi$. This can occur only if both the following hold:

$$\{\mu \in \bar{N} \,|\, s_\mu(\mathcal{N}) = n-1\} = \phi, \text{and}$$
$$\mathcal{N} \text{ fails to satisfy condition } (\star).$$

Now I will derive a contradiction from these statements. From the first one, it follows that:

$$\left|\{\mu \in \bar{N} \setminus \{i\} \,|\, s_\mu^{-i}(\mathcal{N}) \leq n-3\}\right| \geq k.$$

(Otherwise, we cannot assign i's ballot to $\bar{N} \setminus \{i\}$ without making someone's score reach $n-1$.) Indeed, we can further say that the value of the left-hand side is exactly k. Suppose to the contrary that $\left|\{\mu \in \bar{N} \setminus \{i\} \,|\, s_\mu^{-i}(\mathcal{N}) \leq n-3\}\right| = k+1 (= n-1)$. Then, it follows that:

$$\begin{aligned} \sum_{\mu=1}^{n} s_\mu^{-i}(\mathcal{N}) &= s_i^{-i}(\mathcal{N}) + \sum_{\mu \in \bar{N} \setminus \{i\}} s_\mu^{-i}(\mathcal{N}) \\ &\leq n-2 + (n-3)(k+1) \\ &= k(n-1) + 2(n-k) - 5 \\ &< k(n-1) - 1 < k(n-1). \end{aligned}$$

However, we have also that:

$$\sum_{\mu=1}^{n} s_\mu^{-i}(\mathcal{N}) = \left(\sum_{\mu=1}^{n} s_\mu(\mathcal{N})\right) - |N_i| = nk - k = k(n-1).$$

Clearly, this contradicts the inequality above. Thus, we can say that:

$$\left|\{\mu \in \bar{N} \setminus \{i\} \,|\, s_\mu^{-i}(\mathcal{N}) \leq n-3\}\right| = k.$$

Let us denote $\{\mu \in \bar{N} \setminus \{i\} \,|\, s_\mu^{-i}(\mathcal{N}) \leq n-3\} = \{a_1, \ldots, a_k\}$ and $\{j\} = \bar{N} \setminus \{i, a_1, a_2, \ldots, a_k\}$. Because $\mathcal{N}$ does not satisfy the condition $(\star)$, we can say that $\left(s_i^{-i}(\mathcal{N}), s_j^{-i}(\mathcal{N})\right) \neq (n-2, n-2)$. Furthermore, because no one obtains the score of $n-1$ at the ballot profile $\mathcal{N}$, it follows that:

$$s_x^{-i}(\mathcal{N}) \leq n-2 \text{ for all } x \in \{i,j\}, \text{and}$$
$$s_x^{-i}(\mathcal{N}) < n-2 \text{ for some } x \in \{i,j\}.$$

So, we can take the sum of individuals' scores as follows:

$$\sum_{\mu=1}^{n} s_\mu(\mathcal{N}) = |N_i| + s_i^{-i}(\mathcal{N}) + s_j^{-i}(\mathcal{N}) + \sum_{x=1}^{k} s_{a_x}^{-i}(\mathcal{N})$$
$$< k + 2(n-2) + k(n-3) = nk.$$

This contradicts Lemma 8.

To show that for any ballot profile $\mathcal{N} \in \mathfrak{N}^k$, $|\varphi(\mathcal{N})| \leq l$. Take any $\mathcal{N} \in \mathfrak{N}^k$ and suppose $|\varphi(\mathcal{N})| \geq l+1$. I will consider two distinct cases here.

The first case is $\{\mu \in \overline{N} \,|\, s_\mu(\mathcal{N}) = n-1\} \geq l+1$. The sum of the scores is calculated as follows:

$$\sum_{\mu=1}^{n} s_\mu(\mathcal{N}) \geq (n-1)(l+1)$$
$$= nk + (n-l+1) > nk.$$

The last inequality is given by the assumption of $k = l = n-2$. This result contradicts Lemma 8.

The second case is $\{\mu \in \overline{N} \,|\, s_\mu(\mathcal{N}) = n-1\} = l$ and $\mathcal{N}$ satisfies condition $(\star)$. However, it is clear from the condition $(\star)$ that there is at most one individual who obtains the unanimous score of $n-1$. Therefore, it follows that $l = 1$. With $l = n-2$, we have $n = 3$, which contradicts our assumption of $n \geq 4$.

To show that φ *satisfies IMP.* Take any $\mu \in \overline{N}$ and ballot profiles $\mathcal{N} = (N_\mu, N_{-\mu}), \mathcal{N}' = (N'_\mu, N_{-\mu}) \in \mathfrak{N}^k$. If $\mu \in \overline{N} \setminus \{i\}$, it is easy to see that $s_\mu^{-i}(\mathcal{N}) = s_\mu^{-i}(\mathcal{N}')$, which implies $\mu \in \varphi(\mathcal{N}) \Leftrightarrow \mu \in \varphi(\mathcal{N}')$. Suppose $\mu = i$. Note that $i \in \varphi(\mathcal{N})$ if and only if:

$$s_i(\mathcal{N}) = n-1, \text{or}$$
$$\mathcal{N} \text{ satisfies condition } (\star).$$

Now, it is clear that i's choice of N_i or N'_i does not affect these statements:

$$s_i(\mathcal{N}) = s_i(\mathcal{N}'), \text{and}$$
$$\mathcal{N} \text{ satisfies condition } (\star) \Leftrightarrow \mathcal{N}' \text{ satisfies condition } (\star).$$

Therefore, we have $i \in \varphi(\mathcal{N}) \Leftrightarrow i \in \varphi(\mathcal{N}')$.

To show that φ *satisfies WPU.* Because $\{\mu \in \overline{N} \,|\, s_\mu(\mathcal{N}) = n-1\} \subseteq \varphi(\mathcal{N})$ for all $\mathcal{N} \in \mathfrak{N}^k$, the nomination rule φ satisfies WPU. ■

d) Let $l = k$***. IMP, AB, and WPU⇒Impossibility.*** Suppose a nomination rule $\varphi: \mathfrak{N}^k \to \underline{\mathfrak{X}}^l$ exists that satisfies the three axioms. Take any individual $i \in \overline{N}$ and consider an n-tuple of integers $s = (s_1, s_2, \dots, s_n) \in \{0,1,\dots,n-1\}^n$ as follows:

$$s_i = k,$$
$$s_\mu = n-1 \text{ for all } \mu \in \{\overline{\imath+1}, \overline{\imath+2}, \dots, \overline{\imath+k}\}, \text{and}$$

$$s_\mu = 0 \text{ for all } \mu \in \{\overline{\iota+k+1}, \dots, \overline{\iota+n-1}\}.$$

Note that Lemma 8 assures us that $s \in \mathbb{S}[\mathfrak{N}^k]$. Because φ satisfies WPU, it follows that $\{\overline{\iota+1}, \dots, \overline{\iota+k}\} \subseteq \varphi(s)$. Because $|\varphi(s)| \leq l = k$, we can further say that $i \notin \varphi(s) = \{\overline{\iota+1}, \dots, \overline{\iota+k}\}$. With Lemma 7, we can say that $i \notin \varphi(\mathcal{C}^k)$. Because i was arbitrary, it follows that $\varphi(\mathcal{C}^k) = \phi$, which contradicts $\phi \notin \underline{\mathfrak{X}}^l$. ■

e) Let $l > k$. IMP, AB, and WPU. Take a pivotal individual $i \in \overline{N}$. Let $\varphi: \mathfrak{N}^\mathrm{k} \to \underline{\mathfrak{X}}^l$ as follows. For any ballot profile $\mathcal{N} \in \mathfrak{N}^k$,

$$\varphi(\mathcal{N}) = \{i\} \cup \{\mu \in \overline{N} \mid s_\mu(\mathcal{N}) = n-1\}.$$

Because we have $(n-1)k < nk < (n-1)(k+1)$, Lemma 8 shows that there is no $\mathcal{N} \in \mathfrak{N}^k$ such that $|\{\mu \in \overline{N} \mid s_\mu(\mathcal{N}) = n-1\}| \geq k+1$. Therefore, for all $\mathcal{N} \in \mathfrak{N}^k$, we have $1 \leq |\{i\} \cup \{\mu \in \overline{N} \mid s_\mu(\mathcal{N}) = n-1\}| \leq k+1 \leq l$. This shows that the rule is well-defined on the setting $(\mathfrak{N}^k, \underline{\mathfrak{X}}^l)$. Furthermore, we can easily determine if this rule satisfies IMP, AB, and WPU. Note that this rule fails to satisfy NU. ■

Notes on Proposition 13

1) Let $\mathfrak{D} = \mathfrak{N}^{self}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$. IMP but not AB.

$$\varphi(\mathcal{N}) = \begin{cases} N_1 \cap \{2\} & \text{if it is nonempty, and} \\ \{3\} & \text{otherwise.} \end{cases}$$

2) Let $\mathfrak{D} = \mathfrak{N}^{self}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$. AB but not IMP. Let φ be the nomination rule that chooses those who get the highest scores among $\{1,2,\dots,l\}$.

3) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$. IMP, AB, but not NU: The constant rule $con_X(X \in \underline{\mathfrak{X}}^l)$.

4) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$. IMP, NU, but not AB: The bilateral edge scan mechanism.

5) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$. AB, NU, but not IMP.

$$\varphi(\mathcal{N}) = \begin{cases} \varphi^1(\mathcal{N}) & \text{if its cardinality is at most } l \\ \text{those who have the minimum index among } \varphi^1(\mathcal{N}) & \text{otherwise} \end{cases}$$

6) Let $\mathfrak{D} = \mathfrak{N}^{AB}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = 1$. IMP but not AB. Take a pivotal individual $i \in \overline{N}$ and define $\varphi: \mathfrak{N}^{AB} \to \underline{\mathfrak{X}}^l$ as follows:

$$\varphi(\mathcal{N}) = \begin{cases} \{\overline{\iota+1}\} & \text{if } N_i \neq \phi, \text{and} \\ \{\overline{\iota+2}\} & \text{otherwise.} \end{cases}$$

7) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l \leq n-2$. IMP and AB: The constant rule con_X $(X \in \underline{\mathfrak{X}}^l)$.

8) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l \leq n-2$. AB and NU: Let φ be the nomination rule that chooses the individual with the minimum index among $F_{\mathcal{N}}$.

9) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l \leq n-2$. NU and IMP: Let φ be the nomination rule that chooses the individual with the minimum index among N_1.

10) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = n-1$. IMP and AB, but not NU: The rule in 7).

11) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = n - 1$. IMP and NU, but not AB: The rule in 9).

12) Let $\mathfrak{D} = \mathfrak{N}$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = n - 1$. AB and NU, but not IMP: The rule in 8).

13) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = k$ and $n = 3$. IMP: The constant rule con_X $\left(X \in \underline{\mathfrak{X}}^l\right)$.

14) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = k$ and $n = 3$. WPU. Note that the condition imply $l = k = 1$. Clearly, it follows that there is at most one individual that has a score of $n - 1 = 2$. Thus, let

$$\varphi(\mathcal{N}) = \begin{cases} \left\{i \in \overline{N} \,\middle|\, s_i(\mathcal{N}) = 2\right\} & \text{if it is nonempty, and} \\ \{1\} & \text{otherwise.} \end{cases}$$

15) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = k$ and $n \geq 4$. IMP and AB: The constant rule con_X $\left(X \in \underline{\mathfrak{X}}^l\right)$.

16) Let $\mathfrak{D} = \mathfrak{N}^k$ and $\mathfrak{X} = \underline{\mathfrak{X}}^l$, where $l = k$ and $n \geq 4$. AB and WPU. Because there are at most k individuals who have the maximum score $n - 1$ at any score profile, we can assign $\varphi(s)$ for each score profile $s \in \mathbb{S}[\mathfrak{N}^k]$ so that $1 \leq |\varphi(s)| \leq l$ and $\left\{i \in \overline{N} \,\middle|\, s_i(\mathcal{N}) = n - 1\right\} \subseteq \varphi(s)$. Therefore, a nomination rule meeting the requirements can be constructed.

Proof of Proposition 14

Proof of [1]. Let $\mathfrak{D} = \mathfrak{N}, \mathfrak{N}^{self}, \mathfrak{N}^{AB}, \mathfrak{N}^k$ $(2 \leq k \leq n - 3)$ and $\mathfrak{X} = \overline{\mathfrak{X}}$. Suppose a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}$ exists that satisfies IMP, AB, ND, and weak 2CP. Take any $i \in \overline{N}$. According to ND, there exist $\mathcal{N} = (N_i, N_{-i}), \mathcal{N}' = (N_i', N_{-i}) \in \mathfrak{D}$ and $j \in \overline{N}$ such that $j \in \varphi(\mathcal{N})$ and $j \notin \varphi(\mathcal{N}')$. Note that IMP demands $j \neq i$. In addition, considering Lemma 7, there are only two possible cases: (1) $j \in N_i$ and $j \notin N_i'$, and (2) $j \notin N_i$ and $j \in N_i'$, because otherwise it follows that $s_j(\mathcal{N}) = s_j(\mathcal{N}')$, which implies $\mathcal{N} \sim_j \mathcal{N}'$ by Lemma 7. Then by ND and IMP, there is also $\mathcal{M} = \left(M_j, M_{-j}\right), \mathcal{M}' = \left(M_j', M_{-j}\right) \in \mathfrak{D}$ and $l \in \overline{N} \setminus \{j\}$ such that $l \in \varphi(\mathcal{M})$ and $l \notin \varphi(\mathcal{M}')$. here we also have only two possible cases: (a) $l \in M_j$ and $l \notin M_j'$, and (b) $l \notin M_j$ and $l \in M_j'$.

Now there are four possibilities: (1)&(a), (1)&(b), (2)&(a), and (2)&(b). The proof is completed if a contradiction is derived from each of the four. I show this only for (2) and (b), because the other three cases can be demonstrated in a similar way. Let $d_j + 1 \coloneqq s_j(\mathcal{N}')\left(= s_j(\mathcal{N}) + 1\right)$ and $d_l + 1 \coloneqq s_l(\mathcal{M}')(= s_l(\mathcal{M}) + 1)$. Then, we have $1 \leq d_j + 1 \leq n - 1$ and $1 \leq d_l \leq n - 1$. I label the other individuals as $\overline{N} \setminus \{j, l\} = \{a_1, \ldots, a_{n-2}\}$. Because $n \geq 4$, this set $\overline{N} \setminus \{j, l\}$ has at least two elements a_1, a_2.

For any $\mu \in \{a_1, \ldots, a_{n-2}\}$, there exist three subsets $A_\mu^{k-2}, A_\mu^{k-1}, A_\mu^k \subseteq \{a_1, \ldots, a_{n-2}\}$ such that:

$$\mu \notin A_\mu^x \text{ and } \left|A_\mu^x\right| = x \ \ (x = k - 2, k - 1, k).$$

This is because $2 \leq k$ implies $x \geq 0$ and $k \leq n - 2$ implies $|\{a_1, \ldots, a_{n-2}\}| = n - 2 \geq k + 1 > k$. Because $d_j, d_l \leq n - 2$, we can also take two subsets $J, L \subseteq \overline{N} \setminus \{j, l\}$ such that $|J| = d_j$ and

$|L| = d_l$. With these subsets, we can define $\mathcal{N}^1 = (N_1^1, \dots, N_n^1) \in \mathfrak{N}^k \subseteq \mathfrak{N} = \mathfrak{N}^{AB} \cap \mathfrak{N}^{self}$ as follows:

$$N_j^1 = \{l\} \cup \{a_\mu : 1 \leq \mu \leq k-1\},$$
$$N_l^1 = \{j\} \cup \{a_\mu : 1 \leq \mu \leq k-1\},$$
$$N_\mu^1 = \{j, l\} \cup A_\mu^{k-2} \text{ for all } \mu \in J \cap L,$$
$$N_\mu^1 = \{j\} \cup A_\mu^{k-1} \text{ for all } \mu \in J \setminus L,$$
$$N_\mu^1 = \{l\} \cup A_\mu^{k-1} \text{ for all } \mu \in L \setminus J, \text{and}$$
$$N_\mu^1 = A_\mu^k \text{ for all } \mu \in \overline{N} \setminus (J \cup L \cup \{j, l\}).$$

It is clear that $|N_\mu^1| = k$ for all $\mu \in \overline{N}$. Thus, $\mathcal{N}^1$ is a ballot profile. Note that $j \in N_l^1$ and $l \in N_j^1$. With Lemma 7, $j \notin \varphi(\mathcal{N}')$ and $l \notin \varphi(\mathcal{M}')$, we have $j \notin \varphi(\mathcal{N}^1)$ and $l \notin \varphi(\mathcal{N}^1)$. Because $a_k \in \{a_1, \dots, a_{n-2}\} \setminus (N_j^1 \cup N_l^1)$, we can define a ballot profile $\mathcal{N}^2 = (N_1^2, \dots, N_n^2) \in \mathfrak{D}$ as follows:

$$N_j^2 = (N_j^1 \cup \{a_k\}) \setminus \{l\},$$
$$N_l^2 = (N_l^1 \cup \{a_k\}) \setminus \{j\}, \text{and}$$
$$N_\mu^2 = N_\mu^1 \text{ for all } \mu \in \overline{N} \setminus \{j, l\}.$$

Because we have $s_j(\mathcal{N}^2) = s_j(\mathcal{N})$, $s_l(\mathcal{N}^2) = s_l(\mathcal{M})$, and Lemma 7, it follows that $j \in \varphi(\mathcal{N}^2)$ and $l \in \varphi(\mathcal{N}^2)$. The comparison between $\mathcal{N}^1$ and $\mathcal{N}^2$ contradicts weak 2CP. ■

Proof of [2]. Note that if $n = 3$, we have $k = 1 \Leftrightarrow n - 2 = k$. So, the case of $n = 3$ is a special case of [3] (below).

a) IMP, AB, ND, and weak 2CP. First, I propose a nomination rule that satisfies the four axioms. Take any $i \in \overline{N}$ and let:

$$\varphi(\mathcal{N}) = \{i\} \cup \{j \in \overline{N} \setminus \{i\} \,|\, s_j(\mathcal{N}) = n - 1\}.$$

Clearly this rule has the necessary properties. Furthermore, it satisfies WPU. ■

b) IMP, AB, NU, and weak 2CP⇒Impossibility.

b1) The case of $|\varphi(\mathcal{C}^1)| \geq 2$. Let $i, j \in \varphi(\mathcal{C}^1)$ be two distinct winners at $\mathcal{C}^1$. Because $n \geq 4$, there are distinct individuals $\alpha, \beta \in \overline{N} \setminus \{i, j\}$. Let us consider a score profile $s = (s_1, \dots, s_n) \in \{0,1, \dots, n-1\}^n$ as $s_i = s_j = 0$, $s_\alpha = s_\beta = 2$, and $s_\gamma = 1$ for all $\gamma \in \overline{N} \setminus \{i, j, \alpha, \beta\}$. With Lemma 8, $\mathcal{N}^3 \in \mathfrak{N}^k$ exists such that $s(\mathcal{N}^3) = s$. Because $s_i(\mathcal{N}^3) = s_j(\mathcal{N}^3) = 0$, NU demands that $\{i, j\} \cap \varphi(\mathcal{N}^3) = \phi$. Furthermore, $i \notin N_j^3$ and $j \notin N_i^3$. Let $N_i^3 = \{u\}$ and $N_j^3 = \{v\}$. Let $\mathcal{N}^4 \in \mathfrak{N}^k$ be such that:

$$N_i^4 = (N_i^3 \cup \{j\}) \setminus \{u\},$$
$$N_j^4 = (N_j^3 \cup \{i\}) \setminus \{v\}, \text{and}$$
$$N_\mu^4 = N_\mu^3 \text{ for all } \mu \in \overline{N} \setminus \{i, j\}.$$

Then, because $s_i(\mathcal{N}^4) = s_i(\mathcal{C}^1) = s_j(\mathcal{N}^4) = s_j(\mathcal{C}^1) = 1$, Lemma 7 implies that $i \in \varphi(\mathcal{N}^4)$ and $j \in \varphi(\mathcal{N}^4)$. This contradicts weak 2CP.

b2) The case of $|\varphi(\mathcal{C}^1)| = 1$. Let $\varphi(\mathcal{C}^1) = \{i\}$. Consider a ballot profile $\mathcal{N}^5 \in \mathfrak{N}^k$ as:

$$N_{\overline{\iota-1}}^5 = (C_{\overline{\iota-1}}^1 \cup \{\overline{\iota+1}\}) \setminus \{i\}, \text{and}$$

$$N_\mu^5 = C_\mu^1 \text{ for all } \mu \in \overline{N} \setminus \{\overline{\iota - 1}\}.$$

Because $s_\mu(\mathcal{C}^1) = s_\mu(\mathcal{N}^5)$ for all $\mu \in \overline{N} \setminus \{i, \overline{\iota + 1}\}$, Lemma 7 and $\varphi(\mathcal{C}^1) = \{i\}$ show that $\varphi(\mathcal{N}^5) \subseteq \{i, \overline{\iota + 1}\}$. Because NU demands $i \notin \varphi(\mathcal{N}^5)$, we have $\varphi(\mathcal{N}^5) = \{\overline{\iota + 1}\}$. With Lemma 7, we can see that i loses at score 0 and wins at score 1, while $\overline{\iota + 1}$ wins at score 0 but loses at score 1.

Let us consider $\mathcal{N}^6 \in \mathfrak{N}^k$, where $N_{\overline{\iota-1}}^6 = \{\overline{\iota + 1}\}$, $N_i^6 = \{\overline{\iota + 2}\}$, and $N_\mu^6 = C_\mu^1$ for all $\mu \neq \overline{\iota - 1}, i$. Because $s_i(\mathcal{N}^6) = 0$ and $s_{\overline{\iota+1}}(\mathcal{N}^6) = s_{\overline{\iota+1}}(\mathcal{C}^1) = 1$, NU and Lemma 7 show that $i \notin \varphi(\mathcal{N}^6)$ and $\overline{\iota + 1} \notin \varphi(\mathcal{N}^6)$. Next, let $\mathcal{N}^7 \in \mathfrak{N}^k$ be such that $N_i^7 = \{\overline{\iota + 1}\}$, $N_{\overline{\iota+1}}^7 = \{i\}$, and $N_\mu^7 = N_\mu^6$ for all $\mu \neq i, \overline{\iota + 1}$. Because $s_i(\mathcal{N}^7) = s_i(\mathcal{C}^1) = 1$ and $s_{\overline{\iota+1}}(\mathcal{N}^7) = s_{\overline{\iota+1}}(\mathcal{N}^5) = 2$, Lemma 7 shows that $\{i, \overline{\iota + 1}\} \subseteq \varphi(\mathcal{N}^7)$. The comparison between $\mathcal{N}^6$ and $\mathcal{N}^7$ contradicts weak 2CP. ■

Proof of [3]. It is clear that $\varphi^1: \mathfrak{N}^k \to \overline{\mathfrak{X}}$ satisfies IMP, AB, 2CN, NU, and WPU. I will show this for the other axioms.

a) ND. Take any $i \in \overline{N}$. Let $s_i = \cdots = s_{\overline{\iota+n-3}} = n - 1$, $s_{\overline{\iota+n-2}} = k(= n - 2)$, and $s_{\overline{\iota+n-1}} = 0$. Clearly, this makes a score profile in $\mathfrak{N}^k$, where $k = n - 2$. The individual i can change the result by approving $\overline{\iota + n - 1}$ instead of someone else. ■

b) Weak 2CP. Take a distinct $i, j \in \overline{N}$. The proof is complete if we can show that there is no $\mathcal{N} \in \mathfrak{N}^{n-2}$ such that $s_i(\mathcal{N}) = s_j(\mathcal{N}) = 0$. From the score profile s defined above in a), this fact is obvious. ■

Proof of Proposition 15

Let us assume to the contrary that there exists an impartial nomination rule, denoted $\varphi: \mathfrak{D} \to \mathfrak{X}^l$ of rank $n - 1$. I first show the case of $l = 1$, and then I will show the other case of $l \geq 2$.

Proof of Proposition 15 when $\boldsymbol{l = 1}$.

The proof will be done with three steps. The first step is to define a special class of ballot profiles, denoted by $V_m^j \subseteq \mathfrak{D}$, where $V_{n-1}^j \subset V_{n-2}^j \subset \cdots \subset V_m^j \subset \cdots \subset V_1^j \subset V_0^j$. In the second step, I will show that if there exists a nomination rule of rank $n - 1$, then $j \in \varphi(\mathcal{N})$ holds for any ballot profile $\mathcal{N} \in V_m^j$, through a downward induction on m. Finally, I will derive a contradiction in the third step. $|\varphi(\cdot)| = 1$, and so I often write $\varphi(\cdot) = i$ instead of $\varphi(\cdot) = \{i\}$ within this proof.

Step 1: Define the Class of Ballot Profiles $\boldsymbol{V_m^j \subset \mathfrak{D}}$ for any $\boldsymbol{m \in \{0, 1, \dots, n - 1\}}$ and $\boldsymbol{j \in \overline{N}}$

First, I introduce some new notation. For any individual $j \in \overline{N}$, I call a permutation over $\overline{N} \setminus \{j\}$ an $(n - 1)$-tuple

$$\imath = (i_1, \dots, i_{n-1})$$

of individuals in $\bar{N} \setminus \{j\}$, if $\{i_1, \dots, i_{n-1}\} = \bar{N} \setminus \{j\}$. When an integer $m \in \{0,1,\dots,n-1\}$ and a permutation $\imath$ over $\bar{N} \setminus \{j\}$ are given, I write

$$\mathcal{A} = (A_1, \dots, A_T)$$

as a partition of $\bar{N} \setminus \{j, i_1, \dots, i_m\}$ if there exist $a_t, b_t \in \bar{N} \setminus \{j, i_1, \dots, i_m\}$ $(1 \le t \le T)$ such that:

$$a_t \le b_t \ \ (1 \le t \le T),$$

$$a_{t+1} = b_t + 1 \ \ (1 \le t \le T-1), \text{and}$$

$$A_t = \{i_\mu \in \bar{N} \setminus \{j\} \,|\, a_t \le \mu \le b_t\} \ \ (1 \le t \le T).$$

I denote this a_t and b_t as the maximum index and minimum index of the set A_t, respectively. If $m = n-1$, I define $\mathcal{A} = (\phi)$ as the unique partition over $\bar{N} \setminus \{j, i_1, \dots, i_m\}$. Please note that I use the term "partition" with a slightly restricted meaning within this proof.

To make this notation familiar to the reader, I will show an example. Suppose $n = 6$, or $\bar{N} = \{1,2,3,4,5,6\}$, and $j = 3$. Then a permutation $\imath$ over $\bar{N} \setminus \{j\}$ is the way we array the individuals in $\bar{N} \setminus \{j\} = \{1,2,4,5,6\}$. For instance, $\imath = (i_1, i_2, \dots, i_{n-1}) = (i_1, i_2, i_3, i_4, i_5)$ defined as:

$$i_1 = 2,\ i_2 = 4,\ i_3 = 5,\ i_4 = 1, \text{and}\ i_5 = 6$$

makes a permutation over $\bar{N} \setminus \{j\}$. For this permutation and $m = 2$, a partition over $\bar{N} \setminus \{j, i_1, \dots, i_m\} = \bar{N} \setminus \{j, i_1, i_2\} = \{i_3, i_4, i_5\}$ is how we divide the set $\{i_3, i_4, i_5\}$ into pieces without breaking the index order. For example, the following are both partitions over $\bar{N} \setminus \{j, i_1, \dots, i_m\}$:

$$\mathcal{A} = (A_1, A_2, A_3) = (\{i_3\}, \{i_4\}, \{i_5\})$$

$$\mathcal{A}' = (A_1, A_2) = (\{i_3\}, \{i_4, i_5\}).$$

However,

$$\mathcal{A}'' = (A_1, A_2) = (\{i_3, i_5\}, \{i_4\})$$

is NOT a partition over $\bar{N} \setminus \{j, i_1, \dots, i_m\}$. The reason this is not a partition (in our meaning) is that i_4 is skipped in A_1. So, incorporating i_4 into A_1 we get

$$\mathcal{A}''' = (A_1) = (\{i_3, i_4, i_5\})$$

that is actually a partition.

For the second partition given above, $\mathcal{A}'$, the maximum and minimum indices are:

$$a_1' (= \text{the minimum index of } A_1') = 3, b_1' (= \text{the maximum index of } A_1') = 3$$

$$a_2' (= \text{the minimum index of } A_2') = 4, b_2' (= \text{the minimum index of } A_2') = 5.$$

Take any individual $j \in \bar{N}$, a permutation $\imath = (i_1, \dots, i_{n-1})$ over $\bar{N} \setminus \{j\}$, and a partition $\mathcal{A} = (A_1, \dots, A_T)$ of $\bar{N} \setminus \{j, i_1, \dots, i_m\}$ for $m \in \{0,1,\dots,n-1\}$. Now, we define a ballot profile $V_m^j[\imath; \mathcal{A}] \in \mathfrak{D}$. Note that this ballot profile $V_m^j[\imath; \mathcal{A}]$ is made up of $(m+t)$ "rings". For the first m individuals $i_1, \dots, i_m$, each of them and j approves each other. For the rest of the individuals, each subset A_t with j makes a 1-cyclic ballot sub-profile: j approves i_{a_t}, i_{a_t} approves i_{a_t+1}, …, i_{b_s-1} approves i_{b_s}, and i_{b_s} approves j. Formally stated, the ballot profile $V_m^j[\imath; \mathcal{A}] = (N_1, \dots, N_n)$ is defined as follows:

$$N_j = \{i_\mu \,|\, 1 \le \mu \le m\} \cup \{i_{a_1}, \dots, i_{a_T}\},$$

$N_{i_\mu} = \{j\}$ if $1 \le \mu \le m$,
$N_{b_t} = \{j\}$ for all $1 \le t \le T$, and
$N_{i_\mu} = \{i_{\mu+1}\}$ for all $\mu \in \overline{N} \setminus (\{j, b_1, \dots, b_t\} \cup \{\mu \mid 1 \le \mu \le m\})$.

Then we define $V_m^j[\imath]$ as:

$$V_m^j[\imath] \coloneqq \left\{ V_m^j[\imath; \mathcal{A}] \,\middle|\, \mathcal{A} \text{ is a partition of } \{ i_\mu \in \overline{N} \mid m+1 \le \mu \le n \} \right\}.$$

Furthermore, we define V_m^j as:

$$V_m^j \coloneqq \bigcup \left\{ V_m^j[\imath] \,\middle|\, \imath \text{ is a permutation of } \overline{N} \setminus \{j\} \right\}.$$

Step 2: Induction on m

I select an arbitrary individual, fixed as $j \in \overline{N}$ till the end of this step. I am going to show that j is the winner in all of the ballot profiles in V_0^j by an induction on m in a descending order. Thus, this step is made up of two parts: the first is to show the case of $m = n-1$, and the second is to construct the induction.

[1] The Case of $m = n - 1$

I show that $\varphi(\mathcal{N}) = j$ for all ballot profiles $\mathcal{N} \in V_{n-1}^j$. Note that there is only one ballot profile $V_{n-1}^j[\imath] =: (N_1, \dots, N_n) = \mathcal{N}$ because the partition $\mathcal{A}$ is uniquely (ϕ). Suppose to the contrary that $\varphi(\mathcal{N}) = i \ne j$. Consider a ballot profile $\mathcal{N}' \in \mathfrak{D}$ as $N_i' = \overline{N} \setminus \{i\}$ and $N_{-i}' = N_{-i}$. Then, IMP demands $\varphi(\mathcal{N}') = i$. However, i's score ranking at $\mathcal{N}'$ can be found as $r_i(\mathcal{N}') = n$. Contradiction. So, we can conclude that $\varphi(\mathcal{N}) = j$.

[2] Induction Part

Take any $m \in \{0,1,\dots,n-2\}$. Assume that $\varphi(\mathcal{N}) = j$ for all $\mathcal{N} \in V_{m+1}^j (\subseteq V_{m+2}^j \subseteq \cdots \subseteq V_{n-1}^j)$. Then I will show that $\varphi(\mathcal{N}) = j$ for all $\mathcal{N} \in V_m^j$ by negating the other possibilities. Take any permutation $\imath = (i_1, \dots, i_{n-1})$ of $\overline{N} \setminus \{j\}$ and partition $\mathcal{A} = (A_1, \dots, A_T)$ of $\{ i_\mu \in \overline{N} \setminus \{j\} \mid m + 1 \le \mu \le n \}$, where $A_t = \{ i_\mu \mid a_t \le \mu \le b_t \}$ $(1 \le t \le T)$. $m < n-1$, and so we have $T \ge 1$. Let $\mathcal{N}^1 = (N_1, \dots, N_n) \coloneqq V_m^j[\imath; \mathcal{A}]$. If there exists $t^* \in \{1, \dots, T\}$ such that $|A_{t^*}| = 1$, we can regard $\mathcal{N}^1$ as an element in V_{m+1}^j because $\mathcal{N}^1 = V_{m+1}^j[\imath^1; \mathcal{A}^1]$, where:

$\imath^1 = (i_1, \dots, i_m, i_{a_{t^*}}, i_{m+1}, \dots, i_{a_{t^*}-1}, i_{a_{t^*}+1}, \dots, i_{n-1})$, and
$\mathcal{A}^1 = (A_1, \dots, A_{t^*-1}, A_{t^*+1}, \dots, A_T)$.

So, $\varphi(\mathcal{N}^j) = j$ is given by the assumption of the induction. Hereafter, I suppose $|A_t| = 1$ for all $t \in \{1, \dots T\}$.

(1) $\boldsymbol{i_1, \dots, i_m, i_{b_1}, \dots, i_{b_t}}$ are not the winners at $\boldsymbol{\mathcal{N}}$.

Take any $i_\mu \in \{i_1, \dots, i_m, i_{b_1}, \dots, i_{b_t}\}$ and suppose that $\varphi(\mathcal{N}^1) = i_\mu$. Then consider another ballot profile $\mathcal{N}^2 = \left(N^2_{i_\mu}, N^2_{-i_\mu}\right) \in \mathfrak{N}$ as $N^2_{i_\mu} := \overline{N} \setminus \{i_\mu\}$ and $N^2_{-i_\mu} = N^1_{-i_\mu}$. IMP demands $\varphi(\mathcal{N}^2) = i_\mu$. On the other hand, the score profile is given as $s_{i_\mu}(\mathcal{N}^2) = 1 < s_{i_\nu}(\mathcal{N})$ for all $i_\nu \in \overline{N} \setminus \{i_\mu\}$. So, we have $r_{i_\mu}(\mathcal{N}^2) = n$. This contradicts the assumption that φ has rank $n-1$. We can conclude that $\varphi(\mathcal{N}^1) \notin \{i_1, \dots, i_m, i_{b_1}, \dots, i_{b_t}\}$.

(2) $\boldsymbol{i_{a_1}, \dots, i_{a_T}}$ are not the winners, either.

Assume that $\varphi(\mathcal{N}^1) = i_{a_t}$ for some $t \in \{1, \dots, T\}$. I will derive a contradiction. Let $\mathcal{N}^3 \in \mathfrak{D}$, as $N^3_{i_t} = \left(N^1_{i_{a_t}} \cup \{j\}\right) \setminus \{i_{a_t+1}\}$ and $N^3_{-i_{a_t}} = N^1_{-i_{a_t}}$. IMP demands $i_{a_t} = \varphi(\mathcal{N}^3)$, which also means $j \neq \varphi(\mathcal{N}^3)$. Next, let us consider $\mathcal{N}^4 \in \mathfrak{D}$ such that $N^4_j = N^3_j \cup \{i_{a_t}\}$ and $N^4_{-j} = N^3_{-j}$. Here, IMP demands $j \neq \varphi(\mathcal{N}^4)$. However, we can find $\mathcal{N}^4$ in V^j_{m+1}. Indeed, we can check that $\mathcal{N}^4 = V^j_{m+1}[\imath^4; \mathcal{A}^4]$, where:

$$\imath^4 = \left(i_1, \dots, i_m, i_{a_s}, i_{m+1}, \dots, i_{a_s-1}, i_{a_s+1}, \dots, i_n\right), \text{and}$$
$$\mathcal{A}^4 = \left(A_1, \dots, A_t \setminus \{i_{a_t}\}(=: A'_t), \dots, A_T\right).$$

By the assumption of the induction, we already know that $\varphi\left(V^j_{m+1}[\imath^4; \mathcal{A}^4]\right) = j$. This contradicts $\varphi(\mathcal{N}^4) \neq j$. $a_t \in \{a_1, \dots, a_T\}$ was arbitrary, and so it follows that $\varphi(\mathcal{N}^1) \notin \{i_{a_1}, i_{a_2}, \dots, i_{a_T}\}$.

(3) If $\boldsymbol{j}$ is not the winner at $\boldsymbol{\mathcal{M} \in V^j_m}$, there exists another ballot profile $\boldsymbol{\mathcal{M}' = V^j_m[\imath'; \mathcal{A}']}$ such that $\boldsymbol{\mathcal{A}'}$ has strictly smaller width of division and $\boldsymbol{j}$ is not the winner.

Let us denote an individual in $\overline{N} \setminus \{j, i_1, \dots, i_m, a_1, \dots, a_T, b_1, \dots, b_T\}$ as a middle term. I will show that no middle term wins at $\mathcal{N}^1$ by constructing a sequence of ballot profiles in V^j_m that satisfy a certain condition. For a partition $\mathcal{A} = (A_1, \dots, A_T)$, let $\Delta(\mathcal{A})$ be the width of the partition defined as:

$$\Delta(\mathcal{A}) := \sum_{t=1}^{T} |A_t \setminus \{a_t, b_t\}|$$
$$= |\overline{N} \setminus \{j, i_1, \dots, i_m, a_1, \dots, a_T, b_1, \dots, b_T\}|.$$

So, the width simply counts the number of middle terms for a given partition $\mathcal{A}$ of $\overline{N} \setminus \{j, i_1, \dots, i_m\}$.

Now, take any ballot profile $\mathcal{N}^5 := V^j_m[\imath^5; \mathcal{A}^5] \in V^j_m$. I will show that if some middle term wins at $\mathcal{N}^5$, then there exists a ballot profile $\mathcal{M} \in V^j_m$ such that $\Delta(\mathcal{M}) < \Delta(\mathcal{N}^5)$ and $j \neq \varphi(\mathcal{M})$. Suppose a middle term $i_\lambda \in A_t \setminus \{a_t, b_t\} (1 \leq t \leq T)$ wins at $\mathcal{N}^5$, viz. $\varphi(\mathcal{N}^5) = i_\lambda$. Consider a ballot profile $\mathcal{N}^6 \in \mathfrak{D}$ as $N^6_{i_\lambda} = \{j\}$ and $N^6_{-i_\lambda} = N^5_{-i_\lambda}$. IMP demands $\varphi(\mathcal{N}^6) = i_\lambda$. Thus, we have

$j \neq \varphi(\mathcal{N}^6)$. Consider a ballot profile $\mathcal{N}^7 \in \mathfrak{D}$ as $N_j^7 = N_j^6 \cup \{i_{\lambda+1}\}$ and $N_{-j}^7 = N_{-j}^6$. In this case, IMP demands $j \neq \varphi(\mathcal{N}^7)$. Furthermore, we can regard it as $\mathcal{N}^7 = V_m^j[\imath^5; \mathcal{A}^7]$, where:

$$\mathcal{A}^7 = \left(A_1, \dots, A_{t-1}, \{i_{a_t}, i_{a_{t+1}}, \dots, i_\lambda\}, \{i_{\lambda+1}, \dots, i_{b_t}\}, A_{t+1}, \dots, A_T\right).$$

A_t is divided into two parts: $\{i_{a_t}, i_{a_{t+1}}, \dots, i_\lambda\}$ and $\{i_{\lambda+1}, \dots, i_{b_t}\}$, and so we have $\Delta(\mathcal{A}^5) > \Delta(\mathcal{A}^7)$. This is because i_λ is no longer a middle term in the new partition, while the other individuals keep the same status.

When a ballot profile $\mathcal{M}^1 (= \mathcal{N}^5) \in V_m^j$ is given, the argument above shows how we can get a new ballot profile $\mathcal{M}^2 (= \mathcal{N}^7) \in V_m^j$ where the new partition has strictly less width than the original and where $j \neq \varphi(\mathcal{M}^2)$. Furthermore, according to (1) and (2), individuals with the maximum and minimum indices in the new partition cannot win at $\mathcal{M}^2$. Thus, it follows that $\varphi(\mathcal{M}^2)$ is also a middle term (in the new partition). So, we can iterate the argument again to get $\mathcal{M}^3, \mathcal{M}^4, \dots$, all of which are in V_m^j. Let us start this iteration with $\mathcal{M}^1 = \mathcal{N}^1$. $\Delta(\mathcal{A}^1)$ is finite, and so there exists a terminating level x and ballot profile $\mathcal{M}^x = V_m^j[\imath^x; \mathcal{A}^x]$ such that $\Delta(\mathcal{M}^x) = 0$. The width is 0, and so there is no middle term in $\mathcal{A}^x$. I have already shown that $j \neq \varphi(\mathcal{M}^x)$. By (1) and (2), the other individuals also lose at $\mathcal{M}^x$. This contradicts $\phi \notin \mathfrak{X}$. Thus, we can conclude that $\varphi(\mathcal{N}^1)$ is not a middle term. This completes (3).

The arguments in (1), (2), and (3) show that no individual other than j wins at $\mathcal{N}^1$. Thus, we have $\varphi(\mathcal{N}^1) = j$. The permutation $\imath^1$ and the partition $\mathcal{A}^1$ were arbitrary, and so this shows that we have $\varphi(\mathcal{N}) = j$ for all $\mathcal{N} \in V_m^j$.

Step 3: Proof of the Proposition

We know from the last step that for all $m \in \{0,1,\dots,n-1\}$, permutation $\imath$ and partition $\mathcal{A}$, $\varphi\left(V_m^j[\imath; \mathcal{A}]\right) = j$. j was an arbitrary individual in $\overline{N}$ selected at the beginning of the proof and carried throughout, and so the previous sentence holds for any $j \in \overline{N}$. Now, take two adjacent individuals, say $j, \overline{j+1} \in \overline{N}$. We know from the above that for ballot profiles $\mathcal{N}^8, \mathcal{N}^9 \in \mathfrak{N}$ such that:

$$\mathcal{N}^8 \coloneqq V_0^j[(\overline{j+1}, \overline{j+2}, \dots, \overline{j+n-1}); (\overline{N} \setminus \{j\})], \text{and}$$
$$\mathcal{N}^9 \coloneqq V_0^{\overline{j+1}}[(\overline{j+2}, \overline{j+2}, \dots, \overline{j+n}); (\overline{N} \setminus \{\overline{j+1}\})],$$

we obtain $j = \varphi(\mathcal{N}^8)$ and $\overline{j+1} = \varphi(\mathcal{N}^9)$. However, we can easily check that $\mathcal{N}^8 = \mathcal{N}^9 = \mathcal{C}^1$, which directly yields a contradiction. Therefore, our first assumption that there is a nomination rule $\varphi: \mathfrak{D} \to \overline{\mathfrak{X}}$ of rank $(n-1)$ is false. This completes the proof of $l = 1$. ■

Proof of Proposition 15 when $l \geq 2$.

Let us assume that there exists a nomination rule $\varphi: \mathfrak{D} \to \mathfrak{X}^l$ that has rank $n-1$. Take any individual $j \in \overline{N}$, permutation $\imath = (i_1, \dots, i_{n-1})$ over $\overline{N} \setminus \{j\}$, and a partition $\mathcal{A} = (\overline{N} \setminus \{j\})$. Let $\mathcal{N}^{10} \coloneqq V_m^j[\imath; \mathcal{A}]$. $\varphi(\mathcal{N}^{10})$ contains at least two individuals, and so there exists an individual $i_\mu \in \overline{N} \setminus \{j\}$ such that $i_\mu \in \varphi(\mathcal{N}^{10})$. Let us consider a ballot profile $\mathcal{N}^{11} \in \mathfrak{D}$ as $N_{i_\mu}^{11} = \overline{N} \setminus \{i_\mu\}$ and $N_{-i_\mu}^{11} = N_{-i_\mu}^{10}$. IMP demands $i_\mu \in \varphi(\mathcal{N}^{11})$. But i_μ has rank $r_{i_\mu}(\mathcal{N}^{11}) = n$, in contradiction to the rank of φ. ■

Materials for a presentation

1

The Infinite Regress Problem in Choice of a Collective Decision Procedure: An Axiomatic Study (集団的意思決定における手続き正当化を巡る無限後退の解消に関する研究)

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指導教員　堀田昌英　教授

2

二、三章：収束論

－手続き選択の無限後退の問題に対して、「収束」とよぶ新しい見方を提示。「収束」を通じて、特定の選択肢あるいは手続きを合理的に選択できることを意味する一連の定理を示す。

3

準備：スコアリングルール

◇選択肢が3個の場合、配点$[s_1, s_2, s_3]$（一位にs_1点、二位にs_2点、三位にs_3点；$s_1 \geq s_2 \geq s_3 \geq 0$）で特徴づけられる。

$$\begin{cases} \text{個人}1: a \succ b \succ c & +s_1\text{点} \\ \text{個人}2: a \succ b \succ c & +s_1\text{点} \\ \text{個人}3: b \succ a \succ c & +s_2\text{点} \\ \text{個人}4: c \succ b \succ a & +s_3\text{点} \end{cases}$$

aの得点は
$2s_1 + s_2 + s_3$(点)
⇒最も点数の高い選択肢が選ばれる。

◇配点のつけ方は様々：

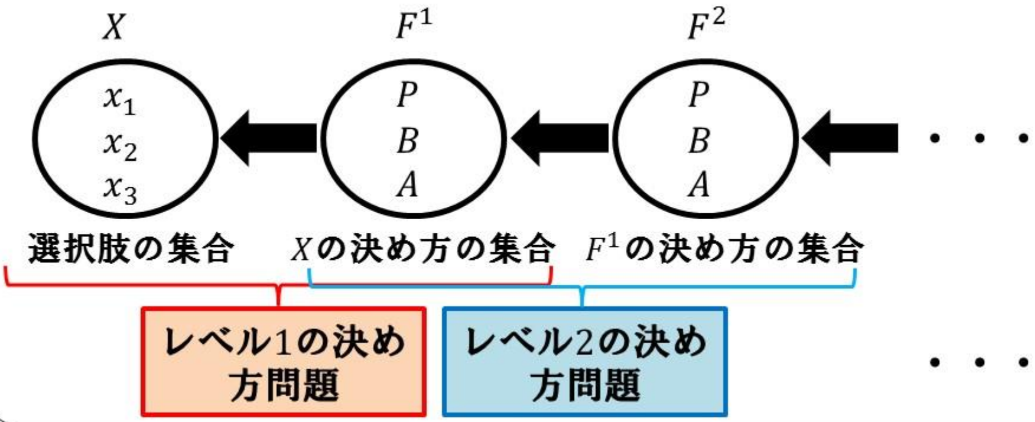
- **Plurality (P)**: [1,0,0]　(一位にのみ1点)
- **Borda (B)**: [2,1,0]　(1点ずつ下げていく)
- **Antiplurality (A)**: [1,1,0]　(最下位以外に1点)

4

基本設定

- $N = \{1,2,\dots,n\}$: **個人の集合（社会）**
- $X = \{x_1, x_2, \dots, x_M\}$: **選択肢の集合** $(3 \le M < \infty)$
- $F = \{f_1, f_2, \dots, f_m\}$: **投票ルールのメニュー**

例：3選択肢、メニュー$F = \{P, B, A\}$での無限後退



5

決め方についての選好によって決め方を決める

◇投票ルールfがself-selective (self-stable) **であるとは、決め方に関する投票でfが自身を選ぶこと** (Koray 2000)

▸ **定理** (Koray 2000): Social Choice Function f **が** neutral, unanimous, universally self-selective **であれば、独裁的である。**

◇投票ルールのメニューFがself-stable**であるとは、任意のプロファイルに対して少なくとも一つのルールは、自身を選ぶ** (Diss and Merlin 2010)

▸ **定理** (Diss and Merlin 2010): The set of $\{P, B, A\}$ is stable with 84.49% (IC) or 84.10% (IAC).

「収束」の考え方 (1/2)

◇ 14人からなる社会が、a案・b案・c案のうちから一つを選択する：$N = \{1,2,\dots,14\}$ and $X = \{a,b,c\}$。

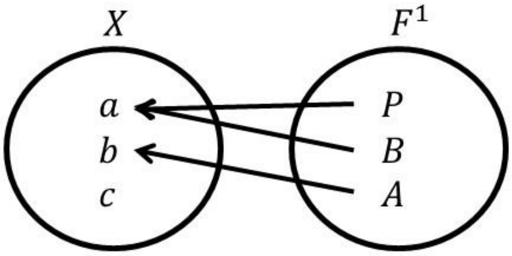
◇ X上の選好プロファイルが次のようになったとする。

$$\begin{cases} 10\text{人}：a \succ b \succ c \\ 4\text{人}:\quad b \succ c \succ a \end{cases}$$

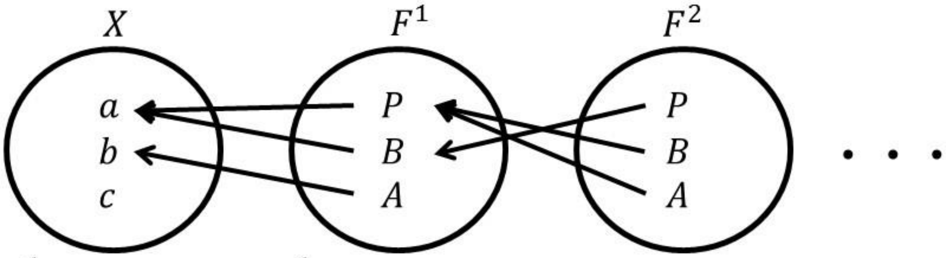
◇ 可能な投票ルールのメニューを

$$F = \{P\ (\text{Plurality}), B\ (\text{Borda}), A\ (\text{Antiplurality})\}$$

とすると、



「収束」の考え方 (2/2)



$$\begin{cases} 10\text{人}：abc \\ 4\text{人}:\quad bca \end{cases} \qquad \begin{cases} 4\text{人}: PBA + 6\text{人}: BPA \\ \qquad 4\text{人}: APB \end{cases}$$: CI プロファイル

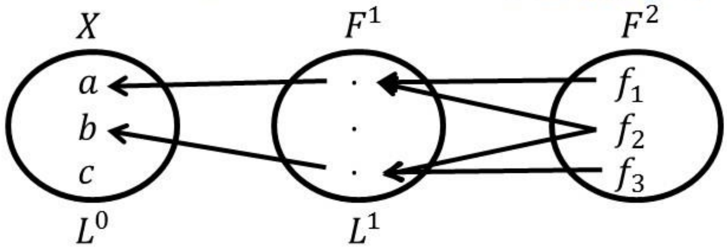
- Consequentialに InduceされるF^1上の選好 (CI プレファレンス) は、

　「10人」　：　$(P \succ B \succ A$または$B \succ P \succ A)$

　「4人」　：　$(A \succ P \succ B$または$A \succ B \succ P)$

- どの「決め方の決め方＝レベル2の決め方」も最終的な行き先が一致＝「収束」

CIプレファレンスの定義 [1/2]



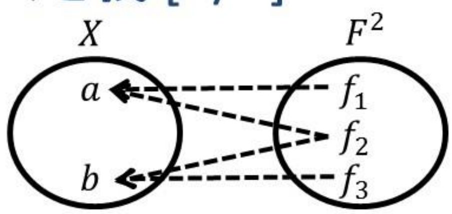
$f \in F^2$**の最終的な行き先をクラス**$C_f[L^0, L^1]$**という。**

- $C_{f^1}[L^0, L^1] = \{a\}, C_{f_2}[L^0, L^1] = \{a, b\}, C_{f_3}[L^0, L^1] = \{b\}$

CI プレファレンスの定義（簡易版）
CIプレファレンスとは、決め方のよしあしを（各々の決め方の）クラスのみによって判断したプレファレンス。

- $\{a\}, \{a, b\}, \{b\} \in \mathfrak{P}(X)$**を各個人はどう判断するか？**

CIプレファレンスの定義 [2/2]



X**上の選好**L_i^0**に「自然に」**$\mathfrak{P}(X)$**上の選好**$e_i(L_i^0)$**を対応付けることを選好の拡張とよぶ。**

- **「自然に」とは**

Extension Rule　: $L_i^0 : a \succ b \Rightarrow e_i(L_i^0) : \{a\} \succ \{b\}$

Gardenfors's Rule　: $L_i^0 : a \succ b \Rightarrow e_i(L_i^0) : \{a, b\} \succ \{b\}$

- **これらを満たす拡張は一般には多数ある**

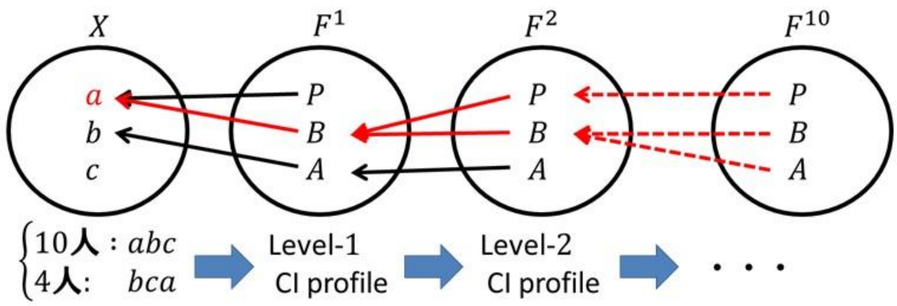
CI プレファレンスの定義
個人i**のCIプレファレンスとは、決め方のよしあしを、各々の決め方のクラスを**$e_i(L_i^0)$**で評価することにより判断したプレファレンス。**

10

弱収束の定義

Definition 11 (p.21) ：弱収束

選択肢の集合X上のプロファイルL^0が弱収束するとは、（任意の選好拡張$\{e_i\}_{i \in N}$に対し）ある階層$k \in \mathbb{N}$までのCIプロファイルの列$L^0, L^1, \dots, L^k$が存在して、全てのlevel-$(k+1)$ルールのクラスが一致すること。



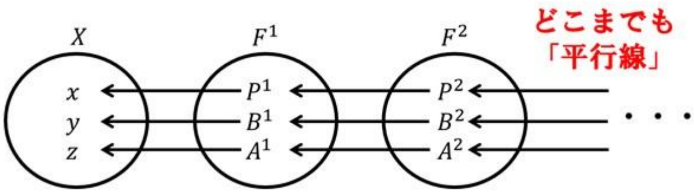
11

自明膠着の定義

Definition 12 (p.22) ：自明膠着

プロファイルL^0が自明膠着にあるとは、任意のCIプロファイルL^1に対し、level-1,2の全ての決め方が相異なる単一の元を選択すること

▶ **全てのルールがばらばらに選択している状況**



※自明膠着下では決して収束しない：Proposition 1, p.22

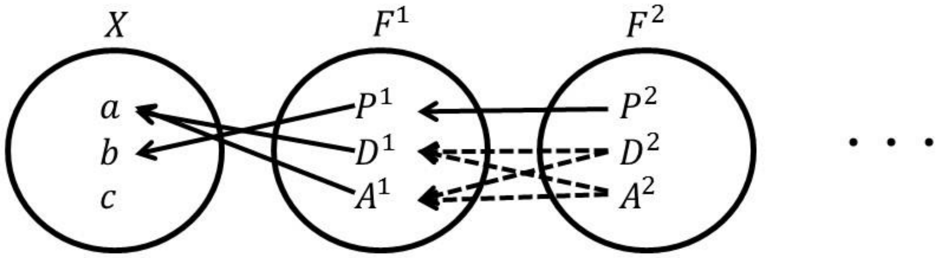
12

弱収束も自明膠着もしない例

Example 10 (p.23)：$n = 17$、$X = \{a, b, c\}$とし、

- $F = \{P, A, D\}$　ただし D: $\left[1, \frac{3}{4}, 0\right]$
- abc (2人)、 acb (3人)、 bac (5人)、 bca (2人)、 cab (3人)、cba (2人)

とすると、このプロファイルは、弱収束も自明膠着もしない。

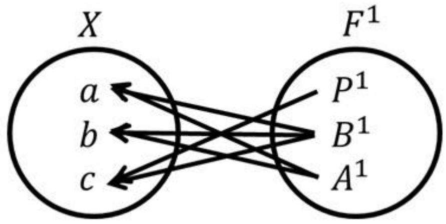


13

弱収束も自明膠着もしない例

Example 11 (p.23)：$n = 1700$、$X = \{a, b, c\}$、$F = \{P, B, A\}$**に対し、次のプロファイルは自明膠着ではないが、弱収束するともいえない。**

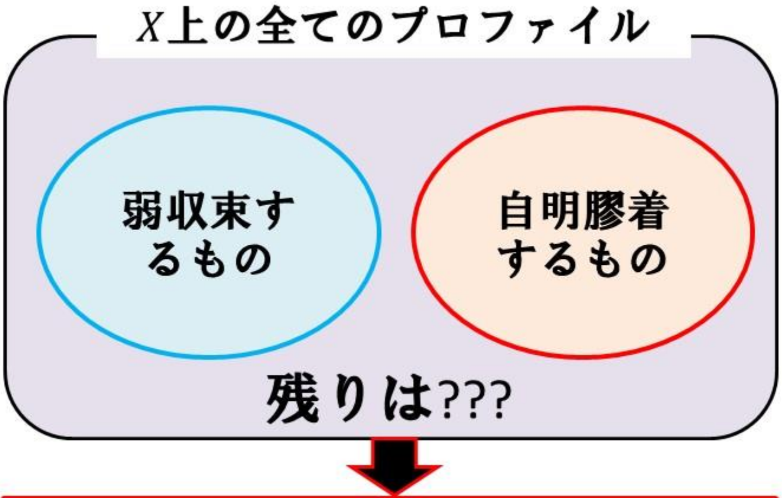
$$
\begin{cases}
L_i^0: abc \text{ if } 1 \leq i \leq 400 \ (\mathrm{I}) \\
L_i^0: acb \text{ if } 401 \leq \mathrm{i} \leq 500 \ (\mathrm{II}) \\
L_i^0: bac \text{ if } 501 \leq i \leq 800 \quad (\mathrm{III}) \\
L_i^0: bca \text{ if } 801 \leq i \leq 1000 \quad (\mathrm{IV}) \\
L_i^0: cab \text{ if } 1001 \leq i \leq 1400 \quad (\mathrm{V}) \\
L_i^0: cba \text{ if } 1401 \leq i \leq 1700 \quad (\mathrm{VI}).
\end{cases}
$$



※選好拡張$\{e_i\}_{i \in N}$**次第で、クラスの一致が見出せることも自明膠着状の状態に帰着することもある。**

14

収束可能性による分類



$\{P, B, A\}$をもつ大規模社会では、残りは「ない」：『二択定理』

15

収束と自明膠着の二択定理

Theorem 3 (p.26): 二択定理

$F = \{f_1, f_2, f_3\}$：3つのスコアリングルールの組。
f_jの配点形式は$[1, s_j, 0]$、$1 \geq s_1 > s_2 > s_3 \geq 0$であり、次を満たすとする：

$$s_3 \geq \frac{1}{2} \text{ or } \left[s_3 < \frac{1}{2} \text{ and } s_2 \leq \frac{1 + s_3}{2 - s_3} \right] \cdots\cdots (\star)$$

このときIACモデルのもとでは、人口$n \to \infty$で

$$\boldsymbol{p_{WC} + p_D \to 1}.$$

（p_{WC}は弱収束可能なプロファイルが生じる確率、p_Dは自明膠着するプロファイルが生じる確率）。

注：特に断らない限り，以降はIACモデルを仮定する

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二択定理の意味

二択定理の概要

3つのスコアリングルールの組Fが$(\star)$条件を満たせば$n \to \infty$で漸近的には収束か自明膠着の二択となる。

- $(\star)$**条件は比較的弱い条件；任意のスコアリングルールXに対し、**$F = \{\text{Plurality}, \text{Borda}, X\}$**は**$(\star)$**を満たす**

  $\Rightarrow F = \{P, B, A\}$**であれば成り立つ**

- **「収束」と「自明膠着」のいずれでも、それ以上遡ることは意味がない**

⇒ したがって、手続き選択の無限後退は実質的に意味を持たない

17

二択定理の確率評価

$F = \{P, B, A\}$**とすると、「二択性」が成り立っていた：**

Corollary 1 (p.26) **：**$\{P, B, A\}$, 3**選択肢での収束確率**

$|X| = 3$、$F = \{P, B, A\}$**とする。**$n \to \infty$**において、弱収束可能なプロファイルが生じる確率**p_{WC}**は、**

$$p_{WC} \to 98.2\% \text{ (IC モデル)}$$
$$p_{WC} \to 98.8\% \text{ (IAC モデル)}$$

cf) **同メニューが**stable**である確率は、**84.49% (IC), 84.10% (IAC) (Diss and Merlin 2010, Diss et al. 2012)

序論 定義 基本定理 考察1 考察2 考察3.1 考察3.2

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二択定理の考察１

二択定理の概要
3つのスコアリングルールの組Fが$(\star)$条件を満たせば$n \to \infty$で漸近的には収束か自明膠着の二択となる。

Q. 二択定理は他の標準的な投票ルールの組でも成り立つか？

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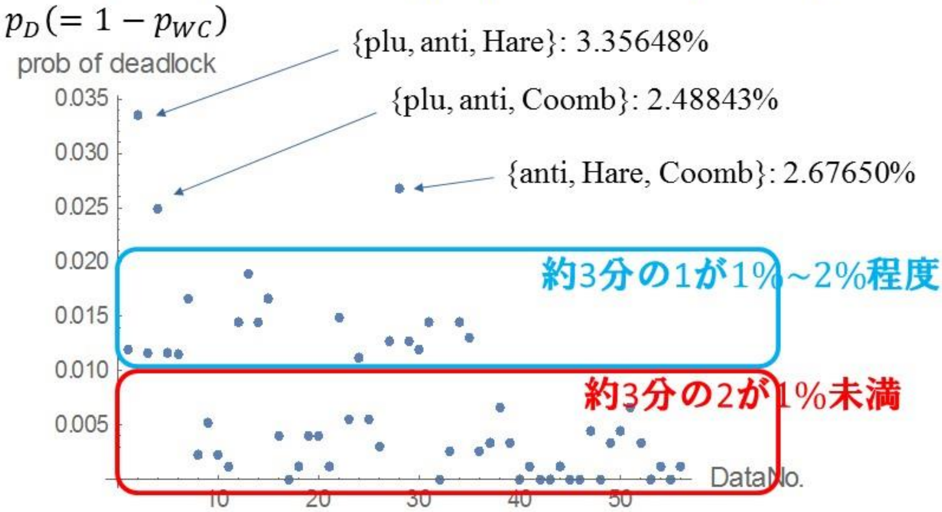
標準的なルール組での収束評価

Theorem 6 (p.30) 他：標準的なルール組での二択定理
Plurality , Borda, Anti-plurality, Hare (f_H), Nanson (f_N), Coomb (f_C), Maximin (f_M), Black (f_{Bl})のいずれか3ルールからなるメニューでは、$n \to \infty$で次が成り立つ： (1) 収束か膠着かの「二択性」 (2) $\lvert X\rvert = 3$のとき、いずれの組でもp_{WC}は96.6%以上 (3) 次の10組では、$\lvert X\rvert$によらずp_{WC}は100%となる： $\{f_P, f_N, f_M\}, \{f_A, f_N, f_M\}, \{f_B, f_H, f_{Bl}\}, \{f_B, f_N, f_M\}, \{f_B, f_N, f_{Bl}\}, \{f_B, f_C, f_{Bl}\}, \{f_B, f_M, f_{Bl}\}, \{f_H, f_N, f_M\}, \{f_N, f_C, f_M\}, \{f_N, f_M, f_{Bl}\}$

二択定理の確率評価 (IAC, $n \to \infty, |X| = 3$：p. 76)

20



56通り全てにおいて、96.6%以上の確率で弱収束。
参考：$\{P, B, A\}$**では**stable (84.10%：Diss, Louichi, Merlin, and Smaoui, 2012)、**弱収束** (98.8%)。

序論 | 定義 | 基本定理 | 考察1 | **考察2** | 考察3.1 | 考察3.2

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二択定理の考察2

二択定理の概要
3つのスコアリングルールの組F**が**$(\star)$**条件を満たせば** $n \to \infty$**で漸近的には収束か自明膠着の二択となる。**

Recall. **弱収束とは、CIプロファイルを適当に取った結果、ある階層で行き先が一致すること。**
Q. **その取り方次第で結果が変わりうる？**
Cf. self-selectivity (Koray 2000)

強収束

Definition 13 (p.27)：強収束

X上のプロファイルL^0が$C \subseteq X$に**強収束**するとは、
(1) Cに弱収束し、
(2) C以外のどんな$C' \subseteq X$にも弱収束しない

- 何百回・何千回と遡ると**無数のCIプロファイル列があるが**、そのうちのどれをとっても**決して他の結果へは収束しない**という条件

- L^0が強収束する⇒L^0が弱収束する
  弱収束よりも論理的に（かなり）強い

強収束による二択定理

Theorem 4 (p.28)：$\{P, B, A\}$の強収束定理

$F = \{P, B, A\}$とする。弱収束・強収束するプロファイルが生じる確率をp_{WC}，p_{SC}とすれば、

$$p_{WC} - p_{SC} \to 0 \text{ as } n \to \infty$$

系：$F = \{P, B, A\}$では、弱収束を強収束に置き換えた形で二択定理が成り立つ。すなわち、

$$p_{SC} + p_D \to 1 \text{ as } n \to \infty$$

であって、$|X| = 3$のとき、

$$p_{SC} \to 98.2\% \text{ (ICモデル)}$$
$$p_{SC} \to 98.8\% \text{ (IACモデル)}$$

二択定理の考察 3.1

二択定理の概要

3つのスコアリングルールの組Fが$(\star)$条件を満たせば$n \to \infty$で漸近的には収束か**自明膠着**の二択となる。

Q. 「いつでも」「収束」できるメニューは存在するか

いつでも収束できるという性質

Definition 14 (p.29) ：収束性

Fをメニューとする。

- Fが**弱収束性**（**強収束性**）を満たすとは、
  任意の$n \geq 3$で、全てのプロファイルが弱収束（強収束）する。
- Fが**漸近弱収束性**（**漸近強収束性**）を満たすとは、
  $n \to \infty$で、$p_{WC} \to 1$（$p_{SC} \to 1$）となる。

▸「いつでも」「収束」できる。

$$\left\{\begin{array}{c} \forall n \geq 3\text{で} \\ \text{or} \\ n \to \infty\text{で} \end{array}\right\} \quad \left\{\begin{array}{c} \text{弱収束} \\ \text{or} \\ \text{強収束} \end{array}\right\}$$

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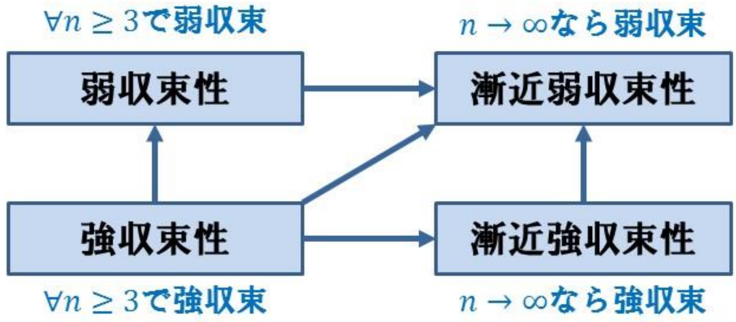
公理間の関係



注1. $\{P, B, A\}$**はいずれも満たさない**

注2. $\{B, \text{Hare}, \text{Black}\}$**等の**6**組は、漸近弱収束性をもつ**

Q. **強収束性をもつメニューは存在するか？**
A. **存在する**

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強収束性を満たすルール組

Theorem 7 (p.31): **強収束性メニューの存在**
（Neutrality, Difference**に加えて）強収束性を満たすルール組**F**が存在する。**

- **実際に次の**$F = \{f_1, f_2, f_3\}$**が条件を満たす。**

  f_1: Plurality SCF in favor of individual 1,

  f_2: Anti-plurality SCF in favor of individual 1, and

$$f_3(\cdot) = \begin{cases} f_1(L) \text{ if } |f_P(L)| \geq 2 \\ f_1(L) \text{ if majority winner exists} \\ f_2(L) \text{ otherwise.} \end{cases}$$

考察 3.1**のまとめ：**$\{B, \text{Hare}, \text{Black}\}$ **(漸近弱収束性)や上のメニューを用いることで、「いつでも」「収束」できる。**

序論 / 定義 / 基本定理 / 考察1 / 考察2 / 考察3.1 / 考察3.2

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二択定理の考察 3.2

二択定理の概要

3つのスコアリングルールの組Fが$(\star)$条件を満たせば$n \to \infty$で漸近的には収束か**自明膠着**の二択となる。

Q. $F = \{P, B, A\}$は漸近収束性をもたないが、なんとか「工夫」できないか。

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「工夫」の意味

Theorem 8 (p.31)：$\{P, B, A\}$の拡大定理

$F = \{P, B, A\}$に対し、次のように定める議長ルールφを加えた$G = F \cup \{\varphi\}$は、漸近弱収束性をもつ：

$$\varphi(L) = G(f_P(L) \cup f_B(L) : L_1)$$

Definition 16 (p.31)：漸近収束拡大

ある$G \supseteq F$が存在して漸近弱収束性をもつとき、Fは**漸近収束拡大**可能であるという

考察 3.1・3.2 の展望

考察 3.1

「いつでも」「収束」できるという公理のもとに、望ましいメニューを提示
⇒**何も制約がなければ**、これらのメニューを使えばよい

考察 3.2

既存のメニュー$\{P, B, A\}$があったとして、これをいかに「**工夫**」するか
⇒ 漸近収束拡大が他のメニューでも可能だとよい

理想
性質 $\boxed{}$ を満たすメニューFは、漸近収束拡大可能である。

大きなF問題〜計算技術の制約〜

- IACモデルの下の投票現象の確率計算は、Ehrhart polynomialの導出に帰着
- Polylib, Barvinok, Normalizなどのソフトウェアがあるが、**選択肢（ルール）3個ないし4個**が現在の限界
- 大きなメニューFでは、現実的には収束する確率が計算できない

~~理想~~ 定理
性質 $\boxed{}$ を満たす**ルールからなる任意の大きさの**メニューFは、漸近収束拡大可能である。

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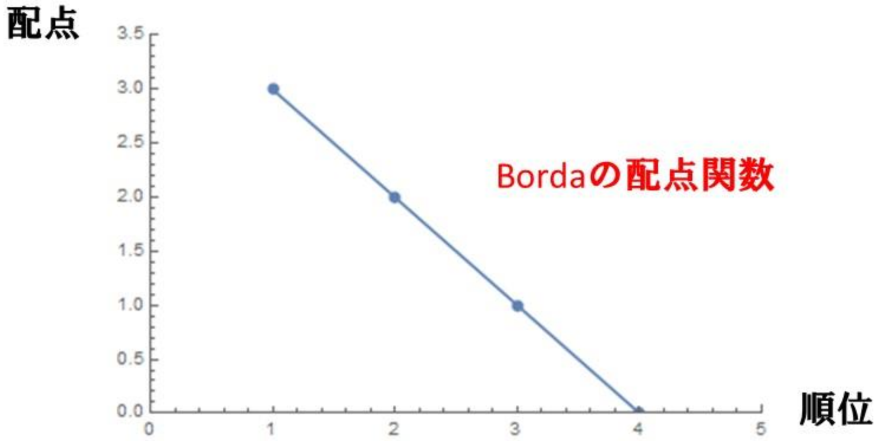
配点関数

- 選択肢が4個の場合、Bordaルールの配点は[3,2,1,0]。
- これをプロットして結んだものを配点関数という。



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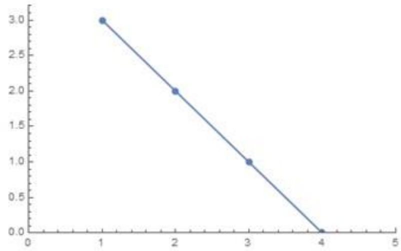
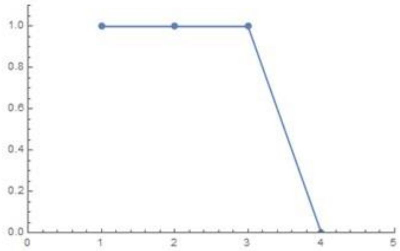
凹ルール

Definition 19 (p.32) ：凹ルール
スコアリングルールfが凹であるとは、$\forall m \geq 3$個の選択肢に対し、配点関数が凹関数 (i.e. 上に凸) となること。

例：Borda, Antipluralityはともに凹ルール。



Borda [3,2,1,0]



Antiplurality [1,1,1,0]

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凹定理

~~理想~~ 定理

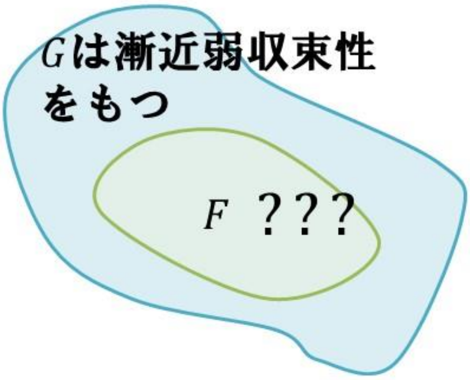
性質 $\boxed{凹}$ を満たすルールからなる任意の大きさのメニューFは、漸近収束拡大可能である。

【凹定理の意義】

・$F=\{$**凹ルール**1000**個**$\}$

・**Fの収束確率は（現実的には）計算できない**

・**ある$G\supseteq F$は$n\to\infty$で収束確率が**100%**だといえる。**



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凹定理の証明は「構成的」

Theorem 9 (p.33)：**凹定理**

有限個の凹ルールのメニューは、漸近収束拡大可能である。

（証明の概要）

◇ $F=\{f_1,f_2,\dots,f_m\}$**に対し、**μ**個の**k**-アプルーバルルール**$E=\{E_1,E_2,\dots,E_\mu\}$**および**$F,E$**の元の「もどき」たち**$C=\tilde{F}\cup\tilde{E}$**を適当に与えると、**$G=F\cup E\cup C$**が漸近収束性をもつことを示す。**

◇ **最大で**$|G|$**階程度遡る過程で、ある都合のよい性質をもった階層**k^***が現れることに注目する。**

◇ **その階層**k^***において、凹性に基づき不等式を簡略化、その成立を示す。**

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四章：指名ルールの公理的分析

－収束論では意思決定主体（の集合）Nは所与
－現実の意思決定では、意思決定主体Nの正当性が問われることもある
－「誰が代表者か（Nに属すべきか）」という判断を集約するためのルールについて分析する

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指名ルール

- $\bar{N}=\{1,2,\ldots,n\}$：ある集団（組織）
- 各個人$i\in\bar{N}$が、代表者と考える人の集合$N_i\subseteq\bar{N}$を票として回答
- $(N_1,N_2,\ldots,N_n)$を票のプロファイルとよぶ。

Definition 22 (p.44)

票のプロファイル$\mathcal{N}=(N_1,N_2,\ldots,N_n)\in\mathfrak{D}$に対し、$\bar{N}$の部分集合$(\phi\neq)\varphi(\mathcal{N})\in\mathfrak{X}$を対応させる関数（対応）を、指名ルール（nomination rule）$\varphi:\mathfrak{D}\to\mathfrak{X}$とよぶ。

☞相互評価に基づき$\varphi(\mathcal{N})\subseteq\bar{N}$を選ぶ
☞通常の投票と異なり、各人が投票者であると同時に候補者でもある。

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指名ルールの例

例（受賞者の選出）

▸**学会内の各個人**$i \in \bar{N}$**が自分たちの中で受賞に相応しいと思う人の集合**N_i**を回答し、それを集約して受賞者（たち）**$\varphi(N_1, N_2, \dots, N_n)$**を決定。**

例：Approval Voting (Brams and Fisuburn, 1978他)

▸**各人は相応しいと思う人を何人でも指名することができる（**$N_i \subseteq \bar{N}$**）**

▸**最も多くの人に指名された人を選出する。**

各投票者が候補者でもあるとき、特殊な戦略的行動が生じる可能性がある

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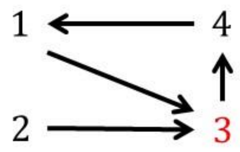
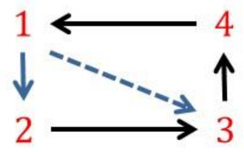
Impartiality

例：$n = 4$**人が**Approval Voting**により受賞者を決める**



個人1**は右のように投票した方が得をする**



Definition (Holzman and Moulin, 2013) (p.45)

指名ルール$\varphi: \mathfrak{D} \to \mathfrak{X}$**が**Impartial (IMP)**であるとは、任意の**$i \in \bar{N}$**、任意の票のプロファイル**$(N_i, N_{-i}) \in \mathfrak{D}$**と**$N_i' \in \mathfrak{D}$**に対し、**$i \in \varphi(N_i, N_{-i}) \Leftrightarrow i \in \varphi(N_i', N_{-i})$**.**

▸**投じる票によって、自分が選ばれるかどうかは変わらないという条件**

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先行研究

(1) Holzman and Moulin (2013)

- $N_i \in \bar{N} \setminus \{i\}$ **（各人は他者を一人指名）**
- $\mathfrak{X} = \bar{N}$ **（勝者ただ一人）**
- (IMP), Positive Unanimity (PU), and Negative Unanimity (NU)⇒**不可能**

(2) Tamura and Ohseto (2014); Tamura (2015)

- $N_i \in \bar{N} \setminus \{i\}$ **（各人は他者を一人指名）**
- $\mathfrak{X} = \mathfrak{P}(\bar{N}) \setminus \{\phi\}$ **（複数の勝者を許す）**
- plurality with runners-up rule
- (IMP), Anonymous Ballots (AB), (PU)⇒ **不可能**

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4章の目的

指名ルール $\varphi\colon \mathfrak{D} \rightarrow \mathfrak{X}$

$\uplus$

票のプロファイル

$$\mathcal{N} = (N_1, N_2, \dots, N_n) \mapsto \varphi(\mathcal{N}) \subseteq \bar{N}$$

- **一般に、どのような票のプロファイルを集計するか（定義域$\mathfrak{D}$）、何人の人を選出するか（終域$\mathfrak{X}$）によって不可能性の程度は異なる（と予想できる）**

4章の目的：様々な定義域・終域の組を考えることにより、「不可能性」を緩和する方法を探る：

（どのような票を集計すればimpartialに加え他の規範的な条件を満たす指名ルールが設計できるか？）

指名ルールの定義域$\mathfrak{D}$（どのような票を許すか）

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1) $\mathfrak{N} = \prod_{i\in\bar{N}} \mathfrak{N}_i$, where $\mathfrak{N}_i = \left\{N_i \,\middle|\, \phi \neq N_i \subseteq \bar{N} \setminus \{i\}\right\}$.

***人数の制限なく各人がよいと思う人を指名（棄権・自身の指名はなし）** cf. Fischer and Klimm (2013)

2) $\mathfrak{N}^{self} = \prod_{i\in\bar{N}} \mathfrak{N}_i^{self}$, where $\mathfrak{N}_i^{self} = \left\{N_i \,\middle|\, \phi \neq N_i \subseteq \bar{N}\right\}$.

*** 自身の指名を許す**

3) $\mathfrak{N}^{ab} = \prod_{i\in\bar{N}} \mathfrak{N}_i^{ab} \setminus \{(\phi, \phi, \dots, \phi)\}$, where $\mathfrak{N}_i^{ab} = \mathfrak{N}_i \cup \{\phi\}$.

*** 棄権を許す** cf. Alon et al. (2011)

指名ルールの定義域$\mathfrak{D}$（どのような票を許すか）

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4) $\mathfrak{N}^k = \prod_{i\in\bar{N}} \mathfrak{N}_i^k$, where $k = 1,2,\dots,n-2$

$$\mathfrak{N}_i^k = \left\{N_i \,\middle|\, N_i \subseteq \bar{N} \setminus \{i\} \text{ and } |N_i| = k\right\}.$$

*** 各人は自身を除いてちょうどk人を指名**

* Holzman and Moulin (2010, 2013), Tamura and Ohseto (2014), Tamura (2015)**などは全て$k = 1$の場合**

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指名ルールの終域$\mathfrak{X}$（≒何人選ぶか）

1) $\overline{\mathfrak{X}} = \mathfrak{P}(\overline{N}) \setminus \{\phi\}$

 ＊ $\overline{N}$**の非空な部分集合を選ぶ**

2) $\mathfrak{X}^l = \left\{ W \in \mathfrak{P}(\overline{N}) \,\middle|\, |W| = l \right\}$

 ＊ **ちょうど**l**人を選ぶ**

3) $\overline{\mathfrak{X}}^l = \left\{ W \in \mathfrak{P}(\overline{N}) \,\middle|\, |W| \geq l \right\}$

 ＊ **少なくとも**l**人を選ぶ**

4) $\underline{\mathfrak{X}}^l = \left\{ W \in \mathfrak{P}(\overline{N}) \,\middle|\, 1 \leq |W| \leq l \right\}$

 ＊ **（一人以上）最大**l**人を選ぶ**

▶ Holzman and Moulin (2010; 2013): $\mathfrak{X}^1$, Tamura and Ohseto (2014) and Tamura (2015): $\overline{\mathfrak{X}}$, Alon et al. (2011): $\mathfrak{X}^l$

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投票者としての平等の条件

Definition: Anonymous Ballots (AB) (p. 45)

指名ルール$\varphi: \mathfrak{D} \to \mathfrak{X}$**が**(AB) **を満たすとは、**$s_i(\mathcal{N}) = s_i(\mathcal{N}') \ \forall i \in \overline{N}$**であれば、**$\varphi(\mathcal{N}) = \varphi(\mathcal{N}')$。

▶ **全ての人のスコア（何人に指名されているか）が同じであれば、結果も同じ。**

▶ **「個人1が個人3を指名した」と「個人4が個人3を指名した」は同じ効果をもつ：投票者として平等。**

$$\varphi\left(\begin{tikzcd} 1 \arrow[d] \arrow[drr, red] & & 4 \arrow[ll] \arrow[dd, phantom] \\ & & \\ 2 \arrow[rr] & & 3 \arrow[uu] \end{tikzcd}\right) = \varphi\left(\begin{tikzcd} 1 \arrow[d] & & 4 \arrow[ll] \arrow[d, bend left, red] \\ 2 \arrow[rr] & & 3 \arrow[u] \end{tikzcd}\right)$$

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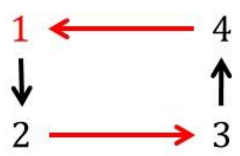
候補者としての平等の条件

Definition: Pairwise Candidate Neutrality (2CN) (p.46)

指名ルール$\varphi: \mathfrak{D} \to \mathfrak{X}$が(2CN)を満たすとは、全ての人が$i, j \in \bar{N}$に関する評価を逆転させたとき、$i, j$についての結果も逆転する。

- **各人が候補者としては平等であるということ。**

例：全ての人が1,3についての評価を逆転する。



1 wins

3 wins

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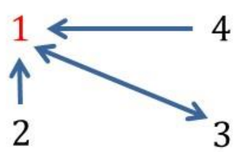
不可能性

Proposition 9 (p. 57)

前述のいずれの定義域・終域のペア$(\mathfrak{D}, \mathfrak{X})$でも、(IMP), (AB), (PU)を満たす指名ルールは存在しない。

- Tamura and Ohseto (2014) **が$(\mathfrak{N}^1, \bar{\mathfrak{X}})$で示した不可能性が、広く成り立つということ。**

- PU (Positive Unanimity)**とは全会一致の条件：**



PU　　：$\varphi(\mathcal{N}) = \{1\}$

Weak Positive Unanimity(WPU)

：$\varphi(\mathcal{N}) \ni 1$

Threshold（閾値）ルールの特徴づけ

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Definition: 閾値t-ルールφ^t

閾値t-ルールφ^tとは、スコアがt票以上の人を全て選ぶ指名ルール。

Proposition 12: φ^1の特徴づけ (p.59)

指名ルール$\varphi: \mathfrak{N} \to \overline{\mathfrak{X}}$が(IMP), (AB), (2CN), (WPU), (NU)を満たす⇔閾値1-ルールφ^1

- Approval Voting（**相対的**に最も高い点数の人を選ぶ）は(IMP)を満たさない（平等、全会一致は満たす）
- **閾値ルール（絶対的な値である閾値に達した人を選ぶ）は（平等、全会一致に加えて）(IMP) を満たす**

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様々な$(\mathfrak{D}, \mathfrak{X})$での比較

Dom$\mathfrak{D}$ \ Cod $\mathfrak{X}$	$\mathfrak{X}^l$	$\overline{\mathfrak{X}}^l$	$\underline{\mathfrak{X}}^l$
$\mathfrak{N}$	IMP, AB⇔Const	$l \geq 2$: IMP, AB, NU⇔φ^1. $l = 1$: IMP, AB, 2CN, NU, WPU⇔φ^1.	$l \leq n-2$: IMP, AB, NU⇒None. $l = n-1$: I, AB, NU⇔φ^1_{-i}.
$\mathfrak{N}^{self}$	IMP, AB⇔Const	IMP, AB⇔Const	IMP, AB⇔Const
$\mathfrak{N}^{ab}$	IMP, AB⇔Const	$l = 1$: IMP, AB, NU⇔φ^1	IMP, AB, NU⇒None
$\mathfrak{N}^k$	IMP, AB⇔Const	$l \leq k+1$: IMP,AB,2CN,NU,WM⇔ $\varphi^1, \ldots, \varphi^m$	$l > k$: IMP, AB, and WPU.

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様々な$(\mathfrak{D},\mathfrak{X})$での

Dom$\mathfrak{D}$ \ Cod $\mathfrak{X}$	$\mathfrak{X}^l$		
$\mathfrak{N}$	IMP, AB⇔Const		
$\mathfrak{N}^{self}$	IMP, AB⇔Const	IMP, AB⇔Const	IMP, AB⇔Const
$\mathfrak{N}^{ab}$	IMP, AB⇔Const	$l=1$: IMP, AB, NU⇔φ^1	IMP, AB, NU⇒None
$\mathfrak{N}^k$	IMP, AB⇔Const	$l \le k+1$: IMP,AB,2CN,NU ,WM⇔ $\varphi^1,\dots,\varphi^m$	$l > k$: IMP, AB, and WPU.

1) **自己指名することが許される場合**
2) **定められた人数を選出する場合**
にはimpartial**な指名ルールの設計が困難となる。**
cf. Holzman and Moulin (2013) showed on $(\mathfrak{N}^1,\mathfrak{X}^1)$.

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様々な$(\mathfrak{D},\mathfrak{X})$での比較

Dom$\mathfrak{D}$ \ Cod $\mathfrak{X}$	$\mathfrak{X}^l$	$\overline{\mathfrak{X}}^l$	$\underline{\mathfrak{X}}^l$
$\mathfrak{N}$	IMP, AB⇔Const	$l \ge 2$: IMP, AB, NU⇔φ^1. $l=1$: IMP, AB, 2CN, NU, WPU⇔φ^1.	$l \le n-2$: IMP, AB, NU⇒None. $l=n-1$: I, AB, NU⇔φ^1_{-i}.
		IMP, AB⇔Const	IMP, AB⇔Const
		$l=1$: IMP, AB, NU⇔φ^1	IMP, AB, NU⇒None
$\mathfrak{N}^k$	IMP, AB⇔Const	$l \le k+1$: IMP,AB,2CN,NU ,WM⇔ $\varphi^1,\dots,\varphi^m$	$l > k$: IMP, AB, and WPU.

比較的多くの人を選ぶことが許される状況（終域）では、閾値ルールが条件の多くを満たす。

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結論

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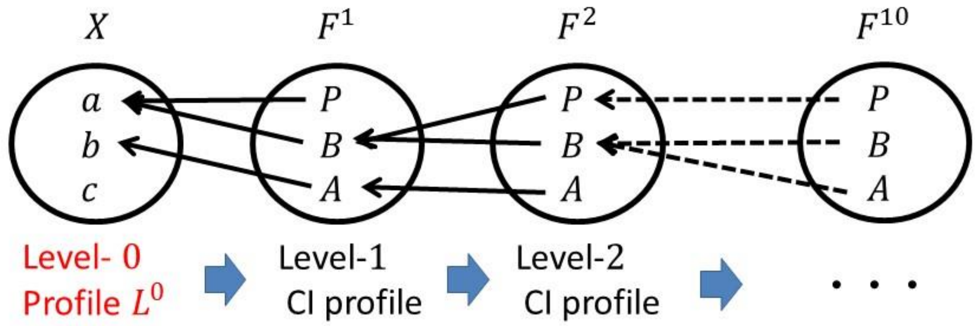
2章、3章：収束論 [1/2]

収束の意義：

- **どこまで遡っても先験的な合意がない状況では、遡って議論する手続きコストも多大なものとなりうるが、**
- **各個人がX上の選好（L^0）さえ提示すれば**
- **ある選択肢（a）を正当化できる（L^0がaに収束）**



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2章、3章：収束論 [2/2]

- Q. $\{P, B, A\}$などではどうか？
- Q. いつでも収束できるメニューはあるのか？

P, B, A, Hare, Nanson, Coomb, Maximin, Blackのうちの3つでも同様

二択定理の概要

3つのスコアリングルールの組Fが$(\star)$条件を満たせば$n \to \infty$で漸近的には収束か自明膠着の二択となる。

$\{P, B, A\}$では漸近的に強収束≒弱収束

◇ いつでも収束できるメニュー
◇ 凹メニューの漸近収束拡大

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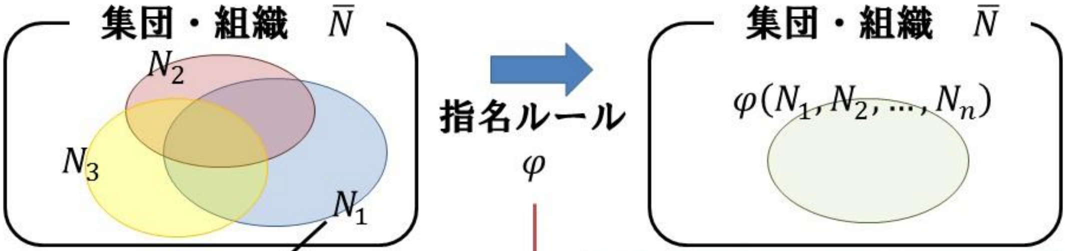
4章：指名ルールの設計



個人1の考える代表者（意思決定者・受賞者）として相応しい人の集合

指名ルール$\varphi: \mathfrak{N} \to \bar{\mathfrak{X}}$が(IMP), (AB), (2CN), (WPU), (NU)を満たす ⇔ 閾値1-ルールφ^1

博士論文全体を通して、「社会」を同定し（：指名ルール）、またその上で無限後退を合理的に解消する（収束論）という、規範的な手続き選択論が得られた。