

博士論文

Stable Matchings on Matroidal Structures

(マトロイド的構造における安定マッチング)

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STABLE MATCHINGS
ON MATROIDAL STRUCTURES

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Preface

The stable matching model, introduced by Gale and Shapley (1962), is a mathematical formulation of two-sided markets, where each agent has a preference on the opposite agent set. Gale and Shapley provided an algorithm to find a stable matching, in which no agent has an incentive to change the current assignment. The algorithm has various applications such as the admission market between colleges and students and the labor market between doctors and hospitals. As the growth of application, however, there arise various factors which cannot be handled by existing models. Then, the development of the stable matching theory shows no sign of slowing down even half a century after its appearance.

In this thesis, we cope with some complicated settings of two-sided market models by extending the matroidal framework proposed by Fleiner (2001). We consider models with certain types of lower quotas, integer- or real- variables, and some complicated preferences. We capture these settings through matroids and extensions. Also, as a novel application of stable matchings, we solve a list coloring problem with supermodular constraints.

We first investigate a stable matching model in which agents have lower quotas in addition to upper quotas. This setting is well captured using generalized matroids. We provide a polynomial-time algorithm which finds a stable matching or reports the nonexistence. We also show properties of stable matchings such as the distributive lattice structure.

Next, we consider stable allocations, real-variable versions of stable matchings. We design the first polynomial-time algorithm to find a stable allocation in polymatroid intersection. The algorithm combines the policy of the Gale–Shapley algorithm with the augmenting path technique, which is common in the matroid literature.

Thirdly, we introduce a new notion “matroidal choice functions” to represent preferences under matroid constraints, which cannot be derived from modular value functions. We find a strong relationship between the greedy algorithm on matroids and the substitutability of choice functions, an essential property for the existence of a stable matching.

In the last part, utilizing a certain generalization of a stable matching, we prove the list supermodular coloring theorem, which generalizes Galvin’s list edge coloring theorem for bipartite graphs. The existence of a stable matching plays a key role in the proof.

Our research deals with different types of generalizations of the stable matching model. In all our models, matroidal structures work effectively when we design algorithms and analyze inherent properties. We can realize that many desirable results, such as the existence of polynomial-time algorithms and the lattice structure of stable matchings, are supported by simple axioms of matroidal structures.

Since there are various kinds of two-sided markets in practice, it is important to establish comprehensive guidelines to handle current and forthcoming problems. Our results suggest that matroidal structures can be useful tools for that purpose.

Acknowledgments

First and foremost, I would like to express my greatest thanks to my supervisor, Satoru Iwata. He supported me all the time in my Ph.D. course by providing right guidance and cheering words. Discussions with him were always full of novel ideas and helpful advice. His deep insight and comprehension on combinatorial optimization enhanced the significances of our results. Without his great support and patience, I could not lead such a happy life in my Ph.D. course.

I am also grateful to my master's supervisor, Kazuo Murota. My interest to matroid theory and discrete convex analysis started when I was learning under him. He continuously provided me fruitful comments which came from his extensive knowledge. He also gave me pointed questions, which made my comprehension deeper.

My thanks go to all current and former members of the laboratories, which I belonged to in my master and Ph.D. courses. It was so lucky for me that I was surrounded by kind and intelligent professors and colleagues. In particular, I thank Tasuku Soma, Yutaro Yamaguchi, Naoki Ito, and Tatsuya Matsuoka, with whom I spend a lot of time enjoying mathematical and daily conversations. Tasuku and Yutaro, my seniors, kindly taught me not only mathematical subjects but also how to submit a paper and how to apply for research grants. In particular, I am thankful to them for serving as good role models for me. To Naoki and Tatsuya, I am grateful for sharing their ideas, which were always thought-provoking for me. I also would like to acknowledge to Professors Kunihiro Sadakane, Akiko Takeda, Kazuhisa Makino, Hiroshi Hirai, Yuji Nakatsukasa, and Yusuke Kobayashi for their helpful comments in numerous seminars.

I am deeply grateful to Professors Akihisa Tamura, Hiroshi Imai, Kunihiro Sadakane, and Hiroshi Hirai for reading this thesis carefully and providing lots of constructive advice.

I also appreciate the financial support by JSPS Research Fellowship for Young Scientists, by JST ERATO Kawarabayashi Large Graph Project, and by JST CREST.

Finally, I am grateful to my family and friends. With warm words, they always supported and encouraged me to pursue my research as I wanted.

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Chapter 1

Introduction

Two-sided matching models are tools to consider matchings between two agent sets, where each agent has a preference on the opposite agent set. Such problems arise in various applications such as the assignment of children to schools and of medical residents to hospitals. In such a situation, a matching, i.e., an assignment, should reflect preferences of all agents, and should be fair and reasonable.

The stability of a matching, introduced by Gale and Shapley [41], is a notion to express the nonexistence of agents who have incentives to deviate from the current assignment. They provided an algorithm which finds a stable matching for any instance. The algorithm is now called the *deferred acceptance algorithm* or the *Gale–Shapley algorithm*. After their work in 1962, the stable matching model and its variants have been studied actively and keep growing until now (e.g., Gusfield and Irving [49], Knuth [64], Manlove [68], Roth and Sotomayer [93]).

Also, researchers, including Roth [88, 89], have made great efforts to apply theoretical results to practice. Nowadays, in the US, a system based on the deferred acceptance algorithm has been used in National Resident Matching Program (NRMP), which produces an assignment between 42,370 medical residents and 30,750 positions in hospitals in 2016 [86]. Also, the deferred acceptance algorithm is implemented in school choice programs of New York City high schools and of Boston Public Schools (Abdulkadiroğlu et al. [1–4]). The importance and influence of this research field was recognized by the award of the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 (known as the Nobel Prize in Economic Sciences) to Alvin Roth and Lloyd Shapley.

The original model of Gale and Shapley has many nice properties from both game-theoretic and algorithmic aspects. This has encouraged practical systems to adopt the deferred acceptance algorithm. As the growth of application, however, there arise various requirements which are not taken into consideration in the original model. For example, in some markets, agents have minimum quotas. In other markets, partnerships of pairs are quantitative, i.e., reals or integers. Therefore, we need to formulate and investigate generalized models to cope with such complicated situations.

This thesis establishes structural and algorithmic results on several generalized stable matching models exploiting their inherent matroidal structures. A matroid, introduced by Whitney [108], is a combinatorial structure well investigated in discrete mathematics (e.g., Tutte [106, 107], Edmonds [21]). Fleiner [26, 27] proposed a matroid generalization of the stable matching model, and showed that many results on the original model can be extended and explained concisely in terms of matroids.

In this thesis, we consider stable matching models which can deal with some types of lower quotas, with integer- or real- variables, and with complicated preferences of agents. These factors can be captured via matroidal structures, such as generalized matroids and polymatroids, or “matroidal choice functions” introduced in this thesis. Recognizing such structures in models allows us to design efficient algorithms utilizing techniques in matroid theory. Furthermore, by analyzing through such abstract structures, we can realize the essence and the limitation to generalize favorable properties of the original model.

The rest of this chapter is organized as follows. We first introduce the original stable matching model and describe some celebrated results for this model in Section 1.1. In Section 1.2, we explain the relationship between stable matchings and matroidal structures to provide the motivation to consider stable matching models on matroidal structures. Finally, Section 1.3 shows summaries of our results and the organization of this thesis.

1.1 The Basic Stable Matching Model

The history of stable matching model started with the remarkable paper “College Admissions and the Stability of Marriage” by Gale and Shapley in 1962 [41]. We first introduce their *college admissions model* with slight extension. (This model is also referred to as the *hospitals/residents model* or as the *many-to-one stable matching model* [49, 93].) We then provide some significant results, which show why their model and algorithm have been so accepted in practice and have attracted many researchers from various fields.

1.1.1 Model Formulation

There are two disjoint agent sets I and J , called *students* and *colleges*. A set $E \subseteq I \times J$ represents a set of student-college pairs which are acceptable for both students and colleges. For each student $i \in I$, denote by E_i the set of acceptable pairs including i , i.e., $E_i = \{e \in E \mid e = (i, j) \text{ for some } j \in J\}$. Similarly, define $E_j \subseteq E$ for each college $j \in J$. Each student $i \in I$ can be assigned to at most one college and has a total order \succ_i on E_i , which represents his preference on acceptable colleges. Here $(i, j) \succ_i (i, j')$ means that i prefers j to j' . Each college $j \in J$ also has a preference on E_j represented by \succ_j , and it has a positive integer q_j , which represents the maximum number of students it can accept.

Thus, an *instance* of this model is represented by $(I, J, E, \{\succ_i\}_{i \in I}, \{(q_j, \succ_j)\}_{j \in J})$. If $|I| = |J|$, $E = I \times J$, and $q_j = 1$ for every $j \in J$, then this model is called the *stable marriage model*, or the *one-to-one stable matching*. The *many-to-many matching model* is a generalization in which each $i \in I$ also has a quota q_i and can be assigned multiple partners.

For an instance $(I, J, E, \{\succ_i\}_{i \in I}, \{(q_j, \succ_j)\}_{j \in J})$ of the college admissions model, a subset X of E is called an *assignment*. A *stable matching* of this instance is defined as follows.

Definition 1.1. An assignment $X \subseteq E$ is a *stable matching* if the following two hold:

1. $|X \cap E_i| \leq 1$ for every student $i \in I$ and $|X \cap E_j| \leq q_j$ for every college $j \in J$.
2. X admits no *blocking pair*, i.e., there is no unmatched pair $e = (i, j) \in E \setminus X$ such that
 - $|X \cap E_i| = 0$ or the unique element $e' \in X \cap E_i$ satisfies $e \succ_i e'$, and
 - $|X \cap E_j| < q_j$ or there is $e'' \in X \cap E_j$ such that $e \succ_j e''$. ■

The first condition requires that the assignment is feasible for all students and colleges. The second condition says that there is no pair of a student and a college that are not matched but want to be matched with each other.

1.1.2 The Deferred Acceptance Algorithm

Gale and Shapley [41] proved the existence of a stable matching by providing the following algorithm, which finds a stable matching for any instance. This algorithm is called the *student-oriented deferred acceptance algorithm*, or *student-oriented Gale–Shapley algorithm*. The algorithm keeps subsets $X, R \subseteq E$ which respectively represent the current assignment and rejected elements.

Algorithm SDA (Student-oriented Deferred Acceptance Algorithm).

1. Let R and X be empty.
2. Let every student i with $X \cap E_i = \emptyset$ apply to the top element in $E_i \setminus R$ with respect to \succ_i . Denote by $P \subseteq E$ the set of applied elements.
3. For every college j , if $|(X \cup P) \cap E_j| > q_j$, then accept the top q_j elements of $(X \cup P) \cap E_j$ with respect to \succ_j and reject others. Otherwise, accept all elements in $(X \cup P) \cap E_j$. Add accepted elements to X and rejected elements to R .
4. If every student i satisfies $X \cap E_i \neq \emptyset$ or $E_i \subseteq R$, return X . Otherwise, go to Step 2.

It was observed by McVitie and Wilson [69, 70] that a variant of this algorithm, in which each round lets only one student make an offer, returns the same assignment.

Theorem 1.2 (Existence). *The algorithm SDA returns a stable matching in $O(|E|)$ time.*

In general, there are multiple stable matchings for a given instance of the college admissions model. We call a stable matching *student optimal* if every student is assigned to the best college among possible colleges across all stable matchings. (There may be students who have no assignment in all stable matchings.) This definition does not presuppose the existence of such a matching, but the output of SDA is, in fact, the student optimal stable matching.

Theorem 1.3 (Student-Optimality). *The output of the algorithm SDA is student-optimal.*

As a counterpart of SDA, we can conceive the college-oriented deferred acceptance algorithm, in which colleges propose and students accept and reject in each iteration. It is known that such an algorithm returns the college-optimal stable matching [49].

The stable matching models are well investigated from the viewpoint of game theory and mechanism design [93, 101]. In this context, the *strategy-proofness* of algorithms becomes a major topic of discussion. For an algorithm which returns a matching for an instance, we call it *strategy-proof for students* if no student can improve his assignment by reporting a false preference. That is, for a strategy-proof algorithm, there is no student $i \in I$ and preference \succ'_i such that the algorithm assigns a better college (w.r.t. \succ_i) to i if the instance is modified by replacing \succ_i with \succ'_i .

Theorem 1.4 (Strategy Proofness). *The algorithm SDA is strategy-proof for students.*

This property eliminates the need for students to bother and to take a gamble when they submit preference lists. This property was one of the reasons for the adoption of SDA based algorithms in school choice systems [1–4, 92]. It should be noted that there is an impossibility theorem [88, 93] stating that no algorithm is strategy-proof for both students and colleges. Indeed, SDA is not strategy-proof for colleges. In application like school choice programs, however, preferences of schools are defined by rules or scores, and usually regarded to be faithful.

1.1.3 Structure of Stable Matchings

As mentioned before, an instance of the college admissions model may have multiple stable matchings. The set of stable matchings has a characteristic structure which has profound significance for both algorithmic theory and application.

We first introduce the result called “*rural hospital theorem*.” The historical importance and the name of this result arose when the SDA algorithm was first introduced to the NRMP program. Certain hospitals (colleges), mainly in rural areas, tended to be assigned significantly less residents (students) than their quotas. Then, there arose the question whether some alternative algorithm can find a stable matching which assigns more residents (students) to hospitals (colleges) in rural areas. The following theorem provides a negative answer to this question.

Theorem 1.5 (Rural Hospital Theorem [42, 89, 91]). *For a given instance, the following properties hold.*

- *The same students are assigned in all stable matchings.*
- *Every college is assigned the same number of students in all stable matchings.*
- *If a college $j \in J$ satisfies $|X \cap E_j| < q_j$ for some stable matching X , then j is assigned exactly the same student set $X \cap E_j$ in all stable matchings.*

This theorem says that a hospital whose quota is unfilled has exactly the same assignment in all stable matchings. Therefore, no matter what algorithms were used to find a stable matching, the rural hospitals could not be better off than SDA.

We next introduce the distributive lattice structure of stable matchings. This structure was first shown for the stable marriage model (Knuth [64] attributes this result to John Conway), and can be generalized for the college admissions model [49]. We prepare some notation. Let \mathfrak{S} be the set of all stable matchings of a given instance. For any $X, Y \in \mathfrak{S}$ and any assigned student $i \in I$, we denote by x_i and y_i the unique elements in $X \cap E_i$ and $Y \cap E_i$, respectively. Define a binary relation \succeq_I on \mathfrak{S} by letting $X \succeq_I Y$ mean that $x_i \succ_i y_i$ for every assigned student $i \in I$. Define assignments $X \vee_I Y$, $X \wedge_I Y \subseteq E$ by

$$\begin{aligned} X \vee_I Y &= \{ \max_{\succ_i} \{x_i, y_i\} \mid i \in I \text{ is assigned in } X \text{ and } Y \}, \\ X \wedge_I Y &= \{ \min_{\succ_i} \{x_i, y_i\} \mid i \in I \text{ is assigned in } X \text{ and } Y \}. \end{aligned}$$

Theorem 1.6 (Distributive Lattice Structure). *The set of stable matchings \mathfrak{S} forms a distributive lattice under the ordering \succeq_I . For any $X, Y \in \mathfrak{S}$, their join and meet are given by $X \vee_I Y$ and $X \wedge_I Y$, respectively.*

The distributive lattice structure has a significant meaning from algorithmic aspects. It is known [55] that the number of stable matchings can be exponential in $|E|$. By Birkhoff's representation theorem [12, 13] for distributive lattices, however, there is a compact representation which encodes all the stable matchings, and yet is of polynomial size [11, 15]. For the stable marriage model, in particular, such a compact representation is called a *rotation poset* and utilized to design efficient algorithms for optimization problems on the set of stable matchings (e.g., Gusfield, Irving, and Leather [47–49, 55, 56]).

1.2 Stable Matchings and Matroidal Structures

In this thesis, we study stable matching models on matroids and extensions, such as generalized matroids and polymatroids. The introduction of matroidal structure to the stable matching literature is due to Fleiner [26, 27] in 2001. He generalized the many-to-many stable matching model by relaxing quota constraints to matroid constraints. More specifically, in Fleiner's model, agents have matroid rank functions that define a quota for

each subset of possible partners. Using notions in matroid theory, he extended algorithmic and structural results of the college admissions model. (He call a stable matching on matroids a *matroid kernel*. See Section 3.3.3 for the precise definition.)

Here we show relationship between the stable matching model and matroidal structures, which supports the matroidal approach to stable matchings.

1.2.1 Matroidal Structures in Markets

There are many two-sided markets in which the feasible region of an agent forms a matroidal structure. Actually, the college admission model is a simple example, in which the family of acceptable sets of a college j forms the independent set family of a *uniform matroid* of rank q_j .

A *laminar matroid* naturally arises if a college has a nested classification of students and puts a quota on each class [29, 54] (see Example 2.3 for the definition of a laminar matroid). The student-project allocation problem is also an example in which laminar matroid constraints arise. This problem is studied by Abraham et al. [5], and indeed there are such centralized allocation processes in practice [20, 60]. In that problem, each student is assigned to a project, and each project is offered by a supervisor, who may be in charge of several projects. Each project has a quota and also each supervisor has a quota for the total number of students assigned to her projects. Then, for each supervisor, the family of acceptable sets of student-project pairs forms a laminar matroid.

A *transversal matroid* arises in the situation where a college has to assign accepted students to positions after their enrollment. If we represent the assignability of each student-position pair by an edge in a bipartite graph between students and positions, then the set of matchable students forms the independent set family of a transversal matroid. Such a matroid is also used in the literature of kidney exchange [88] and combinatorial auction [25].

A *polymatroid* is a polyhedral or integer version of a matroid independent set family. Polymatroids arise when we determine not only which pairs are matched but also how much time each pair spends together. The integer- or real-variable versions of assignments are called *allocations*. In the *stable allocation problem* [10], also called *schedule matching* [9], by Baïou and Balinski [10], each agent's feasible region (i.e., the set of assignments feasible for him) is represented by a truncated hyperrectangle, a special case of a polymatroid. Similarly to matroids, polymatroids arise when a polyhedron is defined by upper quota constraints which stem from laminar structures or bipartite networks.

If we add lower bound constraints to a matroid constraint (resp., a polymatroid constraint) in an appropriate way, then the feasible region forms a *generalized matroid* [103] (resp., a *generalized polymatroid* [30]). Two concrete examples of generalized matroids will appear in Section 3.2 (Examples 3.1 and 3.2).

1.2.2 Preferences on Matroidal Structures

For each of various generalizations of the college admissions model, we have to generalize the notion of stability. We now show that matroid structures are essentially important for the reasonable definition of stability.

Again, let I and J be agent sets and $E \subseteq I \times J$ be the set of acceptable pairs. For each $k \in I \cup J$, let \succ_k be a preference order on E_k and $\mathcal{I}_k \subseteq 2^{E_k}$ be a family of k 's acceptable sets (not yet assumed to be the independent set family of a matroid). Each k wishes to have as large subset as possible in \mathcal{I}_k (with respect to set inclusion) and prioritizes elements with respect to \succ_k . Let us write $X_k := X \cap E_k$ for any $X \subseteq E$ and $k \in I \cup J$. For an assignment X , an agent i has an incentive to take $e = (i, j) \in E \setminus X$ if there exists $Y_i \in \mathcal{I}_i$ such that $Y_i \subseteq X_i + e$ and Y_i is better than X_i with respect to \succ_i . The stability of X then requires that every unmatched pair $e = (i, j) \in E \setminus X$ satisfies:

- X_i is optimal in $\{Y_i \mid Y_i \subseteq X_i + e, Y_i \in \mathcal{I}_i\}$ with respect to \succ_i , or
- X_j is optimal in $\{Y_j \mid Y_j \subseteq X_j + e, Y_j \in \mathcal{I}_j\}$ with respect to \succ_j .

This definition could be ambiguous if \mathcal{I}_i and \mathcal{I}_j are arbitrary families. Here is an easy example. Let $\mathcal{I} = \{\{e\}, \{e'\}, \{e''\}, \{e', e''\}\}$ and $X = \{e', e''\}$ and $e \succ e' \succ e''$. Then, whether $X = \{e', e''\}$ is better than $\{e\}$ or not could depend on how much e is preferred to others.

For matroidal structures, however, the above definition makes sense. Assume that (S, \mathcal{I}) is a matroid (see Section 2.3 for the definition of a matroid). For $X \in \mathcal{I}$ and $e \in E \setminus X$ with $X + e \notin \mathcal{I}$, every maximal members Y_1 and Y_2 of $\{Y \mid Y \subseteq X + e, Y \in \mathcal{I}\}$ satisfy $|Y_1 \setminus Y_2| = |Y_2 \setminus Y_1| = 1$. Hence, a total order on elements suffices to determine whether X is optimal or not. Behind this fact, there is the following property of matroids. For a total order \succ on S , a weight function $w : S \rightarrow \mathbf{R}_{>0}$ is called *consistent* with \succ if $e \succ e'$ implies $w(e) > w(e')$ for every $e, e' \in S$.

Proposition 1.7. *A matroid (S, \mathcal{I}) has the following property. For any total order \succ on S and any subset $X \subseteq S$, there exists a unique subset $C(X) \subseteq X$ which is the unique maximizer of*

$$\max \left\{ \sum_{e \in Y} w(e) \mid Y \subseteq X, Y \in \mathcal{I} \right\}$$

for every $w : S \rightarrow \mathbf{R}_{>0}$ consistent with \succ .

Proposition 1.7 is an easy corollary of the correctness of the standard greedy algorithm (see Rado [87], Edmonds [22], and Theorem 2.4). This proposition says that, given an available set X , we can maximize the sum of individual values only knowing the ordering. Furthermore, this property characterizes matroids as follows. We call (S, \mathcal{I}) an *independence system* if $X \subseteq Y \in \mathcal{I}$ implies $X \in \mathcal{I}$. An independence system (S, \mathcal{I}) is a matroid if and only if it has the property in Proposition 1.7 (see, e.g., [66, 84]).

Proposition 1.7 can be extended to more general matroidal structures: generalized matroids, polymatroids, and generalized polymatroids. They are used to represent feasible

regions of agents in Chapters 3 and 4. A generalized polymatroid, introduced by Frank [30], is a polyhedron defined by a pair of submodular and supermodular functions satisfying the “cross inequalities.” Proposition 1.7 extends to this framework as follows.

Proposition 1.8. *A bounded generalized polymatroid $Q \subseteq \mathbf{R}^S$ has the following property. For any total order \succ on S and any vector x with $\{y \mid y \leq x, y \in Q\} \neq \emptyset$, there exists a unique vector $C(x) \in \mathbf{R}^S$ which is the unique maximizer of*

$$\max \{ \langle w, y \rangle \mid y \leq x, y \in Q \}$$

for every $w : S \rightarrow \mathbf{R}_{>0}$ consistent with \succ , where $\langle w, y \rangle = \sum_{e \in S} w(e) \cdot y(e)$.

Thus, matroidal structures admit such a robust and reasonable definition of optimality.

1.2.3 Nonlinear Preferences on Matroidal Structures

In the argument in Section 1.2.2, we implicitly assume that the value of a subset is defined as the sum of each element’s value. That is, a preference on subsets (resp., on vectors) is defined by a modular function (resp., a linear function). In practice, however, sometimes there are combinatorial factors. Assume that a college cares about the ratio of males to females, i.e., it wants to accept students of males and females as equally as possible. Then, if the college is currently assigned males only, then a female student is likely to be accepted, possibly unseating some male student. In the opposite situation, a male student is preferred. Then, the value of each student depends on other chosen students, and the value function on subsets is not modular.

In the matroid literature, there are some concepts which can be regarded as nonlinear generalizations of weighted matroids. A *valuated matroid*, introduced by Dress and Wenzel [19], is a matroid equipped with a function defined on the base family satisfying a quantitative version of the exchange axiom. Generalizing the idea of valuated matroid, Murota [71] formulated *M-concavity*, a kind of concavity for discrete functions. Then *M[♯]-concavity* was introduced by Murota and Shioura [78] as a variant of M-concavity. The effective domain of an M[♯]-concave function is the set of integer points of a generalized polymatroid [78] (such a set is called an M[♯]-convex set [75]). It is known that M[♯]-concavity has a close relationship with several economic notions [76, 80, 100]. As pointed out by Fujishige and Yang [39], M[♯]-concavity for a set function is equivalent to the *gross substitutes condition*, introduced by Kelso and Crawford [61]. Furthermore, it is also equivalent to the *strong no complementarities property*, introduced by Gul and Stacchetti [45], which has recently been shown by Murota [77].

The origin of the connection between such nonlinear valuations on matroidal structures and the stable matching literature is in Eguchi, Fujishige, and Tamura [24] (see also [37, 38]). They formulated the stable matching model on integer vectors such that the

preference of each agent is represented by an M^\sharp -concave function. They proved the existence of a stable allocation by devising a pseudopolynomial-time algorithm. Furthermore, various nice properties of the college admissions model are extended to this M^\sharp -concave function model: the set of stable allocations forms a distributive lattice [81], and the generalized deferred acceptance algorithm is strategy-proof for students in the many-to-one matching case [65].

For participants of markets, however, to declare ordinal preferences is easier than to declare cardinal ones. In the stable matching literature, a common tool to represent the ordinal preference of an agent is a *choice function*. Given a subset X of possible partners, a choice function C returns the best subset $C(X)$ of X . For models with choice functions, the *substitutability* of choice functions is known to be essential for the existence of a stable matching [90]. In Chapter 5, we introduce “matroidal choice functions,” as substitutable choice functions with matroid constraints. This concept is an ordinal counterpart of the above valuations on matroids.

1.3 The Contribution

This thesis investigates stable matchings with matroidal structures. The results are divided into four parts, which will appear in Chapters 3–6, respectively.

The first and second parts deal with stable matching models on matroidal structures such as generalized matroids and polymatroids. Our interest is in whether nice properties of the college admissions model, described in Section 1.1, can be generalized to our models. The third part investigates the essential property of preferences to behave well under matroid constraints. We also provide a stable matching model with such preferences. In these three topics, we utilize knowledge on matroidal structures to solve the generalized stable matching problems. Intriguingly, there is also a contribution from stable matchings to a different type of combinatorial problem. The fourth topic solves a kind of coloring problem, which we call list supermodular coloring. In the proof, a generalization of a stable matching plays an important role.

Here we give a summary for each of these four topics.

A Generalized Polymatroid Approach to Stable Matchings with Lower Quotas

In the college admissions model and its various generalizations, the feasible regions of agents are defined by upper bound constraints, such as quotas and matroid rank functions. In many practical setting, there also exist lower quotas. The introduction of lower quotas, however, threatens the existence of a stable matching. Huang [54] proposed the classified stable matching model, in which each college has a classification of students and gives upper and lower quotas for each class, and showed that it is NP-hard to decide whether there exists a stable matching or not. On the other hand, he showed the problem is solvable

if the classes form a laminar family. For this case, Fleiner and Kamiyama [29] gave a concise interpretation of the structure of stable matchings in terms of matroids. In Chapter 3, we introduce stable matchings on generalized matroids, extending the model of Fleiner and Kamiyama [29]. We design a polynomial-time algorithm which finds a stable matching or reports the nonexistence. We also show that, the set of stable matchings, if nonempty, forms a distributive lattice and the generalized version of the “rural hospital theorem” holds. Furthermore, we extend these structural results to a polyhedral framework, which we call stable allocations on generalized polymatroids.

This chapter is based on [109]. Reproduced with permission. Copyright, INFORMS, <http://www.informs.org>.

Finding a Stable Allocation in Polymatroid Intersection We consider a stable allocation model in which the feasible regions of agents are given by polymatroids. Our framework includes both integer- and real-variable versions. The real-variable version is a special case of the generalized polymatroid model in Chapter 3, in which only structural results are shown. The integer-variable version corresponds to a special case of the M^\sharp -concave function model due to Eguchi, Fujishige, and Tamura [24], who provided a pseudopolynomial-time algorithm to find a stable allocation. It has been open to develop a polynomial-time algorithm even for our special case. In Chapter 4, we present the first strongly polynomial algorithm for finding a stable allocation in polymatroid intersection. To achieve this, we utilize the augmenting path technique for polymatroid intersection. In each iteration, the algorithm searches for an augmenting path by simulating a chain of proposes and rejects in the deferred acceptance algorithm. The running time of our algorithm is $O(n^3\gamma)$, where n and γ respectively denote the cardinality of the ground set and the time for computing the saturation and exchange capacities. We also show that the output of the algorithm is the optimal stable allocation with respect to the preference order of the proposing side. This chapter is based on [59].

Matroidal Choice Functions In the stable matching model, and also in other game-theoretic models, an agent is supposed to choose a subset of items under a matroid constraint. If the preference of an agent on subsets is represented by a modular function on a matroid, then the choice rule defined by maximizers of that function fulfills the substitutability, an essential property for the existence of a stable matching. In Chapter 5, we introduce a notion of “matroidal choice functions” to capture the entire class of substitutable choice rules under a matroid constraint. For such functions, we provide two characterizations: the one is by the behavior of an online greedy algorithm and the other is by a local condition. We also show that matroidal choice functions are closely related to valuated matroids and M^\sharp -concave set functions. Thus, matroidal choice functions can be regarded as abstracting some combinatorial aspects of maximization algorithms of such nonlinear valuations on matroids. Also, we investigate a stable matching model with

matroidal choice functions and provide an algorithm which outputs the student optimal stable matching and is strategy-proof for students. This chapter is based on [110].

List Supermodular Coloring In Chapter 6, we show that a generalization of a stable matching can be applied to solve a seemingly unrelated combinatorial problem. In 1995, Galvin [43] provided an elegant proof for the list edge coloring conjecture for bipartite graphs, utilizing the existence of a one-to-one stable matching. We generalize Galvin’s result to the setting of supermodular coloring, introduced by Schrijver [96]. That is, we solve a list coloring problem in which a pair of supermodular functions represents a demand, i.e., a required number of colors, for each subset. The main tool for the proof is the monochromatic path theorem, due to Sands, Sauer and Woodrow [94], which is a generalization of the existence of a stable matching (pointed out by Fleiner [27]). For a pair of partial orders, this theorem states the existence of a common antichain dominating every excluded element. Our proof provides a way to construct a list coloring which satisfies all demands. When we define a subset of elements colored by a certain color, the monochromatic path theorem works effectively on the pair of partial orders induced from the given supermodular functions. This chapter is based on [58].

The rest of this thesis is organized as follows. In Chapter 2, we introduce basic definitions and notions related to graphs, orders, matroids, and polymatroids. In particular, for matroids and polymatroids, we provide some key properties. From Chapter 3 to Chapter 6, we present four results summarized above respectively. These chapters are related but could be read almost independently after reading Chapter 2. Finally, Chapter 7 concludes this thesis.

Chapter 2

Preliminaries

We first introduce basic notations used throughout this thesis. Symbols \mathbf{Z} and \mathbf{R} denote the sets of integers and reals, respectively. Also, \mathbf{Z}_+ , \mathbf{R}_+ (resp., $\mathbf{Z}_{>0}$, $\mathbf{R}_{>0}$) are the sets of nonnegative (resp., positive) integers and reals, respectively. For any positive integer k , let $[k]$ denote be the set of first k positive integers, i.e., $[k] := \{1, 2, \dots, k\}$.

For a nonempty finite set S , its power set is denoted by $2^S := \{X \mid X \subseteq S\}$. For a set $X \subseteq S$ and an element $e \in S$, we denote $X \cup \{e\}$ by $X + e$. Also we denote $X \setminus \{e\}$ by $X - e$ when $e \in X$.

For a vector $x = (x(e) \mid e \in S) \in (\mathbf{R} \cup \{-\infty, +\infty\})^S$ and a subset $A \subseteq S$, we write $x(A)$ for $\sum_{e \in A} x(e)$ and let $x(\emptyset) := 0$. We also write $|x|$ for $x(S)$. For two vectors $x, y \in (\mathbf{R} \cup \{-\infty, +\infty\})^S$, the notation $x \leq y$ means $x(e) \leq y(e)$ ($\forall e \in S$), and $x \wedge y$, $x \vee y$ are vectors in $(\mathbf{R} \cup \{\pm\infty\})^S$ whose e -th components are respectively $\min\{x(e), y(e)\}$ and $\max\{x(e), y(e)\}$. For any $A \subseteq S$, the characteristic vector $\chi_A \in \mathbf{R}^S$ is defined by $\chi_A(e) = 1$ ($e \in A$) and $\chi_A(e) = 0$ ($e \in S \setminus A$). We abbreviate $\chi_{\{e\}}$ to χ_e for each $e \in S$.

2.1 Graphs

A *graph* is a pair $G = (V, E)$ where V is a finite set and E is a subset of $\{\{u, v\} \mid u, v \in V\}$, i.e., a set of unordered pairs of elements of V . Each $v \in V$ is called a *vertex* and each $e = \{u, v\} \in E$ is called an *edge*. An edge e is called a *self-loop* at $v \in V$ if $e = \{v, v\}$.

For an edge $e = \{u, v\}$, the vertices u, v are called *end points* of e . For a set $M \subseteq E$ of edges, we denote by $\partial M \subseteq V$ the set of endpoints of edges in M . An edge e is *incident* to v if v is an end point of e . For a vertex v , we denote by $\delta(v) \subseteq E$ the set of edges incident to v . When we want to emphasize the graph we are considering, we write $\delta_G(v)$. Two edges are said to be *adjacent* in G if they have a common end point.

A *walk* in a graph G is a sequence $\{v_1, v_2, \dots, v_k\}$ of vertices with $\{v_i, v_{i+1}\} \in E$ for each $i \in [k-1]$. A *path* is a walk whose vertices are all distinct. A *cycle* is a walk such that v_1, v_2, \dots, v_{k-1} are distinct and $v_k = v_1$.

A *bipartite graph* is a graph $G = (V, E)$ with V partitionable into two sets S and T

in such a way that every edge $e \in E$ has one end point in S and the other in T . We also denote this graph as $G = (S, T; E)$. If we write an edge e of a bipartite graph as an ordered pair $e = (s, t)$, then s is the end point in S and t is the other in T .

A *directed graph* is a pair $G = (V, A)$ where V is a finite set of vertices and A is a subset of $\{(u, v) \mid u, v \in V\}$, i.e., a set of ordered pairs of vertices. Each $a \in A$ is called an *arc*. An arc a is called a *self-loop* at $v \in V$ if $a = (v, v)$. For an arc $a = (u, v)$, we call u the *tail* and v the *head* of a .

A *directed walk* in a directed graph G is a sequence $\{v_1, v_2, \dots, v_k\}$ of vertices with $(v_i, v_{i+1}) \in A$ for each $i \in [k-1]$. A *directed path* is a directed walk whose vertices are all distinct. A *directed cycle* is a directed walk with distinct v_1, v_2, \dots, v_{k-1} and $v_k = v_1$.

2.2 Partial Orders

A *partial order* is a binary relation \succeq on a nonempty ground set, say S (not necessarily finite), such that the following three hold for every $u, v, w \in S$.

- *Reflexivity* : $u \succeq u$.
- *Antisymmetry* : $u \succeq v$ and $v \succeq u$ imply that $u = v$.
- *Transitivity* : $u \succeq v$ and $v \succeq w$ imply that $u \succeq w$.

Two elements $u, v \in S$ are *comparable* if $u \succeq v$ or $v \succeq u$ holds, and otherwise they are *incomparable*. The notation $u \preceq v$ means $v \succeq u$. Also, $u \succ v$ means $u \succeq v$ and $u \neq v$.

A *total order*, or a *linear order*, is a partial order such that every two elements of the ground set are comparable.

A *partially ordered set*, or shortly a *poset*, is a pair $P = (S, \succeq)$, where \succeq is a partial order on a nonempty set S . For a poset $P = (S, \succeq)$, a subset $X \subseteq S$ is called a *chain* if every pair of elements in X is comparable. An *antichain* is a subset in which every pair of distinct elements is incomparable.

A poset $P = (S, \succeq)$ can be represented as a directed graph, in which each element of S is a node and an arc (u, v) represents that $u \succeq v$. A *Hasse diagram* of P is such a graphical representation with self-loops and transitive arcs omitted. More precisely, for any $u, v \in S$, we have an arc (u, v) if and only if $u \succ v$ and no $w \in S$ satisfies $u \succ w \succ v$.

For a poset $P = (S, \succeq)$ and a subset $X \subseteq S$, an element $u \in S$ is an *upper bound* of X if $u \succeq v$ for every $v \in X$. An upper bound u of X is called the *least upper bound* of X if for any upper bound u' of X , we have $u \preceq u'$. We denote the least upper bound of X by $\sup X$, whose uniqueness follows from the antisymmetry. Note that $\sup X$ does not necessarily exist. The *greatest lower bound* of X is defined similarly and denoted by $\inf X$.

A *lattice* is a poset $P = (S, \succeq)$ such that $\sup\{u, v\}$ and $\inf\{u, v\}$ exist for all $u, v \in S$. We shall use the notations

$$u \vee v = \sup\{u, v\}, \quad u \wedge v = \inf\{u, v\},$$

and call the operation \vee *join* and \wedge *meet*. By definition, these binary operations satisfy the following three for every $u, v, w \in S$.

- *Idempotency* : $u \vee u = u, \quad u \wedge u = u.$
- *Commutativity* : $u \vee v = v \vee u, \quad u \wedge v = v \wedge u.$
- *Associativity* : $(u \vee v) \vee w = u \vee (v \vee w), \quad (u \wedge v) \wedge w = u \wedge (v \wedge w).$

Conversely, a pair of binary operations satisfying these three gives rise to a lattice as follows. Let (S, \vee, \wedge) be a triple such that S is a nonempty set and \vee and \wedge are binary operations on S satisfying the above three properties. If we define a partial order \succeq by

$$u \succeq v \iff u \vee v = u,$$

then $P = (S, \succeq)$ is a lattice. We also refer to such an algebra (S, \vee, \wedge) as a *lattice*.

A lattice $L = (S, \vee, \wedge)$ is called *distributive* if, for every $u, v, w \in S$, we have

$$u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w).$$

It is known that there are some equivalent conditions for the distributivity of a lattice. A lattice L is distributive if and only if the following identity holds for every $u, v, w \in S$.

$$u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w).$$

Similarly, L is distributive if and only if the following condition holds for every $u, v, w \in S$.

$$[u \vee v = u \vee w, u \wedge v = u \wedge w] \implies v = w.$$

2.3 Matroids

A pair (S, \mathcal{I}) is called a *matroid* if S is a finite set and $\mathcal{I} \subseteq 2^S$ is a family of subsets satisfying the following (I0)–(I2).

(I0) $\emptyset \in \mathcal{I}.$

(I1) If $X \subseteq Y \in \mathcal{I}$, then $X \in \mathcal{I}.$

(I2) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there exists $e \in Y \setminus X$ such that $X + e \in \mathcal{I}.$

For a matroid $\mathbf{M} = (S, \mathcal{I})$, we call S the *ground set* and \mathcal{I} the *independent set family*. A subset $X \subseteq S$ is called *independent* if $X \in \mathcal{I}$, and otherwise called *dependent*.

An independent set X is called a *base* of \mathbf{M} if X is inclusionwise maximal in \mathcal{I} . We denote by \mathcal{B} the family of bases. That is,

$$\mathcal{B} = \{ B \subseteq S \mid B \in \mathcal{I}, B + e \notin \mathcal{I} (\forall e \in S \setminus B) \}.$$

By (I2), all members of \mathcal{B} have the same cardinality.

A dependent set is called a *circuit* of \mathbf{M} if X is inclusionwise minimal in the family of all dependent sets. Then, the circuit family $\mathcal{C} \subseteq 2^E$ of \mathbf{M} is defined by

$$\mathcal{C} = \{ C \subseteq E \mid C \notin \mathcal{I}, C - e \in \mathcal{I} \ (\forall e \in C) \}.$$

The rank function $r_{\mathbf{M}} : 2^S \rightarrow \mathbf{Z}$ of \mathbf{M} is defined by

$$r_{\mathbf{M}}(X) = \max\{ |Y| : Y \in \mathcal{I}, Y \subseteq X \} \quad (X \subseteq S).$$

We may omit the subscript \mathbf{M} when the matroid is clear from the context. It is known that $r := r_{\mathbf{M}}$ satisfies the following (R0)–(R2):

$$\textbf{(R0)} \quad \forall X \subseteq S, \quad 0 \leq r(X) \leq |X|.$$

$$\textbf{(R1)} \quad \text{If } X \subseteq Y \subseteq S, \text{ then } r(X) \leq r(Y).$$

$$\textbf{(R2)} \quad \forall X, Y \subseteq S, \quad r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y).$$

We call $r(S)$ the rank of \mathbf{M} . Conversely, if a function $r : 2^S \rightarrow \mathbf{Z}$ satisfies conditions (R0)–(R2), then the family $\mathcal{I} \subseteq 2^S$ defined by

$$\mathcal{I} = \{ X \subseteq S : |X| = r(X) \}$$

forms the independent set family of a matroid. We see that this \mathcal{I} is also represented as

$$\mathcal{I} = \{ X \subseteq S : |X \cap A| \leq r(A) \ (\forall A \subseteq S) \}.$$

That is, if $r(A)$ represents an upper bound of the number of elements we can take from each $A \subseteq S$, an independent set can be regarded as a subset satisfying all upper bounds.

For a subset $X \subseteq S$, its *span* (or, *closure*) $\text{span}_{\mathbf{M}}(X)$ with respect to \mathbf{M} is defined by

$$\text{span}_{\mathbf{M}}(X) = \{ e \in S \mid r_{\mathbf{M}}(X) = r_{\mathbf{M}}(X + e) \}.$$

In particular, if $X \in \mathcal{I}$ then it is given by $\text{span}_{\mathbf{M}}(X) = X \cup \{ e \in S \mid X + e \notin \mathcal{I} \}$.

Examples and Operations

Here are some simple examples of matroids. Let S be a nonempty finite set.

Example 2.1 (Free Matroid). The pair $(S, 2^S)$ is a matroid and called the *free matroid*.

Example 2.2 (Uniform Matroid). For a nonnegative integer q , let $\mathcal{I} := \{ X \mid X \subseteq S, |X| \leq q \}$. Then (S, \mathcal{I}) is a matroid and called the *uniform matroid* of rank q .

A family $\mathcal{F} \subseteq 2^S$ is called a *laminar family* if any pair of subsets $F, H \in \mathcal{F}$ satisfies either $F \cap H = \emptyset$, $F \subseteq H$, or $H \subseteq F$.

Example 2.3 (Laminar Matroid). Let \mathcal{F} be a laminar family with a positive integer $q(F)$ associated with each $F \in \mathcal{F}$. Define $\mathcal{I} = \{ X \subseteq S : |X \cap F| \leq q(F) \ (\forall F \in \mathcal{F}) \}$. Then (S, \mathcal{I}) is a matroid. A matroid defined in this way is called a *laminar matroid*.

Here are some operations which induce a matroid from other matroids.

Restriction For a matroid $\mathbf{M} = (S, \mathcal{I})$ and a subset $T \subseteq S$, define $\mathcal{I}_T \subseteq 2^T$ by $\mathcal{I}_T = \{X \subseteq T \mid X \in \mathcal{I}\}$. Then, (T, \mathcal{I}_T) is a matroid called the *restriction* of \mathbf{M} into T .

Direct Sum Let S_1, S_2, \dots, S_k be disjoint finite sets and $\{\mathbf{M}_i = (S_i, \mathcal{I}_i)\}_{i \in [k]}$ be a collection of matroids. Define S and $\mathcal{I} \subseteq 2^S$ by

$$\begin{aligned} S &= S_1 \cup S_2 \cup \dots \cup S_k, \\ \mathcal{I} &= \{X \subseteq S \mid X \cap S_i \in \mathcal{I}_i \ (\forall i \in [k])\}. \end{aligned}$$

Then, the pair (S, \mathcal{I}) is a matroid called *the direct sum of matroids* $\{\mathbf{M}_i\}_{i \in [k]}$.

Induction by Bipartite Graphs Let $G = (S, T; E)$ be a bipartite graph with vertex classes S, T and edge set E . Let (T, \mathcal{I}_T) be a matroid on T and define a family \mathcal{I}_S of subsets of S by

$$\mathcal{I}_S = \{\partial M \cap S \mid M : \text{matching in } G, \partial M \cap T \in \mathcal{I}_T\}.$$

Then, (S, \mathcal{I}_S) is a matroid (due to Perfect [85], see e.g. [84]). In particular, if (T, \mathcal{I}_T) is the free matroid, i.e., $\mathcal{I}_T = 2^T$, then the induced matroid (S, \mathcal{I}_S) is called a *transversal matroid*.

Greedy Algorithm

One of the most important properties of a matroid is that it admits some kinds of greedy algorithms. Consider that we are given a positive weight function $w : S \rightarrow \mathbf{R}_{>0}$ and a subset $X \subseteq S$, and that we want to obtain a maximizer of

$$\max \left\{ \sum_{e \in Y} w(e) \mid Y \subseteq X, Y \in \mathcal{I} \right\}.$$

The following algorithm gives a solution.

Algorithm GA (Greedy Algorithm).

1. Order $X = \{e_1, e_2, \dots, e_k\}$ so that $w(e_1) \geq w(e_2) \geq \dots \geq w(e_k)$, where $k = |X|$.
Set $I \leftarrow \emptyset$.
2. For $i = 1$ to k do: If $I + e_i \in \mathcal{I}$, then $I \leftarrow I + e_i$.
3. Return I .

Theorem 2.4 (Greedy Optimality for Matroids). *If (S, \mathcal{I}) is a matroid, then the algorithm GA returns a maximizer of $\max \{w(Y) \mid Y \subseteq X, Y \in \mathcal{I}\}$ for every positive weight function $w : S \rightarrow \mathbf{R}_{>0}$ and every subset $X \subseteq S$.*

This greedy algorithm also characterizes matroids. A pair (S, \mathcal{I}) is called *independence system* if S is a finite set and $\mathcal{I} \subseteq 2^S$ satisfy (I0) and (I1) but not necessarily (I2).

Theorem 2.5 (Greedy Characterization of Matroids). *An independence system (S, \mathcal{I}) is a matroid if and only if the algorithm GA returns a maximizer of $\max \{ w(Y) \mid Y \subseteq X, Y \in \mathcal{I} \}$ for every positive weight $w : S \rightarrow \mathbf{R}_{>0}$ and every subset $X \subseteq S$.*

2.4 Submodular Functions and Polymatroids

A family $\mathcal{F} \subseteq 2^S$ is called a *ring family* if $A, B \in \mathcal{F}$ implies $A \cup B, A \cap B \in \mathcal{F}$. A function $f : \mathcal{F} \rightarrow \mathbf{R}$ is called *submodular* if its domain \mathcal{F} is a ring family and the *submodular inequality*

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

holds for every $A, B \in \mathcal{F}$. Also, f is called *supermodular* if $-f$ is submodular.

Intersecting-submodularity is an extension of submodularity defined as follows. We say that subsets $A, B \subseteq S$ are *intersecting* if none of $A \cap B$, $A \setminus B$ and $B \setminus A$ is empty. A family $\mathcal{F} \subseteq 2^S$ is called an *intersecting family* if every intersecting $A, B \in \mathcal{F}$ satisfy $A \cup B, A \cap B \in \mathcal{F}$. A function $f : \mathcal{F} \rightarrow \mathbf{R}$ is called *intersecting-submodular* if \mathcal{F} is an intersecting family and f satisfies the submodular inequality for every intersecting $A, B \in \mathcal{F}$. Also, f is called *intersecting-supermodular* if $-f$ is intersecting-submodular.

For any set function $f : \mathcal{F} \rightarrow \mathbf{R}$ with $\emptyset \in \mathcal{F}$ and $f(\emptyset) = 0$, we associate three kinds of polyhedra

$$\begin{aligned} \mathbf{P}(f) &= \{ x \in \mathbf{R}^S \mid x(A) \leq f(A) \ (\forall A \in \mathcal{F}) \}, \\ \mathbf{B}(f) &= \{ x \in \mathbf{R}^S \mid x(A) \leq f(A) \ (\forall A \in \mathcal{F}), \ x(S) = f(S) \}, \\ \mathbf{P}_+(f) &= \mathbf{P}(f) \cap \mathbf{R}_+^S. \end{aligned}$$

In this thesis, whenever we consider such polyhedra, we assume $\emptyset, S \in \mathcal{F}$ and $f(\emptyset) = 0$. For a vector $x \in \mathbf{P}(f)$, a subset $A \in \mathcal{F}$ is said to be *tight* at x if $x(A) = f(A)$. For a submodular function (resp. intersecting-submodular function) $f : \mathcal{F} \rightarrow \mathbf{R}$ and a vector $x \in \mathbf{P}(f)$, the family of tight sets forms a ring family (resp. intersecting family). That is, if A and B are tight (resp. tight and intersecting), then $A \cup B$ and $A \cap B$ are also tight. This is because the submodularity inequality implies

$$x(A) + x(B) = f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \geq x(A \cup B) + x(A \cap B),$$

in which inequalities should be equalities. Polyhedra $\mathbf{P}(f)$ and $\mathbf{B}(f)$ are related in the following way.

Proposition 2.6 (cf. [36, Theorem 2.3]). *If a set function $f : \mathcal{F} \rightarrow \mathbf{R}$ is submodular, then $\mathbf{B}(f)$ is nonempty and coincides with the set of all maximal elements of $\mathbf{P}(f)$ (i.e., we have $\mathbf{B}(f) = \{ x \in \mathbf{P}(f) \mid \nexists y \in \mathbf{P}(f) : y \geq x, y \neq x \}$).*

Polymatroids

We say that a function $f : \mathcal{F} \rightarrow \mathbf{R}$ is *monotone* if $A \subseteq B$ implies $f(A) \leq f(B)$ for every $A, B \in \mathcal{F}$. A function $f : 2^S \rightarrow \mathbf{R}$ is called a *polymatroid rank function* if it is monotone and submodular and satisfies $f(\emptyset) = 0$. A polyhedron $P \subseteq \mathbf{R}^S$ is called a *polymatroid* if $P = \mathbf{P}_+(f)$ for some polymatroid rank function $f : 2^S \rightarrow \mathbf{R}$. In fact, such a function is uniquely determined as follows.

Proposition 2.7 (Edmonds [21]). *For a polymatroid $P \subseteq \mathbf{R}^S$, the function $f : 2^S \rightarrow \mathbf{R}$ defined by $f(A) = \max \{x(A) \mid x \in P\}$ is the unique polymatroid rank function which satisfies $P = \mathbf{P}_+(f)$.*

The function f in Proposition 2.7 is called the *defining function* of P . We also refer to the pair (S, f) of a set and a polymatroid rank function as a *polymatroid*.

For a polymatroid $P = \mathbf{P}_+(f) \subseteq \mathbf{R}^S$, a vector in P is called an *independent vector*. Polymatroids can be regarded as the polyhedral version of matroids. Indeed, for a matroid $\mathbf{M} = (S, \mathcal{I})$, the convex hull of $\{\chi_A \in \{0, 1\}^S \mid A \in \mathcal{I}\}$ is a polymatroid and its defining function is the rank function of \mathbf{M} .

The following proposition says that the intersecting submodularity is sufficient to form a polymatroid.

Proposition 2.8 (Edmonds [21]). *Let $f : \mathcal{F} \rightarrow \mathbf{R}$ be an intersecting-submodular function such that $\mathbf{P}_+(f)$ is nonempty and bounded. Then there exists a monotone submodular function $\hat{f} : 2^S \rightarrow \mathbf{R}$ such that $\mathbf{P}_+(\hat{f}) = \mathbf{P}_+(f)$, and hence $\mathbf{P}_+(f)$ is a polymatroid.*

This proposition immediately implies that, if the domain \mathcal{F} of a function $f : \mathcal{F} \rightarrow \mathbf{R}$ is a laminar family and $\mathbf{P}_+(f) \neq \emptyset$, then $\mathbf{P}_+(f)$ is a polymatroid.

In Section 3.7.3, where we consider the stable allocation model on polymatroids, we use the direct sum of polymatroids defined as follows. Let S_1, S_2, \dots, S_k be disjoint finite sets and, for each $i \in k$, let $P_i \subseteq \mathbf{R}_+^{S_i}$ be a polymatroid whose defining function is $f_i : 2^{S_i} \rightarrow \mathbf{R}$. Define S and $P \subseteq \mathbf{R}^S$ by

$$\begin{aligned} S &= S_1 \cup S_2 \cup \dots \cup S_k, \\ P &= \{ (x_1, x_2, \dots, x_k) \in \mathbf{R}^S \mid x_i \in P_i \ (\forall i \in [k]) \}. \end{aligned}$$

Then, $P \subseteq \mathbf{R}^S$ is a polymatroid and its defining function $f : 2^S \rightarrow \mathbf{R}$ is given by $f(A) = \sum_{i=1}^k f_i(A \cap S_i)$ for each $A \subseteq S$.

Greedy Algorithm on Polymatroids Polymatroids admit the following greedy algorithm. Consider that we are given a positive weight function $w : S \rightarrow \mathbf{R}_{>0}$ and we want to obtain a maximizer of

$$\max \{ \langle x, w \rangle \mid x \in \mathbf{P}_+(f) \},$$

where $\langle x, w \rangle := \sum_{e \in S} w(e) \cdot x(e)$. The following algorithm gives a solution.

Algorithm PGA (Polymatroid Greedy Algorithm).

1. Order $S = \{e_1, e_2, \dots, e_n\}$ so that $w(e_1) \geq w(e_2) \geq \dots \geq w(e_n)$, where $n = |S|$.
Let $S_0 = \emptyset$ and $S_i := \{e_1, e_2, \dots, e_i\}$ for each $i \in [n]$.
2. For $i = 1$ to n , let $x(e_i) = f(S_i) - f(S_{i-1})$.
3. Return x .

Proposition 2.9 (Edmonds [21], Shapley [99]). *For a polymatroid $P = \mathbf{P}_+(f)$, the algorithm PGA returns a maximizer of $\max \{ \langle x, w \rangle \mid x \in \mathbf{P}_+(f) \}$ for every positive weight function $w : S \rightarrow \mathbf{R}_{>0}$.*

Chapter 3

A Generalized Polymatroid Approach to Stable Matchings with Lower Quotas

3.1 Introduction

Since the college admissions model of Gale and Shapley [41], the two-sided stable matching model has been generalized in many different ways [64, 68]. One of these directions is to generalize a feasible region of each agent, which is originally defined as a family of subsets satisfying an upper quota. For example, in the ordered matroid model of Fleiner [26], a quota constraint is generalized to a matroid constraint. Also, in the stable allocation model of Baïou and Balinski [10], variables can take nonnegative reals, i.e., we determine how much time each pair spends together. In both of these models, the following well-known results of the college admissions model have been extended successfully.

1. Every instance has a stable matching.
2. The stable matchings form a distributive lattice.
3. The “rural hospital theorem” holds, i.e., the number of applicants assigned to each college is the same across all stable matchings [42, 91].

In these models, the feasible region of each agent contains zero point (i.e., the empty set or the zero vector). This simple assumption is in fact essential to guarantee the existence of a stable matching. It is known that, if lower quotas are introduced to these models, some instances have no stable matching. There are various approaches to the college admissions model with lower quotas. Hamada, Iwama and Miyazaki [50] considered an optimization version of the problem, i.e., the minimization of the number of blocking pairs subject to upper and lower quotas. They provided an inapproximability result and an exponential-time exact algorithm. Biró et al. [14] considered a variant of the college admissions problem with lower quotas, in which each college should either satisfy its lower quota or have no assignment. A college with no assignment means a closed college. They

showed the NP-completeness of deciding whether there exists a stable matching or not.

The classified stable matching model, proposed by Huang [54], is a one-to-many matching model between academic institutes and applicants. In this model, besides a preference list on applicants, each institute has a classification of applicants based on their expertise and gives upper and lower quotas for each class. Huang showed that it is NP-complete to decide whether there is a stable matching or not in general. On the other hand, he proved that the problem is solvable in polynomial time if classes form a laminar family. This special case is called the laminar classified stable matching (LCSM) problem. Providing a concise interpretation in terms of matroids, Fleiner and Kamiyama [29] gave an algorithm which solves the many-to-many version of the LCSM problem and showed the lattice structure of stable matchings.

In this chapter, we generalize the approach and results of Fleiner and Kamiyama. We introduce the following two models.

Stable Matchings on Generalized Matroids We consider a many-to-many matching model in which the feasible region of each agent is expressed by a generalized matroid. That is, for every agent, the family of acceptable sets of partners forms an independent set family of some generalized matroid, which allows us to deal with various constraints with lower quotas.

A generalized matroid (g-matroid, for short) is a generalization of a matroid defined by the “exchange axioms” [103]. In contrast to a matroid, the independent set family of g-matroid does not have to contain the empty set and hence can express some kinds of lower quotas. Our model is a generalization of the LCSM model [29, 54] since the collection of subsets satisfying upper and lower quotas on a laminar family forms an independent set family of a g-matroid (Example 3.1).

For this model, we design an algorithm which finds a stable matching or reports the nonexistence. This algorithm runs in polynomial time, provided that a membership oracle and an initial independent set are available for each g-matroid. Furthermore, we show that for some concrete examples of g-matroids, we can execute the algorithm in polynomial time without any oracle or independent set (Remark 3.31). We also show that the set of stable matchings, if nonempty, forms a lattice.

The key technique of our analysis is to construct a special matroid for each g-matroid. We call this operation the “lower extension” of g-matroids. Through this operation, we obtain a modified instance in which the independent set family of each g-matroid is extended to be that of a matroid. The modified instance can be treated in the matroidal framework of Fleiner [26, 27], and relates to the original instance through the following dichotomy property. Exactly one of the following statements holds:

- The set of stable matchings of the original instance coincides with that of the modified instance.
- There is no stable matching for the original instance.

This property implies that one can decide whether the original instance has a stable matching or not by finding one stable matching of the modified instance. Also, we see that some structural results of the matroidal framework [26, 27], such as the lattice structure of stable matchings and a generalized “rural hospital theorem,” still hold for our g-matroid framework.

Stable Allocations on Generalized Polymatroids We also consider the polyhedral version of the above model, in which variables take nonnegative reals and the feasible region of each agent is a generalized polymatroid.

A generalized polymatroid (g-polymatroid, for short), introduced by Frank [30], is a polyhedron defined by a pair of submodular and supermodular functions satisfying the “cross inequalities.” (Such a pair of set functions is called paramodular or a strong pair [31].) A g-polymatroid is identical with a polymatroid if it contains the zero vector as the minimum point. Since feasible regions in the stable allocation model of Baïou and Balinski [10] are special cases of polymatroids, our model generalizes their model.

Similarly to the binary case, we construct a modified instance in which each g-polymatroid is extended to be a polymatroid, and show the similar dichotomy property between the original and modified instance. Also, we analyze the modified instance by reducing it to an instance of the choice function model due to Alkan and Gale [9]. More precisely, we show that a choice function induced by a polymatroid satisfies the requirements of their choice function model, such as persistence and size-monotonicity. In this way, we show that the set of stable allocations, if nonempty, is a distributive lattice and satisfies a vector version of the “rural hospital theorem.”

The rest of this chapter is organized as follows. The first three sections are devoted to the stable matching model on generalized matroids. Section 3.2 provides preliminaries on matroids and g-matroids, and Section 3.3 introduces the notions to represent preferences on g-matroids. Section 3.4 investigates stable matchings on g-matroids, and gives algorithmic and structural results. The polyhedral framework starts with Section 3.5, which provides preliminaries on polymatroids and g-polymatroids. In Section 3.6, we define a partial order on vectors, and show how to induce a choice function from a polymatroid. Section 3.7 provides structural properties of stable allocations on g-polymatroids.

3.2 Generalized Matroids

3.2.1 Definition and Examples

A pair (S, \mathcal{J}) is called a *generalized matroid* (*g-matroid*) if S is a finite set and $\mathcal{J} \subseteq 2^S$ is a nonempty family satisfying the following two conditions.

- (J1) If $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$, then $X + e \in \mathcal{J}$ or $\exists e' \in X \setminus Y: X + e - e' \in \mathcal{J}$.
- (J2) If $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$, then $Y - e \in \mathcal{J}$ or $\exists e' \in X \setminus Y: Y - e + e' \in \mathcal{J}$.

We call S the ground set and \mathcal{J} the independent set family of the g-matroid. Conditions (J1) and (J2) are called the *exchange axioms* of g-matroids. One can easily check that a matroid is a special case of a g-matroid.

In our stable matching model in Section 3.4, the feasible region of every agent is given by a g-matroid. For example, an institute has a g-matroid whose independent set family corresponds to the family of acceptable subsets of applicants. Here, we give examples. In Appendix 3.B, we show that these are indeed g-matroids. Recall that a family $\mathcal{F} \subseteq 2^S$ is called a *laminar family* if any $F, H \in \mathcal{F}$ satisfy $F \cap H = \emptyset$ or $F \subseteq H$ or $H \subseteq F$.

Example 3.1 (Laminar Family). Let S be a set of applicants. An institute has a *laminar family* $\mathcal{F} \subseteq 2^S$ and functions $f, g : \mathcal{F} \rightarrow \mathbf{Z}_+$ satisfying $g(F) \leq f(F)$ for every $F \in \mathcal{F}$. Here, \mathcal{F} represents a classification of applicants and $g(F)$ and $f(F)$ are lower and upper quotas, respectively, for each class $F \in \mathcal{F}$. Let $\mathcal{J} \subseteq 2^S$ be a family of subsets satisfying all quotas, i.e.,

$$\mathcal{J} = \{ X \subseteq S \mid g(F) \leq |X \cap F| \leq f(F) \ (\forall F \in \mathcal{F}) \}. \quad (3.1)$$

Then, the pair (S, \mathcal{J}) is a g-matroid if $\mathcal{J} \neq \emptyset$. ■

Example 3.2 (Assignment to Divisions). Let S be a set of applicants. An institute has a set D and functions $\Gamma : D \rightarrow 2^S$ and $p, q : D \rightarrow \mathbf{Z}_+$. The set D represents a set of divisions to which applicants are assigned. For each $d \in D$, the subset $\Gamma(d) \subseteq S$ means acceptable applicants and $p(d)$ and $q(d)$ are lower and upper quotas, respectively. Let $\mathcal{J} \subseteq 2^S$ be the set of applicants assignable to divisions, i.e.,

$$\mathcal{J} = \left\{ X \subseteq S \mid \begin{array}{l} \exists \pi : X \rightarrow D \text{ such that, for every } d \in D, \\ \pi^{-1}(d) \subseteq \Gamma(d), \quad p(d) \leq |\pi^{-1}(d)| \leq q(d) \end{array} \right\}, \quad (3.2)$$

where $\pi^{-1}(d) = \{e \in X \mid \pi(e) = d\}$. Then, the pair (S, \mathcal{J}) is a g-matroid if $\mathcal{J} \neq \emptyset$. ■

In Appendix 3.A, we provide some basic operations under which g-matroids are closed. Combining them, we can construct various g-matroids starting with simple g-matroids or known matroids.

3.2.2 Lower Extension of Generalized Matroids

In this section, we introduce the *lower extension* of g-matroids, which is an operation to construct a matroid as an extension of a given g-matroid. We first show some basic properties of g-matroids. Let (S, \mathcal{J}) be a g-matroid.

Lemma 3.3. *If $X, Z \in \mathcal{J}$ and $X \subseteq Y \subseteq Z$, then $Y \in \mathcal{J}$.*

Proof. Suppose, to the contrary, there is a pair $(X, Z) \in \mathcal{J} \times \mathcal{J}$ and $Y \notin \mathcal{J}$ such that $X \subsetneq Y \subsetneq Z$. Among such pairs, let (X, Z) minimize $|Z \setminus X|$. Apply the exchange axiom

(J1) for $X, Z \in \mathcal{J}$ and $e \in Y \setminus X \subseteq Z \setminus X$. Then, as $X \setminus Z = \emptyset$, we have $X + e \in \mathcal{J}$. As $X + e \subseteq Y \subseteq Z$ and $|Z \setminus (X + e)| < |Z \setminus X|$, this contradicts the minimality of $|Z \setminus X|$. \square

It is known that a g-matroid satisfies the axiom (I2) as follows.

Lemma 3.4 ([103, Lemma 2.4]). *If $X, Y \in \mathcal{J}$ and $|X| < |Y|$, then there is an element $e \in Y \setminus X$ s.t. $X + e \in \mathcal{J}$.*

By Lemmas 3.3 and 3.4, we obtain the following observation.

Observation 3.5. *(S, \mathcal{J}) is a matroid if $\emptyset \in \mathcal{J}$.*

For a g-matroid (S, \mathcal{J}) , let $L(\mathcal{J}) \subseteq 2^S$ be the superfamily of \mathcal{J} defined by

$$L(\mathcal{J}) = \{ X \subseteq S \mid \exists Y \in \mathcal{J} : X \subseteq Y \}.$$

We call the pair $(S, L(\mathcal{J}))$ the *lower extension* of (S, \mathcal{J}) .

Lemma 3.6. *The pair $(S, L(\mathcal{J}))$ is a matroid.¹*

Proof. By definition, the family $(S, L(\mathcal{J}))$ clearly satisfies (I0) and (I1).

To show (I2), assume $X, Y \in L(\mathcal{J})$ and $|X| < |Y|$. Then, there are \tilde{X} and \tilde{Y} s.t. $X \subseteq \tilde{X} \in \mathcal{J}$ and $Y \subseteq \tilde{Y} \in \mathcal{J}$. Let $Z_1 \in \mathcal{J}$ be an independent set s.t. $Z_1 \subseteq X \cup \tilde{Y}$, $|Z_1| \geq |\tilde{Y}|$, and $|Z_1 \cap X|$ is maximal. Then, we have $X \subseteq Z_1$ since otherwise, by the exchange axiom (J1) for $Z_1, \tilde{X} \in \mathcal{J}$ and any $e \in X \setminus Z_1 \subseteq \tilde{X} \setminus Z_1$, we obtain Z_2 with $|Z_2 \cap X| > |Z_1 \cap X|$, which contradicts the maximality. This Z_1 satisfies $(Y \setminus X) \cap Z_1 \neq \emptyset$ as follows: If $(Y \setminus X) \cap Z_1 = \emptyset$ were false, then $X \subseteq Z_1 \subseteq (X \cup \tilde{Y}) \setminus (Y \setminus X)$, and hence $|Z_1| \leq |\tilde{Y}| + |X \setminus \tilde{Y}| - |Y \setminus X| \leq |\tilde{Y}| + |X \setminus Y| - |Y \setminus X| < |\tilde{Y}|$ since $|X| < |Y|$. This contradicts $|Z_1| \geq |\tilde{Y}|$.

Take any $e \in (Y \setminus X) \cap Z_1$. If $e \in \tilde{X}$, then $X + e \subseteq \tilde{X} \in \mathcal{J}$, and we are done. Otherwise, we can apply the exchange axiom (J1) for $\tilde{X}, Z_1 \in \mathcal{J}$ and $e \in Z_1 \setminus \tilde{X}$, and hence $\tilde{X} + e \in \mathcal{J}$ or $\tilde{X} + e - e' \in \mathcal{J}$ for some $e' \in \tilde{X} \setminus Z_1$. Note that $e' \notin X$ since $X \subseteq Z_1$. Thus, we have $X + e \subseteq \tilde{X} + e \in \mathcal{J}$ or $X + e \subseteq \tilde{X} + e - e' \in \mathcal{J}$, and both imply $X + e \in L(\mathcal{J})$ for $e \in Y \setminus X$. \square

Since $\mathbf{M} := (S, L(\mathcal{J}))$ is a matroid, we can define $\text{span}_{\mathbf{M}}(X)$ for every $X \subseteq S$. In particular, for an independent set $X \in L(\mathcal{J})$, it is given by

$$\text{span}_{\mathbf{M}}(X) = X \cup \{ e \in S \mid X + e \notin L(\mathcal{J}) \}.$$

Lemma 3.7. *Let $\mathbf{M} := (S, L(\mathcal{J}))$. For two independent sets $X, Y \in L(\mathcal{J})$, suppose $\text{span}_{\mathbf{M}}(X) = \text{span}_{\mathbf{M}}(Y)$. Then, $X \in \mathcal{J}$ if and only if $Y \in \mathcal{J}$.*

¹In fact, the claim easily follows from the fact that a generalized matroid can be defined by a strong pair (see Section 3.5). Here we provide a proof using exchange axioms (J1) and (J2).

Proof. We show that $X \in \mathcal{J}$ implies $Y \in \mathcal{J}$, which is enough for the proof. As $X, Y \in \mathcal{L}(\mathcal{J})$ and $\text{span}_{\mathbf{M}}(X) = \text{span}_{\mathbf{M}}(Y)$, we have

$$Y + e \notin \mathcal{L}(\mathcal{J}) \quad (\forall e \in X \setminus Y). \quad (3.3)$$

Suppose, to the contrary, $X \in \mathcal{J}$ and $Y \in \mathcal{L}(\mathcal{J}) \setminus \mathcal{J}$. Let \tilde{Y} be such that $Y \subsetneq \tilde{Y} \in \mathcal{J}$ and $|\tilde{Y} \setminus Y|$ is minimal. Since any $e \in \tilde{Y} \setminus Y$ satisfies $Y + e \in \mathcal{L}(\mathcal{J})$, the condition (3.3) implies $\tilde{Y} \setminus Y \subseteq \tilde{Y} \setminus X$. Take $e \in \tilde{Y} \setminus Y \subseteq \tilde{Y} \setminus X$ and apply the exchange axiom (J2) for $X, \tilde{Y} \in \mathcal{J}$ and e . Then, $\tilde{Y} - e \in \mathcal{J}$ or $\tilde{Y} - e + e' \in \mathcal{J}$ for some $e' \in X \setminus \tilde{Y}$. In the former case, as $e \in \tilde{Y} \setminus Y$, $\tilde{Y}_1 := \tilde{Y} - e$ satisfies $Y \subseteq \tilde{Y}_1 \in \mathcal{J}$, which contradicts the minimality. In the latter case, $e' \in X \setminus \tilde{Y} \subseteq X \setminus Y$ and $\tilde{Y}_2 := \tilde{Y} - e + e'$ satisfies $Y + e' \subseteq \tilde{Y}_2 \in \mathcal{J}$, which contradicts (3.3). \square

3.3 Preferences on Generalized Matroids

We call a triple (S, \mathcal{J}, \succ) an *ordered g-matroid* (resp., *ordered matroid*) on S if (S, \mathcal{J}) is a g-matroid (resp., matroid) and \succ is a total order on S . In the matching model in Section 3.4, profiles of agents are represented by ordered g-matroids. This section provides some properties of them.

3.3.1 Dominance Relation

For an ordered g-matroid (S, \mathcal{J}, \succ) , we say that a subset $X \in \mathcal{J}$ *dominates* an element $e \in S \setminus X$ w.r.t. (S, \mathcal{J}, \succ) if the following two conditions hold:

$$\begin{aligned} & X + e \notin \mathcal{J}, \\ & \forall e' \in X: [X + e - e' \in \mathcal{J} \implies e' \succ e]. \end{aligned}$$

Let (S, \mathcal{J}, \succ) be an ordered g-matroid. Then, $(S, \mathcal{L}(\mathcal{J}), \succ)$ is an ordered matroid which we call the *lower extension* of (S, \mathcal{J}, \succ) .

Lemma 3.8. *Let $X \in \mathcal{J}$ and $e \in S \setminus X$. Then, X dominates e w.r.t. (S, \mathcal{J}, \succ) if and only if X dominates e w.r.t. $(S, \mathcal{L}(\mathcal{J}), \succ)$.*

Proof. We show the following two claims, which complete the proof.

- (i) For $X \in \mathcal{J}$ and $e \in S \setminus X$, $X + e \notin \mathcal{J}$ if and only if $X + e \notin \mathcal{L}(\mathcal{J})$.
- (ii) For $X \in \mathcal{J}$ and $e \in S \setminus X$, assume $X + e \notin \mathcal{J}$. Then, for any $e' \in X$, $X + e - e' \in \mathcal{J}$ if and only if $X + e - e' \in \mathcal{L}(\mathcal{J})$.

(i): The “if” part is obvious since $\mathcal{J} \subseteq \mathcal{L}(\mathcal{J})$. For the “only if” part, assume $X + e \in \mathcal{L}(\mathcal{J})$. Then, there is Y with $X + e \subseteq Y \in \mathcal{J}$. By Lemma 3.3, $X \subseteq X + e \subseteq Y$ and $X, Y \in \mathcal{J}$ imply $X + e \in \mathcal{J}$.

(ii): The “only if” part is obvious. For the “if” part, let $X + e - e' \in L(\mathcal{J})$. Then, there is Y with $X + e - e' \subseteq Y \in \mathcal{J}$. Apply the exchange axiom (J1) for $X, Y \in \mathcal{J}$ and $e \in Y \setminus X$. Since $X + e \notin \mathcal{J}$ and $X \setminus Y = \{e'\}$, we should have $X + e - e' \in \mathcal{J}$. \square

3.3.2 Choice Functions Induced by Ordered Matroids

Let $\mathcal{M} = (S, \mathcal{I}, \succ)$ be an ordered matroid with $S = \{e_1, e_2, \dots, e_n\}$ and $e_1 \succ e_2 \succ \dots \succ e_n$. Define a function $C_{\mathcal{M}} : 2^S \rightarrow 2^S$ by letting $C_{\mathcal{M}}(X)$ be the output of the following algorithm for every $X \subseteq S$: Let $Y^0 := \emptyset$ and define Y^l for each $l \in [n]$ by

$$Y^l := \begin{cases} Y^{l-1} + e_l & \text{if } e_l \in X \text{ and } Y^{l-1} + e_l \in \mathcal{I}, \\ Y^{l-1} & \text{otherwise,} \end{cases} \quad (3.4)$$

and then let $C_{\mathcal{M}}(X) := Y^n$.

We call $C_{\mathcal{M}}$ the *choice function induced from $\mathcal{M} = (S, \mathcal{I}, \succ)$* . By definition, for any $X \subseteq S$, we can compute $C_{\mathcal{M}}(X)$ in $O(|S|)$ time provided that the membership oracle of \mathcal{I} is available. If there is a weight function on S consistent with \succ , the definition of $C_{\mathcal{M}}(X)$ is identical to the standard greedy algorithm (see, Section 2.3) applied to X and a positive weight which is consistent with \succ . This fact implies the following proposition.

Proposition 3.9. *Let $w : S \rightarrow \mathbf{R}_{>0}$ be an arbitrary positive weight function such that $e \succ e'$ implies $w(e) > w(e')$. Then, for any subset $X \subseteq S$, the set $C_{\mathcal{M}}(X)$ is the unique solution to $\max \{w(Y) \mid Y \subseteq X, Y \in \mathcal{I}\}$, where $w(Y) := \sum_{e \in Y} w(e)$.*

Corollary 3.10. *For any $X \subseteq S$, the set $C_{\mathcal{M}}(X)$ satisfies $C_{\mathcal{M}}(X) \in \mathcal{I}$ and dominates every $e \in X \setminus C_{\mathcal{M}}(X)$ w.r.t. \mathcal{M} .*

Next, consider the case where an ordered matroid is the lower extension of some ordered g-matroid. Let (S, \mathcal{J}, \succ) be an ordered g-matroid and let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$. We now show that, for any $X \subseteq S$, we can compute $C_{\mathcal{M}}(X)$ with the membership oracle of \mathcal{J} instead of that of $L(\mathcal{J})$. We use the following lemma.

Lemma 3.11. *Assume $Y \subseteq Z \in \mathcal{J}$. For any $e \in S \setminus Y$, we have $Y + e \in L(\mathcal{J})$ if and only if either $e \in Z$ or $Z + e \in \mathcal{J}$ or $Z + e - e' \in \mathcal{J}$ for some $e' \in Z \setminus Y$.*

Proof. The “if” part is obvious by the definition of $L(\mathcal{J})$. To show the “only if” part, let $Y + e \in L(\mathcal{J})$. Then, there is $Z' \subseteq S$ with $Y + e \subseteq Z' \in \mathcal{J}$. If $e \notin Z$, then by the exchange axiom (J1) for $Z, Z' \in \mathcal{J}$ and $e \in Z' \setminus Z$, we have $Z + e \in \mathcal{J}$ or $Z + e - e' \in \mathcal{J}$ for some $e' \in Z \setminus Z' \subseteq Z \setminus Y$. \square

Lemma 3.12. *Let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$ and $Z \in \mathcal{J}$. For any $X \subseteq S$, let $(Y^0, Z^0) := (\emptyset, Z)$*

and define (Y^l, Z^l) for each $l \in [n]$ by

$$(Y^l, Z^l) := \begin{cases} (Y^{l-1} + e_l, Z^{l-1}) & \text{if } e_l \in X \cap Z^{l-1}, \\ (Y^{l-1} + e_l, Z^{l-1} + e_l) & \text{if } e_l \in X \setminus Z^{l-1}, Z^{l-1} + e_l \in \mathcal{J}, \\ (Y^{l-1} + e_l, Z^{l-1} + e_l - e) & \text{if } e_l \in X \setminus Z^{l-1}, Z^{l-1} + e_l \notin \mathcal{J}, \text{ and} \\ & \exists e \in Z^{l-1} \setminus Y^{l-1}: Z^{l-1} + e_l - e \in \mathcal{J}, \\ (Y^{l-1}, Z^{l-1}) & \text{otherwise.} \end{cases} \quad (3.5)$$

Then, Y^n coincides with $C_{\mathcal{M}}(X)$.

Proof. For each $l \in [n]$, we can observe $Y^l \subseteq Z^l \in \mathcal{J}$ and hence $Y^l \in \mathcal{L}(\mathcal{J})$. In the definition (3.5) of Y^l , Lemma 3.11 implies that $Y^l = Y^{l-1} + e_l$ if and only if $e_l \in X$ and $Y^{l-1} + e_l \in \mathcal{L}(\mathcal{J})$. Then, this definition of Y^l corresponds to (3.4) with \mathcal{I} replaced by $\mathcal{L}(\mathcal{J})$. Hence, Y^n coincides with $C_{\mathcal{M}}(X)$. \square

In Lemma 3.12, for each $l \in [n]$, we can define (Y^l, Z^l) from (Y^{l-1}, Z^{l-1}) using the membership oracle of \mathcal{J} at most $|S|$ times, which implies the following proposition.

Proposition 3.13. *Let $\mathcal{M} := (S, \mathcal{L}(\mathcal{J}), \succ)$. For any $X \subseteq S$, one can compute $C_{\mathcal{M}}(X)$ in $O(|S|^2)$ time, provided that a membership oracle of \mathcal{J} and an arbitrary independent set of \mathcal{J} are available.*

3.3.3 Matroid Kernels

Here we introduce the notion of matroid kernels and provide their properties, which were shown by Fleiner [26, 27]. The proofs are given in Appendix 3.D

Let $\mathcal{M}_1 = (S, \mathcal{I}_1, \succ_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2, \succ_2)$ be two ordered matroids on the same ground set S . Also, let $\mathbf{M}_1 := (S, \mathcal{I}_1)$, $\mathbf{M}_2 := (S, \mathcal{I}_2)$. A subset $X \subseteq S$ is called an $\mathcal{M}_1\mathcal{M}_2$ -kernel if it satisfies the following two conditions.

1. $X \in \mathcal{I}_1 \cap \mathcal{I}_2$.
2. Every $e \in E \setminus X$ is dominated by X w.r.t. \mathcal{M}_1 or w.r.t. \mathcal{M}_2 .

An $\mathcal{M}_1\mathcal{M}_2$ -kernel X^* is called \mathcal{M}_1 -optimal if it satisfies $C_{\mathcal{M}_1}(X^* \cup X) = X^*$ for any $\mathcal{M}_1\mathcal{M}_2$ -kernel X .

Theorem 3.14 (Fleiner [26, 27]). *For any ordered matroids \mathcal{M}_1 and \mathcal{M}_2 on the same ground set S , there exists an $\mathcal{M}_1\mathcal{M}_2$ -kernel, and one can find an $\mathcal{M}_1\mathcal{M}_2$ -kernel in $O(|S| \cdot \text{EO}_{\mathcal{M}_1\mathcal{M}_2})$ time, where $\text{EO}_{\mathcal{M}_1\mathcal{M}_2}$ is the time required to compute $C_{\mathcal{M}_1}(X)$, $C_{\mathcal{M}_2}(X)$ for any subset X of S . Furthermore, one can find the \mathcal{M}_1 -optimal $\mathcal{M}_1\mathcal{M}_2$ -kernel.*

Let $\mathfrak{K}_{\mathcal{M}_1\mathcal{M}_2}$ be the set of all $\mathcal{M}_1\mathcal{M}_2$ -kernels.

Theorem 3.15 (Fleiner [26, 27]). *Any two kernels $X, Y \in \mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}$ satisfy $\text{span}_{\mathbf{M}_1}(X) = \text{span}_{\mathbf{M}_1}(Y)$ and $\text{span}_{\mathbf{M}_2}(X) = \text{span}_{\mathbf{M}_2}(Y)$.*

For any $X, Y \subseteq S$, define subsets $X \vee_1 Y$ and $X \wedge_1 Y$ of S by

$$\begin{aligned} X \vee_1 Y &= C_{\mathcal{M}_1}(X \cup Y), \\ X \wedge_1 Y &= C_{\mathcal{M}_2}(X \cup Y). \end{aligned} \tag{3.6}$$

Theorem 3.16 (Fleiner [26, 27]). *The triple $(\mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}, \vee_1, \wedge_1)$ is a distributive lattice.*

3.4 Stable Matchings on Generalized Matroids

In this section, we formulate the matching model where the profile of each agent is given by an ordered g-matroid. We study the structure of the set of stable matchings, and give a polynomial-time algorithm which finds a stable matching or reports the nonexistence.

3.4.1 Model Formulation

Consider two disjoint finite sets I and J of agents. Let $E = I \times J$ be the set of all pairs, and define $E_i = \{(i, j) \mid j \in J\}$ for each $i \in I$ and $E_j = \{(i, j) \mid i \in I\}$ for each $j \in J$. Each agent can have multiple partnerships, and any subset X of $E = I \times J$ is called a *matching*.

The *profile* of each agent $k \in I \cup J$ is an ordered g-matroid $(E_k, \mathcal{J}_k, \succ_k)$, where $\mathcal{J}_k \subseteq 2^{E_k}$ is the set of feasible partnership sets and \succ_k represents the preference of k on the opposite agents. The set $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ of profiles is called an *instance*.

Definition 3.17. A set $X \subseteq E$ is a *stable matching* of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ (or, *stable w.r.t.* $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$) if the following two conditions hold:

1. For every $k \in I \cup J$, $X \cap E_k \in \mathcal{J}_k$.
2. For every $e = (i, j) \in E \setminus X$, $X \cap E_i$ dominates e w.r.t. $(E_i, \mathcal{J}_i, \succ_i)$ or $X \cap E_j$ dominates e w.r.t. $(E_j, \mathcal{J}_j, \succ_j)$. ■

We call this model the *Generalized Matroid Stable Matching (GMSM)* model. By Example 3.1, we see that the GMSM model includes the laminar classified stable matching (LCSM) model [29, 54]. Hence, like the LCSM model, some instances of the GMSM model have no stable matching. Then, the *GMSM problem* is to find a stable matching if it exists, and otherwise to report the nonexistence.

3.4.2 Characterization through Lower Extension

Let $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ be an instance of the GMSM model.

Lemma 3.18. *For $X \subseteq E$, the following two conditions are equivalent:*

1. *X is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$.*
2. *X is a stable matching of $\{(E_k, L(\mathcal{J}_k), \succ_k)\}_{k \in I \cup J}$ and satisfies $X \cap E_k \in \mathcal{J}_k$ for every $k \in I \cup J$.*

Proof. Both conditions require $X \cap E_k \in \mathcal{J}_k$ ($\forall k \in I \cup J$). Then, Lemma 3.8 implies that, for any $k \in I \cup J$ and $e \in E_k \setminus X$, the subset $X \cap E_k$ dominates e w.r.t. $(E_k, \mathcal{J}_k, \succ_k)$ if and only if $X \cap E_k$ dominates e w.r.t. $(E_k, L(\mathcal{J}_k), \succ_k)$. Then, the lemma follows from the definition of the stability. \square

For each agent $k \in I \cup J$, let $\mathcal{M}_k := (E_k, L(\mathcal{J}_k), \succ_k)$. Then, each \mathcal{M}_k is an ordered matroid. Note that $\{E_i\}_{i \in I}$ is a partition of E and let (E, \mathcal{I}_I) be the direct sum of matroids $\{(E_i, L(\mathcal{J}_i))\}_{i \in I}$. Also, let \succ_I be an arbitrary total order on E which satisfies

$$e \succ_i e' \implies e \succ_I e' \quad (\forall i \in I, \forall e, e' \in E_i).$$

Then, $\mathcal{M}_I := (E, \mathcal{I}_I, \succ_I)$ is an ordered matroid and the following fact holds.

Observation 3.19. *Let $X \in \mathcal{I}_I$, $i \in I$, and $e \in E_i \setminus X$. Then, X dominates e w.r.t. \mathcal{M}_I if and only if $X \cap E_i$ dominates e w.r.t. \mathcal{M}_i .*

Since $\{E_j\}_{j \in J}$ is also a partition of E , we similarly define an ordered matroid $\mathcal{M}_J = (E, \mathcal{I}_J, \succ_J)$ from $\{(E_j, L(\mathcal{J}_j), \succ_j)\}_{j \in J}$. Then Observation 3.19 implies the following lemma.

Lemma 3.20. *A set $X \subseteq E$ is a stable matching of $\{(E_k, L(\mathcal{J}_k), \succ_k)\}_{k \in I \cup J}$ if and only if X is an $\mathcal{M}_I \mathcal{M}_J$ -kernel.*

Combining Lemmas 3.18 and 3.20, we obtain the following theorem.

Theorem 3.21. *A set $X \subseteq E$ is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ if and only if X is an $\mathcal{M}_I \mathcal{M}_J$ -kernel satisfying $X \cap E_k \in \mathcal{J}_k$ ($\forall k \in I \cup J$).*

3.4.3 Structure of Stable Matchings

Let \mathfrak{S} be the set of all stable matchings of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ and let $\mathfrak{K}_{\mathcal{M}_I \mathcal{M}_J}$ be the set of all $\mathcal{M}_I \mathcal{M}_J$ -kernels. Then, Theorem 3.21 is rephrased as

$$\mathfrak{S} = \{ X \in \mathfrak{K}_{\mathcal{M}_I \mathcal{M}_J} \mid X \cap E_k \in \mathcal{J}_k \text{ } (\forall k \in I \cup J) \}. \quad (3.7)$$

Therefore, \mathfrak{S} is a subset of $\mathfrak{K}_{\mathcal{M}_I \mathcal{M}_J}$. Moreover, any members of $\mathfrak{K}_{\mathcal{M}_I \mathcal{M}_J}$ satisfy the following significant property.

Lemma 3.22. *For any $\mathcal{M}_I \mathcal{M}_J$ -kernels $X, Y \in \mathfrak{K}_{\mathcal{M}_I \mathcal{M}_J}$ and any $k \in I \cup J$, we have $X \cap E_k \in \mathcal{J}_k$ if and only if $Y \cap E_k \in \mathcal{J}_k$.*

Proof. Without loss of generality, we assume $k = i \in I$. By Theorem 3.15, we have $\text{span}_{\mathbf{M}_I}(X) = \text{span}_{\mathbf{M}_I}(Y)$ where $\mathbf{M}_I = (E, \mathcal{I}_I)$. By the definition of \mathcal{I}_I , this implies $\text{span}_{\mathbf{M}_i}(X \cap E_i) = \text{span}_{\mathbf{M}_i}(Y \cap E_i)$ where $\mathbf{M}_i = (E_i, \mathcal{L}(\mathcal{J}_i))$. Then, Lemma 3.7 implies that $X \cap E_i \in \mathcal{J}_i$ if and only if $Y \cap E_i \in \mathcal{J}_i$. \square

Combining (3.7) and Lemma 3.22 yields the following dichotomy theorem.

Theorem 3.23. *We have either $\mathfrak{S} = \mathfrak{R}_{\mathcal{M}_I, \mathcal{M}_J}$ or $\mathfrak{S} = \emptyset$.*

In the case where each g-matroid (E_k, \mathcal{J}_k) is a matroid, we surely have $\mathfrak{S} = \mathfrak{R}_{\mathcal{M}_I, \mathcal{M}_J}$. We can have $\mathfrak{S} = \emptyset$ only when there are lower quotas.

Theorem 3.23 immediately implies the following corollary.

Corollary 3.24. *If $X \cap E_k \notin \mathcal{J}_k$ for some $X \in \mathfrak{R}_{\mathcal{M}_I, \mathcal{M}_J}$ and $k \in I \cup J$, then $\mathfrak{S} = \emptyset$, i.e., the instance $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ has no stable matching.*

Recall Theorems 3.15 and 3.16, with which Theorem 3.23 implies the following fact.

Corollary 3.25. *If the set \mathfrak{S} of stable matchings of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ is nonempty, then it forms a distributive lattice under the operations \vee_1 and \wedge_1 , which are defined by (3.6) with $C_{\mathcal{M}_1}, C_{\mathcal{M}_2}$ replaced by $C_{\mathcal{M}_I}, C_{\mathcal{M}_J}$, respectively. Also, for any $X, Y \in \mathfrak{S}$ and any $k \in I \cup J$, we have $\text{span}_{\mathbf{M}_k}(X \cap E_k) = \text{span}_{\mathbf{M}_k}(Y \cap E_k)$.*

Remark 3.26. The second claim of Corollary 3.25 is a generalization of the “rural hospital theorem” as follows. Note that the condition $\text{span}_{\mathbf{M}_k}(X \cap E_k) = \text{span}_{\mathbf{M}_k}(Y \cap E_k)$ implies $r_{\mathbf{M}_k}(X \cap E_k) = r_{\mathbf{M}_k}(Y \cap E_k)$. Then $|X \cap E_k| = |Y \cap E_k|$ follows because $X \cap E_k$ and $Y \cap E_k$ are independent in \mathbf{M}_k . Furthermore, if \mathbf{M}_k is a uniform matroid of rank $q_k \in \mathbf{Z}_{>0}$ and $|X \cap E_k| < q_k$ for some $X \in \mathfrak{S}$, then we have $X \cap E_k = \text{span}_{\mathbf{M}_k}(X \cap E_k)$. In this case, $Y \cap E_k = X \cap E_k$ holds for every $Y \in \mathfrak{S}$. \blacksquare

3.4.4 Algorithm for Finding a Stable Matching

For an instance $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ of the GSM problem, let us consider the following algorithm.

Algorithm GSM

Step1. Find an $\mathcal{M}_I, \mathcal{M}_J$ -kernel X .

Step2. If $X \cap E_k \in \mathcal{J}_k$ for every $k \in I \cup J$, then return X . Otherwise, report “There is no stable matching.”

Theorem 3.21 and Corollary 3.24 guarantee the correctness of this algorithm.

Proposition 3.27. *If the algorithm GMSM returns a matching $X \subseteq E$, then X is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$. Otherwise, there is no stable matching.*

We now check the time complexity of this algorithm.

Proposition 3.28. *The algorithm GMSM runs in $O(|E|^3)$ time provided a membership oracle and an independent set of \mathcal{J}_k for each $k \in I \cup J$.*

Proof. By Theorem 3.14, an $\mathcal{M}_I \mathcal{M}_J$ -kernel can be found in $O(|E| \cdot \text{EO}_{\mathcal{M}_I \mathcal{M}_J})$ time, where $\text{EO}_{\mathcal{M}_I \mathcal{M}_J}$ is the time required to compute $C_{\mathcal{M}_I}(X)$, $C_{\mathcal{M}_J}(X)$ for any $X \subseteq E$. Note that, $C_{\mathcal{M}_I}(X)$ is the direct sum of $\{C_{\mathcal{M}_i}(X \cap E_i)\}_{i \in I}$. Since $C_{\mathcal{M}_i}(X \cap E_i)$ is computed in $O(|E_i|^2) = O(|J|^2)$ time by Proposition 3.13, $C_{\mathcal{M}_I}(X)$ is obtained in $O(|I| \cdot |J|^2) = O(|E|^2)$ time. Similar arguments apply to $C_{\mathcal{M}_J}(X)$. Then, $\text{EO}_{\mathcal{M}_I \mathcal{M}_J}$ is $O(|E|^2)$, and Step 1 requires $O(|E|^3)$ time. Also, Step 2 requires $O(|I| + |J|) = O(|E|)$ time. The proof is completed. \square

By Propositions 3.27 and 3.28, we obtain the following main theorem.

Theorem 3.29. *For an instance $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ of the GMSM problem, assume that a membership oracle and an independent set of \mathcal{J}_k are given for each $k \in I \cup J$. One can determine whether a stable matching exists or not in $O(|E|^3)$ time. Also, one can obtain a stable matching simultaneously if it exists.*

Remark 3.30. Recall that we can find the \mathcal{M}_I -optimal $\mathcal{M}_I \mathcal{M}_J$ -kernel by Theorem 3.14. By Theorem 3.23, it is also the \mathcal{M}_I -optimal stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$ if the set \mathfrak{S} of stable matchings is nonempty. That is, when $\mathfrak{S} \neq \emptyset$, we can find $X^* \in \mathfrak{S}$ such that $C_{\mathcal{M}_i}(X_i^* \cup X_i) = X_i^*$ for every $X \in \mathfrak{S}$ and $i \in I$, where $X_i = X \cap E_i$. \blacksquare

Remark 3.31. In some practical settings, we can execute the algorithm GMSM in polynomial time even if we do not have any oracle or independent set. This is because, as is shown in Appendix 3.C, we can reduce the computation of the choice function $C_{\mathcal{M}_k}$ of each $k \in I \cup J$ to the *minimum cost circulation problem* on an associated network.

For example, if each g-matroid (E_k, \mathcal{J}_k) is given by $\mathcal{F}_k \subseteq 2^{E_k}$ and $f_k, g_k : \mathcal{F}_k \rightarrow \mathbf{Z}_+$ as in Example 3.1, GMSM runs in time polynomial in $|E|$ by Propositions 3.72 and 3.74. If each (E_k, \mathcal{J}_k) is given by D_k, Γ_k, p_k and q_k as in Example 3.2, GMSM runs in time polynomial in $|E|$ and $\max\{|D_k| : k \in I \cup J\}$ by Propositions 3.73 and 3.74. \blacksquare

3.5 Generalized Polymatroids

This section introduces generalized polymatroids, the polyhedral versions of g-matroids (see [31, 34] for more information).

3.5.1 Definition

Throughout this section, we suppose that any set function $f : \mathcal{F} \rightarrow \mathbf{R}$ on a family $\mathcal{F} \subseteq 2^S$ satisfies $\emptyset, S \in \mathcal{F}$ and $f(\emptyset) = 0$. Recall that a family of subsets is called a *ring family*

if it is closed under union and intersection. For ring families $\mathcal{F}, \mathcal{G} \subseteq 2^S$ and set functions $f : \mathcal{F} \rightarrow \mathbf{R}$ and $g : \mathcal{G} \rightarrow \mathbf{R}$, we say that the pair (f, g) is a *strong pair* (or that the pair (f, g) is *paramodular*) if

- f is submodular,
- g is supermodular, and
- f and g are *compliant* in the sense that, for any $A \in \mathcal{F}$ and $B \in \mathcal{G}$, we have $A \setminus B \in \mathcal{F}$, $B \setminus A \in \mathcal{G}$, and the following *cross inequality*:

$$f(A) - g(B) \geq f(A \setminus B) - g(B \setminus A).$$

For any set functions $f : \mathcal{F} \rightarrow \mathbf{R}$ and $g : \mathcal{G} \rightarrow \mathbf{R}$, we associate two polyhedra

$$\mathbf{P}(f, g) = \left\{ x \in \mathbf{R}^S \mid \begin{array}{l} x(A) \leq f(A) \quad (\forall A \in \mathcal{F}), \\ x(A) \geq g(A) \quad (\forall A \in \mathcal{G}) \end{array} \right\},$$

$$\mathbf{P}_+(f, g) = \mathbf{P}(f, g) \cap \mathbf{R}_+^S.$$

A polyhedron $Q \subseteq \mathbf{R}^S$ is called a *generalized polymatroid* (*g-polymatroid*) if $Q = \mathbf{P}(f, g)$ for some strong pair (f, g) . Such a strong pair is defined by Q as follows.

Proposition 3.32 (Frank and Tardos [34]). *For a g-polymatroid $Q \subseteq \mathbf{R}^S$, define set functions f and g respectively by $f(A) = \max \{ x(A) \mid x \in Q \}$ and $g(A) = \min \{ x(A) \mid x \in Q \}$, where they are defined only on subsets which make the right-hand side finite. Then, the pair (f, g) is the unique strong pair which satisfies $Q = \mathbf{P}(f, g)$.*

For the strong pair (f, g) in the proposition, we call f the *upper bound function* and g the *lower bound function* of Q .

G-polymatroids are polyhedral versions of g-matroids. We can show that, for any g-matroid (S, \mathcal{J}) , the convex hull of $\{ \chi_A \in \{0, 1\}^S \mid A \in \mathcal{J} \}$ is a g-polymatroid. Note that a g-polymatroid is identical with a polymatroid if it contains the zero vector as the minimum point, and then the defining function coincides with the upper bound function.

We next introduce the notion of intersecting paramodularity, a weaker condition on (f, g) , which still yields g-polymatroids. For set functions $f : \mathcal{F} \rightarrow \mathbf{R}$ and $g : \mathcal{G} \rightarrow \mathbf{R}$ defined on intersecting families $\mathcal{F}, \mathcal{G} \subseteq 2^S$, the pair (f, g) is called an *intersecting paramodular pair* (or *weak pair*, for short) if

- f is intersecting-submodular,
- g is intersecting-supermodular, and
- for each intersecting $A \in \mathcal{F}$ and $B \in \mathcal{G}$, we have $A \setminus B \in \mathcal{F}$, $B \setminus A \in \mathcal{G}$ and the cross inequality holds for A and B .

Proposition 3.33 (Frank [30]). *Let (f, g) be an intersecting paramodular pair with $\mathbf{P}(f, g) \neq \emptyset$. Then, there exists a strong pair (\hat{f}, \hat{g}) such that $\mathbf{P}(\hat{f}, \hat{g}) = \mathbf{P}(f, g)$, and hence $\mathbf{P}(f, g)$ is a g -polymatroid. Also $\mathbf{P}_+(f, g)$ is a g -polymatroid if $\mathbf{P}_+(f, g) \neq \emptyset$.*

A g -polymatroid is known to be a synonym of an \mathbf{M}^\natural -convex polyhedron, which is defined by the exchange property (see [75, Sections 4.7 and 4.8]). This fact gives the following characterization of g -polymatroids. For a real number $\alpha_0 > 0$, we write $[0, \alpha_0] := \{\alpha \in \mathbf{R} \mid 0 \leq \alpha \leq \alpha_0\}$.

Proposition 3.34. *A set $Q \subseteq \mathbf{R}^S$ is a g -polymatroid if and only if it satisfies the following property. For all $x, y \in Q$ and $e \in S$ with $x(e) < y(e)$, either of the following holds for some positive real $\alpha_0 > 0$:*

1. $x + \alpha\chi_e, y - \alpha\chi_e \in Q \quad (\forall \alpha \in [0, \alpha_0])$.
2. *There exists $e' \in S$ with $x(e') > y(e')$ such that, for any $\alpha \in [0, \alpha_0]$, we have $x + \alpha(\chi_e - \chi_{e'}), y - \alpha(\chi_e - \chi_{e'}) \in Q$.*

We introduce an operation on g -polymatroids called the *reduction* by a vector. For any vector $a \in (\mathbf{R} \cup \{+\infty\})^S$, we write $[-\infty, a] := \{x \in \mathbf{R}^S \mid x \leq a\}$.

Proposition 3.35 (Frank and Tardos [34]). *Let $Q \subseteq \mathbf{R}^S$ be a g -polymatroid whose upper bound function is $f : 2^S \rightarrow \mathbf{R}$. For any vector $a \in (\mathbf{R} \cup \{+\infty\})^S$ with $Q \cap [-\infty, a] \neq \emptyset$, the set $Q \cap [-\infty, a]$ is a g -polymatroid whose upper bound function is given by*

$$f^a(A) = \min \{f(B) + a(A \setminus B) \mid B \subseteq A\} \quad (A \subseteq S).$$

3.5.2 Lower Extension of Generalized Polymatroids

Here, we introduce the lower extension of g -polymatroids. This is the polyhedral version of the lower extension of g -matroids.

Let $Q \subseteq \mathbf{R}_+^S$ be a g -polymatroid which is bounded and included in the nonnegative orthant. Also, let $f, g : 2^S \rightarrow \mathbf{R}$ be the upper and lower bound functions of Q , respectively. Then, (f, g) is a strong pair. Recall that polyhedra $\mathbf{P}(f)$ and $\mathbf{B}(f)$ are defined by

$$\begin{aligned} \mathbf{P}(f) &= \{x \in \mathbf{R}^S \mid x(A) \leq f(A) \ (\forall A \subseteq S)\}, \\ \mathbf{B}(f) &= \{x \in \mathbf{R}^S \mid x(A) \leq f(A) \ (\forall A \subseteq S), \ x(S) = f(S)\}. \end{aligned}$$

Lemma 3.36. *We have $\mathbf{B}(f) \subseteq Q$.*

Proof. Since $Q = \mathbf{P}(f, g) = \{x \in \mathbf{P}(f) \mid x(A) \geq g(A) \ (\forall A \subseteq S)\}$ and $\mathbf{B}(f) \subseteq \mathbf{P}(f)$, it suffices to show that $x \in \mathbf{B}(f)$ implies $x(A) \geq g(A) \ (\forall A \subseteq S)$.

Take any $x \in \mathbf{B}(f)$ and $A \subseteq S$. As (f, g) is a strong pair, we have the crossing inequality $f(S) - g(A) \geq f(S \setminus A) - g(\emptyset)$. Substituting $x(S) = f(S)$, $x(S \setminus A) \leq f(S \setminus A)$, and $g(\emptyset) = 0$, we obtain $x(A) \geq g(A)$. \square

Since $\mathbf{B}(f)$ is the set of maximal vectors of $\mathbf{P}(f)$ as stated in Proposition 2.6, we have the following fact.

Corollary 3.37. *If a set function $f : \mathcal{F} \rightarrow \mathbf{R}$ is submodular, then*

$$\mathbf{P}(f) = \{ x \in \mathbf{R}^S \mid \exists y \in \mathbf{B}(f) : x \leq y \}$$

holds.

Define a superset $L(Q) \subseteq \mathbf{R}_+^S$ of $Q \subseteq \mathbf{R}_+^S$ by

$$L(Q) = \{ x \in \mathbf{R}_+^S \mid \exists y \in Q : x \leq y \}.$$

We call $L(Q)$ the *lower extension* of Q .

Lemma 3.38. *The lower extension $L(Q)$ of Q is a polymatroid whose defining function is f , i.e., $L(Q)$ and Q have the same upper bound function.*

Proof. By $Q = \mathbf{P}(f, g) \subseteq \mathbf{P}(f)$ and $Q \subseteq \mathbf{R}_+^S$, we have $Q \subseteq \mathbf{P}_+(f)$. With Lemma 3.36, we obtain $\mathbf{B}(f) \subseteq Q \subseteq \mathbf{P}_+(f)$, which implies

$$\{ x \in \mathbf{R}_+^S \mid \exists y \in \mathbf{B}(f) : x \leq y \} \subseteq L(Q) \subseteq \{ x \in \mathbf{R}_+^S \mid \exists y \in \mathbf{P}_+(f) : x \leq y \}.$$

By Corollary 3.37, the left-hand side coincides with $\mathbf{P}_+(f)$. The right-hand side also coincides with $\mathbf{P}_+(f)$ by definition. Thus, we have $L(Q) = \mathbf{P}_+(f)$. Also, by Proposition 3.32, $Q \subseteq \mathbf{R}_+^S$ implies the monotonicity of the submodular function f , and hence $L(Q) = \mathbf{P}_+(f)$ is a polymatroid. \square

Combining Proposition 3.35 and Lemma 3.38 yields the following lemma.

Lemma 3.39. *For a vector $a \in (\mathbf{R}_+ \cup \{+\infty\})^S$ with $Q \cap [-\infty, a] \neq \emptyset$, sets $Q \cap [-\infty, a]$ and $L(Q) \cap [-\infty, a]$ are g -polymatroids and have the same upper bound function.*

3.6 Preferences on Generalized Polymatroids

We introduce a partial order on vectors, which is used to represent a preference of each agent in the stable allocation model in Section 3.7.

3.6.1 Optimal Points of Generalized Polymatroids

Let \succ be a total order on S . For any element $e \in S$, we denote $S_{\succeq e} := \{ e' \in S \mid e' \succeq e \}$. For two vectors $x, y \in \mathbf{R}^S$, we say that x is \succ -*preferable* to y if

$$\forall e \in S : x(S_{\succeq e}) \geq y(S_{\succeq e}).$$

For a set $Q \subseteq \mathbf{R}^S$, we say that $x \in \mathbf{R}^S$ is a \succ -optimal point of Q (or, \succ -optimal in Q) if $x \in Q$ and x is \succ -preferable to all $y \in Q$.

Observation 3.40. *If we have $x \in Q_2 \subseteq Q_1 \subseteq \mathbf{R}^S$ and x is \succ -optimal in Q_1 , then x is \succ -optimal in Q_2 .*

Since the preference order is a partial order, a general set of vectors may have no \succ -optimal point. A bounded g-polymatroid, however, does have a \succ -optimal point. Moreover, it is obtained by the following greedy algorithm.

Proposition 3.41. *Let $Q \subseteq \mathbf{R}^S$ be a bounded g-polymatroid whose upper bound function is $f: 2^S \rightarrow \mathbf{R}$. Define a vector $x^* \in \mathbf{R}^S$ by*

$$x^*(e) = f(S_{\succeq e}) - f(S_{\succeq e} \setminus \{e\}) \quad (3.8)$$

for each $e \in S$. Then, x^* is \succ -optimal in Q .

Proof. Since this construction of x^* is identical to the greedy algorithm of Proposition 2.9, we have $x^* \in \mathbf{P}(f)$. Also, (3.8) implies $x^*(S_{\succeq e}) = f(S_{\succeq e})$ for each $e \in S$. Since every $x \in \mathbf{P}(f)$ satisfies $x(S_{\succeq e}) \leq f(S_{\succeq e}) = x^*(S_{\succeq e})$ ($\forall e \in S$), the vector x^* is \succ -optimal in $\mathbf{P}(f)$. Also, $x^*(S) = f(S)$ implies $x^* \in Q$ by Lemma 3.36. Then, by Observation 3.40, x^* is \succ -optimal in Q . \square

The important fact observed from Proposition 3.41 is that the \succ -optimal point of Q depends only on the upper bound function and the total order. Then, Lemma 3.39 implies the following fact.

Lemma 3.42. *Let $Q \subseteq \mathbf{R}_+^S$ be a bounded g-polymatroid. For any vector $a \in (\mathbf{R}_+ \cup \{+\infty\})^S$ with $Q \cap [-\infty, a] \neq \emptyset$, sets $L(Q) \cap [-\infty, a]$ and $Q \cap [-\infty, a]$ are g-polymatroids and have the same \succ -optimal point.*

3.6.2 Choice Functions Induced from Ordered Polymatroids

A *choice function* (on vectors) is a function $C: (\mathbf{R}_+ \cup \{+\infty\})^S \rightarrow \mathbf{R}_+^S$ such that $C(x) \leq x$ for every $x \in (\mathbf{R}_+ \cup \{+\infty\})^S$. In the work of Alkan and Gale [9], the following three conditions of choice functions play a central role.

- *Consistency* : $C(x) \leq y \leq x$ implies $C(y) = C(x)$.
- *Persistence* : $x \leq y$ implies $C(y) \wedge x \leq C(x)$.
- *Size-monotonicity* : $x \leq y$ implies $|C(x)| \leq |C(y)|^2$.

An *ordered polymatroid* is a triple (S, P, \succ) such that $P \subseteq \mathbf{R}_+^S$ is a polymatroid and \succ is a total order on S . Here, we show how choice functions arise from ordered polymatroids.

²We use the notation $|x| := \sum_{e \in S} x(e)$ for any vector x . Hence $|C(x)| = \sum_{e \in S} (C(x))(e)$.

Let $\mathcal{P} = (S, P, \succ)$ be an ordered polymatroid and define a choice function $C_{\mathcal{P}}: (\mathbf{R}_+ \cup \{+\infty\})^S \rightarrow \mathbf{R}_+^S$ by letting $C_{\mathcal{P}}(x)$ be the \succ -optimal point of $P \cap [-\infty, x]$ for each $x \in (\mathbf{R}_+ \cup \{+\infty\})^S$. This is well-defined since the set $P \cap [-\infty, x]$ is a nonempty polymatroid and has the \succ -optimal point. We call $C_{\mathcal{P}}$ the *choice function induced from \mathcal{P}* .

Observation 3.43. $C_{\mathcal{P}}(x) = x \iff x \in P \quad (\forall x \in \mathbf{R}_+^S).$

Lemma 3.44. Let $f: 2^S \rightarrow \mathbf{R}$ be the defining function of P . Then, for every $x \in \mathbf{R}_+^S$, we have $|C_{\mathcal{P}}(x)| = \min \{ f(A) + x(S \setminus A) \mid A \subseteq S \}$.

Proof. By Proposition 3.35, the upper bound function of $P \cap [-\infty, x]$ satisfies $f^x(S) = \min \{ f(A) + x(S \setminus A) \mid A \subseteq S \}$. Also, since $C_{\mathcal{P}}(x)$ is the \succ -optimal point of $P \cap [-\infty, x]$, Proposition 3.41 implies $|C_{\mathcal{P}}(x)| = f^x(S)$. \square

We now show that the choice function induced from an ordered polymatroid is consistent, persistent and size-monotone.

Lemma 3.45. $C_{\mathcal{P}}$ is consistent, i.e., $C_{\mathcal{P}}(x) \leq y \leq x$ implies $C_{\mathcal{P}}(y) = C_{\mathcal{P}}(x)$.

Proof. Assume $C_{\mathcal{P}}(x) \leq y \leq x$. Then, $P \cap [-\infty, y] \subseteq P \cap [-\infty, x]$ holds and $C_{\mathcal{P}}(x) \leq y$ implies $C_{\mathcal{P}}(x) \in P \cap [-\infty, y]$. Hence, by Observation 3.40, $C_{\mathcal{P}}(x)$ is \succ -optimal in $P \cap [-\infty, y]$, and hence $C_{\mathcal{P}}(y) = C_{\mathcal{P}}(x)$. \square

Lemma 3.46. $C_{\mathcal{P}}$ is persistent, i.e., $x \leq y$ implies $C_{\mathcal{P}}(y) \wedge x \leq C_{\mathcal{P}}(x)$.

Proof. For $x \leq y$, set $x' = C_{\mathcal{P}}(x)$ and $y' = C_{\mathcal{P}}(y)$. Suppose, to the contrary, $y' \wedge x \not\leq x'$. Then, there is $e \in S$ with $y'(e) > x'(e)$ and $x(e) > x'(e)$. Apply the exchange property of Proposition 3.34 for $x', y' \in P$ and $e \in S$. Then, there is $\alpha_0 > 0$ such that (i) $x' + \alpha\chi_e \in P \quad (\forall \alpha \in [0, \alpha_0])$, or (ii) there is $e' \in S$ with $x'(e') > y'(e')$ s.t. $x' + \alpha(\chi_e - \chi_{e'})$, $y' - \alpha(\chi_e - \chi_{e'}) \in P \quad (\forall \alpha \in [0, \alpha_0])$.

In Case (i), let $\beta := \min\{\alpha_0, x(e) - x'(e)\} > 0$. Then, $x' + \beta\chi_e$ is in $P \cap [-\infty, x]$ and preferable to x' w.r.t. \succ , a contradiction. In Case (ii), let $\beta_1 := \min\{\alpha_0, x(e) - x'(e), x'(e') - y'(e')\}$. As $x(e) > x'(e)$ and $x'(e') > y'(e') \geq 0$,

$$\beta_1 > 0 \quad \text{and} \quad x'' := x' + \beta_1(\chi_e - \chi_{e'}) \in P \cap [-\infty, x].$$

Similarly, let $\beta_2 := \min\{\alpha_0, y'(e), y(e') - y'(e')\}$. As $y'(e) > x'(e) \geq 0$ and $y(e') \geq x(e') \geq x'(e') > y'(e')$, we have

$$\beta_2 > 0 \quad \text{and} \quad y'' := y' - \beta_2(\chi_e - \chi_{e'}) \in P \cap [-\infty, y].$$

If $e \succ e'$, then x'' is preferable to x' which contradicts $x' = C_{\mathcal{P}}(x)$. Otherwise $e' \succ e$, and then y'' is preferable to y' which contradicts $y' = C_{\mathcal{P}}(y)$. \square

Lemma 3.47. $C_{\mathcal{P}}$ is size-monotone, i.e., $x \leq y$ implies $|C_{\mathcal{P}}(x)| \leq |C_{\mathcal{P}}(y)|$.

Proof. This follows from Lemma 3.44 immediately. \square

The following lemma will play an important role in Section 3.7.

Lemma 3.48. *For $x, y \in P$, suppose $|C_{\mathcal{P}}(x \vee y)| = |x| = |y|$. Let $Q \subseteq \mathbf{R}_+^S$ be any g-polymatroid s.t. $L(Q) = P$. Then, $x \in Q$ if and only if $y \in Q$.*

Proof. Let $f: 2^S \rightarrow \mathbf{R}$ be the defining function of P . Then, by Lemma 3.44, the condition $|x| = |C_{\mathcal{P}}(x \vee y)|$ implies

$$x(S) = \min \{ f(A) + (x \vee y)(S \setminus A) \mid A \subseteq S \}.$$

Let $A^* \subseteq S$ be the minimizer of the right-hand side. Then,

$$x(S) = x(A^*) + x(S \setminus A^*) = f(A^*) + (x \vee y)(S \setminus A^*). \quad (3.9)$$

Since $x \in P = \mathbf{P}_+(f)$ implies $x(A^*) \leq f(A^*)$ and $x \leq (x \vee y)$ implies $x(S \setminus A^*) \leq (x \vee y)(S \setminus A^*)$, the condition (3.9) leads to $x(A^*) = f(A^*)$ and $x(S \setminus A^*) = (x \vee y)(S \setminus A^*)$. Similarly we obtain $y(A^*) = f(A^*)$ and $y(S \setminus A^*) = (x \vee y)(S \setminus A^*)$. As $x(S \setminus A^*) = (x \vee y)(S \setminus A^*) = y(S \setminus A^*)$ implies $x(e) = y(e)$ ($\forall e \in S \setminus A^*$), we obtain

$$x(A^*) = y(A^*) = f(A^*) \quad \text{and} \quad x(e) = y(e) \quad (\forall e \in S \setminus A^*). \quad (3.10)$$

For a g-polymatroid Q with $L(Q) = P$, we now show $x \in Q \implies y \in Q$, which completes the proof. By Lemma 3.38, the upper bound function of Q is f . Denote the lower bound function of Q by $g: 2^S \rightarrow \mathbf{R}$. Then (f, g) is a strong pair with $Q = \mathbf{P}(f, g)$. Since $x, y \in P = \mathbf{P}_+(f)$, it suffices to show $y(B) \geq g(B)$ ($\forall B \subseteq S$) assuming $x(B) \geq g(B)$ ($\forall B \subseteq S$). For any $B \subseteq S$,

$$f(A^*) - g(B) \geq f(A^* \setminus B) - g(B \setminus A^*) \quad (3.11)$$

by the cross inequality of (f, g) . By (3.10) and $B \setminus A^* \subseteq S \setminus A^*$, we have $x(B \setminus A^*) = y(B \setminus A^*)$, which implies $-g(B \setminus A^*) \geq -y(B \setminus A^*)$ by assumption. Also, we have $f(A^*) = y(A^*)$ by (3.10) and $f(A^* \setminus B) \geq y(A^* \setminus B)$ by $y \in \mathbf{P}_+(f)$. Substituting these three to (3.11), we obtain $y(B) \geq g(B)$. \square

The argument of the proof of Lemma 3.48 also implies the following claim (see (3.10)).

Lemma 3.49. *Let $f: 2^S \rightarrow \mathbf{R}$ be the defining function of P and suppose $x, y \in P$ and $|C_{\mathcal{P}}(x \vee y)| = |x| = |y|$. For any $e \in S$, if every $A \subseteq S$ with $e \in A$ satisfies $x(A) < f(A)$, then we have $x(e) = y(e)$.*

For an arbitrary choice function $C: (\mathbf{R}_+ \cup \{+\infty\})^S \rightarrow \mathbf{R}_+^S$ and $e \in S$, a vector $x \in \mathbf{R}_+^S$ is said to be *e-satiated* for C if $(C(y))(e) \leq x(e)$ for all $y \geq x$. Let x^e be the vector such that $x^e(e) = \infty$ and $x^e(e') = x(e')$ ($e' \in S \setminus \{e\}$).

Lemma 3.50. *Let $C_{\mathcal{P}}$ be induced from $\mathcal{P} = (S, P, \succ)$. A vector $x \in P$ is e -satiated for $C_{\mathcal{P}}$ if and only if x is \succ -optimal in $P \cap [-\infty, x^e]$.*

Proof. By the definition of $C_{\mathcal{P}}$, x is \succ -optimal in $P \cap [-\infty, x^e]$ if and only if $C_{\mathcal{P}}(x^e) = x$. Then, the “only if” part follows immediately. To show the “if” part, assume $C_{\mathcal{P}}(x^e) = x$ and take any $y \geq x$. Let $z \in \mathbf{R}_+^S$ be the vector s.t. $z(e) = y(e)$ and $z(e') = x(e')$ ($e' \in S \setminus \{e\}$). Since $C_{\mathcal{P}}(x^e) = x \leq z \leq x^e$, the consistency of $C_{\mathcal{P}}$ implies $C_{\mathcal{P}}(z) = x$. Also, the persistence of $C_{\mathcal{P}}$ and $z \leq y$ imply $C_{\mathcal{P}}(y) \wedge z \leq C_{\mathcal{P}}(z) = x$, which yields $(C_{\mathcal{P}}(y))(e) \leq x(e)$ since $(C_{\mathcal{P}}(y))(e) \leq y(e) = z(e)$. \square

3.7 Stable Allocations on Generalized Polymatroids

In this section, we formulate and analyze a stable allocation model on generalized polymatroids. We show that the set of stable allocations, if nonempty, forms a distributive lattice and satisfies a kind of the “rural hospital theorem.”

3.7.1 Model Formulation

Consider two disjoint agent sets I and J , which are interpreted as workers and firms. Let $E = I \times J$ be the set of all worker-firm pairs, and let $E_i = \{(i, j) \mid j \in J\}$ for each $i \in I$ and $E_j = \{(i, j) \mid i \in I\}$ for each $j \in J$.

An *allocation* is a vector $x = (x(i, j) \mid (i, j) \in E) \in \mathbf{R}_+^E$, where $x(i, j)$ represents the amount of contracted labor time of i at j . For an allocation x , we write $x_i := x|_{E_i} = (x(i, j) \mid j \in J)$ for each $i \in I$ and $x_j := x|_{E_j}$ for each $j \in J$. The *profile* of each $k \in I \cup J$ is given by an ordered g-polymatroid (E_k, Q_k, \succ_k) with $Q_k \subseteq \mathbf{R}_+^{E_k}$. Here, Q_k means the set of acceptable vectors of k , and his preference on Q_k is defined based on \succ_k as in Section 3.6.1. The set $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ of profiles is called an *instance*.

Definition 3.51. A vector $x \in \mathbf{R}_+^E$ is a *stable allocation* of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ (or, *stable w.r.t.* $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$) if the following two conditions hold:

1. For every $k \in I \cup J$, $x_k \in Q_k$.
2. For every pair $e = (i, j) \in E$, x_i is \succ_i -optimal in $Q_i \cap [-\infty, x_i^e]$ or x_j is \succ_j -optimal in $Q_j \cap [-\infty, x_j^e]$,

where $x_i^e \in (\mathbf{R}_+ \cup \{+\infty\})^{E_i}$ is defined as $x_i^e(e) = \infty$ and $x_i^e(e') = x_i(e')$ ($e' \in E_i \setminus \{e\}$), and x_j^e is similarly defined. \blacksquare

Condition 1 requires the feasibility of x . Note that the set $Q_i \cap [-\infty, x_i^e]$ in Condition 2 coincides with $\{y_i \in Q_i \mid y_i(e') \leq x_i(e') \text{ } (e' \in E_i \setminus \{e\})\}$. Then, the \succ_i -optimality of x_i in $Q_i \cap [-\infty, x_i^e]$ means that i wishes neither to increase $x_i(e) = x((i, j))$ nor to decrease any component of x_i . Therefore, Condition 2 guarantees that there is no pair such that both of its members have incentives to increase the amount of the contract between them.

Remark 3.52. This model is a generalization of the stable matching model on g-matroids, described in Section 3.4. When each Q_k corresponds to a g-matroid, (i.e., when Q_k is the convex hull of $\{\chi_A \in \{0, 1\}^{E_k} \mid A \in \mathcal{J}_k\}$ for some g-matroid (E_k, \mathcal{J}_k)), we can show that, for any subset $X \subseteq E$, the vector $\chi_X \in \{0, 1\}^E$ is a stable allocation of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ if and only if X is a stable matching of $\{(E_k, \mathcal{J}_k, \succ_k)\}_{k \in I \cup J}$. ■

3.7.2 Choice Function Model

Here, we briefly introduce the results of Alkan and Gale [9], which will be used in the subsequent sections.

In their choice function model, each agent $k \in I \cup J$ has his own choice function $C_k: (\mathbf{R}_+ \cup \{+\infty\})^{E_k} \rightarrow \mathbf{R}_+^{E_k}$ instead of a profile (E_k, Q_k, \succ_k) , and hence an instance is given in the form $\{C_k\}_{k \in I \cup J}$. Recall that, for any $k \in I \cup J$ and $e \in E_k$, a vector $x \in \mathbf{R}_+^{E_k}$ is said to be *e-satiated* for C_k if $(C_k(y))(e) \leq x(e)$ for all $y \geq x$. A vector $x \in \mathbf{R}_+^E$ is said to be a *stable allocation* of $\{C_k\}_{k \in I \cup J}$ if x satisfies the following two conditions:

1. For every $k \in I \cup J$, $C_k(x_k) = x_k$.
2. For every $e = (i, j) \in E$, x_i is *e-satiated* for C_i or x_j is *e-satiated* for C_j .

For two vectors $x, y \in \mathbf{R}_+^E$, define vectors $x \vee_I y$ and $x \wedge_I y$ in \mathbf{R}_+^E by

$$\begin{aligned} (x \vee_I y)|_{E_i} &= C_i(x_i \vee y_i) \in \mathbf{R}_+^{E_i} \quad (i \in I), \\ (x \wedge_I y)|_{E_j} &= C_j(x_j \wedge y_j) \in \mathbf{R}_+^{E_j} \quad (j \in J). \end{aligned} \tag{3.12}$$

That is, $x \vee_I y \in \mathbf{R}_+^E$ is the direct sum of $\{C_i(x_i \vee y_i)\}_{i \in I}$, and $x \wedge_I y \in \mathbf{R}_+^E$ is the direct sum of $\{C_j(x_j \wedge y_j)\}_{j \in J}$.

Theorem 3.53 (Alkan and Gale [9]). *Let \mathfrak{L} be the set of stable allocations of $\{C_k\}_{k \in I \cup J}$. If C_k is consistent, persistent and size-monotone for every $k \in I \cup J$, then \mathfrak{L} satisfies the following properties:*

- (a) $\mathfrak{L} \neq \emptyset$, i.e., there exists a stable allocation.
- (b) The triple $(\mathfrak{L}, \vee_I, \wedge_I)$ is a distributive lattice.
- (c) For every $x, y \in \mathfrak{L}$ and every $k \in I \cup J$, we have $|x_k| = |y_k|$.

3.7.3 Characterization through Lower Extension

In this section, we connect our ordered g-polymatroid model to the choice function model of Alkan and Gale [9] by using lower extensions of g-polymatroids and inductions of choice functions.

Let $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ be an instance of our model, i.e., (E_k, Q_k, \succ_k) is an ordered g-polymatroid with $Q_k \subseteq \mathbf{R}_+^{E_k}$ for each $k \in I \cup J$.

Lemma 3.54. *For $x \in \mathbf{R}_+^E$, the following two conditions are equivalent:*

1. *x is a stable allocation of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$.*
2. *x is a stable allocation of $\{(E_k, L(Q_k), \succ_k)\}_{k \in I \cup J}$ and satisfies $x_k \in Q_k$ for every $k \in I \cup J$.*

Proof. For each $k \in I \cup J$, both conditions require $x_k \in Q_k$, which implies $Q_k \cap [-\infty, x_k^e] \neq \emptyset$. Then, by Lemma 3.42, the \succ_k -optimal points of $Q_k \cap [-\infty, x_k^e]$ and $L(Q_k) \cap [-\infty, x_k^e]$ coincide with each other, and the lemma follows. \square

For each $k \in I \cup J$, the triple $\mathcal{P}_k := (E_k, L(Q_k), \succ_k)$ is an ordered polymatroid by Lemma 3.38. Let $C_{\mathcal{P}_k}: (\mathbf{R}_+ \cup \{+\infty\})^{E_k} \rightarrow \mathbf{R}_+^{E_k}$ be the choice function induced from \mathcal{P}_k for each $k \in I \cup J$. Then, by Observation 3.43 and Lemma 3.50, we obtain the following lemma.

Lemma 3.55. *A vector $x \in \mathbf{R}_+^E$ is a stable allocation of $\{\mathcal{P}_k = (E_k, L(Q_k), \succ_k)\}_{k \in I \cup J}$ if and only if x is a stable allocation of $\{C_{\mathcal{P}_k}\}_{k \in I \cup J}$.*

Combining Lemmas 3.54 and 3.55 gives the following theorem.

Theorem 3.56. *A vector $x \in \mathbf{R}_+^E$ is a stable allocation of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ if and only if x is a stable allocation of $\{C_{\mathcal{P}_k}\}_{k \in I \cup J}$ satisfying $x_k \in Q_k$ for every $k \in I \cup J$.*

3.7.4 Structure of Stable Allocations

Let \mathfrak{S} be the set of stable allocations of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ and let \mathfrak{L} be the set of stable allocations of $\{C_{\mathcal{P}_k}\}_{k \in I \cup J}$, where $\mathcal{P}_k = (E_k, L(Q_k), \succ_k)$. Then, Theorem 3.56 is rephrased as

$$\mathfrak{S} = \{x \in \mathfrak{L} \mid x_k \in Q_k \ (\forall k \in I \cup J)\}. \quad (3.13)$$

Lemma 3.57. *The set \mathfrak{L} is nonempty, and $(\mathfrak{L}, \vee_I, \wedge_I)$ is a distributive lattice, where \vee_I and \wedge_I are defined by (3.12) with C_i and C_j replaced by $C_{\mathcal{P}_i}$ and $C_{\mathcal{P}_j}$, respectively. Also, for any $x, y \in \mathfrak{L}$ and $k \in I \cup J$, we have $|x_k| = |y_k|$.*

Proof. By Lemmas 3.45, 3.46 and 3.47, each choice function $C_{\mathcal{P}_k}$ is consistent, persistent and size-monotone. Then, Theorem 3.53 yields the claim. \square

Lemma 3.58. *For any $x, y \in \mathfrak{L}$ and $k \in I \cup J$, we have $x_k \in Q_k$ if and only if $y_k \in Q_k$.*

Proof. Take any $x, y \in \mathfrak{L}$. By Lemma 3.57, we have $x \vee_I y, x \wedge_I y \in \mathfrak{L}$, and hence $|(x \vee_I y)_i| = |x_i| = |y_i|$ ($i \in I$) and $|(x \wedge_I y)_j| = |x_j| = |y_j|$ ($j \in J$). By the definitions of \vee_I and \wedge_I , these imply $|C_{\mathcal{P}_k}(x_k \vee y_k)| = |x_k| = |y_k|$ for every $k \in I \cup J$. Then, the claim follows from Lemma 3.48. \square

Combining (3.13) and Lemma 3.58 yields the following dichotomy theorem.

Theorem 3.59. *We have $\mathfrak{S} = \mathfrak{L}$ or $\mathfrak{S} = \emptyset$.*

Corollary 3.60. *If the set \mathfrak{S} of stable allocations of $\{(E_k, Q_k, \succ_k)\}_{k \in I \cup J}$ is nonempty, then $(\mathfrak{S}, \vee_I, \wedge_I)$ is a distributive lattice, where \vee_I and \wedge_I are defined by (3.12) with C_i and C_j replaced by $C_{\mathcal{D}_i}$ and $C_{\mathcal{D}_j}$, respectively. Also, for any $x, y \in \mathfrak{S}$ and $k \in I \cup J$, we have $|x_k| = |y_k|$.*

Remark 3.61. Lemma 3.49 and Corollary 3.60 imply a kind of “rural hospital theorem” as follows. Let $f_k : 2^{E_k} \rightarrow \mathbf{R}$ be the upper bound function of Q_k for each $k \in I \cup J$. For any $e \in E_k$ and $x \in \mathfrak{S}$, if every $A \subseteq E_k$ with $e \in A$ satisfies $x_k(A) < f_k(A)$, then, by Lemma 3.49, we have $x_k(e) = y_k(e)$ for every $y \in \mathfrak{S}$. Combined with the second claim of Corollary 3.60, this means a generalization of the “rural hospital theorem.” ■

Remark 3.62. The discrete-concave function model, studied in [24, 37, 38, 81] is a stable allocation model on integer vectors, where preferences of agents are represented by M^\natural -concave functions [78]. An M^\natural -concave function is a kind of discrete concave function defined on an integer g-polymatroid. If an M^\natural -concave function is defined on an integer polymatroid, then it induces a choice function with consistency, persistence, and size monotonicity [37, 81]. M^\natural -concavity is also defined for real variable functions and an M^\natural -concave function on a polymatroid also induces a consistent, persistent, and size monotone choice function [76]. Lemmas 3.45, 3.46 and 3.47 can be regarded as special cases of this fact by setting an appropriate M^\natural -concave function on a given polymatroid.

In [24, 37, 38, 81], algorithmic and structural results are provided assuming that M^\natural -concave functions are defined on integer polymatroids. We can expect to extend such results to the model with M^\natural -concave functions on generalized polymatroids, which also generalizes our models in this chapter. ■

Appendix 3.A Operations on Generalized Matroids

Let S be a finite set. We see that the pair $(S, 2^S)$ is obviously a g-matroid. Let us call it the *free g-matroid* on S . In this section, we introduce some basic operations on g-matroids. By combining them, one can construct various g-matroids from free g-matroids.

Proposition 3.63 (Restriction [103, Lemma 2.7]). *For a g-matroid (S, \mathcal{J}) and any subset $T \subseteq S$, define*

$$\mathcal{J}_T = \{X \cap T \mid X \in \mathcal{J}\}.$$

Then, (T, \mathcal{J}_T) is a g-matroid if $\mathcal{J}_T \neq \emptyset$.

Proposition 3.64 (Truncation [103, Corollary 2.7]). *For a g-matroid (S, \mathcal{J}) and any $k, l \in \mathbf{Z}_+$ with $k \leq l$, define*

$$\mathcal{J}_k^l = \{X \mid X \in \mathcal{J}, k \leq |X| \leq l\}.$$

Then, (S, \mathcal{J}_k^l) is a g -matroid if $\mathcal{J}_k^l \neq \emptyset$.

Proposition 3.65 (Direct Sum). For g -matroids $(S_1, \mathcal{J}_1), \dots, (S_k, \mathcal{J}_k)$ on disjoint ground sets S_1, \dots, S_k , define

$$S = S_1 \cup \dots \cup S_k,$$

$$\mathcal{J} = \{ X \subseteq S \mid X \cap S_i \in \mathcal{J}_i \ (\forall i \in [k]) \}.$$

Then, (S, \mathcal{J}) is a g -matroid. ■

Next, we show that a g -matroid induces another g -matroid via matchings on a bipartite graph. We need the following lemmas.

Lemma 3.66 ([103, Theorem 2.9]). A pair (S, \mathcal{J}) is a g -matroid if and only if \mathcal{J} is written in the form

$$\mathcal{J} = \{ B \cap S \mid B \in \mathcal{B} \}$$

for the base family \mathcal{B} of some matroid whose ground set is a superset of S .

Lemma 3.67. Let $G = (S, T; E)$ be a bipartite graph with vertex classes S, T and edge set E . Let (T, \mathcal{I}_T) be a matroid on T and denote by \mathcal{B}_T its base family. Then, the family defined by

$$\mathcal{B}_S = \{ \partial M \cap S \mid M : \text{matching in } G, \partial M \cap T \in \mathcal{B}_T \},$$

is the base family of a matroid on S if $\mathcal{B}_S \neq \emptyset$.

Proof. The family $\mathcal{I}_S = \{ \partial M \cap S \mid M : \text{matching in } G, \partial M \cap T \in \mathcal{I}_T \}$ is the independent set family of a matroid on S (see Section 2.3). Also, if a matching $M \subseteq E$ satisfies $\partial M \cap T \in \mathcal{B}_T$, then $\partial M \cap S$ is a maximal member of \mathcal{I}_S , i.e., a base of this matroid. \square

We are now ready to show the following proposition.

Proposition 3.68 (Induction by Bipartite Graphs). Let $G = (S, T; E)$ be a bipartite graph with vertex classes S, T and edge set E . Let (T, \mathcal{J}_T) be a g -matroid and define a family \mathcal{J}_S of subsets of S by

$$\mathcal{J}_S = \{ \partial M \cap S \mid M : \text{matching in } G, \partial M \cap T \in \mathcal{J}_T \}.$$

Then, (S, \mathcal{J}_S) is a g -matroid if $\mathcal{J}_S \neq \emptyset$.

Proof. By Lemma 3.66, there is a matroid $(\hat{T}, \mathcal{I}_{\hat{T}})$ s.t. $\hat{T} = T \cup U$ ($T \cap U = \emptyset$) for some set U and its base family $\mathcal{B}_{\hat{T}}$ satisfies $\mathcal{J}_T = \{ B \cap T \mid B \in \mathcal{B}_{\hat{T}} \}$. Let U' be a copy of U and let \hat{G} be a bipartite graph with vertex classes $\hat{S} = S \cup U'$ and $\hat{T} = T \cup U$, and edge set $\hat{E} = E \cup \{ (u', u) \in U' \times U \mid u \in U \}$.

For any matching M in G with $\partial M \cap T \in \mathcal{J}_T$, there is $B \in \mathcal{B}_{\hat{T}}$ with $B \cap T = \partial M \cap T$. Then $\hat{M} := M \cup \{ (u', u) \mid u \in B \cap U \}$ is a matching in \hat{G} with $\partial \hat{M} \cap \hat{T} \in \mathcal{B}_{\hat{T}}$

and $(\partial\hat{M} \cap \hat{S}) \cap S = \partial M \cap S$. On the other hand, for any matching \hat{M} in \hat{G} with $\partial\hat{M} \cap \hat{T} \in \mathcal{B}_T$, $M := \hat{M} \cap E$ satisfies $\partial M \cap T \in \mathcal{J}_T$ and $\partial M \cap S = (\partial\hat{M} \cap \hat{S}) \cap S$. Hence, $\mathcal{J}_S = \{B \cap S \mid B \in \mathcal{B}_{\hat{S}}\}$ holds where

$$\mathcal{B}_{\hat{S}} = \{ \partial\hat{M} \cap \hat{S} \mid \hat{M} : \text{matching in } \hat{G}, \partial\hat{M} \cap \hat{T} \in \mathcal{B}_{\hat{T}} \}.$$

Then, $\mathcal{J}_S \neq \emptyset$ implies $\mathcal{B}_{\hat{S}} \neq \emptyset$. By Lemma 3.67, $\mathcal{B}_{\hat{S}}$ is the base family of a matroid on \hat{S} , and hence (S, \mathcal{J}_S) is a g-matroid by Lemma 3.66. \square

Appendix 3.B Examples of Generalized Matroids

Here we construct two examples of g-matroids. Propositions 3.69 and 3.70 respectively show that Examples 3.1 and 3.2 indeed provide g-matroids.

Proposition 3.69 (Laminar Family). *For a finite set S , let $\mathcal{F} \subseteq 2^S$ be a laminar family and $f, g: \mathcal{F} \rightarrow \mathbf{Z}_+$ be set functions. Define a family \mathcal{J} of subsets of S by (3.1) in Example 3.1. Then, (S, \mathcal{J}) is a g-matroid if $\mathcal{J} \neq \emptyset$.*

Proof. For any member $F \in \mathcal{F}$, define $\mathcal{F}_F := \{H \mid H \in \mathcal{F}, H \subseteq F\}$ and

$$\mathcal{J}_F := \{X \subseteq F : g(H) \leq |X \cap H| \leq f(H) \ (\forall H \in \mathcal{F}_F)\}.$$

Let us call a subset $H \in \mathcal{F}$ a *child* of $F \in \mathcal{F}$ if $H \subsetneq F$ and there is no $H' \in \mathcal{F}$ such that $H \subsetneq H' \subsetneq F$. We will show the following two claims.

- (i) If $F \in \mathcal{F}$ has no child, then (F, \mathcal{J}_F) is a g-matroid.
- (ii) Assume that $F \in \mathcal{F}$ has just k children $H_1, H_2, \dots, H_k \in \mathcal{F}_F$ and (H_i, \mathcal{J}_{H_i}) is a g-matroid for each $i \in [k]$. Then, (F, \mathcal{J}_F) is a g-matroid if $\mathcal{J}_F \neq \emptyset$.

Using these claims, the proposition is shown by induction. Then, what is left is to show the claims. Claim (i) is easily shown by applying Proposition 3.64 for the uniform g-matroid $(F, 2^F)$. We now show Claim (ii). Let $H_{k+1} := F \setminus (H_1 \cup H_2 \cup \dots \cup H_k)$ and define a family $\tilde{\mathcal{J}}_F$ of subsets of F by

$$\tilde{\mathcal{J}}_F = \{X \subseteq F \mid X \cap H_i \in \mathcal{J}_{H_i} \ (\forall i \in [k])\}.$$

Then, $(F, \tilde{\mathcal{J}}_F)$ is the direct sum of $\{(H_i, \mathcal{J}_{H_i})\}_{i \in [k]}$ and $(H_{k+1}, 2^{H_{k+1}})$, and hence is a g-matroid by Proposition 3.65. Also, by definition, we have

$$\mathcal{J}_F = \{X \in \tilde{\mathcal{J}}_F \mid g(F) \leq |X| \leq f(F)\}.$$

Then, by Proposition 3.64, (F, \mathcal{J}_F) is a g-matroid if $\mathcal{J}_F \neq \emptyset$. \square

Proposition 3.70 (Assignment to Divisions). *Let S and D be finite sets. For each $d \in D$, take a subset $\Gamma(d) \subseteq S$ and integers $p(d), q(d) \in \mathbf{Z}_+$ with $p(d) \leq q(d)$. Define a family \mathcal{J} of subsets of S by (3.2) in Example 3.2. Then, (S, \mathcal{J}) is a g-matroid if $\mathcal{J} \neq \emptyset$.*

Proof. Let $T := S \times D$ and $T_d := \{(s, d) \mid s \in S\}$ for each $d \in D$. Define a family \mathcal{J}_T of subsets of T by

$$\mathcal{J}_T = \{ X \subseteq T \mid p(d) \leq |X \cap T_d| \leq q(d) \ (\forall d \in D) \}.$$

Since $\{T_d\}_{d \in D}$ is a laminar family, Proposition 3.69 implies that (T, \mathcal{J}_T) is a g-matroid. Let $G = (S, T; E)$ be a bipartite graph with E defined by $E = \{(s, (s, d)) \in S \times T \mid s \in \Gamma(d)\}$ and define $\mathcal{J}_S \subseteq 2^S$ by

$$\mathcal{J}_S = \{ \partial M \cap S \mid M : \text{matching in } G, \partial M \cap T \in \mathcal{J}_T \}.$$

Then, by Proposition 3.68, (S, \mathcal{J}_S) is a g-matroid if $\mathcal{J}_S \neq \emptyset$. We complete the proof by showing that \mathcal{J}_S coincides with \mathcal{J} defined by (3.2), i.e., we now show that $X \in \mathcal{J}$ if and only if $X \in \mathcal{J}_S$ for any $X \subseteq S$.

If $X \in \mathcal{J}$, there is $\pi : X \rightarrow D$ satisfying (i) $\pi^{-1}(d) \subseteq \Gamma(d)$ and (ii) $p(d) \leq |\pi^{-1}(d)| \leq q(d)$ ($\forall d \in D$). Let $M_\pi := \{(s, (s, d)) \mid s \in X, \pi(s) = d\}$. Then, (i) implies $M_\pi \subseteq E$ and (ii) implies $p(d) \leq |\partial M_\pi \cap T_d| \leq q(d)$ for each $d \in D$, and hence $\partial M_\pi \cap T \in \mathcal{J}_T$. Thus, $X = \partial M_\pi \cap S \in \mathcal{J}_S$.

If $X \in \mathcal{J}_S$, there is a matching $M \subseteq E$ with $\partial M \cap S = X$, $\partial M \cap T \in \mathcal{J}_T$. Define $\pi_M : X \rightarrow D$ by letting $\pi_M(s)$ be the unique $d \in D$ satisfying $(s, (s, d)) \in M$ for each $s \in X$. Then, conditions (i) and (ii) follow from $M \subseteq E$ and $\partial M \cap T \in \mathcal{J}_T$, respectively. Thus, $X \in \mathcal{J}$. \square

Appendix 3.C Computations of Induced Choice Functions

Let (S, \mathcal{J}, \succ) be an ordered g-matroid and let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$. We show that if (S, \mathcal{J}) is given as in Example 3.1 (Laminar Family) or Example 3.2 (Assignment to Divisions) (i.e., as in Proposition 3.69 or 3.70), one can efficiently compute $C_{\mathcal{M}}(X)$ for any $X \subseteq S$. We reduce the computation of $C_{\mathcal{M}}(X)$ to the *minimum cost circulation problem*, which is defined as follows.

Let $G = (V, A)$ be a directed graph with vertex set V and arc set A . Each arc $a \in A$ has lower and upper capacities $l(a), u(a) \in \mathbf{R}$ and a cost $c(a) \in \mathbf{R}$. For this network (G, l, u, c) , a flow $\varphi : A \rightarrow \mathbf{R}$ is called a *feasible circulation* if it satisfies

$$\sum_{t \in V : (t, v) \in A} \varphi((t, v)) - \sum_{t \in V : (v, t) \in A} \varphi((v, t)) = 0 \quad \text{for each } v \in V,$$

$$l(a) \leq \varphi(a) \leq u(a) \quad \text{for each } a \in A.$$

The cost of flow φ is defined as $\langle c, \varphi \rangle := \sum_{a \in A} c(a) \cdot \varphi(a)$. The *minimum cost circulation problem* is to find a feasible circulation with the minimum cost. For this problem, Tardos [104] found the first strongly polynomial algorithm, and Orlin [82] provided the current best algorithm which runs in $O(m \log n(m + n \log n))$ time where $m := |V|$ and $n := |A|$. Also, one can find an integer solution when l and u are integer-valued.

We prepare the following lemma for the reduction. For a weight function $w : S \rightarrow \mathbf{R}$ on a finite set S , we write $w(Y) := \sum_{e \in Y} w(e)$ for any $Y \subseteq S$.

Lemma 3.71. *Let (S, \mathcal{J}, \succ) be an ordered g -matroid and let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$. Take an arbitrary positive weight $w : S \rightarrow \mathbf{R}_{>0}$ such that $e \succ e'$ implies $w(e) > w(e')$. Then, any solution $Z^* \in \mathcal{J}$ to $\max \{w(X \cap Z) \mid Z \in \mathcal{J}\}$ satisfies $X \cap Z^* = C_{\mathcal{M}}(X)$.*

Proof. By Proposition 3.9, $C_{\mathcal{M}}(X)$ is the unique solution to $\max \{w(Y) \mid Y \subseteq X, Y \in L(\mathcal{J})\}$. Note that, for $Y \subseteq X$ with $Y \in L(\mathcal{J})$, there is $Z \in \mathcal{J}$ with $Y \subseteq Z$. Then, $X \cap Z$ satisfies $X \cap Z \subseteq X$, $X \cap Z \in L(\mathcal{J})$. Also, $w(Y) \leq w(X \cap Z)$ since $Y \subseteq X \cap Z$ and w is positive. Therefore, $\max \{w(Y) \mid Y \subseteq X, Y \in L(\mathcal{J})\}$ is equivalent to $\max \{w(X \cap Z) \mid Z \in \mathcal{J}\}$, and the lemma follows. \square

Laminar Family (Example 3.1)

Proposition 3.72. *Let (S, \mathcal{J}, \succ) be an ordered g -matroid where \mathcal{J} is defined as (3.1) by a laminar family $\mathcal{F} \subseteq 2^S$ and $f, g : \mathcal{F} \rightarrow \mathbf{Z}_+$. Let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$. Then, for any $X \subseteq S$, one can compute $C_{\mathcal{M}}(X)$ in time polynomial in $|S|$.*

Proof. Take $w : S \rightarrow \mathbf{R}_{>0}$ such that $e \succ e'$ implies $w(e) > w(e')$. To obtain $C_{\mathcal{M}}(X)$, by Lemma 3.71, it suffices to solve $\max \{w(X \cap Z) \mid Z \in \mathcal{J}\}$. This is equivalent to $\min \{w'(Z) \mid Z \in \mathcal{J}\}$ if $w' : S \rightarrow \mathbf{R}$ is defined by $w'(e) = -w(e)$ for $e \in X$ and $w'(e) = 0$ for $e \in S \setminus X$. We now solve $\min \{w'(Z) \mid Z \in \mathcal{J}\}$ through the minimum cost circulation problem.

Let $G = (V, A)$ be a directed graph with V and A given by

$$\begin{aligned} V &= \{v^+\} \cup V_S \cup V_{\mathcal{F}} \cup \{v^-\} \\ A &= A_0 \cup A_1 \cup A_2 \cup A_3 \cup \{(v^-, v^+)\} \end{aligned}$$

where $V_S = \{v_e \mid e \in S\}$, $V_{\mathcal{F}} = \{v_F \mid F \in \mathcal{F}\}$ and

$$\begin{aligned} A_0 &= \{(v^+, v_e) \in \{v^+\} \times V_S \mid e \in S\}, \\ A_1 &= \{(v_e, v_F) \in V_S \times V_{\mathcal{F}} \mid e \in F \text{ and } F \text{ has no child}\}, \\ A_2 &= \{(v_F, v_H) \in V_{\mathcal{F}} \times V_{\mathcal{F}} \mid F \text{ is a child of } H\}, \\ A_3 &= \{(v_F, v^-) \in V_{\mathcal{F}} \times \{v^-\} \mid F \text{ is inclusionwise maximal in } \mathcal{F}\}. \end{aligned}$$

Let $l(a) = 0$, $u(a) = 1$ for each $a \in A_0 \cup A_1$ and let $l((v_F, v)) = g(F)$, $u((v_F, v)) = f(F)$ for each $(v_F, v) \in A_2 \cup A_3$, and let $l((v^-, v^+)) = 0$, $u((v^-, v^+)) = +\infty$. Also, let $c((v^+, v_e)) =$

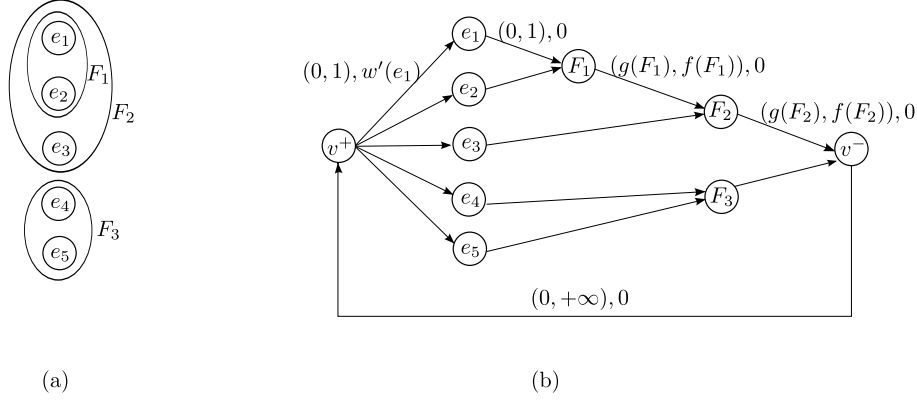


Figure 3.1: (a) An example of a laminar family and (b) the associated network.

$w'(e)$ for each $(v^+, v_e) \in A_0$ and $c(a) = 0$ for each $a \in A \setminus A_0$. In Figure 3.1, (a) represents an example of a laminar family and (b) represents the associated network, in which the numbers attached to an arc $a \in A$ represent $(l(a), u(a)), c(a)$.

For an integer feasible circulation $\varphi : A \rightarrow \mathbf{Z}$, define $Z_\varphi := \{e \in S \mid \varphi((v^+, v_e)) = 1\}$. Then, $g(F) \leq |Z_\varphi \cap F| \leq f(F)$ holds for every $F \in \mathcal{F}$, i.e., $Z_\varphi \in \mathcal{J}$, and we have $w'(Z_\varphi) = \langle c, \varphi \rangle$. Also, every member of \mathcal{J} can be obtained in this way from an integer feasible circulation. Therefore, by finding an integer solution $\varphi^* : A \rightarrow \mathbf{Z}$ to the minimum cost circulation problem of this network, we can obtain the solution Z_{φ^*} to $\min \{w'(Z) \mid Z \in \mathcal{J}\}$, which satisfies $Z_{\varphi^*} \cap X = C_{\mathcal{M}}(X)$.

Note that the laminar family \mathcal{F} has at most $2|S| - 1$ members. By the construction of the network, $|V| = O(|S|)$ and $|A| = O(|S|)$. Then, the problem is solved in time polynomial in $|S|$. For this network, in particular, one can find an integer solution in $O(|S|^2 \log |S|)$ using the successive shortest path algorithm (see, e.g., [6]). \square

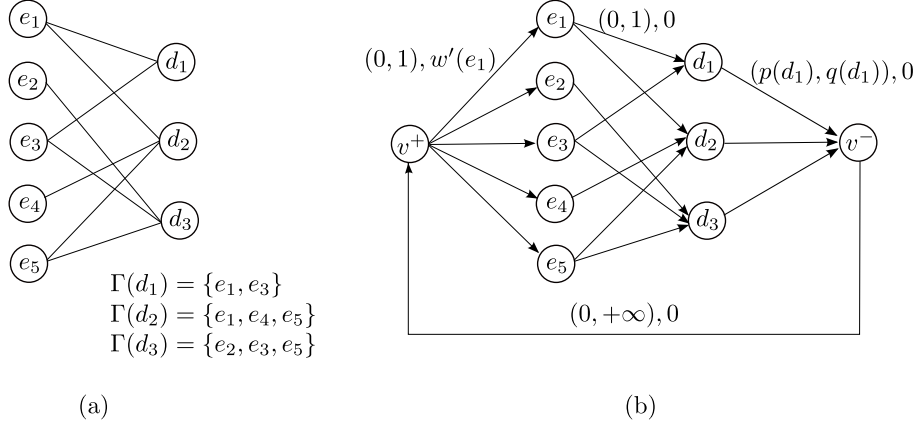
Assignment to Divisions (Example 3.2)

Proposition 3.73. *Let (S, \mathcal{J}, \succ) be an ordered g -matroid where \mathcal{J} is defined as (3.2) by a finite set D and functions $\Gamma : D \rightarrow 2^S$ and $p, q : D \rightarrow \mathbf{Z}_+$. Let $\mathcal{M} := (S, L(\mathcal{J}), \succ)$. Then, for any $X \subseteq S$, one can compute $C_{\mathcal{M}}(X)$ in time polynomial in $|S|$ and $|D|$.*

Proof. Define $w' : S \rightarrow \mathbf{R}$ similarly to the proof of Proposition 3.72. Then, it suffices to solve $\min \{w'(Z) \mid Z \in \mathcal{J}\}$, whose solution Z^* satisfies $Z^* \cap X = C_{\mathcal{M}}(X)$. We reduce this problem to the minimum cost circulation problem.

Let $G = (V, A)$ be a directed graph with V and A given by

$$\begin{aligned} V &= \{v^+\} \cup V_S \cup V_D \cup \{v^-\} \\ A &= A_0 \cup A_1 \cup A_2 \cup \{(v^-, v^+)\} \end{aligned}$$


 Figure 3.2: (a) An example of a function Γ and (b) the associated network.

where $V_S = \{v_e \mid e \in S\}$, $V_D = \{v_d \mid d \in D\}$ and

$$\begin{aligned} A_0 &= \{(v^+, v_e) \in \{v^+\} \times V_S \mid e \in S\}, \\ A_1 &= \{(v_e, v_d) \in V_S \times V_D \mid e \in \Gamma(d)\}, \\ A_2 &= \{(v_d, v^-) \in V_D \times \{v^-\} \mid d \in D\}. \end{aligned}$$

Let $l(a) = 0$, $u(a) = 1$ for each $a \in A_0 \cup A_1$ and let $l((v_d, v^-)) = p(d)$, $u((v_d, v^-)) = q(d)$ for each $(v_d, v^-) \in A_2$, and let $l((v^-, v^+)) = 0$, $u((v^-, v^+)) = +\infty$. Also, let $c((v^+, v_e)) = w'(e)$ for each $(v^+, v_e) \in A_0$ and $c(a) = 0$ for each $a \in A \setminus A_0$. In Figure 3.2, (a) and (b) respectively represent an example of Γ and the associated network, in which $(l(a), u(a)), c(a)$ are attached to an arc $a \in A$.

We can observe that integer feasible circulations and independent sets of \mathcal{J} have one-to-one correspondence as in the proof of 3.72. Therefore, by finding an integer solution $\varphi^* : A \rightarrow \mathbf{Z}$ to the minimum cost circulation problem of this network, we can solve $\min \{w'(Z) \mid Z \in \mathcal{J}\}$.

Since $|V| = O(|S| + |D|)$ and $|A| = O(|S| \cdot |D|)$, an integer minimum cost circulation can be found in time polynomial in $|S|$ and $|D|$. In particular, for this network one can find it in $O(|S|^2|D| + |S|^2 \log |S| + |S| \cdot |D| \log |D|)$ time using the successive shortest path algorithm (see, e.g., [6]). \square

We also show that, if (S, \mathcal{J}) is given as in Examples 3.1 or 3.2, one can efficiently check whether a given subset is contained in \mathcal{J} or not.

Proposition 3.74. *Let (S, \mathcal{J}) be a g -matroid and take $X \subseteq S$ arbitrarily. One can check whether $X \in \mathcal{J}$ or not in time polynomial in $|S|$ if \mathcal{J} is defined by (3.1), and in time polynomial in $|S|$ and $|D|$ if \mathcal{J} is defined by (3.2).*

Proof. In the case of (3.1), by definition, it is enough to check $2|\mathcal{F}| = O(|S|)$ inequalities: $g(F) \leq |X \cap F| \leq f(F)$ ($\forall F \in \mathcal{F}$).

In the case of (3.2), we use the reduction to the feasible circulation problem. Construct a network as in the proof of Proposition 3.73 except that capacities on A_0 are now defined as $l((v^+, v_e)) = u((v^+, v_e)) = 1$ for each $e \in X$ and $l((v^+, v_e)) = u((v^+, v_e)) = 0$ for each $e \in S \setminus X$. Then, we see that $X \in \mathcal{J}$ holds if and only if there exists a feasible circulation on this network. It is known that the existence test of a feasible circulation is solvable in time polynomial in $|V|$ and $|A|$ (see, e.g., [6]), and thus the proof is completed. \square

Appendix 3.D Supplements for Matroid Kernels

For the sake of self-containedness, we provide Fleiner's proofs of Theorems 3.14–3.16.

Let $\mathcal{M}_1 = (S, \mathcal{I}_1, \succ_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2, \succ_2)$ be two ordered matroids on the same ground set S . Also, let $\mathbf{M}_1 := (S, \mathcal{I}_1)$, $\mathbf{M}_2 := (S, \mathcal{I}_2)$. For each $i \in \{1, 2\}$ and $X \in \mathcal{I}_i$, define a superset $D_i(X) \subseteq S$ of X by

$$D_i(X) = X \cup \{e \in S \setminus X \mid e \text{ is dominated by } X \text{ with respect to } \mathcal{M}_i\}.$$

Then, we see that an $\mathcal{M}_1\mathcal{M}_2$ -kernel, introduced in Section 3.3.3, is identical to a subset $X \subseteq S$ satisfying

$$\begin{aligned} X &\in \mathcal{I}_1 \cap \mathcal{I}_2, \\ D_1(X) \cup D_2(X) &= S. \end{aligned}$$

For each $i \in \{1, 2\}$, let $C_{\mathcal{M}_i} : 2^S \rightarrow 2^S$ be defined as in Section 3.3.2. We abbreviate $C_{\mathcal{M}_i}$ to C_i in this section. Then Corollary 3.10 implies

$$D_i(C_i(X)) \supseteq X \quad (\forall X \subseteq S). \quad (3.14)$$

We also see that each C_i fulfills the property called *substitutability* (see Example 5.5 in Chapter 5 for the proof), i.e., it satisfies

$$X \subseteq Y \subseteq S \implies X \setminus C_i(X) \subseteq Y \setminus C_i(Y).$$

This condition is equivalent to

$$X \subseteq Y \subseteq S \implies C_i(Y) \cap X \subseteq C_i(X) \quad (3.15)$$

and implies

$$C_i(D_i(X)) = X \quad (\forall X \in \mathcal{I}_i). \quad (3.16)$$

By the definition of C_i in Section 3.3.2, we can observe that C_i is *consistent*, i.e.,

$$C_i(X) \subseteq Y \subseteq X \implies C_i(Y) = C_i(X).$$

Combining the substitutability and the consistency of C_i yields the *path-independence*, i.e., for every $X, Y \subseteq S$, we have

$$C_i(X \cup Y) = C_i(C_i(X) \cup Y) = C_i(C_i(X) \cup C_i(Y)).$$

Since $|C_i(X)| = r_{\mathbf{M}_i}(X)$ for every $X \subseteq S$, we see that C_i is *size-monotone*, i.e.,

$$X \subseteq Y \subseteq S \implies |C_i(X)| \leq |C_i(Y)|.$$

For each $i \in \{1, 2\}$, define $R_i : 2^S \rightarrow 2^S$ by $R_i(X) = X \setminus C_i(X)$. Then, the substitutability of C_i implies that R_i is *monotone*, i.e., $X \subseteq Y \subseteq S$ implies $R_i(X) \subseteq R_i(Y)$. Let $T : 2^S \rightarrow 2^S$ be a function defined by $T(X) = S \setminus R_2(S \setminus R_1(X))$. Then, the monotonicity of R_1 and R_2 imply that T is also monotone, i.e.,

$$X \subseteq Y \subseteq S \implies T(X) \subseteq T(Y).$$

We first show that any fixed point of T gives an $\mathcal{M}_1\mathcal{M}_2$ -kernel.

Proposition 3.75. *If $Y \subseteq S$ is a fixed point of T , i.e., $T(Y) = Y$, then $C_1(Y)$ is an $\mathcal{M}_1\mathcal{M}_2$ -kernel. Also, if $X \subseteq S$ is an $\mathcal{M}_1\mathcal{M}_2$ -kernel, then $D_1(X)$ is a fixed point of T .*

Proof. Define $U(Y) = S \setminus R_1(Y)$. Then, $T(Y) = S \setminus R_2(U(Y)) = (S \setminus U(Y)) \cup C_2(U(Y))$. Since $S \setminus U(Y)$ and $C_2(U(Y))$ are disjoint, $\{S \setminus U(Y), C_2(U(Y))\}$ is a partition of $T(Y)$. As $S \setminus U(Y) \subseteq Y$, then the condition $T(Y) = Y$ is equivalent to $Y \setminus (S \setminus U(Y)) = C_2(U(Y))$. By $Y \setminus (S \setminus U(Y)) = C_1(Y)$, this is also equivalent to $C_1(Y) = C_2(U(Y))$.

To show the first claim, suppose $T(Y) = Y$, i.e., $C_1(Y) = C_2(U(Y))$. By the definition of C_i , we have $\mathcal{I}_1 \ni C_1(Y) = C_2(U(Y)) \in \mathcal{I}_2$, and hence $C_1(Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$. Also, (3.14) implies $D_1(C_1(Y)) \supseteq Y$ and $D_2(C_1(Y)) = D_2(C_2(U(Y))) \supseteq U(Y) = S \setminus R_1(Y) \supseteq S \setminus Y$, which imply $D_1(C_1(Y)) \cup D_2(C_1(Y)) = S$.

For the second claim, suppose that X is an $\mathcal{M}_1\mathcal{M}_2$ -kernel. By $X \in \mathcal{I}_1 \cap \mathcal{I}_2$, (3.16) implies $C_i(D_i(X)) = C_i(X) = X$ for each $i \in \{1, 2\}$. We also have $D_1(X) \cup D_2(X) = S$, which implies $S \setminus D_1(X) \subseteq D_2(X)$. Then,

$$U(D_1(X)) = S \setminus R_1(D_1(X)) = (S \setminus D_1(X)) \cup C_1(D_1(X)) = (S \setminus D_1(X)) \cup X \subseteq D_2(X),$$

and also $X \subseteq U(D_1(X))$. By $X = C_2(D_2(X))$, then $C_2(D_2(X)) \subseteq U(D_1(X)) \subseteq D_2(X)$. By the consistency of C_2 , this implies $C_2(U(D_1(X))) = C_1(D_1(X))$. Hence, $D_1(X)$ satisfies $T(D_1(X)) = D_1(X)$. \square

Recall that an $\mathcal{M}_1\mathcal{M}_2$ -kernel X^* is said to be \mathcal{M}_1 -optimal if it satisfies $C_1(X^* \cup X) = X^*$ for every $\mathcal{M}_1\mathcal{M}_2$ -kernel X .

Proposition 3.76. *An $\mathcal{M}_1\mathcal{M}_2$ -kernel X^* is \mathcal{M}_1 -optimal if and only if $D_1(X^*) \supseteq D_1(X)$ holds for every $\mathcal{M}_1\mathcal{M}_2$ -kernel X .*

Proof. Take an arbitrary $\mathcal{M}_1\mathcal{M}_2$ -kernel X . It suffices to show that $D_1(X^*) \supseteq D_1(X)$ holds if and only if $C_1(X^* \cup X) = X^*$.

Assume $D_1(X^*) \supseteq D_1(X)$, which implies $D_1(X^*) \supseteq X$. Since C_1 is path-independent and $X^* = C_1(D_1(X^*))$, we obtain $C_1(X^* \cup X) = C_1(D_1(X^*) \cup X) = C_1(D_1(X^*)) = X^*$, where we use (3.16) for the last equality.

Conversely, assume $C_1(X^* \cup X) = X^*$. Since C_1 is path-independence and $X = C_1(D_1(X))$, we have $D_1(X^*) = D_1(C_1(X^* \cup X)) = D_1(C_1(X^* \cup D_1(X))) \supseteq X^* \cup D_1(X)$, where we use (3.14) for the last inclusion. Thus, we obtain $D_1(X^*) \supseteq D_1(X)$. \square

Now we are ready to provide the proof of Theorem 3.14, which states the existence of an $\mathcal{M}_1\mathcal{M}_2$ -kernel and that we can find the \mathcal{M}_1 -optimal one efficiently.

Proof of Theorem 3.14. Since T is monotone, we have $S \supseteq T(S) \supseteq T(T(S)) \supseteq \dots$, and hence $T^{k+1}(S) = T^k(S)$ for some $k \leq |S|$. This means that $T^k(S)$ is a fixed point of T . By Proposition 3.75, then $X^* := C_1(T^k(S))$ is an $\mathcal{M}_1\mathcal{M}_2$ -kernel. By this definition, X^* can be found in $O(|S| \cdot \text{EO}_{\mathcal{M}_1\mathcal{M}_2})$ time, where $\text{EO}_{\mathcal{M}_1\mathcal{M}_2}$ is the time required to compute $C_1(X)$, $C_2(X)$ for any subset X of S .

We now show the \mathcal{M}_1 -optimality of X^* . Take an arbitrary $\mathcal{M}_1\mathcal{M}_2$ -kernel X . By Proposition 3.76, it suffices to show $D_1(X^*) \supseteq D_1(X)$. Since X is a kernel, by the second claim of Proposition 3.75, $D_1(X)$ is a fixed point of T , i.e., $T(D_1(X)) = D_1(X)$. By the monotonicity of T , we have $T^k(S) \supseteq T^k(D_1(X)) = D_1(X)$. On the other hand, (3.14) implies $D_1(X^*) = D_1(C_1(T^k(S))) \supseteq T^k(S)$. Thus, $D_1(X^*) \supseteq D_1(X)$. \square

As in Section 3.3.3, let $\mathfrak{K}_{\mathcal{M}_1\mathcal{M}_2}$ be the set of all $\mathcal{M}_1\mathcal{M}_2$ -kernels. To show Theorems 3.15–3.16, we prepare the following proposition. The proof is based on that of Fleiner [26, 27] and Alkan [8].

Proposition 3.77. *For any kernels $X, Y \in \mathfrak{K}_{\mathcal{M}_1\mathcal{M}_2}$, the following (a)–(c) hold.*

- (a) $C_1(X \cup Y) = C_2(D_2(X) \cap D_2(Y))$, $C_2(X \cup Y) = C_1(D_1(X) \cap D_1(Y))$.
- (b) $\text{span}_{\mathbf{M}_1}(X) = \text{span}_{\mathbf{M}_1}(Y)$, $\text{span}_{\mathbf{M}_2}(X) = \text{span}_{\mathbf{M}_2}(Y)$.
- (c) $C_1(X \cup Y) \cup C_2(X \cup Y) = X \cup Y$, $C_1(X \cup Y) \cap C_2(X \cup Y) = X \cap Y$

Proof. We first show properties (a) and (b). By symmetry of C_1 and C_2 , it suffices to show $C_1(X \cup Y) = C_2(D_2(X) \cap D_2(Y))$ and $\text{span}_{\mathbf{M}_1}(X) = \text{span}_{\mathbf{M}_1}(Y)$.

Let $Z := (X \cup Y) \cap D_2(X) \cap D_2(Y)$. By (3.15) and (3.16),

$$X \cap (D_2(X) \cap D_2(Y)) = C_2(D_2(X)) \cap (D_2(X) \cap D_2(Y)) \subseteq C_2(D_2(X) \cap D_2(Y)).$$

This condition also holds with X and Y interchanged, and hence we obtain

$$Z \subseteq C_2(D_2(X) \cap D_2(Y)).$$

By the path-independence of C_1 and conditions (3.15) and (3.16), $C_1(X \cup Y) \cap D_1(X) = C_1(D_1(X) \cap Y) \cap D_1(X) \subseteq C_1(D_1(X)) = X$. This condition also holds with X and Y interchanged. Therefore,

$$C_1(X \cup Y) \cap (D_1(X) \cap D_2(Y)) \subseteq X \cap D_2(Y), \quad (3.17)$$

$$C_1(X \cup Y) \cap (D_1(Y) \cap D_2(X)) \subseteq Y \cap D_2(X). \quad (3.18)$$

Since X and Y are kernels, we have $D_1(X) \cup D_2(X) = D_1(Y) \cup D_2(Y) = S$, which implies

$$(D_1(X) \cap D_2(Y)) \cup (D_1(Y) \cap D_2(X)) \supseteq X \cup Y.$$

Then, taking union of (3.17) and (3.18), we have $C_1(X \cup Y) \subseteq (X \cap D_2(Y)) \cup (Y \cap D_2(X)) = (X \cup Y) \cap D_2(X) \cap D_2(Y) = Z$. So far, we have obtained

$$C_1(X \cup Y) \subseteq Z \subseteq C_2(D_2(X) \cap D_2(Y)).$$

Since both C_1 and C_2 are size-monotone and $X = C_1(X)$ and $C_2(D_2(X)) = X$,

$$|X| = |C_1(X)| \leq |C_1(X \cup Y)| \leq |Z| \leq |C_2(D_2(X) \cap D_2(Y))| \leq |C_2(D_2(X))| = |X| \quad (3.19)$$

follows. Therefore, the inequalities in (3.19) are in fact equalities, which implies

$$C_1(X \cup Y) = Z = C_2(D_2(X) \cap D_2(Y)).$$

Also, $|C_1(X)| = |C_1(X \cup Y)|$ implies $r_{\mathbf{M}_1}(X) = r_{\mathbf{M}_1}(X \cup Y)$. Since (3.19) holds with X and Y interchanged, $r_{\mathbf{M}_1}(Y) = r_{\mathbf{M}_1}(X \cup Y)$ follows, and hence $\text{span}_{\mathbf{M}_1}(X) = \text{span}_{\mathbf{M}_1}(Y)$.

We now show (c). Recall that Z is defined as $Z := (X \cup Y) \cap D_2(X) \cap D_2(Y)$. By the above argument and its symmetry, we have

$$C_1(X \cup Y) = (X \cup Y) \cap D_2(X) \cap D_2(Y), \quad (3.20)$$

$$C_2(X \cup Y) = (X \cup Y) \cap D_1(X) \cap D_1(Y). \quad (3.21)$$

By $X \subseteq D_i(X)$, $Y \subseteq D_i(Y)$, and $D_1(X) \cup D_2(X) = D_1(Y) \cup D_2(Y) = S$, we also have

$$\begin{aligned} (D_2(X) \cap D_2(Y)) \cup (D_1(X) \cap D_1(Y)) &\supseteq X \cup Y, \\ (D_2(X) \cap D_2(Y)) \cap (D_1(X) \cap D_1(Y)) &\supseteq X \cap Y. \end{aligned}$$

Then, taking union of (3.20) and (3.21) yields $C_1(X \cup Y) \cup C_2(X \cup Y) = X \cup Y$ and taking intersection of (3.20) and (3.21) yields $C_1(X \cup Y) \cap C_2(X \cup Y) \supseteq X \cap Y$. We complete the proof by showing $C_1(X \cup Y) \cap C_2(X \cup Y) \subseteq X \cap Y$. By (3.21), we have $C_1(X \cup Y) \cap C_2(X \cup Y) = C_1(X \cup Y) \cap D_1(X) \cap D_1(Y)$. By the path-independence of C_1 , (3.15), and (3.16) we have

$$\begin{aligned} C_1(X \cup Y) \cap D_1(X) &= C_1(D_1(X) \cup D_1(Y)) \cap D_1(X) \subseteq C_1(D_1(X)) = X, \\ C_1(X \cup Y) \cap D_1(Y) &= C_1(D_1(X) \cup D_1(Y)) \cap D_1(Y) \subseteq C_1(D_1(Y)) = Y. \end{aligned}$$

Then, we obtain $C_1(X \cup Y) \cap C_2(X \cup Y) = C_1(X \cup Y) \cap D_1(X) \cap D_1(Y) \subseteq X \cap Y$. \square

Since Theorem 3.15 is already shown in Proposition 3.77 (b), what is left is to show Theorem 3.16. Recall that, for any $X, Y \subseteq S$, subsets $X \vee_1 Y$ and $X \wedge_1 Y$ are defined as

$$X \vee_1 Y = C_1(X \cup Y), \quad X \wedge_1 Y = C_2(X \cup Y).$$

Theorem 3.16 claims that the triple $(\mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}, \vee_1, \wedge_1)$ is a distributive lattice.

Proof of Theorem 3.16. We first show that, for any $X, Y \in \mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}$, the subset $X \vee_1 Y$ is indeed an $\mathcal{M}_1, \mathcal{M}_2$ -kernel. By definition, $X \vee_1 Y = C_1(X \cup Y) \in \mathcal{I}_1$. Also, Proposition 3.77 (a) implies $X \vee_1 Y = C_2(D_2(X) \cap D_2(Y)) \in \mathcal{I}_2$. Thus, $X \vee_1 Y \in \mathcal{I}_1 \cap \mathcal{I}_2$. By the path-independence of C_1 and (3.16), we have $X \vee_1 Y = C_1(X \cup Y) = C_1(D_1(X) \cup D_1(Y))$, and hence $D_1(X \vee_1 Y) \supseteq D_1(X) \cup D_1(Y)$ by (3.14). Also, $X \vee_1 Y = C_2(D_2(X) \cap D_2(Y))$ implies $D_2(X \vee_1 Y) \supseteq D_2(X) \cap D_2(Y)$. Hence,

$$D_1(X \vee_1 Y) \cup D_2(X \vee_1 Y) \supseteq (D_1(X) \cup D_1(Y)) \cup (D_2(X) \cap D_2(Y)) = S.$$

Therefore, $X \vee_1 Y$ is an $\mathcal{M}_1, \mathcal{M}_2$ -kernel. By a symmetric argument, we can show that $X \wedge_1 Y = C_2(X \cup Y)$ is also an $\mathcal{M}_1, \mathcal{M}_2$ -kernel.

We now show that \vee_1 and \wedge_1 satisfy idempotency, commutativity, associativity, and distributivity. By the definitions of \vee_1 and \wedge_1 , idempotency and commutativity immediately follow. Also, since C_1 is path-independent, for any kernels $X, Y, Z \in \mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}$,

$$(X \vee_1 Y) \vee_1 Z = C_1(C_1(X \cup Y) \cup Z) = C_1(X \cup Y \cup Z) = C_1(X \cup C_1(Y \cup Z)) = X \vee_1 (Y \vee_1 Z).$$

Similarly $(X \wedge_1 Y) \wedge_1 Z = X \wedge_1 (Y \wedge_1 Z)$ follows from the path-independence of C_2 . Therefore, \vee_1 and \wedge_1 are associative. To prove distributivity, it suffices to show the

following condition for any $X, Y, Z \in \mathfrak{K}_{\mathcal{M}_1, \mathcal{M}_2}$ (see Section 2.2).

$$[X \vee_1 Y = X \vee_1 Z, X \wedge_1 Y = X \wedge_1 Z] \implies Y = Z.$$

Assume $X \vee_1 Y = X \vee_1 Z$ and $X \wedge_1 Y = X \wedge_1 Z$. Then, $C_1(X \cup Y) = C_1(X \cup Z)$ and $C_2(X \cup Y) = C_2(X \cup Z)$, and hence

$$\begin{aligned} C_1(X \cup Y) \cup C_2(X \cup Y) &= C_1(X \cup Z) \cup C_2(X \cup Z), \\ C_1(X \cup Y) \cap C_2(X \cup Y) &= C_1(X \cup Z) \cap C_2(X \cup Z). \end{aligned}$$

By Proposition 3.77 (c), these imply $X \cup Y = X \cup Z$ and $X \cap Y = X \cap Z$. Then, $Y = ((X \cup Y) \setminus X) \cup (X \cap Y) = ((X \cup Z) \setminus X) \cup (X \cap Z) = Z$. \square

Chapter 4

Finding a Stable Allocation in Polymatroid Intersection

4.1 Introduction

Since the famous min-max theorem of König [63] in 1931, the bipartite matching has served as a prototype problem in combinatorial optimization. It has been generalized in various directions such as the matroid intersection and transportation problems. In 1970, Edmonds [21] introduced the framework of polymatroid intersection, which unifies these two generalizations as well as various other efficiently solvable combinatorial optimization problems [36, 98].

The primary purpose of this chapter is to shed a new light from the viewpoint of polymatroids on another fundamental problem on bipartite graphs, i.e., the stable matching problem provided by Gale and Shapley [41]. We introduce the concept of stable allocation in polymatroid intersection and design a strongly polynomial algorithm to find it.

Problem Description Let E be a finite set. In this chapter, any polymatroid is provided in the form (E, f) , where $f : 2^E \rightarrow \mathbf{R}$ is a polymatroid rank function. The set of independent vectors is denoted by

$$\mathbf{P}(f) := \mathbf{P}_+(f) = \{x \in \mathbf{R}_{\geq 0}^E \mid \forall A \subseteq E : x(A) \leq f(A)\}.$$

Throughout this chapter, we abbreviate $\mathbf{P}_+(f)$ to $\mathbf{P}(f)$ since we only consider nonnegative vectors. Recall that, at any independent vector, the family of tight sets are closed under union and intersection. For $x \in \mathbf{P}(f)$, we denote by $\text{sat}_f(x)$ the unique maximal tight set. Then, we can observe

$$\text{sat}_f(x) = \{u \in E \mid \forall \alpha > 0 : x + \alpha \chi_u \notin \mathbf{P}(f)\}.$$

For $u \in \text{sat}_f(x)$, denote by $\text{dep}_f(x, u)$ the unique minimal tight set containing u . Then

$$\text{dep}_f(x, u) = \{v \in E \mid \exists \alpha > 0 : x + \alpha(\chi_u - \chi_v) \in \mathbf{P}(f)\}.$$

If $u \in E \setminus \text{sat}_f(x)$, then $\text{dep}_f(x, u)$ is defined to be empty. The functions $\text{sat}_f(\cdot)$ and $\text{dep}_f(\cdot, \cdot)$ are called the *saturation function* and the *dependence function* [36]. If f is the rank function of a matroid and x is the characteristic vector of an independent set I , then $\text{sat}_f(x)$ and $\text{dep}_f(x, u)$ correspond to the closure of I and the unique circuit in $I \cup \{u\}$, respectively.

An ordered polymatroid is represented as a triple (E, f, \succ) , where (E, f) is a polymatroid and \succ is a total order on E . Here, $a \succ e$ means $a \in E$ is preferred to $e \in E$. We denote $a \succeq e$ to mean $a \succ e$ or $a = e$. For a subset $A \subseteq E$, we denote $A \succeq e$ to mean that every $a \in A$ satisfies $a \succeq e$. Let (E, h, \succ_H) and (E, f, \succ_F) be two totally ordered polymatroids on the same ground set E . The concept of stable allocations is defined as follows.

Definition 4.1. A *stable allocation* for a pair of totally ordered polymatroids (E, h, \succ_H) and (E, f, \succ_F) is a common independent vector $x \in \mathbf{P}(h) \cap \mathbf{P}(f)$ such that every $e \in E$ satisfies $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$ or $[e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq_F e]$. ■

This model can represent, for example, a labor allocation model defined as follows¹. We have two disjoint agent sets I and J , which correspond to workers and firms, respectively. Let E be the set of worker-firm pairs, i.e., $E = I \times J$, and define its subsets $E_i = \{(i, j) \mid j \in J\}$ for each $i \in I$ and $E_j = \{(i, j) \mid i \in I\}$ for each $j \in J$. A labor allocation is a vector $x = (x(i, j) \mid (i, j) \in E) \in \mathbf{R}^E$, where $x(i, j)$ means the amount of contracted labor time of i at j . For an allocation x , we write $x_i = x|_{E_i} = (x(i, j) \mid j \in J)$ for each $i \in I$ and $x_j = x|_{E_j}$ for each $j \in J$. The profile of each $i \in I$ is given as an ordered polymatroid (E_i, h_i, \succ_i) . The worker i wishes to have as large allocation as possible in $\mathbf{P}(h_i) \subseteq \mathbf{R}^{E_i}$ with the priority given by \succ_i . Note that $\{E_i\}_{i \in I}$ is a partition of E . Define (E, h, \succ_H) as the direct sum of $\{(E_i, h_i, \succ_i)\}_{i \in I}$, i.e., let $h(A) = \sum_{i \in I} h_i(A \cap E_i)$ ($A \subseteq E$), and let \succ_H be an arbitrary total order on E which is consistent with $\{\succ_i\}_{i \in I}$. Then, (E, h, \succ_H) forms an ordered polymatroid. Similarly, profiles of firms are given as ordered polymatroids $\{(E_j, f_j, \succ_j)\}_{j \in J}$, and an ordered polymatroid (E, f, \succ_F) is defined on the same ground set E .

An allocation $x \in \mathbf{R}^E$ must be feasible for every agent, i.e., $x_i \in \mathbf{P}(h_i)$ ($\forall i \in I$) and $x_j \in \mathbf{P}(f_j)$ ($\forall j \in J$). This means $x \in \mathbf{P}(h) \cap \mathbf{P}(f)$. In addition, x should be stable in the following sense. If $(i, j) \notin \text{sat}_{h_i}(x_i)$ or $(i, j) \succ_i (i, j')$ for some $(i, j') \in \text{dep}_{h_i}(x_i, (i, j))$, then i has incentive to increase $x(i, j)$, possibly at the expense of $x(i, j')$. Similarly, if $(i, j) \notin \text{sat}_{f_j}(x_j)$ or $(i, j) \succ_j (i', j)$ for some $(i', j) \in \text{dep}_{f_j}(x_j, (i, j))$, then j has incentive

¹This labor allocation model is a special case of “the stable allocation model on generalized polymatroids,” investigated in Section 3.7. See Remark 4.8 for the details.

to increase $x(i, j)$, possibly at the expense of $x(i', j)$. If both i and i' have incentives to increase $x(i, j)$, there is no direct way to prevent them from doing so. Definition 4.1 requires x not to admit such a pair $(i, j) \in E$.

Note that the existence of a stable allocation appears to be unclear from the definition. By using the general framework of Alkan and Gale [9], however, it is shown in Section 3.7 that a stable allocation does exist in any setting of polymatroid intersection. This proof of existence does not tell how to find such a solution efficiently.

Our Contribution In this chapter, we present the first strongly polynomial algorithm for finding a stable allocation in the general setting of polymatroid intersection. The algorithm combines the augmenting path technique for polymatroid intersection [35, 95] and the deferred acceptance procedure for stable matching [41]. The correctness of our algorithm provides an alternative proof for the existence of a stable allocation. If h and f are integer-valued functions, then the algorithm finds an integral stable allocation.

The running time of this algorithm is $O(|E|^3\gamma)$, where γ denotes the time for computing the saturation and exchange capacities on the given polymatroids. Assuming an oracle for evaluating the polymatroid rank function, these capacities can be computed in strongly polynomial time via submodular function minimization [44, 57, 67, 83, 97]. Most concrete examples of polymatroids, however, admit more direct ways to design such procedures.

Furthermore, we show that the output of our algorithm is \succ_H -optimal among all stable allocations. This generalizes the fact that we can find the student optimal stable matching in the college admissions model.

Related Stable Matching Models Our framework includes two major generalizations of the stable matching model: matroid kernels in matroid intersection due to Fleiner [26, 27] and stable allocations in bipartite networks due to Baïou and Balinski [10]. On the other hand, it can be regarded as a special case of the discrete-concave function model of Eguchi, Fujishige, and Tamura [24]. We now describe these related results and previous known algorithmic results on polymatroids.

The stable allocation model of Baïou and Balinski [10] is a special case of the above labor allocation model, which is representable by network flow as follows. Let $G = (V, E)$ be a bipartite graph with edge set E and vertex set V partitioned into I and J . Nonnegative capacity functions $b : V \rightarrow \mathbf{R}_+$ and $c : E \rightarrow \mathbf{R}_+$ are associated with G . In addition, each $v \in I \cup J$ has a preference \succ_v on $\delta(v)$. A flow $x : E \rightarrow \mathbf{R}_+$ is said to be feasible if $x(e) \leq c(e)$ holds for every $e \in E$ and $\sum_{e \in \delta(v)} x(e) \leq b(v)$ holds for every $v \in I \cup J$. Define set functions h and f on E by

$$\begin{aligned} h(A) &= \sum_{i \in I} \min \left\{ b(i), \sum_{e \in A \cap \delta(i)} c(e) \right\} \quad (A \subseteq E), \\ f(A) &= \sum_{j \in J} \min \left\{ b(j), \sum_{e \in A \cap \delta(j)} c(e) \right\} \quad (A \subseteq E), \end{aligned}$$

respectively. The set of the feasible flows then coincides with the intersection $\mathbf{P}(h) \cap \mathbf{P}(f)$.

Let \succ_H and \succ_F be arbitrary total orders on E consistent with $\{\succ_i\}_{i \in I}$ and $\{\succ_j\}_{j \in J}$, respectively. Stable allocations in the bipartite network G are exactly the same as stable allocations for the pair of totally ordered polymatroids (E, h, \succ_H) and (E, f, \succ_F) .

Baïou and Balinski [10] presented two algorithms for finding a stable allocation in G . The first is a natural extension of the deferred acceptance algorithm of Gale and Shapley [41] and runs in pseudopolynomial time if all the capacities are integers. However, for instances with real capacities, this algorithm needs exponential time in the worst-case, as shown by Dean, Goemans, and Immorlica [16]. In contrast, the second algorithm finds a stable allocation in $O(|V| \cdot |E|)$ time using augmenting paths. Dean and Munshi [17] improved the latter algorithm to run in $O(|E| \log |V|)$ time.

A matroid kernel, introduced by Fleiner [26, 27], is defined for a pair of ordered matroids $(E, \mathcal{I}_H, \succ_H)$ and $(E, \mathcal{I}_F, \succ_F)$ on the same ground set E . Here (E, \mathcal{I}_H) , (E, \mathcal{I}_F) are matroids and \succ_H , \succ_F are total orders on E . (For the precise definition of a matroid-kernel, see Section 3.3.3.) Let ρ_H and ρ_F be the rank functions of (E, \mathcal{I}_H) and (E, \mathcal{I}_F) , respectively. Then, the common independent sets of (E, \mathcal{I}_H) and (E, \mathcal{I}_F) correspond to the integer vectors in $\mathbf{P}(\rho_H) \cap \mathbf{P}(\rho_F)$. A matroid-kernel coincides with a subset $K \subseteq E$ whose characteristic vector is a stable allocation for (E, ρ_H, \succ_H) and (E, ρ_F, \succ_F) .

Fleiner [26, 27] presented a strongly polynomial algorithm for finding a matroid kernel, extending the deferred acceptance algorithm of Gale and Shapley [41].

The discrete-concave function model, investigated in [24, 37, 38, 81], is a two-sided matching model in which preferences of agents are represented by value functions on integer vectors. The origin of this framework is in Eguchi, Fujishige, and Tamura [24]. In their model, each side has a value function whose effective domain is the set of integer points of a polymatroid and function values satisfy M^\sharp -concavity, a kind of concavity for discrete functions [75]. Our framework can be regarded as a special form of this model in which each function is linear in its effective domain as follows. For an ordered polymatroid (E, h, \succ_H) with an integer-valued rank function, take an arbitrary positive weight $w_H : E \rightarrow \mathbf{R}_+$ such that $a \succ_H e$ implies $w_H(a) > w_H(e)$ for any $a, e \in E$. Define a value function $U_H : \mathbf{Z}^E \rightarrow \mathbf{R} \cup \{-\infty\}$ by

$$U_H(x) = \begin{cases} \langle w_H, x \rangle & \text{if } x \in \mathbf{Z}^E \cap \mathbf{P}(h), \\ -\infty & \text{if } x \in \mathbf{Z}^E \setminus \mathbf{P}(h), \end{cases}$$

where $\langle w_H, x \rangle := \sum_{e \in E} w_H(e)x(e)$. Similarly, define a value function $U_F : \mathbf{Z}^E \rightarrow \mathbf{R} \cup \{-\infty\}$ from an ordered polymatroid (E, f, \succ_F) . Then, an integer allocation $x \in \mathbf{Z}^E$ is stable for (E, h, \succ_H) and (E, f, \succ_F) if and only if it is stable w.r.t. U_H and U_F in the sense of the discrete-concave function model.

Eguchi et al. [24] established the existence of a stable allocation by showing that a simple extension of the deferred acceptance algorithm of Gale and Shapley finds a stable allocation in pseudo-polynomial time. They asked if one can develop a weakly polynomial

algorithm, which has remain open for more than a decade. When applied to our framework, their algorithm still requires pseudo-polynomial time. Indeed, there is a series of examples in which the algorithm requires the numbers of iterations proportional to the width of the effective domain.

Related Polymatroid Algorithms We make comparison of our algorithm with previously known algorithmic results on polymatroids. The concept of polymatroids was introduced by Edmonds [21] as a polyhedral generalization of matroids. He showed that the linear optimization over $\mathbf{P}(f)$ is solved by the greedy algorithm. Let $w : E \rightarrow \mathbf{R}_+$ be an arbitrary weight function that takes distinct positive values, and consider a total order \succ_w on E such that $a \succ_w e$ means $w(a) > w(e)$. The correctness of the greedy algorithm implies that $x \in \mathbf{P}(f)$ maximizes $\langle w, x \rangle$ if and only if $e \in \text{sat}_f(x)$ and $\text{dep}_f(x, e) \succeq_w e$ hold for every $e \in E$. This condition means that x is a stable allocation for (E, f, \succ_w) and (E, f, \succ_w) .

Given a nonnegative vector $z \in \mathbf{R}^E$, one can think of maximizing $\langle w, x \rangle$ subject to $x \in \mathbf{P}(f)$ and $x \leq z$. A feasible solution x is optimal if and only if $e \in \text{sat}_f(x)$ and $\text{dep}_f(x, e) \succeq_w e$ hold for every $e \in E$ with $x(e) < z(e)$. This optimality condition requires x to be a stable allocation in the polymatroid intersection of (E, f, \succ_w) and (E, z, \succ_w) , where z is regarded as a polymatroid rank function defined by $z(A) = \sum_{e \in A} z(e)$. The feasible region of this problem is in fact identical to the associated polytope $\mathbf{P}(f^z)$ with a polymatroid (E, f^z) defined by $f^z(A) := \min\{f(Y) + z(A \setminus Y) \mid Y \subseteq A\}$. A standard method for solving this optimization problem is to apply the polymatroid greedy algorithm to $\mathbf{P}(f^z)$. Our stable allocation algorithm when applied to this setting coincides with this standard method, which runs in $O(|E|\gamma)$ time, where γ denotes the time for computing the saturation and exchange capacities on the given polymatroids.

Edmonds [21] also showed a min-max theorem on the polymatroid intersection problem, i.e., the problem of maximizing $x(E)$ among common independent vectors of two polymatroids on E . Schönsleben [95] invented a technique of using lexicographic shortest augmenting paths to develop an algorithm for solving the polymatroid intersection problem in $O(|E|^5\gamma)$ time. Tardos, Tovey, and Trick [105] improved this algorithm to run in $O(|E|^4\gamma)$ time. Currently the best known running time bound for the polymatroid intersection problem is $O(|E|^3\gamma)$ due to Fujishige and Zhang [40].

The weighted version of this problem, i.e., the weighted polymatroid intersection problem asks for a common independent vector of two polymatroids maximizing a linear function. This problem is also solvable in strongly polynomial time, but the current best running time bound is as high as $O(|E|^6\gamma \log |E|)$. As a special case with extremely distinct weights, we can think of maximizing the components lexicographically with respect to a specified total order on E . An optimal solution of this problem coincides with a stable allocation for the same total order in both polymatroids. Our stable allocation algorithm when applied to this setting determines each component greedily in the specified total order, which requires only $O(|E|\gamma)$ time.

Organization The rest of this chapter is organized as follows. Section 4.2 provides preliminaries on polymatroids by recapitulating properties of saturation and exchange capacities. In Section 4.3, we describe our algorithm and demonstrate it on a small example. Section 4.4 provides invariants which are maintained in the algorithm and play a key role in our analysis. The correctness and complexity of the algorithm are shown in Sections 4.5 and 4.6, respectively. Finally, we show the \succ_H -optimality of the output in Section 4.7.

4.2 Saturation and Exchange Capacities

In this section, we describe fundamental properties of saturation and exchange capacities. Let (E, f) be a polymatroid, i.e., let $f : 2^E \rightarrow \mathbf{R}$ be a polymatroid rank function. For an independent vector $x \in \mathbf{P}(f)$ and an element $u \in E$, define the *saturation capacity* $\hat{c}_f(x, u)$ by

$$\hat{c}_f(x, u) = \max \{ \alpha \in \mathbf{R} \mid x + \alpha \chi_u \in \mathbf{P}(f) \}.$$

For any α with $0 \leq \alpha \leq \hat{c}_f(x, u)$, we have $x + \alpha \chi_u \in \mathbf{P}(f)$. The saturation capacity can also be expressed as

$$\hat{c}_f(x, u) = \min \{ f(A) - x(A) \mid u \in A \subseteq E \}. \quad (4.1)$$

For $x \in \mathbf{P}(f)$ and $u, v \in E$, define the *polymatroid exchange capacity* $\bar{c}_f(x, u, v)$ by

$$\bar{c}_f(x, u, v) = \max \{ \alpha \in \mathbf{R} \mid x + \alpha(\chi_u - \chi_v) \in \mathbf{P}(f) \},$$

where we let $\bar{c}_f(x, u, u) = \infty > 0$ for every $u \in E$. For any α with $0 \leq \alpha \leq \bar{c}_f(x, u, v)$, we have $x + \alpha(\chi_u - \chi_v) \in \mathbf{P}(f)$. For distinct $u, v \in E$, it is known that $\bar{c}_f(x, u, v)$ is also written as

$$\bar{c}_f(x, u, v) = \min \{ \tilde{c}_f(x, u, v), x(v) \}$$

where $\tilde{c}_f(x, u, v)$ is defined for $x \in \mathbf{P}(f)$ and distinct $u, v \in E$ by

$$\tilde{c}_f(x, u, v) = \min \{ f(A) - x(A) \mid u \in A \subseteq E, v \notin A \}. \quad (4.2)$$

It can be easily shown that $\bar{c}_f(x, u, v) = \tilde{c}_f(x, u, v)$ holds if $u \in \text{sat}_f(x)$.

The saturation and dependence functions are now expressed as

$$\begin{aligned} \text{sat}_f(x) &= \{ u \in E \mid \hat{c}_f(x, u) = 0 \} & (x \in \mathbf{P}(f)), \\ \text{dep}_f(x, u) &= \{ v \in E \mid \bar{c}_f(x, u, v) > 0 \} & (x \in \mathbf{P}(f), u \in \text{sat}_f(x)), \end{aligned}$$

where $\text{dep}_f(x, u)$ is defined to be empty for $u \in E \setminus \text{sat}_f(x)$.

Note that, for $x \in \mathbf{P}(f)$ and $u, v \in \text{sat}_f(x)$, the condition $v \in \text{dep}_f(x, u)$ implies

$x(v) > 0$. Also, observe that

$$\bar{c}_f(x, u, v) > 0 \iff x(v) > 0 \text{ and } [u \notin \text{sat}_f(x) \text{ or } v \in \text{dep}_f(x, u)]$$

holds for any $x \in \mathbf{P}(f)$ and $u, v \in E$. This observation implies the following lemma.

Lemma 4.2 (Transitivity of Dependence). *For $x \in \mathbf{P}(f)$ and $s, t, u \in E$, if $\bar{c}_f(x, s, t) > 0$ and $\bar{c}_f(x, t, u) > 0$, then $\bar{c}_f(x, s, u) > 0$.*

Proof. If two or three of s, t, u are the same element, the claim is obvious. We assume that they are all distinct.

Since $\bar{c}_f(x, t, u) > 0$ implies $x(u) > 0$, it suffices to show $u \in \text{dep}_f(x, s)$ assuming $s \in \text{sat}_f(x)$. Note that $s \in \text{sat}_f(x)$ and $\bar{c}_f(x, s, t) > 0$ imply $t \in \text{dep}_f(x, s) \subseteq \text{sat}_f(x)$. Suppose, to the contrary, that we have $u \notin \text{dep}_f(s)$. Then, there is $A \subseteq E$ such that $x(A) = f(A)$, $s \in A$, and $u \notin A$. If $t \notin A$, then A satisfies $s \in A \not\preceq t$, which implies $\bar{c}_f(x, s, t) = 0$ by $s \in \text{sat}_f(x)$, a contradiction. If $t \in A$, then A satisfies $t \in A \not\preceq u$ which implies $\bar{c}_f(x, t, u) = 0$ by $t \in \text{sat}_f(x)$, a contradiction. \square

4.2.1 Moving in Polymatroids

Here we show how saturation and exchange capacities change when we move an independent vector x in the polyhedron $\mathbf{P}(f)$.

Lemma 4.3 (Single Exchange). *For any $x \in \mathbf{P}(f)$ and distinct $u, v \in E$, let $y \in \mathbf{R}^E$ be the vector defined by $y := x + \alpha(\chi_u - \chi_v)$ with $0 \leq \alpha \leq \bar{c}_f(x, u, v)$. Then, we have $y \in \mathbf{P}(f)$ and the following (a1)–(a5).*

- (a1) $\bar{c}_f(y, u, v) = \bar{c}_f(x, u, v) - \alpha$.
- (a2) (i) For $s \in E \setminus \{v\}$ with $\bar{c}_f(x, s, v) = 0$, every $t \in E \setminus \{s\}$ satisfies $\bar{c}_f(y, s, t) = \bar{c}_f(x, s, t)$.
(ii) For $t \in E \setminus \{u\}$ with $\bar{c}_f(x, u, t) = 0$, every $s \in E \setminus \{t\}$ satisfies $\bar{c}_f(y, s, t) = \bar{c}_f(x, s, t)$.
- (a3) For any $s, t \in E \setminus \{u, v\}$, we have $\bar{c}_f(y, s, v) \geq \min\{\bar{c}_f(x, s, v), \bar{c}_f(y, u, v)\}$ and $\bar{c}_f(y, u, t) \geq \min\{\bar{c}_f(x, u, t), \bar{c}_f(y, u, v)\}$.
- (a4) If $u \in \text{sat}_f(x)$, then $\text{sat}_f(x) = \text{sat}_f(y)$ and $\hat{c}_f(y, s) = \hat{c}_f(x, s)$ for every $s \in E \setminus \text{sat}_f(x)$.
- (a5) If $u \notin \text{sat}_f(x)$, then $\hat{c}_f(y, u) \geq \min\{\hat{c}_f(x, u), \bar{c}_f(y, u, v)\}$ and $s \in \text{sat}_f(y)$ for any $s \in \text{sat}_f(x)$ with $\bar{c}_f(x, s, v) = 0$.

Proof. It suffices to consider the case that $\bar{c}_f(x, u, v) > 0$, which means that we can assume $x(v) > 0$ and $[u \notin \text{sat}_f(x) \text{ or } v \in \text{dep}_f(x, u)]$.

(a1): Every $A \subseteq E$ with $u \in A \not\preceq v$ satisfies $y(A) = x(A) + \alpha$, and hence $f(A) - y(A) = f(A) - x(A) - \alpha$. Then, by (4.2), $\bar{c}_f(y, u, v) = \bar{c}_f(x, u, v) - \alpha$. Also, clearly $y(v) = x(v) - \alpha$. Then, $\bar{c}_f(y, u, v) = \min\{\bar{c}_f(y, u, v), y(v)\} = \min\{\bar{c}_f(x, u, v) - \alpha, x(v) - \alpha\} = \bar{c}_f(x, u, v) - \alpha$.

(a2)-(i): Since $x(v) > 0$, the condition $\bar{c}_f(x, s, v) = 0$ implies $s \in \text{sat}_f(x)$ and $v \notin \text{dep}_f(x, s)$. By Lemma 4.2, $\bar{c}_f(x, u, v) > 0$ and $\bar{c}_f(x, s, v) = 0$ lead to $\bar{c}_f(x, s, u) = 0$, which implies $u \notin \text{dep}_f(x, s)$. Thus, $C := \text{dep}_f(x, s)$ satisfies $u, v \notin C$, $s \in C$, and $x(C) = y(C) = f(C)$. Take any $t \in E \setminus \{s\}$ and recall (4.2). Note that now $\bar{c}_f(x, s, t) = \tilde{c}_f(x, s, t)$ because $s \in \text{sat}_f(x)$. For every $A \subseteq E$ with $s \in A$, $t \notin A$, by $x(C) = f(C)$ and $x \in \mathbf{P}(f)$,

$$\begin{aligned} f(A) - x(A) &= f(A) + f(C) - x(C) - x(A) \\ &\geq f(A \cup C) + f(A \cap C) - x(A \cup C) - x(A \cap C) \\ &\geq f(A \cap C) - x(A \cap C). \end{aligned}$$

Also, $A \cap C$ satisfies $s \in A \cap C \not\preceq t$. Hence $\bar{c}_f(x, s, t) = \min \{ f(A) - x(A) \mid A \subseteq C, s \in A \not\preceq t \}$. Similarly, since $y(C) = f(C)$ and $y \in \mathbf{P}(f)$, the above inequality also holds with x replaced by y , and hence $\bar{c}_f(y, s, t) = \min \{ f(A) - y(A) \mid A \subseteq C, s \in A \not\preceq t \}$. Because $A \subseteq C \subseteq E \setminus \{u, v\}$ implies $x(A) = y(A)$, the claim follows.

(a2)-(ii): By $\bar{c}_f(x, u, v) > 0$ and $\bar{c}_f(x, u, t) = 0$, $t \in E \setminus \{u\}$ satisfies $t \neq v$, and hence $y(t) = x(t)$. Then, it suffices to show $\tilde{c}_f(y, s, t) = \tilde{c}_f(x, s, t)$. Assume $x(t) > 0$, since otherwise the claim is obvious. Then, $\bar{c}_f(x, u, t) = 0$ implies $u \in \text{sat}_f(x)$ and $t \notin \text{dep}_f(x, u)$. Also, then $\bar{c}_f(x, u, v) > 0$ implies $v \in \text{dep}_f(x, u)$. Thus, $C := \text{dep}_f(x, u)$ satisfies $u, v \in C$, $t \notin C$, and $x(C) = y(C) = f(C)$. Take any $s \in E \setminus \{t\}$ and recall (4.2). For every $A \subseteq E$ with $s \in A$, $t \notin A$, by $x(C) = f(C)$ and $x \in \mathbf{P}(f)$,

$$\begin{aligned} f(A) - x(A) &= f(A) + f(C) - x(C) - x(A) \\ &\geq f(A \cup C) + f(A \cap C) - x(A \cup C) - x(A \cap C) \\ &\geq f(A \cup C) - x(A \cup C). \end{aligned}$$

Also, $s \in A \cup C \not\preceq t$. Hence $\tilde{c}_f(x, s, t) = \min \{ f(A) - x(A) \mid A \supseteq C, s \in A \not\preceq t \}$. Similarly, by $y(C) = f(C)$ and $y \in \mathbf{P}(f)$, we have $\tilde{c}_f(y, s, t) = \min \{ f(A) - y(A) \mid A \supseteq C, s \in A \not\preceq t \}$. Because $u, v \in C \subseteq A$ implies $x(A) = y(A)$, the claim follows.

(a3): For the first inequality, it suffices to show that $\bar{c}_f(y, s, v) < \bar{c}_f(y, u, v)$ implies $\bar{c}_f(y, s, v) \geq \bar{c}_f(x, s, v)$. By (4.2), $\bar{c}_f(y, s, v) < \bar{c}_f(y, u, v)$ means

$$\min \{ f(A) - y(A) \mid A \subseteq E, s \in A \not\preceq v \} < \min \{ f(A) - y(A) \mid A \subseteq E, u \in A \not\preceq v \}.$$

Then, a minimizer of the left-hand side, say A^* , satisfies $s \in A^* \not\preceq u, v$. This implies $\bar{c}_f(y, s, v) = f(A^*) - y(A^*) = f(A^*) - x(A^*) \geq \min \{ f(A) - x(A) \mid A \subseteq E, s \in A \not\preceq v \} = \bar{c}_f(x, s, v)$.

To show the second inequality similarly, assume $\bar{c}_f(y, u, t) < \bar{c}_f(y, u, v)$. Then

$$\min \{ f(A) - y(A) \mid A \subseteq E, u \in A \not\preceq t \} < \min \{ f(A) - y(A) \mid A \subseteq E, u \in A \not\preceq v \},$$

and a minimizer A^* of the left-hand side satisfies $u, v \in A^*$ and $t \notin A^*$. This implies $\bar{c}_f(y, u, t) = f(A^*) - y(A^*) = f(A^*) - x(A^*) \geq \min \{ f(A) - x(A) \mid A \subseteq E, u \in A \not\subseteq t \} = \bar{c}_f(x, u, t)$.

(a4): As $u \in \text{sat}_f(x)$, the subset $C := \text{sat}_f(x)$ satisfies $u, v \in \text{dep}_f(x, u) \subseteq C$ by $\bar{c}_f(x, u, v) > 0$. Then, $y(C') = x(C')$ for every $C' \supseteq C$, and hence $\text{sat}_f(y) = C = \text{sat}_f(x)$.

Take any $s \in E \setminus \text{sat}_f(x)$ and recall (4.1). Since $x(C) = f(C)$, the inequality in the proof of (a2) holds for any $A \subseteq E$ with $s \in A$. Hence, we have $\hat{c}_f(x, s) = \min \{ f(A) - x(A) \mid A \supseteq C, s \in A \}$. Similarly, $\hat{c}_f(y, s) = \min \{ f(A) - y(A) \mid A \supseteq C, s \in A \}$. Because $u, v \in C \subseteq A$ implies $x(A) = y(A)$, the claim follows.

(a5): When $u \notin \text{sat}_f(x)$, every $s \in \text{sat}_f(x)$ satisfies $u \notin \text{dep}_f(x, s)$. Also, $\bar{c}_f(x, s, v) = 0$ implies $v \notin \text{dep}_f(x, s)$, and hence $y(\text{dep}_f(x, s)) = x(\text{dep}_f(x, s)) = f(\text{dep}_f(x, s))$. Thus, we have $\{ s \in \text{sat}_f(x) \mid \bar{c}_f(x, s, v) = 0 \} \subseteq \text{sat}_f(y)$.

For the inequality, it suffices to show that $\hat{c}_f(y, u) < \bar{c}_f(y, u, v)$ implies $\hat{c}_f(y, u) \geq \hat{c}_f(x, u)$. By (4.1), $\hat{c}_f(y, u) < \bar{c}_f(y, u, v)$ means

$$\min \{ f(A) - y(A) \mid A \subseteq E, u \in A \} < \min \{ f(A) - y(A) \mid A \subseteq E, u \in A \not\subseteq v \}.$$

Then, a minimizer $A^* \subseteq E$ of the left-hand satisfies $u, v \in A^*$. This implies $\hat{c}_f(y, u) = f(A^*) - y(A^*) = f(A^*) - x(A^*) \geq \min \{ f(A) - x(A) \mid A \subseteq E, u \in A \} = \hat{c}_f(x, u)$. \square

For any positive integer $d \in \mathbf{Z}_{>0}$, we write $[d] := \{1, 2, \dots, d\}$. For two nonnegative integers $k, l \in \mathbf{Z}_+$ with $k \leq l$, we write the integer interval by $[k, l] := \{k, k+1, k+2, \dots, l\}$.

Lemma 4.4 (Multiple Exchange). *For $x \in \mathbf{P}(f)$, let u_i, v_i ($i = 1, 2, \dots, d$) be $2d$ distinct elements of E with*

$$\begin{aligned} \bar{c}_f(x, u_i, v_i) &> 0 \quad (i \in [d]), \\ \bar{c}_f(x, u_i, v_j) &= 0 \quad (i, j \in [d] \text{ with } i < j). \end{aligned}$$

For any $\alpha \geq 0$ satisfying $\alpha \leq \bar{c}_f(x, u_i, v_i)$ for every $i \in [d]$, define $y \in \mathbf{R}^E$ by

$$y := x + \alpha \sum_{i=1}^d (\chi_{u_i} - \chi_{v_i}).$$

Then, we have $y \in \mathbf{P}(f)$ and the following (b1)–(b6).

- (b1)** $\bar{c}_f(y, u_i, v_i) = \bar{c}_f(x, u_i, v_i) - \alpha$ for every $i \in [d]$.
- (b2)** For any $s, t \in E$, if we have $\bar{c}_f(x, s, t) \neq \bar{c}_f(y, s, t)$, then there is a pair of indices $(i, j) \in [d]^2$ such that $i \leq j$, $\bar{c}_f(x, s, v_j) > 0$, and $\bar{c}_f(x, u_i, t) > 0$.
- (b3)** For every $s, t \in E$ and $i \in [d]$, we have the following.
 If $\bar{c}_f(x, s, v_j) = 0$ ($\forall j > i$), then $\bar{c}_f(y, s, v_i) \geq \min\{\bar{c}_f(x, s, v_i), \bar{c}_f(y, u_i, v_i)\}$.
 If $\bar{c}_f(x, u_j, t) = 0$ ($\forall j < i$), then $\bar{c}_f(y, u_i, t) \geq \min\{\bar{c}_f(x, u_i, t), \bar{c}_f(y, u_i, v_i)\}$.

- (b4) $\{u_1, u_2, \dots, u_{d-1}\} \subseteq \text{sat}_f(x)$.
- (b5) If $u_d \in \text{sat}_f(x)$, then $\text{sat}_f(x) = \text{sat}_f(y)$ and $\hat{c}_f(y, s) = \hat{c}_f(x, s)$ for every $s \in E \setminus \text{sat}_f(x)$.
- (b6) If $u_d \notin \text{sat}_f(x)$, then $\hat{c}_f(y, u_d) \geq \min\{\hat{c}_f(x, u_d), \bar{c}_f(y, u_d, v_d)\}$ and $s \in \text{sat}_f(y)$ for any $s \in \text{sat}_f(x)$ with $\bar{c}_f(x, s, v_d) = 0$.

Proof. Define $y_0 := x$ and $y_l := y_{l-1} + \alpha(\chi_{u_l} - \chi_{v_l})$ for $l = 1, 2, \dots, d$. Note that $y_d = y$. We first show that for any $l = 1, 2, \dots, d$, a vector y_l satisfies the following conditions:

$$\bar{c}_f(y_l, u_i, v_i) = \bar{c}_f(x, u_i, v_i) - \alpha \quad (i \in [l]), \quad (4.3)$$

$$\bar{c}_f(y_l, u_i, v_i) = \bar{c}_f(x, u_i, v_i) \quad (i \in [l+1, d]), \quad (4.4)$$

$$\bar{c}_f(y_l, u_i, v_j) = \bar{c}_f(x, u_i, v_j) = 0 \quad (i, j \in [d] \text{ with } i < j), \quad (4.5)$$

$$\bar{c}_f(y_l, s, v_i) = \bar{c}_f(x, s, v_i) \quad (i \in [l+1, d], s \in E \setminus \{v_i\}), \quad (4.6)$$

$$\bar{c}_f(y_l, u_i, t) = \bar{c}_f(y_{l-1}, u_i, t) \quad (i \in [l-1], t \in E \setminus \{u_i\}). \quad (4.7)$$

We use induction on l . Assume (4.3)–(4.6) hold for $l-1$; we will prove them for l .

Since $y_l = y_{l-1} + \alpha(\chi_{u_l} - \chi_{v_l})$, we have $\bar{c}_f(y_l, u_l, v_l) = \bar{c}_f(y_{l-1}, u_l, v_l) - \alpha = \bar{c}_f(x, u_l, v_l) - \alpha$ by Lemma 4.3 (a1) and (4.4) for y_{l-1} . Hence, (4.3) for y_l holds.

Note that y_{l-1} satisfies $\bar{c}_f(y_{l-1}, u_i, v_l) = 0$ for $i < l$ and $\bar{c}_f(y_{l-1}, u_l, v_j) = 0$ for $j > l$ by (4.5). Then, Lemma 4.3 (a2) imply that we have $\bar{c}_f(y_l, u_i, v_j) \neq \bar{c}_f(y_{l-1}, u_i, v_j)$ only when $j \leq l \leq i$. Thus, (4.4) and (4.5) hold for y_l .

By (4.5) for y_{l-1} , every $i \in [d]$ with $i > l$ satisfies $\bar{c}_f(y_{l-1}, u_l, v_i) = 0$. By Lemma 4.3 (a2)-(ii), then $\bar{c}_f(y_l, s, v_i) = \bar{c}_f(y_{l-1}, s, v_i)$ for every $s \in E \setminus \{v_i\}$. Note that $i > l$ implies $i \in [l'+1, d]$ for every $l' \geq l$. Hence (4.6) holds by induction.

By (4.5) for y_{l-1} , every $i \in [d]$ with $i < l$ satisfies $\bar{c}_f(y_{l-1}, u_i, v_l) = 0$. By Lemma 4.3 (a2)-(i), then $\bar{c}_f(y_l, u_i, t) = \bar{c}_f(y_{l-1}, u_i, t)$ for every $t \in E \setminus \{u_i\}$, and hence (4.7) holds.

(b1): This is already shown by (4.3) for $y_d = y$.

(b2): For $l = 1, 2, \dots, d$, we show the following claim: For $s, t \in E$ with $0 = \bar{c}_f(x, s, t) < \bar{c}_f(y_l, s, t)$, there is (i, j) such that $1 \leq i \leq j \leq l$ and $\bar{c}_f(x, s, v_j) > 0$ and $\bar{c}_f(x, u_i, t) > 0$. Assume that the claim holds for y_{l-1} ; we will prove it for y_l .

In the case $\bar{c}_f(x, s, t) < \bar{c}_f(y_{l-1}, s, t)$, the claim immediately follows from induction. Hence, suppose $0 = \bar{c}_f(x, s, t) = \bar{c}_f(y_{l-1}, s, t) < \bar{c}_f(y_l, s, t)$. Then, by Lemma 4.3 (a2), we have $\bar{c}_f(y_{l-1}, s, v_l) > 0$ and $\bar{c}_f(y_{l-1}, u_l, t) > 0$. The former means $\bar{c}_f(x, s, v_l) > 0$ by (4.6). The latter implies $\bar{c}_f(x, u_l, t) > 0$ or $\bar{c}_f(y_{l-1}, u_l, t) > \bar{c}_f(x, u_l, t) = 0$, which yields $\bar{c}_f(x, u_i, t) > 0$ for some $i \leq l-1$ by the inductive hypothesis for y_{l-1} . In both cases, we obtain $\bar{c}_f(x, s, v_l) > 0$ and $\bar{c}_f(x, u_i, t) > 0$ for some $i \leq l$.

(b3): Fix $s^*, t^* \in E$ and $i^* \in [d]$. Suppose $s^* \neq v_{i^*}$ and $t^* \neq u_{i^*}$, since otherwise the

claim is obvious. By $y_{i^*} = y_{i^*-1} + \alpha(\chi_{u_{i^*}} - \chi_{v_{i^*}})$ and Lemma 4.3 (a3),

$$\bar{c}_f(y_{i^*}, s^*, v_{i^*}) \geq \min\{\bar{c}_f(y_{i^*-1}, s^*, v_{i^*}), \bar{c}_f(y_{i^*}, u_{i^*}, v_{i^*})\}, \quad (4.8)$$

$$\bar{c}_f(y_{i^*}, u_{i^*}, t^*) \geq \min\{\bar{c}_f(y_{i^*-1}, u_{i^*}, t^*), \bar{c}_f(y_{i^*}, u_{i^*}, v_{i^*})\}. \quad (4.9)$$

For the first claim, assume $\bar{c}_f(x, s^*, v_j) = 0$ ($\forall j > i^*$). Since $\bar{c}_f(y_{j-1}, s^*, v_j) = \bar{c}_f(x, s^*, v_j)$ by (4.6), we obtain $\bar{c}_f(y_{j-1}, s^*, v_j) = 0$ ($\forall j > i^*$). As $y_j = y_{j-1} + \alpha(\chi_{u_j} - \chi_{v_j})$, apply Lemma 4.3 (a2)-(i) with u, v, s, t replaced by u_j, v_j, s^*, v_{i^*} , respectively. Then $\bar{c}_f(y_j, s^*, v_{i^*}) = \bar{c}_f(y_{j-1}, s^*, v_{i^*})$ for every $j > i^*$, and hence $\bar{c}_f(y_{i^*}, s^*, v_{i^*}) = \bar{c}_f(y, s^*, v_{i^*})$. Also, we have $\bar{c}_f(y_{i^*-1}, s^*, v_{i^*}) = \bar{c}_f(x, s^*, v_{i^*})$ by (4.6) for $l = i^* - 1$ and $\bar{c}_f(y_{i^*}, u_{i^*}, v_{i^*}) = \bar{c}_f(y, u_{i^*}, v_{i^*})$ by (4.3). Substituting these three into (4.8), we obtain $\bar{c}_f(y, s^*, v_{i^*}) \geq \min\{\bar{c}_f(x, s^*, v_{i^*}), \bar{c}_f(y, u_{i^*}, v_{i^*})\}$.

For the second claim, assume $\bar{c}_f(x, u_j, t^*) = 0$ ($\forall j < i^*$). Then (a2)-(ii) implies $\bar{c}_f(y_j, u_{i^*}, t^*) = \bar{c}_f(y_{j-1}, u_{i^*}, t^*)$ for every $j < i^*$, and we have $\bar{c}_f(y_{i^*-1}, u_{i^*}, t^*) = \bar{c}_f(x, u_{i^*}, t^*)$. Note that every $l \in [i^* + 1, d]$ satisfies $i^* \in [l - 1]$. Then repeated application of (4.7) yields $\bar{c}_f(y_{i^*}, u_{i^*}, t^*) = \bar{c}_f(y, u_{i^*}, t^*)$. Substituting these two and $\bar{c}_f(y_{i^*}, u_{i^*}, v_{i^*}) = \bar{c}_f(y, u_{i^*}, v_{i^*})$ into (4.9), we obtain $\bar{c}_f(y, u_{i^*}, t^*) \geq \min\{\bar{c}_f(x, u_{i^*}, t^*), \bar{c}_f(y, u_{i^*}, v_{i^*})\}$.

(b4): Every $l \in [d - 1]$ satisfies $\bar{c}_f(x, u_l, v_d) = 0$ by $l < d$, and hence $u_l \in \text{sat}_f(x)$.

(b5): If $u_d \in \text{sat}_f(x)$, as we have (b4), $u_l \in \text{sat}_f(x)$ holds for every $l \in [d]$. Then, by repeating application of Lemma 4.3 (a4) d times, we obtain $\text{sat}_f(y_l) = \text{sat}_f(x)$ and $\hat{c}_f(y_l, s) = \hat{c}_f(x, s)$ ($\forall s \in E \setminus \text{sat}_f(x)$) for each y_l , and so for $y = y_d$.

(b6): As shown in the proof of (b4), we have $\text{sat}_f(y_{d-1}) = \text{sat}_f(x)$ and $\hat{c}_f(y_{d-1}, s) = \hat{c}_f(x, s)$ ($\forall s \in E \setminus \text{sat}_f(x)$). Also, $\bar{c}_f(y_{d-1}, s, v_d) = \bar{c}_f(x, s, v_d)$ ($\forall s \in E \setminus \{v_d\}$) by (4.6). If $u_d \notin \text{sat}_f(x)$, by applying Lemma 4.3 (a5) for y_{d-1} , we obtain the claim. \square

Lemma 4.5 (Simple Augmentation). For any $x \in \mathbf{P}(f)$ and $u \in E$, define $y \in \mathbf{R}^E$ by $y := x + \alpha\chi_u$ with $0 \leq \alpha \leq \hat{c}_f(x, u)$. Then, we have $y \in \mathbf{P}(f)$ and the following (c1)–(c3).

(c1) $\hat{c}_f(y, u) = \hat{c}_f(x, u) - \alpha$.

(c2) For every $s \in \text{sat}_f(x)$ and $t \in E \setminus \{s\}$, we have $\bar{c}_f(y, s, t) = \bar{c}_f(x, s, t)$.

(c3) Every $s \in \text{sat}_f(x)$ satisfies $s \in \text{sat}_f(y)$ and $\text{dep}_f(y, s) = \text{dep}_f(x, s)$.

Proof. We need only consider the case that $\hat{c}_f(x, u) > 0$, i.e., $u \notin \text{sat}_f(x)$. By the definition, $y \in \mathbf{P}(f)$ and condition (c1) are obvious. For (c2), let $s \in \text{sat}_f(x)$ and $C := \text{sat}_f(x)$. Then $s \in C$, $u \notin C$, and $x(C) = y(C) = f(C)$. Take any $t \in E \setminus \{s\}$. Then for any $A \subseteq E$ with $s \in A$ and $t \notin A$, we have $s \in A \cap C$ and $t \notin A \cap C$ and $x(A \cap C) = y(A \cap C)$. Hence, as is the case in Lemma 4.3 (a1), we obtain $\bar{c}_f(y, s, t) = \bar{c}_f(x, s, t)$. The statement (c3) follows from $u \notin \text{sat}_f(x)$, which implies $u \notin \text{dep}_f(x, s)$ for any $s \in \text{sat}_f(x)$, and hence $y(A) = x(A)$ for every $A \subseteq \text{dep}_f(x, s)$. \square

4.2.2 Preferences on Polymatroids

We now introduce a partial order on vectors, which is induced from a total order on the ground set. This partial order is regarded as a preference order in our model.

Let \succ be a total order on E . For any element $a \in E$, we denote $E_{\succeq a} := \{e \in E \mid e \succeq a\}$. For two vectors $x, y \in \mathbf{R}^E$, we say that x is \succ -preferable to y if

$$\forall a \in E : x(E_{\succeq a}) \geq y(E_{\succeq a}).$$

A vector x is \succ -optimal in a set $K \subseteq \mathbf{R}^E$ if $x \in K$ and x is \succ -preferable to every $y \in K$.

Lemma 4.6. *Let (E, f, \succ) be an ordered polymatroid. For $x, y \in \mathbf{P}(f)$ if*

$$\forall e \in E : x(e) \geq y(e) \quad \text{or} \quad [e \in \text{sat}_f(x), \text{dep}_f(y, e) \succeq e],$$

then x is \succ -preferable to y .

Proof. Take an arbitrary $a \in E$. We show $x(E_{\succeq a}) \geq y(E_{\succeq a})$. Define

$$C := \bigcup \{ \text{dep}_f(x, e) \mid e \in E_{\succeq a}, [e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq e] \}.$$

Then $C \subseteq E_{\succeq a}$ and every $e \in E_{\succeq a} \setminus C$ satisfies $x(e) \geq y(e)$. Also, since C is a union of tight sets, it is also tight. As $y \in \mathbf{P}(f)$, this implies $x(C) = f(C) \geq y(C)$. Then, we have $x(E_{\succeq a}) = x(C) + x(E_{\succeq a} \setminus C) \geq y(C) + y(E_{\succeq a} \setminus C) = y(E_{\succeq a})$. \square

Lemma 4.7. *Let (E, f, \succ) be an ordered polymatroid. For $x, y \in \mathbf{P}(f)$ and $a \in E$, if we have*

$$\forall e \in E_{\succeq a} : x(e) \geq y(e) \quad \text{or} \quad [e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq e],$$

then we have

$$\forall e \in E_{\succeq a} : y(e) \geq x(e) \quad \text{or} \quad \neg[e \in \text{sat}_f(y), \text{dep}_f(y, e) \succeq e].$$

Proof. Assume that each $e \in E_{\succeq a}$ satisfies $x(e) \geq y(e)$ or $[e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq e]$. Also assume that $e' \in E_{\succeq a}$ satisfies $[e' \in \text{sat}_f(y), \text{dep}_f(y, e') \succeq e']$. We now show $y(e') \geq x(e')$. Let $C_y := \text{dep}_f(y, e')$ and

$$C_x := \bigcup \{ \text{dep}_f(x, e) \mid e \in C_y \setminus \{e'\}, [e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq e] \}$$

and $D := (C_y \setminus C_x) \setminus \{e'\}$. Then, $\{\{e'\}, C_y \cap C_x, D\}$ is a partition of C_y . By definition, $y(C_y) = f(C_y)$ and $x(C_x) = f(C_x)$, and hence the submodularity of f implies

$$y(C_y) - y(C_y \cap C_x) \geq f(C_y) - f(C_y \cap C_x) \geq f(C_y \cup C_x) - f(C_x) \geq x(C_y \cup C_x) - x(C_x).$$

As we have $C_y = \text{dep}_f(y, e') \succeq e' \in E_{\succeq a}$, we obtain $C_y \subseteq E_{\succeq a}$, and hence every $e \in D = (C_y \setminus C_x) \setminus \{e'\}$ satisfies $x(e) \geq y(e)$, which implies $x(D) \geq y(D)$. Then, we have $y(e') = y(C_y) - y(C_y \cap C_x) - y(D) \geq x(C_y \cup C_x) - x(C_x) - x(D) = x(e')$. \square

Remark 4.8. Here we show the equivalence between two definitions of the stability defined in this chapter and in Section 3.7. In Introduction of this chapter, a stable allocation for $\{(E_i, h_i, \succ_i)\}_{i \in I}$ and $\{(E_j, f_j, \succ_j)\}_{j \in J}$ is defined as a vector satisfying

1. $x_i \in \mathbf{P}(h_i)$ for all $i \in I$ and $x_j \in \mathbf{P}(f_j)$ for all $j \in J$.
2. $\forall e = (i, j) \in E : [e \in \text{sat}_{h_i}(x_i), \text{dep}_{h_i}(x_i, e) \succeq_i e] \text{ or } [e \in \text{sat}_{f_j}(x_j), \text{dep}_{f_j}(x_j, e) \succeq_j e]$.

The stability in Section 3.7 also requires Condition 1 above and, instead of Condition 2, the following condition: For every pair $e = (i, j) \in E$,

- x_i is \succ_i -optimal in $P_i := \{y_i \in \mathbf{P}(h_i) \mid y_i(e') \leq x_i(e') \text{ } (e' \in E_i \setminus \{e\})\}$ or
- x_j is \succ_j -optimal in $P_j := \{y_j \in \mathbf{P}(f_j) \mid y_j(e') \leq x_j(e') \text{ } (e' \in E_j \setminus \{e\})\}$.

Therefore, it suffices to show the equivalence between the following two conditions:

(i) $[e \in \text{sat}_{h_i}(x_i), \text{dep}_{h_i}(x_i, e) \succeq_i e]$, and (ii) x_i is \succ_i -optimal in P_i .

We first show (ii) assuming (i). Take any $y_i \in P_i$. Then $y_i(e') \leq x_i(e')$ for every $e' \in E_i \setminus \{e\}$. With (i) and Lemma 4.6, then x_i is \succ_i -preferable to y_i . Thus, (ii) holds.

Next, we show that (ii) fails if (i) fails. If $e \notin \text{sat}_{h_i}(x_i)$, then $x'_i = x_i + \alpha \chi_e \in P_i$ for some $\alpha > 0$. Otherwise, $\text{dep}_{h_i}(x_i, e) \not\succeq_i e$, and hence $x''_i = x_i + \alpha(\chi_e - \chi_{e'}) \in P_i$ for some $\alpha > 0$ and e' with $e \succ_i e'$. Both x'_i and x''_i are \succ_i -preferable to x_i , and hence (ii) fails. \blacksquare

4.3 Algorithm

In this section, we present an algorithm for finding a stable allocation for (E, h, \succ_H) and (E, f, \succ_F) . The algorithm adopts the augmenting path technique for polymatroid intersection [95]. Each iteration searches for an augmenting path by simulating a chain of proposes and rejects in the deferred acceptance algorithm [41].

We first introduce *pointer functions* next_h and next_f with reference to (E, h, \succ_H) and (E, f, \succ_F) , respectively. In our algorithm, they play fundamental roles of suggesting which element to propose or reject at the next step.

For $D \subseteq E$, $x \in \mathbf{P}(h)$, and $v \in E$, define $\text{next}_h(D, x, v) \in E$ by

$$\text{next}_h(D, x, v) := \max_{\succ_H} \{u \in D \setminus \{v\} \mid \bar{c}_h(x, u, v) > 0\}.$$

If the set in the right-hand side is empty, then $\text{next}_h(D, x, v)$ is undefined. Note that $\text{next}_h(D, x, v)$ is the best element with respect to \succ_H in D to increase at an exchange with v .

For $x \in \mathbf{P}(f)$ and $u \in E$, define $\text{next}_f(x, u) \in E$ by

$$\text{next}_f(x, u) := \min_{\succ_F} \{ v \in E \mid \bar{c}_f(x, u, v) > 0 \}.$$

If $u \in \text{sat}_f(x)$, $\text{next}_f(x, u)$ represents the best element w.r.t. \succ_F to decrease at an exchange with u . Note that $\text{next}_f(x, u) = u$ means that every $v \in E$ with $u \succ_F v$ satisfies $\bar{c}_f(x, u, v) = 0$.

For a common independent vector $x \in \mathbf{P}(h) \cap \mathbf{P}(f)$, we introduce capacities of sequences. Let $P = \{u_0, v_1, u_1, \dots, v_k, u_k\}$ be a sequence of $2k + 1$ distinct elements of E . We define the *path-capacity* of P by

$$c(x, P) = \min \left\{ \hat{c}_h(x, u_0), \min_{i:1 \leq i \leq k} \bar{c}_f(x, u_{i-1}, v_i), \min_{i:1 \leq i \leq k} \bar{c}_h(x, u_i, v_i), \hat{c}_f(x, u_k) \right\}.$$

Let $Q = \{u_0, v_1, u_1, \dots, u_{k-1}, v_k\}$ be a sequence of $2k$ distinct elements of E . We define the *cycle-capacity* of Q by

$$c(x, Q) = \min \left\{ \min_{i:1 \leq i \leq k} \bar{c}_f(x, u_{i-1}, v_i), \min_{i:1 \leq i \leq k} \bar{c}_h(x, u_i, v_i) \right\},$$

where we regard $u_k := u_0$.

The algorithm keeps a vector $x \in \mathbf{P}(h) \cap \mathbf{P}(f)$ and updates it repeatedly. The algorithm also maintains two disjoint subsets $D, R \subseteq E$.

We now introduce three procedures which will be used in the algorithm. The procedure **Augment** is applied to odd-length sequences. For a sequence $P = \{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$ (P can be $\{u_0\}$), the procedure **Augment**(P) updates (x, D, R) by

$$\begin{aligned} x &\leftarrow x + c(x, P) \left(\chi_{u_0} + \sum_{j=1}^{k-1} (\chi_{u_j} - \chi_{v_j}) \right), \\ D &\leftarrow D \setminus \{v_1, v_2, \dots, v_{k-1}\}, \\ R &\leftarrow R \cup \{v_1, v_2, \dots, v_{k-1}\}. \end{aligned}$$

The procedure **Cycle** is applied to even-length sequences. For a sequence $Q = \{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ with $l < k$, the procedure **Cycle**(Q) updates (x, D, R) by

$$\begin{aligned} x &\leftarrow x + c(x, Q) \sum_{j=l+1}^k (\chi_{u_{j-1}} - \chi_{v_j}), \\ D &\leftarrow D \setminus \{v_{l+1}, v_{l+2}, \dots, v_k\}, \\ R &\leftarrow R \cup \{v_{l+1}, v_{l+2}, \dots, v_k\}. \end{aligned}$$

The procedure **Self-loop** is applied to an element e of D and moves e from D to R , i.e., $D \leftarrow D \setminus \{e\}$ and $R \leftarrow R \cup \{e\}$, without changing x .

We are now ready to describe the algorithm for finding a stable allocation.

Algorithm 1 Find a stable allocation**Input:** (E, h, \succ_H) and (E, f, \succ_F) ;**Output:** stable allocation;

```

1:  $D \leftarrow \emptyset, R \leftarrow \emptyset, x \leftarrow \mathbf{0}$ ;
2: while  $D \cup R \neq E$  do
3:    $e^* \leftarrow \max_{\succ_H} E \setminus (D \cup R)$ ;
4:    $D \leftarrow D \cup \{e^*\}$ ;
5:   while  $e^* \in D \setminus \text{sat}_h(x)$  do
6:      $u_0 \leftarrow e^*$ ;
7:     for  $k = 1, 2, \dots$  do
8:       if  $u_{k-1} \notin \text{sat}_f(x)$  then Augment( $\{u_0, v_1, \dots, u_{k-1}\}$ ) and break;
9:        $v_k \leftarrow \text{next}_f(x, u_{k-1})$ ;
10:      if  $v_k = u_{k-1}$  then Self-loop( $u_{k-1}$ ) and break;
11:      if  $v_k = v_l (\exists l < k)$  then Cycle( $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ ) and break;
12:       $u_k \leftarrow \text{next}_h(D, x, v_k)$ ;
13:      if  $u_k = u_l (\exists l < k)$  then Cycle( $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ ) and break;
14:    end for
15:  end while
16: end while
17: return  $x$ .
```

We now describe how our algorithm works on an example.

Example 4.9. Consider total orders \succ_H and \succ_F on $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ given by

$$\begin{aligned}
e_1 \succ_H e_2 \succ_H e_3 \succ_H e_4 \succ_H e_5 \succ_H e_6, \\
e_2 \succ_F e_3 \succ_F e_5 \succ_F e_6 \succ_F e_1 \succ_F e_4.
\end{aligned}$$

Let G_H and G_F be bipartite graphs with vertex sets $E \cup V_H$ and $E \cup V_F$ depicted in Figure 4.1, where vertex capacity functions $b_H : V_H \rightarrow \mathbf{R}$ and $b_F : V_F \rightarrow \mathbf{R}$ are also given. Define rank functions $h, f : 2^E \rightarrow \mathbf{R}$ by

$$\begin{aligned}
h(A) &= \sum \{ b_H(v) \mid v \in \Gamma_H(A) \} \quad (A \subseteq E), \\
f(A) &= \sum \{ b_F(v) \mid v \in \Gamma_F(A) \} \quad (A \subseteq E),
\end{aligned}$$

where $\Gamma_H(A)$ and $\Gamma_F(A)$ denote the sets of nodes adjacent to A in G_H and in G_F , respectively.

We now apply the algorithm to (E, h, \succ_H) and (E, f, \succ_F) . Just after Line 4 with $e^* = e_5$, we have $x = (5, 3, 0, 1, 0, 0)$ and its auxiliary graph is depicted in Figure 4.2 (a). Then, the algorithm searches an augmenting path and finds a cycle (e_5, e_4) whose cycle-capacity is 1. Hence, the procedure Cycle($\{e_5, e_4\}$) updates $x \leftarrow (5, 3, 0, 0, 1, 0)$. The updated auxiliary graph is depicted in Figure 4.2 (b). In the next search, the algorithm

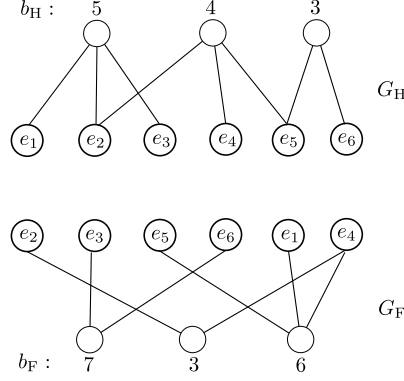
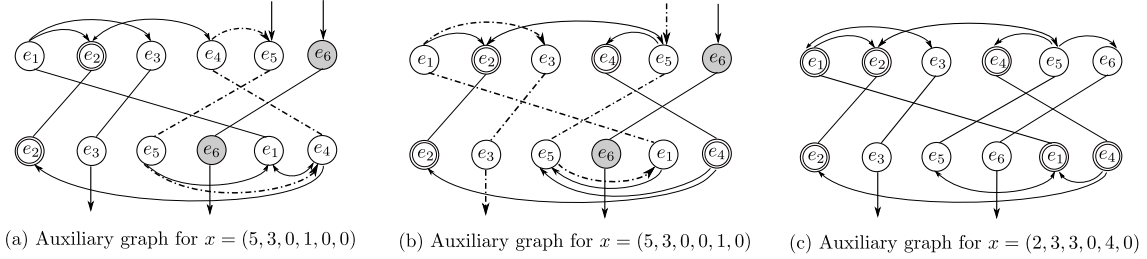
Figure 4.1: Bipartite graphs inducing h and f .

Figure 4.2: Auxiliary graphs for x at some points of the algorithm. Each graph has two lines of nodes. The upper line consists of the elements of E arranged in descending order w.r.t. \succ_H , and so does the lower line w.r.t. \succ_F . In the upper line, an arc from e to e' means $\bar{c}_h(x, e', e) > 0$ and an arc from the outside to e means $\hat{c}_h(x, e) > 0$. In the lower line, an arc from e to e' means $\bar{c}_f(x, e, e') > 0$ and an arc from e to the outside means $\hat{c}_f(x, e) > 0$. We omit some exchangeability arcs incident to unsaturated elements or implied by the transitivity. Elements in $D \subseteq E$ and $R \subseteq E$ are represented by single and double circles, respectively. Other elements are colored gray.

finds an augmenting path (e_5, e_1, e_3) with path-capacity 3, and calls **Augment** $(\{e_5, e_1, e_3\})$. Thus, x is updated as $x \leftarrow (2, 3, 3, 0, 4, 0)$. Then, we see $e_5, e_6 \in \text{sat}_h(x)$, and the algorithm terminates with returning this x , whose auxiliary graph is depicted in Figure 4.2 (c). In this graph, for any $e \in D$ in the upper line, every arc entering e comes from left, which means $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$. Also, for any $e \in R$, every arc leaving e in the lower line goes to left, i.e., $[e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq_F e]$. Since $D \cup R = E$, we may conclude that the output $x = (2, 3, 3, 0, 4, 0)$ is indeed a stable allocation. ■

4.4 Invariants

The main part of our analysis, provided in Section 4.5, is to show that the following invariants are maintained for (x, D, R) throughout the algorithm.

- (P) $\{D, R\}$ is a partition of $E_{\succeq_H e^*}$ and $x(e) = 0$ for every $e \in E \setminus (D \cup R)$, where $e^* := \min_{\succ_H} D$.
- (H) $x \in \mathbf{P}(h)$ holds and every $e \in D \setminus \{e^*\}$ satisfies $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$, where $e^* := \min_{\succ_H} D$. Also, for each $t \in E$, the inequality $\bar{c}_h(x, e^*, t) > 0$ implies $t \succeq_H e^*$.
- (F) $x \in \mathbf{P}(f)$ holds and every $e \in R$ satisfies $[e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq_F e]$.

As will be shown in Section 4.5, if these conditions hold at the end of the algorithm, then the output is a stable allocation.

In this section, we prepare three lemmas related to these invariants. They will be used to analyze the algorithm later in Sections 4.5–4.7.

Lemma 4.10 (Multiple Exchange in $\mathbf{P}(h)$). *For an independent vector $x \in \mathbf{P}(h)$ and $D \subseteq E$, let u_i, v_i ($i = 1, 2, \dots, d$) be $2d$ distinct elements of E such that*

$$(h0) \text{ next}_h(D, x, v_i) = u_i$$

holds for every $i \in [d]$. For any $\alpha \geq 0$ satisfying $\alpha \leq \bar{c}_h(x, u_i, v_i)$ for every $i \in [d]$, define $y \in \mathbf{R}^E$ by $y := x + \alpha \sum_{i=1}^d (\chi_{u_i} - \chi_{v_i})$. Then, we have $y \in \mathbf{P}(h)$ and the following (h1)–(h3), where $E' := E \setminus \{u_1, v_1, \dots, u_d, v_d\}$ and $D' := D \setminus \{v_1, v_2, \dots, v_d\}$.

- (h1) *If (H) holds for x and D , then the same statement holds with x and D replaced by y and D' , respectively.*
- (h2) *For $t \in E \setminus \{u_1, u_2, \dots, u_d\}$, if $\text{next}_h(D, x, t)$ is defined, then $\text{next}_h(D', y, t)$ is undefined or it is defined and satisfies $\text{next}_h(D, x, t) \succeq_H \text{next}_h(D', y, t)$. If $\text{next}_h(D, x, t)$ is undefined, then $\text{next}_h(D', y, t)$ is also undefined.*
- (h3) *For $i \in [d]$, we have $\bar{c}_h(y, u_i, v_i) = \bar{c}_h(x, u_i, v_i) - \alpha$. For $s, t \in E'$ with $\text{next}_h(D, x, t) = s$, we have $\bar{c}_h(y, s, t) = \bar{c}_h(x, s, t)$. For $t \in E'$ and $i \in [d]$ with $\text{next}_h(D, x, t) = u_i$, we have $\bar{c}_h(y, u_i, t) \geq \min\{\bar{c}_h(x, u_i, t), \bar{c}_h(y, u_i, v_i)\}$.*

Proof. Without loss of generality, we can assume $u_1 \succ_H u_2 \succ_H \dots \succ_H u_d$. Then, the definition of next_h and the assumption that (h0) holds for every $i \in [d]$ imply

$$u_i \in D \setminus \{v_1, v_2, \dots, v_d\} \quad (i \in [d]), \quad (4.10)$$

$$\bar{c}_h(x, u_i, v_i) > 0 \quad (i \in [d]), \quad (4.11)$$

$$s \succ_H u_i \implies \bar{c}_h(x, s, v_i) = 0 \quad (i \in [d], s \in D \setminus \{v_i\}). \quad (4.12)$$

In particular, since $i < j$ implies $u_i \succ_H u_j$ for every $i, j \in [d]$, the condition (4.12) yields

$$\bar{c}_h(x, u_i, v_j) = 0 \quad (i, j \in [d] \text{ with } i < j). \quad (4.13)$$

By (4.11) and (4.13), we can apply Lemma 4.4 on multiple exchange to see that $y \in \mathbf{P}(h)$ holds.

(h1): Suppose that (H) holds for x and D . This is equivalent to the combination of the following two conditions.

(H1) $x \in \mathbf{P}(h)$ and $D \setminus \{\min_{\succ_H} D\} \subseteq \text{sat}_h(x)$.

(H2) For any $s \in D$ and $t \in E$, if $\bar{c}_h(x, s, t) > 0$, then $t \succeq_H s$.

We show that (H1) and (H2) hold with x and D replaced by y and D' . We already have $y \in \mathbf{P}(h)$. Fix $e^* := \min_{\succ_H} D$, and note that $D' \subseteq D$ implies $D' \setminus \{\min_{\succ_H} D'\} \subseteq D' \setminus \{e^*\}$.

To obtain (H1) for y and D' , it suffices to show $D' \setminus \{e^*\} \subseteq \text{sat}_h(y)$. Recall that we have (4.10), (4.11), and (4.13). If $u_d \neq e^*$, then $u_d \in D \setminus \{e^*\} \subseteq \text{sat}_h(x)$ and Lemma 4.4 (b5) implies $\text{sat}_h(y) = \text{sat}_h(x)$, and hence $D' \setminus \{e^*\} \subseteq D \setminus \{e^*\} \subseteq \text{sat}_h(x) = \text{sat}_h(y)$. Also, if $u_d = e^*$, then $\text{next}_h(D, x, v_d) = u_d = e^* = \min_{\succ_H} D$. This implies that, for every $s \in D \setminus \{e^*, v_d\}$, we have $\bar{c}_h(x, s, v_d) = 0$, and hence $s \in \text{sat}_h(y)$ by (b6). Thus, $D' \setminus \{e^*\} \subseteq D \setminus \{e^*, v_d\} \subseteq \text{sat}_h(y)$.

We show (H2) for y and D' . Suppose, to the contrary, that there are $s \in D'$ and $t \in E$ such that $\bar{c}_h(y, s, t) > 0$ and $s \succ_H t$. Then $s \in D' \subseteq D$ and $s \succ_H t$ imply $\bar{c}_h(x, s, t) = 0$, because (H2) holds for x and D . As we have $\bar{c}_h(x, s, t) = 0$ and $\bar{c}_h(y, s, t) > 0$, Lemma 4.4 (b2) implies that there are $i, j \in [d]$ such that

$$i \leq j, \quad \bar{c}_h(x, s, v_j) > 0, \quad \bar{c}_h(x, u_i, t) > 0.$$

Since $\text{next}_h(D, x, v_j) = u_j$, conditions $s \in D' \subseteq D \setminus \{v_j\}$ and $\bar{c}_h(x, s, v_j) > 0$ imply $u_j \succeq_H s$. As (H2) holds for x and D , conditions $u_i \in D$ and $\bar{c}_h(x, u_i, t) > 0$ imply $t \succeq_H u_i$. Also, $i \leq j$ implies $u_i \succeq_H u_j$. Thus, we obtain $t \succeq_H u_i \succeq_H u_j \succeq_H s$, which contradicts $s \succ_H t$.

(h2): Suppose, to the contrary, the claim fails for some $t \in E \setminus \{u_1, u_2, \dots, u_d\}$. Then, either of the following holds.

1. Both $s = \text{next}_h(D, x, t)$ and $s' = \text{next}_h(D', y, t)$ are defined and $s' \succ_H s$.
2. $s' = \text{next}_h(D', y, t)$ is defined while $\text{next}_h(D, x, t)$ is undefined.

In Case 1, we have $s \in D \setminus \{t\}$ and $s' \in D' \setminus \{t\} \subseteq D \setminus \{t\}$. By $s' = \text{next}_h(D', y, t)$, the definition of next_h implies $\bar{c}_h(y, s', t) > 0$. Also, by $s' \succ_H s = \text{next}_h(D, x, t)$ and $s' \in D \setminus \{t\}$, we see that $\bar{c}_h(x, s', t) = 0$. By Lemma 4.4 (b2), then there are $i, j \in [d]$ such that

$$i \leq j, \quad \bar{c}_h(x, s', v_j) > 0, \quad \bar{c}_h(x, u_i, t) > 0.$$

Note that $u_i \in D \setminus \{t\}$ by $t \in E \setminus \{u_1, u_2, \dots, u_d\}$. Then, $\bar{c}_h(x, u_i, t) > 0$ and $\text{next}_h(D, x, t) = s$ imply $s \succ_H u_i$. Similarly, by $s' \in D' \subseteq D \setminus \{v_j\}$, $\bar{c}_h(x, s', v_j) > 0$, and $\text{next}_h(D, x, v_j) = u_j$, we have $u_j \succeq_H s'$. Also, $i \leq j$ implies $u_i \succeq_H u_j$. Thus, we obtain $s \succeq_H u_i \succeq_H u_j \succeq_H s'$, which contradicts $s' \succ_H s$.

In Case 2, we also have $\bar{c}_h(y, s', t) > 0$ and $\bar{c}_h(x, s', t) = 0$, and hence there are $i, j \in [d]$ such that $i \leq j$, $\bar{c}_h(x, s', v_j) > 0$, and $\bar{c}_h(x, u_i, t) > 0$. Since $u_i \in D \setminus \{t\}$, the condition $\bar{c}_h(x, u_i, t) > 0$ contradicts the fact that $\text{next}_h(D, x, t)$ is undefined.

(h3): The first claim of (h3) immediately follows from Lemma 4.4 (b1).

To show the second claim by contradiction, suppose $\bar{c}_h(y, s, t) \neq \bar{c}_h(x, s, t)$ for some $s, t \in E'$ with $\text{next}_h(D, x, t) = s$. By (b2), there exist $i, j \in [d]$ such that $i \leq j$, $\bar{c}_h(x, s, v_j) > 0$, and $\bar{c}_h(x, u_i, t) > 0$. Since $\text{next}_h(D, x, v_j) = u_j$, the condition $\bar{c}_h(x, s, v_j) > 0$ implies $u_j \succeq_H s$, and in particular $u_j \succ_H s$ by $s \in E'$. Also, $\text{next}_h(D, x, t) = s$ and $\bar{c}_h(x, u_i, t) > 0$ imply $s \succ_H u_i$, and hence $u_j \succ_H u_i$, which contradicts $i \leq j$.

We show the last claim of (h3). Let $t \in E'$ and $i \in [d]$ be such that $\text{next}_h(D, x, t) = u_i$. Every $j \in [d]$ with $j < i$ satisfies $u_j \succ_H u_i$ and $u_j \in D \setminus \{v_j\}$, and hence the condition $\text{next}_h(D, x, t) = u_i$ implies $\bar{c}_h(x, u_j, t) = 0$. By Lemma 4.4 (b3), then we obtain $\bar{c}_h(y, u_i, t) \geq \min\{\bar{c}_h(x, u_i, t), \bar{c}_h(y, u_i, v_i)\}$. \square

Lemma 4.11 (Multiple Exchange in $\mathbf{P}(f)$). *For an independent vector $x \in \mathbf{P}(f)$, let u_{i-1}, v_i ($i = 1, 2, \dots, d$) be $2d$ distinct elements of E such that*

$$(f0) \quad u_{i-1} \in \text{sat}_f(x), \quad \text{next}_f(x, u_{i-1}) = v_i$$

holds for every $i \in [d]$. For any $\alpha \geq 0$ satisfying $\alpha \leq \bar{c}_h(x, u_i, v_i)$ for every $i \in [d]$, define $y \in \mathbf{R}^E$ by $y := x + \alpha \sum_{i=1}^d (\chi_{u_{i-1}} - \chi_{v_i})$. Then, we have $y \in \mathbf{P}(f)$ and the following (f1)–(f3), where $E' = E \setminus \{u_0, v_1, u_1, \dots, u_{d-1}, v_d\}$.

(f1) *If (F) holds for x and some $R \subseteq E$, then the same statement holds with x and R replaced by y and $R' := R \cup \{v_1, v_2, \dots, v_d\}$, respectively.*

(f2) *For every $s \in \text{sat}_f(x)$, we have $s \in \text{sat}_f(y)$ and $\text{next}_f(y, s) \succeq_F \text{next}_f(x, s)$.*

(f3) *For $i \in [d]$, we have $\bar{c}_f(y, u_{i-1}, v_i) = \bar{c}_f(x, u_{i-1}, v_i) - \alpha$. For $s, t \in E'$ with $\text{next}_f(x, s) = t$, we have $\bar{c}_f(y, s, t) = \bar{c}_f(x, s, t)$. For $s \in E'$ and $i \in [d]$ with $\text{next}_f(x, s) = v_i$, we have $\bar{c}_f(y, s, v_i) \geq \min\{\bar{c}_f(x, s, v_i), \bar{c}_f(y, u_{i-1}, v_i)\}$.*

Proof. Without loss of generality, we can assume $v_1 \succ_F v_2 \succ_F \dots \succ_F v_d$. Then, the definition of next_f and the assumption that (f0) holds for every $i \in [d]$ imply

$$\bar{c}_f(x, u_i, v_i) > 0 \quad (i \in [d]), \quad (4.14)$$

$$v_i \succ_F t \implies \bar{c}_f(x, u_i, t) = 0 \quad (i \in [d], t \in E). \quad (4.15)$$

In particular, since $i < j$ implies $v_i \succ_F v_j$, the condition (4.15) yields

$$\bar{c}_f(x, u_i, v_j) = 0 \quad (i, j \in [d] \text{ with } i < j). \quad (4.16)$$

By (4.14), (4.16), we can apply Lemma 4.4 on multiple exchange. Then, $y \in \mathbf{P}(f)$. Since (f0) says $u_{i-1} \in \text{sat}_f(x)$ for every $i \in [d]$, Lemma 4.4 (b5) implies $\text{sat}_f(y) = \text{sat}_f(x)$. Also,

(f0) implies $v_i \in \text{sat}_f(x)$ for every $i \in [d]$, and hence we have

$$\{v_1, v_2, \dots, v_d\} \subseteq \text{sat}_f(x) = \text{sat}_f(y). \quad (4.17)$$

We show (f2) first, and then (f1) and (f3).

(f2): Since we have $\text{sat}_f(y) = \text{sat}_f(x)$, it suffices to show $\text{next}_f(y, s) \succeq_F \text{next}_f(x, s)$ for every $s \in \text{sat}_f(x)$. Suppose, to the contrary, there is $s \in \text{sat}_f(x)$ such that $t = \text{next}_f(x, s)$, $t' = \text{next}_f(y, s)$ and $t' \prec_F t$. By the definition of next_f , we have $\bar{c}_f(y, s, t') > 0$ and $\bar{c}_h(x, s, t') = 0$. By Lemma 4.4 (b2), then there are $i, j \in [d]$ such that $i \leq j$, $\bar{c}_f(x, s, v_j) > 0$, and $\bar{c}_f(x, u_i, t') > 0$. By $\bar{c}_h(x, u_i, t') > 0$ and $\text{next}_f(x, u_i) = v_i$, we have $t' \succeq_F v_i$. Similarly, $\bar{c}_f(x, s, v_j) > 0$ and $\text{next}_f(x, s) = t$ imply $v_j \succeq_F t$. Also, $i \leq j$ implies $v_i \succeq_F v_j$. Thus, we have $t' \succeq_F v_i \succeq_F v_j \succeq_F t$, a contradiction.

(f1): Suppose that (F) holds for x and R . By the definition of next_f , for any $s \in \text{sat}_f(x)$, the condition $\text{dep}_f(x, s) \succeq_F s$ is equivalent to $\text{next}_f(x, s) = s$. Then, (F) is rephrased as

$$\forall s \in R: s \in \text{sat}_f(x), \text{next}_f(x, s) = s.$$

We show that (F) holds for y and R' . By (F) for x and R , we have $R \subseteq \text{sat}_f(x)$. Then, (4.17) implies $R' = R \cup \{v_1, v_2, \dots, v_d\} \subseteq \text{sat}_f(x) = \text{sat}_f(y)$. By (f2) and the definition of next_f , every $s \in R' \subseteq \text{sat}_f(x)$ satisfies $s \succeq_F \text{next}_f(y, s) \succeq_F \text{next}_f(x, s)$. Hence, it suffices to show $\text{next}_f(x, s) = s$ for every $s \in R' = R \cup \{v_1, v_2, \dots, v_d\}$. For every $s \in R$, we have $\text{next}_f(x, s) = s$ by (F) for x and R . Also, every v_i satisfies $\text{next}_f(x, v_i) = v_i$ as follows. Suppose, to the contrary, we have $\text{next}_f(x, v_i) = t \prec_F v_i$. Then, $\bar{c}_f(x, v_i, t) > 0$. Since we also have $\bar{c}_f(x, u_i, v_i) > 0$ by $\text{next}_f(x, u_i) = v_i$, the transitivity (Lemma 4.2) implies $\bar{c}_f(x, u_i, t) > 0$. This contradicts $\text{next}_f(x, u_i) = v_i \succ_F t$.

(f3): The first claim of (f3) immediately follows from Lemma 4.4 (b1).

To show the second claim by contradiction, suppose $\bar{c}_f(y, s, t) \neq \bar{c}_f(x, s, t)$ for some $s, t \in E'$ with $\text{next}_f(x, s) = t$. By (b2), there exist $i, j \in [d]$ such that $i \leq j$, $\bar{c}_f(x, s, v_j) > 0$, and $\bar{c}_f(x, u_{i-1}, t) > 0$. Since $\text{next}_f(x, u_{i-1}) = v_i$, the condition $\bar{c}_f(x, u_{i-1}, t) > 0$ implies $t \succeq_F v_i$, and in particular $t \succ_F v_i$ by $t \in E'$. Also, $\text{next}_f(x, s) = t$ and $\bar{c}_f(x, s, v_j) > 0$ imply $v_j \succ_F t$, and hence $v_j \succ_F v_i$, which contradicts $i \leq j$.

We show the last claim of (f3). Let $s \in E'$ and $i \in [d]$ be such that $\text{next}_f(x, s) = v_i$. Since every $j \in [d]$ with $j > i$ satisfies $v_i \succ_F v_j$, the condition $\text{next}_f(x, s) = v_i$ implies $\bar{c}_f(x, s, v_j) = 0$. By Lemma 4.4 (b3), then we obtain $\bar{c}_f(y, s, v_i) \geq \min\{\bar{c}_f(x, s, v_i), \bar{c}_f(y, u_{i-1}, v_i)\}$. \square

Lemma 4.12. Assume that (P) holds for $x \in \mathbf{P}(h) \cap \mathbf{P}(f)$ and $D, R \subseteq E$. Let u_0, u_i, v_i ($i = 1, 2, \dots, d$) be (not necessarily distinct) elements such that $u_0 \in D$ and (h0) and (f0) hold for every $i \in [d]$. Define $y := x + \alpha \sum_{i=1}^d (\chi_{u_{i-1}} - \chi_{v_i})$ for an arbitrary $\alpha \geq 0$. Then (P) also holds with (x, D, R) replaced by (y, D', R') , where $D' := D \setminus \{v_1, v_2, \dots, v_d\}$ and $R' := R \cup \{v_1, v_2, \dots, v_d\}$.

Proof. By (f0), we have $x(v_i) > 0$ for every $i \in [d]$. Then $\{v_1, v_2, \dots, v_d\} \subseteq D \cup R$ since x satisfies (P). Therefore, $D' \cup R' = D \cup R \cup \{v_1, v_2, \dots, v_d\} = D \cup R$. By $u_0 \in D$ and (h0), we have $\{u_0, u_1, \dots, u_{d-1}\} \subseteq D \subseteq D' \cup R'$. Then, every $e' \in E \setminus (D' \cup R') = E \setminus (D \cup R)$ satisfies $y(e') = x(e') = 0$. Thus, (P) holds with (x, D, R) replaced by (y, D', R') . \square

4.5 Correctness

In this section, we show that the output of the algorithm is indeed a stable allocation. We first provide basic observations from the description of the algorithm. Recall that $[k, l] = \{k, k+1, \dots, l\}$ for two nonnegative integers $k, l \in \mathbf{Z}_+$ with $k \leq l$.

Claim 4.13. *Just before Line 11, $v_i \neq u_j$ for every $i \in [k]$ and $j \in [0, k-1]$.*

Proof. By Line 10, $v_i \neq u_{i-1}$ for every $i \in [k]$. We then show the case that $j < i-1$ or $j > i-1$. Suppose, to the contrary, that $v_i = u_j$ for such i and j . The definition of next_f implies $\bar{c}_f(x, u_j, v_{j+1}) > 0$ and $\bar{c}_f(x, u_{i-1}, v_i) = \bar{c}_f(x, u_{i-1}, u_j) > 0$. Then, the transitivity of dependence (Lemma 4.2) implies $\bar{c}_f(x, u_{i-1}, v_{j+1}) > 0$. As $\text{next}_f(x, u_{i-1}) = v_i$, this implies $v_{j+1} \succeq_F v_i = u_j$, and hence $v_{j+1} \succeq_F u_j$. On the other hand, by Line 10 for $j \in [0, k-1]$, we have $\text{next}_f(x, u_j) = v_{j+1} \neq u_j$, which implies $u_j \succ_F v_{j+1}$. This contradicts $\text{next}_f(x, u_j) \preceq_F u_j$, which must hold by the definition of next_f . \square

Claim 4.14. *Just before Line 12, $e^* \in D \setminus \{v_k\}$ and $\bar{c}_h(x, e^*, v_k) > 0$ hold. Hence, just after Line 12, $\text{next}_h(D, x, v_k)$ satisfies $\text{next}_h(D, x, v_k) \succeq_H e^*$.*

Proof. Just before Line 12, $x(v_k) > 0$ by $\text{next}_f(x, u_{k-1}) = v_k \neq u_{k-1}$. Also, by Line 5, we have $e^* \in D$ and $e^* \notin \text{sat}_h(x)$, which implies $\bar{c}_f(x, e^*, v_k) > 0$. Also, $e^* = u_0 \neq v_k$ by Claim 4.13. \square

Lemma 4.15. *In the algorithm, the following statements hold.*

- When **Self-loop**(u_{k-1}) is called, $u_{k-1} \in \text{sat}_f(x)$ and $\text{next}_f(x, u_{k-1}) = u_{k-1}$.
- When **Cycle**($\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$) is called, the elements of $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ are all distinct and every $i \in [l+1, k]$ satisfies (h0) and (f0), where $u_k = u_l$.
- When **Augment**($\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$) is called, the elements of $\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$ are all distinct, we have $u_0 \in D \setminus \text{sat}_h(x)$ and $u_{k-1} \notin \text{sat}_f(x)$, and every $i \in [k-1]$ satisfies (h0) and (f0).

Proof. By the algorithm, when **Cycle** is called, we have $u_i \neq u_j$ and $v_i \neq v_j$ for every distinct $i, j \in [l+1, k]$, where $u_k = u_l$. With Lemma 4.13, then all elements in the sequence are distinct. Similarly, when **Augment** is called, $u_i \neq u_j$ for every distinct $i, j \in [0, k-1]$ and $v_i \neq v_j$ for every distinct $i, j \in [k-1]$, and hence all elements in the sequence are distinct. Other statements immediately follow from the algorithm. \square

We prove the correctness of the algorithm by showing that conditions (P), (H), and (F) are maintained throughout the algorithm.

Observation 4.16. *When the algorithm reaches Line 5 for the first time, (P), (H), and (F) hold.*

Lemma 4.17. *Suppose that conditions (P), (H), and (F) hold just after Line 15. If $D \cup R \neq E$, then the conditions also hold when the algorithm reaches Line 5 the next time. If $D \cup R = E$, then the algorithm halts and the current x is a stable allocation.*

Proof. After Line 15, we have $e^* \notin D \setminus \text{sat}_h(x)$ by Line 5, and hence $e^* \notin D$ or $e^* \in \text{sat}_h(x)$. Also, in the latter case, $\text{dep}_h(x, e^*) \succeq_H e^*$ because $x(e) = 0$ for every $e \in E$ with $e^* \succ_H e$. Then, just after Line 15, every $e \in D$ satisfies $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$. This means that (H) holds when the algorithm reaches Line 5 next time. Also, (P) and (F) obviously hold since x and R do not change. If $D \cup R = E$, these conditions mean that x is a stable allocation. \square

By Observation 4.16 and Lemma 4.17, what is left is to show that conditions (P), (H), and (F) are maintained whenever the procedures are applied.

Lemma 4.18. *If (P), (H), and (F) hold when the procedure $\text{Self-loop}(u_{k-1})$ is called, then they also hold just after the procedure.*

Proof. The procedure $\text{Self-loop}(u_{k-1})$ does not change the vector x and replaces (D, R) by $(D \setminus \{u_{k-1}\}, R \cup \{u_{k-1}\})$. Clearly the conditions (P) and (H) are maintained. The element u_{k-1} , which is added to R , satisfies $u_{k-1} \in \text{sat}_f(x)$ by Line 8 and $\text{next}_f(x, u_{k-1}) = u_{k-1}$ by Line 10, which implies $\text{dep}_f(x, u_{k-1}) \succeq_F u_{k-1}$. Thus, (F) is maintained. \square

Lemma 4.19. *If (P), (H), and (F) hold when the procedure $\text{Cycle}(\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\})$ is called, then they also hold just after the procedure.*

Proof. Let Q denote the sequence $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$. The procedure updated the triple (x, D, R) to (y, D', R') , where

$$\begin{aligned} y &= x + c(x, Q) \sum_{i=l+1}^k (\chi_{u_{i-1}} - \chi_{v_i}), \\ D' &= D \setminus \{v_{l+1}, v_{l+2}, \dots, v_k\}, \\ R' &= R \cup \{v_{l+1}, v_{l+2}, \dots, v_k\}. \end{aligned}$$

We now show that conditions (P), (H), and (F) hold with (x, D, R) replaced by (y, D', R') . By Lemma 4.15, $u_l, v_{l+1}, \dots, u_{k-1}, v_k$ are distinct and every $i \in [l+1, k]$ satisfies (h0) and (f0), where $u_k = u_l$. Then by Lemma 4.11 (f1), the condition (F) holds with (x, R) replaced by (y, R') . Note that y is also written as $y = x + c(x, Q) \sum_{i=l+1}^k (\chi_{u_i} - \chi_{v_i})$, and hence Lemma 4.10 (h1) imply that (H) holds with (x, D) replaced by (y, D') . By Lemma 4.12, (P) holds with (x, D, R) replaced by (y, D', R') . \square

Lemma 4.20. *If (P), (H), and (F) hold when the procedure $\text{Augment}(\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\})$ is called, then they also hold just after the procedure.*

Proof. Let P denote the sequence $\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$. The procedure updated the triple (x, D, R) to (y, D', R') , where

$$\begin{aligned} y &= x + c(x, P) \left(\chi_{u_0} + \sum_{i=1}^{k-1} (\chi_{u_i} - \chi_{v_i}) \right), \\ D' &= D \setminus \{v_1, v_2, \dots, v_{k-1}\}, \\ R' &= R \cup \{v_1, v_2, \dots, v_{k-1}\}. \end{aligned}$$

We now show that conditions (P), (H), and (F) hold with (x, D, R) replaced by (y, D', R') . By Lemma 4.15, $u_0, v_1, \dots, v_{k-1}, u_{k-1}$ are all distinct and there hold $u_0 \in D \setminus \text{sat}_h(x)$, $u_{k-1} \notin \text{sat}_f(x)$, and every $i \in [k-1]$ satisfies (h0) and (f0).

We first show (F). Define a vector

$$x' := x + c(x, P) \sum_{i=1}^{k-1} (\chi_{u_{i-1}} - \chi_{v_i}) = y - c(x, P) \chi_{u_{k-1}}.$$

As we have (f*) for each $i \in [k-1]$, Lemma 4.11 (f1) implies that (F) holds for (x', R') . As $\{u_0, u_1, \dots, u_{k-2}\} \subseteq \text{sat}_f(x)$, Lemma 4.4 (b5) and $u_{k-1} \notin \text{sat}_f(x)$ imply $\hat{c}_f(x', u_{k-1}) = \hat{c}_f(x, u_{k-1}) \geq c(x, P)$. Apply Lemma 4.5 to obtain $y = x' + c(x, P) \chi_{u_{k-1}}$. Then Lemma 4.5 (c3) implies that (F) also holds with (x', R') replaced by (y, R') .

The condition (H) can be checked similarly. Define a vector

$$x'' := x + c(x, P) \sum_{i=1}^{k-1} (\chi_{u_i} - \chi_{v_i}) = y - c(x, P) \chi_{u_0}.$$

As we have (h*) for each $i \in [k-1]$, Lemma 4.10 (h1) implies that (H) holds for (x'', D') . As $\{u_1, u_2, \dots, u_{k-1}\} \subseteq D \setminus \{e^*\} \subseteq \text{sat}_h(x)$, Lemma 4.4 (b5) and $u_0 \notin \text{sat}_h(x)$ imply $\hat{c}_h(x'', u_0) = \hat{c}_f(x, u_0) \geq c(x, P)$. Apply Lemma 4.5 to obtain $y = x'' + c(x, P) \chi_{u_0}$. Then Lemma 4.5 (c3) implies that (H) also holds with (x'', D') replaced by (y, D') .

As for (P), Lemma 4.12 implies that it holds with (x, D, R) replaced by (x', D', R') . Also, since $u_{k-1} \in D \subseteq D' \cup R'$ and $y = x' + c(x, P) \chi_{u_{k-1}}$, it also holds for (y, D', R') . \square

Observation 4.16 and Lemmas 4.17–4.20 yield the correctness of the algorithm. Also, the definition of the algorithm implies that, if given functions are integer-valued functions, then x is an integer vector at any time of the algorithm. Thus we obtain the following.

Theorem 4.21. *At the termination of the algorithm, the output x is a stable allocation. If h and f are integer-valued functions, then the output x is an integer vector.*

4.6 Complexity

We now consider the time complexity of the algorithm.

First, observe that each element never moves from D to R in the algorithm. Since the procedure **Self-loop** moves one element from D to R , this yields the following fact.

Lemma 4.22. *The procedure **Self-loop** is called at most $|E|$ times in the algorithm.*

We say that an element $e \in E$ is h -saturated (resp. f -saturated) by a procedure, if we have $e \notin \text{sat}_h(x)$ (resp., $e \notin \text{sat}_f(x)$) just before the procedure and $e \in \text{sat}_h(x)$ (resp., $e \in \text{sat}_f(x)$) just after the procedure.

We say that, $\text{next}_h(D, x, e)$ (resp., $\text{next}_f(x, e)$) *increases* if it becomes larger in \succ_H (resp., in \succ_F) by some update. Similarly, we say that $\text{next}_h(D, x, e)$ (resp., $\text{next}_f(x, e)$) *decreases* if it becomes smaller in \succ_H (resp., in \succ_F). We also say that $\text{next}_h(D, x, e)$ decreases when it turns undefined. Recall that $\text{next}_h(D, x, e)$ may not be defined, while $\text{next}_f(x, e)$ is always defined.

Lemma 4.23. *In the algorithm, the following statements hold for every $e \in E$.*

1. *After e is added to R , e stays in R and $\text{next}_h(D, x, e)$ never increases. Also, if $\text{next}_h(D, x, e)$ once turns undefined and turns defined again, it becomes smaller compared to the last time it is defined.*
2. *After e is f -saturated, e stays in $\text{sat}_f(x)$ and $\text{next}_f(x, e)$ never decreases.*

Proof. We show Condition 1. For any $e \in E$, assume $e \in R$ at some point of the algorithm. Since R is monotone increasing, $e \in R$ is maintained thereafter. By the algorithm, $\text{next}_h(D, x, e)$ may change at Line 4 or when some procedure is applied.

At Line 4, x does not change and the added element e^* is smallest in $D \cup \{e^*\}$ w.r.t. \succ_H . Then, if $\text{next}_h(D, x, e)$ is defined just before Line 4, it remains the same. Otherwise, $\text{next}_h(D, x, e)$ becomes e^* , which is smaller than previous values of $\text{next}_h(D, x, e)$ w.r.t. \succ_H .

When **Self-loop** is applied, x does not change and D becomes a subset of D . When **Cycle** (resp. **Augment**) is called, by Lemma 4.15, we can apply Lemma 4.10 (h2), and then $e \in R$ implies $e \neq u_i$ for each $i \in [0, k-1]$ (resp. for each $i \in [l, k-1]$). In each case, $\text{next}_h(D, x, e)$ never increases, and it does not turn defined if it is undefined just before.

We now show Condition 2. Since $\text{next}_f(x, e)$ is independent from D , it changes only when **Cycle** or **Augment** is applied. By Lemma 4.15 and Lemma 4.11 (f2), once $e \in \text{sat}_f(x)$ holds, then $e \in \text{sat}_f(x)$ remains to hold and $\text{next}_f(x, e)$ never decreases until the end of the algorithm. \square

Lemma 4.24. *When **Cycle**($\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$) is applied in the algorithm, there exists $i \in [l+1, k]$ such that $\text{next}_h(D, x, v_i)$ decreases or $\text{next}_f(x, u_{i-1})$ increases.*

Proof. Let Q denote the sequence $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ and let y denote the vector obtained from x by the procedure. By Lemmas 4.10 (h3) and 4.11 (f3), for every $i \in [l+1, k]$, we have $\bar{c}_h(y, u_i, v_i) = \bar{c}_h(x, u_i, v_i) - c(x, Q)$ and $\bar{c}_f(y, u_{i-1}, v_i) = \bar{c}_f(x, u_{i-1}, v_i) - c(x, Q)$,

where $u_k = u_l$. Also, by the definition of $c(x, Q)$, some i satisfies either $\bar{c}_h(y, u_i, v_i) = 0$ or $\bar{c}_f(y, u_{i-1}, v_i) = 0$. By Lemmas 4.10 (h2) and 4.11 (f2), in the former case $\text{next}_h(D, y, v_i)$ decreases, and in the latter case $\text{next}_f(x, u_i)$ increases. \square

Lemma 4.25. *When $\text{Augment}(\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\})$ is applied in the algorithm, at least one of the following hold: (i) $u_0 = e^*$ is h -saturated, (ii) u_{k-1} is f -saturated, (iii) there is $i \in [k-1]$ such that $\text{next}_h(D, x, v_i)$ decreases or $\text{next}_f(x, u_{i-1})$ increases.*

Proof. Let P denote the sequence $\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$ and let y denote the vector obtained from x by the procedure. As shown in the proof of Lemma 4.20, y is obtained by combining a multiple exchange and a simple augmentation. By Lemmas 4.10 (h3), 4.11 (f3), and 4.5 (c3), then $\hat{c}_h(y, u_0) = \hat{c}_h(x, u_0) - c(x, P)$ and $\hat{c}_f(y, u_{k-1}) = \hat{c}_h(x, u_{k-1}) - c(x, P)$, and every $i \in [k-1]$ satisfies $\bar{c}_h(y, u_i, v_i) = \bar{c}_h(x, u_i, v_i) - c(x, P)$ and $\bar{c}_f(y, u_{i-1}, v_i) = \bar{c}_f(x, u_{i-1}, v_i) - c(x, P)$. By the definition of $c(x, P)$, at least one of their values is 0. The condition $\hat{c}_h(y, u_0) = 0$ implies (i), i.e., u_0 is h -saturated. Similarly, $\hat{c}_f(y, u_{k-1}) = 0$ implies (ii), i.e., u_{k-1} is f -saturated. Otherwise, we have (iii) since, by Lemmas 4.10 (h3) and 4.11 (f3), $\bar{c}_h(y, u_i, v_i) = 0$ implies that $\text{next}_h(D, y, v_i)$ decreases and $\bar{c}_f(y, u_{i-1}, v_i) = 0$ implies that $\text{next}_f(y, u_{i-1})$ increases. \square

Lemma 4.26. *Procedures Cycle and Augment are called at most $2(|E| + |E|^2)$ times in total.*

Proof. By Lines 3–5 of the algorithm, each element is h -saturated at most once in the algorithm, and hence the case (i) of Lemma 4.25 occurs at most $|E|$ times. Also, the case (ii) occurs at most $|E|$ times since $\text{sat}_f(x)$ is monotonically increasing by Lemma 4.23. Note that Cycle and Augment in the case (iii) makes either a decrease of $\text{next}_h(D, x, e)$ with e added to R or an increase of $\text{next}_f(x, e)$ preserving $e \in \text{sat}_f(x)$. By Lemma 4.23, for each $e \in E$, such decreases or increases occur at most $|E|$ times, respectively. Therefore, the total number of applications of procedures Cycle and Augment is at most $2(|E| + |E|^2)$. \square

As shown in Lemma 4.23, for each $e \in E$, after the first call of $\text{next}_h(D, x, e)$, the value of $\text{next}_h(D, x, e)$ is monotone nonincreasing in \succ_H . Similarly, after the first call of $\text{next}_f(x, e)$, the value of $\text{next}_f(x, e)$ is monotone nondecreasing in \succ_F . We exploit this property to compute the functions efficiently. Let the algorithm keep pointers $\text{pt}_h(e)$ and $\text{pt}_f(e)$ for each $e \in E$. Initially, $\text{pt}_h(e)$ and $\text{pt}_f(e)$ are set to be the maximum element in \succ_H and the minimum element in \succ_F , respectively, for every $e \in E$. Each time the algorithm needs $\text{next}_h(D, x, v)$, it proceeds as follows. If $\bar{c}_h(x, s, v) > 0$ and $s \in D \setminus \{v\}$ hold for $s = \text{pt}_h(v)$, then $\text{next}_h(D, x, v) = s$. Otherwise, the algorithm decrement $\text{pt}_h(v)$ in \succ_H and iterates. The algorithm eventually achieves $\bar{c}_h(x, s, v) > 0$ and $s \in D \setminus \{v\}$ for $s = \text{pt}_h(v)$, and then $\text{next}_h(D, x, v) = s$. To find $\text{next}_f(x, u)$, the algorithm repeatedly computes $\bar{c}_h(x, u, t)$ for $t = \text{pt}_h(u)$, incrementing $\text{pt}_h(u)$ in \succ_F in the case of $\bar{c}_h(x, u, t) = 0$. With the aid of these pointers, we obtain the following running time bound of our algorithm.

Recall that γ denotes the time for computing the saturation and exchange capacities on the given polymatroids.

Theorem 4.27. *The algorithm finds a stable allocation in $O(|E|^3\gamma)$ time.*

Proof. By Lemmas 4.22 and 4.26, the algorithm calls procedures **Self-loop**, **Augment**, and **Cycle** $O(|E|^2)$ times in total. Each of these procedures requires $O(|E|\gamma)$ time. To compute the pointer functions next_f and next_h , the algorithm performs additional calls of exchange capacity computation. The total number of such calls during the entire algorithm is $O(|E|^2)$. Thus, the algorithm runs in $O(|E|^3\gamma)$ time. \square

4.7 Optimality

For the stable marriage model of Gale and Shapley [41], it is known that the output of the man-oriented deferred acceptance algorithm is optimal for men among all stable matchings. Likewise, our algorithm finds the \succ_H -optimal stable allocation, which is what we show in this section.

Recall that a vector $x \in \mathbf{R}^E$ is said to be \succ -preferable to $y \in \mathbf{R}^E$ if $x(E_{\succeq a}) \geq y(E_{\succeq a})$ for every $a \in E$. Also, x is called \succ -optimal in a set $K \subseteq \mathbf{R}^E$ if $x \in K$ and x is \succ -preferable to every $y \in K$. In this section, we show that the output of the algorithm is \succ_H -optimal in the set of all the stable allocations. For this purpose, we show that the following condition is also maintained in the algorithm besides (P), (H), and (F).

(R) For each $e \in R$, $x(e) \geq z(e)$ holds for every stable allocation $z \in \mathbf{P}(h) \cap \mathbf{P}(f)$.

Lemma 4.28. *If (x, D, R) satisfies (R) at the end of the algorithm, then x is \succ_H -optimal in the set of all the stable allocations.*

Proof. Take any stable allocation z . At the end of the algorithm, we have $D \cup R = E$ and $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$ holds for every $e \in D$ (as shown in the proof of Lemma 4.17). Then, by (R), every $e \in E$ satisfies $x(e) \geq z(e)$ or $[e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_H e]$. This implies that x is \succ_H -preferable to z by Lemma 4.6. \square

At the beginning of the algorithm, $R = \emptyset$ and hence the condition (R) clearly holds. What is left is to show that (R) is maintained throughout the algorithm. To do so, we prepare the following two lemmas.

Lemma 4.29. *If (P), (H), and (R) hold for (x, D, R) , then any stable allocation z satisfies*

$$\forall e \in E : \quad z(e) \geq x(e) \quad \text{or} \quad [e \in \text{sat}_f(z), \text{dep}_f(z, e) \succeq_F e]. \quad (4.18)$$

Proof. Take an arbitrary stable allocation $z \in \mathbf{P}(h) \cap \mathbf{P}(f)$. Fix $a \in E$ by $a := e^* = \min_{\succ_H} D$ if $x(e^*) > z(e^*)$ and otherwise $a := \min_{\succ_H} \{e \in E \mid e \succ_H e^*\}$. By (P), then

every $e \in E \setminus E_{\succeq_{\text{Ha}}}$ satisfies $z(e) \geq x(e)$. Then, it suffices to show that every $e \in E_{\succeq_{\text{Ha}}}$ satisfies the condition in (4.18). Since (P), (H), and (R) hold, we have

$$\forall e \in E_{\succeq_{\text{Ha}}} : \quad x(e) \geq z(e) \quad \text{or} \quad [e \in \text{sat}_h(x), \text{dep}_h(x, e) \succeq_{\text{H}} e].$$

By Lemma 4.7 for $x, z \in \mathbf{P}(h)$, this implies

$$\forall e \in E_{\succeq_{\text{Ha}}} : \quad z(e) \geq x(e) \quad \text{or} \quad \neg[e \in \text{sat}_h(z), \text{dep}_h(z, e) \succeq_{\text{H}} e].$$

Since z is a stable allocation, $\neg[e \in \text{sat}_h(z), \text{dep}_h(z, e) \succeq_{\text{H}} e]$ implies $[e \in \text{sat}_f(z), \text{dep}_f(z, e) \succeq_{\text{F}} e]$. Thus, we obtain (4.18). \square

For a sequence $W = \{u_0, v_1, u_1, \dots, u_{d-1}, v_d, u_d\}$ of (not necessarily distinct) elements of E , we define its *walk-capacity* by

$$\check{c}(x, W) = \min \left\{ \hat{c}_h(x, u_0), \min_{i:1 \leq i \leq d} \bar{c}_f(x, u_{i-1}, v_i), \min_{i:1 \leq i \leq d} \bar{c}_h(x, u_i, v_i) \right\}.$$

Lemma 4.30. *Suppose that (P), (H), (F) and (R) hold for (x, D, R) . Also, suppose that (not necessarily distinct) elements u_0, u_i, v_i ($i = 1, 2, \dots, d$) satisfies $u_0 \in D$ and (h0) and (f0) for every $i \in [d]$. Then for any stable allocation z , we have $x(v_i) - \check{c}(x, W) \geq z(v_i)$ for each $i \in [d]$.*

Proof. Take an arbitrary stable allocation z . Consider vectors

$$\begin{aligned} x_0 &:= x + \check{c}(x, W)\chi_{u_0}, \\ y_i &:= x + \check{c}(x, W)(\chi_{u_{i-1}} - \chi_{v_i}) \quad (i \in [d]), \\ x_i &:= x + \check{c}(x, W)(\chi_{u_i} - \chi_{v_i}) \quad (i \in [d]). \end{aligned}$$

Also, set $D_i := D \setminus \{v_i\}$ and $R_i := R \cup \{v_i\}$ for $i \in [d]$ and $D_0 := D$, $R_0 := R$. Note that for each $i \in [d]$, the condition $x(v_i) - \check{c}(x, W) \geq z(v_i)$ is equivalent to $y_i(v_i) \geq z(v_i)$. Hence, showing the following three statements completes the proof.

- (i) (P), (H), and (R) hold for (x_0, D_0, R_0) .
- (ii) For each $i \in [d]$, if (P), (H), and (R) hold for $(x_{i-1}, D_{i-1}, R_{i-1})$, then $y_i(v_i) \geq z(v_i)$.
- (iii) For each $i \in [d]$, if $y_i(v_i) \geq z(v_i)$, then (P), (H), and (R) hold for (x_i, D_i, R_i) .

We first show (i). The conditions (P) and (R) are obvious by definition. Since $\check{c}(x, W) \leq \hat{c}_h(x, u_0)$, we can apply Lemma 4.5 (c3) to see that $e \in \text{sat}_h(x_0)$ and $\text{dep}_h(x_0, e) = \text{dep}_h(x, e)$ for every $e \in D \setminus \{e^*\} \subseteq \text{sat}_h(x)$.

We then show (ii). Since both x and x_{i-1} satisfy (P), (H) and (R), by Lemma 4.29, the condition (4.18) holds for both x and x_{i-1} . Note that $y_i(e) \leq \max\{x(e), x_{i-1}(e)\}$ for

every $e \in E$ by definition. Then, the condition $z(e) < y_i(e)$ implies either $z(e) < x(e)$ or $z(e) < x_{i-1}(e)$, each of which implies $[e \in \text{sat}_f(z), \text{dep}_f(z, e) \succeq_F e]$ by (4.18). Therefore, we have

$$\forall e \in E : z(e) \geq y_i(e) \text{ or } [e \in \text{sat}_f(z), \text{dep}_f(z, e) \succeq_F e].$$

By Lemma 4.7, this implies

$$\forall e \in E : y_i(e) \geq z(e) \text{ or } \neg[e \in \text{sat}_f(y_i), \text{dep}_f(y_i, e) \succeq_F e]. \quad (4.19)$$

By $\check{c}(x, W) \leq \bar{c}_f(x, u_{i-1}, v_i)$ and (f0), Lemma 4.11 (f1) implies that the statement (F) holds with (x, R) replaced by $(y_i, R \cup \{v_i\})$, which implies $[v_i \in \text{sat}_f(y_i), \text{dep}_f(y_i, v_i) \succeq_F v_i]$. Then, (4.19) implies $y_i(v_i) \geq z(v_i)$.

Finally, we show (iii). As we have (h0) and (f0), it follows that $u_i, v_i \in D \cup R$, and hence (P) holds. By (h0), we can apply Lemma 4.10 (h1), and hence (H) holds for (x_i, D_i) . We now show (R). Since y_i satisfies $y_i(v_i) \geq z(v_i)$, we have $x_i(v_i) = y_i(v_i) \geq z(v_i)$. Also, $x_i(e) \geq x(e)$ for every $e \in E \setminus \{v_i\}$. Thus, (R) holds for (x_i, R_i) . \square

We now show that each procedure keeps the conditions (P), (H), (F), and (R).

Lemma 4.31. *If (P), (H), (F), and (R) hold when the procedure Self-loop(u_{k-1}) is called, then they also hold just after the procedure.*

Proof. We only consider (R) because other conditions are already shown in Lemma 4.18. Since the procedure adds u_{k-1} to R , it suffices to show $x(u_{k-1}) \geq z(u_{k-1})$. For a stable allocation z , we have (4.18) by Lemma 4.29. Apply Lemma 4.7 to $z, x \in \mathbf{P}(f)$ with respect to (E, f, \succ_F) . Then, each $e \in E$ satisfies $x(e) \geq z(e)$ or $\neg[e \in \text{sat}_f(x), \text{dep}_f(x, e) \succeq_F e]$. Note that $[u_{k-1} \in \text{sat}_f(x), \text{dep}_f(x, u_{k-1}) \succeq_F u_{k-1}]$ follows from $\text{next}_f(x, u_{k-1}) = u_{k-1}$, and hence we have $x(u_{k-1}) \geq z(u_{k-1})$. \square

Lemma 4.32. *If (P), (H), (F), and (R) hold when Augment($\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$) is called, then they also hold just after the procedure.*

Proof. We only consider (R) because other conditions are already shown in Lemma 4.20. Let y denote the vector obtained from x by the procedure. Since (R) is assumed and R is updated to $R \cup \{v_1, v_2, \dots, v_{k-1}\}$, it suffices to show $y(v_i) \geq z(v_i)$ for every $i \in [k-1]$, where z is an arbitrary stable allocation.

Let P denote $\{u_0, v_1, \dots, v_{k-1}, u_{k-1}\}$. By Lemma 4.15, we can apply Lemma 4.30 to x and P , and obtain $x(v_i) - \check{c}(x, P) \geq z(v_i)$ for every $i \in [k-1]$. By the definitions of path and walk capacities, we have $c(x, P) \leq \check{c}(x, P)$. Then $y(v_i) = x(v_i) - c(x, P) \geq x(v_i) - \check{c}(x, P) \geq z(v_i)$ for every $i \in [k-1]$. \square

Lemma 4.33. *If (P), (H), (F), and (R) hold when the procedure Cycle($\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$) is called, then they also hold just after the procedure.*

Proof. Let Q denote the sequence $\{u_l, v_{l+1}, \dots, u_{k-1}, v_k\}$ and define $\chi_Q := \sum_{i=l+1}^k (\chi_{u_{i-1}} - \chi_{v_i})$. The procedure **Cycle** for Q updates the vector x to $y := x + c(x, Q)\chi_Q$. Let W denote the sequence $\{u_0, v_1, \dots, u_{k-1}, v_k, u_k\}$ and set vectors $x_0 := x$ and $x_m := x_{m-1} + \check{c}(x_{m-1}, W)\chi_Q$ for every $m \in \mathbf{Z}_{>0}$. Then

$$x_m = x + \sum_{j=0}^{m-1} \check{c}(x_j, W)\chi_Q \quad (m \in \mathbf{Z}_{>0}).$$

We will show finite convergence of $\{x_m\}_{m \in \mathbf{Z}_{\geq 0}}$ to y .

Note that $u_k = u_l$ when **Cycle** is called, and recall the definitions of capacities:

$$c(x_m, Q) = \min \left\{ \min_{i:l+1 \leq i \leq k} \bar{c}_f(x_m, u_{i-1}, v_i), \min_{i:l+1 \leq i \leq k} \bar{c}_h(x_m, u_i, v_i) \right\},$$

$$\check{c}(x_m, W) = \min \left\{ \hat{c}_h(x_m, u_0), \min_{i:1 \leq i \leq k} \bar{c}_f(x_m, u_{i-1}, v_i), \min_{i:1 \leq i \leq k} \bar{c}_h(x_m, u_i, v_i) \right\}.$$

By these definitions, for every $m \in \mathbf{Z}_{\geq 0}$, we have

$$\check{c}(x_m, W) \leq c(x_m, Q). \quad (4.20)$$

We now show the following conditions for every $m \in \mathbf{Z}_{>0}$. Here, D' and R' are defined as $D' = D \setminus \{v_{l+1}, v_{l+2}, \dots, v_k\}$ and $R' = R \cup \{v_{l+1}, v_{l+2}, \dots, v_k\}$, respectively.

- (i) (x_m, D', R') satisfies (P), (H), (F), and (R).
- (ii) If $x_m \neq y$, then $\check{c}(x_m, W) > 0$, $u_0 \in D'$, and (x_m, D') satisfies (h0), (f0) for every $i \in [k]$. If $x_m = y$, then $\check{c}(x_m, W) = 0$, and hence $x_{m+1} = x_m$.
- (iii) $c(x_m, Q) = c(x, Q) - \sum_{j=0}^{m-1} \check{c}(x_j, W)$.
- (iv) $\check{c}(x_m, W) = c(x_m, Q)$ or $\check{c}(x_m, W) \geq \check{c}(x_{m-1}, W)$, where we put $\check{c}(x_{-1}, W) = 0$.

We show (i)–(iv) by simultaneous induction on $m \geq 0$.

Consider the case that $m = 0$. Clearly (iii) and (iv) hold. Note that x_0 satisfies conditions (i) and (ii) with D, R replaced by D', R' . By applying Lemmas 4.10 (h1), 4.11 (f1), and 4.12 with $\alpha = 0$, we see that (x_0, D', R') still satisfies (P), (H), (F) and (ii). Also, (R) for (x_0, D', R') follows from Lemma 4.30. Thus, all of (i)–(iv) hold in the case that $m = 0$.

We now turn to the case that $m \geq 1$. We first show (i). By the inductive assumption, (x_{m-1}, D', R') satisfies (P), (H), (F), and (R). By applying Lemma 4.30 to x_{m-1} , we obtain $x_{m-1}(v_i) - \check{c}(x_{m-1}, W) \geq z(v_i)$ for any stable allocation z and $i \in [l+1, k]$. Since $x_m(v_i) = x_{m-1}(v_i) - \check{c}(x_{m-1}, W)$, this means that (R) holds for x_m and R' . Also, since (4.20) holds for x_{m-1} , we can apply Lemmas 4.10 (h1), 4.11 (f1), and 4.12, which respectively imply (H), (F), and (P) for (x_m, D', R') . Thus, (i) holds for m . To show conditions (ii)–(iv), let us observe the saturation and exchange capacities of x_m .

1. By Lemma 4.4 (b5) and (b6), the saturation capacity $\hat{c}_h(\cdot, u_0)$ satisfies the following.

(1-1) If $u_0 \neq u_k$, then $\hat{c}_h(x_m, u_0) = \hat{c}_h(x_{m-1}, u_0)$.

(1-2) If $u_0 = u_k$, then $\hat{c}_h(x_m, u_0) \geq \min\{\hat{c}_h(x_{m-1}, u_0), \bar{c}_h(x_m, u_k, v_k)\}$.

2. By Lemma 4.11 (f3), exchange capacity \bar{c}_f satisfies the following.

(2-1) If $1 \leq i < l$, then $\bar{c}_f(x_m, u_{i-1}, v_i) = \bar{c}_f(x_{m-1}, u_{i-1}, v_i)$

(2-2) If $v_l \neq v_k$, then $\bar{c}_f(x_m, u_{l-1}, v_l) = \bar{c}_f(x_{m-1}, u_{l-1}, v_l)$.

(2-3) If $v_l = v_k$, then $\bar{c}_f(x_m, u_{l-1}, v_l) = \bar{c}_f(x_m, u_{l-1}, v_k) \geq \min\{\bar{c}_f(x_{m-1}, u_{l-1}, v_l), \bar{c}_f(x_m, u_{k-1}, v_k)\}$.

(2-4) If $l+1 \leq i \leq k$, then $\bar{c}_f(x_m, u_{i-1}, v_i) = \bar{c}_f(x_{m-1}, u_{i-1}, v_i) - \check{c}(x_{m-1}, W)$.

3. By Lemma 4.10 (h3), exchange capacity \bar{c}_h satisfies the following. Note that $u_k = u_l$.

(3-1) If $1 \leq i < l$, then $\bar{c}_h(x_m, u_i, v_i) = \bar{c}_h(x_{m-1}, u_i, v_i)$

(3-2) If $v_l \neq v_k$, then $\bar{c}_h(x_m, u_l, v_l) = \bar{c}_h(x_m, u_k, v_l) \geq \min\{\bar{c}_h(x_{m-1}, u_{l-1}, v_l), \bar{c}_h(x_m, u_k, v_k)\}$.

(3-3) If $v_l = v_k$, then $\bar{c}_h(x_m, u_l, v_l) = \bar{c}_h(x_m, u_k, v_k)$.

(3-4) If $l+1 \leq i \leq k$, then $\bar{c}_h(x_m, u_i, v_i) = \bar{c}_h(x_{m-1}, u_i, v_i) - \check{c}(x_{m-1}, W)$.

Using these, we first show (iii). By (2-4) and (3-4), every $i \in [l+1, k]$ satisfies $\bar{c}_f(x_m, u_{i-1}, v_i) = \bar{c}_f(x_{m-1}, u_{i-1}, v_i) - \check{c}(x_{m-1}, W)$ and $\bar{c}_h(x_m, u_i, v_i) = \bar{c}_h(x_{m-1}, u_i, v_i) - \check{c}(x_{m-1}, W)$. Hence,

$$c(x_m, Q) = c(x_{m-1}, Q) - \check{c}(x_{m-1}, W) = c(x, Q) - \sum_{j=0}^{m-1} \check{c}(x_j, W),$$

where the last equality is obtained by substituting $c(x_{m-1}, Q) = c(x, Q) - \sum_{j=0}^{m-2} \check{c}(x_j, W)$, which follows from the inductive assumption. Thus, (iii) holds.

We next show (iv). By the definitions of $\check{c}(x_{m-1}, W)$ and $c(x_m, Q)$, we have $\hat{c}_h(x_{m-1}, u_0) \geq \check{c}(x_{m-1}, W)$ and $\bar{c}_h(x_m, u_k, v_k) \geq c(x_m, Q)$. Then, (1-1) and (1-2) above imply

$$\hat{c}_h(x_m, u_0) \geq \min\{\check{c}(x_{m-1}, W), c(x_m, Q)\}. \quad (4.21)$$

Also, we have $\bar{c}_f(x_{m-1}, u_{i-1}, v_i) \geq \check{c}(x_{m-1}, W)$ for every $i \in [k]$ and $\bar{c}_f(x_m, u_{j-1}, v_j) \geq c(x_m, Q)$ for every $j \in [l+1, k]$. Then, the conditions (2-1)–(2-3) imply

$$\bar{c}_f(x_m, u_{i-1}, v_i) \geq \min\{\check{c}(x_{m-1}, W), c(x_m, Q)\} \quad (i \in [k]). \quad (4.22)$$

Similarly, the conditions (3-1)–(3-3) imply

$$\bar{c}_h(x_m, u_i, v_i) \geq \min\{\check{c}(x_{m-1}, W), c(x_m, Q)\} \quad (i \in [k]). \quad (4.23)$$

By (4.21), (4.22) and (4.23), we obtain $\check{c}(x_m, W) \geq \min\{\check{c}(x_{m-1}, W), c(x_m, Q)\}$, i.e., $\check{c}(x_m, W) \geq \check{c}(x_{m-1}, W)$ or $\check{c}(x_m, W) \geq c(x_m, Q)$. As we have (4.20), this implies (iv).

Finally, we show (ii). We first consider the case that $x_m \neq y$. Since (ii) holds for $m-1$ by the inductive assumption, $x_m \neq y$ implies $x_{m-1} \neq y$ and $\check{c}(x_{m-1}, W) >$

0. Also, $x_m \neq y$ implies $\sum_{j=0}^{m-1} \check{c}(x_j, W) \neq c(x, Q)$, which implies $c(x_m, Q) > 0$ by (iii). Thus, we have $\min\{\check{c}(x_{m-1}, W), c(x_m, Q)\} > 0$. By (4.23), every $i \in [k]$ satisfies $\bar{c}_h(x_m, u_i, v_i) > 0$. By the definition of next_h and its monotonicity (Lemma 4.10 (h2)), this implies $\text{next}_f(D', x_m, v_i) = u_i$. Similarly, by (4.22) and Lemma 4.11 (f2), we obtain $\text{next}_f(x_m, u_{i-1}) = v_i$ for every $i \in [k]$. Thus, we have (h0) and (f0) for every $i \in [k]$. Also, $u_0 \in D'$ is clear and $\check{c}(x_m, W) \geq \min\{\check{c}(x_{m-1}, W), c(x_m, Q)\} > 0$. Thus, all requirements in (ii) hold when $x_m \neq y$. In the case that $x_m = y$, we have $\sum_{j=0}^{m-1} \check{c}(x_j, W) = c(x, Q)$, which implies $c(x_m, Q) = 0$ by (iii). As we have (4.20), this implies $\check{c}(x_m, W) = 0$. Hence, $x_{m+1} = x_m + \check{c}(x_m, W)\chi_Q = x_m$.

So far, we have shown that x_m satisfies (i)–(iv) for every $m \in \mathbf{Z}_{\geq 0}$. By (i), it completes the proof to show that there is a finite $m^* \in \mathbf{Z}_{\geq 0}$ such that $x_{m^*} = y$. Let $m^* \in \mathbf{Z}_{\geq 0}$ be the minimum number satisfying

$$m^* \cdot \check{c}(x, W) > c(x, Q).$$

Since $\check{c}(x, W) > 0$ by the definition of the algorithm, m^* is finite. To prove $x_{m^*} = y$ by contradiction, suppose that $x_{m^*} \neq y$. By the second claim of (ii), this implies that $x_m \neq y$ for every $m \in [m^*]$. Then, for every $m \in [m^*]$, we have $\sum_{j=0}^{m-1} \check{c}(x_j, W) \neq c(x, Q)$, which implies $\check{c}(x_{m-1}, W) \neq c(x, Q) - \sum_{j=0}^{m-2} \check{c}(x_j, W) = c(x_{m-1}, Q)$, where the last equality follows from (iii). Therefore, $\check{c}(x_m, W) \neq c(x_m, Q)$ for every $m \in [m^* - 1]$. As we have (iv), this implies

$$\check{c}(x_0, W) \leq \check{c}(x_1, W) \leq \check{c}(x_2, W) \leq \cdots \leq \check{c}(x_{m^*-1}, W).$$

Then (iii) for m^* implies

$$c(x_{m^*}, Q) = c(x, Q) - \sum_{j=0}^{m^*-1} \check{c}(x_j, W) \leq c(x, Q) - m^* \cdot \check{c}(x_0, W) < 0,$$

which contradicts the nonnegativity of $c(x_{m^*}, Q)$. \square

We now obtain the \succ_H -optimality of the output of the algorithm.

Theorem 4.34. *The output of the algorithm is the stable allocation that is \succ_H -preferable to any stable allocation z . That is, the output is \succ_H -optimal in the set of all the stable allocations.*

Proof. At the beginning of the algorithm, $R = \emptyset$ and hence (R) holds. By Observation 4.16 and Lemmas 4.31–4.33, the output of the algorithm satisfies (P), (H), (F), and (R). Then, Lemma 4.28 implies the \succ_H -optimality of the output. \square

Chapter 5

Matroidal Choice Functions

5.1 Introduction

In some game-theoretic models, an agent chooses a subset of offered items under a matroid constraint. For example, in the college admissions model of Gale and Shapley [41], a college takes applicants up to its quota $q \in \mathbf{Z}_{>0}$, which means that it has a uniform matroid constraint of rank q . Also, a laminar matroid naturally arises if a college has a nested classification of students and put a quota on each class [29, 54]. A transversal matroid arises if a college considers an assignment of students to positions.

Under a matroid constraint, if we have a total order on individual applicants, the most naive way of choosing a subset of applicants is to execute the standard greedy algorithm, i.e., we pick available applicants from the highest to the lowest while maintaining the matroid constraint. Fleiner [26] showed that such a choice rule fulfills the “substitutability” (“comonotonicity,” in his terminology), an essential property for the existence of a stable matching in two-sided market models [90].

There are, however, more general substitutable choice rules under a matroid constraint as follows. Assume that a college admits at most two applicants. The applicants are men m_1, m_2 and women w_1, w_2 . The college prefers m_1 to m_2 and w_1 to w_2 , and admits members of both sexes as equally as possible. Hence, its first choice is m_1w_1 . However if the available set is $m_2w_1w_2$ (resp., $m_1m_2w_2$), then the choice is m_2w_1 (resp., m_1w_2). In the case of $m_2w_1w_2$, man m_2 is preferred to woman w_2 , in contrast to the case of $m_1m_2w_2$, in which w_2 is preferred to m_2 . Thus we cannot define a total order on individual applicants, while this choice rule fulfills the substitutability in the sense defined in [26, 90].

The aim of this chapter is to analyze the nature of substitutable choice rules under a matroid constraint. We express choice rules by choice functions, and introduce the notion of “matroidal choice functions” as the whole class of substitutable choice functions under a matroid constraint. We show that each matroidal choice function can be represented by a “de-cycle function,” which is a function on the circuit family of a matroid indicating one element of each circuit. We show that there is a one-to-one correspondence

between matroidal choice functions and de-cycle functions satisfying the “coherency,” and we characterize the coherent de-cycle functions in the following two ways.

The first characterization is by an online greedy algorithm. Assume that a de-cycle function indicates the “worst” element of each circuit, and we are given a subset of the ground set as a sequence of elements. Then, we can naturally conceive the following algorithm: Start with the empty set and add elements of the sequence, but whenever a circuit comes up, eliminate the element indicated by the de-cycle function. This algorithm works well for some kind of de-cycle functions. For example, if there are positive weights on elements and the de-cycle function indicates the minimum weight element in each circuit, the output is the maximum weight independent subset regardless of the order of the sequence. For general de-cycle functions, however, the output of the algorithm differs depending on the order of the sequence. We show that the output of the algorithm is independent of the order if and only if the de-cycle function is coherent. In addition, the output of the algorithm with a coherent de-cycle function is just a subset returned by the corresponding matroidal choice function. Thus, matroidal choice functions are characterized as choice functions computable by the above online greedy algorithm with coherent de-cycle functions.

The other characterization of the coherency is based on a local condition. We introduce the concept of “minimal pair of circuits,” which means a pair of distinct circuits whose union is inclusion-wise minimal among all such unions. While the original definition of the coherency requires a particular condition for all subsets of the ground set, the local characterization is concerned only with minimal pairs of circuits.

Matroidal choice functions are closely related to nonlinear generalizations of weighted matroids, such as valuated matroids and M^\sharp -concave functions. A valuated matroid, introduced by Dress and Wenzel [18, 19], is a matroid equipped with a function defined on the base family satisfying a quantitative version of the exchange axiom. For this structure, various efficient algorithms of weighted matroids have been extended: the greedy algorithm and the valuated matroid intersection algorithm [72, 73]. For valuated matroids, there are equivalent sets of axioms in terms of a function on the independent sets [74] and in terms of vectors on the circuits [79]. Generalizing the idea of valuated matroid, Murota [71] defined M -concave functions on the integer lattice. Then M^\sharp -concave functions were introduced by Murota and Shioura [78] as a variant of M -concave functions. These structures are called discrete concave functions [75].

Our online greedy algorithm mentioned above shows a marked similarity to the maximization algorithm for valuated matroids. Also, it is known that a choice rule which maximizes the value of an M^\sharp -concave function fulfills the substitutability [37, 81]. As suggested by these facts, there is a close relationship between matroidal choice functions and discrete concave functions. We show that a choice function defined by maximizers of an M^\sharp -concave set function is a matroidal choice function under certain conditions. It is also shown that not all matroidal choice functions arise from discrete concave functions. Thus,

the notion of matroidal choice functions can be regarded as an abstraction of combinatorial aspects of maximization algorithms of discrete concave functions.

The rest of this chapter is organized as follows. Section 5.2 introduce the concept of matroidal choice functions and provide some properties. This class of choice functions is characterized in Sections 5.3 and 5.4 by an online greedy algorithm and by a local condition, respectively. Section 5.5 explains how matroidal choice functions are related to valuated matroids and M^\sharp -concave functions. Finally, in Section 5.6, we provide a stable matching model using matroidal choice functions.

5.2 Matroidal Choice Functions

In this section, we introduce the concept of “matroidal choice functions” and provide a representation method with circuit families.

5.2.1 Definition

A *choice function* on a finite set S is a function $F : 2^S \rightarrow 2^S$ such that $F(X) \subseteq X$ for any $X \subseteq S$. We interpret $F(X)$ as the most preferred subset of X . A choice function F is said to be *substitutable* if it satisfies

$$(\text{Sub}) \quad X \subseteq Y \implies F(Y) \cap X \subseteq F(X).$$

One can easily confirm that (Sub) is equivalent to

$$(\text{Sub}^*) \quad X \subseteq Y \implies X \setminus F(X) \subseteq Y \setminus F(Y).$$

This says that an item rejected in some set will also be rejected if the set is expanded. We refer to both (Sub) and (Sub*) as the substitutability.

Let us define a “matroidal choice function” as a substitutable choice function which chooses one maximal subset under a matroid constraint. For a matroid $\mathbf{M} = (S, \mathcal{I})$, we denote by $\mathcal{B}(X)$ the set of maximal independent subsets of X , i.e.,

$$\mathcal{B}(X) = \{ B \subseteq X \mid B \in \mathcal{I}, B + e \notin \mathcal{I} \ (\forall e \in X \setminus B) \},$$

and call each member of $\mathcal{B}(X)$ a *base* of X .

Definition 5.1. For a matroid $\mathbf{M} = (S, \mathcal{I})$, a function $F : 2^S \rightarrow 2^S$ is called a *matroidal choice function* on \mathbf{M} if it satisfies the substitutability and $F(X) \in \mathcal{B}(X)$ for each $X \subseteq S$. We simply say that F is a matroidal choice function if there is such a matroid \mathbf{M} , which is called the *underlying matroid* of F . \blacksquare

For any $X \subseteq S$, we see that $X \in \mathcal{I}$ implies $\mathcal{B}(X) = \{X\}$ and that $X \notin \mathcal{I}$ implies $X \notin \mathcal{B}(X)$. This leads to the following proposition.

Proposition 5.2. *For a matroidal choice function F on $\mathbf{M} = (S, \mathcal{I})$, a subset $X \subseteq S$ satisfies $F(X) = X$ if and only if $X \in \mathcal{I}$. Therefore, given a matroidal choice function F , we can obtain its underlying matroid by setting $\mathcal{I} = \{X \subseteq S \mid F(X) = X\}$.*

Any matroidal choice function satisfies the property called “size-monotonicity,” which means that the cardinality of a chosen subset does not become smaller when a set of available items is expanded.

Proposition 5.3. *A matroidal choice function F is size-monotone, i.e., $X \subseteq Y \subseteq S$ implies $|F(X)| \leq |F(Y)|$.*

Proof. For every $X \subseteq S$, we have $F(X) \in \mathcal{B}(X)$, and hence $|F(X)| = r(X)$ for the rank function r of the underlying matroid of F . The monotonicity of r implies the claim. \square

Remark 5.4. The substitutability is known as an essential condition for the existence of a stable matching in two-sided matching models [61, 90]. Moreover, combined with size-monotonicity (or the “law of aggregate demand”), it yields further results. For example, the set of stable matchings forms a distributive lattice [8, 26, 27], and the deferred acceptance algorithm is strategy-proof for students [53]. Since matroidal choice functions are substitutable and size-monotone, these results hold for matching models with matroidal choice functions. See Section 5.6 for the details. \blacksquare

Here we give some examples of matroidal choice functions.

We first provide the most basic example, which also appeared in Section 3.3. (The choice function $C_{\mathcal{M}}$ defined there by (3.4) is exactly the same with F in this example.)

Example 5.5 (Standard Greedy Algorithm). Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid and \succ be a total order on S such that $e_1 \succ e_2 \succ \cdots \succ e_n$, where $n = |S|$. Define a choice function $F : 2^S \rightarrow 2^S$ by letting $F(X)$ be the output of the following algorithm for every $X \subseteq S$: Let $F^0(X) := \emptyset$ and define $F^i(X)$ for each $i \in \{1, 2, \dots, n\}$ by

$$F^i(X) := \begin{cases} F^{i-1}(X) + e_i & e_i \in X \text{ and } F^{i-1}(X) + e_i \in \mathcal{I}, \\ F^{i-1}(X) & \text{otherwise,} \end{cases} \quad (5.1)$$

and then let $F(X) := F^n(X)$. Then F is a matroidal choice function on \mathbf{M} . Indeed, we can observe that $F(X)$ is a base of X . Also, F is substitutable, as shown by Fleiner [26, 27].

Here is a proof of the substitutability of F . Take $X, Y \subseteq S$ arbitrarily. We show $X \setminus F(X) \subseteq Y \setminus F(Y)$. By (5.1), for every $e_i \in X$, we have $e_i \in X \setminus F(X)$ if and only if $e_i \in \text{span}_{\mathbf{M}}(F^{i-1}(X))$. Then, it suffices to show $(\star) \text{span}_{\mathbf{M}}(F^j(X)) \subseteq \text{span}_{\mathbf{M}}(F^j(Y))$ for each $j \in \{0\} \cup [k-1]$. We use induction on j . Clearly (\star) holds for $j = 0$ since $F^0(X) = F^0(Y) = \emptyset$. For $j > 0$, assume (\star) holds for $j-1$. If $e_j \notin \text{span}_{\mathbf{M}}(F^{j-1}(Y))$, then $e_j \notin \text{span}_{\mathbf{M}}(F^j(X))$ follows, and hence $F^j(X) = F^{j-1}(X) + e_j$ and $F^j(Y) = F^{j-1}(Y) + e_j$, which imply (\star) for j . Also, if $e_j \in \text{span}_{\mathbf{M}}(F^{j-1}(Y))$, then $F^j(X) \subseteq F^{j-1}(X) + e_j \subseteq \text{span}_{\mathbf{M}}(F^{j-1}(Y)) = \text{span}_{\mathbf{M}}(F^j(Y))$, and hence (\star) holds for j . \blacksquare

Next, we provide two practical examples of matroidal choice functions. Each of them represents a criterion of selecting students of some college.

Example 5.6 (Laminar Structure). Assume that the organization of a college is represented by a tree with node set V and root $r \in V$. (For example, if a faculty represented by $u \in V$ consists of two departments, say $v_1, v_2 \in V$, then v_1 and v_2 are children of u in the tree.) Let $L \subseteq V$ be the set of *leaves*, i.e., nodes without children. Each leaf $l \in L$ has a prescribed number p_l of applicants to accept. Then, p_v for each $v \in V \setminus L$ is defined by $p_v := \sum \{p_u \mid u \text{ is a child of } v\}$ inductively. Also, each $v \in V$ has a quota q_v with $q_v \geq p_v$. Let S_l be the set of possible applicants for each $l \in L$. Then, the set of possible applicants for $v \in V \setminus L$ is inductively defined by $S_v := \bigcup \{S_u \mid u \text{ is a child of } v\}$. In particular, we write $S := S_r = \bigcup \{S_l \mid l \in L\}$. Then, the family of acceptable applicant sets is

$$\mathcal{I} := \{ Y \subseteq S \mid |Y \cap S_v| \leq q_v \ (\forall v \in V) \}.$$

The pair (S, \mathcal{I}) is a laminar matroid. The college also has a total order \succ on S , where $e \succ e'$ means that e is better than e' . Assume that the applicants in $X \subseteq S$ apply for the college, i.e., for each l , the applicants in $X \cap S_l$ apply. Then the college decides whom to accept according to the following two steps.

Step 1. Each leaf takes applicants up to its prescribed number: Let $F^0(X)$ be the direct sum of $\{F_l^0(X) \subseteq S_l \mid l \in L\}$, where each $F_l^0(X)$ is the set of top p_l elements of $X \cap S_l$ w.r.t. \succ if $|X \cap S_l| > p_l$ and otherwise $X \cap S_l$. Note that $F^0(X) \in \mathcal{I}$ since $|F^0(X) \cap S_v| \leq p_v \leq q_v$ for every $v \in V$.

Step 2. The college additionally accepts applicants as far as quotas allow: Let $n = |X \setminus F^0(X)|$ and, for each $i \in [n]$, let e_i be the i -th element in $X \setminus F^0(X)$ w.r.t. \succ . Define $F^i(X)$ for each $i \in [n]$ by (5.1), and then let $F(X) := F^n(X)$.

We see that $F(X)$ is a base of X for the matroid (S, \mathcal{I}) . Also, we can check that $F : 2^S \rightarrow 2^S$ satisfies the condition (Sub*). Thus, F is a matroidal choice function on (S, \mathcal{I}) . Note that this example is not a special case of the standard greedy algorithm in Example 5.5, which does not ensure that at least the prescribed number of students could be accepted for each leaf. ■

Example 5.7 (Task Assignment). Let $G = (S, T; E)$ be a bipartite graph with node sets S, T and edge set E . Here, S represents the set of possible applicants for a college and T represents the set of tasks to which accepted applicants are assigned. An edge $(s, t) \in E$ means that s can be assigned to t . Then, the family of acceptable applicant sets is

$$\mathcal{I} := \{ \partial M \cap S \mid M \subseteq E : \text{matching in } G \}.$$

The pair (S, \mathcal{I}) is a transversal matroid. Assume that we have $w : E \rightarrow \mathbf{R}_{>0}$, where $w(s, t)$ denotes a profit obtained by assigning s to t . We write $w(M) = \sum \{w(e) \mid e \in M\}$ for any

$M \subseteq E$ and assume $w(M_1) \neq w(M_2)$ for any $M_1, M_2 \subseteq E$ with $M_1 \neq M_2$. Consider that the applicants in $X \subseteq S$ apply for the college. The college primarily wants to maximize the number of applicants to accept, and also wants to increase the total profit. Let $\mu_G(X)$ be the maximum size of a matching M with $\partial M \cap S \subseteq X$ and define $\mathcal{M}(X) \subseteq 2^E$ by

$$\mathcal{M}(X) := \{ M \subseteq E \mid M : \text{matching, } \partial M \cap S \subseteq X, |M| = \mu_G(X) \}.$$

Then, the chosen set $F(X) \subseteq X$ is defined as $F(X) := \partial M^* \cap S$, where M^* is the unique solution to $\max \{ w(M) \mid M \in \mathcal{M}(X) \}$. By definition, we see that $F(X)$ is a base of X for the matroid (S, \mathcal{I}) . Also, $F : 2^S \rightarrow 2^S$ satisfies the condition (Sub) as follows.

For $X \subseteq Y \subseteq S$, define $F(X)$ and $F(Y)$ as described above. Then, there are $M_X \in \mathcal{M}(X)$ and $M_Y \in \mathcal{M}(Y)$ such that

$$\begin{aligned} F(X) &= \partial M_X \cap S, \quad w(M_X) > w(N) \quad (\forall N \in \mathcal{M}(X) \setminus \{M_X\}), \\ F(Y) &= \partial M_Y \cap S, \quad w(M_Y) > w(N) \quad (\forall N \in \mathcal{M}(Y) \setminus \{M_Y\}). \end{aligned}$$

Their symmetric difference $M_X \triangle M_Y$ is partitioned into alternating paths and cycles. Suppose, to the contrary, that there exists $e \in F(Y) \cap X$ with $e \notin F(X)$. Then, $e \in \partial M_Y \setminus \partial M_X$, and hence e is the end vertex of some alternating path $P \subseteq M_X \triangle M_Y$ with $|P \cap M_Y| \geq |P \cap M_X|$. Let $N_X := M_X \triangle P$ and $N_Y := M_Y \triangle P$, which are matchings in G . In the case $|P|$ is odd, $|N_X| = |M_X| + 1$ and $\partial N_X \cap S = F(X) + e \subseteq X$, which contradicts $|M_X| = \mu_G(X)$. In the case $|P|$ is even, we have $\partial N_X \cap S \subseteq X$ and $\partial N_Y \cap S \subseteq Y$, which imply $N_X \in \mathcal{M}(X)$ and $N_Y \in \mathcal{M}(Y)$, respectively. The former implies $0 < w(M_X) - w(N_X) = w(P \cap M_X) - w(P \cap M_Y)$ whereas the latter implies $0 < w(M_Y) - w(N_Y) = w(P \cap M_Y) - w(P \cap M_X)$, a contradiction. ■

5.2.2 Representation with Circuit Families

Take an arbitrary matroid $\mathbf{M} = (S, \mathcal{I})$ and let \mathcal{C} be its circuit family, i.e.,

$$\mathcal{C} = \{ C \subseteq S \mid C \notin \mathcal{I}, C - e \in \mathcal{I} \quad (\forall e \in C) \}.$$

It is known that \mathcal{C} satisfies the following (C0)–(C2):

(C0) $\emptyset \notin \mathcal{C}$.

(C1) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$, then for every $e \in C_1 \cap C_2$ there exists a circuit $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) - e$.

A function $\vartheta : \mathcal{C} \rightarrow S$ is called a *de-cycle function* if it satisfies $\vartheta(C) \in C$ for each $C \in \mathcal{C}$.

Let $F : 2^S \rightarrow 2^S$ be a matroidal choice function on \mathbf{M} . For each $C \in \mathcal{C}$, we have $F(C) \in \mathcal{B}(C)$, and hence $|F(C)| = r(C) = |C| - 1$. Then, the set $C \setminus F(C)$ is a singleton. Define a de-cycle function $\vartheta_F : \mathcal{C} \rightarrow S$ by letting $\vartheta_F(C)$ be the only element in $C \setminus F(C)$ for each $C \in \mathcal{C}$. That is, identifying a singleton set with its element, a de-cycle function ϑ_F is defined by

$$\vartheta_F(C) = C \setminus F(C) \quad (C \in \mathcal{C}).$$

We call ϑ_F the de-cycle function *associated* with F .

Lemma 5.8. *Let $\vartheta_F : \mathcal{C} \rightarrow S$ be the de-cycle function associated with a matroidal choice function F . Then, for any $X \subseteq S$, we have*

$$F(X) = X \setminus \{ \vartheta_F(C) \mid C \in \mathcal{C}, C \subseteq X \}.$$

Proof. Let $R(X) := \{ \vartheta_F(C) \mid C \in \mathcal{C}, C \subseteq X \}$. For every $e \in R(X)$, there is $C \in \mathcal{C}$ such that $\vartheta_F(C) = e$ and $C \subseteq X$. Then $\{e\} = C \setminus F(C) \subseteq X \setminus F(X)$ by (Sub*). Therefore, $R(X) \subseteq X \setminus F(X)$ which implies $F(X) \subseteq X \setminus R(X)$. Also, the set $X \setminus R(X)$ is independent since $R(X)$ contains at least one element of each circuit. Thus $F(X) \subseteq X \setminus R(X) \in \mathcal{I}$. As $F(X) \in \mathcal{B}(X)$, this means $F(X) = X \setminus R(X)$, and the claim holds. \square

Lemma 5.8 implies that, if a de-cycle function ϑ is associated with some matroidal choice function, then $X \setminus \{ \vartheta(C) \mid C \in \mathcal{C}, C \subseteq X \} \in \mathcal{B}(X)$ holds for every $X \subseteq S$. For a general de-cycle function, however, this property does not hold as is shown in Example 5.9 below. Then Definition 5.10 introduces the “coherency” of a de-cycle function, which guarantees this property.

Example 5.9. Let \mathbf{M} be a uniform matroid of rank 1 on $S = \{e_1, e_2, e_3\}$. Let ϑ be a de-cycle function which indicates e_1, e_2, e_3 for circuits $\{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_3\}$, respectively. Then, we have $S \setminus \{ \vartheta(C) \mid C \in \mathcal{C}, C \subseteq S \} = \emptyset \notin \mathcal{B}(S)$. \blacksquare

Definition 5.10. A de-cycle function $\vartheta : \mathcal{C} \rightarrow S$ is called *coherent* if

$$X \setminus \{ \vartheta(C) \mid C \in \mathcal{C}, C \subseteq X \} \in \mathcal{B}(X)$$

holds for every $X \subseteq S$. \blacksquare

The following theorem gives a one-to-one correspondence between matroidal choice functions on \mathbf{M} and coherent de-cycle functions on the circuit family of \mathbf{M} .

Theorem 5.11. *A de-cycle function $\vartheta : \mathcal{C} \rightarrow S$ is coherent if and only if it is associated with some matroidal choice function on \mathbf{M} . Also, such a matroidal choice function $F : 2^S \rightarrow 2^S$ is uniquely defined by*

$$F(X) = X \setminus \{ \vartheta(C) \mid C \in \mathcal{C}, C \subseteq X \} \quad (X \subseteq S). \quad (5.2)$$

Proof. The “if” part and the second claim follow from Lemma 5.8. For the “only if” part, assume that ϑ is coherent and define F by (5.2). Then, $X \setminus F(X) = \{\vartheta(C) \mid C \in \mathcal{C}, C \subseteq X\}$ for each $X \subseteq S$ and then (Sub*) follows. Also the coherency implies $F(X) \in \mathcal{B}(X)$. Thus, F is a matroidal choice function, and we see that ϑ is associated with F . \square

5.3 Algorithmic Characterization

In this section, we characterize the coherency of de-cycle functions via the behavior of an online algorithm. We call an algorithm *online* if it receives its input piece by piece and makes decisions without future information. Let \mathcal{C} be the circuit family of a matroid $\mathbf{M} = (S, \mathcal{I})$ and recall the following basic facts on circuits [84].

Proposition 5.12. *For an independent set I and an element $e \in S \setminus I$, the subset $I + e$ contains a unique circuit if $I + e \notin \mathcal{I}$.*

When $I + e \notin \mathcal{I}$, we write $C(I|e)$ for this unique circuit contained in $I + e$.

Proposition 5.13. *For any $X \subseteq S$, take $B \in \mathcal{B}(X)$ and $e \in S \setminus B$ arbitrarily. If $B + e \in \mathcal{I}$, then $B + e \in \mathcal{B}(X + e)$. Otherwise, $B + e - f \in \mathcal{B}(X + e)$ for every $f \in C(B|e)$.*

For an arbitrary (not necessarily coherent) de-cycle function $\vartheta : \mathcal{C} \rightarrow S$, let us design an algorithm $\text{MCF}(\vartheta)$ as follows. An input of the algorithm is a sequence of (not necessarily distinct) elements of S of arbitrary length.

Algorithm $\text{MCF}(\vartheta)$.

Input: (e_1, e_2, \dots, e_k)

1. $J \leftarrow \emptyset$.
 2. For $i = 1$ to k , do:
 - (a) If $J + e_i \in \mathcal{I}$, then $J \leftarrow J + e_i$.
 - (b) Otherwise, $J \leftarrow J + e_i - \vartheta(C(J|e_i))$.
 3. Return J .
-

By Propositions 5.12 and 5.13 we can observe that, at any point in the algorithm, J is a base of the set received until then. This implies the following lemma.

Lemma 5.14. *For an input (e_1, e_2, \dots, e_k) , the algorithm $\text{MCF}(\vartheta)$ returns a base of the set $\{e_1, e_2, \dots, e_k\}$.*

In general, the base returned by the algorithm differs depending on the order of the sequence. That is, the output for (e_1, e_2, \dots, e_k) and that for $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k)$ may differ even if $\{e_1, e_2, \dots, e_k\} = \{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k\}$. If ϑ is coherent, however, the output is independent of the order as follows.

Lemma 5.15. *For a sequence (e_1, e_2, \dots, e_k) , let $X := \{e_1, e_2, \dots, e_k\}$. If ϑ is coherent, the algorithm $MCF(\vartheta)$ returns the subset $F(X)$ defined by (5.2).*

Proof. Let J^* be the output of the algorithm. By the algorithm, $e \in X \setminus J^*$ means that $\vartheta(C(J|e_i)) = e$ for some i . Since $C(J|e_i) \subseteq X$, the definition of $F(X)$ implies $e \in X \setminus F(X)$. Thus, $X \setminus J^* \subseteq X \setminus F(X)$, and so $J^* \supseteq F(X)$. This yields $J^* = F(X)$ since both J^* and $F(X)$ are bases of X . \square

Actually, the coherency is not only sufficient but also necessary condition for the independence of outputs from orders.

Theorem 5.16. *A de-cycle function ϑ is coherent if and only if it satisfies the following condition: For any sequences (e_1, e_2, \dots, e_k) and $(\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k)$, $MCF(\vartheta)$ returns the same subset if $\{e_1, e_2, \dots, e_k\} = \{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_k\}$.*

Proof. The “only if” part is shown in Lemma 5.15. We now show the “if” part. Take $X \subseteq S$ arbitrarily and let $J^* \subseteq X$ be the subset returned by $MCF(\vartheta)$ for every sequence (e_1, e_2, \dots, e_k) with $X := \{e_1, e_2, \dots, e_k\}$. For each $C \in \mathcal{C}$ with $C \subseteq X$, consider a sequence $(e_1, e_2, \dots, e_{|X|})$ such that $\{e_1, e_2, \dots, e_{|C|}\} = C$ and $\{e_{|C|+1}, \dots, e_{|X|}\} = X \setminus C$. Then, the output J^* can not contain $\vartheta(C)$ by the algorithm. Thus, $\{\vartheta(C) \mid C \in \mathcal{C}, C \subseteq X\} \subseteq X \setminus J^*$, and hence $J^* \subseteq X \setminus \{\vartheta(C) \mid C \in \mathcal{C}, C \subseteq X\}$. Note that $X \setminus \{\vartheta(C) \mid C \in \mathcal{C}, C \subseteq X\}$ is independent since it has no circuit. Also $J^* \in \mathcal{B}(X)$ by Lemma 5.14. Thus $J^* = X \setminus \{\vartheta(C) \mid C \in \mathcal{C}, C \subseteq X\} \in \mathcal{B}(X)$. \square

The following two lemmas will be used in Section 5.6. The first is obtained by Lemmas 5.8 and 5.15, and the second follows from the first.

Lemma 5.17. *Let F be a matroidal choice function on \mathbf{M} and ϑ_F be the de-cycle function associated with F . For any $X \subseteq S$ and (e_1, e_2, \dots, e_k) with $\{e_1, e_2, \dots, e_k\} = X$, the algorithm $MCF(\vartheta_F)$ returns $F(X)$.*

Lemma 5.18. *Let F be a matroidal choice function on \mathbf{M} and ϑ_F be the de-cycle function associated with F . For any $X', X \subseteq S$ with $X' \subseteq X$, we have $X' = F(X)$ if and only if we have both (i) $X' \in \mathcal{I}$ and (ii) $\forall e \in X \setminus X' : [X' + e \notin \mathcal{I}, \vartheta_F(C(X'|e)) = e]$.*

Proof. Let $k' = |X'|$ and $k = |X|$ and consider the sequence $(e_1, e_2, \dots, e_{k'}, e_{k'+1}, \dots, e_k)$ such that $\{e_1, e_2, \dots, e_{k'}\} = X'$ and $\{e_1, e_2, \dots, e_k\} = X$. By Lemma 5.17, $MCF(\vartheta_F)$ returns $F(X)$ for this sequence. Then, the definition of $MCF(\vartheta_F)$ implies the claim. \square

5.4 Local Characterization

In this section, we characterize the coherency of de-cycle functions by a local property. Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid and \mathcal{C} and r be its circuit family and rank function, respectively. We first introduce the concept of “minimal pair of circuits.”

Definition 5.19. A pair of circuits $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ is said to be *minimal* if it satisfies the following two conditions:

1. $C_1 \neq C_2$.
2. There is no pair $(C'_1, C'_2) \in \mathcal{C} \times \mathcal{C}$ such that $C'_1 \neq C'_2$ and $(C'_1 \cup C'_2) \subsetneq (C_1 \cup C_2)$. ■

For a uniform matroid, a pair of circuits (C_1, C_2) is minimal if and only if $|C_1 \setminus C_2| (= |C_2 \setminus C_1|) = 1$. For a graphic matroid, a pair of circuits (i.e., cycles) (C_1, C_2) is minimal if and only if it satisfies one of the following two conditions: (i) C_1 and C_2 are disjoint; (ii) $C_1 \cup C_2$ forms a theta graph, i.e., a graph which consists of three internally disjoint simple paths that have the same two distinct end vertices.

As will be shown in Lemma 5.22, this minimality can be characterized by the rank function as follows: A pair of circuits (C_1, C_2) is minimal if and only if we have

$$r(C_1 \cup C_2) = |C_1 \cup C_2| - 2. \quad (5.3)$$

Let $\vartheta : \mathcal{C} \rightarrow S$ be an arbitrary de-cycle function. Now we provide the main theorem of this section, whose proof is divided into two subsections.

Theorem 5.20. *ϑ is coherent if and only if ϑ satisfies the following condition:*

(D) *For a minimal pair $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$ and $C_3 \in \mathcal{C}$ with $C_3 \subseteq C_1 \cup C_2$,*

$$|\{\vartheta(C_1), \vartheta(C_2), \vartheta(C_3)\}| \leq 2.$$

In Theorem 5.20, if we regard the minimality of a pair of circuits as a kind of closeness, the condition (D) can be translated as follows: If three circuits are close to each other, then the same element is indicated for at least two of them.

The rest of this section is devoted to the proof of Theorem 5.20. In what follows, we show (1) the “only if” part and (2) the “if” part of Theorem 5.20.

(1) The “only if” part of Theorem 5.20

To show the “only if” part, we prepare the following lemma, which follows from definition.

Lemma 5.21. *A subset $X \subseteq S$ includes two or more circuits if and only if $r(X) \leq |X| - 2$.*

We need the following characterization of a minimal pair of circuits for the proof.

Lemma 5.22. *A pair of circuits (C_1, C_2) is minimal if and only if (5.3) holds.*

Proof. The “if” part: If (C_1, C_2) is minimal, then $C_1 \neq C_2$. Take $e_1 \in C_1 \setminus C_2$ and $e_2 \in C_2 \setminus C_1$ arbitrarily. Then, there is no circuit C such that $C \subseteq (C_1 \cup C_2) \setminus \{e_1, e_2\}$ since such C satisfies $C_1 \neq C$ and $(C_1 \cup C) \subsetneq (C_1 \cup C_2)$ which contradicts the minimality

of (C_1, C_2) . Hence $(C_1 \cup C_2) \setminus \{e_1, e_2\}$ is independent, so $r(C_1 \cup C_2) \geq |C_1 \cup C_2| - 2$. By Lemma 5.21, the equality holds.

The “only if” part: Assume $r(C_1 \cup C_2) = |C_1 \cup C_2| - 2$. For any $e \in C_1 \cup C_2$, we have $r((C_1 \cup C_2) - e) = r(C_1 \cup C_2)$, and hence $r((C_1 \cup C_2) - e) = |(C_1 \cup C_2) - e| - 1$. Then, by Lemma 5.21, $(C_1 \cup C_2) - e$ cannot include two distinct circuits. Therefore, (C_1, C_2) is minimal. \square

Here is the proof of the “only if” part of Theorem 5.20.

Lemma 5.23. *If ϑ is coherent, then ϑ satisfies (D).*

Proof. Let ϑ be coherent and suppose, to the contrary, that $\vartheta(C_1)$, $\vartheta(C_2)$, and $\vartheta(C_3)$ are all distinct for a minimal pair (C_1, C_2) and a circuit $C_3 \subseteq C_1 \cup C_2$. Then $X := C_1 \cup C_2$ satisfies $|X \setminus \{\vartheta(C) : C \in \mathcal{C}, C \subseteq X\}| \leq |X| - 3$ whereas Lemma 5.22 implies $r(X) = |X| - 2$. Thus, $X \setminus \{\vartheta(C) : C \in \mathcal{C}, C \subseteq X\}$ can not be a base of X , and hence ϑ is not coherent. \square

(2) The “if” part of Theorem 5.20

We additionally need the following two lemmas for the proof of the “if” part.

Lemma 5.24. *For any $X \subseteq S$ and $e \in S \setminus X$, we have $r(X + e) = r(X) + 1$ if and only if there is no $C \in \mathcal{C}$ such that $e \in C \subseteq X + e$.*

Lemma 5.25. *For two distinct circuits C_1 and C_0 and arbitrary $e \in C_1$, there exists a circuit C_2 such that $e \notin C_2 \subseteq C_1 \cup C_0$ and (C_1, C_2) is minimal.*

Proof. Let $C_2 \in \mathcal{C}$ such that $e \notin C_2 \subseteq C_1 \cup C_0$ and $|C_1 \cup C_2|$ is minimal. Then (C_1, C_2) is a minimal pair as follows. Suppose, to the contrary, it fails. Then $r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3$ by Lemma 5.21 and Lemma 5.22. Take $e' \in C_2 \setminus C_1$ arbitrarily. Then $r(C_1 \cup C_2 \setminus \{e, e'\}) \leq r(C_1 \cup C_2) \leq |C_1 \cup C_2| - 3 = |C_1 \cup C_2 \setminus \{e, e'\}| - 1$. Hence, $C_1 \cup C_2 \setminus \{e, e'\}$ is dependent and so includes a circuit C'_2 , which satisfies $e \notin C'_2 \subseteq C_1 \cup C_2 \subseteq C_1 \cup C_0$. Also $|C_1 \cup C'_2| \leq |C_1 \cup C_2 - e| < |C_1 \cup C_2|$, a contradiction. \square

Here we prove the “if” part of Theorem 5.20.

Lemma 5.26. *If ϑ satisfies (D), then ϑ is coherent.*

Proof. Assume that ϑ satisfies (D) and define $F(X)$ by (5.2) and let

$$R(X) := X \setminus F(X) = \{\vartheta(C) : C \in \mathcal{C}, C \subseteq X\}$$

for every $X \subseteq S$. We show $F(X) \in \mathcal{B}(X)$ ($\forall X \subseteq S$), i.e., the coherency, using an induction. Clearly $F(\emptyset) = \emptyset \in \mathcal{B}(\emptyset)$. For any $X \subseteq S$ and $e \notin X$, let us show $F(X + e) \in \mathcal{B}(X + e)$ assuming $F(X) \in \mathcal{B}(X)$. By definition of F , the set $F(X + e)$ is independent, and

hence what is left is to show $|F(X + e)| = r(X + e)$. We need consider two cases: (i) $r(X + e) = r(X) + 1$, and (ii) $r(X + e) = r(X)$.

Case (i): By Lemma 5.24, $r(X + e) = r(X) + 1$ implies that there is no $C \in \mathcal{C}$ with $e \in C \subseteq X + e$, and hence $R(X + e) = R(X)$. This implies $F(X + e) = F(X) + e$. Since $|F(X)| = r(X)$ by $F(X) \in \mathcal{B}(X)$, we have $|F(X + e)| = r(X) + 1 = r(X + e)$.

Case (ii): Since $r(X + e) = r(X) = |F(X)|$, we are reduced to show $|F(X + e)| = |F(X)|$, which is equivalent to $|R(X + e)| = |R(X)| + 1$. Also, we have $R(X) \subseteq R(X + e)$ by definition. Then, it suffices to show that $R(X + e) \setminus R(X)$ is a singleton.

Since $F(X) \in \mathcal{B}(X)$ and $r(X) = r(X + e)$, the set $F(X) + e$ includes a circuit C_0 with $e \in C_0$. Since $C_0 \subseteq F(X) + e \subseteq X + e$, we have $\vartheta(C_0) \in R(X + e)$. Also, $C_0 \subseteq F(X) + e = X \setminus R(X) + e$ and $e \in S \setminus X$ implies

$$C_0 \cap R(X) = \emptyset, \quad (5.4)$$

and hence $\vartheta(C_0) \notin R(X)$. Therefore, $\vartheta(C_0) \in R(X + e) \setminus R(X)$. Next, we show that $R(X + e) \setminus R(X)$ contains only $\vartheta(C_0)$.

Suppose, to the contrary, that there is a circuit C_1 such that

$$C_1 \subseteq X + e, \quad \vartheta(C_1) \in R(X + e) \setminus R(X), \quad \vartheta(C_1) \neq \vartheta(C_0). \quad (5.5)$$

Note that $\vartheta(C_1) \notin R(X)$ implies $e \in C_1$ and $\vartheta(C_1) \neq \vartheta(C_0)$ implies $C_1 \neq C_0$. If multiple circuits satisfy these condition, let C_1 be the one which minimizes $|C_1 \cap R(X)|$. By Lemma 5.25 for C_1, C_0 and $e \in C_1$, there exists $C_2 \in \mathcal{C}$ such that $e \notin C_2 \subseteq C_1 \cup C_0$ and (C_1, C_2) is minimal. Then $C_2 \subseteq C_1 \cup C_0 - e \subseteq X$, and hence $\vartheta(C_2) \in R(X)$, which implies $\vartheta(C_2) \notin C_0$ by (5.4). Then, $\vartheta(C_2) \in C_2 \subseteq C_1 \cup C_0$ implies $\vartheta(C_2) \in C_1$. Thus we obtain $\vartheta(C_2) \in C_1 \cap C_2$. By the axiom (C2) of circuits, there is a circuit C_3 such that

$$\vartheta(C_2) \notin C_3 \subseteq C_1 \cup C_2.$$

Since (C_1, C_2) is minimal and $C_3 \subseteq C_1 \cup C_2$, the condition (D) yields

$$|\{\vartheta(C_1), \vartheta(C_2), \vartheta(C_3)\}| \leq 2.$$

This implies $\vartheta(C_1) = \vartheta(C_3)$ since we have $\vartheta(C_1) \neq \vartheta(C_2)$ and $\vartheta(C_2) \neq \vartheta(C_3)$, which follow from $\vartheta(C_1) \notin R(X) \ni \vartheta(C_2)$ and $\vartheta(C_2) \notin C_3$, respectively. The fact $\vartheta(C_1) = \vartheta(C_3)$ implies that (5.5) holds with C_1 replaced by C_3 . Also, $C_3 \subseteq C_1 \cup C_2 \subseteq C_1 \cup C_0$ and (5.4) implies $C_3 \cap R(X) \subseteq C_1 \cap R(X)$, whose equality fails by $\vartheta(C_2) \in (C_1 \cap R(X)) \setminus C_3$. Hence, $|C_3 \cap R(X)| < |C_1 \cap R(X)|$, which contradicts the minimality of $|C_1 \cap R(X)|$. \square

Combining Lemmas 5.23 and 5.26, we obtain Theorem 5.20.

5.5 Relationships with Discrete Concavity

This section shows that matroidal choice functions are closely related to valuated matroids and M^\natural -concave functions.

5.5.1 Valuated Matroids and M^\natural -concave Functions

A *valuated matroid*, introduced by Dress and Wenzel [18, 19], is a pair (\mathcal{B}, ω) such that $\mathcal{B} \subseteq 2^S$ is the base family of a matroid and $\omega : \mathcal{B} \rightarrow \mathbf{R}$ is a function which satisfies the following exchange axiom.

(VM) For any $B_1, B_2 \in \mathcal{B}$ and $e_1 \in B_1 \setminus B_2$, there is $e_2 \in B_2 \setminus B_1$ such that $B_1 - e_1 + e_2, B_2 + e_1 - e_2 \in \mathcal{B}$ and

$$\omega(B_1) + \omega(B_2) \leq \omega(B_1 - e_1 + e_2) + \omega(B_2 + e_1 - e_2).$$

For any set function $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$, denote by $\text{dom } f := \{X \mid f(X) \neq -\infty\}$ the effective domain of f . We say that f is M^\natural -concave [75, 78] if it satisfies the following exchange axiom.

(M^\natural) For any $X_1, X_2 \in \text{dom } f$, $e_1 \in X_1 \setminus X_2$, either of the following holds:

1. $f(X_1) + f(X_2) \leq f(X_1 - e_1) + f(X_2 + e_1)$,
2. $\exists e_2 \in X_2 \setminus X_1 : f(X_1) + f(X_2) \leq f(X_1 - e_1 + e_2) + f(X_2 + e_1 - e_2)$.

The following facts are known (see, e.g., [78]) and can be easily confirmed.

Proposition 5.27. *For an M^\natural -concave function $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ with $\emptyset \in \text{dom } f$, the pair $(S, \text{dom } f)$ is a matroid.*

Proposition 5.28. *For an M^\natural -concave function $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ with $\emptyset \in \text{dom } f$, let \mathcal{B} be the base family of the matroid $(S, \text{dom } f)$ and $f|_{\mathcal{B}} : \mathcal{B} \rightarrow \mathbf{R}$ be the restriction of f to \mathcal{B} . Then, $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid.*

The M^\natural -concavity has the following local characterization.

Lemma 5.29 (Shioura and Tamura [100]). *For $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$, assume that $(S, \text{dom } f)$ is a matroid. Then, f is M^\natural -concave if and only if f satisfies the following conditions for every $X \subseteq S$ and $a, b, c \in S \setminus X$:*

1. $f(X) + f(X \cup \{a, b\}) \leq f(X + a) + f(X + b)$,
2. $f(X + a) + f(X \cup \{b, c\}) \leq \max\{f(X + b) + f(X \cup \{a, c\}), f(X + c) + f(X \cup \{a, b\})\}$,

where we admit the inequality of the form $-\infty \leq -\infty$.

5.5.2 Induced Matroidal Choice Functions

Let $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ be a value function. That is, we regard $X \in \text{dom } f$ as an acceptable set with value $f(X)$ and $X \in 2^S \setminus \text{dom } f$ as unacceptable. We assume $\emptyset \in \text{dom } f$. A value function f is called *unique-selecting* if for each $X \subseteq S$ the maximum value of f among all subsets of X is attained by a unique subset, which we denote by $\arg \max \{f(Y) \mid Y \subseteq X\}$. For a unique-selecting function f , the choice function $F : 2^S \rightarrow 2^S$ defined by

$$F(X) = \arg \max \{f(Y) \mid Y \subseteq X\} \quad (X \subseteq S)$$

is said to be *induced from* f . The following fact is implied by Lemma 5.2 of Fujishige and Tamura [37] (see also Lemma 3.3 of [81]).

Lemma 5.30. *Let f be a unique-selecting M^\natural -concave function. Then, the choice function F induced from f is substitutable.*

We call f *monotone* if $X \subseteq Y$ implies $f(X) \leq f(Y)$ for $X, Y \in \text{dom } f$.

Theorem 5.31. *Let F be a choice function induced from a unique-selecting M^\natural -concave function f . If f is monotone, then F is a matroidal choice function on $(S, \text{dom } f)$.*

Proof. The substitutability follows from Lemma 5.30. The pair $(S, \text{dom } f)$ is a matroid by Proposition 5.27. For any $X \subseteq S$, the monotonicity of f implies $F(X) \in \mathcal{B}(X)$ where $\mathcal{B}(X)$ denotes the family of bases of X . \square

The following theorem generalizes Theorem 5.31 by showing that the induced choice function F may be matroidal even if f is not monotone. For a choice function $F : 2^S \rightarrow 2^S$, we denote by $\mathcal{A}(F)$ the family of accepted subsets, i.e., $\mathcal{A}(F) := \{X \subseteq S \mid F(X) = X\}$.

Theorem 5.32. *Let F be a choice function induced from a unique-selecting M^\natural -concave function f . If $(S, \mathcal{A}(F))$ is a matroid, then F is a matroidal choice function on $(S, \mathcal{A}(F))$.*

Proof. By Lemma 5.33 below, a function $\hat{f} : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by

$$\hat{f}(X) = \begin{cases} f(X) & X \in \mathcal{A}(F), \\ -\infty & \text{otherwise} \end{cases} \quad (5.6)$$

is a monotone M^\natural -concave function with $\text{dom } \hat{f} = \mathcal{A}(F)$. Observe that \hat{f} also induces F . Then, Theorem 5.31 implies that F is a matroidal choice function on $(S, \mathcal{A}(F))$. \square

Lemma 5.33. *Let F be a choice function induced from a unique-selecting M^\natural -concave function f . If $(S, \mathcal{A}(F))$ is a matroid, then a function $\hat{f} : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ defined by (5.6) is a monotone M^\natural -concave function with $\text{dom } \hat{f} = \mathcal{A}(F)$.*

Proof. We have $\text{dom } \hat{f} = \mathcal{A}(F)$ by definition. Then any $X \in \text{dom } \hat{f}$ satisfies $F(X) = X$, which implies $f(Y) < f(X)$ ($\forall Y \subsetneq X$), and hence \hat{f} is monotone. What is left is to show the M^\natural -concavity of \hat{f} . It suffices to show that Conditions 1 and 2 in Lemma 5.29 hold with f replaced by \hat{f} . Take any $X \subseteq S$ and $a, b, c \subseteq S \setminus X$. We first show Condition 1, i.e.,

$$\hat{f}(X) + \hat{f}(X \cup \{a, b\}) \leq \hat{f}(X + a) + \hat{f}(X + b). \quad (5.7)$$

We assume $X \cup \{a, b\} \in \mathcal{A}(F)$, since otherwise $\hat{f}(X \cup \{a, b\}) = -\infty$ and (5.7) holds obviously. Then, $\{X, X + a, X + b, X \cup \{a, b\}\} \subseteq \mathcal{A}(F)$ follows from the axiom (I1) of the matroid $(S, \mathcal{A}(F))$. Then, \hat{f} in the inequality (5.7) can be replaced by f , and then the inequality holds since f is M^\natural -concave.

We next show Condition 2, i.e.,

$$\hat{f}(X + a) + \hat{f}(X \cup \{b, c\}) \leq \max\{\hat{f}(X + b) + \hat{f}(X \cup \{a, c\}), \hat{f}(X + c) + \hat{f}(X \cup \{a, b\})\}. \quad (5.8)$$

We assume that a, b, c are all distinct and $X + a, X \cup \{b, c\} \in \mathcal{A}(F)$, since otherwise (5.8) is obvious. Then, $X + b, X + c \in \mathcal{A}(F)$ by the axiom (I1) of the matroid $(S, \mathcal{A}(F))$. Also, by applying (I2) to $X + a, X \cup \{b, c\} \in \mathcal{A}(F)$, we obtain

$$X \cup \{a, b\} \in \mathcal{A}(F) \quad \text{or} \quad X \cup \{a, c\} \in \mathcal{A}(F). \quad (5.9)$$

Note that the inequality (5.8) with \hat{f} replaced by f holds. Without loss of generality, we assume $f(X + b) + f(X \cup \{a, c\})$ is larger than or equal to $f(X + c) + f(X \cup \{a, b\})$. Then, we have

$$f(X + c) + f(X \cup \{a, b\}) \leq f(X + b) + f(X \cup \{a, c\}), \quad (5.10)$$

$$f(X + a) + f(X \cup \{b, c\}) \leq f(X + b) + f(X \cup \{a, c\}). \quad (5.11)$$

To obtain a contradiction, suppose that (5.8) fails for \hat{f} . Then, we have

$$\hat{f}(X + a) + \hat{f}(X \cup \{b, c\}) > \hat{f}(X + b) + \hat{f}(X \cup \{a, c\}). \quad (5.12)$$

Since we have $\hat{f}(Y) = f(Y)$ for each $Y \in \{X + a, X + b, X \cup \{b, c\}\} \subseteq \mathcal{A}(F)$, (5.11) and (5.12) imply $f(X \cup \{a, c\}) \neq \hat{f}(X \cup \{a, c\})$. Hence $X \cup \{a, c\} \notin \mathcal{A}(F)$. With (5.9), this yields $X \cup \{a, b\} \in \mathcal{A}(F)$. Then, $X \cup \{b, c\}, X \cup \{a, b\} \in \mathcal{A}(F)$ and $X \cup \{a, c\} \notin \mathcal{A}(F)$ respectively imply

$$f(X \cup \{b, c\}) > f(X + b), \quad (5.13)$$

$$f(X \cup \{a, b\}) > f(X + b), \quad (5.14)$$

$$Y^* := \arg \max \{f(Y) \mid Y \subseteq X \cup \{a, c\}\} \neq X \cup \{a, c\}. \quad (5.15)$$

There are three cases (i) $c \notin Y^*$, (ii) $a \notin Y^*$, (iii) $\{a, c\} \subseteq Y^*$, $X \setminus Y^* \neq \emptyset$.

Case (i): $c \notin Y^* \subseteq X \cup \{a, c\}$ implies $Y^* \subseteq X + a$. Since $X + a \in \mathcal{A}(F)$, means $F(X + a) = X + a$, this implies $Y^* = X + a$. Then, (5.15) implies $f(X + a) > f(X \cup \{a, c\})$. Combined with (5.13), this contradicts (5.11).

Case (ii): By a similar argument, $a \notin Y^* \subseteq X \cup \{a, c\}$ implies $f(X + c) > f(X \cup \{a, c\})$. Combined with (5.14), this contradicts (5.10).

Case (iii): We have $\{a, c\} \subseteq Y^* \subseteq X \cup \{a, c\}$. Apply the exchange axiom (M^\natural) of f to Y^* and $X + a$ and $c \in Y^* \setminus (X + a)$. Then, we have

$$f(Y^*) + f(X + a) \leq f(Y^* - c) + f(X \cup \{a, c\}) \quad (5.16)$$

or there is $s \in (X + a) \setminus Y^* = X \setminus Y^*$ such that

$$f(Y^*) + f(X + a) \leq f(Y^* - c + d) + f((X \cup \{a, c\}) - d). \quad (5.17)$$

By $Y^* - c \subseteq X + a$ and $X + a \in \mathcal{A}(F)$, we have $f(X + a) > f(Y^* - c)$. In the case where (5.16) holds, this implies $f(Y^*) < f(X \cup \{a, c\})$, which contradicts (5.15). Similarly, in the case where (5.17) holds, since $Y^* - c + d \subseteq X + a$ and $X + a \in \mathcal{A}(F)$ imply $f(X + a) > f(Y^* - c + d)$, we obtain $f(Y^*) < f((X \cup \{a, c\}) - d)$, which also contradicts (5.15). \square

Lemma 5.33, combined with Proposition 5.28, also implies the following fact, which is used in Section 5.5.3.

Lemma 5.34. *Let F be a choice function induced from a unique-selecting M^\natural -concave function f . If $(S, \mathcal{A}(F))$ is a matroid, then $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid, where \mathcal{B} is the base family of $(S, \mathcal{A}(F))$.*

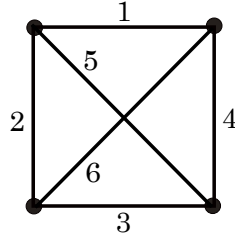
5.5.3 Uninducible Matroidal Choice Function

As shown in Theorem 5.32, an M^\natural -concave function induces a matroidal choice function under certain conditions. Then, one may wonder if every matroidal choice function is obtained in such a way. This is not true in general. We give a counter-example using the following proposition.

Proposition 5.35 (Dress and Wenzel [19]). *Let (\mathcal{B}, ω) be a valuated matroid on a matroid $\mathbf{M} = (S, \mathcal{I})$. If \mathbf{M} is a binary matroid, then ω can be represented as*

$$\omega(B) = \alpha + \sum_{e \in B} \eta(e) \quad (B \in \mathcal{B})$$

for some $\alpha \in \mathbf{R}$ and $\eta : S \rightarrow \mathbf{R}$.

Figure 5.1: A graphical representation of \mathbf{M}

Example 5.36. We give an example of matroidal choice function which cannot be induced from any \mathbf{M}^\sharp -concave set function in the way described in Section 5.5.2. Let $\mathbf{M} = (S, \mathcal{I})$ be a matroid such that $S = \{e_1, e_2, \dots, e_6\}$ and \mathcal{C} consists of the following seven circuits:

$$\begin{aligned} C_1 &= \{e_1, e_2, e_3, e_4\}, & C_2 &= \{e_1, e_3, e_5, e_6\}, & C_3 &= \{e_2, e_4, e_5, e_6\}, \\ C_4 &= \{e_1, e_2, e_6\}, & C_5 &= \{e_2, e_3, e_5\}, & C_6 &= \{e_3, e_4, e_6\}, & C_7 &= \{e_1, e_4, e_5\}. \end{aligned}$$

This is a graphic matroid. (Figure 5.1 shows its graphical representation.)

Define a de-cycle function $\vartheta : \mathcal{C} \rightarrow S$ as

$$\begin{aligned} \vartheta(C_1) &= e_4, \\ \vartheta(C_2) &= \vartheta(C_4) = \vartheta(C_6) = e_6, \\ \vartheta(C_3) &= \vartheta(C_5) = \vartheta(C_7) = e_5. \end{aligned}$$

We can check that ϑ is coherent. Hence, F defined by (5.2) is a matroidal choice function on \mathbf{M} and $\mathcal{A}(F) = \mathcal{I}$ by Proposition 5.2. We now show that there is no unique-selecting \mathbf{M}^\sharp -concave function $f : 2^S \rightarrow \mathbf{R} \cup \{-\infty\}$ which satisfies

$$\forall X \subseteq S : F(X) = \arg \max \{ f(Y) \mid Y \subseteq X \}.$$

Suppose, to the contrary, there exists such a function f . Since f is \mathbf{M}^\sharp -concave and $(S, \mathcal{A}(F)) = \mathbf{M}$ is a matroid, Lemma 5.34 implies that $(\mathcal{B}, f|_{\mathcal{B}})$ is a valuated matroid, where \mathcal{B} is the base family of \mathbf{M} . Since \mathbf{M} is graphic, and hence binary, Proposition 5.35 implies that

$$\forall B \in \mathcal{B} : f(B) = \alpha + \sum_{e \in B} \eta(e)$$

holds for some $\alpha \in \mathbf{R}$ and $\eta : S \rightarrow \mathbf{R}$. Note that, for $C_2 = \{e_1, e_3, e_5, e_6\}$, we have $F(C_2) = C_2 - \vartheta(C_2) = \{e_1, e_3, e_5\}$, which implies $f(\{e_1, e_3, e_5\}) > f(\{e_1, e_3, e_6\})$. As $\{e_1, e_3, e_5\}, \{e_1, e_3, e_6\} \in \mathcal{B}$, this implies $\eta(e_5) > \eta(e_6)$. On the other hand, $C_3 = \{e_2, e_4, e_5, e_6\}$ and $F(C_3) = C_3 - \vartheta(C_3) = \{e_2, e_4, e_6\}$ imply $f(\{e_2, e_4, e_6\}) > f(\{e_2, e_4, e_5\})$, and hence $\eta(e_6) > \eta(e_5)$, a contradiction. \blacksquare

5.6 Matching Model with Matroidal Choice Functions

Here we introduce a many-to-one matching model with matroidal choice functions. As shown in Section 5.2, matroidal choice functions are substitutable and size-monotone, and hence our model is a special case of models of Alkan [8], Fleiner [26, 27], and Hatfield and Milgrom [53]. Therefore, all their results, such as the existence and the lattice structure of stable matchings, automatically carry over to our model. Moreover, we can design an algorithm finding student-optimal stable matching using associated de-cycle functions instead of choice functions.

5.6.1 Model Formulation

We have two disjoint finite sets I and J , which represent students and colleges, respectively. Let $E = I \times J$ be the set of all pairs, and define $E_i = \{(i, j) \mid j \in J\}$ for each $i \in I$ and $E_j = \{(i, j) \mid i \in I\}$ for each $j \in J$.

Each student $i \in I$ is assigned to at most one college, and so his preference is represented by a total order \succ_i on $E_i \cup \{\perp\}$, where the symbol \perp represents unassignment¹. Here, $e \succ_i e'$ means e is better than e' for i . We write $e \succeq_i e'$ to mean $e \succ_i e'$ or $e = e'$. A set $X \subseteq E$ is called *student-feasible* if $|X \cap E_i| \leq 1$ ($\forall i \in I$). For a student-feasible set $X \subseteq E$ and a student $i \in I$, we write x_i for the unique element in $X \cap E_i$ if it is nonempty, and otherwise we let $x_i = \perp$.

Each college $j \in J$ can be assigned to multiple students and its preference is represented by a matroidal choice function $F_j : 2^{E_j} \rightarrow 2^{E_j}$. For a subset $X \subseteq E$ and a college $j \in J$, we denote $X_j = X \cap E_j$.

Thus, an *instance* of this model is represented by $(I, J, E, \{\succ_i\}_{i \in I}, \{F_j\}_{j \in J})$.

Definition 5.37. A student-feasible set $X \subseteq E$ is said to be (pairwise²) *stable matching* if the following two conditions hold:

1. $x_i \succeq_i \perp$ for every $i \in I$ and $F_j(X_j) = X_j$ for every $j \in J$.
2. For every $e = (i, j) \in E \setminus X$, we have $x_i \succ_i e$ or $F_j(X_j + e) = X_j$. ■

Recall that F_j is a matroidal choice function for each $j \in J$. Let $\mathbf{M}_j = (E_j, \mathcal{I}_j)$ be the underlying matroid and ϑ_j be the de-cycle function associated with F_j . For any student-feasible set $X \subseteq E$, let ${}^I\overline{X}$ be the subset of $E \setminus X$ preferred to X by students, i.e.,

$${}^I\overline{X} = \{e = (i, j) \in E \mid e \succ_i x_i\}.$$

By Lemma 5.8, the condition $F_j(X_j) = X_j$ is equivalent to $X_j \in \mathcal{I}_j$. Also, for $X_j \in \mathcal{I}_j$ and $e \in E_j \setminus X_j$, the condition $F_j(X_j + e) = X_j$ is equivalent to $[X_j + e \notin \mathcal{I}_j, \vartheta_j(C(X|e)) = e]$.

¹ In practical setting, such as public school choice programs, students have options outside the program, for example, private education and home schooling. Unassignments correspond to such outside options.

² There is also the notion of “setwise stability,” but it is equivalent to the pairwise stability in this setting. See [53, 102].

Then, by Lemma 5.18, a stable matching defined in Definition 5.37 is characterized as follows.

Lemma 5.38. *A student-feasible set $X \subseteq E$ is a stable matching if and only if every $j \in J$ satisfies $F_j(X_j \cup {}^I\overline{X_j}) = X_j$.*

5.6.2 Matroidal Deferred Acceptance Algorithm

We now describe an algorithm which finds a stable matching using $\{(\mathbf{M}_j, \vartheta_j)\}_{j \in J}$ instead of $\{F_j\}_{j \in J}$. In our algorithm, $X \subseteq E$ represents a temporary matching and $R \subseteq E$ is the set of pairs rejected by colleges until then.

Algorithm MDA.

Input: $(I, J, E, \{\succ_i\}_{i \in I}, \{(\mathbf{M}_j, \vartheta_j)\}_{j \in J})$.

1. $X \leftarrow \emptyset, R \leftarrow \emptyset$.
 2. While there exists $i \in I$ s.t. $x_i = \perp$ and $\exists e \in E_i \setminus R_i : e \succ_i \perp$, repeat the following:
 - (a) Take such i and $e \leftarrow \max_{\succ_i}(E_i \setminus R_i)$, $j \leftarrow$ the end point of e in J .
 - (b) If $X_j + e \in \mathcal{I}_j$, then $X \leftarrow X + e$.
 - (c) Otherwise, $X \leftarrow X + e - \vartheta_j(C(X_j|e))$ and $R \leftarrow R + \vartheta_j(C(X_j|e))$.
 3. Return X .
-

Remark 5.39. This algorithm is a generalization of the “recursive algorithm” of McVitie and Wilson [69], and also a variant of the “cumulative offer process” of Hatfield and Kojima [52]. Similarly to them, our algorithm lets only one student offer to a college in each step. ■

Let X^* and R^* denote X and R , respectively, at the termination of the algorithm. For each $i \in I$, the algorithm takes elements in E_i from the top to the bottom w.r.t. \succ_i , and rejected elements are added to R . Then, for any $e \in E_i$, the condition $e \succ_i x_i^*$ is equivalent to $e \in R^*$. Thus, we obtain the following observation.

Claim 5.40. ${}^I\overline{X^*} = R^*$.

This implies the correctness of the algorithm as follows.

Theorem 5.41. *The output X^* is a stable matching.*

Proof. We can observe that X is student-feasible throughout the algorithm. Hence, by Lemma 5.38, it suffices to show that $F_j(X_j^* \cup {}^I\overline{X_j^*}) = X_j^*$ for any $j \in J$. Note that the way to update X_j in Step 2 can be identified with Step 2 of MCF(ϑ_j). Hence we have $F_j(X_j^* \cup R_j^*) = X_j^*$ by Lemma 5.17. By Claim 5.40, this means $F_j(X_j^* \cup {}^I\overline{X_j^*}) = X_j^*$. □

Define a relation \succeq_I on matchings by $X \succeq_I Y \iff x_i \succeq_i y_i (\forall i \in I)$. Then, it is clearly a partial order.

Theorem 5.42. *The output X^* is a student-optimal stable matching. That is, X^* satisfies $X^* \succeq_I Y$ for every stable matching Y .*

Proof. Let Y be an arbitrary stable matching. Note that $X^* \succeq_I Y$ is equivalent to ${}^I\overline{X^*} \cap Y = \emptyset$, and also this is equivalent to $R^* \cap Y = \emptyset$ by Claim 5.40. Hence it suffices to show that the condition $(\star) R \cap Y = \emptyset$ holds at any time of the algorithm. We prove this by induction on R . Assume that (\star) holds for the current R . To prove that the next updated R still satisfies (\star) , we show that $X_j + e \notin \mathcal{I}_j$ implies $\vartheta_j(C(X_j|e)) \notin Y_j$ for every $j \in J$ and $e \in E_j \setminus R_j$.

By the algorithm, any $i \in I$ and any $e', e'' \in E_i$ satisfy

$$[e' \in X + e, e'' \succ_i e'] \implies e'' \in R. \quad (5.18)$$

Then, any $e' \in (X + e) \cap E_i$ satisfies $e' \succeq_i y_i$, because otherwise (5.18) implies $y_i \in R$ which contradicts (\star) . Thus, we obtain $X + e \subseteq Y \cup {}^I\overline{Y}$. Then, $X_j + e \notin \mathcal{I}_j$ implies $C(X_j|e) \subseteq Y_j \cup {}^I\overline{Y_j}$. By Lemma 5.8, this implies

$$\vartheta_j(C(X_j|e)) \notin F_j(Y_j \cup {}^I\overline{Y_j}). \quad (5.19)$$

On the other hand, since Y is a stable matching, we have $F_j(Y_j \cup {}^I\overline{Y_j}) = Y_j$ by Lemma 5.38. Then, (5.19) means $\vartheta_j(C(X_j|e)) \notin Y_j$. \square

For an algorithm which returns a matching for any instance, we call it *strategy-proof for students* if no student can improve his assignment by reporting a false preference. That is, for a strategy-proof algorithm, there is no student $i \in I$ and preference \succ'_i such that the algorithm assigns a better college (w.r.t. \succ_i) to i if the instance is modified by replacing \succ_i with \succ'_i .

Proposition 5.43. *The algorithm MDA is strategy-proof for students.*

Proof. This is shown by an adaptation of the results of Hatfield and Milgrom [53]. For any algorithm which returns the student-optimal stable matching, they proved that it is strategy-proof if choice functions are substitutable and size-monotone. Since matroidal choice functions satisfy these properties by the definition and Proposition 5.3, the proof is completed by Theorems 5.41 and 5.42. \square

Remark 5.44. There is also the notion of “group strategy-proofness” for matching algorithms. This represents the nonexistence of a group of students such that each of them can improve their assignment by jointly misreporting their preferences. By [51], we can show that the algorithm MDA is also group strategy-proof. \blacksquare

Let us consider the time complexity of the algorithm. We assume that each \mathbf{M}_j is given by an independence oracle \mathcal{O}_j which returns “yes” if a given subset $X \subseteq E_j$ is independent, and otherwise returns some circuit included in X as a certification of the dependency.

Theorem 5.45. *The algorithm MDA runs in $O(|E|)$ time, provided that each oracle call takes constant time.*

Proof. In the algorithm, each $e \in E$ is chosen in Step 2 at most once, and so Step 2 is repeated at most $|E|$ times. Since Step 2 needs constant time, the algorithm runs in $O(|E|)$ in total. Note that $\mathcal{O}_j(X_j + e)$ is either “yes” or the circuit $C(X_j|e)$, since $X_j + e$ includes at most one circuit. \square

Remark 5.46. It may be more common that an independence oracle returns only “no” for a dependent set without giving a certificate circuit. Even with such oracles, MDA needs only $O(|E| \cdot r_{\max})$ time, where $r_{\max} := \max_{j \in J} r(\mathbf{M}_j)$. This is because $C(X_j|e) = \{e' \in X_j + e \mid X_j + e - e' \in \mathcal{I}_j\}$, which is obtained by at most $r_{\max} + 1$ oracle calls. \blacksquare

5.6.3 Structure of Stable Matchings

Alkan [8] and Fleiner [26,27] studied the structure of the set of stable matchings for models with substitutable and size-monotone choice functions. Matroidal choice functions satisfy these two properties by the definition and Proposition 5.3. Therefore, for an instance $(I, J, E, \{\succ_i\}_{i \in I}, \{F_j\}_{j \in J})$ with matroidal choice functions $\{F_j\}_{j \in J}$, the following three propositions immediately follow (see [8,27] for the proofs).

Proposition 5.47. *For any stable matchings X and Y , we have $x_i = \perp \iff y_i = \perp$ for each $i \in I$ and $|X_j| = |Y_j|$ for each $j \in J$.*

Proposition 5.48. *For any stable matchings X and Y , we have $X \succeq_I Y$ if and only if they satisfy $F_j(X_j \cup Y_j) = Y_j$ for each $j \in J$.*

Proposition 5.49. *The set of all stable matchings forms a distributive lattice under the partial order \succeq_I . Moreover, for stable matchings X and Y , their join $X \vee_I Y$ and meet $X \wedge_I Y$ are obtained by*

$$\begin{aligned} X \vee_I Y &= \{ \max_{\succ_i} \{x_i, y_i\} \mid i \in I \text{ is assigned in } X \text{ and } Y \}, \\ X \wedge_I Y &= \{ \min_{\succ_i} \{x_i, y_i\} \mid i \in I \text{ is assigned in } X \text{ and } Y \}. \end{aligned}$$

These propositions imply the following corollary.

Corollary 5.50. *Any stable matchings X and Y satisfy $\text{span}_{\mathbf{M}_j}(X_j) = \text{span}_{\mathbf{M}_j}(Y_j)$ for each $j \in J$, where \mathbf{M}_j is the underlying matroid of F_j .*

Proof. Since stable matchings form a lattice by Proposition 5.49, there is a minimum stable matching with respect to \succeq_I . Let us denote it by Z . For any stable matching X , we have

$X \succeq_I Z$, and hence every $j \in J$ satisfies $F_j(X_j \cup Z_j) = Z_j$ by Proposition 5.48. This means that Z_j is a base of $X_j \cup Z_j$. Then X_j is also a base of $X_j \cup Z_j$ because X_j is independent and $|X_j| = |Z_j|$ by Proposition 5.47. Thus, $\text{span}_{\mathbf{M}_j}(X_j) = \text{span}_{\mathbf{M}_j}(X_j \cup Z_j) = \text{span}_{\mathbf{M}_j}(Z_j)$ holds for any stable matching X . \square

Remark 5.51. Proposition 5.47 and Corollary 5.50 imply a generalization of the “rural hospital theorem” as follows³. If \mathbf{M}_j is a uniform matroid of rank $q_j \in \mathbf{Z}_{>0}$ and we have $|X_j| < q_j$ for some stable matching X , then $X_j = \text{span}_{\mathbf{M}_j}(X_j)$. In this case, Proposition 5.47 and Corollary 5.50 imply that $Y_k = X_k$ holds for any $Y \in \mathfrak{S}$. With the statement of Proposition 5.47, this generalizes the “rural hospital theorem.” \blacksquare

³This argument is quite similar to that of Remark 3.26 in Section 3.4.

Chapter 6

List Supermodular Coloring

6.1 Introduction

A list coloring is a type of coloring in which each of the elements to be colored has its own list of permissible colors. One of the most celebrated results in the study of list coloring is the following theorem of Galvin on edge colorings of bipartite graphs. An *edge coloring* of an undirected graph is an assignment of a color to each edge such that no two adjacent edges have the same color.

Theorem 6.1 (Galvin [43]). *For a bipartite graph that admits an edge coloring with $k \in \mathbf{Z}_{>0}$ colors, if each edge e has a list $L(e)$ of k colors, then there exists an edge coloring such that every edge e is assigned a color in $L(e)$.*

The existence of an edge coloring in a bipartite graph is characterized by König's theorem as follows.

Theorem 6.2 (König [62]). *A bipartite graph admits an edge coloring with k or less colors if and only if each vertex is incident to at most k edges.*

Theorem 6.2 states that the minimum number of colors required for a bipartite edge coloring is equal to the maximum degree of a vertex. Combining Theorems 6.1 and 6.2 we see that, if the size of $L(e)$ for each edge e is at least the maximum degree, there is an edge coloring which assigns a color in $L(e)$ to each edge e .

A surprising aspect of Galvin's proof of Theorem 6.1 is that it utilizes a result of Gale and Shapley [41] on the existence of stable matchings in bipartite graphs. (See also [7] for a beautiful exposition of Galvin's proof.)

The aim of this chapter is to extend this framework to the setting of supermodular coloring, introduced by Schrijver [96]. More precisely, we show a supermodular coloring version of Theorem 6.1 by using “kernels” on a pair of posets, which were introduced by Sands, Sauer and Woodrow [94] and shown by Fleiner [27] to be generalizations of stable matchings. We now introduce the supermodular coloring theorem of Schrijver [96] and then provide our results on list supermodular colorings.

Supermodular Coloring

Let S be a finite set and $g: \mathcal{F} \rightarrow \mathbf{R}$ be an intersecting-supermodular function defined on an intersecting family $\mathcal{F} \subseteq 2^S$. A function $\pi: S \rightarrow [k]$ is called a k -coloring. We say that π *dominates* the function g if $|\pi(X)| \geq g(X)$ holds for every $X \in \mathcal{F}$, where $\pi(X) := \{\pi(u) \mid u \in X\}$. For two intersecting-supermodular functions $g_1: \mathcal{F}_1 \rightarrow \mathbf{Z}$ and $g_2: \mathcal{F}_2 \rightarrow \mathbf{Z}$, a k -coloring π is called a *supermodular k -coloring* for (g_1, g_2) if π dominates both g_1 and g_2 . Schrijver characterized the existence of a supermodular k -coloring in the following theorem, which generalizes Theorem 6.2 as is explained in Remark 6.4.

Theorem 6.3 (Schrijver [96]). *Let $g_1: \mathcal{F}_1 \rightarrow \mathbf{Z}$ and $g_2: \mathcal{F}_2 \rightarrow \mathbf{Z}$ be intersecting-supermodular functions such that each g_i satisfies $|X| \geq g_i(X)$ for every $X \in \mathcal{F}_i$ for $i = 1, 2$. Then, for any $k \in \mathbf{Z}_{>0}$, there exists a supermodular k -coloring for (g_1, g_2) if and only if both $k \geq \max\{g_1(X) \mid X \in \mathcal{F}_1\}$ and $k \geq \max\{g_2(X) \mid X \in \mathcal{F}_2\}$ hold.*

Remark 6.4. We show that an edge coloring of a bipartite graph is a special case of a supermodular coloring. Let $G = (T_1, T_2; E)$ be a bipartite graph. For $i = 1, 2$, define a family $\mathcal{F}_i \subseteq 2^E$ by $\mathcal{F}_i := \{\delta_G(t) \mid t \in T_i\}$ and a function $g_i: \mathcal{F}_i \rightarrow \mathbf{Z}$ by $g_i(\delta_G(t)) := |\delta_G(t)|$ for each $t \in T_i$. Since no two members of \mathcal{F}_i are intersecting, g_i is obviously intersecting-supermodular. Also, we see that a coloring π on E dominates both g_1 and g_2 if and only if no two adjacent edges are assigned the same color. ■

Remark 6.5. Tardos [103] provided an alternative proof for Theorem 6.3 using the generalized matroid intersection theorem. Theorem 6.3 has been extended in [33] to a more general framework including skew-supermodular coloring. ■

List Supermodular Coloring (Our Contribution)

Let us consider the list coloring version of supermodular colorings. Let Σ be a finite set of colors and let each $u \in S$ have a color list $L(u) \subseteq \Sigma$, that is, L is a mapping from S to 2^Σ . For intersecting-supermodular functions $g_1: \mathcal{F}_1 \rightarrow \mathbf{Z}$, $g_2: \mathcal{F}_2 \rightarrow \mathbf{Z}$ and color lists $\{L(u)\}_{u \in S}$, a *list supermodular coloring* is a function $\varphi: S \rightarrow \Sigma$ such that $\varphi(u) \in L(u)$ for every $u \in S$ and φ dominates both g_1 and g_2 . The main result of this chapter is as follows.

Theorem 6.6. *For intersecting-supermodular functions $g_1: \mathcal{F}_1 \rightarrow \mathbf{Z}$ and $g_2: \mathcal{F}_2 \rightarrow \mathbf{Z}$ and $k \in \mathbf{Z}_{>0}$, assume that there exists a supermodular k -coloring for (g_1, g_2) . If L satisfies $|L(u)| = k$ for each $u \in S$, then there exists a list supermodular coloring $\varphi: S \rightarrow \Sigma$ for (g_1, g_2, L) .*

Since a supermodular coloring is a generalization of a bipartite edge coloring, as mentioned in Remark 6.4, Theorem 6.6 is a generalization of Theorem 6.1.

The pair (g_1, g_2) of intersecting-supermodular functions is called *k -choosable* if, for every $L: S \rightarrow 2^\Sigma$ with $|L(u)| = k$ ($\forall u \in S$), there exists a list supermodular coloring for (g_1, g_2, L) . Combining Theorems 6.3 and 6.6 implies the following corollary.

Corollary 6.7. *Let $g_1 : \mathcal{F}_1 \rightarrow \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \rightarrow \mathbf{Z}$ be intersecting-supermodular functions such that each g_i satisfies $|X| \geq g_i(X)$ for every $X \in \mathcal{F}_i$ for $i = 1, 2$. Then, for any $k \in \mathbf{Z}_{>0}$, the pair (g_1, g_2) is k -choosable if and only if both $k \geq \max\{g_1(X) \mid X \in \mathcal{F}_1\}$ and $k \geq \max\{g_2(X) \mid X \in \mathcal{F}_2\}$ hold.*

Remark 6.8. An interesting special case of Schrijver's theorem is Gupta's theorem [46] (described as Theorem 6.37 in Section 6.7), which is a generalization of König's theorem. Theorem 6.6 then naturally derives its list coloring version (Corollary 6.39). In Section 6.7, we also provide an alternative proof which does not rely on Theorem 6.6. ■

The rest of this chapter is organized as follows. In Section 6.2, we introduce the theorem of Sands, Sauer and Woodrow [94] on the existence of a kernel on a pair of posets. In Section 6.3, we introduce a skeleton poset, which is defined for a pair of a coloring and an intersecting-supermodular function. The existence proof of a skeleton poset is postponed to Section 6.5. In Section 6.4, we give a proof of Theorem 6.6 using induction on the size of the ground set. Each step of the induction uses a kernel of the pair of skeleton posets defined for the given intersecting-supermodular functions. Section 6.6 extends Theorem 6.6 to skew-supermodular functions, which solves the conjecture stated in [23]. Finally, in Section 6.7, we show the list coloring version of Gupta's theorem.

6.2 Kernels on a Pair of Posets

As mentioned in Introduction, a key ingredient of our proof of Theorem 6.6 is to use kernels on a pair of posets, which were shown by Fleiner [27] to be generalizations of stable matchings. (To be precise, they are generalizations of one-to-one stable matchings. This will be shown in Remark 6.10.) We now introduce the definition of kernels and their existence theorem due to Sands, Sauer and Woodrow [94] with the terminology of Fleiner and Jankó [28].

Let $P_1 = (S, \preceq_1)$ and $P_2 = (S, \preceq_2)$ be two posets on the same ground set S . A subset $K \subseteq S$ is called a *kernel*¹ if K is a common antichain and every element $u \in S \setminus K$ admits an element $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. Moreover, for any subset $H \subseteq S$, we call its subset $K \subseteq H$ a *kernel of H* if K is a common antichain and every element $u \in H \setminus K$ admits an element $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. Sands et al. provided the following existence theorem of kernels.

Theorem 6.9 (Sands et al. [94]). *Let $P_1 = (S, \preceq_1)$ and $P_2 = (S, \preceq_2)$ be posets on the same ground set S . For any subset $H \subseteq S$, there exists a kernel of H .*

The original statement of Sands et al. was described in terms of directed graphs whose edges are colored with two colors, and the binary relation $v \prec u$ in Theorem 6.9

¹ This notion of kernel is not to be confused with the one of *matroid kernel* in Section 3.3.3. Both of them generalizes one-to-one stable matchings, but neither of them includes the other.

corresponds to the existence of a monochromatic path from a node u to another node v . In fact, their original statement can be applied to more general binary relations. We can, however, deduce it from the statement of Theorem 6.9.

Remark 6.10. We show that a one-to-one stable matching can be represented as a special case of a kernel defined above. Let $G = (T_1, T_2; E)$ be a bipartite graph and, for each $t \in T_1 \cup T_2$, let \prec_t be a total order on $\delta_G(t)$.² We say that a subset $M \subseteq E$ is a (one-to-one) stable matching if M is a matching and every $e = (t_1, t_2) \in E \setminus M$ admits $e' \in M$ such that $[e' \in \delta_G(t_1), e' \prec_{t_1} e]$ or $[e' \in \delta_G(t_2), e' \prec_{t_2} e]$.

Define a poset (E, \preceq_1) so that we have $e' \preceq_1 e$ if and only if there is $t_1 \in T_1$ such that $[e, e' \in \delta_G(t_1), e' \prec_{t_1} e]$. Similarly, define (E, \preceq_2) so that $e' \preceq_2 e$ means that there is $t_2 \in T_2$ such that $[e, e' \in \delta_G(t_2), e' \prec_{t_2} e]$. Then, we see that a subset $M \subseteq E$ is a matching in G if and only if M is a common antichain of (E, \preceq_1) and (E, \preceq_2) . In particular, M is a stable matching if and only if M is a kernel for this pair of posets. ■

6.3 Skeleton Posets

In this section, we introduce a new notion “skeleton posets,” to which we apply Theorem 6.9 in the proof of our main theorem.

Let $g : \mathcal{F} \rightarrow \mathbf{Z}$ be an intersecting-supermodular function on an intersecting family $\mathcal{F} \subseteq 2^S$. For a subset $K \subseteq S$, the *reduction* of g by K is the function $g_K : \mathcal{F}_K \rightarrow \mathbf{Z}$ defined by $\mathcal{F}_K = \{Z \setminus K \mid Z \in \mathcal{F}\}$ and

$$g_K(X) = \max \{ \hat{g}_K(Z) \mid Z \in \mathcal{F}, Z \setminus K = X \} \quad (X \in \mathcal{F}_K),$$

where $\hat{g}_K : \mathcal{F} \rightarrow \mathbf{Z}$ is defined by

$$\hat{g}_K(Z) = \begin{cases} g(Z) - 1 & (Z \in \mathcal{F}, Z \cap K \neq \emptyset), \\ g(Z) & (Z \in \mathcal{F}, Z \cap K = \emptyset). \end{cases}$$

Claim 6.11. *The reduction $g_K : \mathcal{F}_K \rightarrow \mathbf{Z}$ is an intersecting-supermodular function.*

Proof. For every $X, Y \in \mathcal{F}_K$ and $Z_X, Z_Y \in \mathcal{F}$ such that $Z_X \setminus K = X$ and $Z_Y \setminus K = Y$, we have $X \cup Y = (Z_X \setminus K) \cup (Z_Y \setminus K) = (Z_X \cup Z_Y) \setminus K$, and the same holds for the intersection. These imply that \mathcal{F}_K is an intersecting family. We now show the supermodular inequality of g_K for intersecting $X, Y \in \mathcal{F}_K$. Take $Z_X, Z_Y \in \mathcal{F}$ which attain $g_K(X) = \hat{g}_K(Z_X)$ and $g_K(Y) = \hat{g}_K(Z_Y)$. As easily confirmed, $\hat{g}_K : \mathcal{F} \rightarrow \mathbf{Z}$ is intersecting supermodular, and hence $\hat{g}_K(Z_X) + \hat{g}_K(Z_Y) \leq \hat{g}_K(Z_X \cup Z_Y) + \hat{g}_K(Z_X \cap Z_Y)$. Also, we

²In contrast to other sections, where elements larger in a total order are regarded as better elements, here $e' \prec_v e$ means that e' is better than e for v .

have $\hat{g}_K(Z_X \cup Z_Y) \leq g_K(X \cup Y)$ because $Z_X \cup Z_Y \in \mathcal{F}$ and $(Z_X \cup Z_Y) \setminus K = X \cup Y$. Similarly, $\hat{g}_K(Z_X \cap Z_Y) \leq g_K(X \cap Y)$ holds. Combining these inequalities, we obtain $g_K(X) + g_K(Y) \leq g_K(X \cup Y) + g_K(X \cap Y)$. \square

Let $\pi : S \rightarrow [k]$ be a k -coloring. We say that a poset $P = (S, \preceq)$ is *consistent* with π if $u \prec v$ implies $\pi(u) < \pi(v)$ for every $u, v \in S$. For a consistent poset P and a subset $K \subseteq S$, the *reduction* of π by K in P is the k -coloring $\pi_K : S \setminus K \rightarrow [k]$ defined by

$$\pi_K(u) = \begin{cases} \pi(u) - 1 & (\exists v \in K : v \prec u), \\ \pi(u) & (\text{otherwise}). \end{cases}$$

Note that every $u \in S \setminus K$ indeed satisfies $\pi_K(u) \geq 1$ because of the consistency of P .

Definition 6.12. A *skeleton poset* of (π, g) is a poset $P = (S, \preceq)$ which is consistent with π and satisfies the following condition: For every antichain K in P , the reduction of π by K in P dominates the reduction of g by K . \blacksquare

Here, we provide a sufficient condition for the existence of a skeleton poset. We call π a *dominating k -coloring* if it dominates g . A dominating k -coloring π is called *minimal* if there is no dominating k -coloring $\tilde{\pi} : S \rightarrow [k]$ such that $\tilde{\pi}(u) \leq \pi(u)$ for every $u \in S$ and $\tilde{\pi}(v) < \pi(v)$ for some $v \in S$.

Proposition 6.13. *For every intersecting-supermodular function $g : \mathcal{F} \rightarrow \mathbf{Z}$ and every minimal dominating k -coloring $\pi : S \rightarrow [k]$ for g , there exists a skeleton poset P of (π, g) .*

Proposition 6.13 is essential to connect Theorem 6.9 on kernels on posets to a construction of a list supermodular coloring. The proof of Proposition 6.13 is postponed to Section 6.5. Instead, we demonstrate some examples of skeleton posets below.

Example 6.14. Let \mathcal{F} be a partition of S and π be a minimal dominating k -coloring for a function $g : \mathcal{F} \rightarrow \mathbf{Z}$. By the minimality, in each $X \in \mathcal{F}$, exactly one element $u \in X$ satisfies $\pi(u) = j$ for each $j \in \{2, 3, \dots, g(X)\}$ and other elements u satisfy $\pi(u) = 1$.

Define a binary relation \prec on S so that $u, v \in S$ satisfy $u \prec v$ if and only if $u, v \in X$ for some $X \in \mathcal{F}$ and $\pi(u) < \pi(v)$. Let $u \preceq v$ mean $u \prec v$ or $u = v$. Then, $P = (S, \preceq)$ is a skeleton poset of (π, g) . \blacksquare

Example 6.15. Let $S = \{u_1, u_2, u_3, u_4, u_5\}$ and $\mathcal{F} = \{F_1, F_2, S\}$, where $F_1 = \{u_1, u_2, u_3\}$, $F_2 = \{u_4, u_5\}$. Define $g : \mathcal{F} \rightarrow \mathbf{Z}$ by $g(F_1) = 3$, $g(F_2) = 2$, $g(S) = 4$ and $\pi : S \rightarrow [k]$ by $(\pi(u_1), \pi(u_2), \dots, \pi(u_5)) = (1, 2, 4, 1, 3)$, where $k \geq 4$ (see Figure 6.1 (a)). Then, π is a minimal dominating k -coloring for g . Let $P = (S, \preceq)$ be a poset whose Hasse diagram is depicted in Figure 6.1 (b). We can check that P is a skeleton poset of (π, g) . \blacksquare

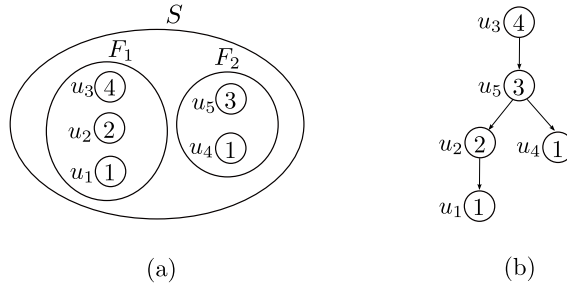


Figure 6.1: (a) The family \mathcal{F} in Example 6.15. The value $\pi(u)$ of each $u \in S$ is written in the circle corresponding to u . (b) The Hasse diagram which defines a skeleton poset of (π, g) in Example 6.15.

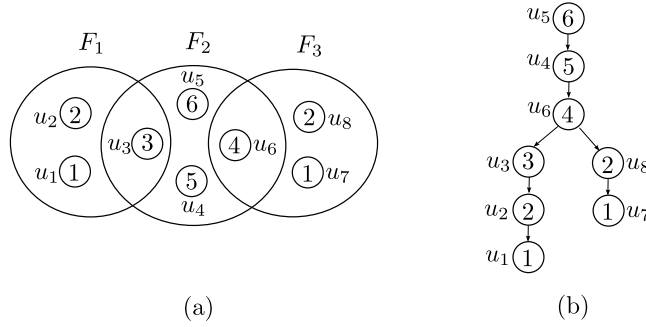


Figure 6.2: (a) The family \mathcal{F} in Example 6.16. The value $\pi(u)$ of each $u \in S$ is written in the circle corresponding to u . (b) The Hasse diagram which defines a skeleton poset of (π, g) in Example 6.16.

Example 6.16. Let $S = \{u_1, u_2, \dots, u_8\}$. Let $F_1 = \{u_1, u_2, u_3\}$, $F_2 = \{u_3, u_4, u_5, u_6\}$, $F_3 = \{u_6, u_7, u_8\}$ and $\mathcal{F} = \{F_1, F_2, F_3, F_1 \cap F_2, F_2 \cap F_3, F_1 \cup F_2, F_2 \cup F_3, S\}$. Define $g : \mathcal{F} \rightarrow \mathbf{Z}$ by $g(F_1) = 3$, $g(F_2) = 2$, $g(F_3) = 3$, $g(F_1 \cap F_2) = g(F_2 \cap F_3) = 1$, $g(F_1 \cup F_2) = g(F_2 \cup F_3) = 4$, and $g(S) = 6$. Define $\pi : S \rightarrow [k]$ by $(\pi(u_1), \pi(u_2), \dots, \pi(u_8)) = (1, 2, 3, 5, 6, 4, 1, 2)$, where $k \geq 6$ (see Figure 6.2 (a)). Then, π is a minimal dominating k -coloring for g . Let $P = (S, \preceq)$ be a poset whose Hasse diagram is depicted in Figure 6.2 (b). We see that P is a skeleton poset of (π, g) . ■

6.4 Proof of the Main Theorem

In this section, we give a proof to Theorem 6.6 applying Theorem 6.9 to skeleton posets, whose existence is shown in Proposition 6.13. Let $g_1 : \mathcal{F}_1 \rightarrow \mathbf{Z}$ and $g_2 : \mathcal{F}_2 \rightarrow \mathbf{Z}$ be intersecting-supermodular functions on $\mathcal{F}_1, \mathcal{F}_2 \subseteq 2^S$ and let $k \in \mathbf{Z}_{>0}$.

Lemma 6.17. *If $\pi_1, \pi_2 : S \rightarrow [k]$ dominate g_1 and g_2 , respectively, then for any nonempty*

subset $H \subseteq S$, there exist nonempty $K \subseteq H$ and k -colorings $\pi'_1, \pi'_2 : S \setminus K \rightarrow [k]$ that satisfy the following conditions.

- (a) For every $u \in S \setminus K$, we have $\pi'_1(u) + \pi'_2(u) \leq \pi_1(u) + \pi_2(u)$.
Moreover, $u \in H \setminus K$ implies $\pi'_1(u) + \pi'_2(u) < \pi_1(u) + \pi_2(u)$.
- (b) For each $i \in \{1, 2\}$, π'_i dominates the reduction of g_i by K .

Proof. For each $i \in \{1, 2\}$, since π_i dominates g_i , there is a minimal dominating k -coloring $\hat{\pi}_i : S \rightarrow [k]$ with $\hat{\pi}_i \leq \pi_i$. By Proposition 6.13, there is a skeleton poset $P_i = (S, \preceq_i)$ of $(\hat{\pi}_i, g_i)$ for each i . Take any nonempty $H \subseteq S$ and apply Theorem 6.9 to P_1, P_2 , and H . Then, we obtain a kernel K of H . That is, $K \subseteq H$ is a common antichain and every $u \in H \setminus K$ admits some $v \in K$ such that $v \prec_1 u$ or $v \prec_2 u$. Let $\pi'_i : S \setminus K \rightarrow [k]$ be a reduction of $\hat{\pi}_i$ by K in P_i . Then $\pi'_1(u) + \pi'_2(u) \leq \hat{\pi}_1(u) + \hat{\pi}_2(u)$ for every $u \in S \setminus K$ and the inequality holds for every $u \in H \setminus K$ by the definition of a kernel. As $\hat{\pi}_1(u) \leq \pi_1(u)$, $\hat{\pi}_2(u) \leq \pi_2(u)$ for every $u \in S$, condition (a) follows. Since P_i is a skeleton poset, π'_i dominates the reduction of g_i by K . \square

Recall that Σ is a set of colors and $L : S \rightarrow 2^\Sigma$ is an assignment of color lists to elements.

Proposition 6.18. For $L : S \rightarrow 2^\Sigma$, assume that there exist k -colorings $\pi_1, \pi_2 : S \rightarrow [k]$ satisfying the following conditions.

- (i) For every $u \in S$, we have $\pi_1(u) + \pi_2(u) - 1 \leq |L(u)|$.
- (ii) For each $i \in \{1, 2\}$, π_i dominates g_i .

Then there exists a list supermodular coloring $\varphi : S \rightarrow \Sigma$ for (g_1, g_2, L) .

Proof. We show this by induction on $|S|$, i.e., the size of the ground set. If $|S| = 1$, the statement is obvious.

If $|S| > 1$, take some $l \in \bigcup \{L(u) \mid u \in S\}$ and let $H := \{u \in S \mid l \in L(u)\}$. By Lemma 6.17, there exist nonempty $K \subseteq H$ and $\pi'_1, \pi'_2 : S \setminus K \rightarrow [k]$ satisfying (a) and (b). For each $i \in \{1, 2\}$, let g'_i denote the reduction of g_i by K . Then, g'_i is intersecting supermodular by Claim 6.11, and π'_i dominates g'_i by (b). Define $L' : S \setminus K \rightarrow 2^\Sigma$ by $L'(u) = L(u) \setminus \{l\}$ for each $u \in S \setminus K$. It then follows from (a) that $\pi'_1(u) + \pi'_2(u) - 1 \leq |L'(u)|$ for every $u \in S \setminus K$. Thus, π'_1, π'_2 satisfy (i) and (ii) with $(S \setminus K, g'_1, g'_2, L')$ in place of (S, g_1, g_2, L) . By the inductive assumption, there exists a list supermodular coloring $\varphi' : S \setminus K \rightarrow \Sigma$ for (g'_1, g'_2, L') . Define $\varphi : S \rightarrow \Sigma$ by

$$\varphi(u) = \begin{cases} \varphi'(u) & (u \in S \setminus K), \\ l & (u \in K). \end{cases}$$

Then, clearly $\varphi(u) \in L(u)$ for every $u \in S$. We see that every $X \in \mathcal{F}$ with $X \cap K \neq \emptyset$ satisfies $|\varphi(X)| = |\varphi'(X \setminus K)| + 1 \geq g'_i(X \setminus K) + 1 \geq g_i(X)$, and every $X \in \mathcal{F}$ with $X \cap K = \emptyset$ satisfies $|\varphi(X)| = |\varphi'(X \setminus K)| \geq g'_i(X \setminus K) \geq g_i(X)$. Thus φ is a list supermodular coloring for (g'_1, g'_2, L) . \square

Proof of Theorem 6.6. Recall that L satisfies $|L(u)| = k$ for every $u \in S$. Also, we are provided a supermodular k -coloring $\pi : S \rightarrow [k]$ which dominates both g_1 and g_2 . Let $\pi_1(u) := \pi(u)$ and $\pi_2(u) := k + 1 - \pi(u)$ for every $u \in S$. They satisfy the condition (i) of Proposition 6.18 as $\pi_1(u) + \pi_2(u) - 1 = k = |L(u)|$ for every u . Also (ii) holds as $|\pi(X)| = |\pi_1(X)| = |\pi_2(X)|$ for every $X \subseteq S$. Proposition 6.18 then implies the statement of Theorem 6.6. \square

6.5 Existence of Skeleton Posets

Let $g : \mathcal{F} \rightarrow \mathbf{Z}$ be an intersecting-supermodular function and $\pi : S \rightarrow [k]$ be a minimal dominating k -coloring for g . In this section, we prove Proposition 6.13 by constructing a skeleton poset $P = (S, \preceq)$ of (π, g) . We first define the poset and then show that it is indeed a skeleton poset of (π, g) .

6.5.1 Poset Construction

We call a subset $X \in \mathcal{F}$ *tight* if $|\pi(X)| = g(X)$ holds. Note that the function $|\pi(\cdot)| : 2^S \rightarrow \mathbf{Z}$ is *submodular*, that is, $|\pi(X)| + |\pi(Y)| \geq |\pi(X \cup Y)| + |\pi(X \cap Y)|$ for any $X, Y \subseteq 2^S$. This implies the following fact.

Claim 6.19. *If $X, Y \in \mathcal{F}$ are tight and intersecting, then $X \cup Y, X \cap Y$ are also tight.*

Proof. Since $|\pi(\cdot)|$ is submodular and π dominates g , we have

$$g(X) + g(Y) = |\pi(X)| + |\pi(Y)| \geq |\pi(X \cup Y)| + |\pi(X \cap Y)| \geq g(X \cup Y) + g(X \cap Y).$$

As g is intersecting-supermodular, $g(X \cup Y) + g(X \cap Y) \geq g(X) + g(Y)$ also holds. Then, the inequalities above are in fact equalities, and hence we obtain $|\pi(X \cup Y)| = g(X \cup Y)$ and $|\pi(X \cap Y)| = g(X \cap Y)$. \square

Claim 6.20. *If $X, Y \in \mathcal{F}$ are tight and intersecting, then $\pi(X) \cap \pi(Y) = \pi(X \cap Y)$.*

Proof. Clearly, $\pi(X \cap Y) \subseteq \pi(X) \cap \pi(Y)$. We then show $|\pi(X \cap Y)| = |\pi(X) \cap \pi(Y)|$ to complete the proof. As shown in the proof of Claim 6.19, $|\pi(X)| + |\pi(Y)| = |\pi(X \cup Y)| + |\pi(X \cap Y)|$. Also, we see $\pi(X) \cup \pi(Y) = \pi(X \cup Y)$. These imply $|\pi(X \cap Y)| = |\pi(X)| + |\pi(Y)| - |\pi(X \cup Y)| = |\pi(X)| + |\pi(Y)| - |\pi(X) \cup \pi(Y)| = |\pi(X) \cap \pi(Y)|$. \square

Claim 6.21. *For any $u \in S$ with $\pi(u) > 1$ and $j \in \{1, \dots, \pi(u) - 1\}$, there exists $F_j \in \mathcal{F}$ which satisfies $u \in F_j$, $|\pi(F_j)| = g(F_j)$, $\pi(F_j - u) \not\supseteq \pi(u)$, and $\pi(F_j) \ni j$.*

Proof. Let $\pi' : S \rightarrow [k]$ be a k -coloring such that $\pi'(v) = \pi(v)$ for every $v \in S \setminus \{u\}$ and $\pi'(u) = j$. Since π is a minimal dominating k -coloring, π' does not dominate g . Hence there exists F_j such that $|\pi'(F_j)| < g(F_j)$. As $|\pi(F_j)| \geq g(F_j)$ holds, we have $|\pi'(F_j)| < |\pi(F_j)|$, which implies the four conditions in the statement. \square

Claim 6.22. *For any $u \in S$ with $\pi(u) > 1$, there exist one or more $F \in \mathcal{F}$ such that*

$$u \in F, \tag{6.1}$$

$$|\pi(F)| = g(F), \tag{6.2}$$

$$\pi(F - u) \not\geq \pi(u), \tag{6.3}$$

$$\pi(F) \supseteq \{1, 2, \dots, \pi(u)\}. \tag{6.4}$$

Furthermore, among all such $F \in \mathcal{F}$, there exists a unique minimal one.

Proof. For each $j \in \{1, 2, \dots, \pi(u) - 1\}$, let $F_j \in \mathcal{F}$ be a subset which satisfies four conditions in Claim 6.21. Then $F := \bigcup \{F_j \mid j = 1, 2, \dots, \pi(u) - 1\}$ satisfies (6.1)–(6.4). Condition (6.2) follows from Claim 6.19 since all F_j contain u . Other three are clear by definition. To show the existence of the minimum, we show that, if both F and F' satisfy (6.1)–(6.4), then so does $F \cap F'$. By definition, (6.1) and (6.3) are clear. Claims 6.19 and 6.20 imply (6.2) and (6.4), respectively. \square

For any $u \in S$ with $\pi(u) > 1$, denote by $D(u)$ the unique minimal $F \in \mathcal{F}$ satisfying (6.1)–(6.4). For $u \in S$ with $\pi(u) = 1$, let $D(u)$ be $\{u\}$. Define \prec by

$$u \prec v \iff [D(u) \subseteq D(v), \pi(u) < \pi(v)]$$

and let $u \preceq v$ mean $u \prec v$ or $u = v$. Then, we see that \preceq is a partial order. Let $P = (S, \preceq)$.

Claim 6.23. *If $D(u) \cap D(v) \neq \emptyset$, then u and v are comparable.*

Proof. Let $u \neq v$ since otherwise the claim is obvious. We assume $\pi(u) \leq \pi(v)$ without loss of generality. In the case $\pi(u) = 1$, we have $D(u) = \{u\}$. Then $D(u) \cap D(v) \neq \emptyset$ implies $D(u) \subseteq D(v)$ and $\pi(v) > 1$, and hence $u \prec v$. Next, consider the case $\pi(u) > 1$. By Claims 6.19 and 6.20, $D(u) \cap D(v)$ is tight and satisfies $\pi(D(u) \cap D(v)) \supseteq \{1, 2, \dots, \pi(u)\}$. The latter implies $D(u) \cap D(v) \ni u$ since $D(u)$ satisfies $\pi(D(u) - u) \not\geq \pi(u)$. Thus, conditions (6.1)–(6.4) for u hold with $F = D(u) \cap D(v)$. By the minimality of $D(u)$, this implies $D(u) \cap D(v) = D(u)$, and hence $D(u) \subseteq D(v)$. Also, as $D(v)$ satisfies $\pi(D(v) - v) \not\geq \pi(v)$, the condition $u \in D(u) \subseteq D(v)$ implies $\pi(u) \neq \pi(v)$, and hence $\pi(u) < \pi(v)$. Thus, $u \prec v$ holds. \square

By Claim 6.23, $D(u) \cap D(v) \neq \emptyset$ implies $D(u) \subseteq D(v)$ or $D(u) \supseteq D(v)$, i.e., the family $\{D(u) \mid u \in S\}$ forms a *laminar family*.

Claim 6.24. *For any $u \in S$ with $\pi(u) > 1$, there exists $v \in S$ with $\pi(v) = \pi(u) - 1$ and $v \prec u$.*

Proof. Since (6.4) holds with $F = D(u)$, there is $v \in D(u)$ with $\pi(v) = \pi(u) - 1$. As $v \in D(v) \cap D(u) \neq \emptyset$ and $\pi(v) < \pi(u)$, Claim 6.23 implies $v \prec u$. \square

Claim 6.25. *If $v \preceq u$, then $v \in D(u)$. Conversely, if $v \in D(u)$, then $v \preceq u$ or $u \prec v$.*

Proof. The condition $v \preceq u$ implies $v \in D(v) \subseteq D(u)$, and the first claim holds. Also, $v \in D(u)$ implies $v \in D(v) \cap D(u) \neq \emptyset$, and hence v is comparable with u by Claim 6.23. \square

Claim 6.26. *If $u \preceq v$ and $u \preceq w$, then v and w are comparable.*

Proof. Since $D(u) \subseteq D(v) \cap D(w) \neq \emptyset$, Claim 6.23 implies the statement. \square

Claim 6.26 implies that the Hasse diagram of $P = (S, \preceq)$ forms a *branching*, i.e., a collection of rooted directed trees.

Claim 6.27. *For each $u \in S$ with $\pi(u) > 1$, let $C(u)$ be a maximal chain included in $D(u)$. Then, the following statements hold:*

- $\pi(C(u)) \supseteq \{1, 2, \dots, \pi(u)\}$,
- $\pi(D(u) \setminus C(u)) \subseteq \{1, 2, \dots, \pi(u) - 1\}$,
- $\pi(C(u)) = \pi(D(u))$ and $g(D(u)) = |C(u)|$.

Proof. The first statement follows from Claims 6.24 and 6.25. The second one follows from Claims 6.25, 6.26, and the maximality of $C(u)$. From these two, we have $\pi(C(u)) = \pi(D(u))$. As $C(u)$ is a chain, we have $|C(u)| = |\pi(C(u))| = |\pi(D(u))|$. Since $D(u)$ is tight, this equals $g(D(u))$. \square

The following fact will be useful later.

Claim 6.28. *Assume that $u, v \in S$ satisfies $\pi(v) = \pi(u) - 1$. If $X \in \mathcal{F}$ is tight and $\{u, v\} \subseteq X$ holds, then we have $D(u) \setminus D(v) \subseteq X$.*

Proof. Note that $D(v)$ is a singleton or a member of \mathcal{F} . Also, we have $v \in D(v) \cap X \neq \emptyset$. Then $D(v) \cup X$ is a member of \mathcal{F} and tight. As $\pi(u) > 1$, the set $D(u)$ is also in \mathcal{F} and tight. Then, the nonempty set $F := (D(v) \cup X) \cap D(u) \ni u$ is also a member of \mathcal{F} and tight. Note that then conditions (6.1)–(6.4) hold, where (6.4) follows from Claim 6.20 and the condition $\pi(v) = \pi(u) - 1$. Therefore, the minimality of $D(u)$ implies $D(u) \subseteq F$, which yields $D(u) \setminus D(v) \subseteq X$. \square

6.5.2 Reduction by an Antichain

We now show that $P = (S, \preceq)$ is indeed a skeleton poset of (π, g) . Clearly P is consistent with π , i.e., $u \prec v$ implies $\pi(u) < \pi(v)$. We now show that, for any antichain K in P , the reduction of π by K dominates the reduction g_K of g by K .

Take an antichain $K \subseteq S$. Let $\pi_K : S \setminus K \rightarrow [k]$ be the reduction of π by K in P , i.e.,

$$\pi_K(u) = \begin{cases} \pi(u) - 1 & (\exists v \in K : v \prec u), \\ \pi(u) & (\text{otherwise}). \end{cases}$$

To prove that π_K dominates g_K , it suffices to show that $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$ holds for every $X \in \mathcal{F}$, where $\hat{g}_K : \mathcal{F} \rightarrow \mathbf{Z}$ is defined by $\hat{g}_K(X) = g(X) - 1$ for $X \in \mathcal{F}$ with $X \cap K \neq \emptyset$ and $\hat{g}_K(X) = g(X)$ for $X \in \mathcal{F}$ with $X \cap K = \emptyset$.

The definition of π_K implies the following observation.

Claim 6.29. *For any chain $C \subseteq S$, exactly one of the following holds.*

1. $|C \cap K| = 1$ and $\pi_K(u) \neq \pi_K(v)$ for every distinct $u, v \in C \setminus K$. In this case, we have $|\pi_K(C \setminus K)| = |C| - 1$.
2. $|C \cap K| = 0$ and $\pi_K(u) \neq \pi_K(v)$ for every distinct $u, v \in C \setminus K$. In this case, we have $|\pi_K(C \setminus K)| = |C|$.
3. $|C \cap K| = 0$ and just one pair of $u, v \in C$ satisfies $\pi_K(u) = \pi_K(v)$. In this case, we have $|\pi_K(C \setminus K)| = |C| - 1$. If $\pi(v) \leq \pi(u)$ for such $u, v \in C$, then $\pi(v) = \pi(u) - 1$ and $(D(u) \setminus D(v)) \cap K \neq \emptyset$.

Proof. The only point to concern is the last statement in the third case. Since C is a chain, the condition $\pi_K(u) = \pi_K(v)$ and the definition of π_K imply $\pi(v) = \pi_K(v) = \pi_K(u) = \pi(u) - 1$. By $\pi_K(u) = \pi(u) - 1$, there exists $w \in K$ with $w \prec u$, which implies $w \in D(u)$ by Claim 6.25. We now prove $w \notin D(v)$ which completes the proof. Note that $w \prec u$ implies $\pi(w) < \pi(u)$, and hence $\pi(w) \leq \pi(u) - 1 = \pi(v)$. Suppose, to the contrary, $w \in D(v)$. If $\pi(w) = \pi(v)$, then $\pi(D(v) - v) \not\geq \pi(v)$ implies $w = v$, which contradicts $w \in K, v \in C$, and $C \cap K = \emptyset$. If $\pi(w) < \pi(v)$, then $w \prec v$ holds, which contradicts $\pi_K(v) = \pi(v)$ by the definition of π_K . Thus, we obtain $w \notin D(v)$. \square

Lemma 6.30. *For $X \in \mathcal{F}$, if there is a chain $C \subseteq X$ with $\pi(C) = \pi(X)$, then we have $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$.*

Proof. Since C is a chain and $\pi(C) = \pi(X)$, we have $|C| = |\pi(C)| = |\pi(X)| \geq g(X)$. Let us consider the three cases described in Claim 6.29.

In the first case, we have $C \cap K \neq \emptyset$ and $|\pi_K(C \setminus K)| = |C| - 1$. Since $X \cap K \supseteq C \cap K \neq \emptyset$ implies $\hat{g}_K(X) = g(X) - 1$, we have $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) - 1 = \hat{g}_K(X)$.

In the second case, we have $C \cap K = \emptyset$ and $|\pi_K(C \setminus K)| = |C|$, which together with $g(X) \geq \hat{g}_K(X)$ imply $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| \geq g(X) \geq \hat{g}_K(X)$.

In the third case, we have $|\pi_K(C \setminus K)| = |C| - 1$, $\pi(v) = \pi(u) - 1$, and $(D(u) \setminus D(v)) \cap K \neq \emptyset$. If X is not tight, then $|C| = |\pi(X)| > g(X)$, and hence $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) \geq \hat{g}_K(X)$. If X is tight, then Claim 6.28 and $\pi(v) = \pi(u) - 1$ imply $D(u) \setminus D(v) \subseteq X$. Combined with $(D(u) \setminus D(v)) \cap K \neq \emptyset$, this implies $X \cap K \neq \emptyset$, and hence $\hat{g}_K(X) = g(X) - 1$. Thus, we obtain $|\pi_K(X \setminus K)| \geq |\pi_K(C \setminus K)| = |C| - 1 \geq g(X) - 1 = \hat{g}_K(X)$. \square

Lemma 6.31. *For $u \in S$ with $\pi(u) > 1$, the set $D(u) \in \mathcal{F}$ satisfies $|\pi_K(D(u) \setminus K)| = \hat{g}_K(D(u))$.*

Proof. By Claim 6.27, a maximal chain $C(u)$ in $D(u)$ satisfies $\pi(C(u)) = \pi(D(u))$ and $|C(u)| = g(D(u))$. Then Lemma 6.30 implies $|\pi_K(D(u) \setminus K)| \geq \hat{g}_K(D(u))$. We then intend to show $|\pi_K(D(u) \setminus K)| \leq \hat{g}_K(D(u))$.

Claim 6.27 says $\pi(D(u) \setminus C(u)) \subseteq \{1, 2, \dots, \pi(u) - 1\}$ and $\pi(C(u)) \supseteq \{1, 2, \dots, \pi(u)\}$, which imply $\pi_K((D(u) \setminus C(u)) \setminus K) \subseteq \{1, 2, \dots, \pi(u) - 1\} \subseteq \pi_K(C(u) \setminus K)$ by the definition of π_K . Hence, we have $\pi_K(D(u) \setminus K) = \pi_K(C(u) \setminus K)$, which yields $|\pi_K(D(u) \setminus K)| = |\pi_K(C(u) \setminus K)| \leq |C(u)| = g(D(u))$. In particular, if $D(u) \cap K = \emptyset$, then $|\pi_K(D(u) \setminus K)| \leq g(D(u)) = \hat{g}_K(D(u))$.

We now consider the case of $D(u) \cap K \neq \emptyset$. As we have $|\pi_K(D(u) \setminus K)| = |\pi_K(C(u) \setminus K)|$ and $g(D(u)) = |C(u)|$, it suffices to show $|\pi_K(C(u) \setminus K)| < |C(u)|$. If $C(u) \cap K \neq \emptyset$, this is clear. Assume $C(u) \cap K = \emptyset$, and then $D(u) \cap K \neq \emptyset$ implies $(D(u) \setminus C(u)) \cap K \neq \emptyset$. As we have $D(u) \setminus C(u) \subseteq \{v \in S \mid v \prec u\}$ by Claims 6.25 and 6.27, we obtain $\{v \in S \mid v \prec u\} \cap K \neq \emptyset$, and hence $\pi_K(u) = \pi(u) - 1$. Since $v \preceq u$ implies $\pi_K(v) \leq \pi_K(u)$ for any v , the subset $C' := \{v \in C(u) \mid v \preceq u\}$ satisfies $\pi_K(C') \subseteq \{1, 2, \dots, \pi(u) - 1\}$. This implies $|\pi_K(C')| < \pi(u) = |C'|$, where the last equality follows from Claim 6.27. Therefore, some pair of distinct $v, w \in C' \subseteq C(u)$ satisfies $\pi_K(v) = \pi_K(w)$, and hence $|\pi_K(C(u) \setminus K)| = |\pi_K(C(u))| < |C(u)|$, which is the desired conclusion. \square

Proposition 6.32. *Every $X \in \mathcal{F}$ satisfies $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$.*

Proof. The proof is by induction w.r.t. set inclusion.

First, consider the case in which $X \in \mathcal{F}$ is minimal, i.e., there is no $Y \in \mathcal{F}$ with $Y \subsetneq X$. Then, every $u \in X$ with $\pi(u) > 1$ satisfies $X \subseteq D(u)$ since otherwise we have $u \in X \cap D(u) \subsetneq X$ and $X \cap D(u) \in \mathcal{F}$, which contradict the minimality of X . Also $u \in X$ with $\pi(u) = 1$ satisfies $D(u) = \{u\} \subseteq X$. Then, every pair of $u, v \in X$ with $\pi(u) < \pi(v)$ satisfies $X \subseteq D(u) \cap D(v) \neq \emptyset$ or $\{u\} \subseteq X \subseteq D(v)$. In either case, we have $u \prec v$ by Claim 6.23. That is, every pair of elements is comparable if their values of π are different. Hence, there is a chain $C \subseteq X$ such that $\pi(C) = \pi(X)$, which implies $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$ by Lemma 6.30.

We now intend to show $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$, assuming inductively that $|\pi_K(Y \setminus K)| \geq \hat{g}_K(Y)$ holds for every $Y \in \mathcal{F}$ with $Y \subsetneq X$.

We start with the case in which every $u \in X$ satisfies $X \subseteq D(u)$. For $u, v \in X$ with $\pi(u) < \pi(v)$, we have $X \subseteq D(u) \cap D(v) \neq \emptyset$, which implies $u \prec v$ by Claim 6.23. Then, there is a chain $C \subseteq X$ such that $\pi(C) = \pi(X)$, and hence $|\pi_K(X \setminus K)| \geq \hat{g}_K(X)$ by Lemma 6.30.

We now consider the case in which some $u \in X$ satisfies $X \not\subseteq D(u)$. Among all such elements, let $u \in X$ maximize $\pi(u)$. Then, every $v \in X$ with $\pi(v) > \pi(u)$ satisfies $u \in X \subseteq D(v)$, and hence $v \succ u$ by Claim 6.25. Recall that every $v \in D(u)$ with $\pi(v) > \pi(u)$ also satisfies $v \succ u$. Then, $C := \{v \in X \cup D(u) \mid \pi(v) > \pi(u)\} \cup \{u\}$ forms a chain whose minimum is u . Let \hat{C} be a maximal chain subject to $C \subseteq \hat{C} \subseteq X \cup D(u)$. The maximality and Claim 6.27 imply $\pi(\hat{C}) \supseteq \{1, 2, \dots, \pi(u)\}$. Therefore, we have $\pi(\hat{C}) \supseteq \pi(C) \cup \{1, 2, \dots, \pi(u)\} \supseteq \pi(X \cup D(u))$. Since $\hat{C} \subseteq X \cup D(u)$, this means $\pi(\hat{C}) = \pi(X \cup D(u))$. Lemma 6.30 then implies $|\pi_K((X \cup D(u)) \setminus K)| \geq \hat{g}_K(X \cup D(u))$. We also have $|\pi_K(D(u) \setminus K)| = \hat{g}_K(D(u))$ by Lemma 6.31 and $|\pi_K((X \cap D(u)) \setminus K)| \geq \hat{g}_K(X \cap D(u))$ by the inductive assumption. Since $|\pi_K(\cdot \setminus K)| : 2^S \rightarrow \mathbf{Z}$ is submodular and \hat{g}_K is intersecting supermodular, we obtain

$$\begin{aligned} |\pi_K(X \setminus K)| &\geq |\pi_K((X \cup D(u)) \setminus K)| + |\pi_K((X \cap D(u)) \setminus K)| - |\pi_K(D(u) \setminus K)| \\ &\geq \hat{g}_K(X \cup D(u)) + \hat{g}_K(X \cap D(u)) - \hat{g}_K(D(u)) \\ &\geq \hat{g}_K(X), \end{aligned}$$

which completes the proof. \square

6.6 Extension to Skew-supermodular Coloring

This section extends Theorem 6.6 to the setting of skew-supermodular coloring. A function $g : 2^S \rightarrow \mathbf{Z} \cup \{-\infty\}$ is called *skew-supermodular* if every pair of $X, Y \subseteq S$ satisfies either the supermodular inequality or the *negamodular inequality*

$$g(X) + g(Y) \leq g(X \setminus Y) + g(Y \setminus X).$$

By definition, skew-supermodularity is a generalization of intersecting supermodularity. It is known that Theorem 6.3 remains true for a pair of skew-supermodular functions [32]. Our main result, Theorem 6.6, is also extended to skew-supermodular functions as follows.

Theorem 6.33. *For skew-supermodular functions $g_1, g_2 : 2^S \rightarrow \mathbf{Z} \cup \{-\infty\}$ and $k \in \mathbf{Z}_{>0}$, assume that there exists a k -coloring which dominates both g_1 and g_2 . If L satisfies $|L(u)| = k$ for each $u \in S$, then there exists a coloring $\varphi : S \rightarrow \Sigma$ such that every $u \in S$ satisfies $\varphi(u) \in L(u)$ and φ dominates both g_1 and g_2 .*

For a skew-supermodular function, define its reduction by a subset $K \subseteq S$ as in

Section 6.3. Then, we can confirm that is again a skew-supermodular. Also, proofs in Section 6.4 do not depend on intersecting supermodularity. Therefore, to extend the proof of Theorem 6.6 to that of Theorem 6.33, it suffices to show the existence of skeleton posets for skew-supermodular functions.

Let $g: 2^S \rightarrow \mathbf{Z} \cup \{-\infty\}$ be a skew-supermodular function and $\pi: S \rightarrow [k]$ be a minimal dominating k -coloring for g . To obtain skeleton posets for skew-supermodular functions, we adjust the arguments in Section 6.5 by using the following three claims.

Claim 6.34. *If two sets $X, Y \subseteq S$ are tight and satisfy the negamodular inequality of g , then $X \setminus Y$ and $Y \setminus X$ are also tight and we have $\pi(X \setminus Y) = \pi(X)$ and $\pi(Y \setminus X) = \pi(Y)$.*

Proof. Since π dominates g and $\pi(X) \supseteq \pi(X \setminus Y)$, $\pi(Y) \supseteq \pi(Y \setminus X)$ obviously hold,

$$g(X) + g(Y) = |\pi(X)| + |\pi(Y)| \geq |\pi(X \setminus Y)| + |\pi(Y \setminus X)| \geq g(X \setminus Y) + g(Y \setminus X).$$

As we have $g(X \setminus Y) + g(Y \setminus X) \geq g(X) + g(Y)$, the inequalities above are in fact equalities. Hence $|\pi(X)| = |\pi(X \setminus Y)| = g(X \setminus Y)$ and $|\pi(Y)| = |\pi(Y \setminus X)| = g(Y \setminus X)$, which imply the required formulas. \square

Claim 6.35. *If the conditions (6.1)–(6.3) hold with $F = X$ and $F = Y$, then X and Y satisfy the supermodular inequality of g .*

Proof. Suppose, to the contrary, the supermodular inequality fails to hold. Since g is skew-supermodular, we then obtain the negamodular inequality. As X, Y are tight by (6.2), Claim 6.34 implies $\pi(X \setminus Y) = \pi(X)$. By (6.1), we have $u \in X \cap Y$, and hence $X \setminus Y \subseteq X - u$. Then (6.3) with $F = X$ implies $\pi(u) \notin \pi(X \setminus Y)$. Thus, we obtain $\pi(u) \in \pi(X) \setminus \pi(X \setminus Y)$, which contradicts $\pi(X \setminus Y) = \pi(X)$. \square

Claim 6.35 enables us to extend Claim 6.22 for skew-supermodular functions. Therefore, we can define $D(u)$ for each $u \in S$ similarly to the case of intersecting-supermodular functions.

Claim 6.36. *For any $u \in S$ and any tight set $X \subseteq S$, we see that $D(u)$ and X satisfy the supermodular inequality of g .*

Proof. If $D(u) \subseteq X$, the claim is obvious. We assume $D(u) \not\subseteq X$, and hence $D(u) \cap X \subsetneq D(u)$. Suppose, to the contrary, the supermodular inequality fails to hold, and then the negamodular inequality holds. Since $D(u)$ and X are tight, Claim 6.34 implies $|\pi(D(u) \setminus X)| = g(D(u) \setminus X)$ and $\pi(D(u) \setminus X) = \pi(D(u))$. The latter implies $u \in D(u) \setminus X$ because we have $\pi(D(u) - u) \not\supseteq \pi(u)$. We then see that the conditions (6.1)–(6.4) holds with $F = D(u) \setminus X \subsetneq D(u)$, where (6.4) follows from $\pi(D(u) \setminus X) = \pi(D(u))$. This contradicts the minimality of $D(u)$. \square

Claim 6.36 says that, for $D(u)$ and tight set X , the skew-supermodularity implies the supermodular inequality. Observe that, in the proofs after Claim 6.22, we apply the supermodular inequality only for such pairs of subsets. Thus, the arguments in Section 6.5 work for skew-supermodular functions and we can obtain their skeleton posets, which completes the proof of Theorem 6.33.

6.7 The List Coloring Version of Gupta's Theorem

While König showed that the minimum number of colors required for a bipartite edge coloring is equal to the maximum degree of a vertex (Theorem 6.2), Gupta showed that the maximum number of disjoint edge covers in a bipartite graph is equal to the minimum degree of a vertex. As a common generalization of these two, Gupta also showed the following theorem.

Theorem 6.37 (Gupta [46]). *For a bipartite graph $G = (T, E)$ and $k \in \mathbf{Z}_{>0}$, there exists a function $\pi : E \rightarrow [k]$ such that $|\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$.*

From Theorem 6.37, we can deduce König's theorem by letting k be the maximum degree and the Gupta's edge cover theorem by letting k be the minimum degree.

Remark 6.38. Theorem 6.37 is a special case of Schrijver's Theorem 6.3 as follows. For a bipartite graph $G = (T, E)$, let (T_1, T_2) be a bipartition of T into two independent sets. For $i = 1, 2$, define a family $\mathcal{F}_i \subseteq 2^E$ by $\mathcal{F}_i := \{\delta_G(t) \mid t \in T_i\}$ and a function $g_i : \mathcal{F}_i \rightarrow \mathbf{Z}$ by $g_i(\delta_G(t)) := \min\{k, |\delta_G(t)|\}$ for each $t \in T_i$. We see that both g_1 and g_2 are intersecting-supermodular functions. We also see that $\pi : E \rightarrow [k]$ is a supermodular coloring for (g_1, g_2) if and only if $|\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$. ■

Since a required coloring in Theorem 6.37 is a special case of a supermodular coloring, Theorem 6.6 naturally derives the list coloring version of Theorem 6.37 as follows.

Corollary 6.39. *For a bipartite graph $G = (T, E)$, assume that every edge e has a color list $L(e) \subseteq \Sigma$ with $|L(e)| = k$. Then, there exists a function $\varphi : E \rightarrow \Sigma$ such that $\varphi(e) \in L(e)$ for every $e \in E$ and $|\varphi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$.*

The statement of Corollary 6.39, in fact, can be shown without using Theorem 6.6. Here we give an alternative proof, which uses Theorem 6.1 rather than Theorem 6.6.

Another Proof of Corollary 6.39. By Theorem 6.37, there is a k -coloring $\pi : E \rightarrow [k]$ such that $|\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}$ for every $t \in T$. From $G = (T, E)$ and π , we derive another bipartite graph $\hat{G} = (\hat{T}, \hat{E})$ and a k -coloring $\hat{\pi} : \hat{E} \rightarrow [k]$. First, set $\hat{G} := G$ and $\hat{\pi} := \pi$. Then, repeat the following procedure as long as \hat{G} has a pair of adjacent edges whose values of $\hat{\pi}$ are the same.

Take a vertex $t \in T$ and an edge $e \in \delta_{\hat{G}}(t)$ such that $\hat{\pi}(e) = \hat{\pi}(e')$ for some $e' \in \delta_{\hat{G}}(t) \setminus \{e\}$. Add a new vertex \hat{t} and replace $e = (t, s)$ by $\hat{e} = (\hat{t}, s)$, where s is another endpoint of e . Set $\hat{\pi}(\hat{e}) := \pi(e)$.

There is a one-to-one correspondence between the final edge set \hat{E} and the original E , and the final vertex set \hat{T} is a superset of T . By the definition of the procedure, $\hat{\pi}$ is an edge coloring of \hat{G} and each $t \in T$ satisfies $|\delta_{\hat{G}}(t)| = |\pi(\delta_G(t))|$. Set $\hat{L}(\hat{e}) := L(e)$ for each corresponding edge pair $(\hat{e}, e) \in \hat{E} \times E$. We then have $|\hat{L}(\hat{e})| = k$ for every $\hat{e} \in \hat{E}$. Since $\hat{\pi}$ is an edge coloring of \hat{G} with k colors, Theorem 6.1 implies that there is $\hat{\varphi} : \hat{E} \rightarrow \Sigma$ such that $\hat{\varphi}(\hat{e}) \in \hat{L}(\hat{e})$ for each $\hat{e} \in \hat{E}$ and $|\hat{\varphi}(\delta_{\hat{G}}(t))| = |\delta_{\hat{G}}(t)|$ for every $t \in \hat{T}$. Define $\varphi : E \rightarrow \Sigma$ by $\varphi(e) := \hat{\varphi}(\hat{e})$ for each corresponding edge pair $(\hat{e}, e) \in \hat{E} \times E$. For each $t \in T$, we have $\delta_G(t) \supseteq \delta_{\hat{G}}(t)$, where edges in E are identified with the corresponding edges in \hat{E} . Therefore, we obtain

$$|\varphi(\delta_G(t))| \geq |\hat{\varphi}(\delta_{\hat{G}}(t))| = |\delta_{\hat{G}}(t)| = |\pi(\delta_G(t))| \geq \min\{k, |\delta_G(t)|\}.$$

We also have $\varphi(e) = \hat{\varphi}(\hat{e}) \in \hat{L}(\hat{e}) = L(e)$ for each corresponding pair $(\hat{e}, e) \in \hat{E} \times E$. Thus, $\varphi : E \rightarrow \Sigma$ satisfies the required conditions. \square

Chapter 7

Conclusion

In this thesis, we have established algorithmic and structural results for stable matching models on matroidal structures (Chapters 3–5). In addition, as a new application of stable matchings, we have solved a coloring problem with supermodular constraints (Chapter 6). We now conclude this thesis by providing a summary of our contribution for each of the four topics and several open problems.

Chapter 3: A Generalized Polymatroid Approach to Stable Matchings with Lower Quotas We have formulated “stable matchings on generalized matroids” and “stable allocations on generalized polymatroids.” For both models, we have shown the distributive lattice structure of stable matchings and the generalized “rural hospital theorem.” In particular, for the former model, we have provided a polynomial-time algorithm which finds a stable matching or reports the nonexistence.

From a perspective of application, we have to find some matching even if there is no stable matching. Then, an algorithm which finds a “nearly stable” matching is required. One natural approach is to minimize the number of blocking pairs. This problem is, however, known to be hard even for a special case [50]. Moreover, such an approach may make some agents significantly dissatisfied. We should consider an alternative concept for a “nearly stable” matching, and devise an algorithm to find it.

Chapter 4: Finding a Stable Allocation in Polymatroid Intersection We have presented the first strongly polynomial-time algorithm to find a stable allocation in polymatroid intersection. Its running time is $O(n^3\gamma)$, where n and γ respectively denote the size of the ground set and the time for computing the saturation and exchange capacities. Also, we have proved that the obtained stable allocation is optimal for the proposing side. Note that, in Chapter 3, it has been shown that the problem to find a stable allocation on g-polymatroids can be reduced to finding that for the associated polymatroids and checking the feasibility for the original g-matroids. Then, we can also solve the g-polymatroid case with this algorithm, provided that oracles for the upper and lower bound functions

of the g -matroids are available.

It is still open whether there is a polynomial-time algorithm for solving further general models, such as the M^\sharp -concave value function model [24].

Chapter 5: Matroidal Choice Functions We have introduced “matroidal choice functions” as substitutable choice functions under matroid constraints. We have characterized them as choice functions computable by a certain online greedy algorithm. We have also provided a characterization by a local condition. It has been shown that a monotone M^\sharp -concave set function induces a matroidal choice function. For the stable matching model with matroidal choice functions, we have designed a variant of the deferred acceptance algorithm, which outputs the student optimal stable matching and is strategy-proof for students.

Chapter 6: List Supermodular Coloring We have shown the list supermodular coloring theorem by utilizing the monochromatic path theorem of Sands, Sauer and Woodrow [94], a generalization of the existence of a stable matching. Our result generalizes Galvin’s [43] list edge coloring theorem on bipartite graphs. We have extended our result to the setting of skew-supermodular coloring, and also have provided the list coloring version of Gupta’s theorem [46], as an immediate consequence of our main theorem.

Our proof can be regarded as a construction of a list coloring satisfying given supermodular constraints. We believe that this construction can be achieved in polynomial time, but have not been successful in proving it. A rigorous analysis of the time complexity is needed.

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