Actions of loop groups on H-surfaces

（H曲面へのループ群の作用）

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ACTIONS OF LOOP GROUPS ON H-SURFACES

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ABSTRACT. In this paper we shall define certain loop groups which act on simply connected H-surfaces* in space forms preserving conformality, and obtain a criterion for these group actions to be equivariant.

Introduction

In recent years, there has been much progress in the theory of harmonic maps from Riemann surfaces into Lie groups or symmetric spaces. For example, the discovery of actions of loop groups on harmonic maps has played an important role [BG], [BP], [GO], [U]. As a special case of the results in [BP], it is shown that a certain loop group of $SO(4)$ acts on harmonic maps from a simply connected Riemann surface into the standard 3-sphere S^3.

Let $\mathfrak{M}^3(c)$ denote the simply connected 3-dimensional space form of curvature c and M a Riemann surface. Since harmonic maps from M to S^3 are H-surfaces in $\mathfrak{M}^3(c)$ for $H = 0$, $c = 1$, it is natural to conjecture judging from the results in [BP] above mentioned, that a certain loop group of the isometry group of $\mathfrak{M}^3(c)$ acts on simply connected H-surfaces in $\mathfrak{M}^3(c)$ naturally, preserving conformality. In this paper, we shall show that this conjecture is true.

On the other hand, in the previous paper [F], the author obtained a criterion for the existence of a natural bijective correspondence between simply connected H-surfaces in $\mathfrak{M}^3(c)$ and simply connected H'-surfaces in $\mathfrak{M}^3(c')$. In this paper, we shall also obtain a criterion for the above

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*Here an H-surface meams a map which satisfies H-surface equations. Hence H-surfaces in the classical sence are conformal H-surfaces.

bijective correspondence to be equivariant with respect to the loop group actions.

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§1 H-surfaces

In the following, we shall describe $\mathfrak{M}^3(c)$ as a Riemannian symmetric space. For $c \in \mathbb{R}$, we set $L(c)$ and $\mathrm{sign}(c)$ as follows.

$$L(c) = \begin{cases} 1 & \text{if } c = 0, \\ \frac{1}{\sqrt{|c|}} & \text{if } c \neq 0 \end{cases}$$

and

$$\mathrm{sign}(c) = \begin{cases} 1 & \text{if } c > 0, \\ 0 & \text{if } c = 0, \\ -1 & \text{if } c < 0. \end{cases}$$

Also for $c \in \mathbb{R}$, we define G_c, K_c, σ_c as follows.

$$G_c = SO(3) \ltimes \mathbb{R}^3 = \left\{ \begin{pmatrix} T & s \\ 0 & 1 \end{pmatrix} \in GL(4, \mathbb{R}); T \in SO(3), s \in \mathbb{R}^3 \right\},$$
$$G_c = SO(4)$$

for $c > 0$, and

$$\begin{aligned} G_c &= SO^+(3,1) \\ &= \left\{ X = (x_{ij}) \in GL(4,\mathbb{R});^t XJX = J, \det X = 1, x_{44} > 0 \right\} \end{aligned}$$

for $c < 0$, where $J = \mathrm{diag}(1,1,1,-1)$.

Set

$$K_c = \left\{ \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix} \in GL(4, \mathbb{R}); T \in SO(3) \right\}.$$

An involutive automorphism of G_c is defined by

$$\sigma_c(X) = JXJ,$$

where $X \in G_c$. Then K_c is the fixed point set of σ_c. So (G_c, K_c, σ_c) is a symmetric pair and its corresponding Cartan decomposition $\mathfrak{g}_c = \mathfrak{k}_c \oplus \mathfrak{m}_c$ is given by

$$\mathfrak{g}_c = \left\{ \begin{pmatrix} A & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix} \in M(4, \mathbb{R}); A \in \mathfrak{so}(3), b \in \mathbb{R}^3 \right\},$$

$$\mathfrak{k}_c = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in M(4,\mathbb{R}); A \in \mathfrak{so}(3) \right\}$$

and

$$\mathfrak{m}_c = \left\{ \begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix} \in M(4,\mathbb{R}); b \in \mathbb{R}^3 \right\}.$$

An $\mathrm{Ad}_{G_c} K_c$-invariant metric on $\mathfrak{m}_c$ is defined by

$$g_c \left(\begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ -\mathrm{sign}(c)^t b' & 0 \end{pmatrix} \right) = L^2(c)\langle b, b' \rangle,$$

where $b, b' \in \mathbb{R}^3$.

Let M be a simply connected Riemann surface and $\pi : G_c \longrightarrow \mathfrak{M}^3(c) = G_c/K_c$ the natural projection. For any map $f : M \longrightarrow \mathfrak{M}^3(c)$, there always exists a map $F : M \longrightarrow G_c$ such that $\pi \circ F = f$. Such a map F is called a framing of f. Then we have a decomposition $F^{-1}dF =: \alpha = \alpha_{\mathfrak{k}_c} + \alpha_{\mathfrak{m}_c}$. Since M is a Riemann surface, we have a type decomposition $\alpha_{\mathfrak{m}_c} = \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c}$, where $\alpha'_{\mathfrak{m}_c}$ is an $\mathfrak{m}_c^{\mathbb{C}}$-valued $(1,0)$-form with complex conjugate $\alpha''_{\mathfrak{m}_c}$. We denote the decomposition $\alpha = \alpha_{\mathfrak{k}_c} + \alpha'_{\mathfrak{m}_c} + \alpha''_{\mathfrak{m}_c}$ simply as $\alpha = \alpha_{\mathfrak{k}} + \alpha'_{\mathfrak{m}} + \alpha''_{\mathfrak{m}}$. If we write $\alpha_{\mathfrak{m}}$ as

$$\alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & b \\ -\mathrm{sign}(c)^t b & 0 \end{pmatrix}$$

for $b \in \Omega^1(M, \mathbb{R}^3)$, we shall define an $\mathfrak{m}_c$-valued 2-form $\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}}$ by

$$\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}} = \begin{pmatrix} 0 & b \times b \\ -\mathrm{sign}(c)^t (b \times b) & 0 \end{pmatrix},$$

where $\times$ is the exterior product on $\mathbb{R}^3$. Then we set

$$n_f = \frac{L}{2} d\pi \left(\mathrm{Ad}F * (\alpha_{\mathfrak{m}} \times \alpha_{\mathfrak{m}}) \right),$$

where $*$ is the Hodge star operator on M. It is easy to see that n_f is independent of the choice of F.

We now fix $p_0 \in M$ and set $o = \{K_c\}$. For $H, c \in \mathbb{R}$, we set

$$\mathcal{H}_{H,c} = \left\{ f : M \longrightarrow \mathfrak{M}^3(c); \frac{1}{2}\mathrm{trace}\nabla df = H n_f, f(p_0) = o \right\},$$

$$\mathcal{C}_{H,c} = \{ f \in \mathcal{H}_{H,c}; f \text{ is weakly conformal} \}.$$

Then $\mathcal{C}_{H,c}$ is the set of based branched conformal immersion with constant mean curvature$= H$.

§2 Definition of loop groups

We fix $0 < \varepsilon < 1$ and partition the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ as follows. Let C_ε and $C_{1/\varepsilon}$ denote the circles of radius ε and $1/\varepsilon$ about $0 \in \mathbb{C}$ and define open sets by

$$I_\varepsilon = \left\{\lambda \in \mathbb{P}^1; |\lambda| < \varepsilon\right\}, I_{1/\varepsilon} = \left\{\lambda \in \mathbb{P}^1; |\lambda| > 1/\varepsilon\right\},$$

$$E^{(\varepsilon)} = \left\{\lambda \in \mathbb{P}^1; \varepsilon < |\lambda| < 1/\varepsilon\right\}.$$

Now put $I^{(\varepsilon)} = I_\varepsilon \cup I_{1/\varepsilon}$ and $C^{(\varepsilon)} = C_\varepsilon \cup C_{1/\varepsilon}$ so that $\mathbb{P}^1 = I^{(\varepsilon)} \cup C^{(\varepsilon)} \cup E^{(\varepsilon)}$. For a map $\xi : C^{(\varepsilon)} \longrightarrow \mathfrak{g}_c^{\mathbb{C}}$, define $A_\xi : C^{(\varepsilon)} \longrightarrow \mathfrak{so}(3)^{\mathbb{C}}$ and $b_\xi : C^{(\varepsilon)} \longrightarrow \mathbb{C}^3$ by

$$\xi(\lambda) = \begin{pmatrix} A_\xi(\lambda) & b_\xi(\lambda) \\ -\mathrm{sign}(c)^t b_\xi(\lambda) & 0 \end{pmatrix},$$

where $\lambda \in C^{(\varepsilon)}$. We define a bijective map $\iota : \mathbb{C}^3 \longrightarrow \mathfrak{so}(3)^{\mathbb{C}}$ by

$$\iota(p) = \begin{pmatrix} 0 & -p^3 & p^2 \\ p^3 & 0 & -p^1 \\ -p^2 & p^1 & 0 \end{pmatrix} \text{ for } p = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix} \in \mathbb{C}^3.$$

First we define a loop group $\Lambda^\varepsilon G_c$ by

$$\Lambda^\varepsilon G_c = \left\{g \in C^\infty\left(C^{(\varepsilon)}, G_c^{\mathbb{C}}\right); \overline{g(\lambda)} = g(1/\overline{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\right\},$$

where conjugation is the Cartan involution of $G_c^{\mathbb{C}}$ fixing G_c. Then the Lie algebra $\Lambda^\varepsilon \mathfrak{g}_c$ of the Lie group $\Lambda^\varepsilon G_c$ is defined by

$$\Lambda^\varepsilon \mathfrak{g}_c = \left\{\xi \in C^\infty\left(C^{(\varepsilon)}, \mathfrak{g}_c^{\mathbb{C}}\right); \overline{\xi(\lambda)} = \xi(1/\overline{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

(For the definition of the manifold structure, we refer to [M], [OMYK], [PS].)

Then the following loop algebras are important for us to define loop groups.

For $c = 0, \rho \in \sqrt{-1}\mathbb{R} \setminus \{0\}$, we set

$$\Lambda^{\varepsilon,\rho} \mathfrak{g}_c = \left\{\xi \in \Lambda^\varepsilon \mathfrak{g}_c; \lambda \frac{dA_\xi}{d\lambda} = \rho\iota(b_\xi) \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

For $c=0, \rho=0$, we set

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c=\left\{\xi\in\Lambda^{\varepsilon}\mathfrak{g}_c; b_{\xi}(\lambda)=0 \text{ for } \lambda\in C^{(\varepsilon)}\right\}.$$

For $c\neq 0, \rho\in\sqrt{-1}\mathbb{R}$, we set

$$\alpha_n=\begin{cases}\frac{1}{2}\left\{(\rho+1)^n+(\rho-1)^n\right\} & \text{if } c>0,\\ \frac{1}{2}\left\{(\rho-\sqrt{-1})^n+(\rho+\sqrt{-1})^n\right\} & \text{if } c<0,\end{cases}$$

$$\beta_n=\begin{cases}\frac{1}{2}\left\{(\rho+1)^n-(\rho-1)^n\right\} & \text{if } c>0,\\ \frac{\sqrt{-1}}{2}\left\{(\rho-\sqrt{-1})^n-(\rho+\sqrt{-1})^n\right\} & \text{if } c<0\end{cases}$$

for $n\in\mathbb{Z}$. Here we put $0^0=1$. Then we set

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c=\left\{\xi\in\Lambda^{\varepsilon}\mathfrak{g}_c; A_{\xi} \text{ and } b_{\xi} \text{ have Fourier series } (*)\right\},$$

where

$$(*)\quad A_{\xi}(\lambda)=\sum_{n\in\mathbb{Z}}\alpha_n\iota(a_n)\lambda^n, b_{\xi}(\lambda)=\sum_{n\in\mathbb{Z}}\beta_n a_n\lambda^n \text{ for } a_n\in\mathbb{C}^3, \lambda\in C_{\varepsilon}.$$

It is easy to see that $\Lambda^{\varepsilon,\rho}\mathfrak{g}_c$ is a Lie subalgebra of $\Lambda^{\varepsilon}\mathfrak{g}_c$. Then we have the following:

Lemma 2.1. *Suppose one of the following conditions are satisfied:*
(i) $c=0$ *(ii)* $c>0$, $(1-\rho^2)\varepsilon^2<1$ *(iii)* $c<0$, $(1+|\rho|)\varepsilon<1$.
Then we have a Lie subgroup $\Lambda^{\varepsilon,\rho}G_c\subset\Lambda^{\varepsilon}G_c$ *such that its Lie algebra is* $\Lambda^{\varepsilon,\rho}\mathfrak{g}_c$.

proof.
Case I: $c=0$, $\rho=0$.
For $g\in\Lambda^{\varepsilon}G_c$, set $T_g\in C^{\infty}\left(C^{(\varepsilon)},SO(3)^{\mathbb{C}}\right)$ and $s_g\in C^{\infty}\left(C^{(\varepsilon)},\mathbb{C}^3\right)$ by

$$g(\lambda)=\begin{pmatrix}T_g(\lambda) & s_g(\lambda)\\ 0 & 1\end{pmatrix},$$

where $\lambda\in C^{(\varepsilon)}$. Then it is obvious to see that

$$\Lambda^{\varepsilon,\rho}G_c=\left\{g\in\Lambda^{\varepsilon}G_c; s_g(\lambda)=0 \text{ for } \lambda\in C^{(\varepsilon)}\right\}.$$

Case II: $c=0$, $\rho\in\sqrt{-1}\mathbb{R}\setminus\{0\}$.
By direct computation, it is easy to see that

$$\Lambda^{\varepsilon,\rho}G_c=\left\{g\in\Lambda^{\varepsilon}G_c; \lambda\frac{dT_g}{d\lambda}=\rho\iota(s_g)T_g \text{ for } \lambda\in C^{(\varepsilon)}\right\}.$$

Case III: $c > 0$, $\rho = 0$.

In this case, we have

$$\Lambda^{\varepsilon,\rho}\mathfrak{g}_c = \left\{\xi \in \Lambda^{\varepsilon}\mathfrak{g}_c; \xi(-\lambda) = \sigma_c\xi(\lambda), \overline{\xi(\lambda)} = \xi(1/\overline{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

Hence we have

$$\Lambda^{\varepsilon,\rho}G_c = \left\{g \in \Lambda^{\varepsilon}G_c; g(-\lambda) = \sigma_c g(\lambda), \overline{g(\lambda)} = g(1/\overline{\lambda}) \text{ for } \lambda \in C^{(\varepsilon)}\right\}.$$

Case IV: $c > 0$, $\rho \in \sqrt{-1}\mathbb{R} \setminus \{0\}$, $(1-\rho^2)\varepsilon^2 < 1$.

It is well-known that the condition $\xi \in C^{\infty}\left(C^{(\varepsilon)}, \mathfrak{g}_c^{\mathbb{C}}\right)$ is equivalent to

$$\sum_{n\in\mathbb{Z}} \left(1+n^2\right)^l \left(|\alpha_n|^2 + |\beta_n|^2\right) \varepsilon^{2n} |a_n|^2 < \infty \tag{2.1}$$

for any $l > 0$. Since $|\alpha_n|^2 + |\beta_n|^2 = \left(1-\alpha^2\right)^n$, (2.1) is equivalent to

$$\sum_{n\in\mathbb{Z}} \left(1+n^2\right)^l \left(\varepsilon\sqrt{1-\rho^2}\right)^{2n} |a_n|^2 < \infty \tag{2.2}$$

for any $l > 0$. We set

$$\psi(\xi)(\lambda) = \begin{pmatrix} \sum_{n\in\mathbb{Z}} {}^t(a_{2n})\lambda^{2n} & \sum_{n\in\mathbb{Z}} a_{2n+1}\lambda^{2n+1} \\ -{}^t\left(\sum_{n\in\mathbb{Z}} a_{2n+1}\lambda^{2n+1}\right) & 0 \end{pmatrix} \tag{2.3}$$

for $\lambda \in C_{\varepsilon}$. Note that from (2.2), ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \longrightarrow \Lambda^{\varepsilon\sqrt{1-\rho^2},0}\mathfrak{g}_1.$$

Then by [OMYK, Theorem 3.2], we have a homomorphism:

$$\varphi : \widetilde{\Lambda^{\varepsilon\sqrt{1-\rho^2},0}G_1} \longrightarrow \Lambda^{\varepsilon}G_c$$

such that $d\varphi = \psi^{-1}$, where we denote the universal covering group of a Lie group G as $\tilde{G}$. Set $\Lambda^{\varepsilon,\rho}G_c = \mathrm{Im}\varphi$. Then $\Lambda^{\varepsilon,\rho}G_c$ is the desired Lie subgroup of $\Lambda^{\varepsilon}G_c$.

Case V: $c < 0$, $(1+|\rho|)\varepsilon < 1$.

Since $|\alpha_n|^2 + |\beta_n|^2 = \frac{1}{2}\left\{(1+|\rho|)^{2n} + (1-|\rho|)^{2n}\right\}$, $\xi \in C^{\infty}\left(C^{(\varepsilon)}, \mathfrak{g}_c^{\mathbb{C}}\right)$ is equivalent to

$$\sum_{n\in\mathbb{Z}} \left(1+n^2\right)^l \left\{(1 \pm |\rho|)\varepsilon\right\}^{2n} |a_n|^2 < \infty \tag{2.4}$$

for any $l > 0$. If we define $\psi(\xi)(\lambda)$ by (2.3), from (2.4), ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \longrightarrow \begin{cases} \Lambda^{(1+|\rho|)\varepsilon,0}\mathfrak{g}_1 \cap \Lambda^{|1-|\rho||\varepsilon,0}\mathfrak{g}_1 & \text{if } |\rho| \neq 1, \\ \Lambda^{(1+|\rho|)\varepsilon,0}\mathfrak{g}_1 & \text{if } |\rho| = 1. \end{cases}$$

Similar to the case IV, we have the desired Lie subgroup $\Lambda^{\varepsilon,\rho}G_c$. □

We identify K_c with $SO(3)$ and fix an Iwasawa decomposition of $SO(3)^{\mathbb{C}}$: $SO(3)^{\mathbb{C}} = SO(3)B$, where B is a Borel subgroup of $SO(3)^{\mathbb{C}}$. We define subgroups of $\Lambda^{\varepsilon,\rho}G_c$ as follows.

$$\Lambda_E^{\varepsilon,\rho}G_c = \left\{g \in \Lambda^{\varepsilon,\rho}G_c; g \text{ extends holomorphically to } g : E^{(\varepsilon)} \to G_c^{\mathbb{C}}\right\},$$

$$\Lambda_I^{\varepsilon,\rho}G_c = \left\{g \in \Lambda^{\varepsilon,\rho}G_c; g \text{ extends holomorphically to } g : I^{(\varepsilon)} \to G_c^{\mathbb{C}}\right\}.$$

It is easy to see that $g \in \Lambda_I^{\varepsilon,\rho}G_c$ satisfies $g(0) \in K_c^{\mathbb{C}}$. We define a subgroup of $\Lambda_I^{\varepsilon,\rho}G_c$ by

$$\Lambda_{I,B}^{\varepsilon,\rho}G_c = \{g \in \Lambda_I^{\varepsilon,\rho}G_c; g(0) \in B\}.$$

Then we obtain the following Iwasawa type decomposition for $\Lambda^{\varepsilon,\rho}G_c$.

Lemma 2.2. *Suppose one of the following conditions are satisfied:*
(i) $c = 0$ *(ii)* $c > 0$, $(1-\rho^2)\varepsilon^2 < 1$ *(iii)* $c < 0$, $(1+|\rho|)\varepsilon < 1$.
Then a map defined by multiplication

$$\Lambda_E^{\varepsilon,\rho}G_c \times \Lambda_{I,B}^{\varepsilon,\rho}G_c \longrightarrow \Lambda^{\varepsilon,\rho}G_c$$

is bijective.

proof.
Case I: $c = 0, \rho = 0$ or $c > 0, \rho = 0$.

This is a special case of the result due to McIntosh [M, Proposition 6.2].

Case II: $c = 0$, $\rho \in \sqrt{-1}\mathbb{R} \setminus \{0\}$.

Similar to the case I, for $g \in \Lambda^{\varepsilon,\rho}G_c$, since T_g satisfies $\overline{T_g(\lambda)} = T_g(1/\overline{\lambda})$ for $\lambda \in C^{(\varepsilon)}$, there exist $T_1, T_2 \in C^\infty\left(C^{(\varepsilon)}, SO(3)^{\mathbb{C}}\right)$ such that (i) $T_g = T_1T_2$, (ii) $\overline{T_i(\lambda)} = T_i(1/\overline{\lambda})$ for $\lambda \in C^{(\varepsilon)}$, $i = 1, 2$, (iii) T_1 extends holomorphically to $T_1 : E^{(\varepsilon)} \longrightarrow SO(3)^{\mathbb{C}}$, (iv) T_2 extends holomorphically to $T_2 : I^{(\varepsilon)} \longrightarrow SO(3)^{\mathbb{C}}$, (v) $T_2(0) \in B$. We define $s_1, s_2 \in C^\infty\left(C^{(\varepsilon)}, \mathbb{C}^3\right)$ by

$$\lambda\frac{dT_i}{d\lambda} = \rho\iota(s_i)T_i \qquad \text{for} \qquad i = 1, 2.$$

By definition, we have (i) $\overline{s_i(\lambda)} = s_i(1/\bar{\lambda})$ for $\lambda \in C^{(\varepsilon)}$, $i = 1, 2$, (ii) s_1 extends holomorphically to $s_1 : E^{(\varepsilon)} \longrightarrow \mathbb{C}^3$, (iii) s_2 extends holomorphically to $s_2 : I^{(\varepsilon)} \longrightarrow \mathbb{C}^3$, (iv) $s_2(0) = 0$. On $C^{(\varepsilon)}$ we have

$$\begin{pmatrix} T_1 & s_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T_2 & s_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} T_1 T_2 & T_1 s_2 + s_1 \\ 0 & 1 \end{pmatrix}.$$

Direct computation shows that

$$\lambda \frac{d(T_1 T_2)}{d\lambda} = \rho\iota(T_1 s_2 + s_1) T_1 T_2.$$

Hence we have the desired decomposition.

Case III: $c > 0$, $\rho \in \sqrt{-1}\mathbb{R} \setminus \{0\}$, $(1 - \rho^2)\varepsilon^2 < 1$.

Using the decomposition of $\Lambda^{\varepsilon\sqrt{1-\rho^2}, 0} G_1$ and the covering map

$$\varphi : \widetilde{\Lambda^{\varepsilon\sqrt{1-\rho^2}, 0} G_1} \longrightarrow \Lambda^{\varepsilon} G_c,$$

we have the following decomposition:

$$\widetilde{\Lambda^{\varepsilon,\rho} G_c} = \widetilde{\Lambda_E^{\varepsilon,\rho} G_c} \widetilde{\Lambda_{I,B}^{\varepsilon,\rho} G_c}.$$

Since any $g \in \Lambda_{I,B}^{\varepsilon,\rho} G_c$ is homotopic to a constant loop by definition of $\Lambda_{I,B}^{\varepsilon,\rho} G_c$, $\Lambda_{I,B}^{\varepsilon,\rho} G$ is simply connected. Hence we have the desired decomposition.

Case IV: $c < 0$, $(1 + |\rho|)\varepsilon < 1$.

Similar to the case III, we have the desired decomposition. $\square$

§3 Extended framings

Let $f \in \mathcal{H}_{H,c}$ and F be a framing of f. Then direct computation shows that $\frac{1}{2}\mathrm{trace}\nabla df = H n_f$ is equivalent to

$$(3.1)\quad (d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]) - (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]) = 2\sqrt{-1} H L(c) \alpha'_{\mathfrak{m}} \times \alpha''_{\mathfrak{m}}.$$

Taking the $\mathfrak{m}_c$- and $\mathfrak{k}_c$-parts of the Maurer-Cartan equations for α, we have

$$(3.2)\qquad (d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}]) + (d\alpha''_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha''_{\mathfrak{m}}]) = 0$$

and

$$(3.3)\qquad d\alpha_{\mathfrak{k}} + \frac{1}{2}[\alpha_{\mathfrak{k}} \wedge \alpha_{\mathfrak{k}}] + [\alpha'_{\mathfrak{m}} \wedge \alpha''_{\mathfrak{m}}] = 0.$$

Equations (3.1) and (3.2) are equivalent to

$$d\alpha'_{\mathfrak{m}} + [\alpha_{\mathfrak{k}} \wedge \alpha'_{\mathfrak{m}}] = \sqrt{-1}HL(c)\alpha'_{\mathfrak{m}} \times \alpha''_{\mathfrak{m}}. \tag{3.4}$$

If we write $\alpha'_{\mathfrak{m}}$ as

$$\alpha'_{\mathfrak{m}} = \begin{pmatrix} 0 & b' \\ -\mathrm{sign}(c)^t b' & 0 \end{pmatrix} \tag{3.5}$$

for $b' \in \Omega^{1,0}(M, \mathbb{C}^3)$, we shall define $\iota(\alpha'_{\mathfrak{m}}) \in \Omega^{1,0}(M, \mathfrak{k}_c^{\mathbb{C}})$ by

$$\iota(\alpha'_{\mathfrak{m}}) = \begin{pmatrix} \iota(b') & 0 \\ 0 & 0 \end{pmatrix}.$$

We define $\iota(\alpha''_{\mathfrak{m}}) \in \Omega^{0,1}(M, \mathfrak{k}_c^{\mathbb{C}})$ similarly. For $\lambda \in \mathbb{C} \setminus \{0\}$, define a $\mathfrak{g}_c^{\mathbb{C}}$-valued 1-form by

$$\begin{aligned} \alpha_\lambda &= \lambda^{-1}\alpha'_{\mathfrak{m}} + \lambda\alpha''_{\mathfrak{m}} \\ &\quad + \alpha_{\mathfrak{k}} - \sqrt{-1}HL(c)(\lambda^{-1} - 1)\iota(\alpha'_{\mathfrak{m}}) + \sqrt{-1}HL(c)(\lambda - 1)\iota(\alpha''_{\mathfrak{m}}). \end{aligned}$$

It is computed in [F] that α satisfies equations (3.3) and (3.4) if and only if

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. Then from the standard fact, there exists a map $F_\lambda : M \longrightarrow G_c^{\mathbb{C}}$, unique up to left translation by a constant element of $G_c^{\mathbb{C}}$, satisfying $F_\lambda^{-1} dF_\lambda = \alpha_\lambda$. Furthermore, we set

$$\Lambda^\rho_{\mathrm{hol}} G_c = \bigcap_{0<\varepsilon<1} \Lambda_E^{\varepsilon,\rho} G_c.$$

These observations lead us to the following definition.

Definition 3.1. *A map $F_\lambda : M \longrightarrow \Lambda^\rho_{\mathrm{hol}} G_c$ is an extended framing if $\lambda F_\lambda^{-1}\partial F_\lambda$ is holomorphic at $\lambda = 0$, where $F_\lambda^{-1}\partial F_\lambda$ is the (1,0)-part of $F_\lambda^{-1} dF_\lambda$.*

Then it is easy to see that $f \in \mathcal{H}_{H,c}$ admits an extended framing $F_\lambda : M \longrightarrow \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c$ such that F_1 is a framing of f. Conversely, if $F_\lambda : M \longrightarrow \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c$ is an extended framing with $F_\lambda(p_0) \in K_c$, then $\pi \circ F_1 \in \mathcal{H}_{H,c}$. We set $\mathcal{E}_{H,c} =$

$$\left\{ F_\lambda : M \to \Lambda^{\sqrt{-1}HL(c)}_{\mathrm{hol}} G_c ; F_\lambda \text{ is an extended framing}, F_\lambda(p_0) \in K_c \right\},$$

$$\mathcal{K}_c = C^\infty(M, K_c).$$

Then we have a bijective correspondence

$$\mathcal{H}_{H,c} \cong \mathcal{E}_{H,c}/\mathcal{K}_c$$

which maps $f \in \mathcal{H}_{H,c}$ to $\{F_\lambda\} \in \mathcal{E}_{H,c}/\mathcal{K}_c$, where $\mathcal{K}_c$ acts by point-wise multiplication on the right.

§4 Action of $\Lambda_I^{\varepsilon,\rho}G_c$ on extended framings

For any $g \in \Lambda^{\varepsilon,\rho}G_c$ such that (i)$c = 0$, or (ii)$c > 0$, $(1-\rho^2)\varepsilon^2 < 1$ or (iii)$c < 0$, $(1+|\rho|)\varepsilon < 1$, we have a unique factorization by Lemma 2.2

$$g = g_E g_I,$$

where $g_E \in \Lambda_E^{\varepsilon,\rho}G_c$, $g_I \in \Lambda_{I,B}^{\varepsilon,\rho}G_c$. Then we define an action of $\Lambda_I^{\varepsilon,\rho}G_c$ on $\Lambda_E^{\varepsilon,\rho}G_c$ by

$$g\#h = (gh)_E,$$

where $g \in \Lambda_I^{\varepsilon,\rho}G_c$, $h \in \Lambda_E^{\varepsilon,\rho}G_c$.

Proposition 4.1. *Let $\rho = \sqrt{-1}HL(c)$, $g \in \Lambda_I^{\varepsilon,\rho}G_c$ and $F_\lambda : M \longrightarrow \Lambda_{\mathrm{hol}}^{\rho}G_c$ be an extended framing. Define $g\#F_\lambda : M \longrightarrow \Lambda_{\mathrm{hol}}^{\rho}G_c$ by*

$$(g\#F_\lambda)(p) = g\#\left(F_\lambda(p)\right),$$

for $p \in M$. Then

(i) $g\#F_\lambda$ is also an extended framing.

(ii) If F_λ is based (that is, $F_\lambda \in \mathcal{E}_{H,c}$) then so is $g\#F_\lambda$.

(iii) If $k \in \mathcal{K}_c$ then

$$g\#(F_\lambda k) = (g\#F_\lambda)\tilde{k}$$

with $\tilde{k} \in \mathcal{K}_c$.

Thus $\Lambda_I^{\varepsilon,\rho}G_c$ acts on $\mathcal{H}_{H,c} = \mathcal{E}_{H,c}/\mathcal{K}_c$.

proof. We just follow directly [BP, Proposition 2.9]. □

Furthermore, we have

Lemma 4.2. *Let $g \in \Lambda_I^{\varepsilon,\sqrt{-1}HL(c)}G_c$ and $F_\lambda \in \mathcal{E}_{H,c}$. If $\pi \circ F_\lambda$ is weakly conformal, so is $\pi \circ (g\#F_\lambda)$.*

proof. Write

$$gF_\lambda = pq,$$

where $p = g\#F_\lambda$, $q : M \longrightarrow \Lambda_{I,B}^{\varepsilon,\sqrt{-1}HL(c)}G_c$. Then

$$p^{-1}dp = \mathrm{Ad}q(F_\lambda^{-1}dF_\lambda - q^{-1}dq).$$

Hence if we write $\alpha'_{\mathfrak{m}}$ as (3.5), the (1,0)-part of the $\mathfrak{m}_c$-part of $p^{-1}dp$ is

$$\begin{pmatrix} 0 & Qb' \\ -\mathrm{sign}(c)^t(Qb') & 0 \end{pmatrix},$$

where

$$q(0) = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}$$

with $Q \in B$. This completes the proof. $\square$

Lemma 4.3. *There exist Lie group isomorphisms*

(i) $\Lambda_I^{\varepsilon,\rho}G_c \cong \Lambda_I^{\varepsilon,0}G_1$, *if* $c = 0$,

(ii) $\Lambda_I^{\varepsilon,\rho}G_c \cong \Lambda_I^{\varepsilon\sqrt{1-\rho^2},0}G_1$ *if* $c > 0$, $\varepsilon\sqrt{1-\rho^2} < 1$,

(iii) $\Lambda_I^{\varepsilon,\rho}G_c \cong \Lambda_I^{\varepsilon(1+|\rho|),0}G_1$ *if* $c < 0$, $\varepsilon(1+|\rho|) < 1$.

proof. By the proof of Lemma 2.1 and Lemma 2.2, we have (ii) and (iii). We have only to show (i).

Case I: $\rho = 0$.

For $\xi \in \Lambda^{\varepsilon,0}\mathfrak{g}_c$, we have a Fourier series of A_ξ:

$$A_\xi(\lambda) = \sum_{n\in\mathbb{Z}} \iota(a_n)\lambda^n,$$

where $a_n \in \mathbb{C}^3$, $\lambda \in C_\varepsilon$. We set $\psi(\xi)(\lambda)$ by (2.3). Then ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,0}\mathfrak{g}_c \longrightarrow \Lambda^{\varepsilon,0}\mathfrak{g}_1.$$

Hence we have the desired isomorphism.

Case II: $\rho \in \sqrt{-1}\mathbb{R} \setminus \{0\}$.

For $\xi \in \Lambda^{\varepsilon,\rho}\mathfrak{g}_c$, it is straightforward to see that we have Fourier series:

$$A_\xi(\lambda) = \iota(a_0) + \sum_{n\neq 0} \frac{\rho}{n}\iota(a_n)\lambda^n, b_\xi(\lambda) = \sum_{n\in\mathbb{Z}} a_n\lambda^n,$$

where $a_n \in \mathbb{C}^3$, $\lambda \in C_\varepsilon$. We set $\psi(\xi)(\lambda)$ by

$$\psi(\xi)(\lambda) = \begin{pmatrix} \iota(a_0) + \sum_{n\neq 0}\frac{\rho}{2n}\iota(a_{2n})\lambda^{2n} & \sum_{n\in\mathbb{Z}}\frac{a_{2n+1}}{2n+1}\lambda^{2n+1} \\ -^t\sum_{n\in\mathbb{Z}}\frac{a_{2n+1}}{2n+1}\lambda^{2n+1} & 0 \end{pmatrix},$$

for $\lambda \in C_\varepsilon$. Then ψ defines a well-defined Lie algebra isomorphism:

$$\psi : \Lambda^{\varepsilon,\rho}\mathfrak{g}_c \longrightarrow \Lambda^{\varepsilon,0}\mathfrak{g}_1.$$

Hence we have the desired isomorphism. $\square$

On the other hand, we have

Theorem 4.4 [F]. *Let $H, H', c, c' \in \mathbb{R}$. If $\operatorname{sign}(H^2+c) = \operatorname{sign}(H'^2+c')$, then there exists a bijective map*

$$\Psi : \mathcal{H}_{H,c} \longrightarrow \mathcal{H}_{H',c'}$$

such that $\Psi(\mathcal{C}_{H,c}) = \mathcal{C}_{H',c'}$.

By Lemma 4.2, Lemma 4.3 and Theorem 4.4, we have the following:

Theorem 4.5. *Let $H, H', c, c' \in \mathbb{R}$. If $\operatorname{sign}(H^2+c) = \operatorname{sign}(H'^2+c')$, then there exists a loop group $\mathcal{G}_{H,c}$ (respectively $\mathcal{G}_{H',c'}$) acting on $\mathcal{H}_{H,c}$ (respectively $\mathcal{H}_{H',c'}$) such that*

(i) $\mathcal{G}_{H,c}$ and $\mathcal{G}_{H',c'}$ are defined by $\mathcal{G}_{H,c} = \Lambda_I^{\varepsilon,\sqrt{-1}HL(c)}G_c$, $\mathcal{G}_{H',c'} = \Lambda_I^{\varepsilon',\sqrt{-1}H'L(c')}G_{c'}$ for suitable ε, ε' with $0 < \varepsilon, \varepsilon' < 1$,

(ii) there exists a Lie group isomorphism

$$\Phi : \mathcal{G}_{H,c} \longrightarrow \mathcal{G}_{H',c'},$$

(iii) the bijective map $\Psi : \mathcal{H}_{H,c} \longrightarrow \mathcal{H}_{H',c'}$ is Φ-equivariant,

(iv) $\mathcal{G}_{H,c}$ (respectively $\mathcal{G}_{H',c'}$) acts on $\mathcal{C}_{H,c}$ (respectively $\mathcal{C}_{H',c'}$) and the bijective map $\Psi|_{\mathcal{C}_{H,c}} : \mathcal{C}_{H,c} \longrightarrow \mathcal{C}_{H',c'}$ is Φ-equivariant.

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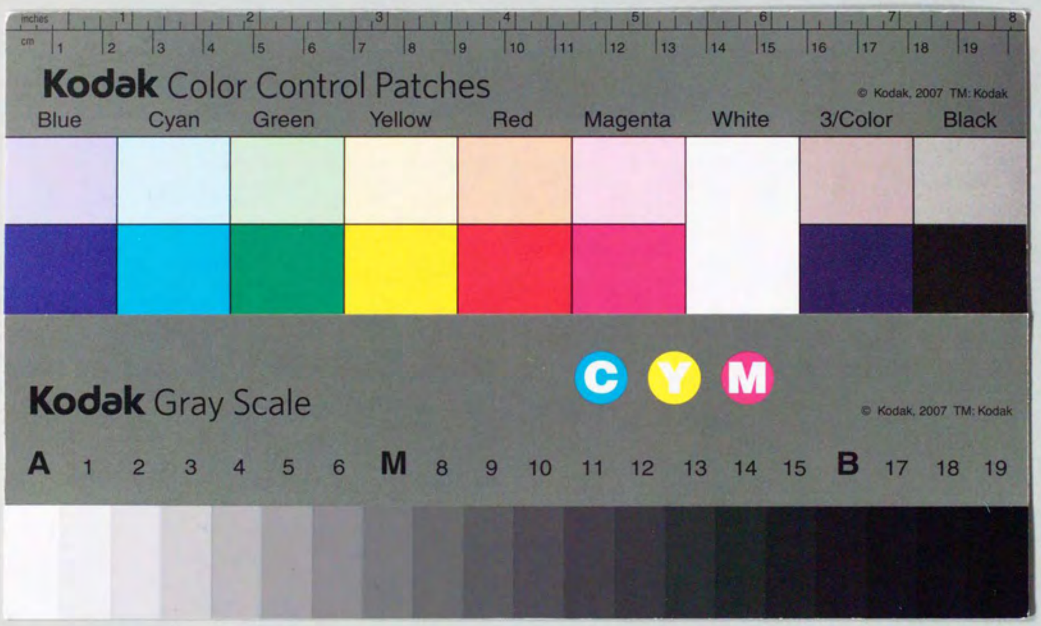
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