博士論文

論文題目　Modular transformation properties of characters of the $\mathcal{N} = 2$ superconformal algebra

（$\mathcal{N} = 2$ 超共形代数の指標のモジュラー変換性）

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MODULAR TRANSFORMATION PROPERTIES OF CHARACTERS OF THE $\mathcal{N}=2$ SUPERCONFORMAL ALGEBRA

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Abstract. We compute the modular transformation formula of the characters for certain families of (finitely or uncountably many) simple modules over the simple $\mathcal{N}=2$ vertex operator superalgebra of central charge $c_{p,p'} = 3\left(1-\frac{2p'}{p}\right)$, where (p,p') is a pair of coprime positive integers such that $p \geq 2$. When $p'=1$, the formula coincides with that of the $\mathcal{N}=2$ unitary minimal series. In addition, we study the corresponding "modular S-matrix" which is no longer a matrix if $p' \geq 2$.

1. Introduction

One of the most remarkable features in representation theory of vertex operator superalgebras (VOSAs) is the modular invariance property of the characters of their modules. The property is firstly established by Y. Zhu in [Zhu96] for rational, C_2-cofinite vertex operator algebras (VOAs) under some natural conditions. See [Miy04] and [VE13] for the generalization to irrational cases and to super cases, respectively. We note that all these previous works are based on the C_2-cofiniteness assumption which is introduced in [Zhu96] and is deeply related to the finite dimensionality of the space of conformal blocks (in the sense of [VE13, Definition 4.2]). See [Zhu96], [Miy04], and [VE13] for more details.

In the present paper, we construct a "modular invariant" family of simple highest weight modules over the simple VOSA $L_{c_{p,p'}}$ associated with the $\mathcal{N}=2$ superconformal algebra of central charge

$$c_{p,p'} := 3\left(1-\frac{2p'}{p}\right).$$

Here (p,p') is a pair of coprime positive integers such that $p \geq 2$. We should note that the VOSA $L_{c_{p,p'}}$ is C_2-cofinite if and only if $p'=1$ (see Corollary 2.3). When the simple $\mathcal{N}=2$ VOSA $L_{c_{p,p'}}$ is not C_2-cofinite, the dimension of the space of conformal blocks is not known to be finite. In fact, the space spanned by the character functions of simple $\frac{1}{2}\mathbb{Z}_{\geq 0}$-gradable $L_{c_{p,p'}}$-modules is not finite dimensional (see Remark 3.6). Therefore, we explain the precise meaning of the "modular invariance" in our case below.

For each pair (p,p'), the modules in the above family are divided into two classes, *atypical* and *typical* ones. Roughly speaking, an atypical (resp. typical) module admits a Bernstein-Gel'fand-Gel'fand-type resolution which is composed of spectral flow twisted generalized Verma modules (resp. spectral flow twisted Verma modules). The atypical modules are indexed by a certain finite set $\mathscr{S}_{p,p'}$ (see Definition 4.9 for the definition) and the typical ones are indexed by $K_{p,p'} \times \mathbb{R}$, where $K_{p,p'}$ is a finite set which parameterizes the Belavin-Polyakov-Zamolodchikov (BPZ) minimal series of central charge $1-\frac{6(p-p')^2}{pp'}$. Note that the sizes of these finite sets are given by

$$|\mathscr{S}_{p,p'}| = \frac{p(p-1)}{2}, \ |K_{p,p'}| = \frac{(p-1)(p'-1)}{2}.$$

To describle the whole picture of the modular invariance in super cases, we need to consider the following 4-types of formal characters:

(1) $\mathsf{ch}^{0,0}(M)$: Neveu-Schwarz twisted character,
(2) $\mathsf{ch}^{0,1}(M)$: Neveu-Schwarz twisted supercharacter,
(3) $\mathsf{ch}^{1,0}(M)$: Ramond twisted character,
(4) $\mathsf{ch}^{1,1}(M)$: Ramond twisted supercharacter.

For $\lambda \in \mathscr{S}_{p,p'}$ and $(\mu, x) \in K_{p,p'} \times \mathbb{R}$, we denote the character functions of the corresponding highest weight modules by

$$\mathbf{A}^{\varepsilon,\varepsilon'}_{\lambda}(\tau,u,t) = \mathsf{ch}^{\varepsilon,\varepsilon'}(\mathcal{L}_{\lambda}), \ \mathbf{T}^{\varepsilon,\varepsilon'}_{\mu,x}(\tau,u,t) = \mathsf{ch}^{\varepsilon,\varepsilon'}(\mathcal{L}_{\mu,x}).$$

Here $(\tau,u,t) \in \mathbb{H}\times\mathbb{C}\times\mathbb{C}$ is a certain coordinate of the Cartan subalgebra of the $\mathcal{N}=2$ superconformal algebra. Then our "modular invariance" in this paper means the establishment of the following modular S-transformation

$$\begin{aligned}
\mathbf{A}^{\varepsilon,\varepsilon'}_{\lambda}\left(-\frac{1}{\tau}, \frac{u}{\tau}, t-\frac{u^2}{6\tau}\right) &= \sum_{\lambda'\in\mathscr{S}_{p,p'}} S^{aa,(\varepsilon,\varepsilon')}_{\lambda,\lambda'} \mathbf{A}^{\varepsilon',\varepsilon}_{\lambda'}(\tau,u,t) \\
&\quad + \sum_{\mu''\in K_{p,p'}} \int_{\mathbb{R}} \mathrm{d}x''\, S^{at,(\varepsilon,\varepsilon')}_{\lambda,(\mu'',x'')} \mathbf{T}^{\varepsilon',\varepsilon}_{\mu'',x''}(\tau,u,t), \\
\mathbf{T}^{\varepsilon,\varepsilon'}_{\mu,x}\left(-\frac{1}{\tau}, \frac{u}{\tau}, t-\frac{u^2}{6\tau}\right) &= \sum_{\mu'\in K_{p,p'}} \int_{\mathbb{R}} \mathrm{d}x'\, S^{tt,(\varepsilon,\varepsilon')}_{(\mu,x),(\mu',x')} \mathbf{T}^{\varepsilon',\varepsilon}_{\mu',x'}(\tau,u,t)
\end{aligned}$$

and the (rather trivial) modular T-transformation. See §3.3 and §4.4 for the details.

At last, we should give some remarks on relationships between our result and the relevant previous works:

- When $p'=1$, the set $K_{p,1}$ is empty and the index set $\mathscr{S}_{p,1}$ bijectively corresponds to the $\mathcal{N}=2$ unitary minimal series of central charge

$$c_{p,1} = 3\left(1-\frac{2}{p}\right).$$

  Therefore, the space

$$\mathsf{span}_{\mathbb{C}}\{\, \mathbf{A}^{\varepsilon,\varepsilon'}_{\lambda} \mid \varepsilon,\varepsilon' \in \{0,1\},\, \lambda\in\mathscr{S}_{p,1}\,\}$$

  forms a finite dimensional $\mathsf{SL}(2,\mathbb{Z})$-invariant space. Then our result recovers the modular transformation of the $\mathcal{N}=2$ minimal unitary characters which is firstly obtained by F. Ravanini and S.-K. Yang in [RY87] and proved by V. G. Kac and M. Wakimoto in [KW94a]. See Appendix D for the details.
- In [STT05], A. M. Semikhatov, A. Taormina, and I. Yu. Tipunin studied the modular property of the characters of simple highest weight modules over the $\widehat{\mathfrak{sl}}_{2|1}$ of level $\widetilde{k} = -1+\frac{p'}{p}$ for $p'\geq 2$. By the quantum BRST reduction (due to the result of T. Arakawa in [Ara05]), the corresponding central charge is given by

$$c = -3(2\widetilde{k}+1) = c_{p,p'}.$$

We can verify that the "admissible $\widehat{\mathfrak{sl}}_{2|1}$ representations" considered in [STT05, §B.3] correspond to principal admissible $\widehat{\mathfrak{sl}}_{2|1}$-modules with spectral flow twists (see [STT05, §B.3] and [KW16, §2]). It is worth noting that the atypical modules over the $\mathcal{N}=2$ superconformal algebra in this paper is obtained from principal admissible $\widehat{\mathfrak{sl}}_{2|1}$-modules by the quantum BRST reduction (see Appendix D.2). We note that the character formula for principal admissible $\widehat{\mathfrak{sl}}_{2|1}$-modules is proved in [GK15] (see [KW16, Lemma 2.1] for the detail).

We should mention that the reduced version of the formula in [STT05, Theorem 4.1] is presented in [Gho03, (4.2.75)]. Then our result gives the expression of [Gho03, (4.2.75)] purely in terms of the character functions.

- By the Kazama-Suzuki coset construction (see [KS89], [HT91], [FST98], and [Sat16]), the modular invariant family of $L_{c_{p,p'}}$-modules in this paper can be regarded as the counterpart of that of $\widehat{\mathfrak{sl}}_2$-modules at the Kac-Wakimoto admissible levels $k=-2+\frac{p}{p'}$ studied in [CR13a]. It is worth noting that the formal characters of "typical" $\widehat{\mathfrak{sl}}_2$-modules which are parameterized by $K_{p,p'}\times\mathbb{R}$ are not convergent to functions defined in the upper half plane $\mathbb{H}$. See Proposition B.2 for the details.

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2. Preliminaries

2.1. **Notation.** Set $(q,z):=(e^{2\pi i\tau},e^{2\pi iu})$, where $(\tau,u)\in\mathbb{H}\times\mathbb{C}$. Then we denote the eta function by

$$\eta(\tau):=q^{\frac{1}{24}}\prod_{n>0}(1-q^n)$$

and the theta functions by

$$\vartheta_{\varepsilon,\varepsilon'}(u;\tau):=z^{\frac{\varepsilon}{2}}q^{\frac{\varepsilon}{8}}\prod_{n>0}(1-q^n)\left(1+(-1)^{\varepsilon'}zq^{n-\frac{1-\varepsilon}{2}}\right)\left(1+(-1)^{\varepsilon'}z^{-1}q^{n-\frac{1+\varepsilon}{2}}\right)$$

for $\varepsilon,\varepsilon'\in\{0,1\}$. We note that

$$\vartheta_{\varepsilon,\varepsilon'}(u;\tau)=i^{-\varepsilon\varepsilon'}\sum_{n\in\mathbb{Z}}e^{\pi i\tau(n+\frac{\varepsilon}{2})^2+2\pi i(u+\frac{\varepsilon'}{2})(n+\frac{\varepsilon}{2})}=z^{\frac{\varepsilon}{2}}q^{\frac{\varepsilon}{8}}\vartheta_{0,0}\left(u+\frac{\varepsilon}{2}\tau+\frac{\varepsilon'}{2};\tau\right).$$

By abuse of notation, we regard $\eta(\tau)$ (resp. $\vartheta_{\varepsilon,\varepsilon'}(u;\tau)$) as a holomorphic function on $\mathbb{H}$ (resp. $\mathbb{C}\times\mathbb{H}$) and also as a convergent series in q (resp. z and q).

2.2. **The $\mathcal{N}=2$ superconformal algebra.** The *Neveu-Schwarz sector of the $\mathcal{N}=2$ superconformal algebra* is the Lie superalgebra

$$\mathfrak{ns}_2=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n\oplus\bigoplus_{n\in\mathbb{Z}}\mathbb{C}J_n\oplus\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^+\oplus\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^-\oplus\mathbb{C}C$$

whose $\mathbb{Z}_2$-grading is given by

$$(\mathfrak{ns}_2)^{\bar{0}}=\bigoplus_{n\in\mathbb{Z}}\mathbb{C}L_n\oplus\bigoplus_{n\in\mathbb{Z}}\mathbb{C}J_n\oplus\mathbb{C}C,\quad(\mathfrak{ns}_2)^{\bar{1}}=\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^+\oplus\bigoplus_{r\in\mathbb{Z}+\frac{1}{2}}\mathbb{C}G_r^-$$

with the following (anti-)commutation relations:

$$\begin{aligned}
&[L_n,L_m]=(n-m)L_{n+m}+\frac{1}{12}(n^3-n)C\delta_{n+m,0},\\
&[L_n,J_m]=-mJ_{n+m},\ [L_n,G_r^{\pm}]=\left(\frac{n}{2}-r\right)G_{n+r}^{\pm},\\
&[J_n,J_m]=\frac{n}{3}C\delta_{n+m,0},\quad[J_n,G_r^{\pm}]=\pm G_{n+r}^{\pm},\\
&[G_r^+,G_s^-]=2L_{r+s}+(r-s)J_{r+s}+\frac{1}{3}\left(r^2-\frac{1}{4}\right)C\delta_{r+s,0},\\
&[G_r^+,G_s^+]=[G_r^-,G_s^-]=0,\quad[\mathfrak{ns}_2,C]=\{0\},
\end{aligned}$$

for $n,m\in\mathbb{Z}$ and $r,s\in\mathbb{Z}+\frac{1}{2}$.

2.3. **The $\mathcal{N}=2$ vertex operator superalgebra.** Let $\mathfrak{ns}_2=(\mathfrak{ns}_2)_+\oplus(\mathfrak{ns}_2)_0\oplus(\mathfrak{ns}_2)_-$ be the triangular decomposition of $\mathfrak{ns}_2$, where

$$\begin{aligned}
(\mathfrak{ns}_2)_+&:=\bigoplus_{n>0}\mathbb{C}L_n\oplus\bigoplus_{n>0}\mathbb{C}J_n\oplus\bigoplus_{r>0}\mathbb{C}G_r^+\oplus\bigoplus_{r>0}\mathbb{C}G_r^-,\\
(\mathfrak{ns}_2)_-&:=\bigoplus_{n<0}\mathbb{C}L_n\oplus\bigoplus_{n<0}\mathbb{C}J_n\oplus\bigoplus_{r<0}\mathbb{C}G_r^+\oplus\bigoplus_{r<0}\mathbb{C}G_r^-,\\
(\mathfrak{ns}_2)_0&:=\mathbb{C}L_0\oplus\mathbb{C}J_0\oplus\mathbb{C}C,
\end{aligned}$$

and set $(\mathfrak{ns}_2)_{\geq0}:=(\mathfrak{ns}_2)_+\oplus(\mathfrak{ns}_2)_0$.

For $(h,j,c)\in\mathbb{C}^3$, let $\mathbb{C}_{h,j,c}$ be the $(1|0)$-dimensional $(\mathfrak{ns}_2)_{\geq0}$-module defined by $(\mathfrak{ns}_2)_+.1:=\{0\}$, $L_0.1:=h$, $J_0.1:=j$ and $C.1:=c$. Then the induced module $\mathcal{M}_{h,j,c}:=\mathrm{Ind}_{(\mathfrak{ns}_2)_{\geq0}}^{\mathfrak{ns}_2}\mathbb{C}_{h,j,c}$ is called the Verma module of $\mathfrak{ns}_2$. Denote by $\mathcal{L}_{h,j,c}$ the simple quotient $\mathfrak{ns}_2$-module of $\mathcal{M}_{h,j,c}$.

We write $V_c=V_c(\mathfrak{ns}_2)\cong\mathcal{M}_{0,0,c}/\big(U(\mathfrak{ns}_2)G^+_{-\frac{1}{2}}|0,0,c\rangle+U(\mathfrak{ns}_2)G^-_{-\frac{1}{2}}|0,0,c\rangle\big)$ for the universal $\mathcal{N}=2$ VOSA, where $|0,0,c\rangle$ is the highest weight vector of $\mathcal{M}_{0,0,c}$. When $c\neq0$, we also write $L_c=L_c(\mathfrak{ns}_2)\cong\mathcal{L}_{0,0,c}$ for its simple quotient VOSA. See [Sat16, §2] for the details.

2.4. **Classification of simple modules.** Let $1\leq r\leq p-1$ and $0\leq s\leq p'-1$. We set $\mathcal{L}_{r,s;\lambda}:=\mathcal{L}_{\Delta_{r,s}-a\lambda^2,2a\lambda,c_{p,p'}}$, where $a:=\frac{p'}{p}$ and $\Delta_{r,s}:=\frac{(ar-s)^2-a^2}{4a}$. We also set $\lambda_{r,s}:=\frac{r-1}{2}-\frac{s}{2a}$. Note that $\Delta_{r,s}=a\lambda_{r,s}(\lambda_{r,s}+1)$.

The following classification is obtained by D. Adamovič via the Kazama-Suzuki coset construction.

Theorem 2.1 ([Ada99, Theorem 7.1 and 7.2])**.** The complete representatives of the isomorphism classes of simple $\frac{1}{2}\mathbb{Z}_{\geq0}$-gradable $L_{c_{p,p'}}$-modules are given as follows:

$$\left\{\mathcal{L}_{r,0;\lambda_{r,0}-\theta}\mid 1\leq r\leq p-1,0\leq\theta\leq r-1,\theta\in\mathbb{Z}\right\}\sqcup\{\mathcal{L}_{r,s;\lambda}\mid(r,s)\in K_{p,p'},\lambda\in\mathbb{C}\},$$

where $K_{p,p'} := \{(m,n) \in \mathbb{Z}^2 \mid 1 \leq m \leq p-1, 1 \leq n \leq p'-1, mp'+np \leq pp'\}$.

Remark 2.2. The index set $K_{p,p'}$ parameterizes the BPZ minimal series of central charge $1 - \frac{6(p-p')^2}{pp'}$ and we have $|K_{p,p'}| = \frac{(p-1)(p'-1)}{2}$.

Corollary 2.3. Let $c \in \mathbb{C} \setminus \{0\}$. Then the simple VOSA L_c is C_2-cofinite if and only if $c = c_{p,1}$ for some $p \in \mathbb{Z}_{\geq 3}$.

Proof. First, the 'if' part follows from the regularity of the VOSA $L_{c_{p,1}}$ proved in [Ada01, Theorem 8.1] and the super-analog of [Li99, Theorem 3.8].

Next, we verify the 'only if' part. We may assume that $c = c_{p,p'}$ for some pair of coprime integers $(p,p') \in \mathbb{Z}_{\geq 2} \times \mathbb{Z}_{\geq 1}$. In fact, otherwise, the simple quotient L_c is isomorphic to the non C_2-cofinite VOSA V_c by [GK07, Corollary 9.1.5 (ii)]. Here we assume that $p' \neq 1$. By the previous theorem and [KW94b, Theorem 1.3], it follows that the Zhu algebra of $L_{c_{p,p'}}$ is infinite-dimensional. Then the infinite-dimensionality of (the even part of) the C_2-algebra of $L_{c_{p,p'}}$ follows from a slight generalization of [ALY14, Proposition 3.3] to the super case (see also [AM11, Introduction]). This completes the proof of the 'only if' part $\square$

In this paper, we introduce the notion of atypical and typical modules as follows.

Definition 2.4. We call a simple $L_{c_{p,p'}}$-module $\mathcal{L}_{r,s;\lambda}$ *typical* if $(r,s) \in K_{p,p'}$ and $\lambda \in \mathbb{C} \setminus S_{r,s}$, where $S_{r,s} := \{\lambda_{r,s}, \lambda_{p-r,p'-s}\} + \mathbb{Z}$. Otherwise, we call $\mathcal{L}_{r,s;\lambda}$ *atypical*.

2.5. **Several formal characters.** Let M be a weight $\big(\mathfrak{ns}_2, (\mathfrak{ns}_2)_0\big)$-module of central charge c, *i.e.* $(\mathfrak{ns}_2)_0$ acts semi-simply on M and C acts as the scalar c. Then the formal characters of M is defined by

$$\mathsf{ch}^{0,0}(M) := q^{-\frac{c}{24}} \sum_{\lambda \in (\mathfrak{ns}_2)_0^*} (\mathsf{dim}\, M_\lambda)\, e^\lambda,$$

$$\mathsf{ch}^{0,1}(M) := q^{-\frac{c}{24}} \sum_{\lambda \in (\mathfrak{ns}_2)_0^*} (\mathsf{sdim}\, M_\lambda)\, e^\lambda,$$

where $M_\lambda := \{v \in M \,|\, X.v = \langle \lambda, X\rangle v \text{ for any } X \in (\mathfrak{ns}_2)_0\}$. We set $(q,z,w) := \big(e^{L_0^*}, e^{J_0^*}, e^{C^*}\big)$, where (L_0^*, J_0^*, C^*) is the dual basis of $(\mathfrak{ns}_2)_0^*$ with respect to the basis (L_0, J_0, C). In what follows, we write $\mathsf{ch}^{0,\varepsilon}(M)(q,z,w) := \mathsf{ch}^{0,\varepsilon}(M)$.

The following lemma is easily verified (see [Sat16, Lemma 5.2]).

Lemma 2.5. For the spectral flow twisted module M^θ (see Appendix A for the definition) and $\varepsilon \in \{0,1\}$, we have

$$\mathsf{ch}^{0,\varepsilon}(M^\theta)(q,z,w) = q^{\frac{c\theta^2}{6}} z^{\frac{c\theta}{3}}\, \mathsf{ch}^{0,\varepsilon}(M)(q, zq^\theta, w).$$

Now we introduce the "half-twisted" characters as follows.

Definition 2.6. For $\varepsilon \in \{0,1\}$, the twisted character of M is deifined by

$$\mathsf{ch}^{1,\varepsilon}(M)(q,z,w) := q^{\frac{c}{24}} z^{-\frac{c}{6}}\, \mathsf{ch}^{0,\varepsilon}(M)(q, zq^{-\frac{1}{2}}, w).$$

3. Typical modular transformation law

In this section, we derive the modular transformation formula for the character functions of typical modules. Throughout this section, we assume that $p' \geq 2$ and $\mathcal{L}_{r,s;\lambda}$ is typical.

3.1. **Character formula for typical modules.**

Theorem 3.1. We have an equality of formal series

$$\mathsf{ch}^{\varepsilon,\varepsilon'}(\mathcal{L}_{r,s;\lambda})(q,z,w) = (-1)^{\varepsilon\varepsilon'} q^{-a(\lambda+\frac{\varepsilon}{2})^2} z^{2a(\lambda+\frac{\varepsilon}{2})} w^{3(1-2a)} \frac{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}{\eta(\tau)^2} \chi_{r,s}(\tau),$$

where

$$\chi_{r,s}(\tau) := \frac{1}{\eta(\tau)} \sum_{n\in\mathbb{Z}} \left(q^{pp'\left(n+\frac{rp'-sp}{2pp'}\right)^2} - q^{pp'\left(n+\frac{-rp'-sp}{2pp'}\right)^2} \right).$$

Proof. By [Sat16, Proposition 7.1], we have $\Omega_\lambda^+(L_{\Delta_{r,s},\lambda,-2+\frac{p}{p'}}) \cong \mathcal{L}_{r,s;\lambda}$ for any $\lambda \in \mathbb{C}$, where $L_{\Delta_{r,s},\lambda,-2+\frac{p}{p'}}$ is the simple relaxed highest weight module over $\widehat{\mathfrak{sl}}_2$ (see Appendix B for the definition) and Ω_λ^+ is the functor defined in [Sat16, Definition 4.1]. When $\lambda \in \mathbb{C} \setminus S_{r,s}$, the character formula for $L_{\Delta_{r,s},\lambda,-2+\frac{p}{p'}}$ is obtained by [CR13a, Corollary 5] (see also Proposition B.2). Therefore we get the required character formula for $\Omega_\lambda^+(L_{\Delta_{r,s},\lambda,-2+\frac{p}{p'}})$ by [Sat16, Theorem 7.8]. By some computations, we also get the characters for $(\varepsilon,\varepsilon') \neq (0,0)$. $\square$

Remark 3.2. (1) We can prove Theorem 3.1 by the character formula for certain simple highest weight modules over $\widehat{\mathfrak{sl}}_{2|1}$ and the quantized Drinfel'd-Sokolov reduction. The detail is discussed in [KS].

(2) The convergent series $\chi_{r,s}(\tau)$ is the normalized character of the corresponding BPZ minimal series. The modular transformation is given as follows:

$$\chi_{r,s}\left(-\frac{1}{\tau}\right) = \sum_{(r',s')\in K_{p,p'}} S_{(r,s),(r',s')} \chi_{r',s'}(\tau),$$

$$\chi_{r,s}(\tau+1) = e^{2\pi i\left(\frac{(rp'-sp)^2}{4pp'} - \frac{1}{24}\right)} \chi_{r,s}(\tau),$$

where

$$S_{(r,s),(r',s')} := \sqrt{\frac{8}{pp'}} (-1)^{(r+s)(r'+s')} \sin\left(\frac{\pi(p-p')rr'}{p}\right) \sin\left(\frac{\pi(p-p')ss'}{p'}\right).$$

See [IK11, Proposition 6.3] for the details.

The next corollary immediately follows from the previous character formula.

Corollary 3.3. For any $\theta \in \mathbb{Z}$, the spectral flow twisted module $(\mathcal{L}_{r,s;\lambda})^\theta$ is isomorphic to another typical module $\mathcal{L}_{r,s;\lambda-\theta}$. In particular, the set of typical modules is closed under the spectral flow.

Now we introduce the corresponding character function.

Definition 3.4. Let $(r,s) \in K_{p,p'}$ and $x \in \mathbb{C}$. Then a *typical character function* is defined as the following holomorphic function on $\mathbb{H} \times \mathbb{C} \times \mathbb{C}$:

$$\mathbf{T}^{\varepsilon,\varepsilon'}_{r,s;x}(\tau,u,t) := (-1)^{\varepsilon\varepsilon'} q^{a\left(x-\frac{i\varepsilon}{2}\right)^2} z^{2ia\left(x-\frac{i\varepsilon}{2}\right)} w^{3(1-2a)} \frac{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}{\eta(\tau)^2} \chi_{r,s}(\tau),$$

where $(q,z,w) = (e^{2\pi i\tau}, e^{2\pi iu}, e^{2\pi it})$.

The next lemma follows from Definition 2.4, Theorem 3.1, and Definition 3.4.

Lemma 3.5. As a function in $(q,z,w)=(e^{2\pi i\tau},e^{2\pi iu},e^{2\pi it})$,

$$\mathbf{T}^{\varepsilon,\varepsilon'}_{r,s;x}(\tau,u,t)=\mathsf{ch}^{\varepsilon,\varepsilon'}(\mathcal{L}_{r,s;ix})(q,z,w)$$

holds if and only if $x\in\mathbb{R}+i(\mathbb{R}\setminus S_{r,s})$.

Remark 3.6. For an appropriate uncountable subset U of $\mathbb{R}+i(\mathbb{R}\setminus S_{r,s})$, the family of holomorphic functions $\{\,\mathbf{T}^{\varepsilon,\varepsilon'}_{r,s;x}\mid x\in U\,\}$ is linearly independent. As a corollary, the space spanned by the character functions of simple $\frac{1}{2}\mathbb{Z}_{\geq 0}$-gradable $L_{c_{p,p'}}$-modules is not finite dimensional.

3.2. **Modular transformation of typical characters.**

Theorem 3.7. For $(r,s)\in K_{p,p'}$ and $x\in\mathbb{C}$, the following hold:

$$\mathbf{T}^{\varepsilon,\varepsilon'}_{r,s;x}\left(-\frac{1}{\tau},\frac{u}{\tau},t-\frac{u^2}{6\tau}\right)=\sum_{(r',s')\in K_{p,p'}}\int_{\mathbb{R}}\mathrm{d}x'\,S^{tt,(\varepsilon,\varepsilon')}_{(r,s;x),(r',s';x')}\mathbf{T}^{\varepsilon',\varepsilon}_{r',s';x'}(\tau,u,t),\tag{3.1}$$

$$\mathbf{T}^{\varepsilon,\varepsilon'}_{r,s;x}(\tau+1,u,t)=e^{2\pi i\left(a\left(x-\frac{i\varepsilon}{2}\right)^2+\frac{(rp'-sp)^2}{4pp'}-\frac{1-\varepsilon}{8}\right)}\mathbf{T}^{\varepsilon,\varepsilon\varepsilon'+(1-\varepsilon)(1-\varepsilon')}_{r,s;x}(\tau,u,t),\tag{3.2}$$

where

$$S^{tt,(\varepsilon,\varepsilon')}_{(r,s;x),(r',s';x')}:=i^{-\varepsilon\varepsilon'}S_{(r,s),(r',s')}\sqrt{2a}e^{-4\pi ia\left(x-\frac{i\varepsilon}{2}\right)\left(x'-\frac{i\varepsilon'}{2}\right)}.$$

Proof. The S-transformaltion (3.1) essentially follows from the Gaussian integral

$$\frac{e^{\frac{2\pi iau^2}{\tau}}}{\sqrt{2a}}\frac{\tilde{q}^{ax^2}\tilde{z}^{2iax}}{\sqrt{-i\tau}}=\int_{\mathbb{R}}dx'\,q^{a(x')^2}z^{2iax'}e^{-4\pi iaxx'},$$

where $(\tilde{q},\tilde{z}):=(e^{-\frac{2\pi i}{\tau}},e^{\frac{2\pi iu}{\tau}})$. The T-transformation (3.2) is clear. $\square$

3.3. **Properties of typical S-data.** The data $S^{tt,(\varepsilon,\varepsilon')}_{(r,s;x),(r',s';x')}$ in the previous subsection satisfies the following conditions.

Proposition 3.8. For $(r,s;x),(r',s';x')\in K_{p,p'}\times\mathbb{R}$, we have

$$S^{tt,(\varepsilon,\varepsilon')}_{(r,s;x),(r',s';x')}=S^{tt,(\varepsilon',\varepsilon)}_{(r',s';x'),(r,s;x)}$$

and

$$\sum_{(r'',s'')\in K_{p,p'}}\int_{\mathbb{R}}\mathrm{d}x''\,S^{tt,(\varepsilon,\varepsilon')}_{(r,s;x),(r'',s'';x'')}\overline{S^{tt,(\varepsilon',\varepsilon)}_{(r'',s'';x''),(r',s';x')}}$$
$$=e^{-4\pi a\varepsilon'\left(x-\frac{i\varepsilon}{2}\right)}\delta_{r,r'}\delta_{s,s'}\delta(x'-x+i\varepsilon),$$

where $\delta(z)$ is the delta distribution.

Proof. The former is obvious. The latter follows from the fact that

$$\sum_{(r'',s'')\in K_{p,p'}}S_{(r,s),(r'',s'')}S_{(r'',s''),(r',s')}=\delta_{r,r'}\delta_{s,s'}\tag{3.3}$$

and

$$2a\int_{\mathbb{R}}\mathrm{d}x''\,e^{-4\pi ia\left(x-\frac{i\varepsilon}{2}\right)\left(x''-\frac{i\varepsilon'}{2}\right)}e^{4\pi ia\left(x''+\frac{i\varepsilon'}{2}\right)\left(x'+\frac{i\varepsilon}{2}\right)}$$
$$=e^{-2\pi a\varepsilon'(x+x')}\times 2a\int_{\mathbb{R}}e^{2\pi i(x'-x+i\varepsilon)\times 2ax''}\mathrm{d}x''$$
$$=e^{-4\pi a\varepsilon'\left(x-\frac{i\varepsilon}{2}\right)}\delta(x'-x+i\varepsilon).$$

For example, see [IK11, (9.10)] for the proof of the equality (3.3). □

4. Atypical modular transformation law

In this section, we present the modular transformation formula for the character functions of atypical modules (Theorem 4.12). Its proof is given in Appendix C. Throughout this section, we assume that $p' \geq 1$.

4.1. **Parameterization of atypical modules.** First, we assign a triple of certain integers to each atypical module as follows.

Lemma 4.1. Let M be a simple highest weight $\mathfrak{ns}_2$-module. Then the following are equivalent:

(1) M is an atypical module.
(2) There exist $1 \leq r \leq p-1$, $0 \leq s \leq p'-1$, and $\theta \in \mathbb{Z}$ such that $M \cong \mathcal{L}(r,s)^\theta$, where $\mathcal{L}(r,s) := \mathcal{L}_{r,s;\lambda_{r,s}} = \mathcal{L}_{a\lambda_{r,s}, 2a\lambda_{r,s}, c_{p,p'}}$.

Proof. It immediately follows from Corollary A.3. □

Remark 4.2. Note that we have $\Omega^+_{\lambda_{r,s}}\big(L(r,s)\big) \cong \mathcal{L}(r,s)$, where $L(r,s)$ is the Kac-Wakimoto admissible $\widehat{\mathfrak{sl}}_2$-module of highest weight $(k_{p,p'} - 2\lambda_{r,s})\Lambda_0 + 2\lambda_{r,s}\Lambda_1$. Here $k_{p,p'} := -2 + \frac{p}{p'}$ and Λ_i stands for the i-th fundamental weight of $\widehat{\mathfrak{sl}}_2$.

Next, we compute the set of equivalence class of all the modules of the form $\mathcal{L}(r,s)^\theta$. When $p'=1$, we have $\mathcal{L}(r,0)^r \cong \mathcal{L}(p-r,0)$ and the periodic isomorphism $\mathcal{L}(r,0)^p \cong \mathcal{L}(r,0)$ for any $1 \leq r \leq p-1$. Therefore, as well known, there are only finitely many inequivalent simple $\frac{1}{2}\mathbb{Z}_{\geq 0}$-gradable $L_{c_{p,1}}$-modules. On the other hand, when $p' \neq 1$, we obtain by Corollary A.3 the following identification:

Lemma 4.3. Suppose that $p' \geq 2$. Then $\mathcal{L}(r,s)^\theta$ is isomorphic to $\mathcal{L}(r',s')$ if and only if one of the following holds:

(1) $(r,s;\theta) = (r,s;0)$ and $(r',s') = (r,s)$,
(2) $(r,s;\theta) = (r,0;r)$ and $(r',s') = (p-r, p'-1)$,
(3) $(r,s;\theta) = (r,p'-1;-p+r)$ and $(r',s') = (p-r,0)$.

As a consequence, if $p' \geq 2$, there exist infinitely many inequivalent atypical modules.

4.2. **Character formula for atypical modules.** In this subsection, we compute the character formula for atypical modules and introduce the corresponding meromorphic functions.

Let (r,s) be a pair of integers such that $1 \leq r \leq p-1$ and $0 \leq s \leq p'-1$. We consider the meromorphic function

$$\Psi_{p,p';r,s}(u;\tau) := \sum_{n \in \mathbb{Z}} \left(\frac{q^{pp'\left(n + \frac{rp'-sp}{2pp'}\right)^2}}{1 - zq^{pn}} - \frac{q^{pp'\left(n + \frac{-rp'-sp}{2pp'}\right)^2}}{1 - zq^{pn-r}} \right)$$

on $\mathbb{C} \times \mathbb{H}$ whose divisor is

$$D_{p,r} := \{(m + pn\tau, \tau),\ (m + (pn+r)\tau, \tau) \,|\, \tau \in \mathbb{H},\ m,n \in \mathbb{Z}\}.$$

Then we define the formal series $\Phi_{p,p';r,s}(y,q)$ as the expansion of $\Psi_{p,p';r,s}(v;\tau)$ with respect to the two variables $(z,q) = (e^{2\pi i u}, e^{2\pi i \tau})$ in the region

$$\mathbb{A} := \left\{(u;\tau) \in \mathbb{C} \times \mathbb{H} \,\middle|\, |q| < |z| < 1\right\} \subset (\mathbb{C} \times \mathbb{H}) \setminus D_{p,r}.$$

That is, the formal series $\Phi_{p,p';r,s}(z,q)$ is given by

$$\Phi_{p,p';r,s}(z,q)=\sum_{n,m\geq 0} z^m\varphi^{n,m}_{p,p';r,s}(q)-\sum_{n,m<0} z^m\varphi^{n,m}_{p,p';r,s}(q),$$

where

$$\varphi^{n,m}_{p,p';r,s}(q):=q^{pp'\left(n+\frac{rp'-sp}{2pp'}\right)^2+pnm}-q^{pp'\left(n+\frac{(p-r)p'+(p'-s)p}{2pp'}\right)^2+(pn+p-r)m}.$$

Theorem 4.4. For any $\theta\in\mathbb{Z}$, we have an equality of formal series

$$\begin{aligned}\mathsf{ch}^{\varepsilon,\varepsilon'}\big(\mathcal{L}(r,s)^{\theta}\big)(q,z,w)=&(-1)^{\varepsilon\varepsilon'}q^{-a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})^2}z^{2a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})}w^{3(1-2a)}\\&\times\frac{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}{\eta(\tau)^3}\Phi_{p,p';r,s}\left((-1)^{1-\varepsilon'}zq^{\theta+\frac{1-\varepsilon}{2}},q\right).\end{aligned}$$

Proof. By [Sat16, Theorem 7.13], we have

$$\mathsf{ch}^{0,0}\big(\mathcal{L}(r,s)\big)(q,z,w)=q^{-a\lambda_{r,s}^2}z^{2a\lambda_{r,s}}w^{3(1-2a)}\frac{\vartheta_{0,0}(u;\tau)}{\eta(\tau)^3}\Phi_{p,p';r,s}\left(-zq^{\frac{1}{2}},q\right).$$

By Lemma 2.5 and some computations, we also have the formula for $\theta\neq 0$ and $(\varepsilon,\varepsilon')\neq(0,0)$. □

Definition 4.5. An *atypical character function* is defined as the following meromorphic function on $\mathbb{H}\times\mathbb{C}\times\mathbb{C}$:

$$\begin{aligned}\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}(\tau,u,t):=&(-1)^{\varepsilon\varepsilon'}q^{-a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})^2}z^{2a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})}w^{3(1-2a)}\\&\times\frac{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}{\eta(\tau)^3}\Psi_{p,p';r,s}\left(u_{\varepsilon,\varepsilon'}+\theta\tau,\tau\right),\end{aligned}$$

where $u_{\varepsilon,\varepsilon'}:=u+\frac{1-\varepsilon}{2}\tau+\frac{1-\varepsilon'}{2}$.

The next lemma holds by definition.

Lemma 4.6. As a function in $(q,z,w)=(e^{2\pi i\tau},e^{2\pi iu},e^{2\pi it})$.

$$\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}(\tau,u,t)=\mathsf{ch}^{\varepsilon,\varepsilon'}\big(\mathcal{L}(r,s)^{\theta}\big)(q,z,w)$$

holds if $|q|<|z|\cdot|q|^{\theta+\frac{1-\varepsilon}{2}}<1$.

Example 4.7. In the trivial case, *i.e.* $(p,p';r,s)=(2,1;1,0)$, we have

$$\frac{\eta(\tau)^3}{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}=(-1)^{\varepsilon\varepsilon'}q^{-\frac{1}{2}(-\theta+\frac{\varepsilon}{2})^2}z^{-\theta+\frac{\varepsilon}{2}}\Psi_{2,1;1,0}\left(u_{\varepsilon,\varepsilon'}+\theta\tau;\tau\right).$$

In particular, when $\theta=0$ and $(\varepsilon,\varepsilon')=(1,1)$, it reproduces the Ramond denominator identity for $\widehat{\mathfrak{gl}}_{1|1}$ (see [KW94a, Example 4.1]):

$$\prod_{i>0}\frac{(1-q^i)^2}{(1-zq^{i-1})(1-z^{-1}q^i)}=\sum_{n\in\mathbb{Z}}\frac{(-1)^nq^{\frac{n(n+1)}{2}}}{1-zq^n},$$

where $|q|<|z|<1$.

4.3. **Discrete spectra.** In this subsection, we define the index set $\mathscr{S}_{p,p'}$ as a finite subset of $\{(r,s;\theta) \mid 1 \le r \le p-1, 0 \le s \le p'-1, \theta \in \mathbb{Z}\}$.

Let (k,ℓ) be the unique pair of integers such that $p' = kp + \ell$ and $1 \le \ell \le p-1$. Since the pair (p,p') is coprime, so is (p,ℓ). Then we obtain the following lemma by easy computations.

Lemma 4.8. For $0 \le m \le p-1$, let (r_m, s_m) be a unique pair of integers such that $0 \le r_m \le p-1$, $0 \le s_m \le \ell - 1$ and $m = \ell r_m - p s_m$. Let $\mathscr{S}^{(m)}_{p,p'}$ be the set of solutions $(r,s;\theta)$ of the equation

$$2m + 2p' - p = 2(r - 2\theta)p' - (2s+1)p \tag{4.1}$$

under the condition

$$1 \le r \le p-1,\ 0 \le s \le p'-1,\ \theta \in \mathbb{Z}.$$

Then we have

$$\mathscr{S}^{(m)}_{p,p'} = \left\{ (r,s;\theta) = (2\theta + 1 + r_m, k r_m + s_m; \theta) \,\middle|\, -\left\lfloor \frac{r_m}{2} \right\rfloor \le \theta \le \left\lfloor \frac{p - r_m}{2} \right\rfloor - 1 \right\}.$$

Definition 4.9. We call

$$\mathscr{S}_{p,p'} := \bigcup_{m=0}^{p-1} \mathscr{S}^{(m)}_{p,p'}$$

the set of *discrete spectra* of central charge $c = c_{p,p'}$.

Lemma 4.10. We have the following:

1. $|\mathscr{S}_{p,p'}| = \frac{p(p-1)}{2}$.
2. For $(r,s;\theta), (r',s';\theta') \in \mathscr{S}_{p,p'}$, the two modules $\mathcal{L}(r,s)^{\theta}$ and $\mathcal{L}(r',s')^{\theta'}$ are isomorphic if and only if $(r,s;\theta) = (r',s';\theta')$.

Proof. (1) is easily verified by calculation. (2) directly follows from Lemma 4.3. □

Example 4.11. We present some examples.

1. Since $(r_0, s_0) = (0,0)$, we have
$$\mathscr{S}^{(0)}_{p,p'} = \left\{ (2i-1, 0; i-1) \,\middle|\, 1 \le i \le \left\lfloor \frac{p}{2} \right\rfloor \right\}.$$
2. If $(p,p') = (p, kp+1)$, we have
$$\mathscr{S}^{(2u-1)}_{p,kp+1} = \left\{ \big(2i, (2u-1)k; i-u\big) \,\middle|\, 1 \le i \le \left\lfloor \frac{p-1}{2} \right\rfloor \right\},$$
$$\mathscr{S}^{(2v)}_{p,kp+1} = \left\{ (2i-1, 2vk; i-1-v) \,\middle|\, 1 \le i \le \left\lfloor \frac{p}{2} \right\rfloor \right\}$$
for $1 \le u \le \left\lfloor \frac{p}{2} \right\rfloor$ and $1 \le v \le \left\lfloor \frac{p-1}{2} \right\rfloor$. In particular, when $k = 0$, the spectra in $\mathscr{S}_{p,1}$ correspond to the $\mathcal{N} = 2$ unitary minimal series (see [Dör98] and [Sat16, Remark 7.12]).

4.4. **Modular transformation of atypical characters.** For $r, r', s, s', \theta, \theta' \in \mathbb{Z}$, we set

$$S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')} := i^{-\varepsilon\varepsilon'}(-1)^{(1-\varepsilon')s+(1-\varepsilon)s'}\frac{2}{p}\sin(\pi a r r')e^{\pi i a(r-2\theta-1+\varepsilon)(r'-2\theta'-1+\varepsilon')}.$$

In addition, if $(r', s') \in K_{p,p'}$, we also set

$$\begin{aligned} S^{at,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';x)} &:= i^{-\varepsilon\varepsilon'}(-1)^{r's+rs'}\frac{2}{p}\sin\bigl(\pi a r r'\bigr) \\ &\quad\times \frac{\sin\left(\frac{\pi s s'}{a}\right)e^{2\pi x} + \sin\bigl(2\pi\lambda_{r',(1+s)s'}\bigr)}{\cosh(2\pi x) - \cos(2\pi\lambda_{r',s'})} e^{-4\pi a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})\left(x-\frac{i\varepsilon'}{2}\right)} \end{aligned}$$

for $x \in \mathbb{R}$.

Now we state the main result of this paper.

Theorem 4.12. The following hold:

$$\begin{aligned} &\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}\left(-\frac{1}{\tau}, \frac{u}{\tau}, t - \frac{u^2}{6\tau}\right) \\ &= \sum_{(r',s';\theta')\in\mathscr{S}_{p,p'}} S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')}\mathbf{A}^{\varepsilon',\varepsilon}_{r',s';\theta'}(\tau,u,t) \\ &\quad + \sum_{(r'',s'')\in K_{p,p'}} \int_{\mathbb{R}} \mathrm{d}x''\, S^{at,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r'',s'';x'')}\mathbf{T}^{\varepsilon',\varepsilon}_{r'',s'';x''}(\tau,u,t), \\ &\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}(\tau+1,u,t) \\ &= e^{2\pi i\left[(ar-s-p')(\theta+\frac{1-\varepsilon}{2})-a(\theta+\frac{1-\varepsilon}{2})^2-\frac{1-\varepsilon}{8}\right]}\mathbf{A}^{\varepsilon,\varepsilon\varepsilon'+(1-\varepsilon)(1-\varepsilon')}_{r,s;\theta}(u;\tau) \end{aligned}$$

for $1 \le r \le p-1$, $0 \le s \le p'-1$, and $\theta \in \mathbb{Z}$.

See Appendix C for the proof of Theorem 4.12.

Example 4.13. When $(p, p') = (3, 2)$ and $(r, s; \theta) = (1, 0; 0)$, we have

$$\begin{aligned} \mathbf{A}^{\varepsilon,\varepsilon'}_{1,0;0}\left(-\frac{1}{\tau}, \frac{u}{\tau}, t-\frac{u^2}{6\tau}\right) = \frac{e^{\frac{\pi i}{6}\varepsilon\varepsilon'}}{\sqrt{3}}\Bigl(&\mathbf{A}^{\varepsilon',\varepsilon}_{1,0;0}(\tau,u,t) - e^{\frac{\pi i}{3}\varepsilon}\mathbf{A}^{\varepsilon',\varepsilon}_{1,1;-1}(\tau,u,t) \\ &- e^{\frac{2\pi i}{3}\varepsilon}\mathbf{A}^{\varepsilon',\varepsilon}_{2,0;0}(\tau,u,t) + \int_{\mathbb{R}} \frac{e^{-\frac{4\varepsilon\pi}{3}y}}{\cosh(2\pi y)}\mathbf{T}^{\varepsilon',\varepsilon}_{1,1;y}(\tau,u,t)\mathrm{d}y\Bigr). \end{aligned}$$

4.5. **Properties of atypical S-data.** In this subsection, we verify the symmetricity and the unitarity of the data $S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')}$.

Lemma 4.14. Let $r, r' \in \mathbb{Z}_{\ge 0}$ such that $r, r' \le p-1$ and $r - r'$ is even.

(1) We have
$$\sum_{j=0}^{p-1}\sin\bigl(\pi a r j\bigr)\sin\bigl(\pi a r' j\bigr) = \frac{p}{2}\delta_{r,r'}.$$

(2) Suppose that p is odd. Then we have
$$\sum_{j=1}^{\lfloor\frac{p-1}{2}\rfloor}\sin\bigl(\pi a r(2j)\bigr)\sin\bigl(\pi a r'(2j)\bigr) = \frac{p}{4}\delta_{r,r'}.$$

Proof. By using $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$ and straightforward calculation, we can verify both (1) and (2). $\square$

Lemma 4.15. Let $(r,s;\theta) \in \mathscr{S}^{(m)}_{p,p'}$ and $(r',s';\theta') \in \mathscr{S}^{(m')}_{p,p'}$. Then we have

$$\sum_{(r'',s'';\theta'')\in\mathscr{S}_{p,p'}} \sin(\pi arr'')\sin(\pi ar'r'')e^{\pi ia(r_m-r_{m'})(r''-2\theta'')} = \frac{p^2}{4}\delta_{r,r'}\delta_{s,s'}\delta_{\theta,\theta'}.$$

Proof. Denote the left hand side by S. It is rewritten as

$$S = \sum_{m''=0}^{p-1} e^{\pi ia(r_m-r_{m'})(r_{m''}+1)} \sum_{(r'',s'';\theta'')\in\mathscr{S}^{(m'')}_{p,p'}} \sin(\pi arr'')\sin(\pi ar'r'').$$

We set

$$S_{\mathrm{odd}} := \sum_{j=1}^{\lfloor \frac{p}{2}\rfloor} \sin\bigl(\pi ar(2j-1)\bigr)\sin\bigl(\pi ar'(2j-1)\bigr),$$
$$S_{\mathrm{even}} := \sum_{j=1}^{\lfloor \frac{p-1}{2}\rfloor} \sin\bigl(\pi ar(2j)\bigr)\sin\bigl(\pi ar'(2j)\bigr).$$

Then, by Lemma 4.8, we have

$$\sum_{(r'',s'';\theta'')\in\mathscr{S}^{(m'')}_{p,p'}} \sin(\pi arr'')\sin(\pi ar'r'') = \begin{cases} S_{\mathrm{odd}} & \text{if } r_{m''} \text{ is even,} \\ S_{\mathrm{even}} & \text{if } r_{m''} \text{ is odd.} \end{cases}$$

By easy computations, we have

$$S_{\mathrm{odd}} = \begin{cases} (-1)^{p'(r_m-r_{m'})}S_{\mathrm{odd}} & \text{if } p \text{ is even,} \\ (-1)^{p'(r_m-r_{m'})}S_{\mathrm{even}} & \text{if } p \text{ is odd} \end{cases}$$

and

$$S_{\mathrm{even}} = \begin{cases} (-1)^{p'(r_m-r_{m'})}S_{\mathrm{even}} & \text{if } p \text{ is even,} \\ (-1)^{p'(r_m-r_{m'})}S_{\mathrm{odd}} & \text{if } p \text{ is odd.} \end{cases}$$

First, we assume that p is odd. Since $\{r_{m''} \mid 0 \le m'' \le p-1\} = \{0,1,\dots,p-1\}$, we have $S = \sum_{\ell=1}^{p} e^{2\pi ia(r_m-r_{m'})\ell}S_{\mathrm{even}}$. Since $r_m = r_{m'}$ holds if and only if $m = m'$, combining with Lemma 4.14 (2), we obtain $S = \frac{p^2}{4}\delta_{r,r'}\delta_{s,s'}\delta_{\theta,\theta'}$.

Next, we assume that p is even. Then p' is odd and it follows that $S = 0$ holds if $r_m - r_{m'}$ is odd. In what follows, we assume that $r_m - r_{m'}$ is even. Then we have

$$\begin{aligned} S &= \sum_{r_{m''}\,:\,\mathrm{odd}} e^{\pi ia(r_m-r_{m'})(r_{m''}+1)}S_{\mathrm{even}} + \sum_{r_{m''}\,:\,\mathrm{even}} e^{\pi ia(r_m-r_{m'})(r_{m''}+1)}S_{\mathrm{odd}} \\ &= \frac{p}{2}\delta_{m,m'}\sum_{\ell=1}^{p-1}\sin\bigl(\pi ar\ell\bigr)\sin\bigl(\pi ar'\ell\bigr) = \frac{p^2}{4}\delta_{r,r'}\delta_{s,s'}\delta_{\theta,\theta'}. \end{aligned}$$

This completes the proof. $\square$

Proposition 4.16. The modular S-matrix $S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')}$ satisfies the following:

$$S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r',s';\theta')} = S^{aa,(\varepsilon',\varepsilon)}_{(r',s';\theta'),(r,s;\theta)}$$

and

$$\sum_{(r'',s'';\theta'')\in\mathscr{S}_{p,p'}} S^{aa,(\varepsilon,\varepsilon')}_{(r,s;\theta),(r'',s'';\theta'')}\overline{S^{aa,(\varepsilon',\varepsilon)}_{(r'',s'';\theta''),(r',s';\theta')}}=\delta_{r,r'}\delta_{s,s'}\delta_{\theta,\theta'}.$$

Proof. Since the former equality is obvious, we verify the latter one. The left hand side is equal to

$$e^{2\pi ia(1-\varepsilon')\left(\lambda_{r,s}-\theta-(\lambda_{r',s'}-\theta')\right)}\times\frac{4}{p^2}\times S.$$

Therefore it immediately follows from the Lemma 4.15. □

5. Conjecture on Verlinde coefficients

Throughout this section, we assume that $p'\geq 2$. In this section, we consider the structure constants of the "Verlinde ring" of the simple VOSA $L_{c_{p,p'}}$ in the spirit of [CR13b]. See [CR13b], [AC14, §5], and references therein for the details.

Let $(r,s;\theta)\in\mathscr{S}_{p,p'}$, $(r_j,s_j)\in K_{p,p'}$, and $x_j\in\mathbb{R}+i(\mathbb{R}\setminus S_{r_j,s_j})$ for $j\in\{1,2\}$. Then we define the (atypical-typical-typical) *Verlinde coefficient* by

$$N^{\mu_2}_{\lambda,\mu_1}:=\sum_{(r',s')\in K_{p,p'}}\int_{\mathbb{R}}\mathrm{d}x'\frac{S^{at,(0,0)}_{\lambda,(r',s';x')}S^{tt,(0,0)}_{\mu_1,(r',s';x')}\overline{S^{tt,(0,0)}_{\mu_2,(r',s';x')}}}{S^{at,(0,0)}_{(1,0;0),(r',s';x')}},$$

where $\lambda:=(r,s;\theta)$ and $\mu_j:=(r_j,s_j;x_j)$ for $j\in\{1,2\}$.

Conjeture 5.1. $N^{\mu_2}_{\lambda,\mu_1}$ can be expressed as a certain delta function with a non-negative integer coffiecient. In addition, the integer coincides with the fusion rule of the corresponding simple $L_{c_{p,p'}}$-modules, *i.e.* the dimension of the space of intertwining operators of type

$$\begin{pmatrix} & \mathcal{L}_{r_2,s_2;ix_2} \\ \mathcal{L}(r,s)^{\theta} & \mathcal{L}_{r_1,s_1;ix_1}\end{pmatrix}.$$

See [KW94b, Definition 1.6] for the definition of intertwining operators.

Example 5.2. When $(p,p')=(3,2)$, we have

$$\mathscr{S}^{(0)}_{3,2}=\{\lambda_0:=(1,0;0)\},\ \mathscr{S}^{(1)}_{3,2}=\{\lambda_1:=(1,1;-1)\},\ \mathscr{S}^{(2)}_{3,2}=\{\lambda_{-1}:=(2,0;0)\}.$$

We also have $K_{3,2}=\{(1,1)\}$ and $S_{1,1}=\{\frac{1}{4}\}+\frac{1}{2}\mathbb{Z}$. For $j\in\{1,2\}$ and $x_j\in\mathbb{R}+i(\mathbb{R}\setminus S_{1,1})$, we set $\mu_j:=(1,1;x_j)$. Then we obtain

$$N^{\mu_2}_{\lambda_\epsilon,\mu_1}=\int_{\mathbb{R}}\mathrm{d}x' e^{2\pi ix'\times(x_2-x_1+\frac{i\epsilon}{2})}=\delta\Big(x_2-x_1+\frac{i\epsilon}{2}\Big)$$

for $\epsilon\in\{0,\pm1\}$. In this case, the conjecture states that

$$\dim I\begin{pmatrix} & \mathcal{L}_{-\frac{1}{8}+\frac{2}{3}(x_2)^2,\frac{4}{3}ix_2,-1} \\ \mathcal{L}_{\frac{1}{3}\epsilon,\frac{2}{3}\epsilon,-1} & \mathcal{L}_{-\frac{1}{8}+\frac{2}{3}(x_1)^2,\frac{4}{3}ix_1,-1}\end{pmatrix}=\begin{cases}1 & \text{if } x_2=x_1-\frac{i\epsilon}{2},\\ 0 & \text{otherwise}\end{cases}$$

holds for ϵ,x_1, and x_2 as above. To the best of our knowledge, the fusion rule between the above modules has not ever appeared in the literature.

Appendix A. Spectral flow automorphisms

A.1. **Definition.** We consider the Lie superalgebra automorphism $U^{\theta}_{\mathfrak{ns}_2}$ of $\mathfrak{ns}_2$ defined by

$$U^{\theta}_{\mathfrak{ns}_2}(L_n) = L_n + \theta J_n + \frac{\theta^2}{6} C\delta_{n,0},\ U^{\theta}_{\mathfrak{ns}_2}(J_n) = J_n + \frac{\theta}{3} C\delta_{n,0},$$
$$U^{\theta}_{\mathfrak{ns}_2}(G^{\pm}_r) = G^{\pm}_{r\pm\theta},\ U^{\theta}_{\mathfrak{ns}_2}(C) = C.$$

It is called the *spectral flow automorphism* of $\mathfrak{ns}_2$.

Let $(U^{\theta}_{\mathfrak{ns}_2})^*\colon \mathfrak{ns}_2\text{-mod} \longrightarrow \mathfrak{ns}_2\text{-mod}$ be the endofunctor induced by the pullback action with respect to $U^{\theta}_{\mathfrak{ns}_2}$. For simplicity, we write M^{θ} for $(U^{\theta}_{\mathfrak{ns}_2})^*(M)$. Since we have $U^{0}_{\mathfrak{ns}_2} = \mathsf{id}_{\mathfrak{ns}_2}$ and $U^{\theta}_{\mathfrak{ns}_2} \circ U^{\theta'}_{\mathfrak{ns}_2} = U^{\theta+\theta'}_{\mathfrak{ns}_2}$ for any $\theta, \theta' \in \mathbb{Z}$, the two functors $(U^{\theta}_{\mathfrak{ns}_2})^*$ and $(U^{-\theta}_{\mathfrak{ns}_2})^*$ are mutually inverse.

Note that we have the following lemma:

Lemma A.1 ([Sat16, Lemma B.4])**.** For any $c \in \mathbb{C}$, the restriction of $(U^{\theta}_{\mathfrak{ns}_2})^*$ gives the categorical isomorphism $L_c\text{-mod} \to L_c\text{-mod};\ M \mapsto M^{\theta}$.

A.2. **Spectral flow on irreducible highest weight modules.** It is easy to verify the next lemma.

Lemma A.2. Let $(h, j, c) \in \mathbb{C}^3$. Then we have

$$(\mathcal{L}_{h,j,c})^1 \cong \begin{cases} \mathcal{L}_{h+j+\frac{c}{6}, j+\frac{c}{3}, c} & \text{if } h = -\frac{j}{2}, \\ \mathcal{L}_{h+j+\frac{c}{6}-\frac{1}{2}, j+\frac{c}{3}-1, c} & \text{if } h \neq -\frac{j}{2}. \end{cases}$$

$$(\mathcal{L}_{h,j,c})^{-1} \cong \begin{cases} \mathcal{L}_{h-j+\frac{c}{6}, j-\frac{c}{3}, c} & \text{if } h = \frac{j}{2}, \\ \mathcal{L}_{h-j+\frac{c}{6}-\frac{1}{2}, j-\frac{c}{3}+1, c} & \text{if } h \neq \frac{j}{2}. \end{cases}$$

By the above lemma and some computations, we obtain the following.

Corollary A.3. Let $1 \leq r \leq p-1$, $0 \leq s \leq p'-1$, and $\theta \in \mathbb{Z}$. Then we have

$$\mathcal{L}(r,s)^{\theta} \cong \begin{cases} \mathcal{L}_{r,s+1;\lambda_{r,s+1}-\theta} & \text{if } s \neq p'-1 \text{ and } \theta < 0 \\ \mathcal{L}(p-r, 0) & \text{if } s = p'-1 \text{ and } \theta = -p+r \\ \mathcal{L}_{r,s+1;\lambda_{r,s+1}-\theta} & \text{if } s = p'-1 \text{ and } -p+r < \theta < 0 \\ \mathcal{L}_{r,s;\lambda_{r,s}-\theta} & \text{if } s = 0 \text{ and } 0 \leq \theta < r, \\ \mathcal{L}(p-r, p'-1) & \text{if } s = 0 \text{ and } \theta = r, \\ \mathcal{L}_{r,s;\lambda_{r,s}-\theta} & \text{if } s \neq 0 \text{ and } 0 \leq \theta, \end{cases}$$

where $\lambda_{m,n} := \frac{m-1}{2} - \frac{n}{2a}$ for $m, n \in \mathbb{Z}$.

Appendix B. Relaxed highest weight modules over $\widehat{\mathfrak{sl}}_2$

B.1. **Definition of relaxed verma module.** Let $\{E, H, F\} \subset \mathfrak{sl}_2$ be the standard triple and $U_0 := \{u \in U(\mathfrak{sl}_2) \mid [H, u] = 0\}$ the centralizer of $\mathfrak{h} = \mathbb{C}H$. For $(h, \lambda, k) \in \mathbb{C}^3$, let $\mathbb{C}_{h,\lambda,k}$ be the 1-dimensional U_0-module defined by $\Omega.1 := 2(k+2)h$ and $H.1 := 2\lambda$, where $\Omega \in U(\mathfrak{sl}_2)$ is the Casimir element. We define the action of $\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K$ on the induced $\mathfrak{sl}_2$-module $\bar{R}_{h,\lambda,k} := \mathrm{Ind}_{U_0}^{U(\mathfrak{sl}_2)} \mathbb{C}_{h,\lambda,k}$ by $X_n := X \otimes t^n \mapsto X\delta_{n,0}$, $K \mapsto k\,\mathsf{id}$ for $X \in \mathfrak{sl}_2$ and $n \geq 0$. Then, we define the relaxed Verma module as the induced module $R_{h,\lambda,k} := \mathrm{Ind}^{\widehat{\mathfrak{sl}}_2}_{\mathfrak{sl}_2 \otimes \mathbb{C}[t] \oplus \mathbb{C}K} \bar{R}_{h,\lambda,k}$. We write $L_{h,\lambda,k}$ for its unique irreducible quotient module.

B.2. **Construction of relaxed highest weight modules.**

Definition B.1. Let $X \in \{E, F\}$ and U_X the localization of $U := U(\widehat{\mathfrak{sl}}_2)$ with respect to the multiplicative set $\{(X_0)^n \mid n \geq 0\}$. For $\mu \in \mathbb{C}$, we define a functor $X^\mu \colon \widehat{\mathfrak{sl}}_2\text{-mod} \to \widehat{\mathfrak{sl}}_2\text{-mod}$ by

$$X^\mu := \mathrm{Res}_U^{U_X} \circ \mathrm{Ad}(X^\mu)^* \circ \mathrm{Ind}_U^{U_X},$$

where $\mathrm{Ad}(X^\mu)$ is a $\mathbb{C}$-algebra automorphism of U_X defined by

$$\mathrm{Ad}(X^\mu)(-) := \sum_{i \geq 0} \binom{\mu}{i} \mathrm{ad}(X_0)^i(-) X_0^{-i}.$$

Suppose that $p' \neq 1$. We define $\mathcal{E}_{r,s;\lambda} := F^{-\lambda+\lambda_{r,s}} L(r,s)$, where $\lambda \in \mathbb{C}$ and $L(r,s)$ is the Kac-Wakimoto admissible $\widehat{\mathfrak{sl}}_2$-module for $(r,s) \in K_{p,p'}$ (see Remark 4.2 for the definition).

Proposition B.2. For any $\lambda \in \mathbb{C}$, we have an equality of formal series

$$\mathsf{tr}_{\mathcal{E}_{r,s;\lambda}} \left(q^{L_0^{\mathsf{Sug}} - \frac{c}{24}} z^{\frac{H_0}{2}} \right) = \frac{\sum_{n \in \mathbb{Z}} z^{\lambda+n}}{\eta(q)^2} \chi_{r,s}(q), \tag{B.1}$$

where the 0-th mode Virasoro operator L_0^{Sug} on $\mathcal{E}_{r,s;\lambda}$ is given by the Sugawara construction and c stands for the corresponding central charge $c_{p,p'}$.

Proof. By applying the exact functor $F^{-\lambda+\lambda_{r,s}}$ to the BGG resolution for $L(r,s)$ [Mal91, Theorem A], we obtain the relaxed Verma resolution for $\mathcal{E}_{r,s;\lambda}$. Thus the character formula follows. □

When $\lambda \in \mathbb{C} \setminus S_{r,s}$, the character (B.1) coincides with $\mathsf{ch}(L_{\Delta_{r,s},\lambda,k_{p,p'}})$ given in [CR13a, Corollary 5]. Therefore $\mathcal{E}_{r,s;\lambda}$ is isomorphic to $L_{\Delta_{r,s},\lambda,k_{p,p'}}$ if $\lambda \in \mathbb{C} \setminus S_{r,s}$.

Appendix C. Proof of the atypical modular transformation law

In this section, we compute the modular (S-)transformation of the atypical character function step by step.

C.1. **Expression in terms of Appell-Lerch sums.** Recall the level K Appell-Lerch sum

$$A_K(u,v;\tau) := z^{\frac{K}{2}} \sum_{n \in \mathbb{Z}} \frac{(-1)^{Kn} y^n q^{\frac{K}{2}n(n+1)}}{1 - zq^n}.$$

Then we obtain the following by Definition 4.5.

Lemma C.1. We have

$$\begin{aligned}
&\frac{\eta(\tau)^3}{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)} \mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}(u;\tau) \\
&= (-1)^{\varepsilon\varepsilon' + (1-\varepsilon')p'} q^{(ar-s-p')(\theta+\frac{1-\varepsilon}{2}) - a(\theta+\frac{1-\varepsilon}{2})^2} z^{(ar-s-p')-2a(\theta+\frac{1-\varepsilon}{2})} \\
&\quad \times \Big(A_{2p'}\big(u_{\varepsilon,\varepsilon'} + \theta\tau, (rp'-sp-pp')\tau; p\tau\big) \\
&\qquad - q^{r(s+p')} A_{2p'}\big(u_{\varepsilon,\varepsilon'} + (\theta - r)\tau, (-rp'-sp-pp')\tau; p\tau\big)\Big).
\end{aligned}$$

C.2. **Modular transformation law of Appell-Lerch sum.** The following modular S-transformation property of the Appell-Lerch sum is a spcecial case of [AC14, Corollary 3.4].

Proposition C.2.

$$A_{2p'}\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{p}{\tau}\right)=\frac{\tau}{p}e^{-2\pi i(p'u^2-uv)/p\tau}\Bigg(A_{2p'}\Big(\frac{u}{p},\frac{v}{p};\frac{\tau}{p}\Big)$$
$$+\frac{e^{2\pi i\left(a-\frac{1}{2p}\right)u}}{4p'}\sum_{m=0}^{2p'-1}\vartheta_{1,1}\Big(f_m(\tau,v);\frac{\tau}{2pp'}\Big)h\Big(\frac{u}{p}-f_m(\tau,v);\frac{\tau}{2pp'}\Big)\Bigg),$$

where $f_m(\tau,v):=\frac{(2p'-1)\tau+2v-2pm}{4pp'}$ and $h(u;\tau):=\int_{\mathbb{R}}\frac{e^{\pi i\tau x^2-2\pi ux}}{\cosh\pi x}dx$ is the Mordell integral.

By Proposition C.2, we see that the S-tansformed character decomposes into the following two factors:

$$\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}\left(-\frac{1}{\tau},\frac{u}{\tau},t-\frac{u^2}{6\tau}\right)=e^{\frac{\pi i(1-2a)u^2}{\tau}}\big((\text{discrete part})+(\text{continuous part})\big).$$

C.3. **Computation of the discrete part - Level reduction.** In this subsection, we write all the objects in terms of the level 1 Appell-Lerch sum A_1.

C.3.1. *S-transformed side.* In what follows, we write

$$\tilde{u}_{\varepsilon,\varepsilon'}:=u+\frac{1-\varepsilon'}{2}\tau-\frac{1-\varepsilon}{2}\equiv u_{\varepsilon',\varepsilon}\ (\mathrm{mod}\ \mathbb{Z}).$$

By Proposition C.2, the discrete part is written in terms of the Appell-Lerch sums:

$$A_{2p'}\left(\frac{\tilde{u}_{\varepsilon,\varepsilon'}-\theta}{p},\frac{-rp'}{p};\frac{\tau}{p}\right)-A_{2p'}\left(\frac{\tilde{u}_{\varepsilon,\varepsilon'}-\theta+r}{p},\frac{rp'}{p};\frac{\tau}{p}\right).\tag{C.1}$$

We can rewrite this in terms of A_1 by the following lemma:

Lemma C.3.

$$A_{2p'}\left(\frac{u}{p},\frac{v}{p};\frac{\tau}{p}\right)=\frac{1}{2p'}\sum_{n,m=0}^{p-1}z^{\frac{2m+2p'-p}{2p}}y^{\frac{n}{p}}q^{\frac{n(p'n+m+p')}{p}-\frac{n}{2}}$$
$$\times\sum_{\ell=0}^{2p'-1}A_1\Big(u+n\tau,\frac{2p'n+m+p'}{2p'}\tau-\frac{p\tau}{4p'}+\frac{v+\ell}{2p'};\frac{p\tau}{2p'}\Big).$$

The proof is straightforward and we omit it.

C.3.2. *Untransformed side.* By [AC14, Proposition 3.3], we have

$$A_{2p'}(u,v;p\tau)=\frac{e^{2\pi i\left(p'-\frac{1}{2}\right)u}}{2p'}\sum_{\ell=0}^{2p'-1}A_1\left(u,\frac{v}{2p'}+\frac{(2p'-1)p}{4p'}\tau+\frac{\ell}{2p'};\frac{p\tau}{2p'}\right).\tag{C.2}$$

Then the (untrasformed) atypical character is also rewritten in terms of A_1 as follows:

Corollary C.4. Let $N, M \in \mathbb{Z}$. We have

$$\begin{aligned}
&\frac{\eta(\tau)^3}{\vartheta_{\varepsilon,\varepsilon'}(u;\tau)}\mathbf{A}^{\varepsilon,\varepsilon'}_{r,s;\theta}(\tau,u,t) \\
&= (-1)^{\varepsilon\varepsilon'}\frac{(-i)^{1-\varepsilon'}}{2p'}q^{\frac{(\theta+Np)\left(\spadesuit+2p'(\theta+Np)\right)}{2p}+\frac{(1-\varepsilon)\left(\spadesuit-p'\right)}{4p}}z^{\frac{\spadesuit}{2p}-(1-\varepsilon)a}w^{3(1-2a)} \\
&\times \sum_{\ell=0}^{2p'-1}\Bigg[A_1\left(u_{\varepsilon,\varepsilon'}+(\theta+Np)\tau, v_+(N); \frac{p\tau}{2p'}\right) \\
&- q^{(Mp-r)(2Np'+Mp'-s-\frac{1}{2})}A_1\left(u_{\varepsilon,\varepsilon'}+\big(\theta-r+(N+M)p\big)\tau, v_-(N+M); \frac{p\tau}{2p'}\right)\Bigg],
\end{aligned}$$

where $\spadesuit := 2(r-2\theta)p' - (2s+1)p$ and

$$v_\pm(n) := \frac{2npp' \pm rp' - sp}{2p'}\tau - \frac{p\tau}{4p'} + \frac{\ell}{2p'}$$

for $n \in \mathbb{Z}$.

Proof. Since $A_1(u+n\tau, v+n\tau;\tau) = (-y)^{-n}q^{-\frac{n^2}{2}}A_1(u,v;\tau)$, we only have to verify the equiality for $N=M=0$. It immediately follows from (C.2). $\square$

C.4. **Computation of the discrete part - Conclusion.**

Proposition C.5.

$$\begin{aligned}
\text{(discrete part)} = i^{-\varepsilon\varepsilon'}\frac{2}{p}&\sum_{(r',s';\theta')\in\mathscr{S}_{p,p'}}(-1)^{(1-\varepsilon')s+(1-\varepsilon)s'} \\
&\times \sin(\pi a r r')e^{\pi i a(r-2\theta-1+\varepsilon)(r'-2\theta'-1+\varepsilon')}\mathbf{A}^{\varepsilon',\varepsilon}_{r',s';\theta'}(u;\tau).
\end{aligned}$$

Proof. The left hand side is equal to (C.1) and can be written in terms of A_1 by Lemma C.3. The right hand side is also written in terms of A_1 by Corollary C.4. Then, by choosing appropriate N and M, we obtain the formula. $\square$

C.5. **Computation of the theta part.** In this subsection, we compute the theta functions in the continuous part. The following lemma is easily verified and we omit the proof.

Lemma C.6. We have

$$\vartheta_{1,1}\left(u;\frac{\tau}{2pp'}\right) = -i\sum_{\ell=0}^{2pp'-1}e^{2\pi i\left(\ell+\frac{1}{2}\right)\left(u+\frac{1}{2}\right)}q^{\frac{\left(\ell+\frac{1}{2}\right)^2}{4pp'}}\vartheta_{0,0}\left(2pp'u+\left(\ell+\frac{1}{2}\right)\tau; 2pp'\tau\right).$$

Corollary C.7. For $0 \le m \le 2p'-1$, we have

$$\begin{aligned}
(\text{theta})_m &:= \vartheta_{1,1}\left(\left(p'-\frac{1}{2}\right)\frac{\tau}{2pp'}+\frac{-rp'+sp-pm+pp'}{2pp'};\frac{\tau}{2pp'}\right) \\
&\quad - e^{2\pi i r(a-\frac{1}{2p})}\vartheta_{1,1}\left(\left(p'-\frac{1}{2}\right)\frac{\tau}{2pp'}+\frac{rp'+sp-pm+pp'}{2pp'};\frac{\tau}{2pp'}\right) \\
&= 2e^{\pi i(ar-s+m)}e^{\pi i\left(\frac{-rp'+sp-pm}{2pp'}\right)}q^{-\frac{(2p'-1)^2}{16pp'}} \\
&\quad\times\sum_{L=p'}^{2pp'+p'-1}e^{\frac{\pi i L(s-m)}{p'}}\sin\left(\frac{\pi r L}{p}\right)\sum_{n\in\mathbb{Z}}q^{pp'\left(n+\frac{L}{2pp'}\right)^2}.
\end{aligned}$$

Proof. By the previous lemma, we have

$$\begin{aligned}
&\vartheta_{1,1}\left(\left(p'-\frac{1}{2}\right)\frac{\tau}{2pp'}+\frac{\mp rp'+sp-pm+pp'}{2pp'};\frac{\tau}{2pp'}\right)\\
&=ie^{\pi i\left(\frac{\mp rp'+sp-pm}{2pp'}\right)}q^{-\frac{(2p'-1)^2}{16pp'}}\sum_{\ell=0}^{2pp'-1}e^{\mp\frac{\pi i\ell r}{p}}e^{\frac{\pi i\ell(s-m)}{p'}}q^{\frac{(\ell+p')^2}{4pp'}}\vartheta_{0,0}\big((\ell+p')\tau;2pp'\tau\big).
\end{aligned}$$

Then the required formula follows. □

Remark C.8. When $p'=1$,

$$\sum_{L=1}^{2p}(-1)^{Lm}\sin\left(\frac{\pi rL}{p}\right)\sum_{n\in\mathbb{Z}}q^{p\left(n+\frac{L}{2p}\right)^2}=0$$

holds for $0\le m\le 1$. In particular, we get $(\mathrm{theta})_m=0$.

C.6. **Computation of the Mordell part.** For $s\in\mathbb{R}$, we have

$$h(u-s\tau;\tau)=q^{\frac{s^2}{2}}z^{-s}\int_{\mathbb{R}-is}\frac{e^{\pi i\tau x^2-2\pi ux}}{\cosh\pi\,(x+is)}\mathrm{d}x.$$

Then, by straightforward calculation, we obtain the following lemma.

Lemma C.9. For $0\le m\le 2p'-1$, we have

$$\begin{aligned}
&(\mathrm{Mordell})_m\\
&:=h\left(\frac{u-\theta-\frac{1-\varepsilon}{2}}{p}-\frac{-rp'+sp-pm+pp'}{2pp'}-\left(\varepsilon'p'-\frac{1}{2}\right)\frac{\tau}{2pp'};\frac{\tau}{2pp'}\right)\\
&=2p'e^{2\pi i\left(\theta+\frac{1-\varepsilon}{2}\right)\left(\varepsilon'a-\frac{1}{2p}\right)}e^{-\pi i\varepsilon'(ar-s+m)}e^{-\pi i\left(\frac{-rp'+sp-pm}{2pp'}\right)}\\
&\quad\times q^{\frac{(2\varepsilon'p'-1)^2}{16pp'}}z^{-\varepsilon'a+\frac{1}{2p}}\int_{\mathbb{R}-i\left(\frac{\varepsilon'}{2}-\frac{1}{4p'}\right)}\frac{e^{2\pi\left[p'-m-2a\left(\lambda_{r,s}-\theta+\frac{\varepsilon}{2}\right)\right]y}}{\sinh\left(2p'\pi y\right)}q^{ay^2}z^{2iay}\mathrm{d}y.
\end{aligned}$$

C.7. **Computation of the continuous part - Conclusion.**

Lemma C.10. We have

$$e^{2p'\pi y}\sum_{m=0}^{2p'-1}\left((-1)^{1-\varepsilon'}e^{-\frac{\pi iL}{p'}}e^{-2\pi y}\right)^m=\frac{2\sinh(2p'\pi y)}{1+e^{-2\pi(y+\frac{i\varepsilon'}{2})-\frac{\pi iL}{p'}}}.$$

Proof. Since

$$-\frac{i\varepsilon'}{2}+\left(n+\frac{1}{2}-\frac{L}{2p'}\right)i\notin\mathbb{R}-i\left(\frac{\varepsilon'}{2}-\frac{1}{4p'}\right)$$

for any $n\in\mathbb{Z}$, the ratio in the left-hand side is not equal to 1. □

Finally we obtain the explicit form of the continuous part as follows.

Proposition C.11. We have

$$
\begin{aligned}
(\text{continuous part}) = {} & i^{1-\varepsilon\varepsilon'}\frac{2}{p}\sum_{(r',s')\in K_{p,p'}}(-1)^{r's+rs'}\sin\bigl(\pi arr'\bigr)\\
& \times\int_{\mathbb{R}}\frac{\sin\left(\frac{\pi ss'}{a}\right)e^{2\pi x'}+\sin\bigl(2\pi\lambda_{r',(1+s)s'}\bigr)}{\cosh(2\pi x')-\cos(2\pi\lambda_{r',s'})}\\
& \times e^{-4\pi a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})\left(x'-\frac{i\varepsilon'}{2}\right)}\mathbf{T}^{\varepsilon',\varepsilon}_{r',s';x'}(u;\tau)\mathrm{d}x'.
\end{aligned}
$$

Proof. By Lemma C.1 and Proposition C.2, the continuous part is equal to $C\times\vartheta_{\varepsilon,\varepsilon'}(u;\tau)\eta(\tau)^{-3}$, where

$$
\begin{aligned}
C := {} & i^{1-\varepsilon\varepsilon'}(-1)^{\varepsilon\varepsilon'}\frac{1}{p}e^{-2\pi ia(1-\varepsilon')\left(\lambda_{r,s}-\theta+\frac{\varepsilon}{2}\right)}\\
& \times q^{-\frac{1-\varepsilon'}{4}a}z^{-(1-\varepsilon')a}\left[\frac{e^{2\pi i(\tilde{u}_{\varepsilon,\varepsilon'}-\theta)(a-\frac{1}{2p})}}{4p'}\sum_{m=0}^{2p'-1}(\text{theta})_m(\text{Mordell})_m\right].
\end{aligned}
$$

By using Lemma C.10 and $e^{2\pi i(\tilde{u}_{\varepsilon,\varepsilon'}-\theta)(a-\frac{1}{2p})}=e^{-2\pi i(\theta+\frac{1-\varepsilon}{2})(a-\frac{1}{2p})}z^{a-\frac{1}{2p}}q^{\frac{1-\varepsilon'}{2}(a-\frac{1}{2p})}$, we have

$$
\begin{aligned}
C = {} & i^{1-\varepsilon\varepsilon'}(-1)^{\varepsilon\varepsilon'}\frac{2}{p}\sum_{L=p'}^{2pp'+p'-1}e^{\frac{\pi iLs}{p'}}\sin\left(\frac{\pi rL}{p}\right)\sum_{n\in\mathbb{Z}}q^{pp'\left(n+\frac{L}{2pp'}\right)^2}\\
& \times\int_{\mathbb{R}-i\left(\frac{\varepsilon'}{2}-\frac{1}{4p'}\right)}\frac{e^{-4\pi a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})y}}{1+e^{-2\pi(y+\frac{i\varepsilon'}{2})-\frac{\pi iL}{p'}}}q^{ay^2}z^{2iay}\mathrm{d}y\\
= {} & i^{1-\varepsilon\varepsilon'}(-1)^{\varepsilon\varepsilon'}\frac{2}{p}\sum_{r'=1}^{p-1}\sum_{s'=1}^{p'-1}(-1)^{r's+rs'}e^{-\frac{\pi iss'}{a}}\sin\bigl(\pi arr'\bigr)\\
& \times\int_{\mathbb{R}-i\left(\frac{\varepsilon'}{2}-\frac{1}{4p'}\right)}\frac{e^{-4\pi a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})y}}{1-e^{-2\pi(y+\frac{i\varepsilon'}{2})-2\pi i\lambda_{r',s'}}}q^{ay^2}z^{2iay}\eta(\tau)\chi_{r',s'}(\tau)\mathrm{d}y\\
= {} & i^{-\varepsilon\varepsilon'}\frac{2}{p}\sum_{(r',s')\in K_{p,p'}}(-1)^{r's+rs'}\sin\bigl(\pi arr'\bigr)\\
& \times\int_{\mathbb{R}+\frac{i}{4p'}}\frac{\sin\left(\frac{\pi ss'}{a}\right)e^{2\pi Y}+\sin\bigl(2\pi\lambda_{r',(1+s)s'}\bigr)}{\cosh(2\pi Y)-\cos(2\pi\lambda_{r',s'})}e^{-4\pi a(\lambda_{r,s}-\theta+\frac{\varepsilon}{2})\left(Y-\frac{i\varepsilon'}{2}\right)}\\
& \times(-1)^{\varepsilon\varepsilon'}q^{a\left(Y-\frac{i\varepsilon'}{2}\right)^2}z^{2ia\left(Y-\frac{i\varepsilon'}{2}\right)}\eta(\tau)\chi_{r',s'}(\tau)\mathrm{d}Y.
\end{aligned}
$$

In the last equality, we put $Y:=y-\frac{i\varepsilon'}{2}$. Finally we shift the region of the integration by the residue theorem and obtain the required formula. $\square$

Appendix D. Comparison with the results of Kac-Wakimoto

D.1. **Modular transformation of the minimal unitary characters.** We write $A^{(p;0)}$ and $A^{(p;\frac{1}{2})}$ for the finite set A_p^+ and A_p^- defined in [KW94a, §6].

Lemma D.1. Let $\varepsilon \in \{0, \frac{1}{2}\}$ and $(j,k) \in A^{(p;\varepsilon)}$. Then we have

$$\mathsf{ch}^{p;\varepsilon}_{j,k}(\tau,u) = \mathbf{A}^{1-2\varepsilon,0}_{j+k,0;k-\varepsilon}(\tau,u,0), \quad \mathsf{sch}^{p;\varepsilon}_{j,k}(\tau,u) = \mathbf{A}^{1-2\varepsilon,1}_{j+k,0;k-\varepsilon}(\tau,u,0),$$

where $\mathsf{ch}^{p;\varepsilon}_{j,k}(\tau,u)$ and $\mathsf{sch}^{p;\varepsilon}_{j,k}(\tau,u)$ are defined in [KW94a, §6].

Proof. Since we have

$$\Psi^{(\varepsilon)+}(\tau,u) = \frac{\eta(\tau)^3}{\vartheta_{1-2\varepsilon,0}\,(u;\tau)}, \ \Psi^{(\varepsilon)-}(\tau,u) = \frac{\eta(\tau)^3}{(-1)^{1-2\varepsilon}\vartheta_{1-2\varepsilon,1}(u;\tau)}$$

in [KW94a], we obtain

$$\mathsf{ch}^{p;\varepsilon}_{j,k}(\tau,u) = q^{\frac{jk}{p}} z^{\frac{j-k}{p}} \frac{\vartheta_{1-2\varepsilon,0}\,(u;\tau)}{\eta(\tau)^3} \left(\sum_{n,m \geq 0} - \sum_{n,m<0} \right) (-1)^{n+m} z^{-n+m} q^{pnm+jn+km},$$

$$\mathsf{sch}^{p;\varepsilon}_{j,k}(\tau,u) = (-1)^{1-2\varepsilon} q^{\frac{jk}{p}} z^{\frac{j-k}{p}} \frac{\vartheta_{1-2\varepsilon,1}(u;\tau)}{\eta(\tau)^3} \left(\sum_{n,m \geq 0} - \sum_{n,m<0} \right) z^{-n+m} q^{pnm+jn+km}.$$

Our character formula gives

$$\begin{aligned}
&\mathbf{A}^{1-2\varepsilon,\varepsilon'}_{j+k,0;k-\varepsilon}(\tau,u,t) \\
&= \mathsf{ch}^{1-2\varepsilon,\varepsilon'}\big(\mathcal{L}(j+k,0)^{k-\varepsilon}\big)(q,z,w) \\
&= (-1)^{(1-2\varepsilon)\varepsilon'} q^{-\frac{(j-k)^2}{4p}} z^{\frac{j-k}{p}} w^{3(1-2a)} \frac{\vartheta_{1-2\varepsilon,\varepsilon'}(u;\tau)}{\eta(\tau)^3} \Phi_{p,1;j+k,0}\left((-1)^{1-\varepsilon'} z q^k, q\right)
\end{aligned}$$

for $\varepsilon' \in \{0,1\}$. We can verify that these formulae coincide by some computations. $\square$

Then we obtain the modular transformation property proved in [KW94a, Theorem 6.1] as a special case of Theorem 4.12 for $p' = 1$.

D.2. **Non-unitary spectra of Kac-Wakimoto.** In this subsection, we compare our spectra $\mathscr{S}_{p,p'}$ with the set of highest weights considered in [KW16, §3]. In what follows, we identify the set of triples $\mathscr{S}_{p,p'}$ with the set of the highest weights of the corresponding $\mathfrak{ns}_2$-modules.

Let $\{\Lambda_0, \Lambda_1, \Lambda_2\}$ be the set of fundamental weights of the affine Kac-Moody Lie superalgebra $\widehat{\mathfrak{sl}}_{2|1}$. We write $\Lambda(s,i)$ for an integral weight $(p'-1-s)\Lambda_0 + s\Lambda_i$, where $0 \leq s \leq p'-1$ and $i \in \{1,2\}$. In [KW16, §3], V. Kac and M. Wakimoto considered a certain family of simple highest weight $\mathfrak{ns}_2$-modules of central charge

$$c = 3(1-2a) = 3\Big(1 - \frac{2p'}{p}\Big)$$

associated with the pair $\big(p, \Lambda(s,i)\big)$ (see [KW16, Lemma 2.1]). The set of the corresponding $\mathfrak{ns}_2$-highest weights $(h,j) = \big(h,j,3(1-2a)\big)$ is give as follows:

$$S\,(a;s,i) := \{(h,j) = \Lambda_{k_1,k_2} \mid k_1,k_2 \in \mathbb{Z}_{\geq 0} \text{ and } k_1+k_2 \leq p-1\},$$

where

$$\Lambda_{k_1,k_2} := \bigg(a\Big(\big(k_1+\frac{1}{2}\big)\big(k_2+\frac{1}{2}\big) - \frac{1}{4}\Big) - s\big(k_1+\frac{1}{2}\big), (-1)^i\big(a(k_2-k_1)-s\big)\bigg).$$

We note that they assume the integer p to be odd. See [KW16, §3] for the details.

Example D.2. Here we present two examples.

(1) If $(p,p')=(3,2)$, we have

$$S\left(\frac{2}{3};0,i\right)=\left\{(0,0),\left(\frac{1}{3},\pm\frac{2}{3}\right)\right\}=\mathscr{S}_{3,2}$$

for $i\in\{1,2\}$.

(2) If $(p,p')=(5,2)$, we have

$$S\left(\frac{2}{5};0,i\right)\cap\mathscr{S}_{5,2}=\left\{(0,0),\left(\frac{1}{5},\pm\frac{2}{5}\right),\left(\frac{2}{5},\pm\frac{4}{5}\right),\left(\frac{4}{5},0\right),\left(\frac{7}{5},\pm\frac{2}{5}\right)\right\},$$

$$S\left(\frac{2}{5};1,i\right)\cap\mathscr{S}_{5,2}=\left\{\left(\frac{1}{10},\frac{(-1)^i}{5}\right)\right\},$$

and

$$\mathscr{S}_{5,2}\subsetneq S\left(\frac{2}{5};0,i\right)\sqcup S\left(\frac{2}{5};1,1\right)\sqcup S\left(\frac{2}{5};1,2\right)$$

for $i\in\{1,2\}$.

References

[AC14] C. Alfes and T. Creutzig. "The mock modular data of a family of superalgebras". *Proc. Amer. Math. Soc.*, 142(7):2265–2280, 2014.

[Ada99] D. Adamović. "Representations of the $N=2$ superconformal vertex algebra". *Int. Math. Res. Not.*, 2(2):61–79, 1999.

[Ada01] D. Adamović. "Vertex algebra approach to fusion rules for $N=2$ superconformal minimal models". *J. Algebra*, 239(2):549–572, 2001.

[ALY14] T. Arakawa, C. H. Lam, and H. Yamada. "Zhu's algebra, C_2-algebra and C_2-cofiniteness of parafermion vertex operator algebras". *Adv. Math.*, 264:261–295, 2014.

[AM11] D. Adamović and A. Milas. "The structure of Zhu's algebras for certain W-algebras". *Adv. Math.*, 227(6):2425–2456, 2011.

[Ara05] T. Arakawa. "Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture". *Duke Math. J.*, 130(3):435–478, 2005.

[CR13a] T. Creutzig and D. Ridout. "Modular data and Verlinde formulae for fractional level WZW models II". *Nucl. Phys. B*, 875(2):423–458, 2013.

[CR13b] T. Creutzig and D. Ridout. "Relating the archetypes of logarithmic conformal field theory". *Nucl. Phys. B*, 872(3):348–391, 2013.

[Dör98] M. Dörrzapf. "The embedding structure of unitary $N=2$ minimal models". *Nucl. Phys. B*, 529(3):639–655, 1998.

[FST98] B. L. Feigin, A. M. Semikhatov, and I. Yu. Tipunin. "Equivalence between chain categories of representations of affine $sl(2)$ and $N=2$ superconformal algebras". *J. Math. Phys.*, 39(7):3865–3905, 1998.

[Gho03] M. Ghominejad. *Higher Level Appell Functions, Modular Transformations and Non-Unitary Characters*. PhD thesis, Durham University, 8 2003.

[GK07] M. Gorelik and V. Kac. "On simplicity of vacuum modules". *Adv. Math.*, 211(2):621–677, 2007.

[GK15] M. Gorelik and V. G. Kac. "Characters of (relatively) integrable modules over affine Lie superalgebras". *Japan. J. Math.*, 10(2):135–235, 2015.

[HT91] S. Hosono and A. Tsuchiya. "Lie algebra cohomology and $N=2$ SCFT based on the GKO construction". *Comm. Math. Phys.*, 136(3):451–486, 1991.

[IK11] K. Iohara and Y. Koga. *Representation theory of the Virasoro algebra*. Springer monographs in mathematics. Springer, 2011.

[KS] Y. Koga and R. Sato. *in preparation*.

[KS89] Y. Kazama and H. Suzuki. "New $N=2$ superconformal field theories and superstring compactification". *Nucl. Phys. B*, 321(1):232–268, 1989.

[KW94a] V.G. Kac and M. Wakimoto. "Integrable highest weight modules over affine superalgebras and number theory". In *"Lie theory and geometry", Progress in Math. Phys.*, volume 123, pages 415–456. Birkhäuser, 1994.

[KW94b] V.G. Kac and W. Wang. "Vertex Operator Superalgebras and Their Representations". In *"Mathematical aspects of conformal and topological field theories and quantum groups", Comtemp. Math.*, volume 175, pages 161–192. American Mathematical Soc., 1994.

[KW16] V. G. Kac and M. Wakimoto. "Representations of affine superalgebras and mock theta functions II". *Adv. Math.*, 300:17–70, 2016.

[Li99] H. Li. "Some finiteness properties of regular vertex operator algebras". *J. Algebra*, 212(2):495–514, 1999.

[Mal91] F. Malikov. "Verma modules over Kac-Moody algebras of rank 2". *Leningrad Math. J*, 2(2):269–286, 1991.

[Miy04] M. Miyamoto. "Modular invariance of vertex operator algebras satisfying C_2-cofiniteness". *Duke Math. J.*, 122(1):51–91, 2004.

[RY87] F. Ravanini and S.-K. Yang. "Modular invariance in $N = 2$ superconformal field theories". *Phys. Lett. B*, 195(2):202–208, 1987.

[Sat16] R. Sato. "Equivalences between weight modules via $\mathcal{N} = 2$ coset constructions". *arXiv preprint arXiv:1605.02343*, 2016.

[STT05] A.M. Semikhatov, A. Taorimina, and I.Yu. Tipunin. "Higher-level Appell functions, modular transformations, and characters". *Comm. Math. Phys.*, 255(2):469–512, 2005.

[VE13] J. Van Ekeren. "Modular Invariance for Twisted Modules over a Vertex Operator Superalgebra". *Comm. Math. Phys.*, 322(2):333–371, 2013.

[Zhu96] Y. Zhu. "Modular invariance of characters of vertex operator algebras". *J. Amer. Math. Soc.*, 9(1):237–302, 1996.

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