博士論文

Studies on algebraic fiber spaces in positive characteristic

(正標数の代数的ファイバー空間に関する研究)

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Contents

Preface

An algebraic fiber space is a family of algebraic varieties, which may have degenerate fibers, such that not only general fibers but also the total and the base space have structures of algebraic varieties. More precisely, it is defined as a separable surjective morphism $f : X \to Y$ of projective varieties with connected fibers. For the classification of algebraic varieties, it is important to consider the relationship between X, Y and the geometric generic fiber F of f. In this thesis, we study their relationship in terms of properties of the (anti-)canonical divisors.

In characteristic zero, there are many significant results on algebraic fiber spaces. For example, Birkar–Chen [9] and Fujino–Gongyo [41, 42] proved that some positivity conditions can be passed from the anti-canonical divisor on X to that on Y when f is smooth. As for properties of the canonical divisors, the semi-positivity theorem, the weak positivity theorem and the partial settlement of Iitaka's $C_{n,m}$ conjecture are established due to Birkar [8], Fujino [34, 35, 38], Fujita [44], Kawamata [66, 67, 68, 69], Kollár [72, 73], Viehweg [108, 109], etc. However, since the proofs of the above results depend on the existence of resolution of singularities, the weak semi-stable reduction theorem or some consequences of the Hodge theory, we cannot use the same arguments in positive characteristic. In this thesis, we overcome this difficulty by applying the methods of F-singularities, singularities defined in terms of the Frobenius morphism, and prove positive characteristic analogs of several results on algebraic fiber spaces in characteristic zero.

This thesis consists of seven chapters.

In Chapter 1, we set up notation and terminology, and recall basic notions such as almost Cartier divisors and the trace of the Frobenius morphism. In particular, we carefully describe the traces of the relative and absolute Frobenius morphisms, which appear repeatedly throughout this thesis.

Chapter 2 is devoted to the study of the Frobenius stable canonical rings introduced by Hacon and Patakfalvi [52]. We compute them in the case of projective Gorenstein curves and of projective varieties with semi-ample canonical divisors. Roughly speaking, if the Frobenius stable canonical ring is large enough, then pathological phenomena in positive characteristic do not occur. Indeed, we see in Chapters 4 and 6 that analogs of some well-known results on algebraic fiber spaces in characteristic zero hold in positive characteristic if the geometric generic fibers have large Frobenius stable canonical rings.

In Chapter 3, we define some positivity conditions of coherent sheaves including the weak positivity, and introduce a numerical invariant which provides a sufficient condition for coherent sheaves to be weakly positive. This invariant plays important roles in the proofs of the main results of Chapters 4 and 5.

Chapter 4 establishes a weak positivity theorem for algebraic fiber spaces whose geometric generic fibers have finitely generated canonical rings and large Frobenius stable canonical rings. We apply it to prove Iitaka's $C_{n,m}$ conjecture in some cases in Chapter 6. For this application, we also show a result on the weak positivity of some subbundles of the direct image sheaf of a relative pluri-canonical bundle.

In Chapter 5, we consider what positivity conditions can be passed from the anti-canonical divisor on the total space of an algebraic fiber space to that on the base space. As a corollary of the mains results of this chapter, we show that in positive characteristic, the image of a weak Fano variety under a smooth morphism is again weak Fano. Another corollary shows that a relatively trivial relative canonical divisor is pseudo-effective if the geometric generic fiber has only F-pure singularities, which are an F-singularity theoretic counterpart of log canonical singularities. This second corollary is used in Chapter 6. Moreover, using modulo p reduction, we obtain some results which are valid in arbitrary characteristic.

In Chapter 6, we discuss Iitaka's $C_{n,m}$ conjecture in characteristic $p > 0$. We prove that it holds true for algebraic fiber spaces whose geometric generic fibers have finitely generated canonical rings and large Frobenius stable canonical rings, under the assumption that the base spaces are of general type or curves. Using this result, we show that the conjecture holds when the total spaces are 3-folds and the characteristic p is greater than 5.

In Chapter 7, using the results in Chapters 4 and 5, we show that the Albanese morphism $a : X \to A$ of a smooth projective variety X in positive characteristic is an algebraic fiber space if one of the following conditions is satisfied: (1) the anti-canonical divisor of X is nef and the geometric generic fiber of a has only F-pure singularities, or (2) the variety X is globally F-split, which means that the affine cone over X has only F-pure singularities. As a consequence, we establish a characterization of (ordinary) abelian varieties.

This thesis is based on the author's papers [27, 28, 29] and a joint paper [31] with Lei Zhang.

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Chapter 1

Preliminaries

In this chapter, we prepare notation, terminologies and fundamental notions which are used throughout this thesis. In particular, we introduce the notion of the trace maps of the (relative) Frobenius morphisms, which plays key roles in the proofs of the main theorems in this thesis.

1.1 Notation and terminology

Let k be a field. A *k-scheme* means a separated scheme of finite type over k. By *variety* over k we mean an integral k-scheme. A morphism $f : X \to Y$ between projective varieties over k is said to be an *algebraic fiber space* (or a *fibration*) if f is separable and the natural morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Let $\varphi : S \to T$ be a morphism of k-schemes and let T' be a T-scheme. Then we denote by $S_{T'}$ and $\varphi_{T'} : S_{T'} \to T'$ respectively the fiber product $S \times_T T'$ and its second projection. For a Cartier or $\mathbb{Q}$-Cartier divisor D on S (resp. an $\mathcal{O}_S$-module $\mathcal{G}$), the pullback of D (resp. $\mathcal{G}$) to $S_{T'}$ is denoted by $D_{T'}$ (resp. $\mathcal{G}_{T'}$) if it is well-defined. Similarly, for a homomorphism of $\mathcal{O}_S$-modules $\alpha : \mathcal{F} \to \mathcal{G}$, the pullback of α to $S_{T'}$ is denoted by $\alpha_{T'} : \mathcal{F}_{T'} \to \mathcal{G}_{T'}$.

Assume that k is of characteristic $p > 0$. We say that k is *F-finite* if the field extension k/k^p is finite. For a k-scheme X, $F_X : X \to X$ is the absolute Frobenius morphism. We often denote the source of ${F_X}^e$ by X^e. Let $f : X \to Y$ be a morphism between schemes of positive characteristic. The same morphism is denoted by $f^{(e)} : X^e \to Y^e$ when we regard X (resp. Y) as X^e (resp. Y^e). We define the e-th relative Frobenius morphism of f to be the morphism $F^{(e)}_{X/Y} := (F^e_X, f^{(e)}) : X^e \to X \times_Y Y^e =: X_{Y^e}$.

1.2 Almost Cartier divisors

Let k be a field of characteristic $p > 0$ and X be a k-scheme of pure dimension satisfying S_2 and G_1. An *AC divisor* (or *almost Cartier divisor*) on X is a coherent $\mathcal{O}_X$-submodule of the sheaf of total quotient rings $K(X)$ which is invertible in codimension one (see [75], [59], or [84]). For any AC divisor D we denote the coherent sheaf defining D by $\mathcal{O}_X(D)$. The set of AC divisors $\mathrm{WSh}(X)$ has the structure of additive group [59, Corollary 2.6]. Let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at $(p) = p\mathbb{Z}$. A $\mathbb{Z}_{(p)}$-*AC divisor* is an element of $\mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. An AC divisor D is said to be *effective* if $\mathcal{O}_X \subseteq \mathcal{O}_X(D)$, and a $\mathbb{Z}_{(p)}$-AC divisor Δ is said to be *effective* if $\Delta = D \otimes r$ for some effective AC divisor D and some $0 \le r \in \mathbb{Z}_{(p)}$. Now we have the following diagram:

$$\begin{tikzcd}
\mathrm{WSh}(X) \arrow[r, "(__)\otimes 1"] & \mathrm{WSh}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \\
\mathrm{CDiv}(X) \arrow[r, "(__)\otimes 1"] \arrow[u, hook] & \mathrm{CDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \arrow[u, hook]
\end{tikzcd}$$

We note that the horizontal homomorphisms are not necessarily injective [75, Page 172]. **Throughout this thesis, given an effective $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Z}_{(p)}$-Cartier) divisor Δ, we fix an effective AC (resp. Cartier) divisor E and an integer $a > 0$ not divisible by p such that $E \otimes 1 = a\Delta$.** The choice of E and a is often represented by $\Delta = E/a$. For every integer m, we regard the $\mathbb{Z}_{(p)}$-AC divisor $am\Delta$ as the AC divisor mE. For instance, the symbol $\mathcal{O}_X(am(D + \Delta))$ denotes the sheaf $\mathcal{O}_X(amD + mE)$, for every AC divisor D. For a morphism (resp. immersion) $\pi : Y \to X$ from a k-scheme Y of pure dimension satisfying S_2 and G_1, we set $\pi^*\Delta = (\pi^*E)/a$ (resp. $\Delta|_Y = E|_Y/a$) if π^*E (resp. $E|_Y$) can be defined.

We also note that if X is a normal variety, then AC divisors are Weil divisors, and the horizontal homomorphisms in the above diagram is injective. In this case, we can choose E and a canonically for an effective $\mathbb{Z}_{(p)}$-divisors Δ: a is the smallest positive integer such that $a\Delta$ is integral and $E := a\Delta$.

Using $\mathbb{Q}$ instead of $\mathbb{Z}_{(p)}$, we define concepts similar to the above.

1.3 Trace maps

In this section, we recall the notion of the trace maps of the absolute and relative Frobenius morphisms. Throughout this section, we work over an F-finite field k of characteristic $p > 0$.

1.3.1 Trace maps of the absolute Frobenius morphisms

We start with recalling the trace maps of finite morphisms. Let $\pi : X \to Y$ be a finite surjective morphism between Gorenstein k-schemes of pure dimension, ω_X and ω_Y be dualizing sheaves of X and Y, respectively. Applying the functor $\mathcal{H}om_{\mathcal{O}_Y}(\_\_, \omega_Y)$ to the natural morphism $\pi^\# : \mathcal{O}_Y \to \pi_*\mathcal{O}_X$, we obtain the morphism $\mathrm{Tr}_\pi : \pi_*\omega_X \to \omega_Y$ of $\mathcal{O}_Y$-modules. This is called the trace map of π. Using this, we define

$$\phi_X^{(1)} := \mathrm{Tr}_{F_X} \otimes \omega_X^{-1} : F_{X*}\omega_X^{1-p} \to \mathcal{O}_X, \quad \text{and}$$
$$\phi_X^{(e+1)} := \phi_X^{(e)} \circ F^e_{X*}(\phi_X^{(1)} \otimes \omega_X^{1-p^e}) : F^{e+1}_{X\ *}\omega_X^{1-p^{e+1}} \to \mathcal{O}_X$$

for each $e > 0$.

Next we extend the definition of $\phi_X^{(e)}$ for pairs. Let X be a Gorenstein k-scheme of pure dimension, $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-Cartier divisor on X and $d > 0$ be the smallest integer such that $a|(p^d - 1)$. Let K_X be a Cartier divisor satisfying $\mathcal{O}_X(K_X) \cong \omega_X$. For each $e > 0$ we define

$$\mathcal{L}^{(de)}_{(X,\Delta)} := \mathcal{O}_X((1-p^{de})(K_X+\Delta)) \subseteq \mathcal{O}_X((1-p^{de})K_X),$$

$$\phi^{(d)}_{(X,\Delta)} : F^d_{X*}\mathcal{L}^{(d)}_{(X,\Delta)} \to F^d_{X*}\mathcal{O}_X((1-p^d)K_X) \xrightarrow{\phi_X^{(d)}} \mathcal{O}_X, \quad \text{and}$$
$$\phi^{(d(e+1))}_{(X,\Delta)} := \phi^{(de)}_{(X,\Delta)} \circ F^{de}_{X}{}_*(\phi^{(d)}_{(X,\Delta)} \otimes \mathcal{L}^{(de)}_{(X,\Delta)}) : F^{d(e+1)}_{X}{}_*\mathcal{L}^{(d(e+1))}_{(X,\Delta)} \to \mathcal{O}_X.$$

We further extend the above notion to a more general case. Let X be a k-scheme of pure dimension satisfying S_2 and G_1, $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-AC divisor on X and d be as above. Let $\iota : U \hookrightarrow X$ be a Gorenstein open subset of X such that $\mathrm{codim} X \setminus U \geq 2$ and that $E|_U$ is Cartier. Then for each $e > 0$ we define

$$\mathcal{L}^{(de)}_{(X,\Delta)} := \iota_*\mathcal{L}^{(de)}_{(U,\Delta|_U)} \quad \text{and} \quad \phi^{(de)}_{(X,\Delta)} := \iota_*(\phi^{(de)}_{(U,\Delta|_U)}) : F^{de}_{X}{}_*\mathcal{L}^{(de)}_{(X,\Delta)} \to \mathcal{O}_X$$

Note that $\phi^{(de)}_{(X,\Delta)}$ is a morphism between reflexive sheaves on X (cf. [59, Proposition 1.11]). Using the trace maps of the Frobenius morphisms, we can define F-pure and strongly F-regular singularities of pairs.

Definition 1.3.1 ([56, Definition 2.1] or [84, Definition 2.6])**.** Let X be a reduced k-scheme of pure dimension satisfying S_2 and G_1. Let Δ be an effective $\mathbb{Q}$-AC divisor on X. (1) The pair (X, Δ) is said to be *F-pure* if for every $e > 0$ and for every effective AC divisor D with $D \leq (p^e - 1)\Delta$, the morphism

$$\phi^{(e)}_{(X,D/(p^e-1))} : F^e_{X*}\mathcal{O}_X((1-p^e)K_X - D) \to \mathcal{O}_X$$

is surjective. We simply say that X is F-pure if $(X, 0)$ is F-pure.
(2) [98, Definition 3.1] Assume that X is a normal variety. (Then Δ is a $\mathbb{Q}$-Weil divisor.) The pair (X, Δ) is said to be *strongly F-regular* if for every effective Cartier divisor D, there exists an $e > 0$ such that

$$\phi^{(e)}_{(X,\lceil (p^e-1)\Delta\rceil + D/p^e-1)} : F^{(e)}_{X}{}_*\mathcal{O}_X(\lfloor (1-p^e)(K_X+\Delta)\rfloor - D) \to \mathcal{O}_X$$

is surjective. Here $\lceil \Delta \rceil$ (resp. $\lfloor \Delta \rfloor$) denotes the round up (resp. down) of Δ. We simply say that X is strongly F-regular if $(X, 0)$ is strongly F-regular.

Remark 1.3.2. (1) With the notation as in Definition 1.3.1, we assume that X is normal and affine. Then Definition 1.3.1 (1) is equivalent to [56, Definition 2.1] (2). Indeed, since $\lfloor (p^e-1)\Delta \rfloor \leq (p^e-1)\Delta$, the condition of Definition 1.3.1 implies that $\phi^{(e)}_{(X,\lfloor (p^e-1)\Delta\rfloor/(p^e-1))}$ is surjective. Conversely, since $\phi^{(e)}_{(X,\lfloor (p^e-1)\Delta\rfloor/(p^e-1))}$ factors through $\phi^{(e)}_{(X,D/(p^e-1))}$ for every effective Weil divisor D with $D \leq \lfloor (p^e-1)\Delta \rfloor$ (or equivalently $D \leq (p^e-1)\Delta$), the surjectivity of $\phi^{(e)}_{(X,\lfloor (p^e-1)\Delta\rfloor/(p^e-1))}$ implies the condition of Definition 1.3.1.
(2) Let (X, Δ) be a strongly F-regular pair and Δ' be an effective $\mathbb{Q}$-divisor on X. Then there exists an $0 < \varepsilon \in \mathbb{Q}$ such that $(X, \Delta + \varepsilon\Delta')$ is again strongly F-regular.

1.3.2 Trace maps of the relative Frobenius morphisms

We recall the notion of the trace maps of the relative Frobenius morphisms and study their properties. See [94] for more details.

Let $f : X \to Y$ be a morphism between Gorenstein k-schemes of pure dimension. We assume that either F_Y is flat (i.e., Y is regular) or f is flat. Then F_Y or f is a Gorenstein morphism, so X_{Y^1} is a Gorenstein k-scheme [57, III, §9]. We define the relative dualizing sheaf $\omega_{X/Y}$ of f to be $\omega_X \otimes f^*\omega_Y^{-1}$. Then we have

$$
\begin{aligned}
\omega_{X_{Y^1}/Y^1} :=& \omega_{X_{Y^1}} \otimes {f_{Y^1}}^*\omega_{Y^1}^{-1} \\
\cong& \omega_{X_{Y^1}} \otimes {f_{Y^1}}^*\omega_{Y^1}^{p-1} \otimes {f_{Y^1}}^*{F_Y}^*\omega_Y^{-1} \\
\cong& {(F_Y)_X}^!\mathcal{O}_X \otimes {(F_Y)_X}^*\omega_X \otimes ({f_{Y^1}}^*{F_Y}^!\mathcal{O}_Y)^{-1} \otimes {(F_Y)_X}^*f^*\omega_Y^{-1} \\
\cong& {(F_Y)_X}^*\omega_{X/Y} = (\omega_{X/Y})_{Y^1}
\end{aligned}
$$

by the assumption. Moreover, for positive integers d, e, we consider the following commutative diagram:

$$
\begin{tikzcd}
X^{de} \arrow[d] \arrow[dd, bend left=60, "F^{(di')}_{X^{di}/Y^{di}}"] \arrow[dddd, bend right=40, "F^{(de)}_{X/Y}"'] \arrow[ddddd, bend right=70, "f^{(de)}"'] \\
\vdots \arrow[d] \\
X^{di}_{Y^{de}} \arrow[d] \arrow[dd, bend left=60, "F^{(di)}_{X_{Y^{di'}}/Y^{di'}}"] \arrow[ddd, bend left=80, "f^{(di)}_{Y^{de}}"] \\
\vdots \arrow[d] \\
X_{Y^{de}} \arrow[d] \\
Y^{de}
\end{tikzcd}
\qquad
\begin{tikzcd}
\ddots \arrow[dr, "F_X^d"] & & & \\
\cdots \arrow[r] & X^{2d} \arrow[d, "F^{(d)}_{X^d/Y^d}"'] \arrow[dr, "F_X^d"] & & \\
\cdots \arrow[r] & X^d_{Y^{2d}} \arrow[r] \arrow[d, "F^{(d)}_{X_{Y^d}/Y^d}"'] & X^d \arrow[d, "F^{(d)}_{X/Y}"'] \arrow[dr, "F_X^d"] & \\
\cdots \arrow[r] & X_{Y^{2d}} \arrow[r] \arrow[d, "f_{Y^{2d}}"'] & X_{Y^d} \arrow[r, "(F_Y^d)_X"'] \arrow[d, "f_{Y^d}"'] & X \arrow[d, "f"'] \\
\cdots \arrow[r, "F_Y^d"'] & Y^{2d} \arrow[r, "F_Y^d"'] & Y^d \arrow[r, "F_Y^d"'] & Y.
\end{tikzcd}
$$

Here, we put $i' := e - i$. Then for each $e > 0$ we define

$$
\begin{aligned}
\phi^{(1)}_{X/Y} &:= \mathrm{Tr}_{F^{(1)}_{X/Y}} \otimes \omega_{X_{Y^1}}^{-1} : {F^{(1)}_{X/Y}}_*\omega_{X^1}^{1-p} \to \mathcal{O}_{X_{Y^1}}, \quad \text{and} \\
\phi^{(e+1)}_{X/Y} &:= \left(\phi^{(e)}_{X/Y}\right)_{Y^{e+1}} \circ {F^{(e)}_{X_{Y^1}/Y^1}}_* \left(\phi^{(1)}_{X^e/Y^e} \otimes \omega^{1-p^e}_{X^e_{Y^{e+1}}}\right) \\
&\quad : {F^{(e+1)}_{X/Y}}_*\omega_X^{1-p^{e+1}} \to \mathcal{O}_{X_{Y^{e+1}}}.
\end{aligned}
$$

Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-AC divisor on X and d be the smallest positive integer satisfying $a|(p^d-1)$. For each $e>0$ we define

$$\mathcal{L}^{(de)}_{(X/Y,\Delta)} := \omega^{1-p^{de}}_{X^{de}}((1-p^{de})\Delta) \subseteq \omega^{1-p^{de}}_{X^{de}},$$
$$\phi^{(d)}_{(X/Y,\Delta)} : F^{(d)}_{X/Y}{}_*\mathcal{L}^{(d)}_{(X/Y,\Delta)} \to F^{(d)}_{X/Y}{}_*\omega^{1-p^d}_{X^d} \xrightarrow{\phi^{(d)}_{X/Y}} \mathcal{O}_{X_{Y^d}},$$

and

$$\begin{aligned}&\phi^{(d(e+1))}_{(X/Y,\Delta)}\\ &:= \left(\phi^{(de)}_{(X/Y,\Delta)}\right)_{Y^{d(e+1)}} \circ F^{(de)}_{X_{Y^d}/Y^d}{}_*\left(\phi^{(d)}_{(X^{de},\Delta)/Y^{de}} \otimes \left(\mathcal{L}^{(de)}_{(X/Y,\Delta)}\right)_{Y^{d(e+1)}}\right)\\ &: F^{(d(e+1))}_{X/Y}{}_*\mathcal{L}^{(d(e+1))}_{(X/Y,\Delta)} \to \mathcal{O}_{X_{Y^{d(e+1)}}}.\end{aligned}$$

Let $f : X \to Y$ be a morphism between k-schemes of pure dimension. Assume that X satisfies S_2 and G_1, Y is Gorenstein, and f or F_Y is flat. Let E be an effective AC divisor on X, and a, d be as above. Let $\iota : U \hookrightarrow X$ be a Gorenstein open subset of X such that $\mathrm{codim} X \setminus U \geq 2$ and that $E|_U$ is Cartier. Then for each $e > 0$ we define

$$\mathcal{L}^{(de)}_{(X/Y,\Delta)} := \iota_{Y^{de}}{}_*\mathcal{L}^{(de)}_{(U/Y,\Delta|_U)}, \quad \text{and}$$
$$\phi^{(de)}_{(X/Y,\Delta)} := \iota_{Y^{de}}{}_*(\phi^{(de)}_{(U/Y,\Delta|_U)}) : F^{(de)}_{X/Y}{}_*\mathcal{L}^{(de)}_{(X/Y,\Delta)} \to \mathcal{O}_{X_{Y^{de}}}.$$

The following lemma is used in the proof of Theorem 5.2.5, which is one of the main theorems of Chapter 5.

Lemma 1.3.3. *Let $f : X \to Y$ be a projective morphism from a pure dimensional k-scheme X satisfying S_2 and G_1 to a variety Y. Let $V \subseteq Y$ be a regular open subset such that $f_V : X_V \to V$ is flat and $U \subseteq X$ be a Gorenstein open subset. Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-AC divisor on X whose support does not contain any component of any fiber over V. Assume that $aK_{X_V} + E_V$ is a Cartier divisor on X_V and that $\mathrm{codim}_{X_{\overline{y}}}(X_{\overline{y}} \setminus U_{\overline{y}}) \geq 2$ for every $y \in V$. Then the following holds.*
(1) [94, Corollary 3.31] *The set $V_0 := \{y \in V | (X_{\overline{y}}, \overline{\Delta|_{U_{\overline{y}}}}) \text{ is } F\text{-pure}\}$ is an open subset of V. Here $\overline{y} := \mathrm{Spec}\ \overline{k(y)}$ and $\overline{\Delta|_{U_{\overline{y}}}}$ is the $\mathbb{Z}_{(p)}$-AC divisor on $X_{\overline{y}}$ obtained as the unique extension of the $\mathbb{Z}_{(p)}$-Cartier divisor $\Delta|_{U_{\overline{y}}}$ on $U_{\overline{y}}$.*
(2) *Assume that V_0 is non-empty. Let A be a Cartier divisor on X such that A_{V_0} is f_{V_0}-ample. Then there exists an $m_0 > 0$ such that*

$$\begin{aligned}&f_{Y^e}{}_*(\phi^{(e)}_{(X/Y,\Delta)} \otimes \mathcal{O}_{Y^e}(mA_{Y^e} + N_{Y^e})) :\\ &f^{(e)}{}_*\mathcal{O}_{X^e}((1-p^e)(K_{X^e/Y^e}+\Delta) + p^e(mA+N)) \to f_{Y^e}{}_*\mathcal{O}_{X_{Y^e}}(mA_{Y^e} + N_{Y^e})\end{aligned}$$

is surjective over V_0 for each $m \geq m_0$, for every Cartier divisor N on X whose restriction N_{V_0} to X_{V_0} is f_{V_0}-nef and for every $e > 0$ with $a|(p^e-1)$.

Proof. Replacing $f : X \to Y$ by $f_V : X_V \to V$, we may assume that Y is regular and f is flat. Take an integer $e > 0$ with $a|(p^e - 1)$ and a point $y \in V$. Then by [94, Lemma 2.18] we get

$$\phi^{(e)}_{(X/Y,\Delta)}|_{U_{\overline{y}}} = \phi^{(e)}_{(U/Y,\Delta|_U)}|_{U_{\overline{y}}} \cong \phi^{(e)}_{\left(U_{\overline{y}}/\overline{y},\Delta|_{U_y}\right)} : F^{(e)}_{U_{\overline{y}}/\overline{y}*}\mathcal{L}^{(e)}_{(U_{\overline{y}}/\overline{y},\Delta|_{U_{\overline{y}}})} \to \mathcal{O}_{U_{\overline{y}^e}}. \tag{1.3.3.1}$$

Let $\iota_{\overline{y}} : U_{\overline{y}} \to X_{\overline{y}}$ be the open immersion, and $\overline{E|_{U_{\overline{y}}}}$ be the unique extension of $E|_{U_{\overline{y}}}$ to $X_{\overline{y}}$. Since $\mathcal{L}^{(e)}_{(X/Y,\Delta)}$ is invertible by the assumption, the natural morphism

$$\mathcal{L}^{(e)}_{(X/Y,\Delta)}|_{X_{\overline{y}}} \to \iota_{\overline{y}*}\mathcal{L}^{(e)}_{(U_{\overline{y}}/\overline{y},\Delta|_{U_{\overline{y}}})} \left(=: \mathcal{L}^{(e)}_{(X_{\overline{y}}/\overline{y},\overline{\Delta|_{U_{\overline{y}}}})} \right)$$

is an isomorphism. Hence, extending the morphism (1.3.3.1) to $X_{\overline{y}}$, we obtain that

$$\phi^{(e)}_{(X/Y,\Delta)}|_{X_{\overline{y}}} \cong \phi^{(e)}_{\left(X_{\overline{y}}/\overline{y},\overline{\Delta|_{U_y}}\right)} : F^{(e)}_{X_{\overline{y}}/\overline{y}*}\mathcal{L}^{(e)}_{(X_{\overline{y}}/\overline{y},\overline{\Delta|_{U_{\overline{y}}}})} \to \mathcal{O}_{X_{\overline{y}^e}}. \tag{1.3.3.2}$$

Then one can show (1) by an argument similar to the proof of [94, Corollary 3.31]. We prove (2). Replacing Y by V_0, we may assume that $V_0 = V = Y$. Then by (1.3.3.2), we have that $\phi^{(e)}_{(X/Y,\Delta)}|_{X_{\overline{y}}}$ is surjective for each $e > 0$ with $a|(p^e - 1)$ and every $y \in Y$, which implies that $\phi^{(e)}_{(X/Y,\Delta)}$ is surjective for each $e > 0$ with $a|(p^e - 1)$. Let $d > 0$ be the minimum integer such that $a|(p^d - 1)$. Note that we have $d|e$ for every integer $e > 0$ with $a|(p^e - 1)$. Applying Keeler's relative Fujita vanishing [71, Theorem 1.5] to the kernel of $\phi^{(d)}_{(X/Y,\Delta)}$, we obtain an integer $m_0 \gg 0$ such that

$$f_{Y^d*}\left(\phi^{(d)}_{(X/Y,\Delta)} \otimes (\mathcal{O}_X(mA+N))_{Y^d}\right) \tag{1.3.3.3}$$

is surjective for each $m \geq m_0$ and every f-nef Cartier divisor N on X. Replacing m_0 by a larger integer if necessary, we may assume that $am_0A - (K_{X/Y} + \Delta)$ is f-nef. We fix an integer $m \geq m_0$ and an f-nef divisor N on X. We show that

$$\psi^{(de)} := f_{Y^{de}*}\left(\phi^{(de)}_{(X/Y,\Delta)} \otimes (\mathcal{O}_X(mA+N))_{Y^{de}}\right)$$

is surjective for every integer $e > 0$ by induction on e. We have already seen that $\psi^{(d)}$ is surjective. We assume that $\psi^{(de)}$ is surjective. By the definition of $\phi^{(d(e+1))}_{(X/Y,\Delta)}$, we have

$$\begin{aligned}
\psi^{(d(e+1))} =& f_{Y^{d(e+1)}*}\left(\phi^{(d(e+1))}_{(X/Y,\Delta)} \otimes (\mathcal{O}_X(mA+N))_{Y^{d(e+1)}}\right) \\
\cong& F_Y^{d*}\left(f_{Y^{de}*}\left(\phi^{(de)}_{(X/Y,\Delta)} \otimes (\mathcal{O}_X(mA+N))_{Y^{de}}\right)\right) \\
&\qquad \circ f^{(de)}_{Y^{d(e+1)}*}\left(\phi^{(d)}_{(X^{de}/Y^{de},\Delta^{de})} \otimes \left(\mathcal{L}^{(de)}_{(X/Y,\Delta)}(p^{de}(mA+N))\right)_{Y^{d(e+1)}}\right) \\
\cong& F_Y^{d*}(\psi^{(de)}) \circ f^{(de)}_{Y^{d(e+1)}*}\left(\phi^{(d)}_{(X^{de}/Y^{de},\Delta^{de})} \otimes \left(\mathcal{L}^{(de)}_{(X/Y,\Delta)}(p^{de}(mA+N))\right)_{Y^{d(e+1)}}\right).
\end{aligned}$$

Since $\psi^{(de)}$ is surjective, $F_Y^{d*}(\psi^{(de)})$ is also surjective. We need to show that the morphism

$$f^{(de)}_{Y^{d(e+1)}*}\left(\phi^{(d)}_{(X^{de}/Y^{de},\Delta^{de})}\otimes\left(\mathcal{L}^{(de)}_{(X/Y,\Delta)}(p^{de}(mA+N))\right)_{Y^{d(e+1)}}\right) \tag{1.3.3.4}$$

is surjective. Here we recall that $f^{(de)}: X^{de}\to Y^{de}$ is nothing but $f: X\to Y$. Let $N_{de,m}$ denote the Cartier divisor

$$\begin{aligned}&(p^{de}-1)a^{-1}(amA-(K_{X^d/Y^d}+\Delta))+p^{de}N\\ =&(p^{de}-1)(m-m_0)A+(p^{de}-1)a^{-1}(am_0A-(K_{X^d/Y^d}+\Delta))+p^{de}N\end{aligned}$$

on X^d. Since $am_0A-(K_{X/Y}+\Delta)$ is f-nef, $N_{de,m}$ is $f^{(d)}$-nef. Now we can rewrite (1.3.3.4) as

$$\begin{aligned}&f_{Y^d*}\left(\phi^{(d)}_{(X/Y,\Delta)}\otimes\left(\mathcal{O}_X((1-p^{de})(K_{X/Y}+\Delta)+p^{de}(mA+N))\right)_{Y^d}\right)\\ =&f_{Y^d*}\left(\phi^{(d)}_{(X/Y,\Delta)}\otimes\left(\mathcal{O}_X(mA+(p^{de}-1)(mA-(K_{X/Y}+\Delta))+p^{de}N)\right)_{Y^d}\right)\\ =&f_{Y^d*}\left(\phi^{(d)}_{(X/Y,\Delta)}\otimes(\mathcal{O}_X(mA+N_{de,m}))_{Y^d}\right).\end{aligned}$$

Hence the required surjectivity follows from the surjectivity of (1.3.3.3). $\square$

1.4 Vector bundles on elliptic curves

In this section, we recall several facts on vector bundles on elliptic curves due to Atiyah and Oda. They are used in Chapters 6 and 7. Throughout this section, let C denote an elliptic curve over an algebraically closed field k of characteristic $p>0$.

Theorem 1.4.1 ([4, 89]). *Let $\mathcal{E}_C(r,d)$ be the set of isomorphism classes of indecomposable vector bundles of rank r and of degree d. Then the following holds:*

(1) ([4, Theorem 10]) *For each $r>0$, there exists a unique element $\mathcal{E}_{r,0}$ of $\mathcal{E}_C(r,0)$ such that $H^0(C,\mathcal{E}_{r,0})\neq 0$. Moreover, for every $\mathcal{E}\in\mathcal{E}_C(r,0)$ there exists an $\mathcal{L}\in\operatorname{Pic}^0(C)=\mathcal{E}_C(1,0)$ such that $\mathcal{E}\cong\mathcal{E}_{r,0}\otimes\mathcal{L}$.*

(2) *For every $\mathcal{E}\in\mathcal{E}_C(r,d)$,*

$$\left(\dim H^0(C,\mathcal{E}),\dim H^1(C,\mathcal{E})\right)=\begin{cases}(d,0) & \text{when } d>0\\ (0,-d) & \text{when } d<0\\ (0,0) & \text{when } d=0 \text{ and } \mathcal{E}\not\cong\mathcal{E}_{r,0}\\ (1,1) & \text{when } \mathcal{E}\cong\mathcal{E}_{r,0}.\end{cases}$$

(3) *Let $\mathcal{E}\in\mathcal{E}_C(r,d)$. If $d>r$ (resp. $d>2r$) then $\mathcal{E}$ is globally generated (resp. ample).*

(4) ([89, Corollary 2.9]) *When C is ordinary, $F_C^*\mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$. When C is supersingular, $F_C^*\mathcal{E}_{r,0} \cong \bigoplus_{1\le i\le \min\{r,p\}} \mathcal{E}_{\lfloor (r-i)/p\rfloor+1,0}$.*

(5) ([89, Theorem 2.16]) *Let $r > 0$ and d be coprime integers and $\mathcal{E}$ be an element of $\mathcal{E}_C(rh, dh)$ for some $h > 0$. When C is ordinary, ${F_C}^*\mathcal{E}$ is indecomposable. When C is supersingular, ${F_C}^*\mathcal{E}$ is indecomposable if and only if either $h = 1$, or $h \neq 1$ and $p \nmid r$.*

(6) ([89, Proposition 2.1]) *Let $\pi : C' \to C$ be an isogeny of degree r and $\mathcal{L}$ be a line bundle of degree d on C'. If r and d are coprime, then $\pi_*\mathcal{L} \in \mathcal{E}_C(r, d)$.*

In characteristic zero, the pullback of $\mathcal{E}_{r,0}$ by a finite morphism from an elliptic curve is again indecomposable. In contrast, in positive characteristic, the lemma below shows that the pullback of $\mathcal{E}_{r,0}$ can be a trivial vector bundle.

Lemma 1.4.2. *For each integer $r \geq 2$, there exists a finite morphism $\pi_r : C_r \to C$ from an elliptic curve C_r such that ${\pi_r}^*\mathcal{E}_{r,0} \cong {\mathcal{O}_{C_r}}^{\oplus r}$ and ${\pi_r}_*\mathcal{O}_{C_r} \cong \mathcal{E}_{p^{r-1},0}$.*

Proof. Recall that $\mathcal{E}_{1,0} \cong \mathcal{O}_C$ and that $\mathcal{E}_{r+1,0}$ is obtained as a non-trivial extension $\xi_r : 0 \to \mathcal{O}_C \to \mathcal{E}_{r+1,0} \to \mathcal{E}_{r,0} \to 0$. We first define $\pi_2 := \pi_{C,2} : C_2 \to C$. If C is ordinary, or equivalently, if ${F_C}^* : H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$ is an isomorphism, then we may assume that ${F_C}^*\xi_1 = \xi_1$. Let $\pi_{C,2} : C_2 \to C$ denote the étale cover defined by ξ_1. Then we have ${\pi_{C,2}}^*\xi_1 = 0$. If C is supersingular, or equivalently ${F_C}^*\xi_1 = 0$, then we set $\pi_{C,2} := F_C$. By the choice of $\pi_{C,2}$, we have ${\pi_{C,2}}^*\mathcal{E}_{2,0} \cong {\mathcal{O}_{C_2}}^{\oplus 2}$. For each $r \geq 2$, we define inductively $\pi_{r+1} := \pi_{C,r+1} := \pi_{C,r} \circ \pi_{C_r,2}$. Since ${\pi_{C,r}}^*\xi_r \in H^1(C_r, {\pi_{C,r}}^*\mathcal{E}_{C_r,0}) \cong H^1(C_r, {\mathcal{O}_{C_r}}^{\oplus r})$, we have $\pi_{C,r+1}^*\xi_r = 0$.

Next we prove the second statement. It is enough to show $\pi_*\mathcal{E}_{p^e,0} \cong \mathcal{E}_{p^{e+1},0}$ for each $e \geq 0$, where $\pi := \pi_2 : C_2 \to C$. Set $\mathcal{F} := \mathcal{F}^{(e)} := \pi_*\mathcal{E}_{p^e,0}$. Since ${\mathcal{E}_{p^e,0}}^* \cong \mathcal{E}_{p^e,0}$, we have $\mathcal{F}^* \cong \mathcal{F}$ by the Grothendieck duality. Let $\mathcal{F}_1, \ldots, \mathcal{F}_l$ be indecomposable vector bundles such that $\mathcal{F} \cong \bigoplus_{1\le i\le l} \mathcal{F}_i$ and r_i be the rank of $\mathcal{F}_i$. We may assume that $H^0(C, \mathcal{F}_1) \cong k$. We show $H^1(C, \mathcal{F}_1) \cong k$ by contradiction. If $H^1(C, \mathcal{F}_1) = 0$, then $\deg \mathcal{F}_1 = 1$, and so $\deg(\mathcal{F}_1 \otimes \mathcal{L}) = 1$ for every $\mathcal{L} \in \mathrm{Pic}^0(C)$. Then

$$0 \neq H^0(C, \mathcal{F}_1 \otimes \mathcal{L}) \subseteq H^0(C, \mathcal{F} \otimes \mathcal{L}) \cong H^0(C_2, \mathcal{E}_{p^e,0} \otimes \pi^*\mathcal{L}).$$

By Theorem 1.4.1 (1), we get $\pi^*\mathcal{L} \cong \mathcal{O}_{C_2}$, which is a contradiction. Since $H^i(C, \mathcal{F}) \cong H^i(C, \mathcal{F}_1) \cong k$ for $i = 0, 1$, we have $\deg \mathcal{F}_1 = \cdots = \deg \mathcal{F}_l = 0$.

In order to prove $l = 1$, we show that $\mathrm{Pic}^0(C) \xrightarrow{\pi^*} \mathrm{Pic}^0(C_2)$ is injective. Take an arbitrary $\mathcal{L} \in \mathrm{Pic}^0(C)$ so that $\pi^*\mathcal{L} \cong \mathcal{O}_{C_2}$. Since $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$ by the projection formula, the group $\{\mathcal{L}^m | m \in \mathbb{Z}\}$ acts on $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$. In addition, taking the determinant of $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$, we get $\mathcal{L}^p \cong \mathcal{O}_C$. Note that we have $\deg \pi = p$. Since $\mathcal{O}_C \to \mathcal{F}_1$ does not split, we see that $r_1 > 1$. Set $e = 0$. Then $\mathcal{F} = \mathcal{F}^{(0)} = \pi_*\mathcal{O}_{C_2}$. Considering ranks, we see that $\mathcal{F}_1 \otimes \mathcal{L}^m \cong \mathcal{F}_1$ for each $m \in \mathbb{Z}$. Then by Theorem 1.4.1 (1) again, we have $\mathcal{L} \cong \mathcal{O}_C$.

We show $l = 1$ for every $e \geq 0$. Now we have $\mathcal{L}_i \in \mathrm{Pic}^0(C)$ with $H^0(C, \mathcal{F}_i \otimes \mathcal{L}_i) \cong k$ for each $1 \leq i \leq l$. By the projection formula, we obtain $H^0(C_2, \mathcal{E}_{p^e,0} \otimes \pi^*\mathcal{L}_i) \cong H^0(C, \mathcal{F} \otimes \mathcal{L}_i) \cong k$, and thus $\pi^*\mathcal{L}_i \cong \mathcal{O}_{C_2}$ and $\mathcal{L}_i \not\cong \mathcal{L}_j$ for $i \neq j$. Hence by the above argument, we see that $l = 1$. ∎

Proposition 1.4.3. *Let $\mathcal{E}$ be a vector bundle on an elliptic curve C. Then there exists a finite morphism $\pi : C' \to C$ from an elliptic curve C' such that $\pi^*\mathcal{E}$ is a direct sum of line bundles.*

Proof. We may assume that for every finite morphism $\varphi : B \to C$ from an elliptic curve B, $\varphi^*\mathcal{E}$ is indecomposable. Set $d := \deg \mathcal{E}$ and $r := \mathrm{rank}\mathcal{E}$. We show that $r = 1$. Let $Q \in C$ be a closed point. Replacing $\mathcal{E}$ by $((r_C)^*\mathcal{E})(-dQ)$, we may assume that $d = 0$. Here $r_C : C \to C$ is the morphism given by multiplication by r. Then by Theorem 1.4.1 (1) and Lemma 1.4.2 there exists a finite morphism from an elliptic curve to C such that the pullback of $\mathcal{E}$ is a direct sum of line bundles. Hence $r = 1$. $\square$

Chapter 2

Stable global sections under traces of Frobenius morphisms

In this chapter, we recall the notion of the Frobenius stable canonical rings, and study its fundamental properties. As we see in Chapter 4, the Frobenius stable canonical rings of the geometric generic fibers are important from the view point of weak positivity theorems. In Section 2.1, after recalling definitions and basic properties, we study the Frobenius stable canonical rings of varieties with ample canonical bundles. We particularly consider the case of Gorenstein projective curves (Corollary 2.1.14). In Section 2.2, we discuss the case of varieties with semi-ample canonical bundles in any dimension (Corollary 2.2.3). To this end, we provide a canonical bundle formula (Theorem 2.2.2). Using this formula, we also study the Frobenius stable canonical rings of surfaces of general type (Corollary 2.2.8).

Throughout this chapter, we work over an algebraically closed field k of characteristic $p > 0$.

2.1 Frobenius stable canonical rings

We fix the following notation.

Notation 2.1.1. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a k-scheme of pure dimension satisfying S_2 and G_1, and let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-AC divisor. Set $d > 0$ be the smallest integer satisfying $a|(p^d - 1)$.

Definition 2.1.2 ([99, §3])**.** In the situation of Notation 2.1.1, let $\mathcal{M}$ be a reflexive sheaf on X of rank one such that invertible in codimension one. Then we define $S^0(X, \Delta, \mathcal{M})$ as

$$\bigcap_{e>0} \mathrm{im}\left(H^0(X, ((F^{de}_{X\,*}\mathcal{L}^{(de)}_{(X,\Delta)}) \otimes \mathcal{M})^{**}) \xrightarrow{H^0(X,(\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{M})^{**})} H^0(X, \mathcal{M}) \right),$$

where $\phi^{(de)}_{(X,\Delta)}$ is the morphism defined in Subsection 1.3.1, and $(\_\_)^{**} := \mathcal{H}om(\mathcal{H}om(\_\_, \mathcal{O}_X), \mathcal{O}_X)$ is the functor of the double dual. For any AC divisor D on X, we denote $S^0(X, \Delta, \mathcal{O}_X(D))$ by $S^0(X, \Delta, D)$. Write $S^0(X, D) := S^0(X, 0, D)$.

Remark 2.1.3. The above definition does not depend on the choice of E and a satisfying $\Delta = E/a$. Indeed, if E' and a' satisfy $\Delta = E'/a'$, then one can check that $\phi^{(eg)}_{(X,E/a)} \cong \phi^{(eg)}_{(X,E'/a')}$ for every $g > 0$ divisible enough.

Example 2.1.4. In the situation of Notation 2.1.1, it is easily seen that the following are equivalent:

(1) (X, Δ) is globally F-split, that is to say, there exists an $e > 0$ divisible enough such that the composite of natural morphisms $\mathcal{O}_X \to F^e_{X*}\mathcal{O}_X$ and $F^e_{X*}\mathcal{O}_X \to F^e_{X*}\mathcal{O}_X((p^e - 1)\Delta)$ splits as a homomorphism of $\mathcal{O}_X$-modules;

(2) $S^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$;

(3) $S^0(X, \Delta, D) = H^0(X, D)$ for every AC divisor D on X.

Definition 2.1.5 ([52, Section 4.1] or [93, Exercise 4.13])**.** In the situation of Notation 2.1.1, let $\mathcal{M}$ be a reflexive sheaf on X of rank one such that invertible in codimension one. Then we define

$$R_S(X, \Delta, \mathcal{M}) := \bigoplus_{n \geq 0} S^0(X, \Delta, \mathcal{M}^{[n]}) \subseteq R(X, \mathcal{M}) := \bigoplus_{n \geq 0} H^0(X, \mathcal{M}^{[n]}),$$

where $\mathcal{M}^{[n]} := (\mathcal{M}^{\otimes n})^{**}$. For any AC divisor D, we denote $R(X, \mathcal{O}_X(D))$ and $R_S(X, \Delta, \mathcal{O}_X(D))$ respectively by $R(X, D)$ and $R_S(X, \Delta, D)$. We call $R_S(X, \Delta, a(K_X + \Delta))$ the *Frobenius stable canonical ring*, where K_X is an AC divisor such that $\mathcal{O}_X(K_X)$ is isomorphic to the dualizing sheaf ω_X of X.
When D is a $\mathbb{Q}$-Weil divisor on a normal variety X, we define

$$R_S(X, \Delta, D) := \bigoplus_{n \geq 0} S^0(X, \Delta, \lfloor nD \rfloor) \subseteq R(X, D) := \bigoplus_{n \geq 0} H^0(X, \lfloor nD \rfloor).$$

Lemma 2.1.6 ([52, Lemma 4.1.1])**.** *$R_S(X, \Delta, D)$ is an ideal of $R(X, D)$.*

Proof. This follows from an argument similar to the proof of [52, Lemma 4.1.1]. □

Notation 2.1.7. We denote by $R/R_S(X, \Delta, D)$ the quotient ring of $R(X, D)$ modulo $R_S(X, \Delta, D)$.

We recall that assumption (ii) of the main theorem (Theorem 4.1.1): there exists an $m_0 > 0$ such that $S^0(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, am(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) = H^0(X_{\overline{\eta}}, am(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ for every $m \geq m_0$. This is equivalent to the condition that there exists an integer $m_0 > 0$ such that the degree m part of $R/R_S(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, a(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ is zero for every $m \geq m_0$. Note that the existence of such m_0 is equivalent to the finiteness of the dimension of k-vector space $R/R_S(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, a(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$

Definition 2.1.8. In the situation of Notation 2.1.1, assume that each connected component of X is integral. An AC divisor D is said to be *finitely generated* if $R(X, D)$ is a finitely generated k-algebra. A $\mathbb{Z}_{(p)}$-AC (resp. $\mathbb{Q}$-AC) divisor Γ is said to be *finitely generated* if there exists a finitely generated AC divisor D such that $\Gamma = D \otimes \lambda$ for some $0 < \lambda \in \mathbb{Z}_{(p)}$ (resp. $\mathbb{Q}$).

Lemma 2.1.9. *Let $R = \bigoplus_{m \geq 0} R_m$ be a graded ring. Assume that R is a domain and R_0 is a field.*

(1) *If the n-th Veronese subring $R^{(n)} := \bigoplus_{m\geq 0} R_{mn}$ is a finitely generated R_0-algebra for some $n > 0$, then so is R.*

(2) *Let $\mathfrak{a} \subseteq R$ be a nonzero homogeneous ideal, and suppose that R is a finitely generated R_0-algebra. If $R^{(n)}/\mathfrak{a}^{(n)}$ is a finite dimensional R_0-vector space for some $n > 0$, then so is $R/\mathfrak{a}$, where $\mathfrak{a}^{(n)} := \bigoplus_{m\geq 0} \mathfrak{a}_{mn}$.*

Proof. For the proof of (1) we refer the proof of [51, Lemma 5.68]. For (2), let $l > 0$ be an integer divisible enough. Then there exists $n_0 > 0$ such that $\mathfrak{a}_{l+n} \subseteq R_{l+n} = R_n \cdot R_l = R_n \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{l+n}$ for each $n \geq n_0$, and hence $\mathfrak{a}_m = R_m$ for each $m \gg 0$, which is our claim. □

As mentioned after Notation 2.1.7, assumption (ii) of the main theorem (Theorem 4.1.1) satisfied if and only if $R/R_S(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, a(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ is finite dimensional as k-vector space. This condition is equivalent to the condition that $R/R_S(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, an(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ is finite dimensional for an integer $n > 0$ by (2) of the above lemma.

Definition 2.1.10. In the situation of Notation 2.1.1, we denote the kernel of $\phi^{(de)}_{(X,\Delta)} : F^{de}_{X\,*}\mathcal{L}^{(de)}_{(X,\Delta)} \to \mathcal{O}_X$ by $\mathcal{B}^{de}_{(X,\Delta)}$ for every integer $e > 0$. When $\Delta = 0$, we denote $\mathcal{B}^e_{(X,0)}$ by $\mathcal{B}^e_X$.

Example 2.1.11. In the situation of Notation 2.1.1, assume that X is projective, and $(p^c - 1)(K_X + \Delta)$ is Cartier for some $c > 0$ divisible by d. Let H be an ample Cartier divisor. We show that $R/R_S(X, \Delta, H)$ is finite dimensional if and only if (X, Δ) is F-pure. By the Fujita vanishing theorem, there is an $m > 0$ such that $H^1(X, \mathcal{B}^c_{(X,\Delta)}(mH + N)) = 0$ for every nef Cartier divisor N. We may assume that $mH - (K_X + \Delta)$ is nef. If (X, Δ) is F-pure, or equivalently if $\phi^{(c)}_{(X,\Delta)}$ is surjective, then so is the morphism $H^0(X, \phi^{(c)}_{(X,\Delta)} \otimes \mathcal{O}_X(mH + N))$. Furthermore we see that $H^0(X, \phi^{(ce)}_{(X,\Delta)} \otimes \mathcal{O}_X(mH + N))$ is also surjective for each $e > 0$, because of the definition of $\phi^{(ce)}_{(X,\Delta)}$ and the following isomorphisms

$$
\begin{aligned}
&\left(F^{ce}_{X\,*}(\phi^{(c)}_{(X,\Delta)} \otimes \mathcal{L}^{(ce)}_{(X,\Delta)})\right) \otimes \mathcal{O}_X(mH + N) \\
\cong &F^{ce}_{X\,*}\left(\phi^{(c)}_{(X,\Delta)} \otimes \mathcal{O}_X(mH + (p^{ce} - 1)(mH - (K_X + \Delta)) + p^{ce}N)\right).
\end{aligned}
$$

This implies that $S^0(X, \Delta, mH+N) = H^0(X, mH+N)$ and that $R/R_S(X, \Delta, H)$ is finite dimensional. Conversely it is clear that if $R/R_S(X, \Delta, H)$ is finite dimensional, then $\phi^{(c)}_{(X,\Delta)}$ is surjective, or equivalently, (X, Δ) is F-pure.

The above example shows that if $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure and $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is an ample $\mathbb{Z}_{(p)}$-Cartier divisor, then assumption (ii) (and (i)) of the main theorem (Theorem 4.1.1) holds. We next consider the value of such m_0 in the case when $X_{\overline{\eta}}$ is a curve. Corollary 2.1.14 provides a value of such m_0 effectively when $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is ample.

Lemma 2.1.12. *Let X be a Gorenstein projective curve, and let H be an ample Cartier divisor such that $H - K_X$ is nef. Then for each integer $e, m \geq 1$,*

$$H^1(X, \mathcal{B}^e_X \otimes \mathcal{O}_X(K_X + mH)) = 0.$$

Moreover if X is F-pure, then

$$S^0(X, K_X + mH) = H^0(X, K_X + mH).$$

Proof. Clearly the second statement follows from the first and the long exact sequence of cohomology induced from the surjective morphism $\phi^{(e)}_{(X,\Delta)} \otimes \mathcal{O}_X(K_X + mH)$. We prove the first statement. Let $\nu : C \to X$ be the normalization. Then a commutative diagram of varieties

$$\begin{tikzcd} C \arrow[r, "F^e_C"] \arrow[d, "\nu"'] & C \arrow[d, "\nu"] \\ X \arrow[r, "F^e_X"'] & X \end{tikzcd}$$

induces a commutative diagram of $\mathcal{O}_X$-modules:

$$\begin{tikzcd} 0 \arrow[r] & \nu_*\mathcal{B}^e_C(K_C) \arrow[r] \arrow[d, "\alpha"] & \nu_* F^e_{C*}\omega_C \arrow[r, "\nu_*\mathrm{Tr}_{F^e_C}"] \arrow[d, "F^e_{X*}\mathrm{Tr}_\nu"] & \nu_*\omega_C \arrow[r] \arrow[d, "\mathrm{Tr}_\nu"] & 0 \\ 0 \arrow[r] & \mathcal{B}^e_X(K_X) \arrow[r] & F^e_{X*}\omega_X \arrow[r, "\mathrm{Tr}_{F^e_X}"'] & \omega_X & \end{tikzcd}$$

Since each vertical morphism is an isomorphism on some dense open subset of X, the kernel and the cokernel of α are torsion $\mathcal{O}_X$-modules. Furthermore since $\mathcal{B}^e_C$ has no torsion, we see that α is injective. For each $m > 0$, the following exact sequence

$$0 \to (\nu_*\mathcal{B}^e_C(K_C))(mH) \xrightarrow{\alpha \otimes \mathcal{O}_X(mH)} \mathcal{B}^e_X(K_X + mH) \to \mathrm{coker}(\alpha) \to 0,$$

induces a surjection

$$H^1(C, \mathcal{B}^e_C(K_C + m\nu^*H)) \cong H^1(X, (\nu_*\mathcal{B}^e_C(K_C))(mH)) \twoheadrightarrow H^1(X, \mathcal{B}^e_X(K_X + mH)).$$

Moreover, since ν^*H is ample and

$$\nu^*H - K_C = \nu^*(H - K_X) + \nu^*K_X - K_C \sim \nu^*(H - K_X) + E$$

is nef, where E is effective divisor on C defined by the conductor ideal, we may assume that X is smooth. Then we have $H^1(X, mpH) = H^0(X, K_X - mpH) = 0$ for each $m \geq 1$ by the Serre duality. For each $m \geq 1$ there exists an exact sequence

$$0 \to \mathcal{O}_X(mH) \to F_{X*}\mathcal{O}_X(mpH) \to \mathcal{B}^1_X(K_X + mH) \to 0$$

induced by Cartier operator, which shows that $H^1(X, \mathcal{B}^1_X(K_X + mH)) = 0$. This implies $H^0(X, \phi^{(1)}_X \otimes \mathcal{O}_X(K_X))$ is surjective, and thus $H^0(X, \phi^{(e)}_X \otimes \mathcal{O}_X(K_X + mH))$

is also surjective for every $e, m \geq 1$ because of the definition of $\phi_X^{(e)}$. Hence the exact sequence

$$0 \to \mathcal{B}_X^e(K_X + mH) \to F_{X*}^e \mathcal{O}_X(K_X + mp^e H) \xrightarrow{\phi_X^{(e)} \otimes \mathcal{O}_X(K_X+mH)} \mathcal{O}_X(K_X + mH) \to 0$$

induces the following:

$$H^1(X, \mathcal{B}_X^e(K_X + mH)) \hookrightarrow H^1(X, \mathcal{O}_X(K_X + mp^e H)) \cong H^0(X, -mp^e H) = 0.$$

□

Proposition 2.1.13. *In the situation of Notation* 2.1.1*, let X be a projective curve, let $K_X + \Delta$ is nef and let H be a Cartier divisor. Assume either that* (i) $H + (a-1)K_X$ *is ample and* $H + (a-2)K_X$ *is nef, or that* (ii) $X \cong \mathbb{P}^1$ *and H is ample. Then for each $e > 0$,*

$$H^1(X, \mathcal{B}_{(X,\Delta)}^{de} \otimes \mathcal{O}_X(a(K_X + \Delta) + H)) = 0.$$

Moreover if (X, Δ) is F-pure, then

$$S^0(X, \Delta, a(K_X + \Delta) + H) = H^0(X, a(K_X + \Delta) + H).$$

Proof. Clearly the second statement follows from the first and the long exact sequence of cohomology induced from the surjective morphism $\phi_{(X,\Delta)}^{(e)} \otimes \mathcal{O}_X(K_X + mH)$. We prove the first statement. Let E' be an effective Cartier divisor satisfying $\mathcal{O}_X(E') \subseteq \mathcal{O}_X(E)$ and $\Delta' := E'/a$. For each $e > 0$ there is a commutative diagram

$$\begin{tikzcd} F_{X*}^{de} \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}(a(K_X + \Delta) + H)) \arrow[r, "\phi_{(X,\Delta)}^{(de)} \otimes \mathcal{O}_X(a(K_X+\Delta)+H)"] & \mathcal{O}_X(a(K_X + \Delta) + H) \\ F_{X*}^{de} \mathcal{L}_{(X,\Delta)}^{(de)}(p^{de}(a(K_X + \Delta') + H)) \arrow[r, "\phi_{(X,\Delta')}^{(de)} \otimes \mathcal{O}_X(a(K_X+\Delta')+H)"] \arrow[u, hook] & \mathcal{O}_X(a(K_X + \Delta') + H), \arrow[u, hook] \end{tikzcd}$$

where the vertical morphisms are natural inclusion. This induces the injective morphism

$$\mathcal{B}_{(X,\Delta')}^{de}(a(K_X + \Delta') + H) \to \mathcal{B}_{(X,\Delta)}^{de}(a(K_X + \Delta) + H)$$

whose cokernel is a torsion $\mathcal{O}_X$-module. Hence it suffices to prove that $H^1(X, \mathcal{B}_{(X,\Delta)}^{de}(a(K_X + \Delta') + H)) = 0$. When (i) holds, we set $E' = 0$. By the previous lemma we have $H^1(X, \mathcal{B}_X^{de}(aK_X + H)) = 0$. When (ii) holds, we may assume $a(K_X + \Delta') \sim 0$. Then it is easily seen that $\dim H^1(X, \mathcal{B}_{(X,\Delta')}^{de}) \leq 1$. Since every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles, we have $H^1(X, \mathcal{B}_{(X,\Delta')}^{de}(H)) = 0$. This completes the proof. □

The following corollary will be used to prove weak positivity theorem for fibrations of relative dimension one (Corollary 4.2.4).

Corollary 2.1.14. *In the situation of Notation* 2.1.1*, assume that X is a projective curve and (X, Δ) is F-pure. If $K_X + \Delta$ is ample (resp. K_X is ample and $a \geq 2$), then for each $m \geq 2$ (resp. $m \geq 1$),*

$$S^0(X, \Delta, am(K_X + \Delta)) = H^0(X, am(K_X + \Delta)).$$

Proof. We note that a Gorenstein curve has nef dualizing sheaf unless it is isomorphic to $\mathbb{P}^1$. Hence the statement follows from the above proposition. □

Remark 2.1.15. F-pure singularities of curves are completely classified [48]. For example, nodes are F-pure singularities, but cusps are not.

2.2 A canonical bundle formula

We next study the Frobenius stable canonical rings of varieties with semi-ample canonical bundles in any dimension. To this end, we establish a canonical bundle formula (Theorem 2.2.2) for algebraic fiber spaces with globally F-split geometric generic fibers. As a corollary, we obtain a criterion of the finiteness of the dimension of R/R_S in terms of singularities of the canonical models (Corollary 2.2.3).

Throughout this section, we fix an algebraically closed field k of characteristic $p > 0$.

In order to formulate the problem, we start with an observation of the Iitaka fibrations.

Observation 2.2.1. Let X be a normal projective variety, and let Δ be an effective $\mathbb{Z}_{(p)}$-Weil divisor on X such that $K_X + \Delta$ is a semi-ample $\mathbb{Q}$-Cartier divisor. Let

$$f : X \to Y := \mathrm{Proj}\ R(X, K_X + \Delta)$$

be the Iitaka fibration. Then there exists an ample $\mathbb{Q}$-Cartier divisor H on Y satisfying $f^*H \sim_{\mathbb{Q}} K_X + \Delta$. Let $Y_0 \subseteq Y$ be an open subset such that $f_0 := f|_{X_0} : X_0 \to Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$.
(I) Assume that $R_S(X, \Delta, K_X + \Delta) \neq 0$. Then there exists an integer $m > 0$ such that $m\Delta$ is integral and $S^0(X, \Delta, m(K_X + \Delta)) \neq 0$. This implies that

$$S^0(X, \Delta, ((m-1)p^{e'} + 1)(K_X + \Delta)) \neq 0$$

for some $e' > 0$ divisible enough. Since $p \nmid (m-1)p^{e'} + 1$, there exists an $e > 0$ such that $S^0(X, \mathcal{O}_X((p^e - 1)(K_X + \Delta))) \neq 0$. We set $R' := (1 - p^e)(K_X + \Delta)$. Let η be the generic point of Y. By the assumption, $\mathcal{O}_X(-R')|_{X_\eta}$ is a torsion line bundle on X_η with nonzero global sections, and thus it is trivial. Hence $\mathcal{O}_X(R')|_{X_\eta}$ is also trivial, and $f_*\mathcal{O}_X(R')$ is a torsion free sheaf on Y of rank one. Then there exists an effective Weil divisor B supported on $X \setminus X_0$ such that $f_*\mathcal{O}_X(R' + B) \cong \mathcal{O}_Y(S)$ for some Weil divisor S on Y. We set $R := R' + B = (1 - p^e)(K_X + \Delta) + B$. Then

$$R = K_X + \Delta + B - p^e(K_X + \Delta) \sim_{\mathbb{Q}} K_X + \Delta + B - p^e f^*H.$$

Replacing e, we may assume that p^eH is $\mathbb{Z}_{(p)}$-Cartier, and thus there exists an integer $a > 0$ not divisible by p such that $a\Delta$ is integral and $H' := ap^eH$ is Cartier. Then we have

$$aR \sim a(K_X + \Delta) + aB - f^*H'.$$

(II) In the situation of (I), after replacing e by its multiple, we assume that $(p^e - 1)(K_X + \Delta)$ is base point free. Then we may take $(p^e - 1)H$ as Cartier. In this case, we have $\mathcal{O}_X(R') \cong f^*\mathcal{O}_Y((1 - p^e)H)$, and thus we may choose $B = 0$, $R = R'$ and $S = (1 - p^e)H$ by the projection formula. In particular we have $R \sim f^*S$.

In a more general situation than the above, we prove the following theorem which is a kind of canonical bundle formula (see [25, Theorem B] for a related result).

Theorem 2.2.2. *Let $f : X \to Y$ be an algebraic fiber space between normal varieties, let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p, and let Y_0 be a smooth open subset of Y such that* $\mathrm{codim} Y \setminus Y_0 \geq 2$ *and $f_0 := f|_{X_0} : X_0 \to Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$. Further assume that the following conditions:*

(i) *The pair $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is globally F-split, where $\overline{\eta}$ is geometric generic point of Y.*

(ii) *There exists a Weil divisor R on X, such that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y(S)$ for some Weil divisor S on Y and $aR \sim a(K_X + \Delta) + B - f^*C$ for some effective Weil divisor B supported on $X \setminus X_0$ and for some Cartier divisor C on Y.*

Then, there exists an effective $\mathbb{Q}$-Weil divisor Δ_Y on Y, which satisfies the following conditions:

(1) *For some integer $a' > 0$ divisible by a but not by p, $a'\Delta_Y$ is integral, and*

$$\mathcal{O}_Y(a'(K_Y + \Delta_Y - S)) \cong \mathcal{O}_Y(a'a^{-1}C) \cong f_*\mathcal{O}_X(a'(K_X + \Delta + a^{-1}B - R)).$$

(2) *For every effective Weil divisor B' supported on $X \setminus X_0$ and for every Cartier divisor D on Y,*

$$S^0(X, \Delta, B' + f^*D + R) \cong S^0(Y, \Delta_Y, D + S).$$

(3) *If f is a birational morphism, then $\Delta_Y = f_*\Delta$.*

(4) *Suppose that X_0 is Gorenstein and $R|_{X_0}$ is Cartier. Let Γ be an effective Cartier divisor on X_0 defined by the image of the natural morphism*

$$\mathcal{O}_{X_0}(-R|_{X_0}) \otimes {f_0}^*(f_{0*}\mathcal{O}_{X_0}(R|_{X_0})) \to \mathcal{O}_{X_0},$$

and let y be a point of Y_0. Then the following conditions are equivalent:

(a) $\mathrm{Supp}\, \Delta$ *does not contain any irreducible component of $f^{-1}(y)$, and $(X_{\overline{y}}, \Delta_{\overline{y}})$ is globally F-split, where $\overline{y}$ is the algebraic closure of y;*

(b) *y is not contained in* $f(\mathrm{Supp}\,\Gamma) \cup \mathrm{Supp}\,\Delta_Y$*.*

Note that if R is linearly equivalent to the pullback of a Cartier divisor on Y, then replacing C, we may assume that $R = 0$, $S = 0$ and $\Gamma = 0$.

For varieties with semi-ample canonical bundles, Corollary 2.2.3 provides a criterion of the finiteness of the dimension of R/R_S in terms of the singularity of the canonical models. As explained after Notation 2.1.7, the finiteness of R/R_S is equivalent to assumption (ii) of the main theorem (Theorems 4.1.1 or 4.2.1). We remark that for such varieties, assumption (i) of the main theorem, that is the finitely generation of canonical rings, is always satisfied.

Corollary 2.2.3. *In the situation of Observation 2.2.1* (I)*, the following holds:*

(1) *The pair* $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ *is globally F-split, where* $\overline{\eta}$ *is the geometric generic point of* Y*. In particular,* f *is separable.*

(2) *Let* Δ_Y *be as in Theorem* 2.2.2*. In the situation of Observation 2.2.1* (II) (i.e. $l(K_X+\Delta)$ *is Cartier for an integer* $l > 0$ *not divisible by* p)*,* $R/R_S(X, \Delta, K_X + \Delta)$ *is a finite dimensional k-vector space if and only if* (Y, Δ_Y) *is F-pure.*

Before the proof of Theorem 2.2.2 and Corollary 2.2.3, we observe the morphisms obtained by pushing forward of the trace maps of the relative Frobenius morphisms.

Observation 2.2.4. Let $f : X \to Y$ be a projective morphism from a Gorenstein variety X to a smooth variety Y. Let $\Delta = E/a$ be an effective $\mathbb{Z}_{(p)}$-AC divisor on X whose support does not contain any irreducible component of any fiber of f. Let d be the smallest positive integer satisfying $a|(p^d - 1)$. Let $e \geq 0$ be an integer.
(I) For every $y \in Y$, we have the following diagram:

$$\begin{tikzcd}
(X_{\overline{y}})^{de} \arrow[r, equal] & X^{de}_{\overline{y}^{de}} \arrow[r] \arrow[d, "F^{(de)}_{X_{\overline{y}}/\overline{y}}"'] & X^{de} \arrow[d, "F^{(de)}_{X/Y}"'] \arrow[dd, bend left, "f^{(de)}"] \\
 & X_{\overline{y}^{de}} \arrow[r] \arrow[d] & X_{Y^{de}} \arrow[d, "f_{Y^{de}}"'] \\
 & \overline{y}^{de} \arrow[r] & Y^{de}
\end{tikzcd}$$

Let R be a Cartier divisor on X. We denote by $\theta^{(de)}$ the morphism

$$f_{Y^{de}*}(\phi^{(de)}_{(X,\Delta)/Y} \otimes \mathcal{O}_X(R)_{Y^{de}}) : f^{(de)}{}_*\mathcal{L}^{(de)}_{(X,\Delta)/Y}(p^{de}R) \to f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(R)_{Y^{de}}.$$

Here we recall that $\mathcal{L}^{(de)}_{(X,\Delta)/Y} := \mathcal{O}_{X^{de}}((1-p^{de})(K_{X^{de}/Y^{de}} + \Delta))$.
(II) Let $Y_0 \subseteq Y$ be an open subset such that $f_0 := f|_{X_0} : X_0 \to Y_0$ is flat, where $X_0 := f^{-1}(Y_0)$. Assume that $y \in Y_0$ and that $E|_{Y_0}$ is Cartier. Since f_0 is a Gorenstein morphism, $X_{\overline{y}}$ is Gorenstein. Set $\Delta_{\overline{y}} = E|_{X_{\overline{y}}}/a$. Then $\mathcal{L}^{(de)}_{(X,\Delta)/Y}|_{(X_{\overline{y}})^{de}} \cong \mathcal{L}^{(de)}_{(X_{\overline{y}}/\overline{y}, \Delta_{\overline{y}})}$

and we have the following diagram of $k(\overline{y}^{de})$-vector spaces for every $e > 0$:

$$\begin{tikzcd}
H^0((X_{\overline{y}})^{de}, \mathcal{L}^{(de)}_{(X_{\overline{y}}/\overline{y}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(p^{de}R)|_{(X_{\overline{y}})^{de}}) \arrow[d, "H^0(X_{\overline{y}^{de}}, \phi^{(de)}_{(X_{\overline{y}}/\overline{y}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(p^{de}R)|_{X_{\overline{y}^{de}}})"'] & \left(f^{(de)}{}_*\mathcal{L}^{(de)}_{(X,\Delta)/Y}(p^{de}R)\right) \otimes k(\overline{y}^{de}) \arrow[l] \arrow[d, "\theta^{(de)} \otimes k(\overline{y}^{de})"] \\
H^0(X_{\overline{y}^{de}}, \mathcal{O}_X(R)|_{X_{\overline{y}^{de}}}) & (f_{Y^{de}*}\mathcal{O}_X(R)_{Y^{de}}) \otimes k(\overline{y}^{de}) \arrow[l]
\end{tikzcd}$$

(Ⅲ) Let $Y_1 \subseteq Y_0$ be an open subset such that $\dim H^0(X_y, \mathcal{O}_X(R)|_{X_y})$ is a constant function on Y_1 with value h. If $y \in Y_1$, then the horizontal morphisms in the above diagram are isomorphisms by [58, Corllary12.9]. Hence for every $e > 0$ we have

$$\begin{aligned}
&\dim_{k(\overline{y})} \mathrm{im}(H^0(X_{\overline{y}}, \phi^{(de)}_{(X_{\overline{y}}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(R)|_{X_{\overline{y}}})) \\
=&\dim_{k(\overline{y}^{de})} \mathrm{im}(H^0(X_{\overline{y}^{de}}, \phi^{(de)}_{(X_{\overline{y}}/\overline{y}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(R)|_{X_{\overline{y}^{de}}})) \\
=&\dim_{k(\overline{y}^{de})} \mathrm{im}(\theta^{(de)} \otimes k(\overline{y}^{de})) \\
=&h - \dim_{k(\overline{y}^{de})} \mathrm{coker}\left(\theta^{(de)} \otimes k(\overline{y}^{de})\right) \\
=&h - \dim_{k(\overline{y}^{de})} \left(\mathrm{coker}(\theta^{(de)})\right) \otimes k(\overline{y}^{de}).
\end{aligned}$$

Here, the last equality follows from the right exactness of the tensor functor.
(Ⅳ) Assume that $(p^d - 1)(K_{X/Y} + \Delta - R)|_{X_1} \sim f_1^*C$ for some Cartier divisor C on Y_1, where $X_1 := f^{-1}(Y_1)$ and $f_1 := f|_{X_1} : X_1 \to Y_1$. Then

$$\mathcal{L}^{(de)}_{(X_{\overline{y}}, \Delta_{\overline{y}})} \otimes \mathcal{O}_{X^{de}}(p^{de}R)|_{(X_{\overline{y}})^{de}} \cong \mathcal{O}_{X^{de}}(R)|_{(X_{\overline{y}})^{de}}$$

for every $y \in Y_1$. Thus we can regard $H^0(X_{\overline{y}}, \phi^{(de)}_{(X_{\overline{y}}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(R)|_{X_{\overline{y}}})$ as the e-th iteration of the (p^{-d}-linear) morphism

$$\begin{aligned}
\tau := H^0(X_{\overline{y}}, \phi^{(d)}_{(X_{\overline{y}}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(R)|_{X_{\overline{y}}}) :& \\
&H^0(X_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}}) \to H^0(X_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}}).
\end{aligned}$$

If $e \geq h$, then $\mathrm{im}(\tau^e) = \mathrm{im}(\tau^h)$, and thus

$$\mathrm{im}(H^0(X_{\overline{y}}, \phi^{(de)}_{(X_{\overline{y}}, \Delta_{\overline{y}})} \otimes \mathcal{O}_X(R)|_{X_{\overline{y}}})) = S^0(X_{\overline{y}}, \Delta_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}}).$$

Hence by (3), we see that

$$\dim_{k(\overline{y})} S^0(X_{\overline{y}}, \Delta_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}}) = h - \dim_{k(\overline{y}^{de})} \left(\mathrm{coker}(\theta^{(de)})\right) \otimes k(\overline{y}^{de}).$$

In particular, since the function $\dim_{k(\overline{y}^{de})}(\mathrm{coker}(\theta^{(de)})) \otimes k(y^{de})$ on Y^{de} is upper semicontinuous, the function $\dim_{k(\overline{y})} S^0(X_{\overline{y}}, \Delta_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}})$ on Y_1 is lower semicontinuous.

Proof of Theorem 2.2.2. Let $d > 0$ be an integer such that $a|(p^d - 1)$.
Step 1. We define Δ_Y and we show that this is independent of the choice of d. We first note that, for each $e \geq 0$ there exist isomorphisms

$$\begin{aligned}
&f_*\mathcal{O}_X((1-p^{de})(K_X+\Delta)+p^{de}R) \\
\cong&f_*\mathcal{O}_X((1-p^{de})(K_X+\Delta+a^{-1}B-R)+(p^{de}-1)a^{-1}B+R) \\
\cong&\mathcal{O}_Y((1-p^{de})a^{-1}C)\otimes f_*\mathcal{O}_X((p^{de}-1)a^{-1}B+R) \\
\cong&\mathcal{O}_Y((1-p^{de})a^{-1}C)\otimes f_*\mathcal{O}_X(R) \\
\cong&\mathcal{O}_Y((1-p^{de})a^{-1}C+S).
\end{aligned}$$

Since Y is normal, to define Δ_Y we may assume $Y = Y_0$ and X is smooth. Then for each $e > 0$ we have

$$\begin{aligned}
f_*^{(de)}\mathcal{L}^{(de)}_{(X,\Delta)/Y}(p^{de}R) &\cong \mathcal{O}_{Y^{de}}((1-p^{de})(a^{-1}C-K_{Y^{de}})+S) \quad \text{and} \\
f_{Y^{de}*}\mathcal{O}_{X_Y^{de}}(R_{Y^{de}}) &\cong F_Y^{de*}f_*\mathcal{O}_X(R)\cong\mathcal{O}_{Y^{de}}(p^{de}S),
\end{aligned}$$

thus

$$\begin{aligned}
\theta^{(de)} &:= f_{Y^{de}*}\phi^{(de)}_{(X,\Delta)/Y}\otimes\mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}}) \\
&: f_*^{(de)}\mathcal{L}^{(de)}_{(X,\Delta)/Y}(p^{de}R)\to f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}})
\end{aligned}$$

is a homomorphism between line bundles. By the assumption of global F-splitting of $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$, we see that the left vertical morphism of the diagram in Observation 2.2.4 (II) (for $y = \eta$) is surjective, and hence by Observation 2.2.4 (III) $\theta^{(de)}$ is generically surjective for every $e > 0$. Thus $\theta^{(de)}$ defines an effective Cartier divisor $E^{(de)}$ on Y. Then for every $e > 1$ we have $E^{(de)} = p^d E^{(d(e-1))} + E^{(d)}$, because relations between morphisms

$$\begin{aligned}
\theta^{(de)} :=&f_{Y^{de}*}\phi^{(de)}_{(X,\Delta)/Y}\otimes\mathcal{O}_{X_{Y^{de}}}(R_{Y^{de}}) \\
=&f_{Y^{de}*}\left(\phi^{(d(e-1))}_{(X,\Delta)/Y}\otimes\mathcal{O}_{X_{Y^{d(e-1)}}}(R_{Y^{d(e-1)}})\right)_{Y^{de}} \\
&\circ f^{(d(e-1))}{}_{Y^{de}*}\phi^{(d)}_{(X^{d(e-1)},\Delta)/Y^{d(e-1)}}\otimes\left(\mathcal{L}^{(d(e-1))}_{(X,\Delta)/Y}\right)_{Y^{de}}(p^{d(e-1)}R_{Y^{de}}) \\
\cong&(F_Y^{d*}\theta^{(d(e-1))})\circ(\theta^{(d)}\otimes\mathcal{O}_{Y^{de}}((p^{d(e-1)}-1)(a^{-1}C-K_{Y^{de}})))
\end{aligned}$$

implies that $E^{(de)} = (p^{d(e-1)} + \cdots + p + 1)E^{(d)} = (p^{de} - 1)(p^d - 1)^{-1}E^{(d)}$ for every $e > 0$. We define $\Delta_Y := (p^d - 1)^{-1}E^{(d)}$, this is independent of the choice of d by the above. Note that by this definition

$$\begin{aligned}
f_*\mathcal{O}_X((p^d-1)(K_{X/Y}+\Delta-R)) \cong&\mathcal{O}_Y((p^d-1)(a^{-1}C-K_Y)) \\
\cong&\mathcal{O}_Y((p^d-1)(\Delta_Y-S)),
\end{aligned}$$

which proves (1).

Step 2. We show that for each $e > 0$ there exists a commutative diagram

$$\begin{tikzcd} F^{de}_{Y\,*}\mathcal{L}^{(de)}_{(Y,\Delta_Y)} \arrow[r, "(\phi^{(de)}_{(Y,\Delta_Y)}\otimes\mathcal{O}_Y(S))^{**}"] \arrow[d, "\cong"'] & \mathcal{O}_Y(S) \arrow[d, "\cong"] \\ f_*F^{de}_{X\,*}\mathcal{L}^{(de)}_{(X,\Delta)}(p^{de}R) \arrow[r, "\psi^{(de)}"] & f_*\mathcal{O}_X(R), \end{tikzcd}$$

where $\psi^{(de)} := f_*((\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_X(R))^{**})$. It is clear that each object of the above diagram is a reflexive sheaf, so we may assume that $Y = Y_0$ and X is smooth. Since $F^d_X = (F^d_Y)_X \circ F^{(d)}_{X/Y}$, we have

$$\begin{aligned} &\phi^{(d)}_{(X,\Delta)} \otimes \mathcal{O}_X(R) \\ :=&\mathrm{Tr}_{F^d_X} \otimes \omega_X^{-1}(R) \cong \left(\mathrm{Tr}_{(F^d_Y)_X} \circ (F^d_Y)_{X*}\mathrm{Tr}_{F^{(d)}_{X/Y}}\right) \otimes \omega_X^{-1}(R) \\ \cong&\left(((f^*\mathrm{Tr}_{F_Y}) \otimes \omega_{X/Y}) \circ (F^d_Y)_{X*}\mathrm{Tr}_{F^{(d)}_{X/Y}}\right) \otimes \omega_X^{-1}(R) \\ \cong&(f^*\phi^{(d)}_Y \otimes \mathcal{O}_X(R)) \circ (F^d_Y)_{X*}\left(\phi^{(d)}_{(X,\Delta)/Y} \otimes f^*_{Y^d}\omega_{Y^d}^{1-p^d}(R_{Y^d})\right). \end{aligned}$$

We note that $\phi^{(d)}_Y$ is a morphism between vector bundles on Y, thus $\psi^{(d)}$ is decomposed into

$$\psi^{(d)} \cong(\phi^{(d)}_Y \otimes \mathcal{O}_Y(S)) \circ F^d_{Y*}(\theta^{(d)} \otimes \omega_{Y^d}^{1-p^d}).$$

On the other hand, by the definition of Δ_Y, there exists a commutative diagram

$$\begin{tikzcd} \mathcal{O}_Y((1-p^d)\Delta_Y + p^dS) \arrow[r] \arrow[d, "\cong"'] & \mathcal{O}_{Y^d}(p^dS) \arrow[d, "\cong"] \\ f^{(d)}{}_*\mathcal{L}^{(d)}_{(X,\Delta)/Y}(p^dR) \arrow[r, "\theta^{(d)}"] & f_{Y^d*}\mathcal{O}_{X_{Y^d}}(R_{Y^d}). \end{tikzcd}$$

Applying the functor $F^d_{Y*}((\_\_) \otimes \omega_Y^{1-p^d})$ to this diagram, we have the following:

$$\begin{tikzcd} (F^d_{Y*}\mathcal{O}_Y((1-p^d)(K_Y+\Delta_Y))) \otimes \mathcal{O}_Y(S) \arrow[r] \arrow[d, "\cong"'] & (F^d_{Y*}\omega_Y^{1-p^d}) \otimes \mathcal{O}_Y(S) \arrow[d, "\cong"] \\ F^d_{Y*}f^{(d)}{}_*\mathcal{L}^{(d)}_{(X,\Delta)}(p^dR) \arrow[r, "F^d_{Y*}(\theta^{(d)}\otimes\omega_Y^{1-p^d})"] & (F^d_{Y*}\omega_Y^{1-p^d}) \otimes f_*\mathcal{O}_X(R). \end{tikzcd}$$

Hence by the decomposition of $\psi^{(d)}$ and the definition of $\phi^{(d)}_{(Y,\Delta_Y)}$, the claim is proved in the case when $e = 1$. Furthermore, for each $e > 0$ we have

$$\begin{aligned}
&\psi^{(d(e+1))}\\
\cong&(f_*(\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_X(R))) \circ f_* F^{de}_{X\ *}(\phi^{(d)}_{(X^{de},\Delta)} \otimes \mathcal{L}^{(de)}_{(X,\Delta)}(p^{de}R))\\
\cong&\psi^{(de)} \circ F^{de}_{Y\ *} f^{(de)}{}_*(\phi^{(d)}_{(X^{de},\Delta)} \otimes \mathcal{O}_{X^{de}}(R + (1-p^{de})(K_{X^{de}} + \Delta - R)))\\
\cong&\phi^{(de)}_{(Y,\Delta_Y)} \otimes \mathcal{O}_Y(S) \circ F^{de}_{Y\ *}(\phi^{(d)}_{(Y^{de},\Delta_Y)} \otimes \mathcal{O}_{Y^{de}}(S + (1-p^{de})a^{-1}C))\\
\cong&\phi^{(de)}_{(Y,\Delta_Y)} \otimes \mathcal{O}_Y(S) \circ F^{de}_{Y\ *}(\phi^{(d)}_{(Y^{de},\Delta_Y)} \otimes \mathcal{O}_{Y^{de}}((1-p^{de})(K_Y + \Delta_Y) + p^{de}S))\\
\cong&\phi^{(de)}_{(Y,\Delta_Y)} \otimes \mathcal{O}_Y(S) \circ F^{de}_{Y\ *}(\phi^{(d)}_{(Y^{de},\Delta_Y)} \otimes \mathcal{L}^{(de)}_{(Y,\Delta_Y)}(p^{de}S))\\
\cong&\phi^{(d(e+1))}_{(Y,\Delta_Y)} \otimes \mathcal{O}_Y(S).
\end{aligned}$$

This is our claim.

Step 3. We prove statements (2)-(4). (3) is obvious. We show (2). By the definition of $\phi^{(d)}_{(X,\Delta)}$, we may assume that X is smooth. Then there is a commutative diagram

$$\begin{tikzcd}
(F^{de}_{Y\ *} f^{(de)}{}_* \mathcal{L}^{(de)}_{(X,\Delta)}(p^{de}R))(D) \arrow[r, "\psi^{(de)} \otimes \mathcal{O}_Y(D)"] \arrow[d, "\cong"'] & (f_*\mathcal{O}_X(R))(D) \arrow[d, "\cong"] \\
F^{de}_{Y\ *} f^{(de)}{}_* \mathcal{L}^{(de)}_{(X,\Delta)}(p^{de}(f^*D + R)) \arrow[r, "f_*(\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_Y(f^*D+R))"] \arrow[d, "\cong"'] & f_*\mathcal{O}_X(f^*D + R) \arrow[d, "\cong"] \\
F^{de}_{Y\ *} f^{(de)}{}_* \mathcal{L}^{(de)}_{(X,\Delta)}(p^{de}(f^*D + B' + R)) \arrow[r, "f_*(\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_X(f^*D+B'+R))"] & f_*\mathcal{O}_X(f^*D + B' + R).
\end{tikzcd}$$

Thus by Step2,

$$\begin{aligned}
&H^0(X, \phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_X(f^*D + B' + R))\\
\cong&H^0(Y, f_*(\phi^{(de)}_{(X,\Delta)} \otimes \mathcal{O}_X(f^*D + B' + R)))\\
\cong&H^0(Y, (\phi^{(de)}_{(Y,\Delta_Y)} \otimes (D + S))^{**}),
\end{aligned}$$

which implies (2). For (4), we may assume that $Y = Y_0$. Then, since $f_*\mathcal{O}_X(R)$ is a line bundle, we only need to show that the case when $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y$ and $R = \Gamma \geq 0$. In this case, since R and $(p^d - 1)(K_X + \Delta)$ are Cartier, and since f is flat projective, we have $H^0(X_y, \mathcal{O}_X(R)|_{X_y}) \neq 0$ and

$$H^0(X_y, \mathcal{O}_X((1-p^d)R)|_{X_y}) = H^0(X_y, \mathcal{O}_{X_y}((1-p^d)(K_{X_y} + \Delta_y))) \neq 0$$

for every $y \in Y$ by assumptions and upper semicontinuity [58, Theorem 12.8]. In particular, if X_y is reduced then $\mathcal{O}_X(R)|_{X_y} \cong \mathcal{O}_{X_y}$, because every nonzero endomorphism of a line bundle on a connected reduced projective scheme over a field is an isomorphism. Hence the isomorphism $\mathcal{O}_Y \cong f_*\mathcal{O}_X(R)$ shows that the support of R is contained a union of nonreduced fibers. Set $Y_1 := \{y \in Y | H^0(X_y, \mathcal{O}_X(R)_y) \cong k(y)\}$. Then we have

$$\mathrm{Supp}\ \Delta_Y|_{Y_1} = \mathrm{Supp}\ \mathrm{coker}(\theta^{(d)})|_{Y_1} = \{y \in Y_1 | S^0(X_{\overline{y}}, \Delta_{\overline{y}}, \mathcal{O}_X(R)|_{X_{\overline{y}}}) = 0\},$$

where the first (resp. the second) equality follows from the definition of Δ_Y (resp. Observation 2.2.4 (IV)). Now we prove (a)$\Rightarrow$(b). In the situation of (a) X_y is reduced, and so $y \in Y \setminus f(\mathrm{Supp}\ R)$. We recall Example 2.1.4, which shows that the global F-splitting of $(X_{\overline{y}}, \Delta_{\overline{y}})$ is equivalent to the equality $S^0(X_{\overline{y}}, \Delta_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) = H^0(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}})$. Thus it is enough to show that $y \in Y_1$. Let $\overline{\{y\}}$ be the closure in Y of the set $\{y\}$ with the reduced induced subscheme structure. Let Y' be a smooth open subset of $\overline{\{y\}}$ such that $R_{Y'} = 0$ and that $\mathrm{Supp}\ \Delta$ does not contain any irreducible component of any fiber over Y'. Then for a general closed point $y' \in Y'$,

$$\begin{aligned}\dim_k S^0(X_{y'}, \Delta_{y'}, \mathcal{O}_{X_{y'}}) &= \dim_{k(\overline{y})} S^0(X_{\overline{y}}, \Delta_{y'}, \mathcal{O}_{X_{\overline{y}}}) \\ &= \dim_{k(\overline{y})} H^0(X_{\overline{y}}, \mathcal{O}_{X_{\overline{y}}}) = \dim_k H^0(X_{y'}, \mathcal{O}_{X_{y'}}),\end{aligned}$$

where the first (resp. the third) equality follows from lower semicontinuity proved in Observation 2.2.4 (IV) (resp. upper semicontinuity). Thus $(X_{y'}, \Delta_{y'})$ is globally F-split, and in particular $X_{y'}$ is reduced. Since k is algebraically closed, we have $H^0(X_{y'}, \mathcal{O}_{X_{y'}}) \cong k$, and hence $H^0(X_y, \mathcal{O}_{X_y}) \cong k(y)$, or equivalently, $y \in Y_1$. To prove (b)$\Rightarrow$(a), we replace Y by its affine open subset contained in $Y \setminus (\mathrm{Supp}\ \Delta_Y \cup f(\mathrm{Supp}\ R))$. Then the surjectivity of $\theta^{(d)}$ shows that $\phi^{(d)}_{(X,\Delta)/Y} : F^{(d)}_{X/Y*}\mathcal{L}^{(d)}_{(X,\Delta)/Y} \to \mathcal{O}_{X_{Y^d}}$ is split, and thus so is $\phi^{(d)}_{(X,\Delta)/Y}|_{X_{\overline{y}^d}} : F^{(d)}_{X_{\overline{y}}/\overline{y}*}(\mathcal{L}^{(d)}_{(X,\Delta)/Y}|_{(X_{\overline{y}})^d}) \to \mathcal{O}_{X_{\overline{y}^d}}$. This means that Δ does not contain any irreducible component of $f^{-1}(y)$, so $\Delta_{\overline{y}}$ is well-defined, and we have $\phi^{(d)}_{(X,\Delta)/Y}|_{X_{\overline{y}^d}} \cong \phi^{(d)}_{(X_{\overline{y}},\Delta_{\overline{y}})/\overline{y}}$, which completes the proof.

□

Proof of Corollary 2.2.3. We use the notation of Observation 2.2.1. Let $l > 0$ be an integer such that $l(K_X + \Delta)$ is Cartier and base point free. We replace Y by its smooth locus Y_{sm}, and X by the smooth locus of $f^{-1}(Y_{\mathrm{sm}})$. As in the proof of Theorem 2.2.2, we set

$$\psi^{(e)} := f_*(\phi^{(e)}_{(X,\Delta)} \otimes \mathcal{O}_X(R)) \text{ and } \theta^{(e)} := f_{Y^e*}(\phi^{(e)}_{(X,\Delta)/Y} \otimes \mathcal{O}_X(R)_{Y^e})$$

for every $e > 0$ divisible enough. Since $S^0(X, \Delta, (p^d - 1)(K_X + \Delta)) \neq 0$, we have

$$\begin{aligned}0 \neq& S^0(X, \Delta, (l-1)(p^d-1)(K_X+\Delta)) \\ \hookrightarrow& S^0(X, \Delta, (l-1)(p^d-1)(K_X+\Delta) + B) = S^0(X, \Delta, f^*l(p^d-1)H + R).\end{aligned}$$

This implies the morphism

$$f_*(\phi^{(e)}_{(X,\Delta)} \otimes \mathcal{O}_X(f^*l(p^d-1)H + R)) \cong \psi^{(e)} \otimes \mathcal{O}_Y(l(p^d-1)H)$$

is nonzero for an $e > 0$ divisible enough, where the isomorphism follows from the projection formula. Hence $\psi^{(e)}$ is also nonzero. By an argument similar to Step2, we can factor $\psi^{(e)}$ into $(\phi_Y^{(e)} \otimes \mathcal{O}_Y(S)) \circ F^e_{Y*}(\theta^{(e)} \otimes \omega_Y^{1-p^e})$, and hence $\theta^{(e)}$ is nonzero. Thus $\theta^{(e)} \otimes k(\overline{\eta}) \cong H^0(X_{\overline{\eta}}, \phi^{(e)}_{(X_{\overline{\eta}}/\overline{\eta}, \Delta_{\overline{\eta}})})$ is nonzero, or equivalently, $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is globally F-split. In particular $X_{\overline{\eta}}$ is reduced, and this means that f is separable. We show (2). First note that $R/R_S(X, \Delta, K_X + \Delta)$ is finite dimensional if and only if so is $R/R_S(X, \Delta, l(K_X + \Delta))$ by Lemma 2.1.9. Let $m \geq 0$ be an integer with $l|m$. Then we have $H^0(X, lm(K_X+\Delta)) \cong H^0(Y, lm(K_Y+\Delta))$. Furthermore, by Theorem 2.2.2 (2), we have

$$\begin{aligned} S^0(X, \Delta, m(K_X + \Delta)) &= S^0(X, \Delta, f^*(mH - S) + R) \\ &= S^0(Y, \Delta_Y, mH - S + S) = S^0(Y, \Delta_Y, mH). \end{aligned}$$

Thus $R/R_S(X, \Delta, l(K_X+\Delta)) \cong R/R_S(Y, \Delta_Y, lH)$. By Example 2.1.11, this k-vector space is finite dimensional if and only if (Y, Δ_Y) is F-pure, which is our claim. □

Example 2.2.5. Let $f : X \to Y$ be a relatively minimal elliptic fibration. In other words, let f be a generically smooth morphism from a smooth projective surface X to a smooth projective curve Y, whose fibers are connected curves having arithmetic genus one and do not contain (-1)-curves of X. Then by the canonical bundle formula [12, Theorem 2], we have

$$K_X \sim f^*D + \sum_{i=1}^{r} l_i F_i,$$

where D is a divisor on Y, $m_i F_i = X_{y_i}$ is a multiple fiber with the multiplicity m_i, and $0 \leq l_i < m_i$. Let m be the least common multiple of $m_1, \ldots, m_r$, and let $a, e \geq 0$ be integers such that $m = ap^e$ and $p \nmid a$. We set

$$R := \sum_{i-1}^{r} \left\{ \frac{(1-p^d)l_i}{m_i} \right\} m_i F_i$$

for some $d \geq e$ satisfying $a|(p^d - 1)$. Here, recall that for every $s \in \mathbb{Q}$, $\{s\}$ is the fractional part $s - \lfloor s \rfloor$ of s. It is easily seen that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y$ and

$$\begin{aligned} aK_X - aR \sim & af^*D + a\sum_{i=1}^{r} l_i F_i - a\sum_{i=1}^{r}(1-p^d)l_i F_i + a\sum_{i=1}^{r} \lfloor \frac{(1-p^d)l_i}{m_i} \rfloor m_i F_i \\ = & f^*(aD + \sum_{i=1}^{r}(\frac{a l_i p^d}{m_i} + a\lfloor \frac{(1-p^d)l_i}{m_i} \rfloor) y_i). \end{aligned}$$

Thus, a and R satisfy condition (ii) of Theorem 2.2.2. Furthermore, assume that the geometric generic fiber of f is globally F-split, or equivalently, is an elliptic curve with nonzero Hasse invariant. Then by Theorem 2.2.2 there exists an effective $\mathbb{Z}_{(p)}$-divisor Δ_Y on Y such that

$$S^0(X, f^*D' + R) = S^0(Y, \Delta_Y, D')$$

for every divisor D' on Y, and $y_1, \ldots, y_r \in f(R) \cup \Delta_Y$. Remark that if $p \nmid m_i$ for each i, then $m_i|(p^d - 1)$, and so $R = 0$.

Finally, applying Theorem 2.2.2, we show that for a smooth projective surface X of general type, $R/R_S(X, K_X)$ is finite dimensional (Corollary 2.2.8).

Corollary 2.2.6. *Let $f : X \to Y$ be a birational morphism between normal projective varieties, let Δ be an effective $\mathbb{Q}$-Weil divisor on X and let $\Delta_Y := f_*\Delta$. Assume that $a(K_Y + \Delta_Y)$ is Cartier for some $a > 0$ not divisible by p and (Y, Δ_Y) is canonical. Then for each $m > 0$,*

$$S^0(Y, \Delta_Y, am(K_Y + \Delta_Y)) \cong S^0(X, \Delta, am(K_X + \Delta)).$$

Proof. Since (Y, Δ_Y) is canonical, $R := a(K_X + \Delta) - f^*a(K_Y + \Delta_Y)$ is an effective Weil divisor on X supported on the exceptional locus of f. Note that $f_*\mathcal{O}_X(R) \cong \mathcal{O}_Y$. We set $B := (a-1)R$ and $B'_m := (m-1)R$ for each $m \geq 1$. Then we have

$$aR = R + (a-1)R = a(K_X + \Delta) - f^*a(K_Y + \Delta_Y) + B.$$

Thus, by Theorem 2.2.2, we have

$$\begin{aligned} S^0(Y, \Delta_Y, am(K_Y + \Delta_Y)) &\cong S^0(X, \Delta, B'_m + f^*am(K_Y + \Delta_Y) + R) \\ &\cong S^0(X, \Delta, am(K_X + \Delta)). \end{aligned}$$

□

Corollary 2.2.7 ([93, Excercise 5.15])**.** *Let $\varphi : Y \dashrightarrow Y'$ be a birational map between normal projective varieties, let Δ be an effective $\mathbb{Q}$-Weil divisor on Y and let $\Delta' := \varphi_*\Delta$. Assume that $a(K_Y + \Delta)$ and $a(K_{Y'} + \Delta')$ are Cartier for some $a > 0$ not divisible by p, and that (Y, Δ) and (Y', Δ') are canonical. Then, for each $m > 0$,*

$$S^0(Y, \Delta, am(K_Y + \Delta)) \cong S^0(K_{Y'}, \Delta', am(K_{Y'} + \Delta')).$$

Proof. This follows directly from Corollary 2.2.6. □

The following corollary is used to prove the weak positivity theorem when geometric generic fibers are normal projective surfaces of general type with only rational double point singularities (Corollary 4.2.5). Recall that the finiteness of the dimension of R/R_S is equivalent to assumption (ii) of the main theorem (Theorems 4.1.1 or 4.2.1).

Corollary 2.2.8. *Let X be a normal projective surface of general type with only rational double point singularities. If $p > 5$, then $R/R_S(X, K_X)$ is a finite dimensional vector space.*

Proof. By Corollary 2.2.7, we may assume that X is a smooth projective surface of general type which has no (-1)-curve. Then for each $n \gg 0$, nK_X is base point free, and $Y := \mathrm{Proj}\, R(X, K_X)$ has only rational double point singularities [5, Theorem 9.1]. When $p > 5$, Y is F-pure, because of the classification of rational double points [2, Section 3], and of Fedder's criterion [32]. Hence the statement follows from Corollary 2.2.3. □

Chapter 3

Positivity conditions and a numerical invariant

In this chapter, we introduce some positivity conditions of coherent sheaves on normal varieties over a field k (Section 3.1). In order to give a sufficient condition for coherent sheaves to have such positivity conditions, we also introduce a numerical invariant when k is an F-finite field of positive (Section 3.2).

3.1 Positivity conditions of coherent sheaves

Definition 3.1.1. Let Y be a quasi-projective normal variety over a field k, let $\mathcal{G}$ be a coherent sheaf on Y and let H be an ample Cartier divisor. Let $V \subseteq Y$ be the largest open subset such that $\mathcal{G}|_V$ is locally free and S be a non-empty subset of V.

(i) We say that $\mathcal{G}$ is *globally generated over* S if the natural morphism $H^0(Y, \mathcal{G}) \otimes_k \mathcal{O}_Y \to \mathcal{G}$ is surjective over S.

(ii) We say that $\mathcal{G}$ is *weakly positive over* S if for every integer $a > 0$, there exists an integer $b > 0$ such that $(S^{ab}\mathcal{G})^{**} \otimes \mathcal{O}_Y(bH)$ is globally generated over S. Here $S^{ab}(\_)$ and $(\_)^{**}$ denote the ab-th symmetric product and the double dual, respectively.

(iii) We say that $\mathcal{G}$ is *big over* S if there exists an integer $a > 0$ such that $(S^a\mathcal{G})(-H)$ is weakly positive over S.

We simply say that $\mathcal{G}$ is globally generated (resp. weakly positive, big) over y when $S = \{y\}$ for a point $y \in V$. We say that $\mathcal{G}$ is *generically globally generated* (resp. *weakly positive*, *big*) if it is globally generated (resp. weakly positive, big) over the generic point η of Y.

Remark 3.1.2. The notion of weak positivity is first introduced by Viehweg as a generalization of nefness of vector bundles, when S is an open subset [110]. In [72] and [92] (resp. [88]), this notion is also defined in the case when $S = \{\eta\}$ (resp. $S = \{y\}$ for a point $y \in Y$).

Remark 3.1.3. Let $Y, \mathcal{G}, V, S$ and H be as above.
(1) The above definition is independent of the choice of H (cf. [110, Lemma 2.14]).
(2) Let $Y_0 \subseteq Y$ be an open subset containing S such that $\mathrm{codim}_Y(Y \setminus Y_0) \geq 2$ and let $i : Y_0 \to Y$ be the open immersion. Then we have

$$(S^m\mathcal{G})^{**}(nH) \cong i_* \left(((S^m\mathcal{G})^{**}(nH))|_{Y_0}\right) \cong i_* \left((S^m(\mathcal{G}|_{Y_0}))^{**}(nH|_{Y_0})\right)$$

for each integers m, n with $m > 0$. Therefore, we see that $\mathcal{G}$ is weakly positive (resp. big) over S if and only if so is $\mathcal{G}|_{Y_0}$.
(3) The natural morphism $\mathcal{G} \to \mathcal{G}^{**}$ induces the morphism $(S^m\mathcal{G})^{**}(nH) \to (S^m\mathcal{G}^{**})^{**}(nH)$ for each integers m, n with $n > 0$, which is an isomorphism because of the normality of Y. In particular, $\mathcal{G}$ is weakly positive (resp. big) over S if and only if so is $\mathcal{G}^{**}$.
(4) Assume that $\mathcal{G}$ is a vector bundle and that Y is projective. Then $\mathcal{G}$ is weakly positive (resp. big) over Y if and only if $\mathcal{G}$ is nef (resp. ample). For details, see for example [79, §6].
(5) Assume that $\mathcal{G}$ is a line bundle and that Y is projective. Then $\mathcal{G}$ is weakly positive (resp. big) over the generic point η of Y if and only if $\mathcal{G}$ is pseudo-effective (resp. big).

Observation 3.1.4. (1) With the notation as Definition 3.1.1, we define

$$T'_S(\mathcal{G}, H) := \left\{ \varepsilon \in \mathbb{Q} \;\middle|\; \begin{array}{c} \text{there exist } a, b \in \mathbb{Z} \text{ such that} \\ \varepsilon = a/b,\, b > 0, \text{ and } (S^b\mathcal{G})^{**}(-aH) \text{ is} \\ \text{globally generated over } S. \end{array} \right\}, \text{ and}$$
$$t'_S(\mathcal{G}, H) := \sup T'_S(\mathcal{G}, H).$$

We first prove that $T'_S(\mathcal{G}, H)$ is equal to $\mathbb{Q} \cap (-\infty, t'_S(\mathcal{G}, H))$ or $\mathbb{Q} \cap (-\infty, t'_S(\mathcal{G}, H)]$. By Remark 3.1.3 (3), we have $T'_S(\mathcal{G}, H) = T'_S(\mathcal{G}^{**}, H)$. Furthermore, similarly to Remark 3.1.3 (2), we see that $T'_S(\mathcal{G}, H) = T'_S(\mathcal{G}|_V, H|_V)$, where $V \subseteq Y$ is the maximum open subset such that $\mathcal{G}^{**}|_V$ is locally free. Hence we may assume that $\mathcal{G}$ is locally free. If $(S^b\mathcal{G})(-aH)$ is globally generated over S for integers a, b with $b > 0$, then $(S^{bc}\mathcal{G})(-acH)$ is also globally generated over S for every $c > 0$, because of the natural morphism

$$S^c((S^b\mathcal{G})(-aH)) \to (S^{bc}\mathcal{G})(-acH)$$

which is surjective over S. Then $(S^{bc}\mathcal{G})((-ac + d)H)$ is also globally generated over S for every $d > 0$ such that dH is free. Hence we see that $(ac - d)/(bc) = a/b - d/(bc) \in T'_S(\mathcal{G}, H)$, which proves our claim.
(2) Next, we show that $\mathcal{G}$ is weakly positive (resp. big) over S if and only if $t'_S(\mathcal{G}, H) \geq 0$ (resp > 0). By an argument similar to the above, we may assume that $\mathcal{G}$ is locally free. The definition of the weak positivity of $\mathcal{G}$ over S is equivalent to that $-1/a \in T'_S(\mathcal{G}, H)$ for all $a > 0$, which is also equivalent to $t'_S(\mathcal{G}, H) \geq 0$ because of (1). If $\mathcal{G}$ is big, then there exists an integer $c > 0$ such that $(S^c\mathcal{G})(-H)$ is weakly positive. Then for an $a > 0$ there exists a $b \gg 0$ such that $(S^{ab}(S^c\mathcal{G}))(-abH + aH)$

is globally generated over S, and so is $(S^{abc}\mathcal{G})(a(1-b)H)$ as seen in (1). Hence $t'_S(\mathcal{G},H) \geq (ab-a)/(abc) = 1/c - 1/(bc) > 0$. Conversely, if $t'_S(\mathcal{G},H) > 0$ then by (1) we have integers $a, b > 0$ such that $(S^b\mathcal{G})(-aH)$ is globally generated over S, and hence $\mathcal{G}$ is big.
(3) For a line bundle $\mathcal{L}$ on Y and an integer $m > 0$, it is easily seen that $t'_S(\mathcal{L}^m, H) = mt'_S(\mathcal{L}, H)$. Hence we see that $\mathcal{L}$ is weakly positive over S if and only if so is $\mathcal{L}^m$.

By Observation 3.1.4 (3), we can define the weak positivity of a $\mathbb{Q}$-Weil divisor.

Definition 3.1.5. With the notation as Definition 3.1.1, let D be a $\mathbb{Q}$-Weil divisor on Y, $m > 0$ be an integer such that mD is integral and $\mathcal{G}$ be isomorphic to $\mathcal{O}_Y(mD)$. Then D is said to be weakly positive (resp. big) over S if so is $\mathcal{G}$.

Note that this definition is independent of the choice of m by Observation 3.1.4 (3).

3.2 A numerical invariant of coherent sheaves

Next we introduce a numerical invariant of coherent sheaves which measures positivity. Throughout this section, we fix an F-finite field k of characteristic $p > 0$.

Definition 3.2.1. Let $Y, \mathcal{G}, V, S$ be as in Definition 3.1.1, and assume that the characteristic of k is $p > 0$. Let D be a $\mathbb{Q}$-Cartier divisor. Then we define

$$T_S(\mathcal{G}, D) := \left\{ \varepsilon \in \mathbb{Q} \;\middle|\; \begin{array}{c} \text{there exists an } e > 0 \text{ such that} \\ p^e\varepsilon D \text{ is Cartier and } ({F_Y^e}^*\mathcal{G})(-p^e\varepsilon D) \text{ is} \\ \text{globally generated over } S. \end{array} \right\}, \text{ and}$$
$$t_S(\mathcal{G}, D) := \sup T_S(\mathcal{G}, D) \in \mathbb{R} \cup \{-\infty, +\infty\}.$$

When S is the singleton $\{\eta\}$ of the generic point $\eta \in Y$, we often denote $T_S(\mathcal{G}, D)$ (resp. $t_S(\mathcal{G}, D)$) by $T(\mathcal{G}, D)$ (resp. $t(\mathcal{G}, D)$).

Lemma 3.2.2. *Under the same assumption as above, let $\mathcal{F}$ be a coherent sheaf on Y.*

(1) *If there exists a morphism $\mathcal{F} \to \mathcal{G}$ which is surjective over S, then $t_S(\mathcal{F}, D) \leq t_S(\mathcal{G}, D)$.*

(2) *Assume that $\{t_S(\mathcal{F}, D), t_S(\mathcal{G}, D)\} \neq \{-\infty, +\infty\}$. Then*

$$t_S(\mathcal{F}, D) + t_S(\mathcal{G}, D) \leq t_S(\mathcal{F} \otimes \mathcal{G}, D).$$

(3) *For each $e > 0$, $t_S({F_Y^e}^*\mathcal{G}, D) = p^e t_S(\mathcal{G}, D)$.*

(4) *If the rank of $\mathcal{G}$ is positive, and $t_S(\mathcal{G}, D) = +\infty$, then $-D$ is weakly positive over S.*

Proof. (1)–(3) follow directly from the definition. We prove (4). Recall that for every $y \in S$, the stalk $\mathcal{G}_y$ of $\mathcal{G}$ at y is free of positive rank. From this we see that the natural morphism $\mathcal{G}^* \otimes \mathcal{G} \to \mathcal{O}_Y$ is surjective over S. Furthermore, there is an ample Cartier divisor H such that $\mathcal{G}^*(H)$ is globally generated, so we have a surjective morphism $\mathcal{O}_Y^{\oplus h} \to \mathcal{G}^*(H)$ for some $h > 0$. Hence we get a morphism

$$\mathcal{G}^{\oplus h} \to \mathcal{G}^* \otimes \mathcal{G}(H) \to \mathcal{O}_Y(H)$$

which is surjective over S. By (1) we have $t_S(\mathcal{O}_Y(H), D) = +\infty$, and thus there exists a sequence $\{\varepsilon_n > 0\}_{n \geq 1}$ of elements of $T_S(\mathcal{O}_Y(H), D)$ such that $\varepsilon_n \xrightarrow{n \to +\infty} +\infty$. By the definition of $t_S(\mathcal{O}_Y(H), D)$, for every $n \geq 1$ there exists an $e \geq 1$ such that

$$(F_Y^{e\,*}\mathcal{O}_Y(H))(-\varepsilon_n p^e D) \cong \mathcal{O}_Y(p^e(H - \varepsilon_n D))$$

is globally generated over S. Set $\mathcal{G} := \mathcal{O}_Y(-bD)$ for an integer $b > 0$ such that bD is Cartier. Then for an integer $l_n > 0$ such that $\varepsilon_n l_n p^e$ is an integer,

$$(S^{\varepsilon_n l_n p^e}\mathcal{G})^{**} \otimes \mathcal{O}_Y(bl_n p^e H) \cong \mathcal{O}_Y(\varepsilon_n l_n p^e(-bD) + bl_n p^e H) \cong S^{bl_n}(\mathcal{O}_Y(p^e(H - \varepsilon_n D)))$$

is also globally generated over S. Using the notation of Observation 3.1.4, we see that $(-bl_n p^e)/(\varepsilon_n l_n p^e) = -b/\varepsilon_n \leq t'_S(\mathcal{O}_Y(-bD), H)$, and so $0 \leq t'_S(\mathcal{O}_Y(-bD), H)$. As shown in Observation 3.1.4 (2), this implies that $\mathcal{O}_Y(-bD)$ is weakly positive over S. $\square$

Proposition 3.2.3. *Let Y be a projective n-dimensional variety over a field of characteristic $p > 0$. Let $Y, \mathcal{G}, V, S$ be as in Definition 3.1.1, H be an ample Cartier divisor on Y and Δ be an effective $\mathbb{Q}$-Weil divisor on Y such that $K_Y + \Delta$ is $\mathbb{Q}$-Cartier and* $\mathrm{Supp}\Delta \cap S = \emptyset$. *Let Y_0 be a regular open subset $Y_0 \subseteq Y$ satisfying* $\mathrm{codim}(Y \setminus Y_o) \geq 2$, *and set $t := t_{S_0}(\mathcal{G}|_{Y_0}, H|_{Y_0})$, where $S_0 := S \cap Y_0$. If D is a Cartier divisor such that*

$$D - (K_Y + \Delta) - nA + tH$$

is ample for a base point free ample divisor A on Y, then $\mathcal{G}^{**}(D)$ is globally generated over S_0.

Proof. Since $t_{S_0}(\mathcal{G}|_{Y_0}, H|_{Y_0})$ is the supremum, there exists an $\varepsilon \in T_{S_0}(\mathcal{G}|_{Y_0}, H|_{Y_0})$ such that $B := D - (K_Y + \Delta) - nA + \varepsilon H$ is ample. We fix such an ε. By the definition, for every $e \gg 0$, $p^e\varepsilon \in \mathbb{Z}$ and there is a morphism $\alpha_e : \bigoplus \mathcal{O}_Y \to (F_Y^{e\,*}\mathcal{G})^{**}(-p^e\varepsilon H)$ which is surjective over S_0. Let $l > 0$ be an integer such that $l(K_Y + \Delta)$ is Cartier and $l\varepsilon \in \mathbb{Z}$. For every $e \geq 0$, we denote by q_e and r_e respectively the quotient and the remainder of the division of $p^e - 1$ by l. Hence we have following morphisms which are surjective over S_0:

$$\begin{aligned}
&\bigoplus F^e_{Y*}\mathcal{O}_Y(q_el(B+nA)+(r_e+1)(D-K_Y+\varepsilon H)+K_Y)\\
\cong &F^e_{Y*}\bigoplus \mathcal{O}_Y(q_el(B+nA)+(r_e+1)(D-K_Y+\varepsilon H)+K_Y)\\
\to &F^e_{Y*}((F^{e*}_Y\mathcal{G})^{**}\otimes\mathcal{O}_Y(-p^e\varepsilon H+q_el(B+nA)+(r_e+1)(D-K_Y+\varepsilon H)+K_Y))\\
\cong &F^e_{Y*}((F^{e*}_Y\mathcal{G})^{**}\otimes\mathcal{O}_Y(q_el(B+nA-\varepsilon H)+(r_e+1)(D-K_Y)+K_Y))\\
= &F^e_{Y*}((F^{e*}_Y\mathcal{G})^{**}\otimes\mathcal{O}_Y(q_el(D-(K_Y+\Delta))+(r_e+1)(D-K_Y)+K_Y))\\
= &F^e_{Y*}((F^{e*}_Y\mathcal{G})^{**}\otimes\mathcal{O}_Y(-q_el\Delta+p^e(D-K_Y)+K_Y))\\
\cong &(\mathcal{G}^{**}(D)\otimes F^e_{Y*}\mathcal{O}_Y((1-p^e)K_Y-q_el\Delta))^{**}\\
\to &(\mathcal{G}^{**}(D)\otimes F^e_{Y*}\mathcal{O}_Y((1-p^e)K_Y))^{**}\\
\to &\mathcal{G}^{**}(D).
\end{aligned}$$

Here the morphism in the third (resp. ninth) line is induced by α_e (resp. $\phi^{(e)}_Y$), and the isomorphism in the seventh line follows from the projection formula. Therefore, it is sufficient to show that

$$F^e_{Y*}\mathcal{O}_Y(q_el(B+nA)+(r_e+1)(D-K_Y+\varepsilon H)+K_Y)$$

is globally generated for each $e\gg 0$. Since $0\le r_e<l$, by the Serre vanishing theorem, we have

$$\begin{aligned}
&H^i(Y,\mathcal{O}_Y(-iA)\otimes F^e_{Y*}\mathcal{O}_Y(q_el(B+nA)+(r_e+1)(D-K_Y+\varepsilon H)+K_Y))\\
\cong &H^i(Y,F^e_{Y*}\mathcal{O}_Y(q_el(B+(n-i)A)+(r_e+1)(D-K_Y+\varepsilon H-iA)+K_Y)=0
\end{aligned}$$

for each $i>0$ and $e\gg 0$. Hence our claim follows from the Castelnuovo-Mumford regularity ([79, Theorem 1.8.5]). □

Proposition 3.2.4. *Let $Y, Y_0, H, \mathcal{G}, V, S$ be as above. If $t_{S_0}(\mathcal{G}|_{Y_0}, H|_{Y_0})\ge 0$, where $S_0 := S\cap Y_0$, then $\mathcal{G}$ is weakly positive over S_0.*

Proof. By the hypothesis and Lemma 3.2.2, we have

$$t_{S_0}((S^l\mathcal{G})^{**}|_{Y_0}, H|_{Y_0})\ge t_{S_0}(\mathcal{G}^{\otimes l}|_{Y_0}, H|_{Y_0})\ge lt_{S_0}(\mathcal{G}|_{Y_0}, H|_{Y_0})\ge 0$$

for every $l>0$. Applying the previous proposition, we obtain an ample Cartier divisor D such that $(S^l\mathcal{G})^{**}(D)$ is globally generated over S_0 for every $l>0$, which is our claim. □

Chapter 4

Weak positivity theorems

4.1 Summary

Let $f : X \to Y$ be an algebraic fiber space, and let X and Y be smooth projective varieties. The positivity of the direct image sheaf $f_*\omega^m_{X/Y}$ of the relative pluricanonical bundle is an important property. In characteristic zero, there are numerous known results. Fujita has proved that $f_*\omega_{X/Y}$ is a nef vector bundle when $\dim Y = 1$ [43]. Kawamata generalized this to the case when $m \geq 2$ [67] and to the case when $\dim Y \geq 2$ [66] (see also [38]). Viehweg has shown that $f_*\omega^m_{X/Y}$ is weakly positive for each $m \geq 1$ [109] (see also [72], [13], and [36]). There are several significant consequences of these results. One of them is Iitaka's conjecture in some special cases, which we discuss in Chapter 6. Other consequences include some moduli problems in [73] and [34] (see also [110]), where results of [43], [66], and [67] are generalized to the case when X is reducible (see also [69], [39], and [40]).

On the other hand, in positive characteristic, it is known that there are counterexamples to the above results. For example, Moret-Bailly constructed a semi-stable fibration $g : S \to \mathbb{P}^1$ from a surface S to $\mathbb{P}^1$ such that $g_*\omega_{S/\mathbb{P}^1}$ is not nef [86]. For other examples, see [96, 114] (or Remark 4.2.2 in this chapter). Hence it is natural to ask under what additional conditions analogous results hold in positive characteristic. Kollár has shown that $f_*\omega^m_{X/Y}$ is a nef vector bundle for each $m \geq 2$ when X is a surface, Y is a curve, and the general fiber of f has only nodes as singularities [73, 4.3. Theorem]. Patakfalvi has proved that $f_*\omega^m_{X/Y}$ is a nef vector bundle for each $m \gg 0$ when Y is a curve, $X_{\overline{\eta}}$ has only normal F-pure singularities, and $\omega_{X/Y}$ is f-ample [91, Theorem 1.1].

In this chapter, we consider the weak positivity of $f_*\omega^m_{X/Y}$ in positive characteristic under a condition on the canonical ring and the Frobenius stable canonical ring of the geometric generic fiber. Recall that for a Gorenstein variety V, the canonical ring of V is the section ring of the dualizing sheaf of V, and the Frobenius stable canonical ring of V is its homogeneous ideal whose degree m subgroup is $S^0(V, mK_V)$ (see Definition 2.1.2).

From now on we work over an algebraically closed field k of characteristic $p > 0$. The following theorem is the main result of this chapter.

Theorem 4.1.1 (Theorem 4.2.1)**.** *Let $f : X \to Y$ be an algebraic fiber space, X and Y be smooth projective varieties and Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p. Let $\overline{\eta}$ be the geometric generic point of Y. Assume that*

(i) *the $k(\overline{\eta})$-algebra $\bigoplus_{m\geq 0} H^0(X_{\overline{\eta}}, am(K_X + \Delta)_{\overline{\eta}})$ is finitely generated, and*

(ii) *there exists an integer $m_0 > 0$ such that for each $m \geq m_0$,*

$$S^0(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, am(K_X + \Delta)_{\overline{\eta}}) = H^0(X_{\overline{\eta}}, am(K_X + \Delta)_{\overline{\eta}}).$$

Then $f_*\mathcal{O}_X(am(K_{X/Y} + \Delta))$ is weakly positive for each $m \geq m_0$.

This theorem is proved in Section 4.2 under a more general situation. Condition (ii) of the theorem holds, for example, in the case where $X_{\overline{\eta}}$ is a curve of arithmetic genus at least two which has only nodes as singularities, $\Delta = 0$, and $m_0 = 2$ (Corollary 2.1.14), or in the case where the pair $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ has only F-pure singularities, $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is ample, and $m_0 \gg 0$ (Example 2.1.11). Thus Theorem 4.1.1 can be viewed as a generalization of [73, 4.3. Theorem] and [91, Theorem 1.1].

Theorem 4.1.1 should be compared with another result of Patakfalvi [92, Theorem 6.4], which states that if $S^0(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}) = H^0(X_{\overline{\eta}}, K_{X_{\overline{\eta}}})$ then $f_*\omega_{X/Y}$ is weakly positive (see also [63]). These two results imply that $S^0(X_{\overline{\eta}}, mK_{X_{\overline{\eta}}})$ is closely related to the positivity of $f_*\omega_{X/Y}^m$ for each $m \geq 1$. In order to prove Theorem 4.1.1, we generalize the method of the proof of [92, Theorem 6.4] using a numerical invariant introduced in Section 3.2.

When the relative dimension of f is one, we obtain the following theorem as a corollary of Theorem 4.1.1.

Theorem 4.1.2 (Corollary 4.2.4)**.** *Let $f : X \to Y$ be an algebraic fiber space of relative dimension one, X and Y be smooth projective varieties, and Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p. Let $\overline{\eta}$ be the geometric generic point of Y. If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure and $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is ample, then $f_*\mathcal{O}_X(am(K_{X/Y} + \Delta))$ is weakly positive for each $m \geq 2$. In particular, if $X_{\overline{\eta}}$ is a smooth curve of genus at least two, then $f_*\omega_{X/Y}^m$ is weakly positive for each $m \geq 2$.*

When the relative dimension of f is two, we also obtain the following theorem as a corollary of Theorem 4.1.1.

Theorem 4.1.3 (Corollary 4.2.5)**.** *Assume that $p > 5$. Let $f : X \to Y$ be an algebraic fiber space and let X and Y be smooth projective varieties. If the geometric generic fiber is a surface of general type with only rational double points as singularities, then $f_*\omega_{X/Y}^m$ is weakly positive for each $m \gg 0$.*

Unfortunately, we cannot necessarily apply Theorem 4.1.1 to the case when the geometric generic fiber $X_{\overline{\eta}}$ is a smooth projective surface not of general type, even if the total space is a 3-fold. However, we can prove the following result.

Theorem 4.1.4. *Assume that $p > 5$. Let $f : X \to Y$ be an algebraic fiber space, X be a smooth projective 3-fold and Y be a smooth projective curve. Suppose that the geometric generic fiber $X_{\overline{\eta}}$ has only rational double points as singularities. If $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}}) = 1$, then there exists a real number $c > 0$ such that $f_*\omega_{X/Y}^m$ contains a nef subbundle of rank at least cm for sufficiently divisible $m > 0$.*

This theorem may be not sufficient to be regarded as a kind of weak positivity theorems, but it is useful enough to study Iitaka's $C_{n,m}$ conjecture for 3-folds. Indeed, in Chapter 6, Theorems 4.1.3 and 4.1.4 are used to show the conjecture for 3-folds in the case when the Kodaira dimensions of the geometric generic fibers are two and one, respectively. Theorem 4.1.4 is proved in Section 4.3 as a consequence of the minimal model program for 3-folds in characteristic $p > 5$ developed by several mathematicians including Birkar, Cascini, Hacon, Tanaka, Waldron and Xu.

4.2 Proof of the main theorem

As in the summary, we fix an algebraically closed field k of characteristic $p > 0$. We prove Theorem 4.1.1 in a more general situation. This theorem is a generalization of a result in the author's master thesis, which deals with algebraic fiber spaces whose total and base spaces are smooth. As applications of Theorem 4.2.1, we show weak positivity theorems for certain surjective morphisms of relative dimension zero, one and two (Corollaries 4.2.3, 4.2.4 and 4.2.5 respectively).

Theorem 4.2.1. *Let $f : X \to Y$ be a separable surjective morphism between normal projective varieties, Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p, and $\overline{\eta}$ be the geometric generic point of Y. Assume that*

(i) *$K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is finitely generated in the sense of Definition 2.1.8, and*

(ii) *there exists an integer $m_0 > 0$ such that*

$$S^0(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, am(K_X + \Delta)_{\overline{\eta}}) = H^0(X_{\overline{\eta}}, am(K_X + \Delta)_{\overline{\eta}})$$

for each $m \geq m_0$.

Then $f_*\mathcal{O}_X(am(K_X + \Delta)) \otimes \omega_Y^{-am}$ is a weakly positive sheaf for every $m \geq m_0$.

Proof of Theorem 4.2.1. We first note that $X_{\overline{\eta}}$ is a $k(\overline{\eta})$-scheme of pure dimension satisfying S_2 and G_1, and that each connected component of $X_{\overline{\eta}}$ is integral. Let $d > 0$ be an integer satisfying $a|(p^d - 1)$.

Step 1. In this step, we reduce to the case where X and Y are smooth. Let H be an ample Cartier divisor on Y. By Proposition 3.2.4, it suffices to prove that $t((f_*\mathcal{O}_X(am(K_X + \Delta)))|_{Y_0} \otimes \omega_{Y_0}^{-am}, H|_{Y_0}) \geq 0$ for each $m \geq m_0$, where $Y_0 \subseteq Y$ is an open subset satisfying $\mathrm{codim}(Y \setminus Y_0) \geq 2$. Hence, replacing X and Y by their smooth loci, we may assume that f is a dominant morphism between smooth varieties (the projectivity of f may be lost, but we do not use it).

We set $t(m) := t(f_*\mathcal{O}_X(am(K_{X/Y}+\Delta)), H)$ for each $m > 0$.

Step 2. We show that there exist integers $l, n_0 > m_0$ such that $t(l) + t(n) \le t(l+n)$ for each $n \ge n_0$. By the hypothesis (i) and Lemma 2.1.9, $R(X_{\overline{\eta}}, a(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ is a finitely generated $k(\overline{\eta})$-algebra. Hence for every $l > m_0$ divisible enough there exists an $n_0 > m_0$ such that the natural morphism

$$H^0(X_{\overline{\eta}}, al(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) \otimes H^0(X_{\overline{\eta}}, an(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) \to H^0(X_{\overline{\eta}}, a(l+n)(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$$

is surjective. This shows that the natural morphism

$$f_*\mathcal{O}_X(al(K_{X/Y}+\Delta)) \otimes f_*\mathcal{O}_X(an(K_{X/Y}+\Delta)) \to f_*\mathcal{O}_X(a(l+n)(K_{X/Y}+\Delta))$$

is generically surjective, thus we have $t(l) + t(n) \le t(l+n)$ by Lemma 3.2.2 (1) and (2).

Step 3. We show that $t(mp^{de} - a^{-1}(p^{de}-1)) \le p^{de}t(m)$ for each $e > 0$ and for each $m \ge m_0$. By the hypothesis (ii) and Observation 2.2.4 (Ⅲ), there exist generically surjective morphisms

$$\begin{aligned} {f^{(de)}}_*\mathcal{O}_{X^{de}}(((am-1)p^{de}+1)(K_{X^{de}/Y^{de}}+\Delta)) & \\ \xrightarrow{\alpha} & f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}}+\Delta_{Y^{de}})) \\ \cong & F_Y^{de*} f_*\mathcal{O}_X(am(K_{X/Y}+\Delta)), \end{aligned}$$

where $\alpha := f_{Y^{de}*}(\phi^{(de)}_{(X,\Delta)/Y} \otimes \mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}}+\Delta_{Y^{de}})))$, and the isomorphism follows from the flatness of F_Y. Hence by Lemma 3.2.2, we have

$$\begin{aligned} t(mp^{de} - a^{-1}(p^{de}-1)) & \overset{\text{Lemma 3.2.2 (1)}}{\le} t(F_Y^{de*} f_*(\omega^{am}_{X/Y}(am\Delta)), H) \\ & \overset{\text{Lemma 3.2.2 (3)}}{=} p^{de}t(m). \end{aligned}$$

Step 4. We prove the theorem. Set $m \ge m_0$. If $am = 1$, then $t(1) \le p^d t(1)$ by Step3, which gives $t(1) \ge 0$. Thus we may assume $am_0 \ge 2$. Let $q_{m,e}$ be the quotient of $mp^{de} - a^{-1}(p^{de}-1) - n_0$ by l and let $r_{m,e}$ be the remainder for $e \gg 0$. We note that $q_{m,e} > 0$ since $m \ge m_0 \ge 2a^{-1} > a^{-1}$, and that $p^{de} - q_{l,e} \xrightarrow{e\to\infty} \infty$. Then

$$q_{m,e}t(l) + t(r_{m,e}+n_0) \overset{\mathit{Step2}}{\le} t(mp^{de} - a^{-1}(p^{de}-1)) \overset{\mathit{Step3}}{\le} p^{de}t(m),$$

and so $c := \min\{t(r+n_0) | 0 \le r < l\} \le p^{de}t(m) - q_{m,e}t(l)$. By substituting l for m, we have $c \le (p^{de} - q_{l,e})t(l)$ for each $e \gg 0$, which means $t(l) \ge 0$. Hence $c \le p^{de}t(m)$ for each $e \gg 0$, and consequently $t(m) \ge 0$. This completes the proof.

□

Remark 4.2.2. There exists an algebraic fiber space $g : S \to C$ from a smooth projective surface S to a smooth projective curve C such that $g_*\omega^m_{S/C}$ is not nef for any $m > 0$ [96][114, Theorem 3.6]. This algebraic fiber space does not satisfy condition (ii) of Theorem 4.2.1. Indeed, the geometric generic fiber of g is a Gorenstein curve which has a cusp, hence by [48] it is not F-pure. Since the dualizing sheaf of a Gorenstein curve not isomorphic to $\mathbb{P}^1$ is trivial or ample, the claim follows from Examples 2.1.4 and 2.1.11.

Corollary 4.2.3. *Let $f : X \to Y$ be a surjective morphism between normal projective varieties, Δ be an effective $\mathbb{Q}$-Weil divisor on X, and $a > 0$ be an integer such that $a\Delta$ is integral. If f is separable and generically finite, then $f_*\mathcal{O}_X(a(K_X + \Delta)) \otimes \omega_Y^{-a}$ is weakly positive.*

Proof. Since f is generically finite, the natural morphism

$$f_*\mathcal{O}_X(aK_X) \otimes \omega_Y^{-a} \to f_*\mathcal{O}_X(a(K_X + \Delta)) \otimes \omega_Y^{-a}$$

is an isomorphism at the generic point of Y. Thus it is enough to show the case of $\Delta = 0$. Since the geometric generic fiber $X_{\overline{\eta}}$ is a reduced $k(\overline{\eta})$-scheme of dimension zero, the assertion follows directly from Theorem 4.2.1. □

Corollary 4.2.4. *Let $f : X \to Y$ be a separable surjective morphism of relative dimension one between normal projective varieties, Δ be an effective $\mathbb{Q}$-Weil divisor on X, and $a > 0$ be an integer not divisible by p such that $a\Delta$ is integral. Assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, where $\overline{\eta}$ is the geometric generic point of Y. If $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is ample (resp. $K_{X_{\overline{\eta}}}$ is ample and $a \ge 2$), then $f_*\mathcal{O}_X(am(K_X + \Delta)) \otimes \omega_Y^{-am}$ is weakly positive for each $m \ge 2$ (resp. $m \ge 1$). In particular, if every connected component of $X_{\overline{\eta}}$ is a smooth curve of genus at least two, then $f_*\omega_X^m \otimes \omega_Y^{-m}$ is weakly positive for each $m \ge 2$.*

Proof. Let $U \subseteq X$ be a Gorenstein open subset such that $\operatorname{codim}(X \setminus U) \ge 2$ and that $a\Delta|_U$ is Cartier. Since $\dim(X \setminus U) \le \dim X - 2 = \dim Y - 1$, $X \setminus U$ does not dominate Y. Thus there exists an open subset $Y_0 \subseteq Y$ such that $f|_{X_0} : X_0 \to Y_0$ is a Gorenstein morphism and that $a\Delta|_{X_0}$ is Cartier, where $X_0 := f^{-1}(Y_0)$. In particular, $X_{\overline{\eta}}$ is Gorenstein and $(a\Delta)_{\overline{\eta}}$ is Cartier. Thus the statement follows directly from Corollary 2.1.14 and Theorem 4.2.1. □

Corollary 4.2.5. *Let $f : X \to Y$ be a separable surjective morphism of relative dimension two between normal projective varieties. Assume that every connected component of the geometric generic fiber is a normal surface of general type with only rational double points. Assume in addition that $p > 5$. Then $f_*\omega_X^m \otimes \omega_Y^{-m}$ is weakly positive for each $m \gg 0$.*

Proof. We note that in this case $K_{X_{\overline{\eta}}}$ is finitely generated (cf. [5, Corollary 9.10]). Hence the assertion follows from Corollary 2.2.8 and Theorem 4.2.1. □

4.3 Algebraic fiber spaces whose total spaces are 3-folds

In this section, we prove Theorem 4.1.4. To this end, we recall several results on the minimal model program for 3-folds of characteristic $p > 5$.

4.3.1 Results on the minimal model program for 3-folds

The existence of (log) minimal models of 3-folds in positive characteristic $p > 5$ has been first proved for canonical singularities by Hacon and Xu [55], and in general by Birkar [8] (see [112] for the lc case). The existence of Mori fiber spaces has been first proved for terminal singularities by Cascini, Tanaka and Xu [17], and in general by Birkar and Waldron [11].

Theorem 4.3.1. *Let k be an algebraically closed field of characteristic $p > 5$. Let $f : X \to Y$ be a contraction from a normal 3-fold, and Δ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X.*

(1) *If either (X, Δ) is klt and $K_X + \Delta$ is pseudo-effective over Y, or (X, Δ) is lc and $K_X + \Delta$ has a weak Zariski decomposition (i.e., there exists a birational projective morphism $\mu : W \to X$ such that $\nu^*(K_X + \Delta) = P + M$, where P is nef over Y and M is effective), then (X, Δ) has a log minimal model over Y.*

(2) *If (X, Δ) is a dlt pair and Y is a smooth projective curve with $g(Y) \geq 1$, then every step of LMMP in* [8, Sec. 3.5-3.7] *starting from (X, Δ) is over Y.*

Proof. For (1) please refer to [8, Theorem 1.2 and Proposition 8.3]. For (2), since (X, Δ) is dlt, every $K_X + \Delta$-extremal ray is generated by a rational curve by cone theorem [11, Theorem 1.1], which is contracted by f since $g(Y) \geq 1$. So for an extremal contraction $X \to \bar{X}$, if there is a divisorial contraction or a flip $\sigma : X \dashrightarrow X^+$ as in [8, Sec. 3.5-3.7], there exist natural morphisms $\bar{f} : \bar{X} \to Y$ and $f^+ : X^+ \to Y$ fitting into the following commutative diagram

$$\begin{tikzcd}
X \arrow[rr, dotted] \arrow[dr] \arrow[ddr, "f"'] & & X^+ \arrow[dl] \arrow[ddl, "f^+"] \\
& \bar{X} \arrow[d, "\bar{f}"] & \\
& Y &
\end{tikzcd}.$$

Note that $(X^+, \Delta^+ = \sigma_*\Delta)$ is a dlt pair. We can show this assertion by induction. $\square$

Lemma 4.3.2. *Let k be an algebraically closed field of characteristic $p > 5$. Let (X, Δ) be a normal $\mathbb{Q}$-factorial lc 3-fold (not necessarily projective). Let C be a projective lc center of (X, Δ) and $\tilde{C}$ be the normalization of C. If $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial, then $(K_X + \Delta)|_{\tilde{C}}$ is $\mathbb{Q}$-trivial.*

Proof. By [8, Lemma 7.5], we can take a crepant partial resolution $\mu : X' \to X$ such that

$$K_{X'} + D + \Delta' \sim_{\mathbb{Q}} \mu^*(K_X + \Delta),$$

where D is a reduced irreducible divisor dominant over C and $\Delta' \geq 0$ is a $\mathbb{Q}$-divisor on X' such that $(X', D + \Delta')$ is dlt. By [8, Lemma 5.2], we see that D is normal.

Considering the restriction of the above $\mathbb{Q}$-divisors to D, we obtain an effective $\mathbb{Q}$-Weil divisor Δ_D on D such that

$$K_D + \Delta_D \sim_{\mathbb{Q}} \mu^*(K_X + \Delta)|_D$$

by the adjunction formula [70, 5.3]. Then we see that (D, Δ_D) is lc by [8, Lemma 5.2] again. Applying [105, Theorem 1.2], we have that $K_D + \Delta_D$ is semi-ample. Thus $\mu^*(K_X + \Delta)|_D$ is $\mathbb{Q}$-trivial since $(K_X + \Delta)|_{\tilde{C}}$ is numerically trivial. We can conclude that $(K_X + \Delta)|_{\tilde{C}}$ is $\mathbb{Q}$-trivial by the lemma below. □

Lemma 4.3.3. *Let $g : V \to W$ be a surjective morphism between proper varieties over a field. Assume that W is normal. Then a Cartier divisor L on W is $\mathbb{Q}$-linearly trivial if and only if so is g^*L.*

Proof. The "only if" part is obvious. We prove the "if" part. Assume that $m(g^*L) \sim 0$. Then by the projection formula, we have $(g_*\mathcal{O}_V) \otimes \mathcal{O}_W(mL) \cong g_*\mathcal{O}_V(m(g^*L)) \cong g_*\mathcal{O}_V$. Taking the determinants, we obtain $\det(g_*\mathcal{O}_V) \otimes \mathcal{O}_W(\mathrm{rank}(g_*\mathcal{O}_V) \cdot mL) \cong \det(g_*\mathcal{O}_V)$, and thus $(m \cdot \mathrm{rank}(g_*\mathcal{O}_V))L \sim 0$. □

4.3.2 Proof of Theorem 4.1.4

The next lemma is a consequence of Tanaka's vanishing theorem for surfaces [106].

Lemma 4.3.4. *Let $g : S \to C$ be a generically smooth surjective morphism from a smooth projective surface to a smooth projective curve. Let H be a nef and g-big divisor on S. Then $g_*\mathcal{O}_S(K_{S/C} + lH)$ is a nef vector bundle for every $l \gg 0$.*

Proof. Set $\mathcal{G}(l) := g_*\mathcal{O}_S(K_{S/C} + lH)$ for each $l \in \mathbb{Z}$. Let A be an ample divisor on C with $\deg A \geq \deg K_C + 2$ and $c \in C$ be a general closed point. Then $A - K_C - c$ is ample, where we regard c as a divisor on C. Note that $\nu(H) \geq 1$ and $H + g^*(A - K_C - c)$ is nef and big. Denote by S_c the fiber of g over c. By [106, Theorem 2.6] we see that

$$\begin{aligned}&H^1(S, K_S + H + g^*(A - K_C) + (l-1)H - S_c)\\ &\qquad\qquad = H^1(S, K_S + H + g^*(A - K_C - c) + (l-1)H) = 0\end{aligned}$$

for $l \gg 0$. Thus for a closed point $c \in C$, by the long exact sequence arising from taking cohomology of the exact sequence

$$\begin{aligned}0 \to \mathcal{O}_S(K_{S/C} + g^*A + lH - S_c) &\to \mathcal{O}_S(K_{S/C} + g^*A + lH)\\ &\to \mathcal{O}_S(K_{S/C} + g^*A + lH)|_{S_c} \to 0,\end{aligned}$$

we conclude that the restriction

$$H^0(S, K_{S/C} + g^*A + lH) \to H^0(S_c, (K_{S/C} + g^*A + lH)|_{S_c})$$

is surjective. This implies that $g_*\mathcal{O}_S(K_{S/C} + g^*A + lH)$, which is isomorphic to $\mathcal{G}(l) \otimes \mathcal{O}_C(A)$, is generically globally generated. Hence we obtain $t(\mathcal{G}(l), A) \geq -1$

(see Definition 3.2.1). On the other hand, since $H|_{S_{\overline{\eta}}}$ is ample, where $S_{\overline{\eta}}$ is the geometric generic fiber of g, we get an $l_0 > 0$ such that

$$g_{C^e *}(\phi^{(e)}_{S/C} \otimes \mathcal{O}_{S_{C^e}}((K_{S/C} + lH)_{C^e})) : \\ g^{(e)}{}_* \mathcal{O}_S(K_{S/C} + lp^e H) \to g_{C^e *}\mathcal{O}_{S_{C^e}}((K_{S/C} + lH)_{C^e})$$

is generically surjective for each $l \geq l_0$ and $e > 0$ by Lemma 1.3.3. Since F_C is flat, the target is isomorphic to $F_C^{e*}\mathcal{G}(l)$. Applying Lemma 3.2.2, we get

$$-1 \leq t(\mathcal{G}(lp^e), A) \overset{\text{Lemma 3.2.2 (1)}}{\leq} t(F_C^{e*}(g_*\mathcal{G}(l), A) \overset{\text{Lemma 3.2.2 (3)}}{=} p^e t(\mathcal{G}(l), A)$$

which means that $t(g_*\mathcal{O}_S(K_{S/C} + lH)) \geq 0$. By Proposition 3.2.4, we conclude that $g_*\mathcal{O}_S(K_{S/C} + lH)$ is nef. $\square$

In order to prove Theorem 4.1.4, we recall the following two results without proofs.

Theorem 4.3.5 ([20, 3.2]). *Let $f : X \to Y$ be an algebraic fiber space such that X and Y are smooth projective varieties over an algebraically closed field of positive characteristic. Assume that the geometric generic fiber of f is a smooth elliptic curve. Then $\kappa(X, K_{X/Y}) \geq 0$.*

The following lemma is also used in Chapter 6.

Lemma 4.3.6 ([111, Lemma 3.2]). *Let $f : X \to Y$ be a fibration between normal quasi-projective varieties. Let L be an f-nef $\mathbb{Q}$-Cartier divisor on X such that $L_\eta \sim_{\mathbb{Q}} 0$, where η is the generic point of Y. Assume* $\dim Y \leq 3$. *Then there exist a commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "\phi"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

with ϕ, ψ projective birational, and a $\mathbb{Q}$-Cartier divisor D on Y' such that $\phi^* L \sim_{\mathbb{Q}} f'^* D$. Furthermore, if f is flat and Y is $\mathbb{Q}$-factorial, then we can take $X' = X$ and $Y' = Y$.

Proof of Theorem 4.1.4. Let W be a minimal model of X over Y. Let $\rho : X_{\overline{\eta}} \to W_{\overline{\eta}}$ be the induced morphism. Since $\rho_*\mathcal{O}_{X_{\overline{\eta}}} \cong \mathcal{O}_{W_{\overline{\eta}}}$, $W_{\overline{\eta}}$ is normal. Furthermore, since W is terminal, we have $K_{X_{\overline{\eta}}} \geq \rho^* K_{W_{\overline{\eta}}}$, and hence $W_{\overline{\eta}}$ has at most canonical singularities. Replacing X with a minimal model with loss of smoothness, we may assume that $K_{X/Y}$ is f-nef. Then by [105, Theorem 1.2], $K_{X_{\overline{\eta}}}$ is semi-ample, and since $p > 5$, the geometric generic fiber of the Iitaka fibration $I_{\overline{\eta}} : X_{\overline{\eta}} \to C_{\overline{\eta}}$ is a smooth elliptic curve over $k(\overline{\eta})$ by [5, Theorem 7.18]. For the generic fiber X_η and a sufficiently divisible positive integer n, since $H^0(X_{\overline{\eta}}, nK_{X_{\overline{\eta}}}) \cong H^0(X_\eta, nK_{X_\eta}) \otimes_{k(\eta)} k(\overline{\eta})$, we see that the Iitaka fibration $I_{\overline{\eta}} : X_{\overline{\eta}} \to C_{\overline{\eta}}$ coincides with the Iitaka fibration $I_\eta : X_\eta \to C_\eta$ tensoring with $k(\overline{\eta})$. Thus the geometric generic fiber of I_η is a smooth elliptic curve. Considering the relative Iitaka fibration of $f : X \to Y$, whose geometric generic fiber is a smooth elliptic curve, we get a birational morphism $u : X' \to X$, a fibration $g : S \to Y$ from a smooth projective surface S, and an elliptic fibration $h : X' \to S$ fitting into the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & Y. \end{array}$$

Note that the geometric generic fiber $S_{\overline{\eta}}$ of $g : S \to Y$ is normal, and hence smooth. By Lemma 4.3.6, we may assume that $u^*K_{X/Y} \sim_{\mathbb{Q}} h^*H$ for a nef g-big $\mathbb{Q}$-Cartier divisor on S. By Theorem 4.3.5, we have $\kappa(X', K_{X'/S}) \geq 0$, and hence there exists an injective homomorphism $h^*\omega^m_{S/Y} \to \omega^m_{X'/Y}$ for sufficiently divisible $m > 0$. Let $l \gg 0$ be an integer such that lH is Cartier and $u^*lK_{X/Y} \sim h^*lH$. Then we have natural homomorphisms

$$\begin{aligned} (g_*\mathcal{O}_S(K_{S/Y} + lH))^{\otimes m} &\to g_*\mathcal{O}_S(m(K_{S/Y} + lH)) \\ &\cong g_*h_*\mathcal{O}_{X'}(mh^*(K_{S/Y} + lH)) \\ &\hookrightarrow f_*u_*\mathcal{O}_{X'}(mK_{X'/Y} + u^*lmK_{X/Y}) \\ &\cong f_*\mathcal{O}_X(m(l+1)K_{X/Y}). \end{aligned}$$

Replacing l if necessary, we may assume that the first homomorphism is generically surjective. By Lemma 4.3.4, $g_*\mathcal{O}_S(K_{S/Y}+lH)$ is nef, and hence so is $g_*\mathcal{O}_S(m(K_{S/Y}+lH))$. This completes the proof. $\square$

Chapter 5

Positivity of anti-canonical divisors

5.1 Summary

Let $f : X \to Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field k. Kollár, Miyaoka and Mori [74, Corollary 2.9] have proved that, under the assumption that f is smooth, if X is a Fano variety, that is, $-K_X$ is ample, then so is Y. It follows from an analogous argument that, under the same assumption, if $-K_X$ is nef, then so is $-K_Y$ (cf. [85], [42, Theorem 1.1] and [26, Corollary 3.15 (a)]). Based on these results, Yasutake asked: "what positivity condition is passed from $-K_X$ to $-K_Y$?" Some answers to this question are known in characteristic 0. Fujino and Gongyo [41, Theorem 1.1] have proved that, under the assumption that f is smooth, if X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y. Birkar and Chen [9, Theorem 1.1] have shown that, under the same assumption, if $-K_X$ is semi-ample, then so is $-K_Y$. Furthermore, similar but weaker results hold even if f is not smooth (but the characteristic of k is still zero). For example, a result of Prokhorov and Shokurov [95, Lemma 2.8] (cf. [41, Corollary 3.3]) implies that if $-K_X$ is nef and big, then $-K_Y$ is big. Chen and Zhang [19, Main theorem] have also proved that if $-K_X$ is nef, then $-K_Y$ is pseudo-effective.

In contrast, little was known about the positive characteristic case. In this chapter, assuming that the geometric generic fiber has only F-pure or strongly F-regular singularities, we prove that (generalizations of) the statements above hold in positive characteristic, except the one about semi-ampleness. F-purity and strong F-regularity are mild singularities defined in terms of Frobenius splitting properties (Definition 1.3.1), which have a close connection to log canonical and Kawamata log terminal singularities, respectively.

Suppose that k is an algebraically closed field of characteristic $p > 0$. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties, let Δ be an effective $\mathbb{Q}$-divisor on X which is $\mathbb{Q}$-Cartier with index m, and let D be a $\mathbb{Q}$-divisor on Y which is $\mathbb{Q}$-Cartier with index n. Then our main theorem is stated as follows.

Theorem 5.1.1 (Theorem 5.2.5). *Let S be a subset of Y such that the following conditions hold for every $y \in S$:*

(i) $\dim X_y = \dim X - \dim Y$;

(ii) *the support of Δ does not contain any irreducible component of X_y;*

(iii) $(X_{\overline{y}}, \Delta_{\overline{y}})$ *is F-pure, where $\overline{y}$ is given by* $\mathrm{Spec}\,\overline{k(y)}$.

If $p \nmid m$ and $-(K_X + \Delta + f^*D)$ is nef, then $\mathcal{O}_Y(-n(K_Y + D))$ is weakly positive over an open subset of Y containing S.

In Section 5.2, Theorem 5.2.5 is proved in a more general setting, which is used to prove Iitaka's conjecture $C_{3,1}$ (Theorem 6.1.2).

The following two theorems are corollaries of Theorem 5.1.1.

Theorem 5.1.2 (Corollary 5.2.9). *Assume that f is flat, the support of Δ does not contain any component of any fiber, and (X_y, Δ_y) is F-pure for every closed point $y \in Y$.*

(1) *If $p \nmid m$ and $-(K_X + \Delta + f^*D)$ is nef, then so is $-(K_Y + D)$.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then so is $-(K_Y + D)$.*

Theorem 5.1.3 (Corollary 5.2.10). *Assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, where $\overline{\eta}$ is the geometric generic point of Y.*

(1) *If $p \nmid m$ and $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is pseudo-effective.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big.*

(3) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is strongly F-regular and $-(K_X + \Delta + f^*D)$ is nef and big, then $-(K_Y + D)$ is big.*

Theorem 5.1.2 is a generalization of [74, Corollary 2.9] and [26, Corollary 3.15] in positive characteristic. We can also recover [74, Corollary 2.9] in characteristic zero from Theorem 5.1.2 by standard reduction to characteristic p techniques. Our proof relies on a study of the positivity of direct image sheaves for f in terms of the Grothendieck trace of the relative Frobenius morphism. This is completely different from the proof of Kollár, Miyaoka and Mori which relies on a detailed study of rational curves on varieties. Theorem 5.1.3 should be compared with [95, Lemma 2.8] and [19, Main Theorem].

The following two theorems are direct consequences of Theorems 5.1.2 and 5.1.3.

Theorem 5.1.4 (Corollary 5.2.11). *Assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, where $\overline{\eta}$ is the geometric generic point of Y. If $p \nmid m$ and $K_X + \Delta$ is numerically equivalent to $f^*(K_Y + L)$ for some $\mathbb{Q}$-divisor L on Y, then L is pseudo-effective.*

Theorem 5.1.5 (Corollary 5.2.14)**.** *Assume that f is flat, that every closed fiber is F-pure, and that the geometric generic fiber is strongly F-regular. If X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y.*

Theorem 5.1.5 is a positive characteristic counterpart of [41, Thorem 1.1].

For another application of Theorem 5.1.1, we return to the situation where k is of arbitrary characteristic. Suppose that $f : X \to Y$ is a generically smooth surjective morphism between smooth projective varieties over an algebraically closed field of arbitrary characteristic and in addition that the dimension of Y is positive.

Theorem 5.1.6 (Corollary 5.2.15 and Theorem 5.3.5)**.** *$-K_{X/Y}$ is not nef and big.*

Theorem 5.1.7 (Corollary 5.2.16 and Theorem 5.3.6)**.** *Assume that $\omega_{X_{\overline{\eta}}}^{-m}$ is globally generated for an integer $m > 0$, where $\overline{\eta}$ be the geometric generic point of Y. Then $f_*\omega_{X/Y}^{-m}$ is not big in the sense of Definition 3.1.1.*

In both of the theorems, the characteristic zero case is proved by reduction to positive characteristic. Theorem 5.1.6 improves a result of Kollár, Miyaoka and Mori [74, Corollary 2.8] which states that $-K_{X/Y}$ is not ample. Theorem 5.1.7 includes a result of Miyaoka [85, COROLLARY 2'] which states that if $\omega_{X/Y}^{-1}$ is f-ample and $\omega_{X/Y}^{-m}$ is f-free for an integer $m > 0$, then $f_*\omega_{X/Y}^{-m}$ is not an ample vector bundle.

5.2 Main theorems in positive characteristic

The purpose of this section is to prove Theorems 5.2.5 and 5.2.6. First we prepare three lemmas. The following lemma is used in the proof of Theorem 5.2.5.

Lemma 5.2.1. *Let k be a field. Let $f : X \to Y$ be a surjective projective morphism from a k-scheme X to a variety Y, let A be an f-ample Cartier divisor on X and let $\mathcal{F}$ be a coherent sheaf on X. Then there exists an integer $m_0 > 0$ such that for each integer $m \geq m_0$ and for every nef Cartier divisor N on X, $f_*(\mathcal{F}(mA+N))$ is locally free over V, where $V \subseteq Y$ is an open subset such that $\mathcal{F}_V$ is flat over V.*

Proof. By Keeler's relative Fujita vanishing ([71, Theorem 1.5]), there exists an integer $m_0 > 0$ such that

$$Rf_*^i\mathcal{F}(mA + N) = 0$$

for each $m \geq m_0$, for every nef Cartier divisor N on X and for each $i > 0$. Fix an integer $m \geq m_0$ and a nef Cartier divisor N. For each $i \geq 0$, we define the function h^i on V by $h^i(y) := \dim_{k(y)} H^0(X_y, \mathcal{F}(mA + N)|_{X_y})$. Since $\dim X_y \leq \dim X$ for every $y \in V$, we see that $h^i = 0$ for each $i > \dim X$. Let $i \geq 2$ be an integer such that $h^i = 0$. By applying [58, Theorem III 12.11] to the morphism $f_V : X_V \to V$ and $\mathcal{F}(mA + N)_V$, we obtain that $h^{i-1} = 0$. From this, we see that $h^i = 0$ for each $i \geq 1$, and hence $\chi(\mathcal{F}(mA + N)|_{X_y}) = h^0(y)$ for every $y \in V$. Since the left hand side is constant on V by [58, Theorem III 9.9] and its proof, h^0 is also constant on V. Applying [58, Corollary III 12.9], we obtain that $f_*\mathcal{F}(mA + N)$ is locally free over V, which is our claim. $\square$

Lemma 5.2.2. *With the notation as in Lemma 5.2.1, assume that X and Y are projective and A is ample. Then there exists an integer $m_0 > 0$ such that for each integer $m \geq m_0$ and for every nef Cartier divisor N on X, $f_*(\mathcal{F}(mA + N))$ is globally generated.*

Proof. Let H be an ample and free Cartier divisor on Y, and let $m_1 > 0$ be an integer such that $m_1 A - \dim Y f^*H$ is nef. By the Fujita vanishing ([44, Theorem (1)], [45, Section 5]) and Keeler's relative Fujita vanishing ([71, Theorem 1.5]), there exists an integer $m_2 > 0$ such that

$$H^i(X, \mathcal{F}(mA + N)) = 0 \text{ and } R^i f_*(\mathcal{F}(mA + N)) = 0$$

for each $m \geq m_2$, for every nef Cartier divisor N on X and for each $i > 0$. Since $R^i f_*(\mathcal{F}(mA + N)) = 0$, by the Leray spectral sequence, we have

$$\begin{aligned} H^i(Y, (f_*(\mathcal{F}((m+m_1)A + N)))(-iH)) &\cong H^i(Y, f_*(\mathcal{F}((m+m_1)A - if^*H + N))) \\ &\cong H^i(X, \mathcal{F}(mA + m_1A - if^*H + N)) \end{aligned}$$

for each $m \geq m_0$ and for each $i > 0$. Since $m_1A - if^*H + N$ is nef for $0 < i \leq \dim Y$, the right hand side vanishes. This implies that $f_*(\mathcal{F}((m+m_1)A + N))$ is 0-regular with respect to H, and hence it is globally generated by the Castelnuovo-Mumford regularity [79, Theorem 1.8.5]. Defining $m_0 := m_1 + m_2$, we obtain the assertion. $\square$

The next lemma is a consequence of the relative Castelnuovo-Mumford regularity [79, Example 1.8.24], and is used in the proof of Theorems 5.2.5 and 5.2.6.

Lemma 5.2.3. *Let k, f, X, Y and $\mathcal{F}$ be as in Lemma 5.2.1. Let $W \subseteq Y$ be an open subset. Let L be a Cartier divisor on X.*

(1) *If L_W is f_W-free, then there exists an integer $n_0 > 0$ such that for each $n \geq n_0$ and each $m > 0$, the natural morphism*

$$f_*\mathcal{O}_X(mL) \otimes f_*(\mathcal{F}(nL)) \to f_*(\mathcal{F}((m+n)L))$$

is surjective over W.

(2) *If L_W is f_W-ample and f_W-free, then there exists an integer $n_0 > 0$ such that for each $n \geq n_0$, for each $m > 0$ and for every Cartier divisor N on X whose restriction N_W to X_W is f_W-nef, the natural morphism*

$$f_*\mathcal{O}_X(mL) \otimes f_*(\mathcal{F}(nL + N)) \to f_*(\mathcal{F}((n+m)L + N))$$

is surjective over W.

Proof. Replacing $f : X \to Y$ by $f_W : X_W \to W$, we may assume that $W = Y$. Set $\mathcal{L} := \mathcal{O}_X(L)$. We first show that (2) implies (1). Indeed, since $\mathcal{L}$ is f-free, there exist surjective projective morphisms $g : X \to Z$ and $h : Z \to Y$ with $h \circ g = f$

such that $\mathcal{L} \cong g^*\mathcal{M}$ for an h-ample and h-free line bundle $\mathcal{M}$ on Z. Then we have the following commutative diagram of natural morphisms:

$$\begin{tikzcd}
(h_*\mathcal{M}^m) \otimes h_*((g_*\mathcal{F}) \otimes \mathcal{M}^n) \arrow[r] \arrow[d] & h_*((g_*\mathcal{F}) \otimes \mathcal{M}^{m+n}) \arrow[dd, "\cong"] \\
(h_*g_*g^*\mathcal{M}^m) \otimes h_*((g_*\mathcal{F}) \otimes \mathcal{M}^n) \arrow[d, "\cong"] & \\
h_*g_*\mathcal{L}^m \otimes h_*g_*(\mathcal{F} \otimes \mathcal{L}^n) \arrow[r] & h_*g_*(\mathcal{F} \otimes \mathcal{L}^{m+n})
\end{tikzcd}$$

Here the isomorphisms follow from the projection formula. The surjectivity of the upper horizontal morphism induces that of the lower horizontal morphism. Hence we see that it is enough to prove (2).

We show (2). We first prove the case $m = 1$. In this case, by Keeler's relative Fujita vanishing ([71, Theorem 1.5]), there exists an integer $n_0 > 0$ such that for each $n \geq n_0$ and every f-nef line bundle $\mathcal{N}$ on X, $\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}$ is 0-regular with respect to $\mathcal{L}$ and f, and hence the surjectivity follows from the relative Castelnuovo-Mumford regularity [79, Example 1.8.24]. Next we show the case when $m \geq 2$. In this case, we have the following commutative diagram of natural morphisms:

$$\begin{tikzcd}
(f_*\mathcal{L})^{\otimes m} \otimes f_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}) \arrow[r] \arrow[d] & f_*\mathcal{L}^m \otimes f_*(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{N}) \arrow[ddd] \\
(f_*\mathcal{L})^{\otimes m-1} \otimes f_*(\mathcal{F} \otimes \mathcal{L}^{n+1} \otimes \mathcal{N}) \arrow[d] & \\
\vdots \arrow[d] & \\
f_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \mathcal{N}) \arrow[r, equal] & f_*(\mathcal{F} \otimes \mathcal{L}^{m+n} \otimes \mathcal{N})
\end{tikzcd}$$

By the above argument, we see that the left vertical morphisms are surjective, and hence so is the right vertical morphism, which is our claim. □

Notation 5.2.4. Let k be an F-finite field and $f : X \to Y$ be a surjective projective morphism from a pure dimensional quasi-projective k-scheme X satisfying S_2 and G_1 to a normal quasi-projective variety Y. Let $a > 0$ be an integer and E be an effective AC-divisor on X such that $aK_X + E$ is Cartier. Set $\Delta := E/a$. Let $U \subset X$ be a Gorenstein open subset and let $S \subseteq Y$ be a non-empty subset such that the following conditions hold for every $y \in S$:

(i) Y is regular in a neighborhood of y and f is flat at every point in $f^{-1}(y)$;

(ii) $\mathrm{codim}_{X_{\overline{y}}}(X_{\overline{y}} \setminus U_{\overline{y}}) \geq 2$;

(iii) the support of E does not contain any irreducible component of $f^{-1}(y)$;

(iv) $(X_{\overline{y}}, \overline{\Delta|_{U_{\overline{y}}}})$ is F-pure, where $\overline{y} := \operatorname{Spec} \overline{k(y)}$ and $\overline{\Delta|_{U_{\overline{y}}}}$ is the $\mathbb{Z}_{(p)}$-AC divisor on $X_{\overline{y}}$ obtained as the unique extension of the $\mathbb{Z}_{(p)}$-Cartier divisor $\Delta|_{U_{\overline{y}}}$ on $U_{\overline{y}}$.

Let D be a $\mathbb{Q}$-Cartier divisor on Y.

In this setting, we show the following theorems.

Theorem 5.2.5. *With the notation as in 5.2.4, assume that X is projective and $K_X + \Delta$ is $\mathbb{Q}$-Cartier.*

(1) *If $p \nmid a$ and $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is weakly positive over an open subset of Y containing S.*

(2) *If K_X is $\mathbb{Q}$-Cartier and $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big over an open subset of Y containing S.*

Theorem 5.2.6. *With the notation as in 5.2.4, let $b > 0$ be an integer such that bD is Cartier. Set $L := abK_X + bE + af^*(bD)$. Assume that the natural morphism*

$$f^* f_* \mathcal{O}_X(-L) \to \mathcal{O}_X(-L)$$

is surjective over $f^{-1}(S)$. If $p \nmid a$ and $f_*\mathcal{O}_X(-L)$ is globally generated over S, then $-(K_Y + D)$ is weakly positive over S.

Remark 5.2.7. In the case when X is a normal variety and S is the singleton $\{\eta\}$ of the generic point η of Y, conditions (i)–(iii) above hold. However, condition (iv) does not necessarily hold even if X is smooth and $\Delta = 0$.

Proof of Theorem 5.2.5. First we show that (1) implies (2). By the assumption of (2), Δ is $\mathbb{Q}$-Cartier. Let $m > 0$ and $c \geq 0$ be integers such that $a = mp^c$ and $p \nmid m$. Take an integer $e \gg 0$. Set $a' := m(p^e + 1)$, $E' := p^{e-c}E$ and $\Delta' := (p^{e-c}E) \otimes a'^{-1}$. Then $p \nmid a'$, and E' satisfies assumption (iii). Furthermore, Δ' satisfies assumption (iv). Indeed, we have

$$\begin{aligned}\Delta - \Delta' &= a'E \otimes \frac{1}{aa'} - ap^{e-c}E \otimes \frac{1}{aa'} \\ &= (m(p^e+1) - mp^c p^{e-c})E \otimes \frac{1}{aa'} = mE \otimes \frac{1}{aa'} = \frac{m}{a'}\Delta = \frac{1}{p^e+1}\Delta \geq 0,\end{aligned}$$

and hence $\Delta'_{\overline{y}} \leq \Delta_{\overline{y}}$ for every $y \in S$, which implies that $(X_{\overline{y}}, \Delta'_{\overline{y}})$ is also F-pure for every $y \in S$. Replacing e by a larger integer if necessary, we may assume that

$$-(K_X + \Delta' + f^*D) = -(K_X + \Delta + f^*D) - \frac{1}{p^e+1}\Delta$$

is an ample $\mathbb{Q}$-Cartier divisor. Then $-(K_X + \Delta' + f^*(D' + \varepsilon H))$ is nef for an ample Cartier divisor H on Y and an $\varepsilon \in \mathbb{Q}$ with $0 < \varepsilon \ll 1$. By (1) we see that $-(K_Y + D + \varepsilon H)$ is weakly positive over an open subset $Y_1 \subseteq Y$ containing S, and hence $-(K_Y + D)$ is big over Y_1.

Next we show (1). The proof is divided into seven steps.

Step 1. In this step, we prove that we may assume that S is an open subset. In other words, there exists an open subset $S' \subseteq Y$ containing S such that conditions (i)–(iv) hold for every $y \in S'$. Note that weak positivity over S' implies weak positivity over S. Set $r := \dim X - \dim Y$. Let Y' be the subset of points satisfying condition (i). Then $Y' \subseteq Y$ is an open subset containing S, and $X_{\overline{y}}$ is an S_2 scheme of pure dimension r for every $y \in Y'$. Since the function $\dim(X \setminus U)_{\overline{y}}$ on Y is upper semicontinuous by Chevalley's theorem ([50, Corollaire 13.1.5]), we obtain that the function $\operatorname{codim}_{X_{\overline{y}}}(X_{\overline{y}} \setminus U_{\overline{y}}) = r - \dim((X \setminus U)_{\overline{y}})$ on Y' is lower semicontinuous. Therefore the subset of points y in Y' with $\operatorname{codim}_{X_{\overline{y}}}(X_{\overline{y}} \setminus U_{\overline{y}}) \geq 2$ is open, and contains S. By an argument similar to the above, we see that the subset of points y in Y' with $\operatorname{codim}_{X_{\overline{y}}}(E_{\overline{y}}) \geq 1$ is open, and contains S. Hence, shrinking Y' if necessarily, we may assume that every $y \in Y'$ satisfies conditions (i)–(iii). Applying Lemma 1.3.3 (1) (set $S' = V$), we obtain an open subset $S' \subseteq Y'$ containing S such that conditions (i)–(iv) hold for every $y \in S'$.

Step 2. In this step, we reduce the proof to a numerical condition. By Step 1, we may assume that $S \subseteq Y$ is open. Let A be an ample and free Cartier divisor on X. By Lemma 5.2.2, we have an integer $m' > 0$ such that $f_*\mathcal{O}_X(mA)$ is globally generated for each integer $m \geq m'$. Let V denote the regular locus of Y. Note that $S \subseteq V$ by assumption (i). By Lemma 5.2.1, we have an integer $m'' > 0$ such that $f_{V*}\mathcal{O}_{X_V}(N + mA_V)$ is locally free over S for each integer $m \geq m''$ and every Cartier divisor N on X_V whose restriction N_S to X_S is f_S-nef. Replacing A by $\max\{m', m''\}A$ if necessary, we may assume that $m' = m'' = 1$. For simplicity, we set

$$\begin{aligned} D' &:= D + K_Y, \\ \mathcal{G}(l,m) &:= f_{V*}\mathcal{O}_{X_V}(l(K_{X_V/V} + \Delta_V) + mA_V) \quad \text{and} \\ t(l,m) &:= t_S(\mathcal{G}(l,m), D'|_V) \end{aligned}$$

for every integers l, m with $a|l$. Then, since $f_*\mathcal{O}_X(mA)$ is globaly generated, we have $t(0,m) \geq 0$ for each $m > 0$. Furthermore, since $-l(K_{X_V/V} + \Delta_V)_S$ is f_S-nef for each $l \geq 0$ with $a|l$ by the assumption, $\mathcal{G}(-l,m)$ is locally free over S for each $l \geq 0$ with $a|l$ and $m > 0$. Our goal is to prove that $t(0,m) = +\infty$ for an integer $m \geq 0$. If this is shown, then by Lemma 3.2.2 we obtain that $-D'|_V = -(K_Y + D)|_V$ is weakly positive over S. As mentioned in Remark 3.1.3 (2), this is equivalent to the weak positivity of $-(K_Y + D)$. In order to get a contradiction, we assume that $t(0,m) \neq +\infty$ for each integer $m > 0$. Then we have $0 \leq t(0,m) \in \mathbb{R}$.

Step 3. Let $d > 0$ be an integer divisible by a such that dD and $d(K_X + \Delta)$ is Cartier. Let q_l and r_l denote the quotient and the remainder of the division of an integer l by d, respectively. In this step, we show that there exists an integer $m_0 > 0$ such that

$$dq_l \leq t(-l,m)$$

for each $m \geq m_0$ and each $l \geq 0$ with $a|l$. Since $aK_X + E$ is Cartier and S is contained in the regular locus V of Y, we have that for each $0 \leq r < d$ with $a|r$,

$$\mathcal{M}_r := f^*\omega_Y^{\otimes r}(-r(K_X + \Delta))$$

is invertible in a neighborhood of $f^{-1}(S)$. Hence we see that $\mathcal{M}_r$ is flat over Y at every point of $f^{-1}(S)$ by assumption (i). As shown in Lemmas 5.2.1 and 5.2.2, there exists an integer $m_0 > 0$ such that for every integer $m \geq m_0$, for every nef Cartier divisor N on X and for each $0 \leq r < d$ with $a|r$, $f_*(\mathcal{M}_r(N + mA))$ is locally free over S and globally generated over Y. Set $L := -(dK_X + d\Delta + f^*(dD))$. Then by the assumption of (1), L is a nef Cartier divisor on X. For each $m \geq m_0$ and each $l \geq 0$ with $a|l$, we have the following isomorphisms:

$$\begin{aligned}
&(\mathcal{G}(-l, m))\,(-dq_l D'|_V) \\
&= \big(f_{V*}\mathcal{O}_{X_V}(-l(K_{X_V/V} + \Delta_V) + mA_V)\big)\,(-dq_l D'|_V) \\
&\cong \big(f_{V*}\mathcal{O}_{X_V}((-r_l - dq_l)(K_{X_V/V} + \Delta_V) + mA_V)\big)\,(-q_l(dD'|_V)) \\
&\cong f_{V*}\mathcal{O}_{X_V}(-r_l(K_{X_V/V} + \Delta_V) - q_l(dK_{X_V/V} + d\Delta_V + f_V{}^*(dD'|_V)) + mA_V) \\
&\cong f_{V*}\mathcal{O}_{X_V}(-r_l(K_{X_V/V} + \Delta_V) - q_l(dK_{X_V} + d\Delta_V + f_V{}^*(dD|_V)) + mA_V) \\
&\cong f_{V*}\mathcal{O}_{X_V}(r_l f_V{}^* K_V - r_l(K_X + \Delta)_V - q_l(dK_X + d\Delta + f^*(dD))_V + mA_V) \\
&\cong f_{V*}\left((\mathcal{M}_{r_l}|_{X_V})(q_l L_V + mA_V)\right) \cong \left(f_*\mathcal{M}_{r_l}(q_l L + mA)\right)|_V.
\end{aligned}$$

Here the isomorphisms in the forth and the last line follow from the projection formula and the flatness of $V \to Y$, respectively. Note that since $a|l$ and $a|d$, we have $a|r_l$. By the choice of m_0, we see that $f_*(\mathcal{M}_{r_l}(q_l L + mA))$ is globally generated, and hence so is $(\mathcal{G}(-l, m))\,(-dq_l D'|_V)$. This implies that $dq_l \leq t(-l, m)$.

Step 4. The assumption of the projectivity of X is used only in Steps 2 and 3. Hence, replacing $f : X \to Y$ by $f_V : X_V \to V$, we may assume that $V = Y$ in the steps below. Then

$$\mathcal{G}(l, m) = f_*\mathcal{O}_X(l(K_{X/Y} + \Delta) + mA) \quad \text{and} \quad t(l, m) = t_S(\mathcal{G}(l, m), D')$$

for every integers l, m with $a|l$.

Step 5. In this step, we show that there exists an integer $n_0 > 0$ such that

$$t(0, m) + t(-l, n) \leq t(-l, m + n)$$

for each $n \geq n_0$, each $m > 0$ and each $l \geq 0$ with $a|l$. By Step 1 we may assume that $S \subseteq Y$ is open. By the choice of A, we have that A_S is f_S-ample and f_S-free. Applying Lemma 5.2.3 (2), we get an integer $n_0 > 0$ such that for each $n \geq n_0$, each $m > 0$ and every Cartier divisor N on X whose restriction N_S to X_S is f_S-nef,

$$f_{S*}\mathcal{O}_{X_S}(mA_S) \otimes f_{S*}\mathcal{O}_{X_S}(N_S + nA_S) \to f_{S*}\mathcal{O}_{X_S}(N_S + (m + n)A_S)$$

is surjective. Since $-(aK_{X/Y} + E)_S$ is an f_S-nef Cartier divisor, it follows that for each $n \geq n_0$ and each $l \geq 0$ with $a|l$, the natural morphism

$$\mathcal{G}(0, m) \otimes \mathcal{G}(-l, n) \to \mathcal{G}(-l, m + n)$$

is surjective over S. Hence we obtain that

$$\begin{aligned} t(0,m)+t(-l,n) &= t_S(\mathcal{G}(0,m),D')+t_S(\mathcal{G}(-l,n),D') \\ &\overset{\text{Lemma 3.2.2 (2)}}{\le} t_S(\mathcal{G}(0,m)\otimes\mathcal{G}(-l,n),D') \\ &\overset{\text{Lemma 3.2.2 (1)}}{\le} t_S(\mathcal{G}(-l,m+n),D')=t(-l,m+n). \end{aligned}$$

Here note that since $t(0,m)\neq+\infty,-\infty$ as mentioned in Step 2, we can use Lemma 3.2.2 (2).

Step 6. In this step, we prove that there exists an integer $m_1>0$ such that

$$t(1-p^e,mp^e)\le p^e t(0,m)$$

for each $m\ge m_1$ and each $e\ge 0$ with $a|(p^e-1)$. Now we have the morphism

$$f_{Y^e*}\phi^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{O}_{X_{Y^e}}(mA_{Y^e}):\mathcal{G}(1-p^e,mp^e)\to f_{Y^e*}\mathcal{O}_{X_{Y^e}}(mA_{Y^e})\cong F_Y^{e*}\mathcal{G}(0,m)$$

for each integers $e,m\ge 0$ with $a|(p^e-1)$. Here the isomorphism follows from the flatness of F_Y^e. Applying Lemma 1.3.3 (2), we see that there exists an integer $m_1>0$ such that the morphism

$$(f_S)_{S^e*}\left(\phi^{(e)}_{(X_S/S,\Delta_S)}\otimes\mathcal{O}_{X_{S^e}}(mA_{S^e})\right)\cong\left(f_{Y^e*}\phi^{(e)}_{(X/Y,\Delta)}\otimes\mathcal{O}_{X_{Y^e}}(mA_{Y^e})\right)|_S$$

is surjective for each $m\ge m_1$ and each $e\ge 0$ with $a|(p^e-1)$. Therefore we obtain that

$$\begin{aligned} t(1-p^e,mp^e)=t_S(\mathcal{G}(1-p^e,mp^e),D') &\overset{\text{Lemma 3.2.2 (1)}}{\le} t_S(F_Y^{e*}\mathcal{G}(0,m),D') \\ &\overset{\text{Lemma 3.2.2 (3)}}{=} p^e t_S(\mathcal{G}(0,m),D')=p^e t(0,m). \end{aligned}$$

Step 7. In this step, we obtain a contradiction. Fix an integer $\mu>0$ such that $m_1\mu\ge\max\{m_0,n_0\}$. Then by Steps 5 and 6, we see that

$$(p^e-\mu)t(0,m_1)+t(1-p^e,m_1\mu)\overset{\text{Step 5}}{\le} t(1-p^e,m_1p^e)\overset{\text{Step 6}}{\le} p^e t(0,m_1)$$

for each $e\ge 0$ with $a|(p^e-1)$ and $p^e\ge\mu$. Therefore, we obtain that $t(1-p^e,m_1\mu)\le\mu t(0,m_1)$. In particular, $t(1-p^e,m_1\mu)$ is bounded from above. However, as shown in Step 3 we have $dq_{p^e-1}\le t(1-p^e,m_1\mu)$, which implies that $t(1-p^e,m_1\mu)$ goes to infinity as e goes to infinity. This is a contradiction.

□

Proof of Theorem 5.2.6. The strategy of the proof is very similar to the one of Theorem 5.2.5 (1).

Step 1. In this step, we show that we may assume that Y is regular. Let V be the regular locus of Y. Then we have $S\subseteq V$. As mentioned in Remark 3.1.3 (2), the weak positivity of $-(K_Y+D)$ is equivalent to the weak positivity of $-(K_Y+D)|_V$. Hence, replacing $f:X\to Y$ by $f_V:X_V\to V$, we may assume that Y is regular.

Step 2. In this step, we show that we may assume that $S \subseteq Y$ is open. As shown in Step 1 of the proof of Theorem 5.2.5, there exists an open subset $S' \subseteq Y$ containing S such that conditions (i)–(iv) hold for every $y \in S'$. Let $B \subset X$ and $C \subset Y$ be, respectively, the supports of the cokernels of the natural morphisms

$$f^*f_*\mathcal{O}_X(-L) \to \mathcal{O}_X(-L) \quad \text{and} \quad H^0(Y, f_*\mathcal{O}_X(-L)) \otimes \mathcal{O}_Y \to f_*\mathcal{O}_X(-L).$$

Then $S \cap f(B) = S \cap C = \emptyset$ by the assumption. Hence we may replace S by the open subset $S' \setminus (f(B) \cup C)$.

Step 3. In this step, we reduce the proof to a numerical condition. Let A' be an f-ample and f-free Cartier divisor on X. Let H be an ample and free Cartier divisor on Y. We take an integer $c > 0$ and set $A := A' + cf^*H$. Applying Lemma 5.2.1 and Lemma 5.2.3 (2), we get an integer $n' > 0$ such that $f_*\mathcal{O}_X(N + nA)$ is locally free over S and the natural morphism

$$f_*\mathcal{O}_X(mA) \otimes f_*\mathcal{O}_X(N + nA) \to f_*\mathcal{O}_X(N + (m + n)A)$$

is surjective over S for each $n \geq n'$, each $m > 0$ and every Cartier divisor N whose restriction N_S to X_S is f_S-nef. Note that since the statement is local on Y, n' is independent of the choice of c. Replacing A by $n'A$, we may assume that $n' = 1$. For simplicity, we set

$$\begin{aligned} D' &:= D + K_Y, \\ \mathcal{G}(l, m) &:= f_*\mathcal{O}_X(l(K_{X/Y} + \Delta) + mA) \quad \text{and} \\ t(l, m) &:= t_S(\mathcal{G}(l, m), D') \end{aligned}$$

for every integers l, m with $a|l$. Note that since $-l(K_{X/Y}+\Delta)_S$ is f_S-nef for each $l \geq 0$ by the assumption, we see that $\mathcal{G}(-l, m)$ is locally free over S for each $l \geq 0$ with $a|l$ and $m > 0$. Our goal is to prove that $t(-l, m) = +\infty$ for some integers $l, m \geq 0$ with $a|l$. If this is shown, then the assertion follows from Lemma 3.2.2 (4). Note that $\mathcal{G}(-l, m)$ is of positive rank, because of the f_S-freeness of $(-l(K_X+\Delta)+mA)_S$. In order to get a contradiction, we assume that $t(-l, m) \neq +\infty$ for every integers $l, m \geq 0$ with $a|l$.

Step 4. Set $d := ab$. Let q_l and r_l denote the quotient and the remainder of the division of an integer l by d, respectively. In this step, we show that there exists an $l_0 > 0$ (independent of the choice of c) such that for each $l \geq l_0$ with $a|l$,

$$dq_{l-l_0} + t(-r_{l-l_0} - l_0, 1) \leq t(-l, 1).$$

By the projection formula, we have

$$\begin{aligned} (\mathcal{G}(-d, 0))\,(-dD') &\cong f_*\mathcal{O}_X(-b(aK_{X/Y} + E) - dD') \\ &= f_*\mathcal{O}_X(-b(aK_X + E + af^*D)). \end{aligned}$$

The right hand side is globally generated over S by the assumption, and hence so is the left hand side. This implies that

$$d \leq t(-d, 0) \tag{5.2.6.1}$$

On the other hand, by the assumption, $-b(aK_{X/Y}+E)|_{X_S} = -L_S + abf_S{}^*(D|_Y + K_Y)|_S$ is f_S-free. Applying Lemma 5.2.3 (1), we have an integer $l_0 > 0$ such that for each $l \geq l_0$, the natural morphism

$$\begin{aligned} f_*\mathcal{O}_X(-d(K_{X/Y}+\Delta)) \otimes & f_*\mathcal{O}_X(-l(K_{X/Y}+\Delta)+A) \\ & \to f_*\mathcal{O}_X(-(d+l)(K_{X/Y}+\Delta)+A) \end{aligned}$$

is surjective over S. Note that since the statement is local on Y, l_0 is independent of the choice of c. Then by an argument similar to Step 5 of the proof of Theorem 5.2.5, we get

$$t(-d,0)+t(-l,1) \leq t(-d-l,1). \tag{5.2.6.2}$$

Consequently, we obtain

$$\begin{aligned} dq_{l-l_0} + t(-r_{l-l_0}-l_0,1) & \overset{(5.2.6.1)}{\leq} q_{l-l_0}t(-d,0) + t(-r_{l-l_0}-l_0,l) \\ & \overset{(5.2.6.2)}{\leq} t(-dq_{l-l_0}-r_{l-l_0}-l_0,1) = t(-l,1). \end{aligned}$$

Step 5. We show that for each $m,n>0$ and $l \geq 0$ with $a|l$,

$$t(0,m)+t(-l,n) \leq t(-l,m+n).$$

As mentioned in the previous step, $-b(aK_{X/Y}+E)|_{X_S}$ is f_S-free, and so it is f_S-nef. By an argument similar to Step 5 of the proof of Theorem 5.2.5, we obtain the claimed inequality.

Step 6. By the same argument as Step 6 of the proof of Theorem 5.2.5, we see that there exists an $m_1 > 0$ such that for each $m \geq m_1$ and each $e \geq 0$ with $a|(p^e-1)$,

$$t(1-p^e, mp^e) \leq p^e t(0,m).$$

Step 7. In this step, we obtain a contradiction. Let l_0 be as in Step 4. Recall that $A := A' + cf^*H$ and l_0 is independent of the choice of c. Replacing c by a larger integer, we may assume that for each integer l with $0 \leq l \leq d(l_0+1)$ and $a|l$,

$$\mathcal{G}(l,1) = f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+A'+cf^*H) \cong \left(f_*\mathcal{O}_X(l(K_{X/Y}+\Delta)+A')\right)(cH)$$

is globally generated, which implies that

$$0 \leq t(-l,1). \tag{5.2.6.3}$$

Using the inequalities in the previous steps, we obtain that

$$\begin{aligned} & (p^e-1)t(0,m_1) + dq_{p^e-1-l_0} \\ \overset{(5.2.6.3)}{\leq} & (p^e-1)t(0,m_1) + (m_1-1)t(0,1) + t(-r_{p^e-1-l_0}-l_0,1) + dq_{p^e-1-l_0} \\ \overset{\text{Step 4}}{\leq} & (p^e-1)t(0,m_1) + (m_1-1)t(0,1) + t(1-p^e,1) \\ \overset{\text{Step 5}}{\leq} & (p^e-1)t(0,m_1) + t(1-p^e,m_1) \overset{\text{Step 5}}{\leq} t(1-p^e,m_1p^e) \overset{\text{Step 6}}{\leq} p^e t(0,m_1). \end{aligned}$$

for each $e \geq 0$ with $p^e - 1 \geq l_0$ and $a|(p^e - 1)$. Hence by the transposition we obtain $dq_{p^e-1-l_0} \leq t(0, m_1)$. Note that since $0 \leq m_1 t(0,1) \leq t(0, m_1) < +\infty$ by Step 5, we have $t(0, m_1) \in \mathbb{R}$. However, $dq_{p^e-1-l_0}$ goes to infinity as e goes to infinity, which is a contradiction.

□

In the remaining part of this section, we give some corollaries of the above theorems in the following situation:

Notation 5.2.8. Let $f : X \to Y$ be a surjective morphism between regular projective varieties over an F-finite field, and let Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral. Let D be a $\mathbb{Q}$-divisor on Y. Let $\overline{\eta}$ be the geometric generic point of Y.

Corollary 5.2.9. *With the notation as in 5.2.8, assume that f is flat, that the support of Δ does not contain any component of any fiber, and that $(X_{\overline{y}}, \Delta_{\overline{y}})$ is F-pure for every point $y \in Y$.*

(1) *If $p \nmid a$ and $-(K_X + \Delta + f^*D)$ is nef, then so is $-(K_Y + D)$.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then so is $-(K_Y + D)$.*

Proof. This follows from Theorem 5.2.5 and Remark 3.1.3 immediately. □

Corollary 5.2.10. *With the notation as in 5.2.8, assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure.*

(1) *If $p \nmid a$ and $-(K_X + \Delta + f^*D)$ is nef, then $-(K_Y + D)$ is pseudo-effective.*

(2) *If $-(K_X + \Delta + f^*D)$ is ample, then $-(K_Y + D)$ is big.*

(3) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is strongly F-regular and $-(K_X + \Delta + f^*D)$ is nef and big, then $-(K_Y + D)$ is big.*

Proof. By remark 3.1.3, (1) and (2) of the corollary follow from (1) and (2) of Theorem 5.2.5, respectively. We prove (3). By Kodaira's lemma, there exists a $\mathbb{Q}$-divisor $\Delta' \geq \Delta$ on X such that $-(K_X + \Delta' + f^*D)$ is ample and $(X_{\overline{\eta}}, \Delta'_{\overline{\eta}})$ is again strongly F-regular. Hence (3) follows from (2). □

Corollary 5.2.11. *With the notation as in 5.2.8, assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure. If $p \nmid a$ and $K_X + \Delta$ is numerically equivalent to $f^*(K_Y + L)$ for a $\mathbb{Q}$-divisor L on Y, then L is pseudo-effective.*

Proof. Set $D := -(K_Y + L)$. Then $K_X + \Delta + f^*D$ is numerically trivial, and so it is nef. Hence by Corollary 5.2.10 (1), we obtain the assertion. □

Remark 5.2.12. Assume that $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + L)$ for a $\mathbb{Q}$-divisor. It is known that if $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is globally F-split, then L is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor on Y (see Theorem 2.2.2 or [25, Theorem B]). However, $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is not necessary globally F-split even if $X_{\overline{\eta}}$ is a smooth curve and $\Delta = 0$. Incidentally, Chen and Zhang have proved that relative canonical divisors of elliptic fibrations are $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor on X [20, 3.2].

Remark 5.2.13. In the case when $\dim Y = 1$, Corollary 5.2.11 follows from a result of Patakfalvi [91, Theorem 1.6].

Corollary 5.2.14. *With the notation as in 5.2.8, assume that f is flat and every geometric fiber is F-pure.*

(1) *If X is a Fano variety, that is, $-K_X$ is ample, then so is Y.*

(2) *If the geometric generic fiber of f is strongly F-regular and if X is a weak Fano variety, that is, $-K_X$ is nef and big, then so is Y.*

Proof. This follows from Corollaries 5.2.9 (2) and 5.2.10 (3) by setting $D = \Delta = 0$. □

Corollary 5.2.15. *With the notation as in 5.2.8, assume that Y is not a point.*

(1) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, then $-(K_{X/Y} + \Delta)$ is not ample.*

(2) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is strongly F-regular, then $-(K_{X/Y} + \Delta)$ cannot be nef and big.*

Proof. Set $D := -K_Y$. Then $-(K_X+\Delta+f^*D) = -(K_{X/Y}+\Delta)$. Since $-(K_Y+D) = 0$ is not big, the assertions follow from Corollary 5.2.10 (2) and (3). □

Corollary 5.2.16. *With the notation as in 5.2.8, assume that $\mathcal{O}_X(-m(K_X+\Delta))|_{X_\eta}$ is globally generated for an $m > 0$ with $a|m$ and that Y is not a point. If $p \nmid a$ and $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, then $f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G}(n) := f_*\mathcal{O}_X(n(K_{X/Y} + \Delta))$ for every $n \in \mathbb{Z}$ with $a|n$. Let H be an ample and free divisor on Y. We show that the bigness of $\mathcal{G}(-m)$ induces the pseudo-effectivity of $-H$, which contradicts to the assumption that Y is not a point. Recall that the pseudo-effectivity of $-H$ is equivalent to its weak positivity. Since $\mathcal{G}(-m)$ is torsion free, there exists an open subset $V \subseteq Y$ such that $\mathcal{G}(-m)|_V$ is locally free and $\mathrm{codim}_Y(Y \setminus V) \geq 2$. As explained in Remark 3.1.3 (2), $\mathcal{G}(-m)$ (resp. $-H$) is weakly positive if and only if so is $\mathcal{G}(-m)|_V$ (resp. $-H|_V$). Replacing $f : X \to Y$ by $f_V : X_V \to V$, we may assume that $\mathcal{G}(-m)$ is locally free. (Here we lose the projectivity of X and Y.) We assume that $\mathcal{G}(-m)$ is big. Then by Observation 3.1.4 (1) and (2), we see that $(S^l\mathcal{G}(-m))(-H)$ is generically globally generated for some $l > 0$. Since $\mathcal{O}_X(-m(K_X + \Delta))|_{X_\eta}$ is globally generated, we have an $n_0 > 0$ such that the natural morphism

$$(S^l\mathcal{G}(-m))^{\otimes n-n_0} \otimes \mathcal{G}(-lmn_0) \to \mathcal{G}(-lmn)$$

is generically surjective for every $n \geq n_0$, by Lemma 5.2.3 (1). Let $n_1 > 0$ be an integer such that $(\mathcal{G}(-lmn_0))(n_1H)$ is globally generated. Tensoring the above morphism with $\mathcal{O}_X((n_0+n_1-n)H)$, we have the generically surjective morphism

$$\left((S^l\mathcal{G}(-m))(-H)\right)^{\otimes n-n_0} \otimes (\mathcal{G}(-lmn_0))(n_1H) \to (\mathcal{G}(-lmn))((n_0+n_1-n)H).$$

From this we see that $(\mathcal{G}(-lmn))((n_0+n_1-n)H)$ is generically globally generated. Since

$$\begin{aligned}(\mathcal{G}(-lmn))((n_0+n_1-n)H) &\cong f_*\mathcal{O}_X(-lmn(K_{X/Y}+\Delta)+f^*(n_0+n_1-n)H) \\ &\cong f_*\mathcal{O}_X(-lmn(K_{X/Y}+\Delta+f^*H_n)) \\ &\cong f_*\mathcal{O}_X(-lmn(K_X+\Delta+f^*(H_n-K_Y))),\end{aligned}$$

where $H_n := (n-n_0-n_1)(lmn)^{-1}H$, we can apply Theorem 5.2.6, and then we get that $-(K_Y+(H_n-K_Y)) = -H_n$ is weakly positive for every $n \geq n_0$. Hence $-H$ is weakly positive, which completes the proof. □

Remark 5.2.17. We cannot remove the assumption of F-purity of fibers in Corollaries 5.2.11 and 5.2.16. Indeed, the quasi-elliptic fibration $g : X \to C$ introduced in Remark 4.2.2 satisfies $K_{S/C} \sim_{\mathbb{Q}} g^*L$ for a $\mathbb{Q}$-divisor L on C with $\deg L < 0$. Note that general fibers of g have cuspidal singularities, which are not F-pure [48].

5.3 Results in arbitrary characteristic

In this section, we generalize several results in Section 5.2 to arbitrary characteristic by using reduction to positive characteristic. In particular, we prove characteristic zero counterparts of Corollaries 5.2.15 and 5.2.16 (Theorems 5.3.5 and 5.3.6). In the last of this section, we deal with morphisms which are special but not necessarily smooth, and show that images of Fano varieties are again Fano varieties. First we recall some classes of singularities in characteristic zero defined by reduction to positive characteristic.

Definition 5.3.1. Let X be a normal variety over a field k of characteristic zero, and Δ be an effective $\mathbb{Q}$-Weil divisor on X. Let (X_R, Δ_R) be a model of (X, Δ) over a finitely generated $\mathbb{Z}$-subalgebra R of k. (X, Δ) is said to be of *dense F-pure type* (resp. *strongly F-regular type*) if there exists a dense (resp. dense open) subset $S \subseteq \operatorname{Spec} R$ such that (X_μ, Δ_μ) is *F-pure* (resp. strongly F-regular) for all closed points $\mu \in S$.

Remark 5.3.2. The above definition can be generalized in an obvious way to the case where X is a finite disjoint union of varieties over k.

Theorem 5.3.3 ([104, Corollary 3.4])**.** *Let X be a normal variety over a field of characteristic zero, and let Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Then (X, Δ) is klt if and only if it is of strongly F-regular type.*

Notation 5.3.4. Let $f : X \to Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field of characteristic zero, and Δ be an effective $\mathbb{Q}$-divisor on X.

Theorem 5.3.5. *With the notation as in 5.3.4, assume that Y is not a point. If (X_y, Δ_y) is of dense F-pure type (resp. klt) for a general closed point $y \in Y$, then $-(K_{X/Y} + \Delta)$ is not ample (resp. not nef and big). In particular, $-K_{X/Y}$ is not nef and big.*

Proof. Assume that (X_y, Δ_y) is of dense F-pure type for a general closed point $y \in Y$. Let X_R, Δ_R, Y_R, y_R and f_R be respectively models of X, Δ, Y, y and f over a finitely generated $\mathbb{Z}$-algebra R. We may assume that $(X_R)_{y_R}$ is a model of X_y over R. Then there exists a dense subset $S \subseteq \operatorname{Spec} R$ such that $((X_y)_\mu, \Delta_\mu)$ is F-pure for every $\mu \in S$. Note that $(X_y)_\mu \cong (X_\mu)_{y_\mu}$ and $(\Delta_y)_\mu = (\Delta_\mu)_{y_\mu}$. Thus by Corollary 5.2.15, we see that $-(K_{X_\mu/Y_\mu} + \Delta_\mu)$ is not ample. This implies that $-(K_{X/Y} + \Delta)$ is not ample. Next, we assume that (X_y, Δ_y) is klt for a general closed point $y \in Y$. If $-(K_{X/Y} + \Delta)$ is nef and big, then by Kodaira's lemma, there exists $\Delta' \geq \Delta$ such that (X_y, Δ'_y) is klt for a general closed point $y \in Y$ and $-(K_{X/Y} + \Delta')$ is ample. However, by Theorem 5.3.3, (X_y, Δ'_y) is of dense F-pure type, which contradicts to the above arguments. $\square$

Theorem 5.3.6. *With the notation as in 5.3.4, assume that Y is not a point and that (X_y, Δ_y) is of dense F-pure type for a general closed point $y \in Y$. Let $\overline{\eta}$ be a geometric generic point of Y. If $\mathcal{O}_X(-m(K_{X/Y} + \Delta))|_{X_{\overline{\eta}}}$ is globally generated for some $m > 0$ such that $m\Delta$ is integral, then $f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ is not big.*

Proof. Set $\mathcal{G} := f_*\mathcal{O}_X(-m(K_{X/Y} + \Delta))$ and $r := \operatorname{rank} \mathcal{G}$. Since $y \in Y$ is general, f is flat at every point in $f^{-1}(y)$ and $\dim H^0(X_y, -m(K_{X_y} + \Delta_y)) = r$. Let X_R, Δ_R, Y_R, y_R and f_R be respectively models of X, Δ, Y, y and f. By replacing R if necessary, we may assume that $f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R} + \Delta_R))$ and $(X_R)_{y_R}$ are respectively models of $\mathcal{G}$ and X_y, and that for every $\mu \in \operatorname{Spec} R$, $\dim H^0((X_\mu)_{y_\mu}, -m(K_{X_\mu} + \Delta_\mu)_{y_\mu}) = r$. Hence by [58, Corollary 12.9], the natural morphism

$$\mathcal{G}_\mu = f_{R*}\mathcal{O}_{X_R}(-m(K_{X_R/Y_R} + \Delta_R))|_{Y_\mu} \to f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu} + \Delta_\mu))$$

is surjective over y_μ. Since $f_{\mu*}\mathcal{O}_{X_\mu}(-m(K_{X_\mu/Y_\mu} + \Delta_\mu))$ is not big as shown in Corollary 5.2.16, $\mathcal{G}_\mu$ is also not big. Thus the lemma below completes the proof. $\square$

Lemma 5.3.7. *Let $\mathcal{G}$ be a torsion free coherent sheaf on a smooth projective variety Y over an algebraically closed field of characteristic zero. Let Y_R and $\mathcal{G}_R$ be models of Y and $\mathcal{G}$ respectively over a finitely generated $\mathbb{Z}$-algebra. If $\mathcal{G}$ is big, then there exists a dense open subset $S \subseteq \operatorname{Spec} R$ such that $\mathcal{G}_\mu$ is big for every $\mu \in S$.*

Proof. Let H be an ample and free divisor on Y. Replacing Y by its appropriate open subset, we may assume that $\mathcal{G}$ is locally free. By Observation 3.1.4 (1) and (2), we have an integer $c > 0$ such that $(S^c\mathcal{G})(-H)$ is generically globally generated. In other words, there exists a generically surjective morphism $\varphi : \bigoplus \mathcal{O}_Y(H) \to S^c\mathcal{G}$. Let $y \in Y$ be a general closed point. Then φ is surjective in a neighborhood of y. Let φ_R, H_R and y_R be models of φ, H and y over R, respectively. By replacing R if necessary, we may assume that φ_R is surjective over y_R. Thus for every closed point $\mu \in \operatorname{Spec} R$, the morphism $\varphi_\mu : \bigoplus \mathcal{O}_{X_\mu}(H_\mu) \to S^c\mathcal{G}_\mu$ obtained as the restriction of φ_R is surjective over y_μ. This implies that $\mathcal{G}_\mu$ is big, since H_μ is ample. □

Kollár, Miyaoka and Mori [74, Corollary 2.9] (cf. [85, THEOREM 3]) have proved that images of Fano varieties under smooth morphisms are again Fano varieties. The next theorem shows that the same statement holds when morphisms are not necessary smooth.

Theorem 5.3.8. *Let $f : X \to Y$ be a surjective morphism between smooth projective varieties over an algebraically closed field k of any characteristic $p \geq 0$, and let Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$. Assume that for every closed point $x \in X$, there exist a neighborhood $U \subseteq X$ (resp. $V \subseteq Y$) of x (resp. $f(x)$) and a commutative diagram*

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "(f_V)|_U"'] & \mathbb{A}^m \arrow[r, equal] \arrow[d, "\varphi"] & \operatorname{Spec} k[u_1, \ldots, u_m] \\ V \arrow[r, "\beta"] & \mathbb{A}^n \arrow[r, equal] & \operatorname{Spec} k[v_1, \ldots, v_n] \end{tikzcd}$$

whose horizontal morphisms are étale, that the morphism φ is defined as

$$\varphi(a_1, \ldots, a_m) = \left(\prod_{0<i\leq l_1} a_i, \prod_{l_1<i\leq l_2} a_i, \ldots, \prod_{l_{n-1}<i\leq l_n} a_i \right) \text{ with } 0 < l_1 < \cdots < l_n \leq m,$$

and that

$$\Delta|_U = \alpha^* \left(\sum_{l_n<i\leq m} d_i \operatorname{div}(u_i) \right) \text{ with } d_{l_n+1}, \ldots, d_m \in \mathbb{Z}_{(p)} \cap [0,1].$$

In this situation, if $-(K_X + \Delta + f^*D)$ is ample for some $\mathbb{Q}$-Cartier divisor D on Y, then so is $-(K_Y + D)$.

Proof. When the characteristic of k is zero, one can easily check that the entire setting can be reduced to characteristic $p \gg 0$. Thus we only need to consider the case when $p > 0$. Let $x \in X$ be a closed point, and let U, V, α, β and φ be as above. Set $y := f(x)$, $\boldsymbol{a} := \alpha(x) = (a_1, \ldots, a_m) \in \mathbb{A}^m$ and $\boldsymbol{b} := \beta(y) = (b_1, \ldots, b_n) \in \mathbb{A}^n$. Set $u'_i := u_i - a_i$ and $v'_j := v_j - b_j$ for each i and j. Then

$$\varphi^* v'_j = \prod_{l_{j-1}<i\leq l_j} (u'_i + a_i) - \prod_{l_{j-1}<i\leq l_j} a_i$$

for $j = 1, \ldots, n$, where $l_0 := 0$. Set $g := \prod_{l_n<i\leq m} u_i$. It is easy to check that the sequence $\varphi^* v'_1, \ldots, \varphi^* v'_n, g$ is $k[u_1, \ldots, u_m]$-regular. In particular, $Z :=$

$Z(\varphi^* v'_1, \ldots, \varphi^* v'_n) \subseteq \mathbb{A}^m$ is equi-dimensional of dimension $m-n$. Note that Z is the fiber of φ over $\boldsymbol{b}$. Since étaleness is stable under base change, the morphism $\alpha_{\boldsymbol{b}} : U_{\boldsymbol{b}} \to Z$ obtained by the restriction of α to the fibers over $\boldsymbol{b}$ is again étale. Replacing V and U if necessary, we may assume that $\beta^{-1}(\boldsymbol{b}) = \{y\}$. Then $U_y \cong U_{\boldsymbol{b}}$. This implies that every closed fiber of f is equi-dimensional, in particular f is flat.

Claim 5.3.9. (X_y, Δ_y) is F-pure for every closed point $y \in Y$.

If this claim holds, then the theorem follows from Corollary 5.2.9, because by the assumption the support of Δ does not contain any component of any fiber. Since $\sum_{l_n < i \le m} d_i \mathrm{div}(u_i) \le \mathrm{div}(g)$, it suffice to show that the pair $(Z, \mathrm{div}(g)|_Z)$ is F-pure around $\boldsymbol{a}$. Let $\mathfrak{m}_{\boldsymbol{a}}$ be the maximal ideal of $\boldsymbol{a}$. Then it is easily seen that

$$(\varphi^* v'_1 \cdots \varphi^* v'_n)^{q-1} \cdot g^{q-1} \notin \mathfrak{m}_{\boldsymbol{a}}^{[q]}$$

for every $q = p^e$. Thus by [56, Corollary 2.7], the pair $(Z, \mathrm{div}(g)|_Z)$ is F-pure, which completes the proof. $\square$

Example 5.3.10. When $\Delta = 0$, the assumptions of Theorem 5.3.8 hold for a flat toric morphism with reduced closed fibers between smooth projective toric varieties over an algebraically closed field.

Example 5.3.11. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. For integers $m, n \ge 0$, we define $\boldsymbol{v}_{m,n} := (1, m, n) \in \mathbb{R}^3$. Let $\Sigma_{m,n}$ be the fan consisting of all the faces of the following cones:

$$\begin{aligned}
&< \boldsymbol{v}_{m,n}, \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_3 >, < \boldsymbol{v}_{m,n}, \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_3 >, \\
&< \boldsymbol{v}_{m,n}, -\boldsymbol{e}_2, \boldsymbol{e}_3 >, < \boldsymbol{v}_{m,n}, \boldsymbol{e}_2, -\boldsymbol{e}_3 >, < \boldsymbol{v}_{m,n}, -\boldsymbol{e}_2, -\boldsymbol{e}_3 >, \\
&< -\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_2 + \boldsymbol{e}_3 >, < -\boldsymbol{e}_1, \boldsymbol{e}_2 + \boldsymbol{e}_3, \boldsymbol{e}_3 >, \\
&< -\boldsymbol{e}_1, -\boldsymbol{e}_2, \boldsymbol{e}_3 >, < -\boldsymbol{e}_1, \boldsymbol{e}_2, -\boldsymbol{e}_3 >, < -\boldsymbol{e}_1, -\boldsymbol{e}_2, -\boldsymbol{e}_3 > .
\end{aligned}$$

Let $X_{m,n}$ be the smooth toric 3-fold corresponding to the fan $\Sigma_{m,n}$ with respect to the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. Then $X_{m,n}$ is a Fano variety if and only if $m, n \in \{0, 1\}$. The projection $\mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$ induces a toric morphism $f : X_{m,n} \to Y_m$ from $X_{m,n}$ to the Hirzebruch surface $Y_m := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m))$. Set $\Delta = 0$. Then one can check that f satisfies the assumptions of Theorem 5.3.8, but is not smooth. Hence by Theorem 5.3.8, we see that Y_m is a Del Pezzo surface if $m = 0, 1$. In fact, it is well known that Y_m is a Del Pezzo surface if and only if $m = 0, 1$.

Chapter 6

Iitaka's $C_{n,m}$ conjecture

6.1 Summary

The Kodaira dimension is one of the most important birational invariants and plays a key role in the birational classification of algebraic varieties. For an algebraic fiber space, we have the following conjecture on Kodaira dimensions, which has been proposed by Iitaka in characteristic zero.

Conjecture 6.1.1 ($C_{n,m}$)**.** *Let $f : X \to Y$ be an algebraic fiber space, and let X and Y be smooth projective varieties of dimension n and m respectively over an algebraically closed field k. Suppose that the geometric generic fiber $X_{\overline{\eta}}$ is smooth. Then*

$$\kappa(X) \geq \kappa(X_{\overline{\eta}}) + \kappa(Y).$$

In characteristic zero, many results related to this conjecture are known [7, 14, 16, 18, 35, 37, 66, 67, 68, 72, 76, 77, 108, 109]. In particular, this conjecture has been reduced to problems in the minimal model program by Kawamata [68, Corollary 1.2].

In positive characteristic, Conjecture 6.1.1 has been proved in some cases recently. From now on we work over an algebraically closed field k of characteristic $p > 0$. Chen and Zhang have shown $C_{n,n-1}$ [20, Theorem 1.2]. Patakfalvi has proved the conjecture in the case where Y is of general type and $S^0(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}}) \neq 0$ [92]. Under the assumption that $p > 5$, $C_{3,1}$ has been shown when $k = \overline{\mathbb{F}_p}$ by Birkar, Chen and Zhang [10, Theorem 1.2] and when the genus of Y is at least two by Zhang [116, Corollary 1.9]. Note that [20, 10, 92, 116] dealt with algebraic fiber spaces with singular geometric generic fibers.

One of the main results of this chapter is the theorem below.

Theorem 6.1.2. *Conjecture $C_{3,m}$ holds when $p > 5$.*

The cases when $\kappa(X_{\overline{\eta}})$ is equal to 0 and 1 are proved in Subsections 6.3.1 and 6.3.2, respectively. The proofs rely on results of the minimal model program for 3-folds in characteristic $p > 5$, which are summarized in Subsection 4.3.1. The case when $\kappa(X_{\overline{\eta}}) = 2$ is a direct consequence of the next theorem, which is another of the main results of this chapter.

Theorem 6.1.3 (Theorems 6.2.2 and 6.2.4)**.** *Let $f : X \to Y$ be an algebraic fiber space, X and Y be smooth projective varieties and Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p. Let $\overline{\eta}$ be the geometric generic point of Y. Assume that*

(i) *the $k(\overline{\eta})$-algebra $\bigoplus_{m\ge 0} H^0(X_{\overline{\eta}}, m(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}}))$ is finitely generated,*

(ii) *there exists an integer $m_0 \ge 0$ such that*

$$S^0(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, m(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}})) = H^0(X_{\overline{\eta}}, m(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}}))$$

for each $m \ge m_0$, and

(iii) *either that Y is of general type or Y is a curve.*

Then

$$\kappa(X, K_X + \Delta) \ge \kappa(Y) + \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}).$$

Note that, as shown in the proof of Corollary 6.2.6, conditions (i) and (ii) are satisfied if $X_{\overline{\eta}}$ is smooth, $\Delta_{\overline{\eta}} = 0$, $p > 5$ and $\kappa(X_{\overline{\eta}}) = 2$.

Theorem 6.1.3 is proved in Section 6.2 as an application of Theorem 4.1.1.

6.2 Algebraic fiber spaces with large $R_S(X_{\overline{\eta}}, \omega_{X_{\overline{\eta}}})$

In this section, we prove Theorem 6.1.3. Throughout this section, we use the following notation.

Notation 6.2.1. Let X and Y be smooth projective varieties over an algebraically closed field k of characteristic $p > 0$, and Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for some integer $a > 0$ not divisible by p. Let $\overline{\eta}$ be the geometric generic point of Y. Let $f : X \to Y$ be a separable surjective morphism such that

(i) $K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}$ is finitely generated in the sense of Definition 2.1.8, and

(ii) there exists an integer $m_0 > 0$ such that for every integer $m \ge m_0$,

$$S^0(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, m(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}})) = H^0(X_{\overline{\eta}}, m(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}})).$$

Here condition (i) and (ii) are the same as in Theorem 4.2.1. We first prove the case where Y is of general type, by using an argument similar to [92, §4] and the proof of [92, Theorem 1.7].

Theorem 6.2.2. *In the situation of Notation* 6.2.1*, assume that Y is of general type. Then*

$$\kappa(X, K_X + \Delta) \ge \kappa(Y) + \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}).$$

Proof. We may assume that $\kappa(X_{\overline{\eta}}, K_{\overline{\eta}} + \Delta_{\overline{\eta}}) \geq 0$.

Step 1. Set $S' := \{\varepsilon \in \mathbb{Q} | \kappa(X, K_{X/Y} + \Delta - \varepsilon f^*H) \geq \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) + \kappa(Y)\}$, where H is an ample divisor on Y. We show that S' is nonempty. By assumption (i), there exists an integer $b > 0$ such that $R(X_{\overline{\eta}}, ab(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$ is generated by $H^0(X_{\overline{\eta}}, ab(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}))$. By the projection formula, there exists an integer $c > 0$ such that $f_*\mathcal{O}_X(ab(K_{X/Y} + \Delta) + cf^*H)$ is globally generated. Thus for every $m > 0$, the natural morphism

$$\bigotimes^m f_*\mathcal{O}_X(ab(K_{X/Y} + \Delta) + cf^*H) \to f_*\mathcal{O}_X(abm(K_{X/Y} + \Delta) + cmf^*H)$$

is generically surjective, and hence $f_*\mathcal{O}_X(abm(K_{X/Y} + \Delta) + cmf^*H)$ is generically globally generated. This implies

$$\begin{aligned} &\dim_k H^0(X, abm(K_X + \Delta) + cmf^*H) \\ &\qquad \geq \dim_{k(\overline{\eta})} H^0(X_{\overline{\eta}}, abm(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) + \dim_k H^0(Y, abmK_Y). \end{aligned}$$

Hence for $\varepsilon_0 := -c/(ab)$, we have $\kappa(X, K_X + \Delta - \varepsilon_0 f^*H) \geq \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) + \kappa(Y)$.

Step 2. Set $S := \{\varepsilon \in \mathbb{Q} | \kappa(X, K_{X/Y} + \Delta - \varepsilon f^*H) \geq 0\}$. We show that $\sup S = \sup S'$. Since $S \supseteq S'$ we have the inequality $\geq$. We show the inequality $\leq$. For an $\varepsilon \in S$, $K_{X/Y} + \Delta - \varepsilon f^*H$ is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor. Thus for every $0 < \delta \in \mathbb{Q}$ and $\varepsilon_0 \in S'$,

$$\begin{aligned} \kappa(X, (1+\delta)(K_{X/Y} + \Delta) - (\varepsilon + \delta\varepsilon_0)f^*H) &\geq \kappa(X, \delta(K_{X/Y} + \Delta - \varepsilon_0 f^*H)) \\ &\geq \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) + \kappa(Y). \end{aligned}$$

This implies $(\varepsilon + \delta\varepsilon_0)/(1+\delta) \leq \sup S'$. Since $\lim_{\delta \to 0}(\varepsilon + \delta\varepsilon_0)/(1+\delta) = \varepsilon$, we have $\varepsilon \leq \sup S'$, and hence $\sup S \leq \sup S'$.

Step 3. We show that $\sup S \geq 0$. For simplicity of notation, we denote $f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ by $\mathcal{G}(m)$ for each $m \in \mathbb{Z}$ with $a|m$. By the proof of 4.2.1, we have $t(\mathcal{G}(m), H) \geq 0$ for each $m \geq m_0$. We fix an $m \geq m_0$ such that $\mathcal{G}(m) \neq 0$, where such m exists by the assumption that $\kappa(X_{\overline{\eta}}, \Delta_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) \geq 0$. Let $d > 0$ be an integer such that $a|(p^d - 1)$. Then for every $\varepsilon \in T(\mathcal{G}(m), H)$ there exists an $e > 0$ such that $p^{de}e \in \mathbb{Z}$ and $(F_Y^{de*}\mathcal{G}(m))(-p^{de}\varepsilon H)$ has a nonzero global section. On the other hand, since $X_{Y^{de}}$ is a variety, the natural morphism $F_{X/Y}^{(de)\#} : \mathcal{O}_{X_{Y^{de}}} \to F^{(de)}_{X/Y\,*}\mathcal{O}_{X^{de}}$ is injective, which induces an injective $\mathcal{O}_{Y^{de}}$-module homomorphism

$$\begin{aligned} F_Y^{de*}\mathcal{G}(m) \cong& f_{Y^{de}*}\mathcal{O}_{X_{Y^{de}}}(am(K_{X_{Y^{de}}/Y^{de}} + \Delta_{Y^{de}})) \\ \hookrightarrow& f^{(de)}{}_*\mathcal{O}_{X^{de}}(amp^{de}(K_{X^{de}/Y^{de}} + \Delta)). \end{aligned}$$

Note that the reducedness of $X_{Y^{de}}$ follows from the separability of f and the flatness of F_Y. From this

$$H^0(X, amp^{de}(K_{X/Y} + \Delta) - p^{de}\varepsilon f^*H) \neq 0,$$

and hence we have $(amp^{de})^{-1}p^{de}\varepsilon = (am)^{-1}\varepsilon \leq \sup S$, and so

$$0 \leq \frac{t(\mathcal{G}(m), H)}{am} \leq \sup S.$$

Step 4. We show the assertion. By the assumption and Step3, there exists an $\varepsilon \in S'$ such that $K_Y - \varepsilon H$ is linearly equivalent to an effective $\mathbb{Q}$-divisor. Then

$$\begin{aligned}\kappa(X, K_X + \Delta) =&\kappa(X, K_{X/Y} + \Delta + f^*K_Y)\\ \geq&\kappa(X, K_{X/Y} + \Delta + \varepsilon f^*H) \geq \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}) + \kappa(Y).\end{aligned}$$

This is our claim.

□

Next, we show that Iitaka's conjecture when Y is an elliptic curve (Theorem 6.2.4). To this end, we recall the following result without the proof.

Theorem 6.2.3 ([62, Theorem 10.5]). *Let $f : X \to Y$ be a surjective morphism between smooth complete varieties, D be a divisor on Y, and E be an effective divisor on X such that* $\operatorname{codim}(f(E)) \geq 2$. *Then $\kappa(X, f^*D + E) = \kappa(Y, D)$.*

Theorem 6.2.4. *In the situation of Notation* 6.2.1*, assume that Y is an elliptic curve. Then*

$$\kappa(X, K_X + \Delta) \geq \kappa(Y) + \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}}).$$

Proof. Set $\mathcal{G}(m) := f_*\mathcal{O}_X(m(K_{X/Y} + \Delta))$ for each $m \in \mathbb{Z}$ with $a|m$.

Step 1. By Theorem 4.1.1, we see that there exists an $m_0 > 0$ such that $\mathcal{G}(m)$ is weakly positive for each $m \geq m_0$ with $a|m$. Let $M \geq m_0$ be an integer. Applying Proposition 1.4.3, we obtain a finite morphism $\alpha_M : Y' \to Y$ from an elliptic curve Y' such that for each integer m with $M \geq m \geq m_0$, ${\alpha_M}^*\mathcal{G}(m)$ is isomorphic to a direct sum of line bundles. Therefore, replacing α_M if necessary, we may assume that ${\alpha_M}^*\mathcal{G}(m) \cong \mathcal{G}^+(m) \oplus \mathcal{G}^0(m)$, where $\mathcal{G}^+(m)$ (resp. $\mathcal{G}^0(m)$) is a direct sum of very ample (resp. numerically trivial) line bundles.

Step 2. We show that $\kappa(X_{Y'}, K_{X_{Y'}} + \Delta_{Y'}) = \kappa(X, K_X + \Delta)$. Obviously, we need only consider when α is separable and when α is purely inseparable. If $\alpha : Y' \to Y$ is separable then it is étale, thus so is $\alpha_X : X_{Y'} \to X$, in particular $X_{Y'}$ is a smooth variety. Hence the claim follows from Theorem 6.2.3, where we note that $K_{X_{Y'}} \sim K_{X_{Y'}/Y'} \sim (K_{X/Y})_{Y'} \sim (K_X)_{Y'}$. If $\alpha = F^{(e)}_{Y/k}$ for some $e > 0$, then there is a commutative diagram

$$\begin{tikzcd}
X^e \arrow[d, "F^{(e)}_{X/Y}"'] \arrow[dr, "F^{(e)}_{X/k}"] & & \\
X_{Y^e} \arrow[r, "(F^{(e)}_{Y/k})_X"'] \arrow[d, "f_{Y^e}"'] & X_{k^e} \arrow[r, "\cong"'] \arrow[d, "f_{k^e}"] & X \arrow[d, "f"] \\
Y^e \arrow[r, "F^{(e)}_{Y/k}"'] & Y_{k^e} \arrow[r, "\cong"'] & Y.
\end{tikzcd}$$

Since X_{Y^e} is a variety (cf. [92, Lemma 5.2]), we have injective morphisms $\mathcal{O}_{X_{k^e}} \to {(F^{(e)}_{Y/k})_X}_*\mathcal{O}_{X_{Y^e}} \to {F^{(e)}_{X/k}}_*\mathcal{O}_{X^e}$, which induce injective morphisms

$$H^0(X, am(K_X + \Delta)) \hookrightarrow H^0(X_{Y^e}, am(K_{X_{Y^e}} + \Delta_{Y^e})) \hookrightarrow H^0(X^e, amp^e(K_{X^e} + \Delta))$$

for every $m > 0$. Thus $\kappa(X, K_X + \Delta) = \kappa(X_{Y^e}, K_{X_{Y^e}} + \Delta_{Y^e})$ as claimed.

Step 3. We complete the proof. Let $l, n_0 > m_0$ be as in the proof of Theorem 4.2.1. By Steps 1 and 2, we may assume that $\mathcal{G}(m) \cong \mathcal{G}^+(m) \oplus \mathcal{G}^0(m)$ for each $m \in \{l\} \cup \{n_0 + i\}_{1 \le i < l}$, where $\mathcal{G}^+(m)$ and $\mathcal{G}^0(m)$ are as in Step 1. Let $S_m \subseteq \mathrm{Pic}^0(Y)$ be the set of line bundles which is a direct summand of $\mathcal{G}^0(m)$.

Claim 6.2.5. The subgroup G of $\mathrm{Pic}^0(Y)$ generated by S_l is a finite group.

Proof of Claim 6.2.5. Let $d, q_{l,e}, r_{l,e}$ be as in the proof of Theorem 4.2.1 for each $e \gg 0$. Set $S_l = \{\mathcal{L}_1, \ldots, \mathcal{L}_h\}$. Then for each $i = 1, \ldots, h$, there exist generically surjective morphisms

$$\begin{aligned}&(\mathcal{G}^+(l) \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_h)^{\otimes q_{l,e}} \otimes (\mathcal{G}^+(n_0 + r_{l,e}) \oplus \mathcal{G}^0(n_0 + r_{l,e}))\\ \cong &\mathcal{G}(l)^{\otimes q_{l,e}} \otimes \mathcal{G}(n_0 + r_{l,e}) \to \mathcal{G}(lp^{de} + a^{-1}(1 - p^{de})) \to F_Y^{de*}\mathcal{G}(l) \to \mathcal{L}_i^{p^{de}}\end{aligned}$$

as in the proof of Theorem 4.2.1. It follows that there exists a nonzero morphism $\mathcal{L}_1^{t_1} \otimes \cdots \otimes \mathcal{L}_h^{t_h} \otimes \mathcal{L} \to \mathcal{L}_i^{p^{de}}$ for some integers $t_1, \ldots, t_h \ge 0$ satisfying $\sum_{i=1}^h t_i = q_{l,e}$ and for some $\mathcal{L} \in \bigcup_{r=0}^{l-1} S_{n_0+r}$. Since this is a nonzero morphism between line bundles of degree zero on a smooth projective curve, this is an isomorphism, in particular $\mathcal{L} \in G$. For each $i = 1, \ldots, h$ we denote $\mathcal{L}_i^{-1}$ by $\mathcal{L}_{i+h}$, and for each $m > 0$ we set

$$G(m) := \{\textstyle\bigotimes_{i=1}^{2h} \mathcal{L}_i^{m_i} \mid 0 \le m_i \text{ and } \sum_{i=1}^{2h} m_i \le m\ \} \subseteq G.$$

Let $c > 0$ be an integer satisfying $\{\mathcal{L} \in G | \mathcal{L} \text{ or } \mathcal{L}^{-1} \text{ is in } \bigcup_{r=0}^{l-1} S_{n_0+r}\} \subseteq G(c)$. Then by the above argument $\mathcal{L}_1^{p^{de}}, \ldots, \mathcal{L}_{2h}^{p^{de}} \in G(q_{l,e} + c)$. Since $p^{de} > q_{l,e} + c$ for some $e \gg 0$, there exists an $N > 0$ such that $G = G(N)$, which is our claim. □

By the claim, there exists an $n > 0$ such that $n_Y^*\mathcal{L} \cong \mathcal{L}^n \cong \mathcal{O}_Y$ for each $\mathcal{L} \in S_l$. Hence, replacing f by its base change with respect to n_Y, we may assume that $\mathcal{G}(l)$ is globally generated. Then, for each $b \gg 0$, $\mathcal{G}(bl)$ is generically globally generated, because the natural morphism $\mathcal{G}(l)^{\otimes b} \to \mathcal{G}(bl)$ is generically surjective as in the proof of Theorem 4.2.1. Thus we have

$$\begin{aligned}\dim_k H^0(X, abl(K_{X/Y} + \Delta)) &= \dim_k H^0(Y, \mathcal{G}(bl))\\ &\ge \dim_{k(\overline{\eta})} (\mathcal{G}(bl))_{\overline{\eta}}\\ &= \dim_{k(\overline{\eta})} H^0(X_{\overline{\eta}}, bl(aK_{X_{\overline{\eta}}} + (a\Delta)_{\overline{\eta}}))\end{aligned}$$

for each $b \gg 0$, and so $\kappa(X, K_X + \Delta) \ge \kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})$.

□

Corollary 6.2.6. *Let $f : X \to Y$ be an algebraic fiber space such that X is a smooth projective 3-fold and Y is a smooth projective curve. Assume that the geometric generic fiber $X_{\overline{\eta}}$ is a normal surface of general type with only rational double point singularities, and $p > 5$. Then*

$$\kappa(X) \ge \kappa(Y) + \kappa(X_{\overline{\eta}}).$$

Proof. We note that in this case $K_{X_{\overline{\eta}}}$ is finitely generated (cf. [5, Corollary 9.10]). Thus the result follows from Corollary 2.2.8 and Theorems 6.2.2 and 6.2.4. □

6.3 Iitaka's conjecture $C_{3,1}$

This section is devoted to the proof of Theorem 6.1.2. As in the summary, we work over an algebraically closed field k of characteristic $p > 0$.

6.3.1 Proof in the case $\kappa(X_{\overline{\eta}}) = 0$

In this subsection, we prove Theorem 6.1.2 in the case when the Kodaira dimension of the geometric generic fiber is equal to zero. It is proved as a consequence of the next theorem.

Theorem 6.3.1. *Let $f : X \to Y$ be a surjective morphism from a normal projective variety X over an algebraically closed field k of characteristic $p > 0$ to an elliptic curve Y, and Δ be an effective $\mathbb{Q}$-divisor on X such that $a\Delta$ is integral for an integer $a > 0$ not divisible by p. Assume that $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-pure, where $\overline{\eta}$ is the geometric generic point of Y. If $K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + L)$ for some $\mathbb{Q}$-divisor L on Y, then L is semi-ample.*

Proof. By Theorem 5.2.5 and Remark 5.2.7, we have $\deg L \geq 0$. We may assume that $\deg L = 0$, and it suffices to show that $L \sim_{\mathbb{Q}} 0$. Since $(K_X + \Delta)_{\overline{\eta}} \sim_{\mathbb{Q}} 0$, there is an ample Cartier divisor A on X such that $l(K_X + \Delta)_{\overline{\eta}} + A_{\overline{\eta}}$ is ample and free for every $l \in a\mathbb{Z}$. Recall that $0 < a \in \mathbb{Z} \setminus p\mathbb{Z}$ and $a\Delta$ is integral. By Fujita's vanishing theorem, there exist some $m_0 > 0$ such that for every nef Cartier divisor N on $X_{\overline{\eta}}$, $\mathcal{O}_{X_{\overline{\eta}}}((m_0 - 1)A_{\overline{\eta}} + N)$ is 0-regular with respect to $l(K_X + \Delta)_{\overline{\eta}} + A_{\overline{\eta}}$ for every $l \in a\mathbb{Z}$. Then the natural homomorphism

$$\begin{aligned} &H^0(X_{\overline{\eta}}, l(K_X + \Delta)_{\overline{\eta}} + mA_{\overline{\eta}}) \otimes H^0(X_{\overline{\eta}}, (m' - 1)A_{\overline{\eta}}) \otimes H^0(X_{\overline{\eta}}, l'(K_X + \Delta)_{\overline{\eta}} + A_{\overline{\eta}}) \\ &\qquad \to H^0(X_{\overline{\eta}}, (l + l')(K_X + \Delta)_{\overline{\eta}} + (m + m')A_{\overline{\eta}}) \end{aligned}$$

is surjective for every $l, l' \in a\mathbb{Z}$ and $m, m' \geq m_0$. Thus

$$\begin{aligned} &H^0(X_{\overline{\eta}}, l(K_X + \Delta)_{\overline{\eta}} + mA_{\overline{\eta}}) \otimes H^0(X_{\overline{\eta}}, l'(K_X + \Delta)_{\overline{\eta}} + m'A_{\overline{\eta}}) \\ &\qquad \to H^0(X_{\overline{\eta}}, (l + l')(K_X + \Delta)_{\overline{\eta}} + (m + m')A_{\overline{\eta}}) \end{aligned}$$

is also surjective, and hence the natural homomorphism

$$\mathcal{G}(l, m) \otimes \mathcal{G}(l', m') \to \mathcal{G}(l + l', m + m')$$

is generically surjective, where $\mathcal{G}(l, m) := f_*\mathcal{O}_X(l(K_{X/Y} + \Delta) + mA)$. By Example 2.1.11, replacing m_0 if necessary, we may assume that

$$\begin{aligned} &H^0(X_{\overline{\eta}}, \phi^{(e)}_{(X_{\overline{\eta}}, \Delta_{\overline{\eta}})} \otimes \mathcal{O}_{X_{\overline{\eta}}}(N + m_0 A_{\overline{\eta}})) : \\ &\qquad H^0(X_{\overline{\eta}}, (1 - p^e)(K_X + \Delta)_{\overline{\eta}} + p^e(N + m_0 A_{\overline{\eta}})) \to H^0(X_{\overline{\eta}}, N + m_0 A_{\overline{\eta}}) \end{aligned}$$

is surjective for every $e > 0$ with $a|(p^e - 1)$ and for every nef Cartier divisor N on $X_{\overline{\eta}}$. Since $l(K_X + \Delta)_{\overline{\eta}}$ is nef,

$$\begin{aligned} &f_{Y^e *}(\phi^{(e)}_{(X/Y, \Delta)} \otimes \mathcal{O}_{X_{Y^e}}(l(K_{X/Y} + \Delta)_{Y^e} + m_0 A_{Y^e})) : \\ &\qquad \mathcal{G}((l - 1)p^e + 1, m_0 p^e) \to f_{Y^e *}\mathcal{O}_{X_{Y^e}}(l(K_{X/Y} + \Delta)_{Y^e} + m_0 A_{Y^e}) \cong F_Y^{e*}\mathcal{G}(l, m_0) \end{aligned}$$

is generically surjective. Let $b > 0$ be an integer such that $a|b$, bL is integral and $b(K_X + \Delta)$ is linearly equivalent to bf^*L. By Proposition 1.4.3, there exists a finite morphism $\pi : Y' \to Y$ from an elliptic curve Y' such that $\pi^*\mathcal{G}(r, m_0)$ is a direct sum of line bundles for each $0 \le r < b$ with $a|r$. By Lemma 4.3.3, we may replace L and $\mathcal{G}(r, m_0)$ respectively with their pullbacks by π. Set

$$\begin{aligned}
\mathcal{F} :=& \bigoplus_{0\le r<b,\ a|r} \mathcal{G}(r, m_0),\\
\mu :=& \min\{\deg \mathcal{M} | \mathcal{M} \in \operatorname{Pic}(Y) \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\}, \text{ and}\\
T :=& \{\mathcal{M} \in \operatorname{Pic}(Y) | \deg \mathcal{M} = \mu \text{ and } \mathcal{M} \text{ is a direct summand of } \mathcal{F}\}\\
=& \{\mathcal{M}_1, \dots, \mathcal{M}_\lambda\}.
\end{aligned}$$

Then for every $\mathcal{M}_i \in T$, there exists an $0 \le s < b$ with $a|s$ such that the composition

$$\begin{aligned}
&\mathcal{G}(s, m_0)^{\otimes p^e-1} \otimes \mathcal{G}(r_{i,e}, m_0) \otimes \mathcal{O}_Y(-q_{i,e}bL)\\
&\qquad \to \mathcal{G}((s-1)p^e + 1, p^e m_0) \to F_Y^{e\,*}\mathcal{G}(s, m_0) \twoheadrightarrow \mathcal{M}_i^{p^e}
\end{aligned}$$

is generically surjective for every $e > 0$ with $a|(p^e - 1)$. Here $q_{i,e}$ and $r_{i,e}$ are integers satisfying $1 + s - p^e = -q_{i,e}b + r_{i,e}$ and $0 \le r_{i,e} < b$. Then there exists a line bundle $\mathcal{M}$ which is a direct summand of $\mathcal{G}(s, m_0)^{p^e-1} \otimes \mathcal{G}(r_{i,e}, m_0)$ and a non-zero morphism $\mathcal{M} \to \mathcal{M}_i^{p^e}(q_{i,e}bL)$. By considering the degree of the line bundles, we see that $\mathcal{M}_i^{p^e}(q_{i,e}bL) \cong \mathcal{M} \in T^{p^e}$, where

$$T^n := \{\textstyle\bigotimes_{1\le i\le\lambda} \mathcal{M}_i^{n_i} \in \operatorname{Pic}(Y) | n_i \ge 0,\ \sum_{1\le i\le\lambda} n_i = n\}.$$

Fix an integer $e > 0$ such that $a|p^e - 1$. Set $n := \lambda(p^e - 1) + 1$. For every $\mathcal{N} \in T^n$, there exist $n_1, \dots, n_\lambda \ge 0$ such that $\mathcal{N} \cong \bigotimes_{1\le i\le\lambda} \mathcal{M}_i^{n_i}$ and $n'_j := n_j - p^e \ge 0$ for at least one j. Then

$$\mathcal{N}(q_{j,e}bL) \cong (\bigotimes_{i\ne j} \mathcal{M}_i^{n_i}) \otimes \mathcal{M}_j^{n'_j} \otimes \mathcal{M}_j^{p^e}(q_{j,e}bL).$$

Since $\mathcal{M}_j^{p^e}(q_{j,e}bL) \in T^{p^e}$, we have $\mathcal{N}(q_{j,e}bL) \in T^n$. Hence for every $m \ge q := \max\{q_{1,e}, \dots, q_{\lambda,e}\}$,

$$\mathcal{N}(mbL) \in \{\mathcal{M}(kbL) \in \operatorname{Pic}(Y) | \mathcal{M} \in T^n,\ 0 \le k < q\}.$$

Since T^n is a finite set, there are integers $m > m' > 0$ such that $\mathcal{N}(mbL) \cong \mathcal{N}(m'bL)$, and hence $(m - m')bL \sim 0$. □

Proof of Theorem 6.1.2: the case $\kappa(X_{\overline{\eta}}) = 0$. As in the proof of Theorem 4.1.4, we may assume that X is minimal over Y and $K_{X_{\overline{\eta}}}$ is semi-ample, so $K_{X_{\overline{\eta}}} \sim_{\mathbb{Q}} 0$. By Lemma 4.3.6, K_X is $\mathbb{Q}$-linearly equivalent to the pullback of $K_Y + L$ for some $\mathbb{Q}$-divisor L on Y. In particular $\kappa(X, K_X) = \kappa(Y, K_Y + L)$. It is enough to show that $\kappa(Y, K_Y + L) \ge \kappa(Y)$. By Theorem 5.2.5 and Remark 5.2.7, we see that L is nef. Note that since $p > 5$ and $X_{\overline{\eta}}$ has only rational double points as singularities, $X_{\overline{\eta}}$ is F-pure by [2, Section 3] and [32]. When Y is of general type, we have $K_Y + L$ is big, thus $\kappa(Y, K_Y + L) = \dim Y = \kappa(Y)$. When Y is an elliptic curve, by Theorem 6.3.1, we have $\kappa(Y, K_Y + L) \ge \kappa(Y)$, and the proof is complete. □

6.3.2 Proof in the case $\kappa(X_{\overline{\eta}}) = 1$

In this subsection, we consider the case when the Kodaira dimension of the geometric generic fiber is one.

Proof of Theorem 6.1.2*: the case* $\kappa(X_{\overline{\eta}}) = 1$. Let $f : X \to Y$ be a surjective morphism from a smooth projective 3-fold to a smooth projective curve of genus at least one, and let $\overline{\eta}$ be the geometric generic point of Y. Suppose that $\kappa(X_{\overline{\eta}}) = 1$. With loss of smoothness, by Theorem 4.3.1 (2) we may assume that X is a minimal model. Then $X_{\overline{\eta}}$ has canonical singularities by the proof of Theorem 4.1.4.

If $g(Y) > 1$, then since $f_*\omega^m_{X/Y}$ contains a nef subbundle of rank $\geq cm$ for some $c > 0$ and any sufficiently divisible m (Theorem 4.1.4), by some standard arguments (see, for example, the proof of [10, Proposition 5.1]), we can conclude that

$$\kappa(X) \geq 2 = \kappa(Y) + \kappa(X_{\overline{\eta}}).$$

From now on, we assume $g(Y) = 1$. Then $\omega_X = \omega_{X/Y}$. We break the proof into several steps.

Step 1. By considering the relative Iitaka fibration and applying Lemma 4.3.6, we get the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y, \end{tikzcd}$$

where Z is a smooth projective surface, such that $\sigma^*K_X \sim_{\mathbb{Q}} h^*D$ for a nef g-big divisor D on Z. Here h is a fibration with geometric fiber being a smooth elliptic curve by the proof of Theorem 4.1.4h If D is big, then we are done. From now on, we assume the numerical dimension $\nu(K_X) = \nu(D) = 1$.

Claim 6.3.2. If X has a fibration $f' : X \to W$ to a normal projective curve W such that $K_{F'}$ is numerically trivial, where F' denotes the generic fiber of f'. Assume moreover that there exists an $L \in \mathrm{Pic}^0(Y)$ and an integer $m > 0$ such that $\dim H^0(X, mK_X + f^*L) > 0$. Then K_X is semi-ample.

Proof of the claim. Take an effective divisor $D_L \sim mK_X + f^*L$. Since D_L is nef, effective and $D_L|_{F'} \sim_{num} 0$, we have

$$(mK_X + f^*L)|_{F'} \sim D_L|_{F'} \sim 0.$$

By Lemma 4.3.6 we can assume $D_L \sim_{\mathbb{Q}} f'^*A$, where A is a divisor on W, which is ample since $D_L \neq 0$. We only need to show that $L \sim_{\mathbb{Q}} 0$.

Since X has only terminal singularities, it is smooth in codimension one, so F' is a regular surface over the function filed $K(W)$ of W. Applying [107, Theorem 0.2], we have $K_{F'} \sim_{\mathbb{Q}} 0$. Therefore, we conclude that

$$f^*L|_{F'} \sim_{\mathbb{Q}} mK_{F'} + f^*L|_{F'} \sim_{\mathbb{Q}} (mK_X + f^*L)|_{F'} \sim_{\mathbb{Q}} D_L|_{F'} \sim_{\mathbb{Q}} 0.$$

On the other hand, since F' is dominant over the curve $Y \otimes_k K(W)$, we see that L is torsion by Lemma 4.3.3. $\square$

Step 2. By Theorem 4.1.4, there exists an integer $c > 0$ such that for sufficiently divisible m_1, $f_*\omega_X^{m_1}$ contains a nef subbundle V of rank $r_{m_1} \geq cm_1$. If $\deg V > 0$, then we are done by some standard arguments ([10, Propostion 5.1]). Therefore, we assume that $\deg V = 0$. By Proposition 1.4.3 there exists a finite morphism $\pi : Y_1 \to Y$ from an elliptic curve Y_1 such that $\pi^*V = \oplus_{i=1}^n \mathcal{L}_i$, where $\mathcal{L}_i \in \mathrm{Pic}^0(Y_1)$. Let X_1 be the normalization of $X \times_Y Y_1$. Then we get the following commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "\pi_1"] \arrow[d, "f_1"'] & X \arrow[d, "f"] \\ Y_1 \arrow[r, "\pi"] & Y, \end{tikzcd}$$

where π_1 and f_1 denote the natural morphisms. We have that $\pi^* f_*\omega_X^{m_1} \subset f_{1*}\pi_1^*\omega_X^{m_1}$ by [58, Proposition 9.3], and hence

$$\pi^*V = \oplus_{i=1}^n \mathcal{L}_i \subset f_{1*}\pi_1^*\omega_X^{m_1}.$$

Thus we conclude that $\dim H^0(X_1, \pi_1^*\omega_X^{m_1} \otimes f_1^*\mathcal{L}_i^{-1}) \geq 1$, and if $\mathcal{L}_i = \mathcal{L}_j$ for some $j \neq i$ then the strict inequality holds. Since $\pi^* : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(Y_1)$ is surjective, there exists an $\mathcal{L}'_i$ such that $\mathcal{L}_i \cong \pi^*\mathcal{L}'_i$, so we have

$$\pi_1^*\omega_X^{m_1} \otimes f_1^*\mathcal{L}_i^{-1} \cong \pi_1^*(\omega_X^{m_1} \otimes f^*(\mathcal{L}'_i)^{-1}).$$

Applying Theorem 6.2.3, we can find a sufficiently divisible integer $l > 0$ such that $\dim H^0(X, l(m_1K_X - f^*L'_i)) \geq 1$. Let $m = lm_1$ and L_i be a divisor on Y with $\mathcal{O}_Y(L_i) \cong (\mathcal{L}'_i)^l$. Then $\dim H^0(X, mK_X - f^*L_i) \geq 1$. If $\dim H^0(X, mK_X - f^*L_i) > 1$, then $\dim H^0(Z, mD - g^*L_i) > 1$ by the construction in Step 1. Since $mD - g^*L_i$ is nef and $\nu(mD - g^*L_i) = 1$, the movable part of $|mD - g^*L_i|$ has no base point, and hence it induces a fibration $g' : Z \to W'$ on Z to a curve W'. The Stein factorization of the composite morphism $g' \circ h : X' \to W'$ induces a fibration $f'' : X' \to W$ from X' to a normal curve W, which is defined by the base point free linear system $|\mu^*l(mK_X - f^*L_i)|$ for sufficiently divisible integer $l > 0$. Since $\sigma : X' \to X$ is a birational morphism such that $\sigma_*\mathcal{O}_{X'} = \mathcal{O}_X$, we conclude that $|\mu^*l(mK_X - f^*L_i)| = \mu^*|l(mK_X - f^*L_i)|$, and thus $|l(mK_X - f^*L_i)|$ has no base point, hence defines such a fibration $f' : X \to W$ as in Claim of Step 1. Thus K_X is semi-ample, and this completes the proof in this case.

From now on, we can assume $\dim H^0(X, mK_X - f^*L_i) = 1$ and $\dim H^0(X_1, \pi_1^*(mK_X - f^*L_i)) = 1$. For every i, we have an effective divisor $B_i \sim mK_X - f^*L_i$. By the construction, we can assume $\pi_1^*B_i \neq \pi_1^*B_j$ if $i \neq j$, thus $L_i \neq L_j$. We only need to show that at least two of L_i are torsion.

Step 3. For every j, we have the unique effective divisor $B_j \sim mK_X - f^*L_j$. Let B' be the reduced divisor supported on the union of components of $\sum_j B_j$. Take a smooth log resolution $\mu : \tilde{X} \to X$ of the pair (X, B'). Denote by $\tilde{f} : \tilde{X} \to Y$ the natural morphism. Let $\tilde{B}$ be the reduced divisor supported on the union components of $\sum_j \mu^* B_j$. Consider the dlt pair $(\tilde{X}, \tilde{B})$. Since X has terminal singularities, there exists an effective μ-exceptional divisor E on $\tilde{X}$ such that

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \mu^* K_X + E.$$

Hence $K_{\tilde{X}} + \tilde{B} \sim_{\mathbb{Q}} \mu^* K_X + E + \tilde{B}$ has a weak Zariski decomposition. By Theorem 4.3.1, $(\tilde{X}, \tilde{B})$ has a minimal model $(\hat{X}, \hat{B})$ which is dlt, and there exists a natural morphism $\hat{f} : \hat{X} \to Y$. By the construction, we have the following:
(i) Note that $B_j|_{X_{\bar{\eta}}}$ is contained in finitely many fibers of the Iitaka fibration $I_{\bar{\eta}} : X_{\bar{\eta}} \to C_{\bar{\eta}}$, which implies that $\kappa(\tilde{X}_{\bar{\eta}}, (K_{\tilde{X}} + \tilde{B})|_{\tilde{X}_{\bar{\eta}}}) = 1$. Since the restriction $(K_{\hat{X}} + \hat{B})|_{\hat{X}_{\bar{\eta}}}$ is semi-ample by [105, Theorem 1.2], it induces an elliptic fibration on $\hat{X}_{\bar{\eta}}$ by the construction. Hence applying Lemma 4.3.6 again, we get the following commutative diagram

$$\begin{array}{ccc} \hat{X}' & \xrightarrow{\hat{\sigma}} & \hat{X} \\ \downarrow{\scriptstyle \hat{h}} & & \downarrow{\scriptstyle \hat{f}} \\ \hat{Z} & \xrightarrow{\hat{g}} & Y, \end{array}$$

where $\hat{Z}$ is a smooth projective surface and $\hat{h}$ is an elliptic fibration such that $\hat{\sigma}^*(K_{\hat{X}} + \hat{B}) \sim_{\mathbb{Q}} \hat{h}^* \hat{D}$ for a nef and $\hat{g}$-big divisor $\hat{D}$ on $\hat{Z}$.
(ii)We claim that $\nu(K_{\hat{X}} + \hat{B}) = \nu(\hat{D}) = 1$. Indeed, otherwise $\hat{D}$ is big. Note that the divisor $\mu^* \sum_j B_j - \tilde{B}$ is effective and $\mu^* \sum_j B_j \sim \mu^* nmK_X - \sum_j \tilde{f}^* L_j$. Then applying Theorem 6.2.3 we can get a contradiction as follows:

$$\begin{aligned} 2 =& \kappa(\hat{Z}, \hat{D}) = \kappa(\hat{X}', \hat{\sigma}^*(K_{\hat{X}} + \hat{B})) = \kappa(\hat{X}, K_{\hat{X}} + \hat{B}) = \kappa(\tilde{X}, K_{\tilde{X}} + \tilde{B}) \\ \leq& \kappa(\tilde{X}, K_{\tilde{X}} + \mu^* nmK_X - \sum_j \tilde{f}^* L_j) \\ =& \kappa(\tilde{X}, \mu^* K_X + E + \mu^* nmK_X - \sum_j \mu^* f^* L_j) \\ =& \kappa(X, (nm+1)K_X - \sum_j f^* L_j) = 1. \end{aligned}$$

(iii) For sufficiently divisible M and every $1 \leq i \leq n$, we get an effective Cartier divisor

$$\tilde{\Gamma}_i = M(\mu^* B_i + mE) + Mm\tilde{B} \sim M(mK_{\tilde{X}} - \tilde{f}^* L_i) + Mm\tilde{B}.$$

Denote by $\nu : \tilde{X} \dashrightarrow \hat{X}$ the natural birational map. Let $\hat{\Gamma}_i = \nu_* \tilde{\Gamma}_i$. Then

$$\hat{\Gamma}_i \sim M(mK_{\hat{X}} - \hat{f}^* L_i) + Mm\hat{B} \sim Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^* L_i.$$

Since E is contained in finitely many fibers of $\tilde{f}$, $\nu_* E$ is contracted by $\hat{f}$. Therefore, if a component of $\hat{\Gamma}_i$ is dominant over Y then it is contained in $\hat{B}$.
(iv) Take an effective divisor $\hat{D}_i \sim Mm\hat{D} - M\hat{g}^* L_i$ for each i. Since $\hat{D}$ is nef and $\hat{D}^2 = 0$, so is $\hat{D}_i$. Considering connected components of $\hat{D}_1, \dots, \hat{D}_n$, we see that there exist nef effective Cartier divisors $\hat{D}'_1, \dots, \hat{D}'_k$ satisfying the conditions below:

- $\operatorname{Supp}(\hat{D}'_j)$ is connected for each j, and $\operatorname{Supp}(\hat{D}'_j)\cap\operatorname{Supp}(\hat{D}'_l) = \emptyset$ for each $j \neq l$;
- $(\hat{D}'_1)^2 = \cdots = (\hat{D}'_k)^2 = 0$;
- the greatest common divisor of the coefficients of every $\hat{D}'_j$ is equal to one;
- for each i, there exist $a_{i1}, \ldots, a_{ik} \geq 0$ such that $\hat{D}_i = a_{i1}\hat{D}'_1 + \cdots + a_{ik}\hat{D}'_k$.

Note that at least one of the $\hat{D}'_j$ is dominant over Y, and hence intersects every fiber of $\hat{f}$. From this we see that every $\hat{D}'_j$ is dominant over Y. Indeed, if a $\hat{D}'_j$ is contained in one fiber, then the support of $\hat{D}'_j$ is equal to the whole of the fiber as shown by [6, VIII.4], which contradicts to the first condition above. Now we have $\hat{\sigma}^*\hat{\Gamma}_i = \hat{h}^*\hat{D}_i$ by the construction. Hence $\hat{h}^*\hat{D}'_1, \ldots, \hat{h}^*\hat{D}'_k$ are disjoint connected components of $\hat{\sigma}^*(\sum \hat{\Gamma}_i)$. Let $\hat{G}_j := \hat{\sigma}_*\hat{h}^*\hat{D}'_j$. Then we have that $\operatorname{Supp}(\hat{G}_j)\cap\operatorname{Supp}(\hat{G}_l) = \emptyset$ for each $j \neq l$.
(v) Take two divisors $\hat{D}_1, \hat{D}_2$. Since $\hat{D}_1 \neq \hat{D}_2$, we may assume that $a_{11} > a_{21} \geq 0$. We may further assume that $a_{22} > a_{12} \geq 0$, because of $\hat{D}_1 \sim_{num} \hat{D}_2$. We can write that

$$\hat{\Gamma}_1 = a_{11}\hat{G}_1 + a_{12}\hat{G}_2 + \hat{G}'_3 \text{ and } \hat{\Gamma}_2 = a_{21}\hat{G}_1 + a_{22}\hat{G}_2 + \hat{G}''_3,$$

where neither of $\hat{G}_1$ and $\hat{G}_2$ intersects $\hat{G}'_3 \cup \hat{G}''_3$.
Step 4. Take two reduced, irreducible and dominant over Y components $\hat{C}_1, \hat{C}_2$ of $\hat{G}_1, \hat{G}_2$, respectively. Then $\hat{C}_1, \hat{C}_2$ are contained in $\hat{B}$ by the construction of $\hat{\Gamma}_i$ in Step 3 (iii). Since $(\hat{X}, \hat{B})$ is dlt and $\hat{B}$ is a reduced integral divisor, so $\hat{C}_1, \hat{C}_2$ are log canonical centers of $(\hat{X}, \hat{B})$. By Step 3 (iv), since $\hat{D}$ is nef and $\hat{D} \cdot \hat{D}_i = 0$, we have $\hat{D}|_{\hat{D}_i} \sim_{num} 0$. For $j = 1, 2$, since $\hat{h}(\hat{\sigma}^{-1}\hat{C}_j)$ is a component of some $\hat{D}_i$, we conclude that

$$\hat{\sigma}^*(K_{\hat{X}} + \hat{B})|_{\hat{\sigma}^{-1}\hat{C}_j} \sim_{\mathbb{Q}} \hat{h}^*\hat{D}|_{\hat{\sigma}^{-1}\hat{C}_j} \sim_{num} 0.$$

Denote by $\hat{C}'_i$ the normalization of $\hat{C}_i$. Then $(K_{\hat{X}}+\hat{B})|_{\hat{C}'_i} \sim_{num} 0$, so $(K_{\hat{X}}+\hat{B})|_{\hat{C}'_i} \sim_{\mathbb{Q}} 0$ by Lemma 4.3.2. Therefore,

$$\begin{aligned} -a_{21}M\hat{f}^*L_1|_{\hat{C}'_1} &\sim_{\mathbb{Q}} a_{21}(Mm(K_{\hat{X}} + \hat{B}) - M\hat{f}^*L_1)|_{\hat{C}'_1} \\ &\sim_{\mathbb{Q}} a_{21}\hat{\Gamma}_1|_{\hat{C}'_1} \sim_{\mathbb{Q}} a_{11}a_{21}\hat{G}_1|_{\hat{C}'_1} \\ &\sim_{\mathbb{Q}} a_{11}\hat{\Gamma}_2|_{\hat{C}'_1} \sim_{\mathbb{Q}} -a_{11}M\hat{f}^*L_2|_{\hat{C}'_1}. \end{aligned} \tag{6.1}$$

By Lemma 4.3.3, this implies that

$$a_{21}ML_1 \sim_{\mathbb{Q}} a_{11}ML_2.$$

In the same way, restricting on $\hat{C}'_2$ gives

$$a_{22}ML_1 \sim_{\mathbb{Q}} a_{12}ML_2.$$

Finally by the conditions $a_{11} > a_{21}$ and $a_{12} < a_{22}$, we conclude that $L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} 0$, and this completes the proof.

□

Chapter 7

When is the Albanese morphism an algebraic fiber space?

7.1 Summary

The Albanese morphism is an important tool in the study of varieties with non-positive Kodaira dimension. In characteristic zero, Kawamata has proved that the Albanese morphism of a smooth projective variety with Kodaira dimension zero is an algebraic fiber space [66, Theorem 1]. Zhang has shown the same statement for a smooth projective variety with nef anti-canonical divisor [117, Corollary 2]. Under the same assumption, Cao has recently proved that the Albanese morphism is locally isotrivial [15, 1.2. Theorem]. In positive characteristic, Hacon and Patakfalvi have proved that the Albanese morphism of a smooth projective variety X is surjective if the S-Kodaira dimension $\kappa_S(X)$ of X is zero [52, Theorem 1.1.1]. Here, S-Kodaira dimension is an analog of usual Kodaira dimension defined by using the trace maps of the Frobenius morphisms. Wang has shown that the Albanese morphism of a threefold with semi-ample anti-canonical divisor is surjective if the general fiber is F-pure [113, Theorem B]. In this chapter, we generalize his result to varieties of arbitrary dimension, which can be viewed as a positive characteristic counterpart of the above result of Zhang.

Theorem 7.1.1. *Let X be a normal projective variety over an algebraically closed field of characteristic $p > 0$, and Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $-m(K_X + \Delta)$ is a nef Cartier divisor for an integer $m > 0$ not divisible by p. Let $a : X \to A$ be the Albanese morphism of X, and $X_{\overline{\eta}}$ be the geometric generic fiber over the image of a. If $(X_{\overline{\eta}}, \Delta|_{X_{\overline{\eta}}})$ is F-pure, then a is an algebraic fiber space.*

We also study the relation between the Albanese morphisms and Frobenius splitting. The notion of an F-split variety has been introduced by Mehta and Ramanathan as a variety with splitting of the Frobenius morphism [82], which are considered to be related to varieties of Calabi–Yau type [46, 47, 90, 100]. As a generalization of F-splitting of varieties, we consider a notion of the F-splitting of a pair (f, Γ) consisting of a morphism $f : V \to W$ and an effective $\mathbb{Q}$-Weil divisor Γ

on V (Definition 7.3.1). In this chapter, we focus on the F-splitting of the Albanese morphism. Let X be a normal projective variety over an algebraically closed field k of characteristic $p > 0$, Δ be an effective $\mathbb{Q}$-Weil divisor on X, and $a : X \to A$ be the Albanese morphism of X. Then there is the following relationship between the F-splitting of a and that of X.

Theorem 7.1.2. *(X, Δ) is F-split if and only if (a, Δ) is F-split and A is ordinary.*

We study the Albanese morphism a under the assumption that (a, Δ) is locally F-split (Definition 7.3.1), which is weaker than the assumption that it is F-split. For instance, a flat morphism with normal F-split fibers is locally F-split, but not necessarily F-split. The next theorem shows that the local F-splitting of (a, Δ) requires that a is an algebraic fiber space and that Δ and fibers satisfy some geometric properties.

Theorem 7.1.3. *Assume that (a, Δ) is locally F-split. Then a is an algebraic fiber space. Furthermore, if $m\Delta$ is Cartier for an integer $m > 0$ not divisible by p, then the following holds:*

(1) *The support of Δ does not contain any irreducible component of any fiber.*

(2) *For every scheme-theoretic point $z \in A$, $(X_{\overline{z}}, \Delta_{\overline{z}})$ is F-split, where $\overline{z}$ is the algebraic closure of z. In particular, $X_{\overline{z}}$ is reduced.*

(3) *The morphism a is smooth in codimension one. In other words, there exists an open subset U of X such that $\operatorname{codim}(X \setminus U) \geq 2$ and $a|_U : U \to A$ is a smooth morphism. In particular, the general geometric fiber of a is normal.*

This theorem recovers the result of Hacon and Patakfalvi when K_X is numerically trivial, because the condition $\kappa_S(X) = 0$ is equivalent to the F-splitting of X in that case. As a corollary of this theorem, we provide a new characterization of abelian varieties. Before stating the precise statement, we recall that the first Betti number $b_1(X)$ of X is defined as a dimension of the $\mathbb{Q}_l$-vector space $H^1_{\text{ét}}(X, \mathbb{Q}_l)$ for a prime $l \neq p$ and is equal to $2 \dim A$.

Theorem 7.1.4. *Assume that (a, Δ) is locally F-split (resp. (X, Δ) is F-split). Then $b_1(X) \leq 2 \dim X$. Furthermore, the equality holds if and only if X is an abelian variety (resp. ordinary abelian variety) and $\Delta = 0$.*

As an application of Theorem 7.1.4, we give a necessary and sufficient condition for a normal projective variety to have F-split Albanese morphism (Theorem 7.4.6). We conclude this chapter with a classification of minimal surfaces with F-split or locally F-split Albanese morphisms (Theorem 7.5.1).

7.2 Varieties with nef anti-canonical divisors

In this section, we prove Theorem 7.1.1 which states that the Albanese morphism of a normal projective variety with nef anti-canonical divisor is an algebraic fiber space if the geometric generic fiber is F-pure. Throughout this section, we work over an algebraically closed field k of characteristic $p > 0$.

We first recall that the *Albanese morphism* of X is defined as a morphism $a : X \to A$ to an Abelian variety A (called the *Albanese variety*) such that every morphism $b : X \to B$ to an abelian variety B factors uniquely through a. The existence of the Albanese morphism for a normal projective variety is proved for instance in [33, §9]. We must remark that **the above definition of the Albanese morphisms is different in general from the standard notion of the Albanese maps defined by using the Albanese morphisms of resolutions of singularities** (see [113, Example 2.3] for an example of variety whose Albanese morphism and Albanese map are different).

Theorem 7.1.1 is proved as an application of Theorem 5.2.5 and the theorem below.

Theorem 7.2.1 ([53, Theorem 0.2])**.** *Let X be a normal projective variety with $\kappa(X, K_X) = 0$. Let $a : X \to A$ be the Albanese morphism of X. If $a : X \to \mathrm{Im}(a)$ is generically finite and separable, then a is surjective.*

The following lemma is also used in the proof of Theorem 7.1.1.

Lemma 7.2.2. *Let D be an effective Weil divisor on a normal projective variety Y. If $\mathcal{O}_Y(-D)$ is weakly positive, then $D = 0$.*

Proof. Let $\pi : Y' \to Y$ be the blowing-up of Y along D. Then we have the natural surjection $\pi^*\mathcal{O}_Y(-D) \to \mathcal{O}_{Y'}(-D')$, where D' is the exceptional divisor of π. Since $\mathcal{O}_Y(-D)$ is weakly positive, so is $\pi^*\mathcal{O}_Y(-D)$. Then by the above surjection, we see that $\mathcal{O}_{Y'}(-D')$ is also weakly positive. Since the weak positivity of a line bundle is equivalent to the pseudo-effectivity, we see that $-D'$ is pseudo-effective. Hence $D' = 0$, and so $D = 0$. □

Proof of Theorem 7.1.1. Let Z be the normalization of $\mathrm{Im}(a)$ and $f : X \to Z$ be the induced morphism. Now we have the natural morphism $\Omega^1_A|_Z \to \Omega^1_Z$ which is generically surjective. Hence $H^0(Z, \omega_Z) \neq 0$. Furthermore, by Theorem 5.2.5, we obtain that ω_Z^{-1} is weakly positive. Therefore we have $\omega_Z \cong \mathcal{O}_Z$ by Lemma 7.2.2. By Theorem 7.2.1 we see that a is surjective, or equivalently $Z = A$. Let $a : X \xrightarrow{g} Y \xrightarrow{h} A$ be the Stein factorization of a. Since the geometric generic fiber of a is F-pure, it is reduced, and hence a is separable. This implies that h is also separable, and therefore we have an injection $\mathcal{O}_Y \cong h^*\omega_A \to \omega_Y$. By the same argument as before we see that $\omega_Y \cong \mathcal{O}_Y$, and by the Zariski-Nagata purity we obtain that a is an étale morphism. Hence we see that Y is an abelian variety by [87, Section 18, Theorem] and h is an isomorphism. Consequently we obtain $a_*\mathcal{O}_X \cong \mathcal{O}_A$, which is our assertion. □

7.3 Splittings of Relative Frobenius

In this section, we introduce and study the notion of F-split morphisms. We fix an algebraically closed field k of characteristic $p > 0$.

Definition 7.3.1. Let X be a normal variety and Δ be an effective $\mathbb{Q}$-Weil divisor on X. Let $f : X \to Z$ be a projective morphism to a smooth variety Z. We say that f is *sharply F-split* (*F-split* for short) with respect to Δ if there exists an $e > 0$ such that the composite

$$\mathcal{O}_{X_{Z^e}} \xrightarrow{{F_{X/Z}^{(e)}}^{\sharp}} {F_{X/Z}^{(e)}}_* \mathcal{O}_{X^e} \hookrightarrow {F_{X/Z}^{(e)}}_* \mathcal{O}_{X^e}(\lceil (p^e - 1)\Delta \rceil) \qquad (7.3.1.1)_e$$

of the natural homomorphism ${F_{X/Y}^{(e)}}^{\sharp}$ and the natural inclusion ${F_{X/Z}^{(e)}}_* \mathcal{O}_{X^e} \hookrightarrow {F_{X/Z}^{(e)}}_* \mathcal{O}_{X^e}(\lceil (p^e - 1)\Delta \rceil)$ is injective and splits as an $\mathcal{O}_{X_{Z^e}}$-module homomorphism. We say that f is *locally sharply F-split* (*locally F-split* for short) with respect to Δ if there exists an open covering $\{V_i\}$ of Z such that $f|_{f^{-1}(V_i)} : f^{-1}(V_i) \to V_i$ is F-split with respect to $\Delta|_{f^{-1}(V_i)}$ for every i.

We often say that the pair (f, Δ) is F-split (resp. locally F-split) if f is F-split (resp. locally F-split) with respect to Δ. We simply say f is F-split (resp. locally F-split) if so is $(f, 0)$.

Remark 7.3.2. (1) If the morphism $(7.3.1.1)_e$ splits, then $(7.3.1.1)_{ne}$ also splits for every integer $n > 0$.
(2) When $Z = \operatorname{Spec} k$, it is easily seen that (f, Δ) is F-split if and only if (X, Δ) is F-split. Note that we now assume that k is algebraically closed.
(3) Let Δ' be an effective $\mathbb{Q}$-divisor on X with $\Delta' \leq \Delta$. If (f, Δ) is F-split (resp. locally F-split), then so is (f, Δ').
(4) Hashimoto has dealt with morphisms with local splittings of $(7.3.1.1)_e$ in [60].

Example 7.3.3. Let X, Δ, Z and f be as in Definition 7.3.1. Assume that X is the projective space bundle $\mathbb{P}(\mathcal{E})$ associated with a locally free coherent sheaf $\mathcal{E}$ and that $f : X \to Z$ is its projection. Then f is locally F-split. Furthermore, if $\mathcal{E}$ is the direct sum of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$ on Z, then f is F-split. The first statement follows from the second. We assume that $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_i$. For every $m \geq 0$, there exists the natural injective morphism

$$\psi_m : \bigoplus_{m_1 + \cdots + m_n = m} \mathcal{L}_i^{m_i p} \cong {F_Z}^* S^m \mathcal{E} \to S^{mp} \mathcal{E}.$$

Then obviously the image of ψ_m is $\bigoplus_{m_1 + \cdots + m_n = m} \mathcal{L}_i^{m_i p} \subseteq S^{mp}\mathcal{E}$, and hence ψ_m splits. The morphism $\mathcal{O}_{X_{Z^1}} \to {F_{X/Z}^{(1)}}_* \mathcal{O}_{X^1}$ corresponds to the morphism

$$\psi := \bigoplus_{m \geq 0} \psi_m : \bigoplus_{m \geq 0} S^m {F_Z^1}^* \mathcal{E} \to \bigoplus_{m \geq 0} S^{mp} \mathcal{E} \subseteq \bigoplus_{m \geq 0} S^m \mathcal{E}.$$

Since ψ_m splits for every $m \geq 0$, ψ also splits, and hence $\mathcal{O}_{X_{Z^1}} \to F^{(1)}_{X/Z*}\mathcal{O}_{X^1}$ splits. Note that as we see in Theorem 7.5.1, there exists an indecomposable vector bundle $\mathcal{E}$ on an elliptic curve Z such that $\mathbb{P}(\mathcal{E}) \to Z$ is not F-split.

We first prove that F-split morphisms are surjective.

Lemma 7.3.4. *Let X, Δ, Z and f be as in Definition 7.3.1. Assume that f is locally F-split. Then there exists an $e > 0$ such that for each $i \geq 0$, $\mathcal{G}^i := R^i f_* \mathcal{O}_X$ is a vector bundle satisfying $F_Z^{e*}\mathcal{G}^i \cong \mathcal{G}^i$. In particular, f is surjective.*

Proof. Applying the functor $R^i f_{Z^e *}$ to $\mathcal{O}_{X_{Z^e}} \to F^{(e)}_{X/Z*}\mathcal{O}_{X^e}$, we obtain the morphism $R^i f_{Z^e *}\mathcal{O}_{X_{Z^e}} \to R^i f^{(e)}{}_* \mathcal{O}_{X^e} = \mathcal{G}^i$ which is injective and splits locally. Since F_Z is flat, we have $R^i f_{Z^e *}\mathcal{O}_{X_{Z^e}} \cong F_Z^{e*} R^i f_* \mathcal{O}_X = F_Z^{e*}\mathcal{G}^i$. Hence we obtain the morphism $F_Z^{e*}\mathcal{G}^i \to \mathcal{G}^i$ which is injective and splits locally. It is easily seen that this morphism is an isomorphism. By the lemma below, we see that $\mathcal{G}^i$ is locally free. $\square$

Lemma 7.3.5 ([81, Lemma 1.4])**.** *Let M be a finitely generated module over a regular local ring R of positive characteristic. If $F_R^{e*}M \cong M$ for some $e > 0$, then M is free.*

The following proposition shows that locally F-splitting requires some conditions on boundaries and fibers.

Proposition 7.3.6. *Let X, Δ, Z and f be as in Definition 7.3.1. Assume that Δ is $\mathbb{Z}_{(p)}$-Cartier and (f, Δ) is locally F-split. Then the following holds:*

1. *The support of Δ does not contain any irreducible component of any fiber.*
2. *For every $z \in Z$, $(X_{\overline{z}}, \Delta_{\overline{z}})$ is F-split, where $\overline{z}$ is the algebraic closure of z. In particular, $X_{\overline{z}}$ is reduced.*
3. *There exists an open subset $U \subseteq X$ such that* $\operatorname{codim}(X \setminus U) \geq 2$ *and $f|_U : U \to Y$ is a smooth morphism. In particular, general geometric fibers of f are normal.*

Note that f is surjective as shown by Lemma 7.3.4.

Proof. Let $z \in Z$. Restricting the homomorphism $(7.3.1.1)_e$ to $X_{\overline{z}^e}$, we obtain the homomorphism of $\mathcal{O}_{X_{\overline{z}^e}}$-modules

$$\mathcal{O}_{X_{\overline{z}^e}} \xrightarrow{F^{(e)}_{X_{\overline{z}}/\overline{z}}{}^{\sharp}} F^{(e)}_{X_{\overline{z}}/\overline{z}*}\mathcal{O}_{(X_{\overline{z}})^e} \to F^{(e)}_{X_{\overline{z}}/\overline{z}*}(\mathcal{O}_{(X_{\overline{z}})^e}((p^e-1)\Delta))|_{(X_{\overline{z}})^e}$$

which is injective and splits for some $e > 0$. This implies that the homomorphism $\mathcal{O}_{X_{\overline{z}}} \to (\mathcal{O}_X(p^e - 1\Delta))|_{X_{\overline{z}}}$ is not zero over each irreducible component. Hence the support of Δ does not contain any component of $X_{\overline{z}}$, and $(X_{\overline{z}}, \Delta_{\overline{z}})$ is F-split. Thus (1) and (2) hold. We show (3). Let $\pi : Y \to X_{Z^e}$ be the normalization of X_{Z^e}. Then $F^{(e)}_{X/Y} : X^{(e)} \to X_{Y^e}$ factors through Y, and we have morphisms

$$\mathcal{O}_{X_{Z^e}} \to \pi_*\mathcal{O}_Y \to F^{(e)}_{X/Y*}\mathcal{O}_{X^e}$$

of $\mathcal{O}_{X_{Z^e}}$-modules. Therefore the morphism $\mathcal{O}_{X_{Z^e}} \to \pi_*\mathcal{O}_Y$ splits. Since $(\pi_*\mathcal{O}_Y)/\mathcal{O}_{X_{Z^e}}$ is a torsion module and $\pi_*\mathcal{O}_Y$ is torsion free, we see that $(\pi_*\mathcal{O}_Y)/\mathcal{O}_{X_{Z^e}} = 0$. Hence X_{Z^e} is normal. Since $F^{(e)}_{X/Z_*}\mathcal{O}_{X^e}$ is torsion free, there exists an open subset $U \subseteq X$ such that $F^{(e)}_{X/Z_*}\mathcal{O}_{X^e}|_{U_{Z^e}} \cong F^{(e)}_{U/Z_*}\mathcal{O}_{U^e}$ is locally free over U_{Z^e}. From this, we see that $F^{(e)}_{U_{\overline{z}}/\overline{z}_*}\mathcal{O}_{U_{\overline{z}}} \cong F^{(e)}_{U/Z_*}\mathcal{O}_{U^e}|_{U_{Z^e}}$ is locally free for every $z \in Z$. Consequently, we deduce that $U_{\overline{z}}$ is regular by Kunz's theorem, and thus $f|_U : U \to Z$ is smooth. □

On the contrary to the above, f is not necessarily F-split even if every fiber is F-split (see Theorem 7.5.1 for example). However, if K_X is $\mathbb{Z}_{(p)}$-linearly trivial over Z, then the converse holds as seen in the next theorem. This is used in the proofs of Proposition 7.4.9 and Theorem 7.5.1.

Theorem 7.3.7 (A special case of Theorem 2.2.2)**.** *Let $f : X \to Z$ be an algebraic fiber space, and let X and Z be normal varieties. Let Δ be an effective $\mathbb{Z}_{(p)}$-Weil divisor on X such that $K_X + \Delta \sim_{\mathbb{Z}_{(p)}} f^*C$ for some Cartier divisor C on Z. Let $\overline{\eta}$ be the geometric generic point of Z.*

(i) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is not F-split, then so is $(X_{\overline{z}}, \Delta_{\overline{z}})$ for general $z \in Z$.*

(ii) *If $(X_{\overline{\eta}}, \Delta_{\overline{\eta}})$ is F-split, then there exists an effective $\mathbb{Z}_{(p)}$-Weil divisor Δ_Z on Z such that the following holds:*

  (1) *The divisor $(K_Z + \Delta_Z)$ is $\mathbb{Z}_{(p)}$-linearly equivalent to C.*

  (2) *The pair (X, Δ) is F-split if and only if so is (Z, Δ_Z).*

  (3) *The following are equivalent:*

   (3-1) *(f, Δ) is F-split;*

   (3-2) *(f, Δ) is locally F-split;*

   (3-3) *$(X_{\overline{z}}, \Delta|_{X_{\overline{z}}})$ is F-split for every codimension one point $z \in Z$, where $\overline{z}$ is the algebraic closure of z;*

   (3-4) *$\Delta_Z = 0$.*

Proof. (i) follows from Observation 2.2.4. (1) of (ii) follows directly from Theorem 2.2.2. Theorem 2.2.2 (2) shows that $S^0(X, \Delta, \mathcal{O}_X) \cong S^0(Z, \Delta_Z, \mathcal{O}_Z)$. Hence (2) of the theorem follows from the fact that (X, Δ) is F-split if and only if $S^0(X, \Delta, \mathcal{O}_X) = H^0(X, \mathcal{O}_X)$. To prove (3), we recall the construction on Δ_Z. Replacing X and Z by its smooth locus respectively, we may assume that X and Z are smooth. For an $e > 0$ with $a|(p^e - 1)$, we have

$$f^{(e)}{}_*\mathcal{L}^{(e)}_{(X/Z,\Delta)} = f^{(e)}{}_*\mathcal{O}_{X^e}((1-p^e)(K_{X^e/Z^e} + \Delta)) \cong \mathcal{O}_{Z^e}((1-p^e)(C - K_{Z^e}))$$

by the projection formula. Then we set

$$\theta^{(e)} : \mathcal{O}_{Z^e}((1-p^e)(C - K_{Z^e})) \cong f^{(e)}{}_*\mathcal{O}_{X^e}\mathcal{L}^{(e)}_{(X/Z,\Delta)} \xrightarrow{f_{Z^e*}\phi^{(e)}_{(X/Z,\Delta)}} \mathcal{O}_{X_{Z^e}}.$$

Since $(f_{Z^e*}\phi^{(e)}_{(X/Z,\Delta)}) \otimes k(\overline{\eta}^e) \cong H^0(X_{\overline{\eta}^e}, \phi^{(e)}_{(X_{\overline{\eta}}/\overline{\eta},\Delta_{\overline{\eta}})})$ is surjective because of the assumption, $\theta^{(e)}$ is nonzero. Hence there exists an effective divisor E on Z such that $\mathcal{O}_Z(-E)$ is equal to the image of $\theta^{(e)}$. We define $\Delta_Z := (p^e-1)^{-1}E$. By definition, $\Delta_Z = 0$ if and only if $\theta^{(e)}$ is surjective. Furthermore, by the argument similar to the above, we see that for a codimension one point $z \in Z$, $(X_{\overline{z}}, \Delta|_{X_{\overline{z}}})$ is F-split if and only if $\theta^{(e)} \otimes k(\overline{z})$ is non-zero, or equivalently, Δ is zero around z. Now we prove (3). (3-1)⇒(3-2) is obvious. (3-2)⇒(3-3) follows from Proposition 7.3.6. (3-3)⇒(3-4) follows from the above argument. If $\theta^{(e)}$ is surjective, or equivalently is an isomorphism, then $H^0(X_{Z^e}, \phi^{(e)}_{(X/Z,\Delta)}) \cong H^0(Z^e, \theta^{(e)})$ is also surjective, and hence $\phi^{(e)}_{(X/Z,\Delta)}$ splits. This proves (3-4)⇒(3-1). □

When $f : X \to Z$ is F-split with respect to Δ, there exists a $\mathbb{Z}_{(p)}$-Weil divisor $\Delta' \geq \Delta$ on X such that $K_{X/Z} + \Delta' \sim_{\mathbb{Z}_{(p)}} 0$ as explained below.

Observation 7.3.8. Let X, Δ, Z and f be as in Definition 7.3.1. Assume that (f, Δ) is F-split. Then there exists an $e > 0$ such that $\phi^{(e)}_{(X,\Delta)} : F^{(e)}_{X/Z*}\mathcal{L}^{(e)}_{(X/Z,\Delta)} \to \mathcal{O}_{X_{Z^e}}$ splits as a homomorphism of $\mathcal{O}_{X_{Z^e}}$-module. Here, we recall that

$$\mathcal{L}^{(e)}_{(X/Z,\Delta)} := \mathcal{O}_{X^e}(\lfloor (1-p^e)(K_{X^e/Z^e} + \Delta) \rfloor).$$

Then there exists an element $s \in H^0(X^e, \lfloor (1-p^e)(K_{X^e/Z^e} + \Delta) \rfloor)$ such that $\phi^{(e)}_{(X/Z,\Delta)}$ sends s to 1. Let E be an effective Weil divisor on X^e defined by s. Set $\Delta' := (p^e-1)^{-1}\lceil (p^e-1)\Delta + E \rceil \geq \Delta$. Then by the choice of E we have

$$\mathcal{L}^{(e)}_{(X,\Delta')} := \mathcal{O}_{X^e}((1-p^e)(K_{X^e/Z^e} + \Delta')) = \mathcal{O}_{X^e}(\lfloor (1-p^e)(K_{X^e/Z^e} + \Delta) - E \rfloor) \cong \mathcal{O}_{X^e},$$

and $\phi^{(e)}_{(X/Z,\Delta)} : F^{(e)}_{X/Z*}\mathcal{L}^{(e)}_{(X/Z,\Delta')} \to \mathcal{O}_{X_{Z^e}}$ splits.

Next we consider the case of finite morphisms.

Proposition 7.3.9. *Let X, Δ, Z and f be as in Definition 7.3.1. Assume that $\dim X = \dim Z$. Then the following conditions are equivalent:*

(1) *(f, Δ) is F-split;*

(2) *(f, Δ) is locally F-split;*

(3) *f is étale and $\Delta = 0$.*

In the case when $\Delta = 0$, the proposition has been shown in [60, 2.19 Theorem.].

Proof. (1)⇒(2) is obvious. Let f be étale and $\Delta = 0$. Then $F^{(e)}_{X/Z} : X^e \to X_{Z^e}$ is a finite morphism of degree one between normal varieties, and hence it is an isomorphism, which implies (3)⇒(1). We show (2)⇒(3). By Lemma 7.3.4 and Proposition 7.3.6, f is a separable surjective morphism, and hence we obtain that f is generically finite by the assumption. Let $e > 0$ be an integer such that the morphism

$$\mathcal{O}_{X_{Z^e}} \to F^{(e)}_{X/Z*}\mathcal{O}_{X^e}(\lceil (p^e-1)\Delta \rceil)$$

splits. Since $F^{(e)}_{X/Z}$ is a finite morphism of degree zero, $F^{(e)}_{X/Z}{}_*\mathcal{O}_{X^e}(\lceil (p^e-1)\Delta\rceil)$ is a torsion free sheaf of rank one. Note that as f is separable X_{Z^e} is a variety. Therefore the cokernel of the above morphism is zero, or equivalently, the above morphism is an isomorphism. Hence $\Delta = 0$ and $F^{(e)}_{X/Z}$ is an isomorphism. Then for every $z \in Z$, $F^{(e)}_{X_{\overline{z}}/\overline{z}}$ is also an isomorphism, where $\overline{z}$ is the algebraic closure of $z \in Z$. This implies that $X_{\overline{z}}$ is isomorphic to a disjoint union of copies of the spectrum of $k(\overline{z})$, and thus f is finite. Since $f_*\mathcal{O}_X$ is locally free as shown by Lemma 7.3.4, f is flat. Hence the smoothness of $X_{\overline{z}}$ implies that f is étale. $\square$

The following lemma is used in the proofs of Proposition 7.4.9 and Theorem 7.5.1.

Lemma 7.3.10. *Let X, Δ, Z and f be as in Definition 7.3.1. Assume that Δ is a $\mathbb{Z}_{(p)}$-Weil divisor and that (f, Δ) is locally F-split. Then the Iitaka-Kodaira dimension $\kappa(X, K_{X/Z} + \Delta)$ of $K_{X/Z} + \Delta$ is non-positive. Furthermore, if (f, Δ) is F-split, then $\kappa(X, -(K_{X/Z} + \Delta)) \geq 0$.*

Proof. The second statement follows from Observation 7.3.8. By Lemma 7.3.4, f is surjective. Assume that $\kappa(X, K_{X/Z} + \Delta) \geq 0$. Then $\kappa(X_{\overline{\eta}}, K_{X_{\overline{\eta}}/\overline{\eta}} + \Delta_{\overline{\eta}}) \geq 0$, where $\overline{\eta}$ is the geometric generic point of Z. Since $X_{\overline{\eta}}$ is F-split, we have $H^0(X_{\overline{\eta}}, (1 - p^e)(K_{X_{\overline{\eta}}} + \Delta_{\overline{\eta}})) \neq 0$ for some $e > 0$, and hence $(1 - p^e)(K_{X_{\overline{\eta}}} + \Delta) \sim 0$. Then the morphism

$$f_{Z^e}{}_*\phi^{(e)}_{(X/Z,\Delta)} : f^{(e)}{}_*\mathcal{O}_{X^e}((1-p^e)(K_{X^e/Z^e} + \Delta)) \to f_{Z^e}{}_*\mathcal{O}_{X_{Z^e}}$$

is a surjective morphism between torsion free coherent sheaves of the same rank, and thus it is an isomorphism. Hence $H^0(X, (1-p^e)(K_{X/Z}+\Delta)) \neq 0$, which implies that $\kappa(X, K_{X/Z}) = 0$. This is our assertion. $\square$

In the rest of this section, we consider the composition of morphisms in the next proposition, which is used frequently in Section 7.4.

Proposition 7.3.11. *Let X, Δ, Z and f be as in Definition 7.3.1, and Y be a normal variety. Assume that $f : X \to Z$ can be factored into projective morphisms $g : X \to Y$ with $g_*\mathcal{O}_X \cong \mathcal{O}_Y$ and $h : Y \to Z$.*

(1) *If (f, Δ) is F-split, then so is h.*

(2) *Assume that Y is smooth. If (g, Δ) and h are F-split, then so is (f, Δ).*

(3) *The converse of (2) holds if $K_Y \sim_{\mathbb{Z}_{(p)}} h^*K_Z$.*

Proof. Let $e > 0$ be an integer. Now we have the following commutative diagram:

$$\begin{tikzcd}
X^e \arrow[r, "F^{(e)}_{X/Y}"] \arrow[rr, bend left, "F^{(e)}_{X/Z}"] \arrow[dr, "g^{(e)}"'] & X_{Y^e} \arrow[r, "\pi^{(e)}"] \arrow[d, "g_{Y^e}"] & X_{Z^e} \arrow[r, "(F^e_Z)_X"] \arrow[d, "g_{Z^e}"] & X \arrow[d, "g"'] \arrow[dd, bend left, "f"] \\
& Y^e \arrow[r, "F^{(e)}_{Y/Z}"] \arrow[dr, "h^{(e)}"'] & Y_{Z^e} \arrow[r, "(F^e_Z)_Y"] \arrow[d, "h_{Z^e}"] & Y \arrow[d, "h"'] \\
& & Z^e \arrow[r, "F^e_Z"] & Z.
\end{tikzcd}$$

Here $\pi^{(e)} := (F^{(e)}_{Y/Z})_X$. We first show (1). The above diagram induces a commutative diagram of $\mathcal{O}_{Y_{Z^e}}$-modules

$$\begin{tikzcd}
\mathcal{O}_{Y_{Z^e}} \arrow[r] \arrow[d, "\cong"'] & F^{(e)}_{Y/Z*}\mathcal{O}_{Y^e} \arrow[d, "\cong"] \\
g_{Z^e*}\mathcal{O}_{X_{Z^e}} \arrow[r] & g_{Z^e*}F^{(e)}_{X/Z*}\mathcal{O}_{X^e}.
\end{tikzcd}$$

Here the left vertical morphism is an isomorphism because of the flatness of $(F^e_Z)_Y$. Since the lower horizontal morphisms splits, so is the upper one.

Next we show (2) and (3). As explained in Observation 7.3.8, if (g, Δ) (resp. (f, Δ)) is F-split, then there exists an effective $\mathbb{Z}_{(p)}$-Weil divisor $\Delta' \geq \Delta$ on X such that $K_{X/Y} + \Delta'$ (resp. $K_{X/Z} + \Delta'$) is $\mathbb{Z}_{(p)}$-linearly trivial and that (g, Δ') (resp. (f, Δ')) is also F-split. Thus we may assume that Δ is a $\mathbb{Z}_{(p)}$-Weil divisor and that $(p^e - 1)(K_{X/Y} + \Delta) \sim 0$ (resp. $\sim (p^e - 1)(f^*K_Z - g^*K_Y)$) for every $e > 0$ divisible enough. In particular, $\mathcal{L}^{(e)}_{(X/Y,\Delta)}$ (resp. $\mathcal{L}^{(e)}_{(X/Z,\Delta)}$) is isomorphic to the pullback by $g^{(e)}$ of a line bundle on $Y^{(e)}$.

Let $V \subseteq Y$ be an open subset such that g is flat at every point in $X_V := g^{-1}(V)$ and $\mathrm{codim}(Y \setminus V) \geq 2$. Let $u : U \to X_V$ be the open immersion of the regular locus of X_V. Set $g' := g \circ u : U \to Y$. Then we have ${g'}_*\mathcal{O}_U \cong g_*\mathcal{O}_X \cong \mathcal{O}_Y$ because of the assumptions. Additionally, by the flatness of F^e_Z, we see that ${g'}_{Z^e*}\mathcal{O}_{U_{Z^e}} \cong \mathcal{O}_{Y_{Z^e}}$. Thus by the projection formula, we see that

$$H^0(U_{Z^e}, ({g_{Z^e}}^*\mathcal{L})|_{U_{Z^e}}) \cong H^0(Y_{Z^e}, {g'}_{Z^e*}({g'_{Z^e}}^*\mathcal{L})) \cong H^0(Y_{Z^e}, \mathcal{L}) \cong H^0(X_{Z^e}, {g_{Z^e}}^*\mathcal{L})$$

for every line bundle $\mathcal{L}$ on Y_{Z^e}. Hence there exists the following commutative diagram:

$$\begin{tikzcd}
H^0(U^e, \mathcal{L}^{(e)}_{(X/Z,\Delta)}|_{U^e}) \arrow[r, "H^0(U_{Z^e}{,}\phi^{(e)}_{(U/Z,\Delta|_U)})"] \arrow[d, "\cong"'] & H^0(U_{Z^e}, \mathcal{O}_{U_{Z^e}}) \arrow[d, "\cong"] \\
H^0(X^e, \mathcal{L}^{(e)}_{(X/Z,\Delta)}) \arrow[r, "H^0(X_{Z^e}{,}\phi^{(e)}_{(X/Z,\Delta)})"] & H^0(X_{Z^e}, \mathcal{O}_{X_{Z^e}}).
\end{tikzcd}$$

Note that in particular, $H^0(U_{Y^e}, \mathcal{O}_{U_{Y^e}}) \cong H^0(Y^e, \mathcal{O}_{Y^e}) \cong k$. Clearly, the splitting of $\phi^{(e)}_{(X/Z,\Delta)}$ is equivalent to the surjectivity of $H^0(X_{Z^e}, \phi^{(e)}_{(X/Z,\Delta)})$. From this we see that the F-splitting of (f, Δ) is equivalent to the F-splitting of $(f|_U : U \to Z, \Delta|_U)$. By an argument similar to the above, we also see that the F-splitting of (g, Δ) is equivalent to the F-splitting of $(g|_U, \Delta|_U)$.

Assume that we can choose $V = Y$ and $U = X$, in other words, X and Y are regular and g is flat. Let $e > 0$ be an integer. By the flatness of g, we have the following commutative diagram:

$$\begin{tikzcd} {g_{Z^e}}^*\mathcal{O}_{Y_{Z^e}} \arrow[r, "{g_{Z^e}}^*\left({F^{(e)}_{Y/Z}}^\sharp\right)"] \arrow[d, "\cong"'] & {g_{Z^e}}^*{F^{(e)}_{Y/Z}}_*\mathcal{O}_{Y^e} \arrow[d, "\cong"] \\ \mathcal{O}_{X_{Z^e}} \arrow[r, "{\pi^{(e)}}^\sharp"] & {\pi^{(e)}}_*\mathcal{O}_{X_{Y^e}}. \end{tikzcd}$$

This implies that

$$\mathcal{H}om({\pi^{(e)}}^\sharp, \mathcal{O}_{X_{Z^e}}) \cong {g_{Z^e}}^*\mathcal{H}om({F^{(e)}_{Y/Z}}^\sharp, \mathcal{O}_{V_{Z^e}}) = {g_{Z^e}}^*\phi^{(e)}_{Y/Z}.$$

Applying the functor $\mathcal{H}om(\_\_, \mathcal{O}_{X_{Z^e}})$ and the Grothendieck duality to the natural morphism

$$\mathcal{O}_{X_{Z^e}} \xrightarrow{{\pi^{(e)}}^\sharp} {\pi^{(e)}}_*\mathcal{O}_{X_{Y^e}} \to {F^{(e)}_{X/Z}}_*\mathcal{O}_{X^e}(\lceil (p^e-1)\Delta\rceil),$$

we obtain the morphism

$$\phi^{(e)}_{(X/Z,\Delta)} : {F^{(e)}_{X/Z}}_*\mathcal{L}^{(e)}_{(X/Z,\Delta)} \xrightarrow{{\pi^{(e)}}_*\phi^{(e)}_{(X/Y,\Delta)}\otimes\omega_{\pi^{(e)}}} {g_{Z^e}}^*{F^{(e)}_{Y/Z}}_*\mathcal{L}^{(e)}_{Y/Z} \xrightarrow{{g_{Z^e}}^*\phi^{(e)}_{Y/Z}} \mathcal{O}_{X_{Z^e}}.$$

Note that $\omega_{\pi^{(e)}} \cong \omega_{X_{Y^e}} \otimes {\pi^{(e)}}^*\omega_{X_{Z^e}} \cong {g_{Z^e}}^*\omega^{1-p^e}_{Y^e/Z^e}$.

Now we prove the assertion. If (g, Δ) is F-split and h is F-split, then both of $\phi^{(e)}_{(X/Y,\Delta)}$ and $\phi^{(e)}_{Y/Z}$ split for every $e > 0$ divisible enough. Hence $\phi^{(e)}_{(X/Z,\Delta)}$ also splits, or equivalently, (f, Δ) is F-split. Conversely, assume that (f, Δ) is F-split and that $(p^e-1)K_{Y/Z} \sim 0$ for an $e > 0$. Then $\omega_{\pi^{(e)}} \cong \mathcal{O}_{X_{Y^e}}$. Since for every $e > 0$ divisible enough $H^0(X_{Z^e}, \phi^{(e)}_{(X/Z,\Delta)})$ is surjective, $H^0(X_{Z^e}, {\pi^{(e)}}_*\phi^{(e)}_{(X/Y,\Delta)})$ is a nonzero morphism, and thus so is $H^0(X_{Y^e}, \phi^{(e)}_{(X/Y,\Delta)})$. This is surjective because its target space $H^0(X_{Y^e}, \mathcal{O}_{X_{Y^e}}) \cong H^0(Y^e, \mathcal{O}_{Y^e}) \cong k$. Hence $\phi^{(e)}_{(X/Y,\Delta)}$ splits, and so (g, Δ) is F-split. Note that the F-splitting of h follows directly from (1). $\square$

7.4 Varieties with F-split Albanese morphisms

In this section, we prove Theorems 7.1.2, 7.1.3, 7.1.4 and 7.4.6. Throughout this section, we denote by X and Δ respectively a normal projective variety over an algebraically closed field k of characteristic $p > 0$ and an effective $\mathbb{Q}$-Weil divisor on X.

Proof of Theorem 7.1.3. Assume that (a, Δ) is locally F-split. The surjectivity of a follows from Lemma 7.3.4. Let $X \xrightarrow{f} Z \xrightarrow{g} A$ be the Stein factorization of a. As seen in Proposition 7.3.11 (1), g is F-split, and hence we see that g is étale by Proposition 7.3.9. Therefore [87, Section 18, Theorem] shows that Z is an abelian variety, and hence g is an isomorphism and $a_*\mathcal{O}_X \cong g_*\mathcal{O}_Z \cong \mathcal{O}_A$. (1)-(3) follows directly from Proposition 7.3.6. $\square$

The next lemma is used to prove Theorems 7.1.2 and 7.1.4.

Lemma 7.4.1. *Let $\mathcal{F}$ be a coherent sheaf of rank r on a normal variety Y. Let $\mathcal{F}'$ be an indecomposable direct summand of $\mathcal{F}$ of rank r'. Set $I := \{\mathcal{L} \in \mathrm{Pic}(Y) | \mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}\}$ and $I' := \{\mathcal{L} \in I | \mathcal{F}' \otimes \mathcal{L} \cong \mathcal{F}'\}$. Then $\bigoplus_{[\mathcal{L}] \in I/I'} \mathcal{F}' \otimes \mathcal{L}$ is a direct summand of $\mathcal{F}$. In particular, $\#(I/I') \leq r/r'$.*

Proof. For every $\mathcal{L} \in I$, $\mathcal{F}' \otimes \mathcal{L}$ is again a direct summand of $\mathcal{F}$. Furthermore, $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F} \otimes \mathcal{L}'$ if and only if $\mathcal{L}' \otimes \mathcal{L}^{-1} \in I$. Hence by Krull-Schmidt theorem [3], we see that $\bigoplus_{[\mathcal{L}] \in I/I'} \mathcal{F}' \otimes \mathcal{L}$ is a direct summand of $\mathcal{F}$. This implies $r' \#(I/I') \leq r$, which is our claim. □

To prove Theorem 7.1.2, we recall a characterization of ordinary abelian varieties due to Sannai and Tanaka.

Theorem 7.4.2 ([97, Theorem 1.1])**.** *Let Y be a normal projective variety over an algebraically closed field k of characteristic $p > 0$. Then Y is an ordinary abelian variety if and only if K_Y is pseudo-effective and ${F^e_Y}_*\mathcal{O}_Y$ is isomorphic to a direct sum of line bundles for infinitely many $e > 0$.*

Remark 7.4.3. In [30], it is proved that we only need to check ${F^e_Y}_*\mathcal{O}_X$ for $e = 1, 2$ in the above theorem.

For convenience, we use the following notation.

Notation 7.4.4. Let $\varphi : S \to T$ be a morphism of schemes. We denote by $\mathrm{Pic}(S)[\varphi]$ (resp. $\mathrm{Pic}^0(S)[\varphi]$) the kernel of the induced homomorphism $\varphi^* : \mathrm{Pic}(T) \to \mathrm{Pic}(S)$ (resp. $\varphi^* : \mathrm{Pic}^0(T) \to \mathrm{Pic}^0(S)$). Then for every $e > 0$, $\mathrm{Pic}(X)[F^e_X]$ is the set of p^e-torsion line bundles. We denote it by $\mathrm{Pic}(X)[p^e]$.

Proof of Theorem 7.1.2. We first prove that if (X, Δ) is F-split, then (a, Δ) is F-split and A is ordinary. We have the following commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X) & \xrightarrow{{F_X}^*} & H^1(X, \mathcal{O}_X) \\ {\scriptstyle \alpha^*}\uparrow & & \uparrow{\scriptstyle \alpha^*} \\ H^1(A, \mathcal{O}_A) & \xrightarrow{{F_A}^*} & H^1(A, \mathcal{O}_A). \end{array}$$

Since X is F-split, the upper horizontal arrow is bijective. Furthermore, by [83, Lemma(1.3)] we see that the vertical arrows are injective. (Note that although X is assumed to be smooth in [83, Lemma(1.3)], the smoothness of X is not needed in the proof.) Hence the lower horizontal arrow is injective, and thus A is ordinary. Let $X \xrightarrow{f} Z \xrightarrow{g} A$ be the Stein factorization of a. Then as shown by Proposition 7.3.11 (1), Z is F-split, or equivalently, $\mathcal{O}_Z$ is a direct summand of $\mathcal{F}^{(e)} := {F^e_Z}_*\mathcal{O}_{Z^e}$ for every $e > 0$. Since $a^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is bijective, $g^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(Z)$ is injective. Hence

$$p^{e \cdot \dim A} = \#\mathrm{Pic}^0(A)[F^e_A] \leq \#\mathrm{Pic}^0(Z)[F^e_Z].$$

Then by the projection formula and Lemma 7.4.1 (set $\mathcal{F} := \mathcal{F}^{(e)}$ and $\mathcal{F}' := \mathcal{O}_Z$), we obtain

$$p^{e\cdot\dim A} \leq \#\{\mathcal{L} \in \mathrm{Pic}(Z) | \mathcal{F}^{(e)} \otimes \mathcal{L} \cong \mathcal{F}^{(e)}\} \leq \mathrm{rank}\, \mathcal{F}^{(e)} = p^{e\cdot\dim Z}.$$

This implies $\dim Z = \dim A$ and that $\bigoplus_{\mathcal{L}\in\mathrm{Pic}(Z)[p^e]} \mathcal{L} \subseteq \mathcal{F}^{(e)}$ is a direct summand of maximum rank. Since $\mathcal{F}^{(e)}$ is torsion free, the inclusion is an isomorphism. Therefore F_Z^e is flat, or equivalently, Z is smooth. Now it is enough to show that ω_Z is pseudo-effective. Indeed, if it holds, then Theorem 7.4.2 shows that Z is an ordinary abelian variety, since $\mathcal{F}^{(e)}$ is a direct sum of line bundles for every $e > 0$. Then $g : Z \to A$ is an isomorphism, and by Proposition 7.3.11 (3), we see that (a, Δ) is F-split, which is our assertion. We show the pseudo-effectivity of ω_Z. Fix an $e > 0$. Now we have $(\mathcal{F}^{(e)})^* \cong \mathcal{F}^{(e)}$ and $F_Z^{e*}\mathcal{F}^{(e)} \cong \bigoplus \mathcal{O}_{Z^e}$. Furthermore, by the Grothendieck duality, we obtain

$$F^e_{Z*}\omega_{Z^e}^{1-p^e} \cong \mathcal{H}om(F^e_{Z*}\mathcal{O}_{Z^e}, \mathcal{O}_Z) = (\mathcal{F}^{(e)})^* \cong \mathcal{F}^{(e)}.$$

Hence there exists a surjection $F_Z^{e*}\mathcal{F}^{(e)} \cong F_Z^{e*}F^e_{Z*}\omega_{Z^e}^{1-p^e} \to \omega_{Z^e}^{1-p^e}$, which implies that $\omega_Z^{1-p^e}$ is globally generated. Since $H^0(Z^e, \omega_{Z^e}^{1-p^e}) \cong H^0(Z, \mathcal{F}^{(e)}) \cong k$, we get $\omega_Z^{1-p^e} \cong \mathcal{O}_Z$, or equivalently $\omega_Z^{p^e-1} \cong \mathcal{O}_Z$, and thus ω_Z is pseudo-effective.

The converse follows directly from Proposition 7.3.11. $\square$

Proof of Theorem 7.1.4. Assume that (a, Δ) is locally F-split. By Theorem 7.1.3, a is surjective with $a_*\mathcal{O}_X \cong \mathcal{O}_X$, and hence the first statement follows. We show the second statement. The "if" part is obvious. For the "only if" part, we assume $\dim A = \dim X$. Then by Proposition 7.3.9, we see that a is an isomorphism and $\Delta = 0$. $\square$

Remark 7.4.5. For a smooth projective variety V over an algebraically closed field of characteristic zero, we have $b_1(V)/2 = h^{1,0}(V) := \dim H^0(V, \Omega_V^1)$. However, in positive characteristic, we only have the inequality $b_1(V)/2 \leq h^{1,0}(V)$. Igusa constructed a smooth projective surface S with $b_1(S) = h^{1,0}(S) = 2$ [61]. In [30], ordinary abelian varieties of odd characteristic are characterized as smooth projective F-split varieties V with $h^{1,0}(V) = \dim V$.

The purpose of the remainder of this section is to prove the next theorem.

Theorem 7.4.6. *Let γ_A be the p-rank of A. Assume that there exists a morphism $f : X \to B$ to an abelian variety B of p-rank γ_B such that (f, Δ) is F-split. Then (a, Δ) is F-split and $\gamma_A = \gamma_B + \dim A - \dim B$. In particular, if B is ordinary, then (X, Δ) is F-split.*

To prove this, we need to prove Proposition 7.3.9, which is an application of Theorem 7.1.4. We first observe line bundles whose pullbacks by the relative Frobenius morphisms are trivial.

Observation 7.4.7. Let $f : X \to Z$ be an algebraic fiber space and X, Z be smooth projective varieties. (1) We consider the following commutative diagram of Picard groups:

$$\begin{tikzcd}
\mathrm{Pic}(X^e) & \\
\mathrm{Pic}(X_{Z^e}) \arrow[u, "{F^{(e)}_{X/Z}}^*"] & \mathrm{Pic}(X) \arrow[ul, "{F^e_X}^*"'] \arrow[l, "{(F^e_Z)_X}^*"] \\
\mathrm{Pic}(Z^e) \arrow[u, "{f_{Z^e}}^*"] & \mathrm{Pic}(Z). \arrow[l, "{F^e_Z}^*"'] \arrow[u, "f^*"']
\end{tikzcd}$$

Clearly, f^* induces an injective morphism $\mathrm{Pic}(Z)[p^e] \xrightarrow{f^*} \mathrm{Pic}(X)[(F^e_Z)_X]$. We show that this is an isomorphism. Let $\mathcal{L} \in \mathrm{Pic}(X)[(F^e_Z)_X]$. Then by the flatness of F^e_Z, we have

$$F^{e\,*}_Z f_*\mathcal{L} \cong f_{Z^e\,*}\mathcal{L}_{Z^e} \cong f_{Z^e\,*}\mathcal{O}_{X_{Z^e}} \cong F^{e\,*}_Z f_*\mathcal{O}_X \cong \mathcal{O}_{Z^e}.$$

Hence $f_*\mathcal{L}$ is a p^e-torsion line bundle on Z, and the natural morphism $f^*f_*\mathcal{L} \to \mathcal{L}$ is an isomorphism. Therefore we deduce that the above homomorphism is surjective. Note that a non-zero homomorphism between numerically trivial line bundles on a projective variety is an isomorphism.

By the above argument, we have the following exact sequence

$$0 \to \mathrm{Pic}(Z)[p^e] \xrightarrow{f^*} \mathrm{Pic}(X)[p^e] \to \mathrm{Pic}(X_{Z^e})[F^{(e)}_{X/Z}].$$

(2) Set $\mathcal{F} := F^{(e)}_{X/Z\,*}\mathcal{O}_{X^e}$ and $I := \{\mathcal{L} \in \mathrm{Pic}(X_{Z^e}) | \mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}\}$. Then we have $\mathrm{Pic}(X_{Z^e})[F^{(e)}_{X/Z}] \subseteq I$ by the projection formula. Let $\mathcal{F}'$ be an indecomposable direct summand of $\mathcal{F}$ and let $I' := \{\mathcal{L} \in \mathrm{Pic}(X_{Z^e}) | \mathcal{F}' \otimes \mathcal{L} \cong \mathcal{F}'\}$. Then by Lemma 7.4.1, we obtain that $\bigoplus_{[\mathcal{L}] \in I/I'} \mathcal{F}' \otimes \mathcal{L}$ is a direct summand of $\mathcal{F}$. In particular,

$$\mathrm{rank}\,\mathcal{F}' \cdot \#(I/I') \leq \mathrm{rank}\,\mathcal{F} = p^{e(\dim X - \dim Z)}.$$

The following lemma is used to prove Proposition 7.4.9.

Lemma 7.4.8. *Let $f : X \to Z$ be an F-split morphism to a smooth projective variety Z. Let X_z be the general closed fiber of f. Then $h^1(X, \mathcal{O}_X) \leq h^1(X_z, \mathcal{O}_{X_z}) + h^1(Z, \mathcal{O}_Z)$.*

Proof. Set $\mathcal{G}^i := R^if_*\mathcal{O}_X$. Then we have $\mathrm{rank}\,\mathcal{G}^i = h^i(X_z, \mathcal{O}_{X_z})$ and $F^{e\,*}_Z\mathcal{G}^i \cong \mathcal{G}^i$ for some $e > 0$ by Lemma 7.3.4. As shown by [78, 1.4. Satz], there exists an étale cover $\pi : Z' \to Z$ such that $\pi^*\mathcal{G}^i \cong \bigoplus \mathcal{O}_{Z'}$ for each i, and hence

$$\dim H^0(Z, \mathcal{G}^i) \leq \dim H^0(Z', \pi^*\mathcal{G}^i) = \mathrm{rank}\,\mathcal{G}^i = h^i(X_z, \mathcal{O}_{X_z}).$$

Therefore by the Leray spectral sequence, we have

$$h^1(X, \mathcal{O}_X) \leq h^0(Z, \mathcal{G}^1) + h^1(Z, \mathcal{O}_Z) \leq h^1(X_z, \mathcal{O}_{X_z}) + h^1(Z, \mathcal{O}_Z).$$

$\square$

Proposition 7.4.9. *Let $f : X \to Z$ be an F-split morphism to an abelian variety Z. Suppose that the Albanese morphism $a : X \to A$ of X is a finite morphism. Then a is an isomorphism, or equivalently, X is an abelian variety.*

Proof. Let $f : X \xrightarrow{f'} Z' \xrightarrow{\pi} Z$ be the Stein factorization. As shown by Proposition 7.3.11, π is F-split. Hence we see that π is étale by Proposition 7.3.9. This implies that Z' is also an abelian variety by [87, Section 18, Theorem] and that (f', Δ) is F-split by Proposition 7.3.11. Replacing Z by Z', we may assume that $f_*\mathcal{O}_X \cong \mathcal{O}_Z$. We can factor f into $f : X \xrightarrow{a} A \xrightarrow{g} Z$. Let $z \in Z$ be a general closed point. Then as shown by Proposition 7.3.6, X_z is integral, normal and F-split. We recall that a is a finite morphism by the assumption. Then the induced morphism $X_z \to (A_z)_{\mathrm{red}}$ is a finite morphism to an abelian variety, and therefore X_z is an ordinary abelian variety by Theorem 7.1.4. Hence by Lemma 7.4.8, we have

$$\dim A \leq h^1(X, \mathcal{O}_X) \leq h^1(X_z, \mathcal{O}_{X_z}) + h^1(Z, \mathcal{O}_Z) = \dim X_z + \dim Z = \dim X.$$

This means that a is surjective. Since f is F-split, it is separable, and hence so is g, which implies that A_z is reduced. We may assume $X_z \to A_z$ is an isogeny of abelian varieties. Considering p-torsion points, we see that A_z is also ordinary. Therefore the p-rank γ_A of A is equal to

$$\gamma_{A_z} + \gamma_Z = \dim A_z + \gamma_Z = \dim A - \dim Z + \gamma_Z,$$

and hence g is F-split because of Theorem 7.3.7 (ii).

Claim 7.4.10. The morphism $a : X \to A$ is separable.

If the claim holds, then $0 \sim a^*K_A \leq K_X$. Since f is F-split, we have $\kappa(X, K_{X/Z}) = \kappa(X, K_X) \leq 0$ by Lemma 7.3.10, and hence $K_X = 0$. Applying the Zariski-Nagata purity, we see that a is an étale morphism. Hence we obtain that X is an abelian variety by [87, Section 18, Theorem]. This is our assertion. $\square$

Proof of Claim 7.4.10. We factor $a : X \to A$ into two finite morphisms $i : X \to Y$ and $s : Y \to A$ such that i is purely inseparable and s is separable. We show that i is an isomorphism. We fix an $e > 0$ such that there exists a morphism $b : Y^e \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X^e \arrow[r, "F_X^e"] \arrow[d, "i^{(e)}"'] & X \arrow[d, "i"] \\ Y^e \arrow[r, "F_Y^e"'] \arrow[ur, "b"] & Y. \end{tikzcd}$$

This induces the following commutative diagram:

$$\begin{tikzcd} X^e \arrow[r, "F_{X/Z}^{(e)}"] \arrow[d, "i^{(e)}"'] & X_{Z^e} \arrow[r, "(F_Z^e)_X"] \arrow[d, "i_{Z^e}"] & X \arrow[d, "i"] \\ Y^e \arrow[r, "F_{Y/Z}^{(e)}"'] \arrow[ur, "b_{Z^e}"] & Y_{Z^e} \arrow[r, "(F_Z^e)_Y"'] & Y. \end{tikzcd}$$

Note that since $f : X \to Z$ and $g \circ s : Y \to Z$ are separable, X_{Z^e} and Y_{Z^e} are varieties. Since $\mathcal{O}_{X_{Z^e}} \to F^{(e)}_{X/Z_*}\mathcal{O}_{X^e}$ splits, $\mathcal{O}_{X_{Z^e}} \to b_{Z^e*}\mathcal{O}_{Y^e}$ also splits. From this, the coherent sheaf $\mathcal{F} := F^{(e)}_{Y/Z_*}\mathcal{O}_{Y^e}$ on Y_{Z^e} has $i_{Z^e*}\mathcal{O}_{X_{Z^e}}$ as a direct summand. Let $\mathcal{F}'$ be the indecomposable direct summand of $i_{Z^e*}\mathcal{O}_{X_{Z^e}}$ with $H^0(Y_{Z^e}, \mathcal{F}') \neq 0$. Set

$$I := \{\mathcal{L} \in \mathrm{Pic}(Y_{Z^e}) | \mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}\} \text{ and } I' := \{\mathcal{L} \in I | \mathcal{F}' \otimes \mathcal{L} \cong \mathcal{F}'\}.$$

Let $\mathcal{L}$ be a p^e-torsion line bundle on A. Now we have the morphisms $Y \xrightarrow{s} A \xrightarrow{g} Z$. Set $\mathcal{M}$ to be the p^e-torsion line bundle $(s^*\mathcal{L})_{Z^e}$ on Y_{Z^e}. We show that $\mathcal{M} \in I$ and that $\mathcal{M} \in I'$ if and only if $\mathcal{L} \in g^*\mathrm{Pic}(Z)[p^e]$. By the projection formula, we have

$$\mathcal{F} \otimes \mathcal{M} = (F^{(e)}_{Y/Z_*}\mathcal{O}_{Y^e}) \otimes (s^*\mathcal{L})_{Z^e} \cong F^{(e)}_{Y/Z_*}(F_Y^{e*}(s^*\mathcal{L})) \cong F^{(e)}_{Y/Z_*}(s^*\mathcal{L}^{p^e}) \cong \mathcal{F},$$

and hence $\mathcal{M} \in I$. If $\mathcal{L} \cong g^*\mathcal{N}$ for an $\mathcal{N} \in \mathrm{Pic}(Z)[p^e]$, then $\mathcal{M} \cong s_{Z^e}{}^* g_{Z^e}{}^* F_Z^{e*}\mathcal{N} \cong \mathcal{O}_{Y_{Z^e}} \in I'$. Conversely, if $\mathcal{M} \in I'$, then again by the projection formula, we have

$$\begin{aligned} 0 \neq H^0(Y_{Z^e}, \mathcal{F}') &\cong H^0(Y_{Z^e}, \mathcal{F}' \otimes \mathcal{M}) \\ &\subseteq H^0(Y_{Z^e}, (i_{Z^e*}\mathcal{O}_{X^e}) \otimes \mathcal{M}) \\ &\cong H^0(X_{Z^e}, i_{Z^e}{}^*\mathcal{M}) \cong H^0(X_{Z^e}, (a^*\mathcal{L})_{Z^e}). \end{aligned}$$

Therefore $(a^*\mathcal{L})_{Z^e} \cong \mathcal{O}_{X_{Z^e}}$. By Observation 7.4.7 (1), we get $a^*\mathcal{L} \in f^*(\mathrm{Pic}(Z)[p^e]) = a^*g^*(\mathrm{Pic}(Z)[p^e])$. Since $a^* : \mathrm{Pic}^0(A) \to \mathrm{Pic}^0(X)$ is an isomorphism, we have $\mathcal{L} \in g^*(\mathrm{Pic}(Z)[p^e])$. From the argument above, we have the following injective morphism

$$G := \mathrm{Pic}(A)[p^e]/g^*\mathrm{Pic}(Z)[p^e] \xrightarrow{(s^*(\underline{\quad}))_{Z^e}} I/I'.$$

Let r' be the rank of $\mathcal{F}'$. Since the number of p^e-torsion line bundles on A (resp. Z) is equal to $p^{e\cdot\gamma_A}$ (resp. $p^{e\cdot\gamma_Z}$), we have

$$p^{e(\dim A - \dim Z)}r' = p^{e(\gamma_A - \gamma_Z)}r' \leq \#G \cdot r' \leq \#(I/I') \cdot r' \leq p^{e(\dim A - \dim Z)}$$

by Observation 7.4.7 (2). This implies that $r' = 1$ and that $\bigoplus_{[\mathcal{L}]\in G} \mathcal{F}' \otimes (s^*\mathcal{L})_{Z^e} \subseteq \mathcal{F}$ is a direct summand of maximal rank. Since $\mathcal{F}$ is torsion free, this inclusion is an isomorphism. Since $i_{Z^e*}\mathcal{O}_{X_{Z^e}}$ is a direct summand of $\mathcal{F}$, there exists a subset $H \subseteq G$ such that $\bigoplus_{[\mathcal{L}]\in H} \mathcal{F}' \otimes (s^*\mathcal{L})_{Z^e} \cong i_{Z^e*}\mathcal{O}_{X_{Z^e}}$. Then the rank $i_{Z^e*}\mathcal{O}_{X_{Z^e}} = \#H \cdot r' = \#H$. We show $\#H = 1$. For $\mathcal{L} \in \mathrm{Pic}(A)[p^e]$ with $[\mathcal{L}] \in H$, we have

$$\begin{aligned} 0 \neq H^0(Y_{Z^e}, \mathcal{F}') &= H^0(Y_{Z^e}, \mathcal{F}' \otimes (s^*\mathcal{L})_{Z^e} \otimes (s^*\mathcal{L}^{-1})_{Z^e}) \\ &\subseteq H^0(Y_{Z^e}, (i_{Z^e*}\mathcal{O}_{X_{Z^e}}) \otimes (s^*\mathcal{L}^{-1})_{Z^e}) = H^0(X_{Z^e}, (a^*\mathcal{L}^{-1})_{Z^e}). \end{aligned}$$

By an argument similar to the above, we see that $\mathcal{L}^{-1} \in g^*\mathrm{Pic}(Z)[p^e]$, and thus $[\mathcal{L}] = [\mathcal{O}_A] \in G$. Hence $H = \{[\mathcal{O}_A]\}$. Since $\deg i = \deg i_{Z^e} = \mathrm{rank}\, i_{Z^e*}\mathcal{O}_{X_{Z^e}} = 1$, we see that i is an isomorphism, which is our assertion. $\square$

Proof of Theorem 7.4.6. Let $X \xrightarrow{\pi} X' \xrightarrow{g'} A$ be the Stein factorization of a. Then we can factor f into $f : X \xrightarrow{\pi} X' \xrightarrow{g'} A \xrightarrow{h} B$. By Proposition 7.3.11 (1), $h \circ g'$ is F-split. Since the finite morphism $g' : X' \to A$ is the Albanese morphism of X', we see that g' is an isomorphism by Proposition 7.4.9. Therefore Proposition 7.3.11 (3) shows that (a, Δ) is F-split. Since $h : A \to B$ is an F-split morphism whose closed fibers A_z are ordinary abelian varieties, we obtain

$$\gamma_A = \gamma_{A_z} + \gamma_B = \dim A_z + \gamma_B = \dim A - \dim B + \gamma_B.$$

$\square$

7.5 Minimal surfaces with F-split Albanese morphisms

The aim of this section is to specify minimal surfaces over an algebraically closed field k of characteristic $p > 0$ such that the Albanese morphisms are F-split or locally F-split. Note that if a smooth projective surface has F-split (resp. locally F-split) Albanese morphism, then so are its minimal surfaces. Indeed, let S_1 be a smooth projective surface and $\pi : S_1 \to S_2$ be the contraction of a (-1)-curve. Then it is easily seen that the induced morphism $\mathrm{Alb}(\pi) : \mathrm{Alb}(S_1) \to \mathrm{Alb}(S_2)$ is an isomorphism. Hence if S_1 has F-split (resp. locally F-split) Albanese morphism, then so does S_2 by Proposition 7.3.11 (1).

Throughout this section, we denote by X a smooth projective minimal surface and by $a : X \to A$ the Albanese morphism of X.

Theorem 7.5.1. *If a is locally F-split, then one of the following holds:*

(0) $b_1(X) = 0$ *and X is F-split;*

(1-1) $b_1(X) = 2$, $\kappa(X) = -\infty$ *and X is the projective space bundle $\mathbb{P}(\mathcal{E})$ associated with a rank two vector bundle $\mathcal{E}$ on A;*

(1-2) $b_1(X) = 2$, $\kappa(X) = 0$ *and X is a hyperelliptic surface such that every closed fiber of a is an ordinary elliptic curve;*

(2) *X is an abelian surface.*

Furthermore, in the case of (1-1)*, the morphism a is F-split if and only if either*

(a) *$\mathcal{E}$ is decomposable,*

(b) *$\mathcal{E}$ is indecomposable, $p > 2$ and $\deg \mathcal{E}$ is odd, or*

(c) *$\mathcal{E}$ is indecomposable, $p = 2$ and A is ordinary.*

In the case of (1-2)*, the morphism a is F-split.*

Note that the first Betti number $b_1(X)$ is equal to $2\dim A$. By Theorem 7.1.4, we see that $b_1(X) = 0$, 2 or 4.

- If $b_1(X) = 0$, then the F-splitting of a is equivalent to the F-splitting of X.
- If $b_1(X) = 4$, then X is an abelian surface as shown by Theorem 7.1.4.
- The case when $b_1(X) = 2$ is dealt with in the remainder of this section. As shown by Lemma 7.3.10, we have $\kappa(X) \le 0$. We consider the cases $\kappa(X) = -\infty$ and $\kappa(X) = 0$ respectively in Subsections 7.5.1 and 7.5.2.

7.5.1 The case $b_1(X) = 2$ and $\kappa(X) = 0$

In this case, by Bombieri and Mumford's classification of minimal surfaces with Kodaira dimension zero [12], we see that X is a hyperelliptic or quasi-hyperelliptic surface. If a is locally F-split, then a has normal geometric generic fiber as shown by Proposition 7.3.6, and hence X is hyperelliptic. In particular, there exist two elliptic curves E_0 and E_1 such that $X \cong E_1 \times E_0/B$, where B is a finite subgroupscheme of E_1 [12, Theorem 4]. Furthermore, every closed fiber of a is isomorphic to E_0, and $A \cong E_1/B$.

Proposition 7.5.2. *The following are equivalent:*

(1) *a is F-split;*

(2) *a is locally F-split;*

(3) *E_0 is ordinary.*

Proof. (1)$\Rightarrow$(2) is obvious. If a is locally F-split, then the general fibers are F-split by Proposition 7.3.6, and hence E_0 is F-split. Thus (2)$\Rightarrow$(3) holds. We prove (3)$\Rightarrow$(1). Assume that E_0 is F-split. Since a is flat and every fiber has the trivial canonical bundle, $K_X \sim a^*C$ for a Cartier divisor C on A. Then by Theorem 7.3.7 (ii), we obtain an effective $\mathbb{Q}$-divisor Δ_A on A such that $C \sim_{\mathbb{Z}_{(p)}} K_A + \Delta_A \sim \Delta_A$. Since $K_X \sim_{\mathbb{Q}} 0$, we have $\Delta_A = 0$, and hence a is F-split as shown by Theorem 7.3.7 (ii)-(3). This is our claim. $\square$

7.5.2 The case $b_1(X) = 2$ and $\kappa(X) = -\infty$

In this case, X is a ruled surface over an elliptic curve. We start with recalling some facts on elliptic curves. In the following lemmas, we denote by C an elliptic curve.

Lemma 7.5.3. *Let $\mathcal{F}$ be a vector bundle on C of rank r and $\mathcal{L}$ be a line bundle such that $\mathcal{F} \otimes \mathcal{L} \cong \mathcal{F}$. Then $\mathcal{L}^r \cong \mathcal{O}_C$.*

Proof. This follows from $(\det \mathcal{F}) \otimes \mathcal{L}^r \cong \det(\mathcal{F} \otimes \mathcal{L}) \cong \det \mathcal{F}$. $\square$

Lemma 7.5.4. *Let $\pi : C' \to C$ be a finite morphism of degree d from an elliptic curve C'. Let $\mathcal{L}$ be a line bundle on C such that $\pi^*\mathcal{L} \cong \mathcal{O}_{C'}$. Then $\mathcal{L}^d \cong \mathcal{O}'_C$.*

Proof. By the projection formula, we have $(\pi_*\mathcal{O}_{C'})\otimes\mathcal{L} \cong \pi_*\mathcal{O}_{C'}$. Hence the assertion follows from Lemma 7.5.3. $\square$

Lemma 7.5.5. *The m-th symmetric product $S^m\mathcal{E}_{2,0}$ of $\mathcal{E}_{2,0}$ is isomorphic to a direct sum of vector bundles of the form $\mathcal{E}_{r,0}$.*

Proof. Let $\mathcal{F}$ be an indecomposable direct summand of $S^m\mathcal{E}_{2,0}$ of rank r. By Theorem 1.4.1, we may write $\mathcal{F} \cong \mathcal{E}_{r,0}\otimes\mathcal{L}$ for an $\mathcal{L} \in \mathrm{Pic}^0(C)$. Let $\pi := \pi_2 : C_2 \to C$ be as in Lemma 1.4.2. Since $\pi^*S^m\mathcal{E}_{2,0}$ is trivial, we have $\pi^*\mathcal{L} \cong \mathcal{O}_{C_2}$. By Lemma 7.5.4, we see that $\mathcal{L}^p \cong \mathcal{O}_C$. Since supersingular elliptic curves have no non-trivial p-torsion line bundle, we may assume that C is ordinary. Then since ${F_C}^*\mathcal{E}_{r,0} \cong \mathcal{E}_{r,0}$, we get that ${F_C}^*\mathcal{F} \cong \mathcal{E}_{r,0} \otimes \mathcal{L}^p \cong \mathcal{E}_{r,0}$ and ${F_C}^*S^m\mathcal{E}_{2,0} \cong S^m\mathcal{E}_{2,0}$. Hence we conclude that $\mathcal{L} \cong \mathcal{O}_C$. $\square$

Now we return to study the F-splitting of the Albanese morphism $a : X \to A$ of X. We may regard X and a respectively as $\mathbb{P}(\mathcal{E})$ for a vector bundle on A of rank two and its projection. If $\mathcal{E}$ is decomposable, then a is F-split as seen in Example 7.3.3. Assume that $\mathcal{E}$ is indecomposable. We only need to consider the two cases: $\deg\mathcal{E} = 0$ and $\deg\mathcal{E} = 1$.

The case $\deg\mathcal{E} = 0$.

In this case, we may assume that $\mathcal{E} = \mathcal{E}_{2,0}$ by Theorem 1.4.1 (1). Then we have a finite morphism $\pi : A' \to A$ from an elliptic curve A' such that $\pi^*\mathcal{E}_{2,0} \cong \mathcal{O}_{A'}^{\oplus 2}$, as seen in Lemma 1.4.2. In particular, $X_{A'} \cong \mathbb{P}(\pi^*\mathcal{E}_{2,0}) \cong \mathbb{P}^1 \times A^1$. We show the following:

Proposition 7.5.6. *$a : X \to A$ is F-split if and only if A is ordinary and $p = 2$.*

To prove Proposition 7.5.6, we prepare the following claims.

Claim 7.5.7. There exists an algebraic fiber space $g : X \to Y \cong \mathbb{P}^1$ such that $g^*\mathcal{O}_Y(1) \cong \mathcal{O}_X(p)$.

Claim 7.5.8. The p-th symmetric power $S^p\mathcal{E}_{2,0}$ of $\mathcal{E}_{2,0}$ is isomorphic to $\mathcal{E}_{p,0} \oplus \mathcal{O}_A$.

Proof of Claims 7.5.7 and 7.5.8. Since the Iitaka-Kodaira dimensions of line bundles are preserved under the pullback by any surjective projective morphism [62, Theorem 10.5], we have

$$\kappa(X, \mathcal{O}_X(1)) = \kappa(X_{A'}, \mathcal{O}_{X_{A'}}(1)) = \kappa(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) = 1.$$

Since $\nu(X, \mathcal{O}_X(1))$ is also equal to one, we deduce that $\mathcal{O}_X(1)$ is semi-ample. Let $g : X \to Y$ be the Iitaka fibration associated to $\mathcal{O}_X(1)$. Then $Y \cong \mathbb{P}^1$ obviously. Let B be the general fiber of g. Then B is an elliptic curve. Now we have the following commutative diagram:

$$\begin{tikzcd}
B_{A'} \arrow[r] \arrow[d] & B \arrow[d] & \\
X_{A'} \arrow[r, "\pi_X"] \arrow[d, "a_{A'}"'] & X \arrow[r, "g"] \arrow[d, "a"] & Y \\
A' \arrow[r, "\pi"'] & A. &
\end{tikzcd}$$

By the construction, we have $\mathcal{O}_X(B) \cong \mathcal{O}_X(m) \otimes a^*\mathcal{L}$ for an $m > 0$ and a torsion line bundle $\mathcal{L}$ on A. We consider the exact sequence

$$0 \to \mathcal{O}_X(l) \otimes \mathcal{O}_X(-B) \to \mathcal{O}_X(l) \to \mathcal{O}_B \to 0$$

for $l \in \mathbb{Z}$. Taking the direct image, we obtain exact sequences

$$0 \to \mathcal{L}^* \to S^m\mathcal{E}_{2,0} \to a_*\mathcal{O}_B \to 0 \quad \text{and} \quad 0 \to S^{m-1}\mathcal{E}_{2,0} \to a_*\mathcal{O}_B \to 0$$

when $l = m$ and $l = m - 1$, respectively. By the first exact sequence and by Lemma 7.5.5, we see that $\mathcal{L} \cong \mathcal{O}_A$, or equivalently $\mathcal{O}_X(B) \cong \mathcal{O}_X(m)$. By the second one and Lemma 7.5.5, we obtain that $a_*\mathcal{O}_B \cong S^{m-1}\mathcal{E}_{2,0} \cong \mathcal{E}_{m,0}$. Hence $\pi^*a_*\mathcal{O}_B \cong \mathcal{O}_{A'}^{\oplus m}$. Since $\pi : A' \to A$ and $(a|_B) : B \to A$ are flat, we obtain

$$\begin{aligned}
\dim H^0(B, (a|_B)^*\mathcal{E}_{p,0}) &= \dim H^0(B, (a|_B)^*\pi_*\mathcal{O}_{A'}) \\
&= \dim H^0(B_{A'}, \mathcal{O}_{B_{A'}}) = \dim H^0(A', \pi^*a_*\mathcal{O}_B) = m.
\end{aligned}$$

Since $a : X \to A$ is a non-trivial projective space bundle, we have $2 \leq m \leq p$. For an $\mathcal{N} \in \mathrm{Pic}^0(A)[a|_B]$, by the projection formula, we have $\mathcal{E}_{m,0} \cong \mathcal{E}_{m,0} \otimes \mathcal{N}$. Then $\mathcal{N} \cong \mathcal{O}_A$ by Theorem 1.4.1 (1). Thus we get $\mathrm{Pic}^0(A)[a|_B] = \{\mathcal{O}_A\}$, which means that m is a power of p, so $m = p$. Since $H^0(A, S^p\mathcal{E}_{2,0}) \cong H^0(X, \mathcal{O}_X(p)) \cong H^0(Y, \mathcal{O}_Y(1)) = k^{\oplus 2}$, we see that the first exact sequence splits, which implies that $S^p\mathcal{E}_{2,0} \cong \mathcal{E}_{p,0} \oplus \mathcal{O}_A$. $\square$

Now we start the proof of Proposition 7.5.6.

Proof of Proposition 7.5.6. We use the same notation as the proof of Claim 7.5.7. First we prove the "if" part. We show that (a, B) is F-split. Now we have $\omega_X \otimes \mathcal{O}_X(B) \cong \mathcal{O}_X(-2) \otimes \mathcal{O}_X(p) \cong \mathcal{O}_X$. Hence by Theorem 7.3.7 (ii)-(3), it is enough to show that $(X_z, B|_{X_z})$ is F-split for a fiber $X_z \cong \mathbb{P}^1$ of a. Since A is ordinary, $\pi : A' \to A$ is étale. Since $\pi^*a_*\mathcal{O}_B$ is a trivial vector bundle of rank two on A', we see that $B_{A'}$ is a disjoint union of sections of $a_{A'} : X_{A'} \to A'$. This implies that the divisor $B|_{X_z}$ is a sum of two distinct points. Using the assumption that $p = 2$, we conclude that $(X_z, B|_{X_z})$ is F-split.

Next we prove the "only if" part. We first show that A is ordinary by contradiction. Assume that A is supersingular. Then $\pi = F_A$. In this case, we see that $B_{A'} = pS$ as divisors, where S is a section of $a_{A'} : X_{A'} \to A'$. Set

$$\psi^{(e)} := H^0(X_{A'^1}, \phi^{(1)}_{X_{A'}/A'} \otimes \omega_{X_{A'^1}}^{1-p^{e-1}}) : H^0((X_{A'})^1, \omega_{X_{A'}}^{1-p^e}) \to H^0(X_{A'^1}, \omega_{X_{A'^1}}^{1-p^{e-1}}).$$

Then by Claim 7.5.7, we have

$$\begin{aligned} H^0(X, \omega_X^{1-p^e}) &= H^0(X, \mathcal{O}_X(2p^e-2)) \\ &= (g^* H^0(Y, \mathcal{O}_Y(2p^{e-1}-1))) \cdot (H^0(X, \mathcal{O}_X(p-2))). \end{aligned}$$

Since there exists a section S of $a_{A'}$ such that every fiber of $g \circ \pi_X$ is equal to pS as a divisor, we see that for every $s \in {\pi_X}^* g^* H^0(\mathcal{O}_Y(2p^{e-1}-1))$ there exists a $t \in H^0(\mathcal{O}_{X_{A'}}(2p^{e-1}-1))$ with $s = t^p$. Hence we have $\psi^{(e)}(s \cdot r) = \psi^{(e)}(r) \cdot t = 0$ for every $r \in {\pi_X}^* H^0(X, \mathcal{O}_X(p-2))$. We then deduce that $\phi^{(e)}_{X_{A'}/A'}$ sends $s \cdot r$ to 0. Since $\phi^{(e)}_{X_{A'}/A'}$ is obtained as the pullback of $\phi^{(e)}_{X/A}$, we conclude that $H^0(X_{A^e}, \phi^{(e)}_{X/A})$ is the zero map. Therefore a is not F-split, which is a contradiction. Thus A is ordinary. We show $p = 2$. By the assumption we see that the morphism $\mathcal{O}_{X_{A^1}} \to F^{(1)}_{X/A_*}\mathcal{O}_{X^1}$ splits. Applying the functor $a_{A^1*}(\underline{\quad} \otimes \mathcal{O}_{X_{A^1}}(1))$ to this, we obtain the morphism

$$\mathcal{E}_{2,0} \cong {F_A}^* \mathcal{E}_{2,0} \cong a_{A^1*}\mathcal{O}_{X_{A^1}}(1) \to {a^{(1)}}_* \mathcal{O}_{X^1}(p) \cong S^p \mathcal{E}_{2,0}$$

which splits as $\mathcal{O}_{A^1}$-modules. Since $S^p\mathcal{E}_{2,0} \cong \mathcal{E}_{p,0} \oplus \mathcal{O}_A$ as shown by Claim 7.5.8, we get $\mathcal{E}_{p,0} \cong \mathcal{E}_{2,0}$ and thus $p = 2$. □

The case $\deg \mathcal{E} = 1$**.**

The following proposition is the conclusion of this case.

Proposition 7.5.9. *If* $\deg \mathcal{E} = 1$*, then a is F-split if and only if A is ordinary or* $p > 2$*.*

Proof. We first prove the "if" part. When $p > 2$, we take the étale cover $\rho : A' \to A$ of degree two corresponding to a torsion line bundle $\mathcal{L}$ of order two. Then $\rho_* \mathcal{O}_{A'} \cong \mathcal{O}_A \oplus \mathcal{L}$ and

$$\rho_{A'*}\mathcal{O}_{A' \times_A A'} \cong \rho^* \rho_* \mathcal{O}_{A'} \cong \mathcal{O}_{A'} \oplus \mathcal{O}_{A'}.$$

Here the first isomorphism follows from the flatness of ρ. Hence $A' \times_A A'$ is a disjoint union of two copies of A'. By Theorem 1.4.1 (1) and (6), there exists a line bundle $\mathcal{M}$ of degree one such that $\rho_* \mathcal{M} \cong \mathcal{E}$. Then

$$\rho^* \mathcal{E} \cong \rho^* \rho_* \mathcal{M} \cong \rho_{A'*}\mathcal{M}_{A'} \cong \mathcal{M} \oplus \mathcal{M}.$$

Therefore $X' := X_{A'} \cong \mathbb{P}(\mathcal{M} \oplus \mathcal{M})$ is F-split over A'. We now have the following commutative diagrams:

$$\begin{tikzcd} X'^e \arrow[r] \arrow[d, "F^{(e)}_{X'/A'}"'] & X^e \arrow[d, "F^{(e)}_{X/A}"] \\ {X'}_{A'^e} \arrow[r, "(\rho^{(e)})_X"'] & X_{A^e} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{O}_{X_{A^e}} \arrow[r] \arrow[d] \arrow[dr] & (\rho^{(e)})_{X*}\mathcal{O}_{{X'}_{A'^e}} \arrow[d] \\ F^{(e)}_{X/A_*}\mathcal{O}_{X^e} \arrow[r] & (\rho^{(e)})_{X*}F^{(e)}_{X'/A'_*}\mathcal{O}_{X'^e}. \end{tikzcd}$$

Since ρ is a finite étale morphism of degree not divisible by p, the upper horizontal morphism splits. Then the diagonal morphism also splits, and hence so is the left morphism. Consequently, we see that X is F-split over A. When $p = 2$ and A is ordinary, $F_A^*\mathcal{E} \in \mathcal{E}_A(2,2)$ as shown by Theorem 1.4.1 (5). Then by Proposition 7.5.6, we see that $a_{A^1} : X_{A^1} \to A^1$ is F-split. Replacing ρ by F_A, we can prove the assertion by the same argument as the above.

Next we prove the "only if" part by contraposition. Assume that $p = 2$ and A is supersingular. Then Theorem 1.4.1 (5) shows that ${F_A}^*\mathcal{E} \in \mathcal{E}_A(2,2)$. Hence as seen in Proposition 7.5.6, $a_{A^1} : X_{A^1} \to A^1$ is not F-split. This requires that $a : X \to A$ is also not F-split, which completes the proof. $\square$

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