# 博士論文

# Classical BV action for NS sector of open superstring field theory in the large Hilbert space

（ラージヒルベルト空間における開いた超弦の場の理論の NS セクターに対する古典 BV 作用）

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# 1 Introduction

String theory is expected to play an important role to construct quantum gravity theory. However, string theory is not completed yet. One of the problems is that the theory is defined as a perturbative theory, in which one can investigate only around a fixed background. The relationships between backgrounds are not clear. Construction of string field theory is an approach to make nonperturbative string theory. For example, Witten made a covariant open bosonic string field theory [1] and this theory succeeded in analyzing tachyon condensation [5]. Tachyon condensation describes D-brane decayed background from a background which has unstable D-brane.

String theory consists of not only bosonic string theory but also supersymmetric string theory, which is called superstring theory. There are two types of construction of superstring theory: Ramond-Neveu-Schwarz (RNS) form and Green-Schwartz form. Open superstring in RNS formalism consists of Neveu-Schwarz (NS) sector, which gives spacetime bosons, and Ramond sector, which gives space-time fermions. We consider only NS sector in this thesis. Witten constructed an open superstring field theory as well as bosonic theory [2]. However, Witten's open superstring field theory has a singularity which cause a divergence in the four point amplitude. Erler, Konopka and Sachs modified this theory by the approach explored by Iimori, Noumi, Okawa and Torii and resolve the singularity [6][7]. The action constructed by Erler, Konopka and Sachs has a structure called $A_\infty$ algebra. Berkovits provided another formulation open superstring field theory [3] [4]. These two theories use different Hilbert spaces. Berkovits formulation is based on large Hilbert space. In the large Hilbert space, superconformal ghost sector is described in terms of $\eta(z), \xi(z), \phi(z)$. Erler-Konopka-Sachs (EKS) formulation is also constructed in the large Hilbert space. However, its dynamical fields and gauge transformation is restricted to the small Hilbert space, where superconformal ghost sector is described in terms of $\beta(z), \gamma(z)$ ghosts. $\beta(z)$ and $\gamma(z)$ are related to $\eta(z), \xi(z), \phi(z)$ as $\beta(z) = e^{-\phi(z)}\partial\xi(z)$, $\gamma(z) = e^{\phi(z)}\eta(z)$. Since $\beta(z)$ does not depend on the zero mode of $\xi(z)$, the Hilbert space of $\beta\gamma$ ghosts is smaller.

EKS formulation and Berkovits formulation are related by embedding EKS action into the large Hilbert space and redefining fields [9][10][11]. However, embedding EKS action increases gauge degrees of freedom and makes the gauge fixing more difficult.

Since superstring field theories have complicated gauge structures, the Batalin-Vilkovisky (BV) formalism is used to fix gauge transformation. To quantize string field theory, it is necessary to fix gauge by constructing BV master action which includes ghost fields and antifields. BV action satisfies master equation $\frac{1}{2}\{S, S\} = i\hbar\triangle S$ where $\{\ ,\ \}$ is antibracket which is defined in section 2 and $\triangle$ is an operator. Constructing classical BV action which satisfies classical master equation $\{S, S\} = 0$ is the first step to achieve second quantization of string field theory. A classical master action can be constructed from EKS action in the small Hilbert space easily because of the $A_\infty$ structure. However, when EKS action is embedded into the large Hilbert space, we cannot construct a classical BV action in a straightforward way. The goal of this thesis is constructing classical BV master action in the large Hilbert space.

This thesis is organized as follows. In section 2 we review the BV formalism briefly. We see general gauge structures and how to fix gauge transformations with fields and antifields. In section 3 we review construction of EKS action. We see the definition of $A_\infty$ algebra and the form of EKS action. We review how to construct the string products in the action. In section 4 we check the gauge transformations of EKS action. Embedding EKS action into the large Hilbert space, we see the changes of gauge degrees of freedom. In section 5 we try to solve master equation in the large Hilbert space naively. However, we fail to construct master action in this way. We show the details of the calculation. In section 6 we propose a action which satisfy the master equation in the large Hilbert space. We double fields and antifields in this approach. However, this action does not satisfy a boundary condition. In section 7 we propose another BV action. We add ghost fields and antifields and impose constraints.

# 2 BV formalism

In some gauge theories, gauge transformations are dependent on each other. These theories are called irreducible. When all gauge transformations are independent, the theory is reducible. Open superstring field theory is a reducible gauge theory. Quantization of such theories is complicated.

The BV formalism can quantize reducible gauge theories in a covariant way [17][18]. In this thesis, we use only the classical BV formalism. We review gauge fixing procedure using field and antifield in this section. This review is written in [8].

## 2.1 Gauge transformation

We consider a classical action $S_0[\phi]$, which depends on $n$ different fields $\phi^i(x)$, $(i = 1, \cdots, n)$. Let $\epsilon(\phi^i) = \epsilon_i$ denote the Grassmann parity of $\phi^i$. Each $\phi^i$ is either a commuting field ($\epsilon_i = 0$) or an anticommuting field ($\epsilon_i = 1$).

Let us assume that the action is invariant under a set of $m_0$ non-trivial gauge transformations

$$\delta\phi^i(x) = \left(R^i_\alpha(\phi)\varepsilon^\alpha\right)(x) \tag{2.1}$$

where $\alpha = 1, 2, \cdots, m_0$. Here, $\varepsilon^\alpha(x)$ are infinitesimal gauge parameters and $R^i_\alpha$ are the generators of gauge transformations. We use a generalized summation convention in which a repeated discrete index implies not only a sum over that index but also an integration over the corresponding space-time variable. As a simple example, consider the multiplication of two matrices $g$ and $h$, written with explicit matrix indices. In compact notation,

$$f^A{}_B = g^A{}_C h^C{}_B \tag{2.2}$$

represents

$$f^A{}_B(x,y) = \sum_C \int dz g^A{}_C(x,z) h^C{}_B(z,y) \tag{2.3}$$

in conventional notation. In other words, the index $A$ in eq(2.2) stands for $A$ and $x$ in eq(2.3). Likewise, $B$ and $C$ in eq(2.2) represents $\{B, y\}$ and $\{C, z\}$. The generalized summation convention for $C$ in compact notation yields a sum over the discrete index $C$ and an integration over $z$ in eq(2.3).

With this convention, the transformation laws

$$\delta\phi^i(x) = \sum_\alpha \int dy R^i_\alpha(x,y)\varepsilon^\alpha(y) \tag{2.4}$$

can be written as

$$\delta\phi^i = R^i_\alpha \varepsilon^\alpha. \tag{2.5}$$

Let $S_{0,i}(\phi, x)$ denote the variation of the action with respect to $\phi^i(x)$:

$$S_{0,i}(\phi,x) \equiv \frac{\partial_r S_0}{\partial\phi^i(x)} \tag{2.6}$$

where $\partial_r$ indicates that the derivative is to be taken from the right. If the subscript is $l$, the derivative is taken from the left. The right derivative is related to the left derivative

$$\frac{\partial_r F}{\partial\phi} = (-)^{\epsilon(\phi)(\epsilon(F)+1)} \frac{\partial_l F}{\partial\phi}. \tag{2.7}$$

The statement that the action is invariant under the gauge transformation in eq(2.1) means the Neother identity

$$S_{0,i} R^i_\alpha = 0. \tag{2.8}$$

Assume that all gauge invariances of a theory are known and that the regularity condition

$$\operatorname{rank}\left(\frac{\partial_l \partial_r S_0}{\partial\phi^i \partial\phi^j}\right)\bigg|_\Sigma = n_{\mathrm{dof}} \tag{2.9}$$

is satisfied, where $\Sigma$ is the stationary surface defined implicitly by

$$S_{0,i}|_\Sigma = 0 \tag{2.10}$$

and $n_{\mathrm{dof}}$ is the number of fields that enter dynamically in $S_0$.

The most general solution to the Noether identities (2.8) is a gauge transformation, up to terms proportional to the equations of motion:

$$S_{0,i}\lambda^i = 0 \Leftrightarrow \lambda^i = R^i_{0\alpha_0}\lambda'^{\alpha_0} + S_{0,j}T^{ij} \tag{2.11}$$

where $T^{ij}$ must satisfy the graded symmetry property

$$T^{ij} = -(-)^{\epsilon_i \epsilon_j} T^{ji}. \tag{2.12}$$

The $R^i_{0\alpha_0}$ are the gauge generators. The second term is a trivial gauge transformation.

If the functionals $R^i_{0\alpha_0}$ are independent on-shell, the theory is irreducible. In such a case,

$$\text{rank} R^i_{0\alpha_0}|_\Sigma = m_0, \tag{2.13}$$

where $m_0$ is the number of gauge transformations. The rank of the hessian is

$$\text{rank}\left(\frac{\partial_l \partial_r S_0}{\partial\phi^i \partial\phi^j}\right)\bigg|_\Sigma = n - \text{rank} R^i_\alpha|_\Sigma = n - m_0. \tag{2.14}$$

Then for an irreducible theory $n_{\text{dof}} = n - m_0$ since there $m_0$ gauge degrees of freedom.

If there are dependences among the gauge generators, and the rank of the generators is less than their number

$$\text{rank} R^i_{0\alpha_-}|_\Sigma < m_0, \tag{2.15}$$

the theory is irreducible. If $m_0 - m_1$ of the generators are independent on-shell, then there are $m_1$ relations among them and there exist $m_1$ functionals $R^{\alpha_0}_{1\alpha_1}$ such that

$$R^i_{0\alpha_0} R^{\alpha_0}_{1\alpha_1} = S_{0,j} V^{ji}_{1\alpha_1}, \quad (\alpha_1 = 1, \ldots, m_1) \tag{2.16}$$

for some $V^{ji}_{1\alpha_1}$, satisfying $V^{ji}_{1\alpha_1} = -(-)^{\epsilon_i \epsilon_j} V^{ij}_{1\alpha_1}$.

The $R^{\alpha_0}_{1\alpha_1}$ are the on-shell null vectors for $R^i_{0\alpha_0}$ since $R^i_{0\alpha_0} R^{\alpha_0}_{1\alpha_1}|_\Sigma = 0$. If $\varepsilon^\alpha = R^\alpha_{1\alpha_1}\varepsilon^{\alpha_1}$, $\delta\phi^i$ in eq(2.1) is zero on-shell. Then, no gauge transformation is produced. $\varepsilon^{\alpha_1}$ is called level one gauge parameter. The Grassmann parity of $R^{\alpha_0}_{1\alpha_1}$ is

$$\epsilon(R^{\alpha_0}_{1\alpha_1}) = \epsilon_{\alpha_0} + \epsilon_{\alpha_1} \tag{2.17}$$

where $\epsilon_{\alpha_1}$ is the Grassmann parity of the level one gauge parameter. $R^{\alpha_0}_{1\alpha_1}$ also constitute a complete set

$$R^i_{0\alpha_0} \lambda^{\alpha_0} = S_{0,j} M^{ji}_0 \tag{2.18}$$

$$\Rightarrow \lambda^{\alpha_0} = R^{\alpha_0}_{1\alpha_1} \lambda'^{\alpha_1} + S_{0,j} T^{j\alpha_0}_0, \tag{2.19}$$

for some $\lambda'^{\alpha_1}, T^{j\alpha_0}_0$ and $M^{ji}_0$ satisfying $M^{ji}_0 = -(-)^{\epsilon_i \epsilon_j} M^{ij}_0$.

If the functionals $R^{\alpha_0}_{1\alpha_1}$ are independent on-shell

$$\text{rank} R^{\alpha_0}_{1\alpha_1}|_\Sigma = m_1, \tag{2.20}$$

the theory is first stage reducible. The rank of $R^i_{0\alpha_0}$ is

$$\text{rank} R^i_{0\alpha_0}, \tag{2.21}$$

and the net number of degrees of freedom in the theory is $n - m_0 + m_1$.

If the functionals $R^{\alpha_0}_{1\alpha_1}$ are not all independent on-shell, relations exist among them and the theory is second or higher stage reducible. Then, there are higher level gauge parameters and the on-shell null vectors of $R^{\alpha_0}_{1\alpha_1}$ exist.

If the theory is $L$-th stage reducible, there are functionals

$$R^{\alpha_{s-1}}_{s\alpha_s} \quad (\alpha_s = 1, \dots, m_s, \quad s = 0, \dots, L), \tag{2.22}$$

such that $R^i_{0\alpha_0}$ satisfies $S_{0,i}R^i_{0\alpha_0} = 0$, and that, at each stage, the $R^{\alpha_{s-1}}_{s\alpha_s}$ constitute a complete set, i.e.,

$$R^{\alpha_{s-1}}_{s\alpha_s}\lambda^{\alpha_s} = S_{0,j}M^{j\alpha_{s-1}}_s \tag{2.23}$$

$$\Rightarrow \lambda^{\alpha_s} = R^{\alpha_s}_{s+1,\alpha_{s+1}}\lambda'^{\alpha_{s+1}} + S_{0,j}T^{j\alpha_s}_s, \tag{2.24}$$

$$R^{\alpha_{s-2}}_{s-1,\alpha_{s-1}}R^{\alpha_{s-1}}_{s\alpha_s} = S_{0,i}V^{i\alpha_{s-2}}_{s\alpha_s}, \quad (s = 1, \dots, L), \tag{2.25}$$

$$\mathrm{rank} R^{\alpha_{s-1}}_{s\alpha_s}|_\Sigma = \sum_{t=s}^{L}(-1)^{t-s}m_t \quad (s = 0, \dots L), \tag{2.26}$$

where we have defined $\alpha_{-1} \equiv i$. The $R^{\alpha_{s-1}}_{s\alpha_s}$ are the on-shell null vectors for $R^{\alpha_{s-2}}_{s-1,\alpha_{s-1}}$. The Grassmann parity of $R^{\alpha_{s-1}}_{s\alpha_s}$ is

$$R^{\alpha_{s-1}}_{s\alpha_s} = \epsilon_{\alpha_{s-1}} + \epsilon_{\alpha_s} \tag{2.27}$$

where $\epsilon_{\alpha_s}$ is the Grassmann parity of the $s$ level gauge transformation parameter associated with the index $\alpha_s$. Finally,

$$n_{\mathrm{dof}} = n - \sum_{s=0}^{L}(-1)^s m_s \tag{2.28}$$

is the net number of degrees of freedom.

The gauge transformations (2.24) contain trivial gauge transformations $S_{0,j}T^{j\alpha_s}_s$. We will discuss their role.

Suppose that the finite invertible gauge transformations satisfy the group axioms, their infinitesimal counterparts necessarily form an algebra. Besides the usual gauge transformations (2.1), there are the trivial gauge transformations, defined as

$$\delta_\mu\phi^i = S_{0,j}\mu^{ji}, \quad \mu^{ji} = -(-)^{\epsilon_i\epsilon_j}\mu^{ji} \tag{2.29}$$

where $\mu^{ji}$ are arbitrary functions. Such gauge transformations appear in commutators of two non-trivial gauge transformations, so we need to take into consideration trivial gauge transformations.

The commutator of a trivial gauge transformation $\delta_\mu$ with another transformation $\delta_r$ is

$$[\delta_\mu, \delta_r]\phi^i = r^i{}_{,k}S_{0,j}\mu^{jk} - S_{0,j}\mu^{ji}{}_{,k}r^k - S_{0,jk}r^k\mu^{ji} \tag{2.30}$$

where

$$\delta_r \phi^i = r^i. \tag{2.31}$$

Given that $\delta_r$ is a symmetry transformation of $S_0$, it follows by differentiation by $\phi^j$ that

$$S_{0,k} r^k = 0 \tag{2.32}$$

$$\Rightarrow S_{0,jk} r^k + S_{0,k} r^k{}_{,j} = 0 \tag{2.33}$$

then the commutator becomes

$$[\delta_\mu, \delta_r]\phi^i = S_{0,j} \left( r^j{}_{,k} \mu^{ki} - (-)^{\epsilon_i \epsilon_j} r^i{}_{,k} \mu^{kj} - \mu^{ji}{}_{,k} r^k \right) = S_{0,j} \tilde{\mu}^{ji}. \tag{2.34}$$

The commutator of a trivial transformation with any other transformation is a trivial transformation. Hence, the trivial transformations are a subgroup $H$ of the full group of gauge transformations $\bar{G}$. The trivial gauge transformations have no physical significance. We can consider the gauge theory on $G = \bar{G}/H$.

## 2.2 The field antifield formalism

The ultimate goal is to quantize this theory in a covariant way. The field antifield formalism, which is called the BV formalism was developed to achieve this aim.

Suppose a theory is irreducible with $m_0$ gauge invariances. At the quantum level, $m_0$ ghost fields are needed. It is useful to introduce these ghost fields at the classical level. Hence, the field set $A$ is $A = \{\phi^i, \mathcal{C}_0^{\alpha_0}\}$ where $\alpha_0 = 1, \ldots m_0$. If the theory is first stage reducible, there are gauge invariances for gauge parameters and there are ghosts for ghosts. If there are $m_1$ level one gauge invariances, there are ghost-for-ghost fields $\mathcal{C}_1^{\alpha_1}$ where $\alpha_1 = 1, \ldots, m_1$ in addition to the above set. If the theory is $L$-th stage reducible, the set of fields is

$$A = \{\phi^i, \mathcal{C}_s^{\alpha_s}\} \quad (s = 0, \ldots, L;\ \alpha_s = 1, \ldots, m_s). \tag{2.35}$$

An additive conserved charge, which is called ghost number, is assigned to each of these fields. The classical fields $\phi^i$ have ghost number zero, whereas ordinary ghosts have ghost number one. Ghosts for ghosts, i.e., level one ghosts, have ghost number one, and so on. Similarly, ghosts have opposite Grassmann parity of the corresponding gauge parameter, but ghosts for ghosts have the same Grassmann parity as the corresponding gauge parameter. In general,

$$\mathrm{gh}[\mathcal{C}_s^{\alpha_s}] = s + 1, \tag{2.36}$$

$$\epsilon(\mathcal{C}_s^{\alpha_s}) = \epsilon_{\alpha_s} + s + 1 \pmod 2. \tag{2.37}$$

We set fields $\phi^A = \phi^i, \mathcal{C}_s^{\alpha_s}$ and introduce antifields $\phi^*_A$ for each fields $\phi^A$. The ghost number and Grassmann parity of antifields are

$$\mathrm{gh}[\phi^*_A] = -\mathrm{gh}[\phi^A] - 1, \tag{2.38}$$

$$\epsilon(\phi^*_A) = \epsilon(\phi^A) + 1 \pmod 2. \tag{2.39}$$

In the space of fields and antifields, the antibracket is defined by

$$\{X, Y\} \equiv \sum_A \left( \frac{\partial_r X}{\partial \phi^A} \frac{\partial_l Y}{\partial \phi^*_A} - \frac{\partial_r X}{\partial \phi^*_A} \frac{\partial_l Y}{\partial \phi^A} \right). \tag{2.40}$$

The properties of the antibracket are

$$\{Y, X\} = -(-)^{(\epsilon_X+1)(\epsilon_Y+1)}\{X, Y\}, \tag{2.41}$$

$$\{\{X, Y\}, Z\} + (-)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}\{\{Y, Z\}, X\} + (-)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}\{\{Z, X\}, Y\} = 0, \tag{2.42}$$

$$\mathrm{gh}[\{X, Y\}] = \mathrm{gh}[X] + \mathrm{gh}[Y] + 1, \tag{2.43}$$

$$\epsilon[\{X, Y\}] = \epsilon_X + \epsilon_Y + 1 \pmod 2. \tag{2.44}$$

The first equation means that the antibracket is graded antisymmetric. The second equation shows that the antibracket satisfies a graded Jacobi identity. The antibracket carries ghost number one and is Grassmann odd.

From these properties and the definition of right and left derivatives, one concludes that

$$\{B, B\} = 2\frac{\partial_r B}{\partial \phi^A} \frac{\partial_l B}{\partial \phi^*_A}, \tag{2.45}$$

$$\{F, F\} = 0, \tag{2.46}$$

$$\{\{X, X\}, X\} = 0, \tag{2.47}$$

where $B$ is bosonic and $F$ is fermionic.

The classical master equation is

$$\{S, S\} = \sum_A 2\frac{\partial_r S}{\partial \phi^A} \frac{\partial_l S}{\partial \phi^*_A} = 0. \tag{2.48}$$

The BV action satisfies the master equation and it is on the boundary condition

$$S_{\mathrm{BV}}[\phi, \phi^*]|_{\phi^*=0} = S_0 \tag{2.49}$$

where $S_0$ is the original action.

# 3 $A_\infty$ superstring field theory

Recently, an open superstring field theory based on $A_\infty$ algebra is developed [6]. This theory can be related to Berkovits superstring field theory by field redefinition[9][10][11]. We review the construction of the action based on $A_\infty$ algebra in this section.

### 3.1 $A_\infty$ algebra

In this subsection, we review a construction of $A_\infty$ algebra which is called bar construction [12]. This review is written in [13]. An element of an $A_\infty$ algebra belong to a $\mathbb{Z}$-graded vector space. First, we provide the definition of coalgebra and operators on the coalgebra to define $A_\infty$ algebra.

1. Let $C$ be a graded vector space. When a coproduct $\triangle : C \to C \otimes C$ is defined on $C$ and it is coassociative, i.e.
$$(\triangle \otimes \mathbb{I})\triangle = (\mathbb{I} \otimes \triangle)\triangle \tag{3.1}$$
then $C$ is called a coalgebra.

2. A linear operator $m : C \to C$ raising the degree of $C$ by one is called coderivation when
$$\triangle m = (m \otimes \mathbb{I})\triangle + (\mathbb{I} \otimes m)\triangle \tag{3.2}$$
is satisfied. Here, for $x, y \in C$, the sign is defined as
$$(\mathbb{I} \otimes m)(x \otimes y) = (-)^{\deg(x)}(x \otimes m(y)). \tag{3.3}$$

3. Given two coalgebras $C$ and $C'$, a cohomomorphism or coalgebra homomorphisim $\mathcal{F}$ from $C$ to $C'$ is a map of degree zero satisfying the condition
$$\triangle \mathcal{F} = (\mathcal{F} \otimes \mathcal{F})\triangle. \tag{3.4}$$

Let $\mathcal{H}$ be a $\mathbb{Z}$-graded vector space. We consider the tensor algebra
$$T\mathcal{H} = \mathcal{H}^{\otimes 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\otimes 2} \oplus \cdots \tag{3.5}$$
as a coalgebra $C(\mathcal{H})$. Here $\mathcal{H}^{\otimes 0}$ consists of the identity of the tensor algebra $\mathbf{1}$, satisfying
$$\mathbf{1} \otimes A = A \otimes \mathbf{1} = A \tag{3.6}$$
for any $A \in T\mathcal{H}$.

Then the coassociative product $\triangle : T\mathcal{H} \to T\mathcal{H} \otimes T\mathcal{H}$ is uniquely determined. For $o_1, \ldots, o_n \in \mathcal{H}$, it is given by
$$\triangle(o_1 \otimes \cdots \otimes o_n) = \sum_{k=0}^{n} (o_1 \otimes \cdots \otimes o_k) \otimes' (o_{k+1} \otimes \cdots \otimes o_n) \tag{3.7}$$
where the term for $k = 0$ is $\mathbf{1} \otimes' (o_1 \otimes \cdots \otimes o_n)$ and the term for $k = n$ is $(o_1 \otimes \cdots \otimes o_n) \otimes' \mathbf{1}$. Here $\otimes$ represents the tensor product of $\mathcal{H}$ and $\otimes'$ is the tensor product of $T\mathcal{H}$ in this equation.

The form of the coderivation corresponding to this coproduct is also given as follows. Let $\{c_k : \mathcal{H}^{\otimes k} \to \mathcal{H} \quad (k \geq 0)\}$ be multilinear maps which are

$$c_k : o_1 \otimes \cdots \otimes o_k \mapsto c_k(o_1, \ldots o_k). \tag{3.8}$$

The degree of this state is

$$\deg c_n(o_1, \ldots, o_n) = \deg(c_n) + \deg o_1 + \cdots + \deg o_n. \tag{3.9}$$

The tensor product of $b_{k,m} : \mathcal{H}^{\otimes m} \to \mathcal{H}^{\otimes k}$ and $c_{l,n} : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes l}$ is

$$\begin{aligned}&(b_{k,m} \otimes c_{l,n})(o_1 \otimes \cdots \otimes o_{m+n})\\ &= (-)^{\deg(c_{l,n})(\deg(\Psi_1)+\cdots+\deg(\Psi_m))} b_{k,m}(o_1, \ldots, o_m) \otimes c_{l,n}(o_{m+1}, \ldots, o_{m+n})\end{aligned} \tag{3.10}$$

The operations of $c_k$ on $T\mathcal{H}$ are given as

$$\begin{aligned}&\mathbf{c}_k(o_1 \otimes \cdots \otimes o_n)\\ &= \sum_{p=1}^{n-k+1} (-)^{\deg(c_k)(\deg o_1+\cdots+\deg o_{p-1})} o_1 \otimes \cdots \otimes o_{p-1} \otimes c_k(o_p, \ldots, o_{p+k-1}) \otimes o_{p+k} \otimes \cdots \otimes o_n\end{aligned} \tag{3.11}$$

for $n \geq k$. If $n < k$,

$$\mathbf{c}_k(o_1 \otimes \cdots \otimes o_n) = 0. \tag{3.12}$$

The commutator of the operators is defined as

$$[\mathbf{b}_k, \mathbf{c}_l] = \mathbf{b}_k(\mathbf{c}_l(o_1 \otimes \cdots \otimes o_n)) - (-)^{\deg(b_k)\deg(c_l)} \mathbf{c}_l(\mathbf{b}_k(o_1 \otimes \cdots \otimes o_n)) \tag{3.13}$$

Suppose that all $c_n$ have the same degree parity. $\mathbf{c}$ is defined as

$$\mathbf{c} = \mathbf{c}_0 + \mathbf{c}_1 + \mathbf{c}_2 + \cdots, \tag{3.14}$$

and this $\mathbf{c}$ is the coderivation. The coderivation on the coalgebra $T\mathcal{H}$ is always written in this form.

Moreover, the form of a cohomomorphism $\mathcal{F} : T\mathcal{H} \to T\mathcal{H}'$ is determined by a collection of degree zero multilinear maps $\{f_k : \mathcal{H}^{\otimes k} \to \mathcal{H}' \quad (k \geq 0)\}$. For $o_1, \ldots, o_n \in \mathcal{H}$, it is given as

$$\begin{aligned}&\mathcal{F}(o_1 \otimes \cdots \otimes o_n)\\ &= \sum_{i=1}^{n} \sum_{1 \leq k_1 < k_2 < \cdots < k_i = n} \frac{1}{1-f_0} \otimes f_{k_1}(o_1, \ldots, o_{k_1}) \otimes \frac{1}{1-f_0} \otimes f_{k_2-k_1}(o_{k_1+1}, \ldots, o_{k_2}) \otimes \cdots\\ &\qquad \cdots \otimes \frac{1}{1-f_0} \otimes f_{n-k_{i-1}}(o_{k_{i-1}+1}, \ldots, o_n) \otimes \frac{1}{1-f_0}\end{aligned} \tag{3.15}$$

where $f_0 \in \mathcal{H}'$ and $\frac{1}{1-f_0}$ is defined by

$$\frac{1}{1-f_0} \equiv \mathbf{1} + f_0 + f_0 \otimes f_0 + \cdots . \tag{3.16}$$

A weak $A_\infty$ algebra is a coalgebra $C(\mathcal{H})$ with a coderivation $\mathbf{c} = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \cdots$ satisfying

$$(\mathbf{c})^2 = 0. \tag{3.17}$$

We denote the weak $A_\infty$-algebra by $(\mathcal{H}, \mathbf{c})$. In particular, $(\mathcal{H}, \mathbf{c})$ is called an $A_\infty$-algebra if $c_0 = 0$.

For an $A_\infty$-algebra $(\mathcal{H}, \mathbf{c})$, if we act $\mathbf{c}^2$ on $o_1 \otimes \cdots \otimes o_n$, its image belongs to $\mathcal{H}^{\otimes 1} \oplus \cdots \oplus \mathcal{H}^{\otimes n}$. Using the projection on $\mathcal{H}^{\otimes 1}$, the equation is

$$\begin{aligned} 0 =& \pi_1(\mathbf{c})^2(o_1 \otimes \cdots \otimes o_n) \\ =& \sum_{k+l=n+1} \sum_{j=0,\dots k-1} (-)^{\deg(c)(\deg o_1 + \cdots + \deg o_j)} c_k(o_1, \dots, o_j, c_l(o_{j+1}, \dots, o_{j+l}), o_{j+l+1}, \dots, o_n) \end{aligned} \tag{3.18}$$

Here $\pi_n$ is a projection operator $\pi_n : T\mathcal{H} \to \mathcal{H}^{\otimes n}$ that is

$$\pi_n \left( \mathbf{1} + \Psi + \Psi \otimes \Psi + \Psi \otimes \Psi \otimes \Psi + \cdots \right) = \Psi^{\otimes n} \tag{3.19}$$

Consider an odd constant symplectic structure $\omega$ on the graded vector space $\mathcal{H}$

$$\omega : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}. \tag{3.20}$$

If $(\mathcal{H}, \mathbf{c})$ is an $A_\infty$ algebra and $\mathbf{c}$ is cyclic with respect to $\omega$, that is,

$$\omega(o_1, c_n(o_2, \dots, o_{n+1})) = -(-)^{\deg(c_n)\deg o_1} \omega(c_n(o_1, \dots, o_n), o_{n+1}), \tag{3.21}$$

then $(\mathcal{H}, \omega, \mathbf{c})$ is called cyclic $A_\infty$ algebra.

Given two weak $A_\infty$ algebras $(\mathcal{H}, \mathbf{c})$ and $(\mathcal{H}', \mathbf{c}')$, a cohomomorphism $\mathcal{F} : \mathcal{TH} \to \mathcal{TH}'$ satisfying

$$\mathcal{F}\mathbf{c} = \mathbf{c}'\mathcal{F}. \tag{3.22}$$

is a weak $A_\infty$ morphism $\mathcal{F} : (\mathcal{H}, \mathbf{c}) \to (\mathcal{H}', \mathbf{c}')$. In particular, if $(\mathcal{H}, \mathbf{c})$ and $(\mathcal{H}', \mathbf{c}')$ are $A_\infty$ algebras and $f_0 = 0$, a weak $A_\infty$ morphism $\mathcal{F} : (\mathcal{H}, \mathbf{c}) \to (\mathcal{H}', \mathbf{c}')$ is called an $A_\infty$ morphism.

Suppose that $(\mathcal{H}, \omega, \mathbf{c})$ and $(\mathcal{H}', \omega', \mathbf{c}')$ are cyclic $A_\infty$ algebras and there exists an $A_\infty$ morphism $\mathcal{F} : (\mathcal{H}, \mathbf{c}) \to (\mathcal{H}', \mathbf{c}')$. $\mathcal{F}$ is called cyclic $A_\infty$ morphism when

$$\omega'(f_1(o), f_1(o')) = \omega(o, o'), \tag{3.23}$$

for any $o, o' \in \mathcal{H}$ and for fixed $n \geq 3$,

$$\sum_{k=1}^{n-1} \omega'(f_k(o_1, \dots, o_k), f_{n-k}(o_{k+1}, \dots, o_n)) = 0. \tag{3.24}$$

### 3.2 EKS action

Erler, Konopka and Sachs proposed an open superstring field theory based on $A_\infty$ algebra in the small Hilbert space[6]. We review the construction of EKS action.

Let $\mathcal{H}_S$ be small string field space in NS sector. An open superstring field $\Psi \in \mathcal{H}_S$ has ghost number 1 and picture number $-1$, and

$$\eta\Psi = 0 \tag{3.25}$$

where $\eta$ is the zero mode of $\eta$ field. $\Psi$ has ghost number 1 and picture number $-1$. If $\eta\Phi$ does not vanish, $\Phi$ belongs to the large Hilbert space $\mathcal{H}_L$. The degree of this string field is defined as

$$\deg \Psi = \epsilon(\Psi) + 1 = 0 \pmod 2 \tag{3.26}$$

where $\epsilon(\Psi)$ is the Grassmann number. Then, the small Hilbert space is a $\mathbb{Z}$-graded vector space. The inner product of the small Hilbert space $\langle\ ,\ \rangle_S$ is provided by the BPZ inner product. This inner product does not vanish only when the total ghost number in the inner product is 3 and the total picture number is $-2$.

The EKS action takes the form

$$\begin{aligned} S =& \frac{1}{2}\langle\Psi, Q\Psi\rangle_S + \frac{1}{3}\langle\Psi, M_2(\Psi,\Psi)\rangle_S + \frac{1}{4}\langle\Psi, M_n(\Psi,\Psi,\Psi)\rangle_S + \cdots \\ =& \sum_{n=1}^{\infty} \frac{1}{1+n}\langle\Psi, M_n(\Psi,\ldots,\Psi)\rangle_S \end{aligned} \tag{3.27}$$

$M_1 = Q$ is the BRST operator. This is nilpotent and anticommutes with $\eta$ i.e.

$$Q^2 = 0, \tag{3.28}$$

$$[Q,\eta] = Q\eta + \eta Q = 0. \tag{3.29}$$

$M_2, M_3, \ldots$ are multi-string products of odd degree. The products are in the small Hilbert space

$$[\boldsymbol{\eta}, \mathbf{M}_n] = 0 \tag{3.30}$$

where $\boldsymbol{\eta}$ is an operator

$$\begin{aligned} \boldsymbol{\eta}(\Psi_1 \otimes \Psi_2 \otimes \cdots \otimes \Psi_n) =& \eta\Psi_1 \otimes \Psi_2 \otimes \cdots \otimes \Psi_n + \Psi_1 \otimes \eta\Psi_2 \otimes \cdots \otimes \Psi_n \\ &+ \cdots + \Psi_1 \otimes \Psi_2 \otimes \cdots \otimes \eta\Psi_n, \end{aligned} \tag{3.31}$$

$\Psi_k$ is in the small or large Hilbert space. The product $M_{n+1}$ carries picture number $n$ and ghost number $1-n$. We review how to construct these multi-string products in the next subsection. The coderivation $\mathbf{M}$ is

$$\mathbf{M} = \sum_{n=1}^{\infty} \mathbf{M}_n \tag{3.32}$$

An symplectic structure $\omega_S : \mathcal{H}_S \otimes \mathcal{H}_S \to \mathbb{C}$ is defined as

$$\omega_S(A, B) = (-)^{\deg A}\langle A, B\rangle_S. \tag{3.33}$$

This is degree graded antisymmetric

$$\omega_S(A, B) = -(-)^{\deg A \deg B}\omega_S(B, A). \tag{3.34}$$

Then $(\mathcal{H}_S, \omega_S, \mathbf{M})$ is a cyclic $A_\infty$ algebra, i.e.

$$(\mathbf{M})^2 = 0 \tag{3.35}$$

and

$$\omega_S(\Psi_1, M_n(\Psi_2, \ldots, \Psi_{n+1})) = -(-)^{\deg(M_n)\deg\Psi_1}\omega_S(M_n(\Psi_1, \ldots, \Psi_n), \Psi_{n+1}). \tag{3.36}$$

The equation of motion is

$$\pi_1 \mathbf{M}\frac{1}{1-\Psi} = 0. \tag{3.37}$$

### 3.3 Construction of Multi-string products

The string product $\mathbf{M}$ is determined by recursive equations and can be written by BRST operator $Q$ and a cohomomorphism $\mathbf{G}$

$$\mathbf{M} = \mathbf{G}^{-1}\mathbf{Q}\mathbf{G}. \tag{3.38}$$

Note that $\mathbf{G}$ is in the large Hilbert space $\mathcal{H}_L$

$$[\boldsymbol{\eta}, \mathbf{G}] \neq 0. \tag{3.39}$$

The BPZ inner product in the large Hilbert space $\langle\ ,\ \rangle$ is

$$\langle A, B\rangle_S = \langle \xi A, B\rangle. \tag{3.40}$$

This large Hilbert space inner product does not vanish only on states whose ghost number adds to 2 and picture number adds to $-1$. $\xi$ is an operator constructed from $\xi$ ghost, which has ghost number $-1$ and picture number 1

$$\xi = \oint_{|z|=1}\frac{dz}{2\pi i}f(z)\xi(z). \tag{3.41}$$

$\xi$ is BPZ even in the large Hilbert space and

$$[\eta, \xi] = g_0. \tag{3.42}$$

where $g_0$ is the open string coupling constant. We set the coupling constant $g_0$ to 1. The symplectic structure $\omega_L : \mathcal{H}_L \otimes \mathcal{H}_L \to \mathbb{C}$ is defined as

$$\omega_L(A, B) = (-)^{\deg A} \langle A, B \rangle \tag{3.43}$$

The products $M_n$ are defined by a set of recursive equations. We introduce three products. $M_{n+1}$ is product whose degree is odd and picture number is $n$. $m_{n+2}$ is called bare product whose degree is odd and picture number is $n$. $\mu_{n+2}$ is gauge product whose degree is even and picture number is $n+1$. The products $M_{n+1}$ start with $M_1 = Q$. The bare products $m_{n+2}$ start with $m_2$ which is the star product [1] with a sign factor

$$m_2(A, B) = (-)^{\deg A} A * B. \tag{3.44}$$

The gauge products $\mu_{n+2}$ also start with 2-string multiplication. The recursive equations for these products are described by promoting these products to coderivations and defining generating functions

$$\mathbf{M}(t) = \sum_{n=0}^{\infty} t^n \mathbf{M}_{n+1}, \tag{3.45}$$

$$\mathbf{m}(t) = \sum_{n=0}^{\infty} t^n \mathbf{m}_{n+2}, \tag{3.46}$$

$$\boldsymbol{\mu}(t) = \sum_{n=0}^{\infty} t^n \boldsymbol{\mu}_{n+2}. \tag{3.47}$$

The generating functions satisfy

$$\frac{d}{dt}\mathbf{M}(t) = [\mathbf{M}(t), \boldsymbol{\mu}(t)], \tag{3.48}$$

$$\frac{d}{dt}\mathbf{m}(t) = [\mathbf{m}(t), \boldsymbol{\mu}(t)], \tag{3.49}$$

$$[\boldsymbol{\eta}, \boldsymbol{\mu}(t)] = \mathbf{m}(t). \tag{3.50}$$

Expanding these relations in powers of $t$, the $n$-th relations are

$$\mathbf{M}_{n+2} = \frac{1}{n+1} \sum_{k=0}^{n} [\mathbf{M}_{n-k+1}, \boldsymbol{\mu}_{k+2}], \tag{3.51}$$

$$\mathbf{m}_{n+3} = \frac{1}{n+1} \sum_{k=0}^{n} [\mathbf{m}_{n-k+2}, \boldsymbol{\mu}_{k+2}], \tag{3.52}$$

$$[\boldsymbol{\eta}, \boldsymbol{\mu}_{n+2}] = \mathbf{m}_{n+2}. \tag{3.53}$$

These equations define higher products in terms of commutators of lower products. However, the solution of the eq(3.50) is not unique since one can add an $\eta$-exact term to $\mu_{n+2}$. We make a specific choice to make $(\mathcal{H}_S, \mathbf{M})$ an $A_\infty$ algebra

$$\mu_{n+2} = \frac{1}{n+3}\left(\xi m_{n+2} - m_{n+2}\left(\sum_{k=0}^{n+1} \mathbb{I}^{\otimes k} \otimes \xi \otimes \mathbb{I}^{\otimes n-k+1}\right)\right). \tag{3.54}$$

$\boldsymbol{\mu}$ is cyclic on $\omega_L$.

Consider the cohomomorphism

$$\begin{aligned} \mathbf{G}(t) =& \mathcal{P}\left[\exp\left(\int_0^t dt_1 \boldsymbol{\mu}(t_1)\right)\right] && (3.55) \\ \equiv& \mathbb{I} + \int_0^t dt \boldsymbol{\mu}(t) + \sum_{n=2}^{\infty}\left(\int_0^t dt_1 \boldsymbol{\mu}(t_1)\right)\left(\int_{t_1}^t dt_2 \boldsymbol{\mu}(t_2)\right)\cdots\left(\int_{t_{n-1}}^t dt_n \boldsymbol{\mu}(t_n)\right). && (3.56) \end{aligned}$$

We can express the generating functions with this cohomomorphism

$$\boldsymbol{\mu}(t) = \mathbf{G}(t)^{-1}\frac{d}{dt}\mathbf{G}(t), \tag{3.57}$$

$$\mathbf{M}(t) = \mathbf{G}(t)^{-1}\mathbf{Q}\mathbf{G}(t), \tag{3.58}$$

$$\mathbf{m}(t) = \mathbf{G}(t)^{-1}\mathbf{m}_2\mathbf{G}(t). \tag{3.59}$$

where

$$\begin{aligned} \mathbf{G}(t)^{-1} =& \mathcal{P}^{-1}\left[\exp\left(-\int_0^t dt_1 \boldsymbol{\mu}(t_1)\right)\right] && (3.60) \\ \equiv& \mathbb{I} - \int_0^t dt \boldsymbol{\mu}(t) + \sum_{n=2}^{\infty}\left(-\int_0^t dt_1 \boldsymbol{\mu}(t_1)\right)\left(-\int_0^{t_1} dt_2 \boldsymbol{\mu}(t_2)\right)\cdots\left(-\int_0^{t_{n-1}} dt_n \boldsymbol{\mu}(t_n)\right). && (3.61) \end{aligned}$$

The coderivation $\mathbf{M}$ is provided by

$$\mathbf{M} = \mathbf{G}^{-1}\mathbf{Q}\mathbf{G} \tag{3.62}$$

where

$$\mathbf{G} \equiv \mathbf{G}(1). \tag{3.63}$$

# 4 Gauge invariance of EKS action

There are gauge invariances in EKS action which is defined in the small Hilbert space. These gauge degrees of freedom can be fixed with the BV formalism. When we embed EKS action into the large Hilbert space, the gauge invariance also changes. We review the gauge invariances in the small and large Hilbert space.

## 4.1 String fields and CFT basis

First, we see the composition of string field theory. A string field $\Psi_{s;g,p}$ consists of a set of space-time fields $A^r_{s,p}$ and a set of CFT basis $B^r_{g,p}$.

$$\Psi_{s;g,p} \equiv \sum_r A^r_{s,p} B^r_{g,p} \tag{4.1}$$

$A^r_{s,p}$ has space-time ghost number $s$. Its $p$ is just a label. $B^r_{g,p}$ has world sheet ghost number $g$ and picture number $p$. The $r$-label distinguishes different states which have same ghost number and picture number.

The Grassmann parity of the string field $G(\Psi_{s;g,p})$ is

$$G(\Psi_{s;g,p}) = s + g \tag{4.2}$$

and the Grassmann parity of the space-time field is

$$G(A^s_{s,p}) = s. \tag{4.3}$$

We define the degree of the string field by

$$\deg(\Psi_{s;g,p}) = s + g - 1. \tag{4.4}$$

Let $\{Y^r_{g,p}\}$ be a basis of the small Hilbert space such that

$$\langle Y^{r^C}_{g,p}, Y^s_{h,q}\rangle_S = \delta_{r,s}\delta_{g+h,3}\delta_{p+q,-2} \tag{4.5}$$

This inner product is BPZ inner product of the small Hilbert space. This satisfies

$$\langle Y^r_{g,p}, Y^s_{h,q}\rangle_S = \langle Y^s_{h,q}, Y^r_{g,p}\rangle_S. \tag{4.6}$$

We define the dual basis $\{Y^{r*}_{g,p}\}$ by

$$Y^{r*}_{3-g,-2-p} \equiv Y^{r^C}_{g,p} \tag{4.7}$$

which satisfy the usual orthogonal relation

$$\langle Y^r_{g,p}, Y^{s*}_{h,q}\rangle_S = \langle Y^{s*}_{h,q}, Y^r_{g,p}\rangle_S = \delta_{r,s}\delta_{g,h}\delta_{p,q}, \tag{4.8}$$

$Y^{r*}_{3-g,-2-p}$ has world-sheet ghost number $g$ and picture number $p$. The basis has completeness

$$\sum_{t,f,u} \langle Y^r_{g,p}, Y^{t*}_{f,u}\rangle_S \langle Y^t_{f,u}, Y^s_{h,q}\rangle_S = \langle Y^r_{g,p}, Y^s_{h,q}\rangle_S. \tag{4.9}$$

The inner products of string fields are defined by

$$\left\langle \sum_s A^s_{a,p} Y^s_{g,p}, \sum_t A^t_{b,q} Y^t_{h,q} \right\rangle_S \equiv \sum_{s,t} (-)^{a+b(g+1)} A^s_{a,p} A^t_{b,q} \langle Y^s_{g,p}, Y^t_{h,q} \rangle_S \tag{4.10}$$

$$= \sum_{s,t} (-)^{a(g+h)+bh} \langle Y^s_{g,p}, Y^t_{h,q} \rangle_S A^s_{a,p} A^t_{b,q}. \tag{4.11}$$

We consider the basis of the large Hilbert space next. Let $\{Z^r_{g,p}\}$ be a basis of the large Hilbert space such that

$$\langle Z^{r^C}_{g,p}, Z^s_{h,q} \rangle = (-)^{hq} \delta_{r,s} \delta_{g+h,2} \delta_{p+q,-1} \tag{4.12}$$

This inner product is BPZ inner product of the large Hilbert space. This has graded symmetry

$$\langle Z^r_{g,p}, Z^s_{h,q} \rangle = (-)^g \langle Z^s_{h,q}, Z^r_{g,p} \rangle. \tag{4.13}$$

We define the dual basis $\{Z^{r*}_{g,p}\}$ by

$$Z^{r*}_{2-g,-1-p} \equiv (-)^{gp} Z^{r^C}_{g,p} \tag{4.14}$$

which satisfy the usual orthogonal relation

$$\langle Z^r_{g,p}, Z^{s*}_{h,q} \rangle = \delta_{r,s} \delta_{g,h} \delta_{p,q}, \quad \langle Z^{r*}_{g,p}, Z^s_{h,q} \rangle = (-)^g \delta_{r,s} \delta_{g,h} \delta_{p,q}. \tag{4.15}$$

$Z^{r*}_{2-g,-1-p}$ has world-sheet ghost number $g$ and picture number $p$. The orthogonal relations of the complete basis provides simple decompositions of the unit.

$$\sum_{t,f,u} \langle Z^r_{g,p}, Z^{t*}_{f,u} \rangle \langle Z^t_{f,u}, Z^s_{h,q} \rangle = \langle Z^r_{g,p}, Z^s_{h,q} \rangle \tag{4.16}$$

$$\sum_{t,f,u} \langle Z^r_{g,p}, Z^t_{f,u} \rangle \langle Z^{t*}_{f,u}, Z^s_{h,q} \rangle = (-)^g \langle Z^r_{g,p}, Z^s_{h,q} \rangle \tag{4.17}$$

The inner products of string fields are defined by

$$\left\langle \sum_s A^s_{a,p} Z^s_{g,p}, \sum_t A^t_{b,q} Z^t_{h,q} \right\rangle \equiv \sum_{s,t} (-)^{bg} A^s_{a,p} A^t_{b,q} \langle Z^s_{g,p}, Z^t_{h,q} \rangle \tag{4.18}$$

$$= \sum_{s,t} (-)^{a(g+h)+bh} \langle Z^s_{g,p}, Z^t_{h,q} \rangle A^s_{a,p} A^t_{b,q}. \tag{4.19}$$

## 4.2 The classical BV action in the small Hilbert space

The EKS action (3.27) is $\infty$-th stage reducible. We review this fact and construct the master action for the $A_\infty$ superstring field theory in the small Hilbert in this subsection

The string field $\Psi$ has world sheet ghost number 1 and picture number $-1$

$$\Psi = \Psi_{1,-1} = \sum_r \psi^r_{0,-1} |Y^r_{1,-1}\rangle_S. \tag{4.20}$$

The gauge transformation of EKS action (3.27) is

$$\delta_1 \Psi_{1,-1} = \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{0,-1} \otimes \frac{1}{1-\Psi_{1,-1}}. \tag{4.21}$$

where

$$\Lambda_{0,-1} = \sum_r \lambda^r_0 |Y^r_{0,-1}\rangle_S, \tag{4.22}$$

$\lambda^r_0$ is gauge parameters, which are Grassmann even. The gauge transformation of EKS action is

$$\delta_1 S = \left\langle \delta_1 \Psi_{1,-1}, \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1.-1}} \right\rangle_S = -\left\langle \Lambda_{0,-1}, \pi_1 \mathbf{M}^2 \frac{1}{1-\Psi_{1,-1}} \right\rangle_S = 0. \tag{4.23}$$

Consider a transformation of this gauge parameter $\delta_2 \lambda^r_0$ such that

$$\delta_2 \Lambda_{0,-1} = \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{-1,-1} \otimes \frac{1}{1-\Psi_{1,-1}}. \tag{4.24}$$

The transformation of the gauge transformation is

$$\begin{aligned}
\delta_2\delta_1\Psi =& \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \pi_1 \left( \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{-1,-1} \otimes \frac{1}{1-\Psi_{1,-1}} \right) \otimes \frac{1}{1-\Psi_{1,-1}} \\
=& -\pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \left( \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \right) \otimes \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{1,-1} \otimes \frac{1}{1-\Psi_{1,-1}} \\
&+ \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{-1,-1} \otimes \frac{1}{1-\Psi_{1,-1}} \otimes \left( \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \right) \otimes \frac{1}{1-\Psi_{1,-1}}
\end{aligned} \tag{4.25}$$

This vanishes on-shell. Then, $\delta_2$ is level one gauge transformation. Similarly, there are level $g$ gauge transformations

$$\delta_{g+1} \Lambda_{-(g-1),-1} = \pi_1 \mathbf{M} \frac{1}{1-\Psi_{1,-1}} \otimes \Lambda_{-g,-1} \otimes \frac{1}{1-\Psi_{1,-1}} \tag{4.26}$$

where $\Lambda_{-g,-1} = \sum_r \lambda^r_g |Y^r_{-g,-1}\rangle_S$. Therefore, EKS action is $\infty$-th stage reducible.

The set of fields $A^S$ in the BV formalism is

$$A^S = \{\psi^r_{g,-1} | g \geq 0,\ r \in \mathbb{N}\}. \tag{4.27}$$

Fields $\psi^r_{g,p}$ $(g \geq 1)$ are ghost fields corresponding to the gauge parameter $\lambda^r_{g-1}$. Then string fields $\Psi_{1-g,-1}$ are

$$\Psi_{1-g,-1} = \sum_r \psi^r_{g,-1} |Y^r_{1-g,-1}\rangle_S. \tag{4.28}$$

$\Psi_{1-g,-1}$ carries space-time ghost number $g$, world sheet ghost number $1-g$ and picture number $-1$. The anti string field $\Psi^*_{2+g,-1}$ $(g \geq 0)$ is defined as

$$\Psi^*_{2+g,-1} = \sum_r (\psi^r_{g,-1})^* |Y^{r^C}_{2+g,-1}\rangle_S, \tag{4.29}$$

where $(\psi^r_{g,-1})^*$ is the antifield corresponding to $\psi^r_{g,-1}$. The minimal set of the fields and the antifields is

$$\begin{aligned}\mathcal{A}^S_{\min} =& A^S \oplus (A^S)^* \\ =& \{\psi^r_{g,-1}, (\psi^r_{g,-1})^* | g \geq 0,\ r \in \mathbb{N}\}.\end{aligned} \tag{4.30}$$

The definition of antibracket on this set is

$$\{F, G\} = \sum_{g=0}^{\infty} \sum_r \left( \frac{\partial_r F}{\partial \psi^r_{g,-1}} \frac{\partial_l G}{\partial (\psi^r_{g,-1})^*} - \frac{\partial_r F}{\partial (\psi^r_{g,-1})^*} \frac{\partial_l G}{\partial \psi^r_{g,-1}} \right). \tag{4.31}$$

$\Psi^*_{2+g,-1}$ has space-time ghost number $-1-g$, world sheet ghost number $2+g$ and picture number $-1$. String field and anti string field have the same Grassmann parity. We can define string field $\Psi'$

$$\Psi' = \sum_{g=0}^{\infty} \Psi_{1-g,-1} + \sum_{g=0}^{\infty} \Psi^*_{2+g,-1}. \tag{4.32}$$

The master action is

$$S^S_{\mathrm{BV}} = \sum_{n=1}^{\infty} \frac{1}{n+1} \langle \Psi', M_n(\Psi', \dots, \Psi') \rangle_S. \tag{4.33}$$

From the completeness of the basis (4.9), $\frac{1}{2}\{S^S_{\mathrm{BV}},\ S^S_{\mathrm{BV}}\}$ is

$$\begin{aligned}(\mathrm{LHS}) =& \left\langle \pi_1 \mathbf{M} \frac{1}{1-\Psi'},\ \pi_1 \mathbf{M} \frac{1}{1-\Psi'} \right\rangle_S \\ =& \sum_{n=1}^{\infty} \sum_{k=1}^{n} \langle M_k(\Psi', \dots, \Psi'),\ M_{n-k+1}((\Psi', \dots, \Psi') \rangle_S \\ =& \sum_{n=1}^{\infty} \frac{2}{n+1} \left\langle \Psi,\ \pi_1 \mathbf{M}^2 \pi_n \frac{1}{1-\Psi'} \right\rangle_S = 0.\end{aligned} \tag{4.34}$$

Then, the action (4.33) satisfies the master equation

$$\{S_{\rm BV}^S,\ S_{\rm BV}^S\} = \sum_{g=0}^{\infty}\sum_r 2\frac{\partial_r S_{\rm BV}^S}{\partial\psi_{g,-1}^r}\frac{\partial_l S_{\rm BV}^S}{\partial(\psi_{g,-1}^r)^*} = 0. \tag{4.35}$$

Actually, the $A_\infty$ structure implies that the action satisfy the BV master equation [21].

## 4.3 Embedding EKS action into the large Hilbert space

Although the string product $\mathbf{M}$ is constructed from $Q$ in the large Hilbert space, EKS action is defined in the small Hilbert space. We consider string fields $\Phi \in \mathcal{H}_L$ related to $\Psi \in \mathcal{H}_S$ by partial gauge fixing

$$\Psi = \eta\Phi \tag{4.36}$$

and the action which is not restricted to the small Hilbert space [22][23].

The $A_\infty$ type action in the large Hilbert space is

$$\begin{aligned} S =& \sum_{n=1}^{\infty}\langle \xi\eta\Phi, M_n(\eta\Phi,\dots,\eta\Phi)\rangle \\ =& \sum_{n=1}^{\infty}\langle \Phi, M_n(\eta\Phi,\dots,\eta\Phi)\rangle. \end{aligned} \tag{4.37}$$

The ghost number of $\Phi$ is 0 and the picture number is also 0.

$$\Phi = \Phi_{0,0} = \sum_r \phi_{0,0}^r |Z_{0,0}^r\rangle \tag{4.38}$$

Gauge transformation of $\Phi$ is

$$\delta_1\Phi_{0,0} = \pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-1,0}\otimes\frac{1}{1-\eta\Phi_{0,0}} + \eta\Lambda_{-1,1} \tag{4.39}$$

where

$$\Lambda_{-1,0} = \sum_r \lambda_{-1,0}^r|Z_{-1,0}^r\rangle,\quad \Lambda_{-1,1} = \sum_r \lambda_{-1,-1}^r|Z_{-1,1}^r\rangle. \tag{4.40}$$

Consider a transformation of gauge parameters $\delta_2\lambda_{-1,0}^r, \delta_2\lambda_{-1,1}^r$ such that

$$\delta_2\Lambda_{-1,0} = \pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-2,0}\otimes\frac{1}{1-\eta\Phi_{0,0}} + \eta\Lambda_{-2,1}, \tag{4.41}$$

$$\delta_2\Lambda_{-1,1} = \pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-2,1}\otimes\frac{1}{1-\eta\Phi_{0,0}} + \eta\Lambda_{-2,2}. \tag{4.42}$$

Then,

$$\begin{aligned}\delta_2\delta_1\Phi_{0,0} =&\pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\pi_1\left(\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-2,0}\otimes\frac{1}{1-\eta\Phi_{0,0}}\right)\otimes\frac{1}{1-\eta\Phi_{0,0}}\\ &+\pi_1[\mathbf{M},\boldsymbol{\eta}]\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-2,1}\otimes\frac{1}{1-\eta\Phi_{0,0}}+\eta^2\Lambda_{-2.2}.\end{aligned}\tag{4.43}$$

The second and third terms are zero because $\eta^2 = 0$ and $\mathbf{M}$ is in the small Hilbert space. The first vanishes on-shell. Therefore, $\delta_2$ is level one gauge transformation. We obtain higher level gauge transformation in the same way

$$\delta_{g+1}\begin{pmatrix}\Lambda_{-g,0}\\ \Lambda_{-g,1}\\ \vdots\\ \Lambda_{-g,g}\end{pmatrix}=\begin{pmatrix}\pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-(g+1),0}\otimes\frac{1}{1-\eta\Phi_{0,0}}+\eta\Lambda_{-(g+1),1}\\ \pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-(g+1),1}\otimes\frac{1}{1-\eta\Phi_{0,0}}+\eta\Lambda_{-(g+1),2}\\ \vdots\\ \pi_1\mathbf{M}\frac{1}{1-\eta\Phi_{0,0}}\otimes\Lambda_{-(g+1),g}\otimes\frac{1}{1-\eta\Phi_{0,0}}+\eta\Lambda_{-(g+1),g+1}\end{pmatrix}.\tag{4.44}$$

Then the set of field $A^L$ in the BV formalism is

$$A^L=\{\phi^r_{g,p}|g\geq 0,\ 0\leq p\leq g,\ r\in\mathbb{N}\}.\tag{4.45}$$

The minimal set of fields and antifields is

$$\mathcal{A}^L_{\min}=\{\phi^r_{g,p},(\phi^r_{g,p})^*|g\geq 0,\ 0\leq p\leq g,\ r\in\mathbb{N}\}.\tag{4.46}$$

The definition of antibracket on this set is

$$\{F,G\}=\sum_{g\geq 0}\sum_{0\leq p\leq g}\sum_r\left(\frac{\partial_r F}{\partial\phi^r_{g,p}}\frac{\partial_l G}{\partial(\phi^r_{g,p})^*}-\frac{\partial_r F}{\partial(\phi^r_{g,p})^*}\frac{\partial_l G}{\partial\phi^r_{g,p}}\right).\tag{4.47}$$

String fields $\Phi_{-g,p}$ are defined as

$$\Phi_{-g,p}\equiv\sum_r\phi^r_{g,p}|Z^r_{-g,p}\rangle\tag{4.48}$$

$\phi^r_{g,p}$ has space-time ghost number $g$ . Anti string fields $(\Phi_{-g,p})^*$ are conventionally defined [16]

$$(\Phi_{-g,p})^*=\Phi^*_{2+g,-1-p}\equiv\sum_r(\phi^r_{g,p})^*|Z^{r*}_{-g,p}\rangle=\sum_r(-)^{g(p+1)}\phi^{r*}_{-1-g,-1-p}|Z^{r^C}_{2+g,-1-p}\rangle.\tag{4.49}$$

where $(\phi^r_{g,p})^*=\phi^{r*}_{-g-1,-1-p}$ is the antifield of $\phi^r_{g,p}$ and carries ghost number $-g-1$.

## 5 Naive construction of BV action in the large Hilbert space

We construct string field BV action in large Hilbert space from EKS action. The simplest way is expanding the BV action with antifield number and solving the master equation at each antifield numbers. However, it has critical defect. We explain the problem in this section.

### 5.1 Antifield number expansion

We introduce antifield number. In principle, we can solve the classical master equation systematically by using this. The antifield numbers are assigned in the string field theory according to the following rule [14][15].

1. All the fields carry no antifield number.
2. The antifield of the field in the original action $(\phi_{0,0})^* = \phi^*_{-1,-1}$ carries antifield number one.
3. The antifield of the $g$-th ghosts $(\phi_{g,p})^* = \phi^*_{-1-g,-1-p}$ carry antifield number $g+1$.

We expand the BV action by antifield number

$$S = \sum_{n=0}^{\infty} S^{(n)} \tag{5.1}$$

where $S^{(n)}$ $(n > 0)$ denotes the sum of the all terms which have antifield number $n$, with $S^{(0)}$ coinciding with the original action. The antifield number of a term is defined as the total of the antifield numbers of the fields which the term includes. Therefore, antifield numbers are assigned as

$$\mathrm{afn}[\Phi^*_{2+g,-p}] = 1+g, \quad \mathrm{afn}[\Phi_{-g,p}] = 0, \tag{5.2}$$

$$\mathrm{afn}\left[\frac{\partial S^{(a+1)}}{\partial(\phi_{g,p})^*}\right] = a-g, \quad \mathrm{afn}\left[\frac{\partial S^{(a)}}{\partial\phi_{g,p}}\right] = a. \tag{5.3}$$

The master equation can be decomposed into its sub-equations by their antifield numbers. By solving each equations, we can determine $S^{(n)}$ one by one. In some theories, only a finite number of $S^{(n)}$ are nonzero. In this case, we can obtain the master action completely. Open superstring field theory has infinite number of $S^{(n)}$ which are nonzero, but $S^{(n)}$ are determined systematically.

The master equation is

$$\{S,S\} = \sum_{r,g,p}\left(\frac{\partial_r S}{\partial\phi^r_{g,p}}\frac{\partial_l S}{\partial(\phi^r_{g,p})^*} - \frac{\partial_r S}{\partial(\phi^r_{g,p})^*}\frac{\partial_l S}{\partial\phi^r_{g,p}}\right) = 2\sum_{r,g,p}\frac{\partial_r S}{\partial\phi^r_{g,p}}\frac{\partial_l S}{\partial(\phi^r_{g,p})^*} = 0. \tag{5.4}$$

We write

$$\frac{\partial_r F}{\partial \phi^r_{g,p}} = (-)^g \left\langle \frac{\partial_r F}{\partial \Phi_{-g,p}}, Z^r_{-g,p} \right\rangle, \quad \frac{\partial_r F}{\partial (\phi^r_{g,p})^*} = \left\langle \frac{\partial_r F}{\partial (\Phi_{-g,p})^*}, Z^{r*}_{-g,p} \right\rangle, \tag{5.5}$$

$$\frac{\partial_l F}{\partial \phi^r_{g,p}} = \left\langle Z^r_{-g,p}, \frac{\partial_l F}{\partial \Phi_{-g,p}} \right\rangle, \quad \frac{\partial_l F}{\partial (\phi^r_{g,p})^*} = \left\langle Z^{r*}_{-g,p}, \frac{\partial_r F}{\partial (\Phi_{-g,p})^*} \right\rangle, \tag{5.6}$$

so that [16]

$$\sum_r \frac{\partial_r F}{\partial \phi^r_{g,p}} \frac{\partial_l G}{\partial (\phi^r_{g,p})^*} = \left\langle \frac{\partial_r F}{\partial \Phi_{-g,p}}, \frac{\partial_l G}{\partial (\Phi_{-g,p})^*} \right\rangle, \tag{5.7}$$

$$\sum_r \frac{\partial_r F}{\partial (\phi^r_{g,p})^*} \frac{\partial_l G}{\partial \phi^r_{g,p}} = \left\langle \frac{\partial_r F}{\partial (\Phi_{-g,p})^*}, \frac{\partial_l G}{\partial \Phi_{-g,p}} \right\rangle. \tag{5.8}$$

The antifield number $a$ part of the master equation is given by

$$\sum_r \sum_{s=0}^{a} \sum_{g=0}^{s} \sum_{p=0}^{g} \frac{\partial_r S^{(a-s+g)}}{\partial \phi^r_{g,p}} \frac{\partial_l S^{(1+s)}}{\partial (\phi^r_{g,p})^*} = \sum_{s=0}^{a} \sum_{g=0}^{s} \sum_{p=0}^{g} \left\langle \frac{\partial_r S^{(a-s+g)}}{\partial \Phi^r_{-g,p}}, \frac{\partial_l S^{(1+s)}}{\partial (\Phi^r_{-g,p})^*} \right\rangle = 0. \tag{5.9}$$

## 5.2 Naive BV approach

In the naive BV approach, we require the following three properties:

1. Regarding states, the master action consists of only the minimal set of fields and antifields $\mathcal{A}_{\rm min}$ given by
$$\mathcal{A}^L_{\rm min} = \{\phi^r_{g,p},\ (\phi^r_{g,p})^* | 0 \le g,\ 0 \le p \le g,\ r \in \mathbb{N}\}. \tag{5.10}$$

2. Regarding operators and products, the master action consists of the operators and products which appear in the action and its gauge invariance, namely, $\mathbf{M}, \boldsymbol{\eta}$, and the large BPZ inner product only.

3. In the master action, effective change of property 1 or 2 does not arise, and thus explicit insertions of $\xi$ of $\mathbf{M}^{-1}$ are not included.

We carry out the calculation of antifield number expansion in accordance with these requests.

First, the antifield number 0 part is

$$\left\langle \frac{\partial_r S^{(0)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle = 0. \tag{5.11}$$

Here,

$$S^{(0)} = \sum_{n=0}^{\infty} \left\langle \Phi_{0,0}, \frac{1}{1+n} M_n(\eta\Phi_{0,0}, \dots) \right\rangle . \tag{5.12}$$

Then, the equation is

$$\left\langle M, \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle = 0 \tag{5.13}$$

where

$$M = \pi_1 \mathbf{M} \frac{1}{1-\eta\Phi_{0,0}} . \tag{5.14}$$

By using $\mathbf{M}^2 = 0,\ [\mathbf{M}, \boldsymbol{\eta}],\ \boldsymbol{\eta}^2 = 0$, we obtain a solution

$$\frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} = M((\Phi_{-1,0})) + \eta\Phi_{-1,1}, \tag{5.15}$$

$$S^{(1)} = \left\langle \Phi^*_{2,-1}, M((\Phi_{-1,0})) + \eta\Phi_{-1,1} \right\rangle \tag{5.16}$$

where

$$M((A)) = \pi_1 \mathbf{M} \frac{1}{1-\eta\Phi_{0,0}} \otimes A \otimes \frac{1}{1-\eta\Phi_{0,0}} . \tag{5.17}$$

Next, the antifield number 1 part is

$$\left\langle \frac{\partial_r S^{(0)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{-1,0}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} \right\rangle + \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{-1,1}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} \right\rangle = 0. \tag{5.18}$$

Here

$$\frac{\partial_r S^{(1)}}{\partial \Phi_{0,0}} = -\eta M((\Phi_{-1,0}), (\Phi^*_{2,-1})), \tag{5.19}$$

$$\frac{\partial_r S^{(1)}}{\partial \Phi_{-1,0}} = M((\Phi^*_{2,-1})), \quad \frac{\partial_r S^{(1)}}{\partial \Phi_{-1,1}} = \eta\Phi^*_{2,-1}, \tag{5.20}$$

where

$$\begin{aligned} M((A),(B)) =& \pi_1 \mathbf{M} \frac{1}{1-\eta\Phi_{0,0}} \otimes A \otimes \frac{1}{1-\eta\Phi_{0,0}} \otimes B \otimes \frac{1}{1-\eta\Phi_{0,0}} \\ &+ (-)^{\deg(A)\deg(B)} \pi_1 \mathbf{M} \frac{1}{1-\eta\Phi_{0,0}} \otimes B \otimes \frac{1}{1-\eta\Phi_{0,0}} A \otimes \frac{1}{1-\eta\Phi_{0,0}} . \end{aligned} \tag{5.21}$$

$M((A),(B),(C),\dots)$ are defined in the same way, i.e.

$$M((A),(B),(C),\dots) = \pi_1 \mathbf{M} \frac{1}{1-\eta\Phi_{0,0}} \otimes A \otimes \frac{1}{1-\eta\Phi_{0,0}} \otimes B \otimes \cdots \otimes \frac{1}{1-\eta\Phi_{0,0}} + \cdots \tag{5.22}$$

which satisfies $M(\dots,(D),(E)\dots) = (-)^{\deg(D)\deg(E)}M(\dots,(E),(D),\dots)$. $M(\dots,M(\dots),\dots)$ represents $M(\dots,(M(\dots)),\dots)$.

We substitute eq(5.19),eq(5.20) in the left hand side of the eq(5.18)

$$
\begin{aligned}
\text{(LHS)} = &\left\langle M, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle \\
&+ \langle -\eta M((\Phi_{-1,0}),(\Phi^*_{2,-1})), M((\Phi_{-1,0})) + \eta\Phi_{-1,1}\rangle \\
&+ \left\langle M((\Phi^*_{2,-1})), \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} \right\rangle \\
&+ \left\langle \eta\Phi^*_{2,-1}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} \right\rangle .
\end{aligned} \tag{5.23}
$$

The $A_\infty$ relations are

$$M(M((A))) + M(M,(A)) = 0, \tag{5.24}$$

$$M(M((A),(B))) + M(M,(A),(B)) + M(M((A)),(B)) + (-)^{\deg(A)}M((A),M((B))) = 0. \tag{5.25}$$

By using these relations, we obtain the solution of the eq(5.18),

$$\frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} = \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0})) + M((\Phi_{-2,0})) + \eta\Phi_{-2,1}, \tag{5.26}$$

$$\frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} = -\frac{1}{2}M(M((\Phi_{-1,0})),(\Phi_{-1,0})) + M((\Phi_{-2,1})) + \eta\Phi_{-2,2}, \tag{5.27}$$

$$\frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} = \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}),(\Phi_{-1,0})) + M((\Phi^*_{2,-1}),(\Phi_{-2,0})). \tag{5.28}$$

Then, $S^{(2)}$ is

$$
\begin{aligned}
S^{(2)} = &\left\langle \Phi^*_{3,-1}, \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0})) + M((\Phi_{-2,0})) + \eta\Phi_{-2,1} \right\rangle \\
&+ \left\langle \Phi^*_{3,-2}, -\frac{1}{2}M(M((\Phi_{-1,0})),(\Phi_{-1,0})) + M((\Phi_{-2,1})) + \eta\Phi_{-2,2} \right\rangle \\
&+ \left\langle \Phi^*_{2,-1}, \frac{1}{4}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}),(\Phi_{-1,0})) + \frac{1}{2}M((\Phi^*_{2,-1}),(\Phi_{-2,0})) \right\rangle .
\end{aligned} \tag{5.29}
$$

The antifield number 2 part of the master equation is

$$\left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle + \sum_{p=0}^{1} \left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{-1,p}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1-p}} \right\rangle$$
$$+ \left\langle \frac{\partial_r S^{(0)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}} \right\rangle + \sum_{p=0}^{1} \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{-1,p}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-1-p}} \right\rangle + \sum_{p=0}^{2} \left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{-2,p}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1-p}} \right\rangle = 0. \tag{5.30}$$

Here

$$\begin{aligned}\frac{\partial_r S^{(2)}}{\partial \Phi_{0,0}} =& -\frac{1}{2}\eta M((\Phi^*_{3,-1}),(\eta\Phi_{-1,0}),(\Phi_{-1,0})) - \eta M((\Phi^*_{3,-1}),(\Phi_{-2,0}))\\ &+ \frac{1}{2}\eta M((\Phi^*_{3,-2}), M((\Phi_{-1,0})),(\Phi_{-1,0})) - \frac{1}{2}\eta M((\Phi_{-1,0}), M((\Phi^*_{3,-2}),(\Phi_{-1,0})))\\ &- \eta M((\Phi^*_{3,-2}),(\Phi_{-2,1})) - \frac{1}{2}\eta M((\Phi^*_{2,-1})^2,(\eta\Phi_{-1,0}),(\Phi_{-1,0}))\\ &- \eta M((\Phi^*_{2,-1})^2,(\Phi_{-2,0})),\end{aligned} \tag{5.31}$$

$$\begin{aligned}\frac{\partial_r S^{(2)}}{\partial \Phi_{-1,0}} =& \frac{1}{2} M((\eta\Phi_{-1,0}),(\Phi^*_{3,-1})) - \frac{1}{2}\eta M((\Phi_{-1,0}),(\Phi^*_{3,-1}))\\ &- \frac{1}{2} M(M((\Phi_{-1,0})),(\Phi^*_{3,-2})) + \frac{1}{2} M(M((\Phi_{-1,0}),(\Phi^*_{3,-2})))\\ &+ \frac{1}{2} M((\Phi^*_{2,-1})^2,(\eta\Phi_{-1,0})) - \frac{1}{2}\eta M((\Phi^*_{2,-1})^2,(\Phi_{-1,0})),\end{aligned} \tag{5.32}$$

$$\frac{\partial_r S^{(2)}}{\partial \Phi_{-1,1}} = 0, \tag{5.33}$$

$$\frac{\partial_r S^{(2)}}{\partial \Phi_{-2,0}} = M((\Phi^*_{3,-1})) + M((\Phi^*_{2,-1})^2), \tag{5.34}$$

$$\frac{\partial_r S^{(2)}}{\partial \Phi_{-2,1}} = \eta\Phi^*_{3,-1} + M((\Phi^*_{3,-2})), \tag{5.35}$$

$$\frac{\partial_r S^{(2)}}{\partial \Phi_{-2,2}} = \eta\Phi^*_{3,-2}. \tag{5.36}$$

However, eq(5.30) has no solution. We write details of the calculation hereinafter. We calculate the terms which includes $\Phi^*_{3,-2}$ and three $\Phi_{-1,0}$ in eq(5.30).

$$\begin{aligned}
&\left\langle \frac{1}{2}\eta M((\Phi^*_{3,-2}), M((\Phi_{-1,0})), (\Phi_{-1,0})),\ M((\Phi_{-1,0}))\right\rangle\\
&-\left\langle \frac{1}{2}\eta M((\Phi_{-1,0}), M((\Phi^*_{3,-2}), (\Phi_{-1,0}))),\ M((\Phi_{-1,0}))\right\rangle\\
&-\left\langle \frac{1}{2}M((\Phi^*_{3,-2}), M((\Phi_{-1,0}))),\ \frac{1}{2}M((\eta\Phi_{-1,0}), (\Phi_{-1,0}))\right\rangle\\
&+\left\langle \frac{1}{2}M(M((\Phi^*_{3,-2}), (\Phi_{-1,0}))),\ \frac{1}{2}M((\eta\Phi_{-1,0}), (\Phi_{-1,0}))\right\rangle\\
&+\left\langle M, \frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\right\rangle+\left\langle M((\Phi^*_{3,-2})), \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right\rangle+\left\langle \eta\Phi^*_{3,-2}, \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\right\rangle=0
\end{aligned}\tag{5.37}$$

The terms of the left hand side are

$$\begin{aligned}
(\text{First term}) &= \left\langle \Phi^*_{3,-2},\ -\frac{1}{2}M(M((\Phi_{-1,0})), M((\eta\Phi_{-1,0})), (\Phi_{-1,0}))\right\rangle\\
&= \bigg\langle \Phi^*_{3,-2},\ -\frac{1}{4}M(M((\Phi_{-1,0})), M((\eta\Phi_{-1,0})), (\Phi_{-1,0}))\\
&\qquad -\frac{1}{4}\eta M(M((\Phi_{-1,0}))^2, (\Phi_{-1,0}))\\
&\qquad -\frac{1}{4}M(M((\Phi_{-1,0}))^2, (\eta\Phi_{-1,0}))\bigg\rangle,
\end{aligned}\tag{5.38}$$

$$\begin{aligned}
(\text{Second term}) &= \left\langle \Phi^*_{3-2},\ \frac{1}{2}M(M(M((\eta\Phi_{-1,0})), (\Phi_{-1,0})), (\Phi_{-1,0}))\right\rangle\\
&= \bigg\langle \Phi^*_{3,-2},\ -\frac{1}{4}\eta M(M(M((\Phi_{-1,0})), (\Phi_{-1,0})), (\Phi_{-1,0}))\\
&\qquad -\frac{1}{4}M(M(M((\Phi_{-1,0})), (\Phi_{-1,0})), (\eta\Phi_{-1,0}))\\
&\qquad -\frac{1}{4}M(M(M((\eta\Phi_{-1,0}), (\Phi_{-1,0}))), (\Phi_{-1,0}))\\
&\qquad -\frac{1}{4}M(M((\eta\Phi_{-1,0}), M, (\Phi_{-1,0})), (\Phi_{-1,0}))\bigg\rangle,
\end{aligned}\tag{5.39}$$

$$(\text{Third term}) = \left\langle \Phi^*_{3,-2},\ -\frac{1}{4}M(M((\Phi_{-1,0})), M((\eta\Phi_{-1,0}), (\Phi_{-1,0})))\right\rangle,\tag{5.40}$$

$$(\text{Fourth term}) = \left\langle \Phi^*_{3,-2},\ \frac{1}{4}M(M(M((\eta\Phi_{-1,0}), (\Phi_{-1,0}))), (\Phi_{-1,0}))\right\rangle.\tag{5.41}$$

We used cyclicity (3.21), $A_\infty$ relation $\mathbf{M}^2 = 0$ and $[\mathbf{M}, \boldsymbol{\eta}] = 0$. The sum of the terms from first to fourth is

$$\begin{aligned}
\Big\langle \Phi^*_{3,-2},\ &-\frac{1}{4}\eta M(M((\Phi_{-1,0}))^2, (\Phi_{-1,0})) \\
&-\frac{1}{4}\eta M(M(M((\Phi_{-1,0})), (\Phi_{-1,0})), (\Phi_{-1,0})) \\
&-\frac{1}{4}M(M(M, (\Phi_{-1,0}), (\eta\Phi_{-1,0})), (\Phi_{-1,0})) \\
&-\frac{1}{4}M(M((\Phi_{-1,0})), M((\eta\Phi_{-1,0})), (\Phi_{-1,0})) \\
&-\frac{1}{4}M(M((\Phi_{-1,0}))^2, (\eta\Phi_{-1,0})) \\
&-\frac{1}{4}M(M(M((\Phi_{-1,0})), (\Phi_{-1,0})), (\eta\Phi_{-1,0})) \\
&-\frac{1}{4}M(M((\eta\Phi_{-1,0}), (\Phi_{-1,0})), M((\Phi_{-1,0}))) \Big\rangle
\end{aligned} \tag{5.42}$$

$$\begin{aligned}
= &\Big\langle M,\ -\frac{1}{4}M((\Phi_{-1,0}), (\eta\Phi_{-1,0}), M((\Phi^*_{3,-2}), (\Phi_{-1,0}))) - \frac{1}{4}M((\Phi_{-1,0}), M((\Phi_{-1,0}), (\Phi^*_{3,-2}), (\eta\Phi_{-1,0}))) \Big\rangle \\
&+\Big\langle M((\Phi^*_{3,-2})),\ \frac{1}{4}M(M((\Phi_{-1,0})), (\eta\Phi_{-1,0}), (\Phi_{-1,0})) \Big\rangle \\
&+\Big\langle \eta\Phi^*_{3,-2},\ -\frac{1}{4}M(M((\Phi_{-1,0}))^2, (\Phi_{-1,0})) - \frac{1}{4}M(M(M((\Phi_{-1,0})), (\Phi_{-1,0})), (\Phi_{-1,0})) \Big\rangle \\
&+\Big\langle \Phi^*_{3,-2},\ \frac{1}{4}M(M(M((\Phi_{-1,0})), (\eta\Phi_{-1,0})), (\Phi_{-1,0})) \Big\rangle.
\end{aligned} \tag{5.43}$$

The last term does not vanish while the others can be canceled by

$$\left\langle M, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle M((\Phi^*_{3,-2})), \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-2}} \right\rangle + \left\langle \eta\Phi^*_{3,-2}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-3}} \right\rangle.$$

One may think that the remaining term can vanish in a good way, but all the string field derivatives of $S^{(3)}$ are used to cancel the other terms of eq(5.30) as shown in Appendix A. Therefore, the equation is not completed at least. We can carry out a similar calculation by using BV variation, but we fail to construct the BV action as it is written in Appendix B. We need some contrivances to construct the BV action.

# 6 Linear BV approach

In the previous section, we worked on the naive BV approach, which causes the breakdown. As a resolution, we add an extra set of fields and antifields, and give a solution to the master equation.

### 6.1 Modification of requirements

We tried to construct BV action satisfying the conventional requirements which is written in subsection 5.2. There is, however, no solution satisfying the three requirements. We have to modify the properties.

In the naive construction, $S^{(3)}$ do not have degrees of freedom to cancel the terms which include two $\Phi^*_{3,-2}$ and three $\Phi_{-1,0}$. To solve this problem, we add some fields.

Even if two string fields have the same ghost number and picture number, they need not necessarily be the same field. We change the first requirement in the previous section and double the fields and antifields. Let the set of fields and antifields $\mathcal{A}$ be

$$\mathcal{A} = \{\phi^{1,r}_{g,p},\ (\phi^{1,r}_{g,p})^*,\ \phi^{2,r}_{g,p},\ (\phi^{2,r}_{g,p})^* | 0 \leq g,\ 0 \leq p \leq g,\ r \in \mathbb{N}\}. \tag{6.1}$$

The antibracket is defined as

$$\{F, G\} = \sum_{g\geq 0} \sum_{0\leq p\leq g} \sum_{r} \left( \frac{\partial_r F}{\partial \phi^{1,r}_{g,p}} \frac{\partial_l G}{\partial (\phi^{1,r}_{g,p})^*} + \frac{\partial_r F}{\partial \phi^{2,r}_{g,p}} \frac{\partial_l G}{\partial (\phi^{2,r}_{g,p})^*} - \frac{\partial_r F}{\partial (\phi^{1,r}_{g,p})^*} \frac{\partial_l G}{\partial \phi^{1,r}_{g,p}} - \frac{\partial_r F}{\partial (\phi^{2,r}_{g,p})^*} \frac{\partial_l G}{\partial \phi^{2,r}_{g,p}} \right). \tag{6.2}$$

The string fields are defined as

$$\Phi^1_{-g,p} = \sum_r \phi^{1,r}_{g,p} |Z^r_{-g,p}\rangle, \tag{6.3}$$

$$\Phi^2_{-g,p} = \sum_r \phi^{2,r}_{g,p} |Z^r_{-g,p}\rangle, \tag{6.4}$$

$$\Phi^{*1}_{2+g,-1-p} = \sum_r (\phi^{1,r}_{g,p})^* |Z^{r*}_{-g,p}\rangle, \tag{6.5}$$

$$\Phi^{*2}_{2+g,-1-p} = \sum_r (\phi^{2,r}_{g,p})^* |Z^{r*}_{-g,p}\rangle. \tag{6.6}$$

### 6.2 Linear BV approach

In the naive approach, the terms in which there are interactions between ghost fields violated the master equation. Then we solve the master equation as such terms are canceled at each antifield number part.

We expand a action with antifield number

$$S^{\text{linear}} = \sum_{n=0}^{\infty} S^{(n)}, \tag{6.7}$$

and we set

$$S^{(0)} = \sum_{n=1}^{\infty} \frac{1}{n+1} \left\langle \Phi^1_{0,0} + \Phi^2_{0,0},\ \pi_1 \mathbf{M} \pi_n \frac{1}{1 - \eta(\Phi^1_{0,0} + \Phi^2_{0,0})} \right\rangle \tag{6.8}$$

$$\begin{aligned}S^{(1)} =&\langle \Phi^{*1}_{2,-1},\ M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^1_{-1,1}\rangle\\&+\langle \Phi^{*2}_{2,-1},\ -M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^2_{-1,1}\rangle\end{aligned}\tag{6.9}$$

where

$$M((A)) = \pi_1\mathbf{M}\frac{1}{1-\eta(\Phi^1_{0,0}+\Phi^2_{0,0})}\otimes A\otimes\frac{1}{1-\eta(\Phi^1_{0,0}+\Phi^2_{0,0})}.\tag{6.10}$$

The antifield number 0 part of the master equation is complete.

$$\begin{aligned}&\left\langle\frac{\partial_r S^{(0)}}{\partial\Phi^1_{0,0}},\ \frac{\partial_r S^{(1)}}{\partial\Phi^{*1}_{2,-1}}\right\rangle+\left\langle\frac{\partial_r S^{(0)}}{\partial\Phi^2_{0,0}},\ \frac{\partial_r S^{(1)}}{\partial\Phi^{*2}_{2,-1}}\right\rangle\\&=\left\langle\pi_1\mathbf{M}\frac{1}{1-\eta(\Phi^1_{0,0}+\Phi^2_{0,0})},\ M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^1_{-1,1}\right.\\&\qquad\qquad\left.-M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^2_{-1,1}\right\rangle=0.\end{aligned}\tag{6.11}$$

The antifield number 1 part of the master equation is

$$\begin{aligned}&\sum_{m=1}^{2}\left(\left\langle\frac{\partial_r S^{(1)}}{\partial\Phi^m_{0,0}},\ \frac{\partial_l S^{(1)}}{\partial\Phi^{*m}_{2,-1}}\right\rangle+\left\langle\frac{\partial_r S^{(0)}}{\partial\Phi^m_{0,0}},\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*m}_{2,-1}}\right\rangle+\left\langle\frac{\partial_r S^{(1)}}{\partial\Phi^m_{-1,0}},\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*m}_{3,-1}}\right\rangle+\left\langle\frac{\partial_r S^{(1)}}{\partial\Phi^m_{-1,1}},\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*m}_{3,-2}}\right\rangle\right)\\&=\big\langle-\eta M((\Phi^1_{-1,0}+\Phi^2_{-1,0}),(\Phi^{*1}_{2,-1}))+\eta M((\Phi^1_{-1,0}+\Phi^2_{-1,0}),(\Phi^{*2}_{2,-1})),\\&\quad M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^1_{-1,1}-M((\Phi^1_{-1,0}+\Phi^2_{-1,0}))+\eta\Phi^2_{-1,1}\big\rangle\\&\ +\left\langle M,\ \sum_{m=1}^{2}\frac{\partial_l S^{(2)}}{\partial\Phi^{*m}_{2,-1}}\right\rangle\\&\ +\left\langle M((\Phi^{*1}_{2,-1}))-M((\Phi^{*2}_{2,-1})),\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*1}_{3,-1}}+\frac{\partial_l S^{(2)}}{\partial\Phi^{*2}_{3,-1}}\right\rangle\\&\ +\left\langle\eta\Phi^{*1}_{2,-1},\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*1}_{3,-2}}\right\rangle+\left\langle\eta\Phi^{*2}_{2,-1},\ \frac{\partial_l S^{(2)}}{\partial\Phi^{*2}_{3,-2}}\right\rangle=0.\end{aligned}\tag{6.12}$$

The solution is

$$\frac{\partial_l S^{(2)}}{\partial\Phi^{*m}_{2,-1}}=0\quad(m=1,2),\tag{6.13}$$

$$\frac{\partial_l S^{(2)}}{\partial\Phi^{*1}_{3,-1}}=M((\Phi^1_{-2,0}+\Phi^2_{-2,0}))+\eta\Phi^1_{-2,1},\tag{6.14}$$

$$\frac{\partial_l S^{(2)}}{\partial\Phi^{*2}_{3,-1}}=-M((\Phi^1_{-2,0}+\Phi^2_{-2,0}))+\eta\Phi^2_{-2,1},\tag{6.15}$$

$$\frac{\partial_l S^{(2)}}{\partial \Phi^{*1}_{3,-2}} = M((\Phi^1_{-2,1} + \Phi^2_{-2,1})) + \eta \Phi^1_{-2,2}, \tag{6.16}$$

$$\frac{\partial_l S^{(2)}}{\partial \Phi^{*2}_{3,-2}} = -M((\Phi^1_{-2,1} + \Phi^2_{-2,1})) + \eta \Phi^1_{-2,2}. \tag{6.17}$$

Therefore

$$\begin{aligned} S^{(3)} =& \langle \Phi^{*1}_{3,-1},\ M((\Phi^1_{-2,0} + \Phi^2_{-2,0})) + \eta \Phi^1_{-2,1} \rangle \\ &+ \langle \Phi^{*1}_{3,-2},\ M((\Phi^1_{-2,1} + \Phi^2_{-2,1})) + \eta \Phi^1_{-2,2} \rangle \\ &+ \langle \Phi^{*2}_{3,-1},\ -M((\Phi^1_{-2,0} + \Phi^2_{-2,0})) + \eta \Phi^2_{-2,1} \rangle \\ &+ \langle \Phi^{*2}_{3,-2},\ -M((\Phi^1_{-2,1} + \Phi^2_{-2,1})) + \eta \Phi^2_{-2,2} \rangle. \end{aligned} \tag{6.18}$$

After all, the action is

$$S^{\text{linear}} = \sum_{n=0}^{\infty} S^{(n)} \tag{6.19}$$

where

$$S^{(0)} = \sum_{n=1}^{\infty} \frac{1}{n+1} \left\langle \Phi^1_{0,0} + \Phi^2_{0,0},\ \pi_1 \mathbf{M} \pi_n \frac{1}{1 - \eta(\Phi^1_{0,0} + \Phi^2_{0,0})} \right\rangle \tag{6.20}$$

$$\begin{aligned} S^{(k)} =& \sum_{j=1}^{k} \langle \Phi^{*1}_{k+1,-j},\ M((\Phi^1_{-k,j-1} + \Phi^2_{-k,j-1})) + \eta \Phi^1_{-k,j} \rangle \\ &+ \sum_{j=1}^{k} \langle \Phi^{*2}_{k+1,-j},\ -M((\Phi^1_{-k,j-1} + \Phi^2_{-k,j-1})) + \eta \Phi^2_{-k,j} \rangle \quad (k \geq 1). \end{aligned} \tag{6.21}$$

The master equation $\{S^{\text{linear}}, S^{\text{linear}}\} = 0$ is completed.

However, $S^{(0)}$ is not equal to the original action. There are extra dynamical fields.Then this action does not satisfy the boundary condition. We need to find constraints which exclude additional degrees of freedom.

## 7 Constrained BV approach

Linear BV approach can construct a classical master action. However, the physical meaning is not clear. We investigate other approaches in this section [24]. We add extra fields to the minimal set of fields and impose constraints considered by Berkovits [20] in this time. We modify the antibracket with constraints and define Dirac antibracket. We calculate the master equation with this Dirac bracket.

## 7.1 Extra set of fields and antifields

We introduce an extra set of space-time ghosts

$$A_{\rm ex} = \{\phi^r_{-1-g,-p} | 0 \le g,\ 0 \le p \le g,\ r \in \mathbb{N}\} \tag{7.1}$$

which carry negative space-time ghost number. It provides an extra set of ghost string fields $\{\Phi_{g+1,-p} | 0 \le g,\ 0 \le p \le g\}$ via

$$\Phi_{g+1,-p} = \sum_r \phi^r_{-1-g,-p} |Z^r_{g+1,-p}\rangle. \tag{7.2}$$

For these extra space-time ghosts, we introduce their space-time antifields

$$A^*_{\rm ex} = \{(\phi^r_{-1-g,-p})^* | 0 \le g,\ 0 \le p \le g,\ r \in \mathbb{N}\}. \tag{7.3}$$

Unlike ghost string fields in the BV formalism, there is no criteria or rule for how to assemble string antifields. The BV formalism just suggests that how or what kind of ghost string fields are provided from the gauge invariance, that one can introduce their space-time antifields such that the antibracket takes the Darboux form, and that a given master action is proper or not. In general, the anti string field $(\Phi_{-g,p})^*$ takes the following form [24]

$$(\Phi_{-g,p})^* = \sum_r (\phi^r_{g,p})^* |g,p;r\rangle, \tag{7.4}$$

where

$$|g,p;r\rangle = \sum_{h,q,r'} a^{r,r'}_{(-g,p),(-h,q)} |Z^{r'}_{2+h,-1-q}\rangle, \tag{7.5}$$

$a^{r,r'}_{(g,p),(h,q)}$ are constants. For example, eq(4.49) gives

$$a^{r,r'}_{(g,p),(h,q)} = (-)^{g(p+1)} \delta_{r^C,r'} \delta_{g,h} \delta_{p,q}. \tag{7.6}$$

These antifields satisfy $\langle \Phi_{g,p}, (\Phi_{g,p})^* \rangle \neq 0$ formally.

We consider the nonminimal set of fields and antifields

$$\begin{aligned} \mathcal{A} \equiv& \mathcal{A}_{\rm min} \oplus \mathcal{A}_{\rm ex} \\ =& \{\phi^r_{g,p},\ \phi^r_{-1-g,-p},\ (\phi^r_{g,p})^*,\ (\phi^r_{-1-g,-p})^* | 0 \le g,\ 0 \le p \le g,\ r \in \mathbb{N}\}, \end{aligned} \tag{7.7}$$

where

$$\mathcal{A}_{\rm ex} = A_{\rm ex} \oplus A^*_{\rm ex}, \tag{7.8}$$

and define an antibracket acting on this $\mathcal{A}$ by

$$\{F, G\} \equiv \sum_{g \in \mathbb{Z}} \sum_{p,r} \left( \frac{\partial_r F}{\partial \phi^r_{g,p}} \frac{\partial_l G}{\partial (\phi^r_{g,p})^*} - \frac{\partial_r F}{\partial (\phi^r_{g,p})^*} \frac{\partial_l G}{\partial \phi^r_{g,p}} \right). \tag{7.9}$$

Since there are extra degrees of freedom in this phase space, we have to introduce a set of constraints $\{\Gamma_a\}$ which cancel them. When the constraints are second class, the BV antibracket need to be modified using the Dirac procedure [19].

$$\{F,G\}_{\mathrm{D}} \equiv \{F,G\} - \sum_{a,b}\{F,\Gamma_a\}(\{\Gamma,\Gamma\}^{-1})_{ab}\{\Gamma_a,G\}. \tag{7.10}$$

A second class constraint $\Gamma_a$ has constraints $\Gamma_b$ which satisfies $\{\Gamma_a,\Gamma_b\} \neq 0$ . On the other hand, a first class constraint $\Gamma_a$ satisfies $\{\Gamma_a,\Gamma_b\} = 0$ for all constraints $\Gamma_b$. This modified antibracket is called Dirac antibracket. We will construct a master action $S_{\mathrm{BV}}$ based on this redundant set of fields and antifields.

## 7.2 Constrained BV action

Let $\{\Phi_{-g,p}\}$ be a set of dynamical, ghost and extra-ghost string fields. We write $\varphi$ for the sum of fields for brevity,

$$\varphi \equiv \sum_{g,p}\Phi_{-g,p}. \tag{7.11}$$

As proposed by Berkovits[20], we take the following constrained BV action

$$S_{\mathrm{BV}}^{\mathrm{con}} = \int_0^1 dt\Big\langle\varphi,\ \pi_1\mathbf{M}\frac{1}{1-t\eta\varphi}\Big\rangle, \tag{7.12}$$

which has the same form as the original action. The anti string fields $(\Phi_{-g,p})^*$ are introduced into $S_{\mathrm{BV}}$ via constraints.

Note that action (7.12) has special property. One can split fields into $\eta$-exact and $\xi$-exact components as

$$\Phi_{-g,p} = \sum_r \phi^r_{g,p}(\eta\xi+\xi\eta)|Z^r_{-g,p}\rangle = \sum_{r_\eta}\phi^{r_\eta}_{g,p}|Z^{r_\eta}_{-g,p}\rangle + \sum_{r_\xi}\phi^{r_\xi}_{g,p}|Z^{r_\xi}_{-g,p}\rangle, \tag{7.13}$$

where $|Z^{r_\eta}_{-g,p}\rangle$ are $\eta$-exact and $|Z^{r_\xi}_{-g,p}\rangle$ are $\xi$-exact. For any pairs of $(g,p)$, we find

$$\frac{\partial_l S^{\mathrm{con}}_{\mathrm{BV}}}{\partial\phi^{r_\eta}_{g,p}} = \langle Z^{r_\eta}_{-g,p},\ M\rangle = 0, \quad \frac{\partial_r S^{\mathrm{con}}_{\mathrm{BV}}}{\partial\phi^{r_\eta}_{g,p}} = 0. \tag{7.14}$$

We impose the constraint $\Gamma_{g,p}$ [20]

$$\Gamma_{g,p} = (\Phi_{-g,p})^* - \eta\Phi_{1+g,-p}. \tag{7.15}$$

Anti string fields are defined as eq(4.49) with this constraint. This constraint implies first class constraint

$$\eta(\Phi_{-g,p})^* = 0, \tag{7.16}$$

which generates the gauge transformation

$$\delta\Phi_{-g,p} = \eta \sum_r \lambda^r |Z^r_{-g-1,p+1}\rangle. \tag{7.17}$$

So the constraint (7.15) has first class and second class pieces [20]. In other words, splitting fields and antifields into $\eta$-exact and $\xi$-exact components, the space-time fields and antifields of eq(7.15) are

$$(\phi^{r_\eta}_{g,p})^* = 0, \tag{7.18}$$

$$\sum_{r_\xi} (\phi^{r_\xi}_{g,p})^* |Z^{r_\xi *}_{-g,p}\rangle = \sum_{r'_\xi} \eta \phi^{r'_\xi}_{-1-g,-p} |Z^{r'_\xi}_{1+g,-p}\rangle. \tag{7.19}$$

Note that $|Z^{r_\xi *}_{-g,p}\rangle$ are $\eta$-exact and $|Z^{r_\eta *}_{-g,p}\rangle$ are $\xi$-exact. The constraints (7.18) are first class and the constraints (7.19) are second class.

We have two ways to define the Dirac bracket in such a case. The first one is to introduce new constraints which fix the gauge invariance (7.17). For example,

$$\xi\Phi_{-g,p} = 0 \tag{7.20}$$

that is

$$\phi^{r_\eta}_{g,p} = 0. \tag{7.21}$$

In the presence of the new constraints, all of the constraints including (7.16) (7.18) become second class constraints.

The second way is that we do not fix the first class gauge and define the Dirac bracket only for operators which commute with the first class constraints. The Dirac bracket is only defined for operators which are gauge invariant with respect to eq(7.17). In this case, the matrix $(\{\Gamma, \Gamma\})^{-1}_{ab}$ which is appeared in the Dirac bracket (7.10) is defined to be the inverse of the matrix $\{\Gamma_a, \Gamma_b\}$ where $a, b$ range only over the second class constraints. The choice of how to split off these second class constraints from the first class constraints does not cause ambiguities because the operators $F, G$ in the Dirac bracket (7.10) vanishes in the antibracket with the first class constraints.

We discuss only the second way in this thesis. The gauge transformations of space-time fields generated by the first class constraints (7.18) are

$$\delta\phi^{r_\eta}_{g,p} = \lambda^{r_\eta}_{g,p}, \tag{7.22}$$

and $\delta\phi^{r_\xi}_{g,p}$ are zero. Then, operators which are invariant with respect to the gauge transformation (7.17) are $(\Phi_{-g,p})^*$ and $\Phi^\xi_{-g,p}$ where

$$\Phi^\xi_{-g,p} = \sum_{r_\xi} \phi^{r_\xi}_{g,p} |Z^{r_\xi}_{-g,p}\rangle, \quad \Phi^\eta_{-g,p} = \sum_{r_\eta} \phi^{r_\eta}_{g,p} |Z^{r_\eta}_{-g,p}\rangle. \tag{7.23}$$

Because of eq(7.14), $S_{\rm BV}$ is also gauge invariant with respect to the gauge transformation (7.17).

The antibrackets of fields $\Phi^\xi$ and antifields $(\Phi)^*$ without constraints are

$$\{\Phi^\xi_{-g,p},(\Phi_{-g',p'})^*\} = (-)^g\sum_{r_\xi}|Z^{r_\xi}_{-g,p}\rangle|Z^{r_\xi *}_{-g',p'}\rangle\delta_{g,g'}\delta_{p,p'}, \tag{7.24}$$

$$\{(\Phi_{-g,p})^*,\Phi^\xi_{-g',p'}\} = -\sum_{r_\xi}|Z^{r_\xi *}_{-g,p}\rangle|Z^{r_\xi}_{-g',p'}\rangle\delta_{g,g'}\delta_{p,p'}, \tag{7.25}$$

$$\{\Phi^\xi_{-g,p},\Phi^\xi_{-g',p'}\} = 0, \tag{7.26}$$

$$\{(\Phi_{-g,p})^*,(\Phi_{-g',p'})^*\} = 0. \tag{7.27}$$

The antibracket of the constraint is

$$\begin{aligned}\{\Gamma_{g,p},\Gamma_{g',p'}\} =&\{(\Phi_{-g,p})^*,-\eta\Phi_{1+g',-p'}\}+\{-\eta\Phi_{1+g,-p},(\Phi_{-g',p'})^*\}\\ =&\sum_{r_\xi}(-)^g|Z^{r_\xi *}_{-g,p}\rangle\eta|Z^{r_\xi}_{-g,p}\rangle\delta_{g,-(1+g')}\delta_{p,-p'}\\ &+\sum_{r'_\xi}(-)^g\eta|Z^{r'_\xi}_{1+g,-p}\rangle|Z^{r'_\xi *}_{1+g,-p}\rangle\delta_{g,-(1+g')}\delta_{p,-p'}\end{aligned} \tag{7.28}$$

Here if we set

$$|Z^{r_\xi *}_{-g,p}\rangle = \eta|Z^{r'_\xi}_{1+g,-p}\rangle, \tag{7.29}$$

then, by definition,

$$\begin{aligned}1 &= \langle Z^{r_\xi}_{-g,p},Z^{r_\xi *}_{-g,p}\rangle\\ &= \langle Z^{r_\xi}_{-g,p},\eta Z^{r'_\xi}_{1+g,-p}\rangle\\ &= \langle Z^{r'_\xi}_{1+g,-p},\eta Z^{r_\xi}_{-g,p}\rangle\end{aligned} \tag{7.30}$$

$$\Leftrightarrow |Z^{r'_\xi *}_{1+g,-p}\rangle = \eta|Z^{r_\xi}_{-g,p}\rangle. \tag{7.31}$$

Therefore,

$$\{\Gamma_{g,p},\Gamma_{g',p'}\} =2\sum_{r_\xi}(-)^g|Z^{r_\xi *}_{-g,p}\rangle\eta|Z^{r_\xi}_{-g,p}\rangle\delta_{g,-(1+g')}\delta_{p,-p'} \tag{7.32}$$

$$=2\sum_{r'_\xi}(-)^g\eta|Z^{r'_\xi}_{1+g,-p}\rangle|Z^{r'_\xi *}_{1+g,-p}\rangle\delta_{g,-(1+g')}\delta_{p,-p'} \tag{7.33}$$

$$=2\sum_{r'_\xi}(-)^{1+g'}\eta|Z^{r'_\xi}_{-g',p'}\rangle|Z^{r'_\xi *}_{-g',p'}\rangle\delta_{g,-(1+g')}\delta_{p,-p'} \tag{7.34}$$

The inverse of this antibracket is

$$(\{\Gamma,\Gamma\})^{-1}_{(g,p),(g',p')} = \frac{1}{2}\sum_{r_\xi}\langle Z^{r_\xi *}_{-g',p'}|\xi\langle Z^{r_\xi}_{-g',p'}|\delta_{g,-(1+g')}\delta_{p,-p'} \tag{7.35}$$

$$= -\frac{1}{2}\sum_{r_\xi}\langle Z^{r_\xi}_{-g,p}|\langle Z^{r_\xi *}_{-g,p}|\xi\delta_{g,-(1+g')}\delta_{p,-p'}. \tag{7.36}$$

The Dirac brackets of the constraints are

$$\{\Gamma_{g,p},\Gamma_{g',p'}\}_D = 0. \tag{7.37}$$

The Dirac antibrackets of the operators are

$$\{\Phi^{\xi}_{-g,p},(\Phi_{-g',p'})^*\}_{\mathrm{D}} = (-)^g\frac{1}{2}\sum_{r_\xi}|Z^{r_\xi}_{-g,p}\rangle|Z^{r_\xi *}_{-g',p'}\rangle\delta_{g,g'}\delta_{p,p'}, \tag{7.38}$$

$$\{(\Phi_{-g,p})^*,\Phi^{\xi}_{-g',p'}\}_{\mathrm{D}} = -\frac{1}{2}\sum_{r_\xi}|Z^{r_\xi *}_{-g,p}\rangle|Z^{r_\xi}_{-g',p'}\rangle\delta_{g,g'}\delta_{p,p'}, \tag{7.39}$$

$$\{\Phi^{\xi}_{-g,p},\Phi^{\xi}_{-g',p'}\}_{\mathrm{D}} = -\frac{1}{2}|Z^{r_\xi}_{-g,p}\rangle\xi|Z^{r_\xi *}_{-g,p}\rangle\delta_{g',-(1+g)}\delta_{p',-p}, \tag{7.40}$$

$$\{(\Phi_{-g,p})^*,(\Phi_{-g',p'})^*\}_{\mathrm{D}} = (-)^{1+g}\frac{1}{2}|Z^{r_\xi *}_{-g,p}\rangle\eta|Z^{r_\xi}_{-g,p}\rangle\delta_{g',-(1+g)}\delta_{p',-p}. \tag{7.41}$$

Since the master action does not include antifields, the antibracket of the master actions without constraints $\{S^{\mathrm{con}}_{\mathrm{BV}},S^{\mathrm{con}}_{\mathrm{BV}}\}$ is zero, and antibrackets of the master action and constraints are

$$\{S^{\mathrm{con}}_{\mathrm{BV}},\Gamma_{g,p}\} = \{S^{\mathrm{con}}_{\mathrm{BV}},(\Phi_{-g,p})^*\} = (-)^g\sum_{r_\xi}\left\langle\frac{\partial_r S^{\mathrm{con}}_{\mathrm{BV}}}{\partial\Phi^{\xi}_{-g,p}},\ Z^{r_\xi}_{-g,p}\right\rangle|Z^{r_\xi *}_{-g,p}\rangle, \tag{7.42}$$

$$\{\Gamma_{g,p},S^{\mathrm{con}}_{\mathrm{BV}}\} = \{(\Phi_{-g,p})^*,S^{\mathrm{con}}_{\mathrm{BV}}\} = -\sum_{r_\xi}|Z^{r_\xi *}_{-g,p}\rangle\left\langle Z^{r_\xi}_{-g,p},\ \frac{\partial_l S^{\mathrm{con}}_{\mathrm{BV}}}{\partial\Phi^{\xi}_{-g,p}}\right\rangle. \tag{7.43}$$

Then, the master equation with the constraints (7.15) is

$$
\begin{aligned}
&\{S_{\rm BV}^{\rm con}, S_{\rm BV}^{\rm con}\}_{\rm D} \\
&= \frac{1}{2}(-)^{g+1} \sum_{g,p,r_\xi,r'_\xi} \left\langle \frac{\partial_r S_{\rm BV}^{\rm con}}{\partial \Phi^\xi_{-g,p}},\ Z^{r_\xi}_{-g,p} \right\rangle \left\langle Z^{r_\xi *}_{-g,p} \middle| \xi \middle| Z^{r'_\xi *}_{1+g,-p} \right\rangle \left\langle Z^{r'_\xi}_{1+g,-p},\ \frac{\partial_l S_{\rm BV}^{\rm con}}{\partial \Phi^\xi_{1+g,-p}} \right\rangle \\
&= -\frac{1}{2} \sum_{g,p} \left\langle \frac{\partial_r S_{\rm BV}^{\rm con}}{\partial \Phi^\xi_{-g,p}},\ \xi \frac{\partial_l S_{\rm BV}^{\rm con}}{\partial \Phi^\xi_{1+g,-p}} \right\rangle \\
&= -\frac{1}{2} \left\langle \pi_1 \mathbf{M} \frac{1}{1-\eta\varphi},\ \xi \pi_1 \mathbf{M} \frac{1}{1-\eta\varphi} \right\rangle \\
&= -\frac{1}{n+1} \left\langle \varphi,\ \pi_1 \mathbf{M}^2 \pi_n \frac{1}{1-\eta\varphi} \right\rangle = 0.
\end{aligned} \tag{7.44}
$$

We used the completeness (4.16), (4.17). Then the $S_{\rm BV}$ satisfies the master equation. This action does not have kinetic terms for $\Phi_{-g,p}\ \ (g>0, p=g)$

$$
\begin{aligned}
S_{\rm BV}^{\rm con} =& \frac{1}{2} \langle \Phi_{0,0},\ Q\eta \Phi_{0,0} \rangle + \sum_{g\geq 0} \sum_{0 \leq p \leq g} \langle (\Phi_{-g,p})^*,\ Q \Phi_{-1-g,p} \rangle \\
&+ \int dt \sum_{n \geq 2} \left\langle \varphi,\ \pi_1 \mathbf{M} \pi_n \frac{1}{1-t\eta\varphi} \right\rangle .
\end{aligned} \tag{7.45}
$$

Fields $\Phi_{-g,p}\ \ (p=g, g>0)$ are auxiliary fields which act as lagrange multiplier.

# 8 Conclusion

We tried to construct BV action of superstring field theory in the large Hilbert space naively. However, it failed under the condition written in the section 5. We proposed two approaches to overcome this problem:linear BV approach, constrained BV approach.

In linear BV approach, we add extra fields to cancel interaction terms between ghost fields in the master equation. The boundary condition of the master equation $S^{(0)}$ is different from the original action and the physical meaning of the additional fields is not clear.

In constrained BV approach, we introduce additional fields and constraints. These constraints make all the anti string fields $\eta$-exact. It does not affect the master equation since the master action is independent of the space-time fields on $\eta$-exact basis. In this construction, fields $\Phi_{-g,p=g}$ do not have kinetic terms. Then, they are considered as auxiliary fields.

Constructing classical BV action is an important step to quantize the string field. These approach will be helpful to accomplish this purpose.

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# Appendix A

The master equation for antifield number 2 is

$$\left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle + \sum_{p=0}^{1} \left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{-1,p}}, \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1-p}} \right\rangle$$
$$+ \left\langle \frac{\partial_r S^{(0)}}{\partial \Phi_{0,0}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}} \right\rangle + \sum_{p=0}^{1} \left\langle \frac{\partial_r S^{(1)}}{\partial \Phi_{-1,p}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-1-p}} \right\rangle + \sum_{p=0}^{2} \left\langle \frac{\partial_r S^{(2)}}{\partial \Phi_{-2,p}}, \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1-p}} \right\rangle = 0. \quad \text{(A.1)}$$

The terms for one $\Phi^*_{3,-1}$ and three $\Phi_{-1,0}$ in the left hand side are

$$\begin{aligned}
&\left\langle -\frac{1}{2}\eta M((\Phi^*_{3,-1}),(\eta\Phi_{-1,0}),(\Phi_{-1,0})),\ M((\Phi_{-1,0}))\right\rangle \\
&+\left\langle \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{3,-1})) - \frac{1}{2}\eta M((\Phi_{-1,0}),(\Phi^*_{3,-1})),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))\right\rangle \\
&+\left\langle M,\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\Big|_{(\Phi^*_{3,-1}),(\Phi_{-1,0})^3}\right\rangle + \left\langle M((\Phi^*_{3,-1})),\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1}}\Big|_{(\Phi_{-1,0})^3}\right\rangle \\
&+\left\langle \eta\Phi^*_{3,-1},\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-2}}\Big|_{(\Phi_{-1,0})^3}\right\rangle
\end{aligned} \qquad \text{(A.2)}$$

where $\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\big|_{(\Phi^*_{3,-1}),(\Phi_{-1,0})^3}$ represents the terms for one $\Phi^*_{3,-1}$ and three $\Phi_{-1,0}$ in $\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}$. $\frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1}}\big|_{(\Phi_{-1,0})^3}$ and $\frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-2}}\big|_{(\Phi_{-1,0})^3}$ are defined in the same way. The first term plus the second term is

$$
\begin{aligned}
&\left\langle -\frac{1}{2}\eta M((\Phi^*_{3,-1}),(\eta\Phi_{-1,0}),(\Phi_{-1,0})),\ M((\Phi_{-1,0}))\right\rangle \\
&+\left\langle \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{3,-1}))-\frac{1}{2}\eta M((\Phi_{-1,0}),(\Phi^*_{3,-1})),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))\right\rangle \\
=&\frac{1}{4}\langle\Phi^*_{3,-1},\ M((\eta\Phi_{-1,0}),M((\eta\Phi_{-1,0}),(\Phi_{-1,0})))\rangle+\frac{1}{4}\langle\Phi^*_{3,-1},\ M(M((\eta\Phi_{-1,0})^2),(\Phi_{-1,0}))\rangle \\
&\ +\frac{1}{4}\langle\Phi^*_{3,-1},\ M((\eta\Phi_{-1,0}),M((\eta\Phi_{-1,0})),(\Phi_{-1,0}))\rangle+\frac{1}{4}\langle\Phi^*_{3,-1},\ M((\eta\Phi_{-1,0})^2,M((\Phi_{-1,0})))\rangle \\
&\ +\frac{1}{4}\langle\Phi^*_{3,-1},\ \eta M((\eta\Phi_{-1,0}),M((\Phi_{-1,0})),(\Phi_{-1,0}))\rangle \\
=&-\frac{1}{4}\langle\Phi^*_{3,-1},\ M(M((\eta\Phi_{-1,0})^2,(\Phi_{-1,0})))\rangle \\
&\ -\frac{1}{4}\langle\Phi^*_{3,-1},\ M(M,(\eta\Phi_{-1,0})^2,(\Phi_{-1,0}))\rangle \\
&\ +\frac{1}{4}\langle\Phi^*_{3,-1},\ \eta M((\eta\Phi_{-1,0}),M((\Phi_{-1,0})),(\Phi_{-1,0}))\rangle
\end{aligned}
\tag{A.3}
$$

We used $(\mathbf{M})^2=0$. The solution is

$$
\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\right|_{(\Phi^*_{3,-1}),(\Phi_{-1,0})^3}=\frac{1}{4}M((\eta\Phi_{-1,0})^2,(\Phi_{-1,0}),(\Phi^*_{3,-1})),
\tag{A.4}
$$

$$
\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\right|_{(\Phi_{-1,0})^3}=\frac{1}{4}M((\eta\Phi_{-1,0})^2,(\Phi_{-1,0})),
\tag{A.5}
$$

$$
\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right|_{(\Phi_{-1,0})^3}=-\frac{1}{4}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}),M((\Phi_{-1,0}))).
\tag{A.6}
$$

The terms for $\Phi^*_{3,-1},\Phi_{-1,0},\Phi_{-2,0}$ in the master equation are

$$
\begin{aligned}
&\langle-\eta M((\Phi^*_{3,-1}),(\Phi_{-2,0})),\ M((\Phi_{-1,0}))\rangle \\
&+\left\langle\frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}))-\frac{1}{2}\eta M((\Phi_{-1,0}),(\Phi^*_{2,-1})),\ M((\Phi_{-2,0}))\right\rangle \\
&+\left\langle M,\ \left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\right|_{(\Phi^*_{3,-1}),(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle+\left\langle M((\Phi^*_{3,-1})),\ \left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle \\
&+\left\langle\eta\Phi^*_{3,-1},\ \left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle \\
&=0.
\end{aligned}
\tag{A.7}
$$

The first term and the second term of the left hand side in this equation are

$$\begin{aligned}
&\langle -\eta M((\Phi^*_{3,-1}),(\Phi_{-2,0})),\ M((\Phi_{-1,0}))\rangle \\
&+\left\langle \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}))-\frac{1}{2}\eta M((\Phi_{-1,0}),(\Phi^*_{2,-1})),\ M((\Phi_{-2,0}))\right\rangle \\
=&\frac{1}{2}\langle \Phi^*_{3,-1},\ M((\eta\Phi_{-1,0}),M((\Phi_{-2,0})))\rangle-\frac{1}{2}\langle \Phi^*_{3,-1},\ M(\eta M((\Phi_{-2,0})),(\Phi_{-1,0}))\rangle \\
&+\langle \Phi^*_{3,-1},\ M(M((\eta\Phi_{-1,0})),(\Phi_{-2,0}))\rangle \\
=&\frac{1}{2}\langle \Phi^*_{3,-1},\ \eta M(M((\Phi_{-2,0})),(\Phi_{-1,0}))\rangle \\
&+\langle \Phi^*_{3,-1},\ M(M((\Phi_{-2,0}),(\eta\Phi_{-1,0})))\rangle+\langle \Phi^*_{3,-1},\ M(M,(\Phi_{-2,0}),(\eta\Phi_{-1,0}))\rangle
\end{aligned} \tag{A.8}$$

Then, we obtain

$$\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\right|_{(\Phi^*_{3,-1}),(\Phi_{-1,0}),(\Phi_{-2,0})}=M((\eta\Phi_{-1,0}),(\Phi_{-2,0}),(\Phi^*_{3,-1})), \tag{A.9}$$

$$\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,0})}=M((\eta\Phi_{-1,0}),(\Phi_{-2,0})), \tag{A.10}$$

$$\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,0})}=-\frac{1}{2}M((\Phi_{-1,0}),M((\Phi_{-2,0}))). \tag{A.11}$$

The terms for $\Phi^*_{3,-1},\Phi_{-1,0},\Phi_{-2,1}$ are

$$\begin{aligned}
&\frac{1}{2}\langle M((\Phi^*_{3,-1}),(\eta\Phi_{-1,0})),\ \eta\Phi_{-2,1}\rangle \\
&+\left\langle \eta\Phi^*_{3,-1},\ \left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle+\left\langle M((\Phi^*_{3,-1})),\ \left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle=0.
\end{aligned} \tag{A.12}$$

Therefore,

$$\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})}=\frac{1}{2}M((\Phi_{-1,0}),(\eta\Phi_{-2,1})), \tag{A.13}$$

$$\left.\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})}=0. \tag{A.14}$$

The terms for $\Phi^*_{3,-2}, \Phi_{-1,0}, \Phi_{-2,1}$ are

$$\begin{aligned}
&\langle -\eta M((\Phi^*_{3,-2}),(\Phi_{-2,1})),\ M((\Phi_{-1,0}))\rangle \\
&+\left\langle -\frac{1}{2}M(M((\Phi_{-1,0})),(\Phi^*_{3,-2}))+\frac{1}{2}M(M((\Phi_{-1,0}),(\Phi^*_{3,-2}))),\eta\Phi_{-2,1}\right\rangle \\
&+\left\langle M,\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{3,-2}),(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle+\left\langle M((\Phi^*_{3,-2})),\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle \\
&+\left\langle \eta\Phi^*_{3,-2},\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle \\
&=0.
\end{aligned}\tag{A.15}$$

Substituting eq(A.13),

$$\begin{aligned}
(\mathrm{LHS})=&\left\langle \Phi^*_{3,-2},\ -M(\eta M((\Phi_{-1,0})),(\Phi_{-2,1}))+\frac{1}{2}M((\eta\Phi_{-2,1}),M((\Phi_{-1,0})))\right. \\
&\qquad\left.-\frac{1}{2}M(M((\eta\Phi_{-2,1})),(\Phi_{-1,0}))+\frac{1}{2}M(M((\Phi_{-1,0}),(\eta\Phi_{-2,1})))\right\rangle \\
&+\left\langle M,\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{3,-2}),(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle+\left\langle \eta\Phi^*_{3,-2},\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,1})}\right\rangle
\end{aligned}\tag{A.16}$$

Then, the solution is

$$\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{3,-2}),(\Phi_{-1,0}),(\Phi_{-2,1})}=\frac{1}{2}M((\Phi_{-1,0}),(\eta\Phi_{-2,1}),(\Phi^*_{3,-2})),\tag{A.17}$$

$$\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,1})}=-M(M((\Phi_{-1,0})),(\Phi_{-2,1}))\tag{A.18}$$

The terms for $\Phi^*_{3,-2}, \Phi_{-1,0}, \Phi_{-2,0}$ are

$$\begin{aligned}
&\left\langle -\frac{1}{2}M(M((\Phi_{-1,0})),(\Phi^*_{3,-2}))+\frac{1}{2}M(M((\Phi_{-1,0}),(\Phi^*_{3,-2}))),\ M((\Phi_{-2,0}))\right\rangle \\
&+\left\langle M((\Phi^*_{3,-2})),\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle+\left\langle \eta\Phi^*_{3,-2},\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle \\
&+\left\langle M,\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{3,-2}),(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle=0.
\end{aligned}\tag{A.19}$$

Using eq(A.11), the solution is

$$\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-3}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,0})} = 0, \tag{A.20}$$

$$\begin{aligned}\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\right|_{(\Phi^*_{3,-2}),(\Phi_{-1,0}),(\Phi_{-2,0})} =& M((\Phi_{-2,0}), M((\Phi_{-1,0}), (\Phi^*_{3,-2}))) \\ & - \frac{1}{2} M(M((\Phi_{-2,0})), (\Phi_{-1,0}), (\Phi^*_{3,-2})).\end{aligned} \tag{A.21}$$

The terms for two $\Phi^*_{2,-1}$, $\Phi_{-1,0}, \Phi_{-2,1}$ are

$$\begin{aligned}&\left\langle \frac{1}{2} M((\Phi^*_{2,-1})^2, (\eta\Phi_{-1,0}),\ \eta\Phi_{-2,1} \right\rangle \\ &+ \left\langle M((\Phi^*_{2,-1})^2),\ \left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})} \right\rangle + \left\langle M((\Phi^*_{2,-1})),\ \left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-1}}\right|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,1})} \right\rangle \\ &+ \left\langle \eta\Phi^*_{2,-1},\ \left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-2}}\right|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,1})} \right\rangle + \left\langle M,\ \left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\right|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0}),(\Phi_{-2,1})} \right\rangle = 0.\end{aligned} \tag{A.22}$$

$\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1}}\right|_{(\Phi_{-1,0}),(\Phi_{-2,1})}$ is already fixed in eq(A.14). The solution is

$$\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-1}}\right|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,1})} = 0, \tag{A.23}$$

$$\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-2}}\right|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,1})} = -\frac{1}{2} M((\Phi_{-1,0}), (\eta\Phi_{-2,1}), (\Phi^*_{2,-1})), \tag{A.24}$$

$$\left.\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\right|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0}),(\Phi_{-2,1})} = 0. \tag{A.25}$$

The terms for two $\Phi^*_{2,-1}$ and three $\Phi_{-1,0}$ are

$$\begin{aligned}
&\left\langle -\frac{1}{2}\eta M((\Phi^*_{2,-1})^2,(\eta\Phi_{-1,0}),(\Phi_{-1,0})),\ M((\Phi_{-1,0}))\right\rangle \\
&+\left\langle -\eta M((\Phi_{-1,0}),(\Phi^*_{2,-1})),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}),(\Phi_{-1,0}))\right\rangle \\
&+\left\langle \frac{1}{2}M((\Phi^*_{2,-1})^2,(\eta\Phi_{-1,0}))-\frac{1}{2}\eta M((\Phi^*_{2,-1})^2,(\Phi_{-1,0})),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))\right\rangle \\
&+\left\langle M,\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0})^3}\right\rangle+\left\langle M((\Phi^*_{2,-1})),\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{3,-1}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0})^3}\right\rangle \\
&+\left\langle \eta\Phi^*_{2,-1},\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{3,-2}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0})^3}\right\rangle+\left\langle M((\Phi^*_{2,-1})^2),\ \frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-1}}\bigg|_{(\Phi_{-1,0})^3}\right\rangle=0. \qquad \text{(A.26)}
\end{aligned}$$

Because of eq(A.4), $S^{(3)}$ includes the term $\frac{1}{4}\langle\Phi^*_{2,-1},\ M((\eta\Phi_{-1,0})^2,(\Phi_{-1,0}),(\Phi^*_{3,-1}))\rangle$. Then, we obtain

$$\frac{\partial_l S^{(3)}}{\partial\Phi^*_{3,-1}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0})^3}=\frac{1}{4}M((\eta\Phi_{-1,0})^2,(\Phi_{-1,0}),(\Phi^*_{2,-1})). \qquad \text{(A.27)}$$

We substitute this and eq(A.5). Then, the solution is

$$\begin{aligned}
\frac{\partial_l S^{(3)}}{\partial\Phi^*_{3,-2}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0})^3}=&-\frac{1}{4}M(M((\Phi_{-1,0})),(\Phi^*_{2,-1}),(\eta\Phi_{-1,0}),(\Phi_{-1,0})) \\
&-\frac{1}{4}M(M((\Phi_{-1,0}),(\Phi^*_{2,-1})),(\eta\Phi_{-1,0}),(\Phi_{-1,0})) \\
&-\frac{1}{4}M(M((\Phi_{-1,0}),(\Phi^*_{2,-1}),(\eta\Phi_{-1,0})),(\Phi_{-1,0})) \qquad \text{(A.28)}
\end{aligned}$$

$$\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\bigg|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0})^3}=\frac{1}{8}M((\Phi_{-1,0}),(\eta\Phi_{-1,0})^2,(\Phi^*_{2,-1})^2) \qquad \text{(A.29)}$$

The terms for two $\Phi^*_{2,-1}, \Phi_{-1,0}, \Phi_{-2,0}$ are

$$\begin{aligned}
&\langle -\eta M((\Phi^*_{2,-1})^2, (\Phi_{-2,0})),\ M((\Phi_{-1,0}))\rangle \\
&+ \langle -\eta M((\Phi^*_{2,-1}), (\Phi_{-1,0})),\ M((\Phi^*_{2,-1}), (\Phi_{-2,0}))\rangle \\
&+ \left\langle \frac{1}{2} M((\Phi^*_{2,-1})^2, (\eta\Phi_{-1,0})) - \frac{1}{2}\eta M((\Phi^*_{2,-1})^2, (\Phi_{-1,0})),\ M((\Phi_{-2,0}))\right\rangle \\
&+ \left\langle M,\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\bigg|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle + \left\langle M((\Phi^*_{2,-1})),\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-1}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle \\
&+ \left\langle \eta\Phi^*_{2,-1},\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{3,-2}}\bigg|_{(\Phi^*_{2,-1}),(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle + \left\langle M((\Phi^*_{2,-1})^2),\ \frac{\partial_l S^{(3)}}{\partial \Phi^*_{4,-1}}\bigg|_{(\Phi_{-1,0}),(\Phi_{-2,0})}\right\rangle = 0.
\end{aligned} \tag{A.30}$$

We use eq(A.9), eq(A.10), eq(A.21), and the solution is

$$\frac{\partial_l S^{(3)}}{\partial \Phi^*_{2,-1}}\bigg|_{(\Phi^*_{2,-1})^2,(\Phi_{-1,0}),(\Phi_{-2,0})} = -M((\Phi^*_{2,-1})^2, (\eta\Phi_{-1,0}), (\Phi_{-2,0})). \tag{A.31}$$

All the terms are canceled by the derivatives of $S^{(3)}$ except the terms for $\Phi^*_{3,-2}$ and three $\Phi_{-1,0}$. There is no remaining degrees of freedom of $S^{(3)}$ to cancel the left terms, so the master equation (5.30) is not completed.

# Appendix B

BV variation is defined as

$$\delta_{\mathrm{BV}} A \equiv \{S, A\} \tag{B.1}$$

$S$ is the BV action (5.1). We will calculate BV variations of the action expanded by antifield number, and construct the action to satisfy $\delta_{\mathrm{BV}} S = 0$. The BV variation of the original action is

$$\delta_{\mathrm{BV}} S^{(0)} = \langle \delta_{\mathrm{BV}} \Phi_{0,0}|^{(0)}, M\rangle. \tag{B.2}$$

Here

$$\delta_{\mathrm{BV}} \Phi_{0,0}|^{(0)} = \{S, \Phi_{0,0}\}|^{(0)} = -\frac{\partial_r S^{(1)}}{\partial \Phi^*_{2,-1}} = \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}}. \tag{B.3}$$

The antifield number 1 action $S^{(1)}$ includes one $\Phi^*_{2,-1}$ and does not include other antifields. The BV variation of $\Phi^*_{2,-1}$ with antifield number 0 is

$$\delta_{\mathrm{BV}} \Phi^*_{2,-1}|^{(0)} = \frac{\partial_r S^{(0)}}{\partial \Phi_{0,0}} = M. \tag{B.4}$$

Then, the BV variation of $S^{(1)}$ with antifield number 0 is

$$\delta S^{(1)}|^{(0)} = \left\langle \delta_{\rm BV}\Phi^*_{2,-1}|^{(0)},\ \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle = \left\langle M,\ \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle. \tag{B.5}$$

Since the BV variation of $\Phi^*_{g,p},\ (g>2)$ is

$$\delta_{\rm BV}\Phi^*_{g,p}|^{(0)} = \{\Phi^*_{g,p}, S^{(0)}\} = -\frac{\partial_l S^{(0)}}{\partial \Phi_{2-g,-1-p}} = 0 \ \ (g>2), \tag{B.6}$$

we obtain

$$\delta_{\rm BV} S^{(n)}|^{(0)} = \left\langle \delta_{\rm BV}\Phi^*_{n+1,p}|^{(0)},\ \frac{\partial_l S^{(n)}}{\partial \Phi^*_{n+1,p}} \right\rangle = 0 \ \ (n\geq 2). \tag{B.7}$$

Therefore, the BV variation of the BV action with antifield number 0 is

$$\delta_{\rm BV} S|^{(0)} = \delta_{\rm BV}(S^{(0)}+S^{(1)})|^{(0)} = 2\left\langle M,\ \frac{\partial_l S^{(1)}}{\partial \Phi^*_{2,-1}} \right\rangle. \tag{B.8}$$

BV variations of the action with each antifield number vanish. Then

$$\delta_{\rm BV} S|^{(0)} = 0. \tag{B.9}$$

The solution is

$$S^{(1)} = \langle \Phi^*_{2,-1},\ M((\Phi_{-1,0})) + \eta\Phi_{-1,1}\rangle. \tag{B.10}$$

The BV variation of the action with antifield number 1 is

$$\delta_{\rm BV} S^{(0)}|^{(1)} = \langle \delta_{\rm BV}\Phi_{0,0}|^{(1)},\ M\rangle = \left\langle M,\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle, \tag{B.11}$$

$$\begin{aligned}
\delta_{\rm BV} S^{(1)}|^{(1)} =& \langle \delta_{\rm BV}\Phi^*_{2,-1}|^{(1)},\ M((\Phi_{-1,0})) + \eta\Phi_{-1,1}\rangle + \langle \Phi^*_{2,-1},\ M((\delta_{\rm BV}\Phi_{-1,0}|^{(0)}))\rangle \\
& - \langle \Phi^*_{2,-1},\ M((\eta\delta_{\rm BV}\Phi_{0,0}|^{(0)}),(\Phi_{-1,0}))\rangle + \langle \Phi^*_{2,-1},\ \eta\delta_{\rm BV}\Phi_{-1,1}|^{(0)}\rangle \\
=& 2\langle \Phi^*_{2,-1},\ M(M((\eta\Phi_{-1,0})),(\Phi_{-1,0}))\rangle + \left\langle M((\Phi^*_{2,-1})),\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} \right\rangle \\
& + \left\langle \eta\Phi^*_{2,-1},\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} \right\rangle.
\end{aligned} \tag{B.12}$$

A term of $S^{(2)}$ includes two $\Phi^*_{2,-1}$, one $\Phi^*_{3,-1}$ or one $\Phi^*_{3,-2}$, so

$$\begin{aligned}
\delta_{\rm BV} S^{(2)}|^{(1)} =& \left\langle \delta_{\rm BV}\Phi^*_{2,-1}|^{(0)},\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle \delta_{\rm BV}\Phi^*_{3,-1}|^{(1)},\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} \right\rangle + \left\langle \delta_{\rm BV}\Phi^*_{3,-2}|^{(1)},\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} \right\rangle \\
=& \left\langle M,\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{2,-1}} \right\rangle + \left\langle M((\Phi^*_{2,-1})),\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-1}} \right\rangle + \left\langle \eta\Phi^*_{2,-1},\ \frac{\partial_l S^{(2)}}{\partial \Phi^*_{3,-2}} \right\rangle
\end{aligned} \tag{B.13}$$

Here $\delta_{\rm BV}S^{(n)}|^{(1)}=0\ \ (n\geq 3)$ as it is showed in the same way with eq(B.7). Therefore, the BV variation of the action with antifield number 1 is

$$\begin{aligned}\delta_{\rm BV}S|^{(1)}=&2\langle\Phi^*_{2,-1},\ M(M((\eta\Phi_{-1,0})),(\Phi_{-1,0}))\rangle\\&+2\left\langle M,\ \frac{\partial_l S^{(2)}}{\partial\Phi^*_{2,-1}}\right\rangle+2\left\langle M((\Phi^*_{2,-1})),\ \frac{\partial_l S^{(2)}}{\partial\Phi^*_{3,-1}}\right\rangle+2\left\langle \eta\Phi^*_{2,-1},\ \frac{\partial_l S^{(2)}}{\partial\Phi^*_{3,-2}}\right\rangle.\end{aligned}\tag{B.14}$$

The solution of $\delta_{\rm BV}S|^{(1)}=0$ is

$$\begin{aligned}S^{(2)}=&\left\langle\Phi^*_{3,-1},\frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))+M((\Phi_{-2,0}))+\eta\Phi_{-2,1}\right\rangle\\&+\left\langle\Phi^*_{3,-2},-\frac{1}{2}M(M((\Phi_{-1,0})),(\Phi_{-1,0}))+M((\Phi_{-2,1}))+\eta\Phi_{-2,2}\right\rangle\\&+\left\langle\Phi^*_{2,-1},\frac{1}{4}M((\eta\Phi_{-1,0}),(\Phi^*_{2,-1}),(\Phi_{-1,0}))+\frac{1}{2}M((\Phi^*_{2,-1}),(\Phi_{-2,0}))\right\rangle\end{aligned}\tag{B.15}$$

We calculate $\delta_{\rm BV}S|^{(2)}$ in the same way and see the terms which include one $\Phi^*_{3,-2}$ and three $\Phi_{-1,0}$.

$$\begin{aligned}\frac{1}{2}\delta_{\rm BV}S|^{(2)}_{(\Phi^*_{3,-2}),(\Phi_{-1,0})^3}=&\left\langle\frac{1}{2}\eta M((\Phi^*_{3,-2}),M((\Phi_{-1,0})),(\Phi_{-1,0})),\ M((\Phi_{-1,0}))\right\rangle\\&-\left\langle\frac{1}{2}\eta M((\Phi_{-1,0}),M((\Phi^*_{3,-2}),(\Phi_{-1,0}))),\ M((\Phi_{-1,0}))\right\rangle\\&-\left\langle\frac{1}{2}M((\Phi^*_{3,-2}),M((\Phi_{-1,0}))),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))\right\rangle\\&+\left\langle\frac{1}{2}M(M((\Phi^*_{3,-2}),(\Phi_{-1,0}))),\ \frac{1}{2}M((\eta\Phi_{-1,0}),(\Phi_{-1,0}))\right\rangle\\&+\left\langle M,\frac{\partial_l S^{(3)}}{\partial\Phi^*_{2,-1}}\right\rangle+\left\langle M((\Phi^*_{3,-2})),\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-2}}\right\rangle+\left\langle\eta\Phi^*_{3,-2},\frac{\partial_l S^{(3)}}{\partial\Phi^*_{4,-3}}\right\rangle\end{aligned}\tag{B.16}$$

$\delta_{\rm BV}S|^{(2)}_{(\Phi^*_{3,-2}),(\Phi_{-1,0})^3}=0$ is equivalent to eq(5.37). We have no proper $S^{(3)}$.

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