博士論文

論文題目　Validity of formal asymptotic expansions for singularly perturbed competition-diffusion systems

（競争拡散系の特異摂動に対する形式的漸近展開の正当性）

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VALIDITY OF FORMAL ASYMPTOTIC EXPANSIONS FOR SINGULARLY PERTURBED COMPETITION-DIFFUSION SYSTEMS

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Abstract. We consider a two-species competition-diffusion system involving a small parameter $\varepsilon > 0$ and discuss the validity of formal asymptotic expansions of solutions near the sharp interface limit $\varepsilon \approx 0$. We assume that the corresponding ODE system has two stable equilibria. As in the scalar Allen–Cahn equation, it is known that the motion of the sharp interfaces of such systems is governed by the mean curvature flow with a driving force. The formal expansion also suggests that the profile of the transition layers converges to that of a traveling wave solution as $\varepsilon \to 0$. In this paper, we rigorously verify this latter ansatz for a large class of initial data.

The proof relies on a rescaling argument, the super–subsolution method and a Liouville type theorem for eternal solutions of parabolic systems. Roughly speaking, the Liouville type theorem states that any eternal solution that lies between two traveling waves is itself a traveling wave. The same Liouville type theorem was established for the scalar Allen–Cahn equation by Berestycki and Hamel. In view of their importance, we prove the Liouville type theorems in a rather general framework, not only for two-species competition-diffusion systems but also for m-species cooperation-diffusion systems possibly with time periodic or spatially periodic coefficients.

1. Introduction

We consider the following Lotka–Volterra competition-diffusion system:

$$(1.1)\qquad \begin{cases} \varepsilon u_t = \varepsilon D_1 \nabla \cdot (k(x)\nabla u) + \frac{h(x)}{\varepsilon}(R_1 - a_1 u - b_1 v)u, & x \in \Omega,\ t > 0, \\ \varepsilon v_t = \varepsilon D_2 \nabla \cdot (k(x)\nabla v) + \frac{h(x)}{\varepsilon}(R_2 - a_2 u - b_2 v)v, & x \in \Omega,\ t > 0, \\ \partial u/\partial \nu = \partial v/\partial \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where ε is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$, $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$, $R_i, a_i, b_i,\ D_i\ (i = 1, 2)$ are positive constants and $k(x)$, $h(x)$ are positive smooth functions. Our focus is on the behavior of solutions when ε is very small.

In the case of scalar Allen–Cahn equation, its singular limit has been studied by many researchers. It is known that, when ε is very small, solutions starting from rather general initial data develop steep transition layers — or interface — within a very short time (generation of interface), and that the motion of these transition layers is well approximated by the spatially heterogeneous mean curvature flow (motion of interface). There is extensive literature on this subject, particularly on the motion of interface. We do not give a large list of references here. On the other hand, there are much fewer rigorous studies that cover both the generation and the motion of interface; see for example, [1, 2, 7, 15]. In many of those studies, formal asymptotic expansions near the transition layers are used to make a rough approximation of the actual behavior of solutions and are also used to construct super- and subsolutions to establish the limit motion law of the sharp interface rigorously. X. Chen [7] shows that the Hausdorff distance between the layer of the actual solution and the limit interface is of order $O(\varepsilon|\log\varepsilon|)$ for rather general initial data. Alfaro, Matano and Hilhorst [1] improve this interface error estimate to $O(\varepsilon)$.

As regards the profile of interface, Bellettini and Paolini [4] and de Mottoni and Schatzman [16] show that the real solution is well approximated by the formal expansion within an error margin of $O(\varepsilon^2|\log|^2)$ and $O(\varepsilon^2)$, at least on a finite time interval, provided that the initial data is already sufficiently close to the formal expansion. However, whether the actual solutions that start from arbitrary initial data really possess a profile predicted by the formal expansion or not remained long open. In Alfaro and Matano [2], this question was answered affirmatively for a large class of initial data by showing rigorously that the solution converges uniformly to the principal term of the formal expansion as $\varepsilon \to 0$.

In the case of the two-species competition-diffusion system of the form (1.1), its singular limit has been studied by Ei and Yanagida [9] and Hilhorst et al. [11]. Ei and Yanagida [9] prove that the Hausdorff distance between the layer of the actual solution and the limit interface is of order $O(\varepsilon|\log\varepsilon|)$ for a class of initial data which already have steep transition layers. Hilhorst et al. [11] prove that the width of the transition layer is of order $O(\varepsilon)$ and that the interface converges as $\varepsilon \to 0$ to a time-dependent hypersurface whose motion is governed by the mean curvature flow with a driving force. However, to what extent the formal expansion represents the actual profile of the solution was not studied. Our goal is to prove the validity of this formal expansion; namely, we show that the solution profile of (1.1) near the interface converges uniformly to the principal term of the formal expansion for a rather general class of initial data.

Throughout this paper, we assume

$$\frac{a_1}{a_2} < \frac{R_1}{R_2} < \frac{b_1}{b_2}. \tag{1.2}$$

under this assumption, the corresponding ODE system

$$\begin{cases} \dot{u} = f(u,v), & t \in \mathbb{R}, \\ \dot{v} = g(u,v), & t \in \mathbb{R}, \\ u(0;u_0,v_0) = u_0, & v(0;u_0,v_0) = v_0 \end{cases} \tag{1.3}$$

has precisely four equilibria: two stable nodes

$$p^+ := (R_1/a_1, 0),\ p^- := (0, R_2/b_2),$$

a saddle point

$$(u^*, v^*) := \Big(\frac{b_2R_1 - b_1R_2}{a_1b_2 - a_2b_1}, \frac{a_1R_2 - a_2R_1}{a_1b_2 - a_2b_1}\Big)$$

and an unstable node $(0,0)$. Here $\dot{u} = \frac{du}{dt}$ and

$$f(u,v) := (R_1 - a_1u - b_1v)u, \quad g(u,v) := (R_2 - a_2u - b_2v)v.$$

We also assume:

Assumption 1. The following system has a solution:

$$
\begin{cases}
D_1 U'' + f(U,V) = 0, & z \in \mathbb{R}, \\
D_2 V'' + g(U,V) = 0, & z \in \mathbb{R}, \\
(U(-\infty), V(-\infty)) = (R_1/a_1, 0), & (U(+\infty), V(+\infty)) = (0, R_2/b_2).
\end{cases} \tag{1.4}
$$

This assumption implies that the diffusion system

$$
\begin{cases}
U_t = D_1 U_{zz} + f(U,V), & z \in \mathbb{R},\ t \in \mathbb{R}, \\
V_t = D_2 V_{zz} + g(U,V), & z \in \mathbb{R},\ t \in \mathbb{R}
\end{cases} \tag{1.5}
$$

has a stationary wave solution.

The existence and uniqueness of a traveling wave solution of (1.5) are shown by Kan-on [13] under the condition (1.2). The paper also shows continuously dependence of the traveling wave speed on the coefficients of the competition-diffusion system.

As $\varepsilon \to 0$, by a formal asymptotic analysis, the solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) tends to a step function whose values are $(R_1/a_1, 0)$, $(0, R_2/b_2)$ and the boundary $\Gamma(t)$ of the domain in which $(u^\varepsilon, v^\varepsilon)$ converges to $(R_1/a_1, 0)$ moves according to the following equation

$$
V = -C_1(N-1)k(x)\kappa - C_1 \frac{\partial}{\partial n} k(x) - \frac{2k(x)(C_1+C_2)}{K(x)} \frac{\partial}{\partial n} K(x). \tag{1.6}
$$

Here V is the normal velocity, κ is the mean curvature and n is the unit normal vector of $\Gamma(t)$ and $C_1 > 0$ and C_2 are constants defined by (2.16) in Section 2. K(x) is defined by

$$
K(x) = \sqrt{\frac{h(x)}{k(x)}}.
$$

Let S denote the stable manifold of (u^*, v^*) of (1.3), that is,

$$
S := \{(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \lim_{\tau\to\infty} (u(\tau;\xi,\eta), v(\tau;\xi,\eta)) = (u^*, v^*)\},
$$

where $\mathbb{R}_+ := (0, \infty)$ and $(u(\tau;\xi,\eta), v(\tau;\xi,\eta))$ is a solution of (1.3) with initial data (ξ, η). S is called a separatrix and

$$
(\mathbb{R}_+ \times \mathbb{R}_+) \backslash S = \Delta_1 \cup \Delta_2,
$$

where

$$
\Delta_1 := \{(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \lim_{\tau\to\infty} (u(\tau;\xi,\eta), v(\tau;\xi,\eta)) = (R_1/a_1, 0)\},
$$
$$
\Delta_2 := \{(\xi, \eta) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \lim_{\tau\to\infty} (u(\tau;\xi,\eta), v(\tau;\xi,\eta)) = (0, R_2/b_2)\}.
$$

For the proof of this result, see Chapter 12 of Hirsch and Smale [12].

Remark 1.1. The stable manifold S of (u^*, v^*) of (1.3) can be described as follows.

$$
S = \{(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid H(u,v) = 0\}, \tag{1.7}
$$

where $H \in C(\overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+}) \cap C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfies

$$
H(0,0) = 0,\ H_u(u,v) < 0 \ \text{ and } \ H_v(u,v) > 0 \ \text{ for } \ u > 0,\ v > 0.
$$

Moreover there is a function $\zeta \in C(\overline{\mathbb{R}_+}) \cap C^\infty(\mathbb{R}_+)$ such that $\zeta'(u) > 0$ for $u > 0$ and

$$\zeta(u^*) = v^*, \quad H(u,v) = v - \zeta(u) \quad \text{for} \quad u \geq 0, \ v \geq 0 \quad \text{or}$$
$$\zeta(v^*) = u^*, \quad H(u,v) = \zeta(v) - u \quad \text{for} \quad u \geq 0, \ v \geq 0.$$

In fact, by geometric theory of ODE systems, (1.3) has a locally stable manifold

$$\{(u, \zeta(u)) \mid u^* - \varepsilon_0 < u < u^* + \varepsilon_0\},$$

where ζ is a smooth function satisfying $\zeta'(u^*) > 0$, $\zeta(u^*) = v^*$ and

$$\frac{d\zeta}{du}(u) = \frac{g(u,\zeta(u))}{f(u,\zeta(u))}. \tag{1.8}$$

Let $\zeta(u)$, $u \in (U_*, U^*)$ be a function satisfying $\zeta'(u^*) > 0$, $\zeta(u^*) = v^*$ and (1.8) which has the maximal interval of existence. From $\zeta'(u^*) > 0$, $\zeta(u^*) = v^*$, (1.8) and

$$f > 0, \ g > 0 \quad \text{on} \quad (0, u^*) \times (0, v^*),$$
$$f < 0, \ g < 0 \quad \text{on} \quad (u^*, \infty) \times (v^*, \infty),$$

ζ is strictly increasing in (U_*, U^*). Since, for each $(u_0, v_0) \in (0, u^*) \times (0, v^*)$, the solution $(u(t; u_0, v_0), v(t; u_0, v_0))$ of (1.3) tends to $(0, 0)$ as $t \to -\infty$,

$$U_* = 0 \quad \text{and} \quad \lim_{u \to U_*} \zeta(u) = 0.$$

Furthermore it is also easily obtained that

(1) $U^* = \infty$ or
(2) $U^* < \infty$ and $\lim_{u \to U^*} \zeta(u) = \infty$.

In the case that (1) holds, we may put $H(u,v) = v - \zeta(u)$. In the case that (2) holds, we may put $H(u,v) = \zeta^{-1}(v) - u$. Therefore (1.7) holds.

On the other hand, a solution (U, V) of (1.4) satisfies

$$U'(z) < 0, \ V'(z) > 0 \quad (z \in \mathbb{R}).$$

Therefore if $H(u,v) = v - \zeta(u)$, then

$$H(U(-\infty), V(-\infty)) = H(R_1/a_1, 0) = -\zeta(R_1/a_1) < 0,$$
$$H(U(+\infty), V(+\infty)) = H(0, R_2/b_2) = R_2/b_2 > 0,$$

$$\frac{d}{dz} H(U,V) = (H_u(U,V), H_v(U,V)) \cdot (U', V') > 0. \tag{1.9}$$

Hence $(U(z), V(z))$ $(z \in \mathbb{R})$ and $S = \{(u,v) \mid H(u,v) = 0\}$ intersect at exact one point, transversely.

We define Γ_0 as follows:

$$\Gamma_0 := \{x \in \Omega \mid (u_0(x), v_0(x)) \in S\}$$

and we assume:

Assumption 2. u_0, v_0 are continuous on $\overline{\Omega}$ and satisfy $|u_0| + |v_0| > 0$ on $\overline{\Omega}$.

Assumption 3. Γ_0 is a smooth closed hypersurface in Ω and satisfies $\Gamma_0 \cap \partial\Omega = \emptyset$.

Assumption 4. The classical solution $\Gamma(t)$ of (1.6) with initial data $\Gamma(0) = \Gamma_0$ exists on an interval $0 \leq t \leq T$ and is a smooth closed hypersurface in Ω for every $t \in [0, T]$.

Assumption 5. There exists a constant $A_0 > 0$ such that

$$\mathrm{dist}_{\mathbb{R}^2}((u_0(x), v_0(x)), S) \geq A_0 \mathrm{dist}(x, \Gamma_0), \quad x \in \Omega,$$

where

$$\mathrm{dist}_{\mathbb{R}^2}((u, v), S) := \inf_{(\xi,\eta)\in S} |(u - \xi, v - \eta)|, \quad \mathrm{dist}(x, \Gamma_0) := \inf_{y\in\Gamma_0} |x - y|.$$

Remark 1.2. Under Assumption 3, by coordinate transformation and a theorem for a quasilinear parabolic equation in Lunardi [14], there exists $T_{\max} > 0$ such that (1.6) possesses a unique smooth solution $\Gamma(t)$, $0 \leq t < T_{\max}$. In the sequel, we can select any $T \in (0, T_{\max})$ in Assumption 4.

The hypersurface $\Gamma(t)$ divides Ω into two connected components, the inside of $\Gamma(t)$ and the outside of $\Gamma(t)$, denoted by $\Omega_{in}(t)$ and $\Omega_{out}(t)$, respectively. As in [11], we may assume that $(u_0(x), v_0(x))$ satisfies

$$\Omega_{in}(0) = \{x \mid (u_0(x), v_0(x)) \in \Delta_1\}, \quad \Omega_{out}(0) = \{x \mid (u_0(x), v_0(x)) \in \Delta_2\}.$$

Let $(u^\varepsilon, v^\varepsilon)$ be the solution of (1.1) and define $\Gamma^\varepsilon(t)$, $\Omega^\varepsilon_{in}(t)$, $\Omega^\varepsilon_{out}(t)$ as follows:

$$\begin{aligned}
\Gamma^\varepsilon(t) &:= \{x \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in S\},\\
\Omega^\varepsilon_{in}(t) &:= \{x \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \Delta_1\},\\
\Omega^\varepsilon_{out}(t) &:= \{x \mid (u^\varepsilon(x,t), v^\varepsilon(x,t)) \in \Delta_2\}.
\end{aligned}$$

Let $d(x,t)$, $d^\varepsilon(x,t)$ be the signed distance functions associated with $\Gamma(t)$, $\Gamma^\varepsilon(t)$, respectively, that is,

$$d(x,t) := \begin{cases} -\mathrm{dist}(x, \Gamma(t)) \text{ if } x \in \Omega_{in}(t), \\ \mathrm{dist}(x, \Gamma(t)) \text{ if } x \in \Omega_{out}(t), \end{cases}$$

$$d^\varepsilon(x,t) := \begin{cases} -\mathrm{dist}(x, \Gamma^\varepsilon(t)) \text{ if } x \in \Omega^\varepsilon_{in}(t), \\ \mathrm{dist}(x, \Gamma^\varepsilon(t)) \text{ if } x \in \Omega^\varepsilon_{out}(t). \end{cases}$$

Now we state our main theorem.

Theorem 1.3. *Let Assumptions 1,2,3,4 and 5 hold. Let $(u^\varepsilon, v^\varepsilon)$ be the solution of* (1.1) *and let (ϕ_0, ψ_0) be the solution of* (1.4) *satisfying $(\phi_0(0), \psi_0(0)) \in S$. Put*

$$t^\varepsilon := \varepsilon^2 |\log \varepsilon| \quad \text{and} \quad (U_0(\zeta, x), V_0(\zeta, x)) := (\phi_0(K(x)\zeta), \psi_0(K(x)\zeta)).$$

Then there exists a constant $C > 0$ such that the following hold for arbitrary $\mu > 1$.

(i) *If ε is small enough, then, for each $t \in [\mu C t^\varepsilon, T]$, $\Gamma^\varepsilon(t)$ can be expressed as a graph of a smooth function $\eta^\varepsilon(\cdot, t)$ over $\Gamma(t)$ whose norm $\|\eta^\epsilon(\cdot, t)\|_{L^\infty(\Gamma(t))}$ and gradient $\nabla_{\Gamma(t)} \eta^\varepsilon(x,t)$ on $\Gamma(t)$ tend to 0 as $\varepsilon \to 0$ uniformly for $x \in \Gamma(t)$ and $t \in [\mu C t^\varepsilon, T]$.*

(ii) *Let d^ε be the signed distance function associated with Γ^ε. Then*

$$\begin{cases} \lim\limits_{\varepsilon\to 0} \sup\limits_{\mu C t^\varepsilon \leq t \leq T,\, x\in\overline{\Omega}} \Big| u^\varepsilon(x,t) - U_0\Big(\dfrac{d^\varepsilon(x,t)}{\varepsilon}, x\Big)\Big| = 0, \\ \lim\limits_{\varepsilon\to 0} \sup\limits_{\mu C t^\varepsilon \leq t \leq T,\, x\in\overline{\Omega}} \Big| v^\varepsilon(x,t) - V_0\Big(\dfrac{d^\varepsilon(x,t)}{\varepsilon}, x\Big)\Big| = 0. \end{cases} \tag{1.10}$$

(iii) *There exists a family of functions*

$$\theta^\varepsilon : \bigcup_{0\le t\le T}(\Gamma(t)\times\{t\}) \to \mathbb{R}$$

whose L^∞-norms are bounded as $\varepsilon\to 0$, such that

$$(1.11)\qquad \begin{cases} \displaystyle\lim_{\varepsilon\to 0}\sup_{\mu C t^\varepsilon\le t\le T,\, x\in\overline{\Omega}} \Big|u^\varepsilon(x,t) - U_0\Big(\frac{d(x,t)-\varepsilon\theta^\varepsilon(p(x,t),t)}{\varepsilon}, x\Big)\Big| = 0, \\ \displaystyle\lim_{\varepsilon\to 0}\sup_{\mu C t^\varepsilon\le t\le T,\, x\in\overline{\Omega}} \Big|v^\varepsilon(x,t) - V_0\Big(\frac{d(x,t)-\varepsilon\theta^\varepsilon(p(x,t),t)}{\varepsilon}, x\Big)\Big| = 0, \end{cases}$$

where d denotes the signed distance function associated with Γ and $p(x,t)$ denotes a point on $\Gamma(t)$ satisfying $\mathrm{dist}(x,\Gamma(t)) = |x-p(x,t)|$.

The statement (i) means that the interface $\Gamma^\epsilon(t)$ of the solution converges to the hypersurface $\Gamma(t)$ as $\epsilon\to 0$ in the C^1 topology, where $\Gamma(t)$ is the classical solution of (1.6) given in Assumption 4. The statement (iii) implies that the principal term of the formal expansion gives uniform approximation of the real solution. In the proof of Theorem 1.3, we use an idea similar to what is found in [2] for the scalar Allen–Cahn equation. Namely, the proof is based on a rescaling argument, the super–subsolution method and a Liouville type result for eternal solutions of some competition-diffusion systems. However, in our problem, the interface $\Gamma^\epsilon(t)$ of the solution is defined as the inverse image of the one-dimensional separatrix S rather than that of a single point, which makes the estimates of the distance between $\Gamma^\epsilon(t)$ and $\Gamma(t)$ more involved than in the scalar Allen–Cahn case. We also need to extend the Liouville type results in Berestycki and Hamel [5, 6] to parabolic systems.

The rest of the paper is organized as follows. In Section 2, we formally derive the interface equations corresponding to the system (1.1). In Section 3, we state the Liouville type theorems for eternal solutions of parabolic systems which play a key rule in proving the main theorem. Though what we need in the proof of Theorem 1.3 is the Liouville type theorem for a two-species conpetition-diffusion equation, in view of the importance of such Liouville type theorems, we present the results in a more general setting, namely for m-species cooperation-diffusion systems possibly with spatially periodic or time periodic coefficients. In Section 4, we state two lemmas, Lemmas 4.1 and 4.2 that are used in the proof of Theorem 1.3 and we prove Theorem 1.3. At the end of Section 4, we prove Lemmas 4.1 and 4.2. In Section 5, we prove the Liouville type theorems.

2. A Formal derivation of the interface equation

In this section, for the sake of completeness, we formally derive the interface motion equation for the competition-diffusion system (1.1). When $\varepsilon > 0$ is very small, for a short time in the first stage, the effect of diffusion is negligible and the solution $(u^\varepsilon, v^\varepsilon)$ is approximated by the solution of the ordinary differential equation:

$$\varepsilon u_t = \frac{h(x)}{\varepsilon}(R_1 - a_1 u - b_1 v)u, \quad \varepsilon v_t = \frac{h(x)}{\varepsilon}(R_2 - a_2 u - b_2 v)v$$

at each point $x\in\Omega$. Thus the value of $(u^\varepsilon, v^\varepsilon)$ is quickly attracted by p^+ if $(u^\varepsilon(x,0), v^\varepsilon(x,0))\in\Delta_1$ and attracted by p^- if $(u^\varepsilon(x,0), v^\varepsilon(x,0))\in\Delta_2$. Consequently, $(u^\varepsilon, v^\varepsilon)$ develops a steep transition layer between the two regions $\{(u^\varepsilon, v^\varepsilon)\approx p^+\}$ and $\{(u^\varepsilon, v^\varepsilon)\approx p^-\}$, which is located near the hypersurface $\Gamma_0 = \{x \mid$

$(u^\varepsilon(x,0), v^\varepsilon(x,0)) \in S\}$. In the second stage, the effect of diffusion is large enough near the interface Γ_0 and the transition layer starts to move according to the equation (1.6). In what follows we derive this interface equation (1.6) by using a formal asymptotic expansion which is done in Hilhorst et al. [11].

Now we assume that the solution $(u^\varepsilon, v^\varepsilon)$ has the expansions

$$
\begin{aligned}
u^\varepsilon(x,t) &= \widetilde{U}_0(x,t) + \varepsilon \widetilde{U}_1(x,t) + \varepsilon^2 \widetilde{U}_2(x,t) + \cdots, \\
v^\varepsilon(x,t) &= \widetilde{V}_0(x,t) + \varepsilon \widetilde{V}_1(x,t) + \varepsilon^2 \widetilde{V}_2(x,t) + \cdots
\end{aligned} \tag{2.1}
$$

away from the transition layer $\Gamma^\varepsilon(t)$ (outer expansion) and

$$
\begin{aligned}
u^\varepsilon(x,t) &= U_0(\zeta,x,t) + \varepsilon U_1(\zeta,x,t) + \varepsilon^2 U_2(\zeta,x,t) + \cdots, \\
v^\varepsilon(x,t) &= V_0(\zeta,x,t) + \varepsilon V_1(\zeta,x,t) + \varepsilon^2 V_2(\zeta,x,t) + \cdots
\end{aligned} \tag{2.2}
$$

near $\Gamma^\varepsilon(t)$ (inner expansion), where $\zeta = d(x,t)/\varepsilon$ and $d(x,t)$ is the signed distance function with respect to the interface $\Gamma(t)$.

To make the inner and outer expansions consistent, we require that

$$
\begin{aligned}
(U_k(-\infty,x,t), V_k(-\infty,x,t)) &= (\widetilde{U}_k(x,t), \widetilde{V}_k(x,t)) \quad \text{if} \quad x \in \Omega_{in}(t), \\
(U_k(\infty,x,t), V_k(\infty,x,t)) &= (\widetilde{U}_k(x,t), \widetilde{V}_k(x,t)) \quad \text{if} \quad x \in \Omega_{out}(t)
\end{aligned} \tag{2.3}
$$

for all (x,t) near $\Gamma = \bigcup_{t\geq 0} \Gamma(t) \times \{t\}$ and all $k \geq 0$.

Substituting (2.1) into (1.1) and collecting the terms of ε^{-1} and ε^0 respectively, we get

$$
f(\widetilde{U}_0, \widetilde{V}_0) = 0, \quad g(\widetilde{U}_0, \widetilde{V}_0) = 0, \tag{2.4}
$$

$$
\begin{pmatrix} f_u(\widetilde{U}_0, \widetilde{V}_0) & f_v(\widetilde{U}_0, \widetilde{V}_0) \\ g_u(\widetilde{U}_0, \widetilde{V}_0) & g_v(\widetilde{U}_0, \widetilde{V}_0) \end{pmatrix} \begin{pmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.5}
$$

in $Q_{in} = \bigcup_{t\geq 0} \Omega_{in}(t) \times \{t\}$ or in $Q_{out} = \bigcup_{t\geq 0} \Omega_{out}(t) \times \{t\}$. Since the regions Q_{in} and Q_{out} correspond to the regions $\{(u^\varepsilon, v^\varepsilon) \approx p^+\}$ and $\{(u^\varepsilon, v^\varepsilon) \approx p^-\}$ respectively, (2.4) implies

$$
(\widetilde{U}_0, \widetilde{V}_0) = p^+ \quad \text{in} \quad Q_{in}, \quad (\widetilde{U}_0, \widetilde{V}_0) = p^- \quad \text{in} \quad Q_{out}. \tag{2.6}
$$

Thus (2.5) implies

$$
(\widetilde{U}_1, \widetilde{V}_1) = (0,0) \quad \text{in} \quad Q_{in} \cup Q_{out} \tag{2.7}
$$

since the matrix in (2.5) is equal to

$$
\begin{pmatrix} -R_1 & -\frac{b_1}{a_1} R_1 \\ 0 & R_2 - \frac{a_2}{a_1} R_1 \end{pmatrix} \quad \text{in} \quad Q_{in}, \quad \begin{pmatrix} R_1 - \frac{b_1}{b_2} R_2 & 0 \\ -\frac{a_2}{b_2} R_2 & -R_2 \end{pmatrix} \quad \text{in} \quad Q_{out},
$$

both of which are invertible by (1.2). Next substituting (2.2) into (1.1) and collecting the terms of ε^{-1} and ε^0 respectively, we get

$$
k(x) \begin{pmatrix} D_1 U_{0,\zeta\zeta} \\ D_2 V_{0,\zeta\zeta} \end{pmatrix} + h(x) \begin{pmatrix} f(U_0,V_0) \\ g(U_0,V_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.8}
$$

$$
\begin{aligned}
&k(x) \begin{pmatrix} D_1 U_{1,\zeta\zeta} \\ D_2 V_{1,\zeta\zeta} \end{pmatrix} + h(x) \begin{pmatrix} f_u(U_0,V_0) & f_v(U_0,V_0) \\ g_u(U_0,V_0) & g_v(U_0,V_0) \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} \\
&= d_t \begin{pmatrix} U_{0,\zeta} \\ V_{0,\zeta} \end{pmatrix} - \nabla \cdot (k(x) \nabla d) \begin{pmatrix} D_1 U_{0,\zeta} \\ D_2 V_{0,\zeta} \end{pmatrix} - 2k(x) \begin{pmatrix} D_1 \nabla U_{0,\zeta} \cdot \nabla d \\ D_2 \nabla V_{0,\zeta} \cdot \nabla d \end{pmatrix}.
\end{aligned} \tag{2.9}
$$

Both (2.8) and (2.9) are ordinary differential equations with parameters x, t. From (2.3), (2.6) and (2.8), we find

$$U_0(\zeta, x) = \phi_0(K(x)\zeta), \quad V_0(\zeta, x) = \psi_0(K(x)\zeta) \tag{2.10}$$

for all $\zeta \in \mathbb{R}$ and (x, t) near Γ, where $K(x) = \sqrt{h(x)/k(x)}$ and (ϕ_0, ψ_0) is the solution of (1.4) satisfying $(\phi_0(0), \psi_0(0)) \in S$. The following lemma gives estimates of ϕ_0, ψ_0 and their derivatives:

Lemma 2.1 (Lemma 2.1 of [11]). *There exist constants $C > 0$, $M > 0$ such that*

$$\begin{gathered} 0 < \phi_0(0) < Ce^{-M|z|}, \quad 0 < R_2 - \psi_0(z) < Ce^{-M|z|} \quad \text{for} \quad z \geq 0, \\ 0 < R_1 - \phi_0 < Ce^{-M|z|}, \quad 0 < \psi_0(z) < Ce^{-M|z|} \quad \text{for} \quad z < 0, \\ \phi_0'(z) > 0, \quad \psi_0'(z) < 0 \quad \text{for} \quad z \in \mathbb{R}, \\ |\phi_0^{(j)}(z)| < Ce^{-M|z|}, \quad |\psi_0^{(j)}(z)| < Ce^{-M|z|} \quad \text{for} \quad z \in \mathbb{R}, \quad j = 1, 2. \end{gathered}$$

Substituting (2.10) into (2.9), we get

(2.11)
$$\begin{aligned} k(x) \begin{pmatrix} D_1 U_{1,\zeta\zeta} \\ D_2 V_{1,\zeta\zeta} \end{pmatrix} + h(x) \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} = K(x) d_t \begin{pmatrix} \phi_0' \\ \psi_0' \end{pmatrix} \\ - K(x) \nabla \cdot (k(x) \nabla d) \begin{pmatrix} D_1 \phi_0' \\ D_2 \psi_0' \end{pmatrix} - 2k(x) \nabla d \cdot \nabla K(x) \begin{pmatrix} D_1 \{\phi_0' + \zeta K(x) \phi_0''\} \\ D_2 \{\psi_0' + \zeta K(x) \psi_0''\} \end{pmatrix}. \end{aligned}$$

The following two lemmas imply a solvability condition of the equation (2.11).

Lemma 2.2 (Lemma 2.2 of [11]). *There exists a solution $(\phi^*(z, x), \psi^*(z, x))$ of the equation*

$$\begin{pmatrix} D_1 \phi^*_{zz} \\ D_2 \psi^*_{zz} \end{pmatrix} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} \phi^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (z \in \mathbb{R}), \tag{2.12}$$

satisfying $\phi^* > 0$ and $\psi^* < 0$. Moreover, the solution of (2.12) *is unique up to multiplication of a constant.*

Lemma 2.3 (Lemma 2.3 of [11]). *Let $(A_1(z, x, t), A_2(z, x, t))$ be given and assume that $A_j(z, x, t) = O(e^{-\delta|z|})$ as $|z| \to \infty$ for some $\delta > 0$ for $j = 1, 2$. Then for each fixed (x, t), the following equation*

$$\begin{pmatrix} D_1 \phi_{zz} \\ D_2 \psi_{zz} \end{pmatrix} + \begin{pmatrix} f_u(U_0, V_0) & f_v(U_0, V_0) \\ g_u(U_0, V_0) & g_v(U_0, V_0) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} A_1(z, x, t) \\ A_2(z, x, t) \end{pmatrix} \tag{2.13}$$

has a solution if and only if

$$\int_{\mathbb{R}} \{\phi^*(z, x) A_1(z, x, t) + \psi^*(z, x) A_2(z, x, t)\} dz = 0.$$

In addition, the solution, if it exists, is unique under the normalization condition $\phi(0, x, t) = 0$ and satisfies

$$\phi(z, x, t) = O(e^{-\hat{\delta}|z|}), \quad \psi(z, x, t) = O(e^{-\hat{\delta}|z|}) \tag{2.14}$$

for some $\hat{\delta} \in (0, \delta]$ as $|z| \to \infty$.

By Lemma 2.3, the solvability condition for (2.11) is rewritten as

$$Kd_t\int_{\mathbb{R}}\{\phi^*\phi_0'+\psi^*\psi_0'\}dz-(K\nabla\cdot(k(x)\nabla d)+2k\nabla d\cdot\nabla K)\int_{\mathbb{R}}\{D_1\phi^*\phi_0'+D_2\psi^*\psi_0'\}dz$$
$$-2k\nabla d\cdot\nabla K\int_{\mathbb{R}}\{D_1z\phi^*\phi_0''+D_2z\psi^*\psi_0''\}dz=0.$$

Lemma 2.2 assures that $\int_{\mathbb{R}}\{\phi^*\phi_0'+\psi^*\psi_0'\}dz<0$, which implies

$$d_t-C_1\nabla(k(x)\nabla d)=\frac{2(C_1+C_2)k\nabla d\nabla K}{K}, \tag{2.15}$$

$$C_1=\frac{\int_{\mathbb{R}}\{D_1\phi^*\phi_0'+D_2\psi^*\psi_0'\}dz}{\int_{\mathbb{R}}\{\phi^*\phi_0'+\psi^*\psi_0'\}dz}>0,\quad C_2=\frac{\int_{\mathbb{R}}\{D_1z\phi^*\phi_0''+D_2z\psi^*\psi_0''\}dz}{\int_{\mathbb{R}}\{\phi^*\phi_0'+\psi^*\psi_0'\}dz}. \tag{2.16}$$

Let us derive the equation of the interface from (2.15). Since ∇d coincides with the outward normal unit vector of the interface Γ_t, one easily sees that $-d_t=V$, where V is the normal velocity of the interface Γ_t. It is also known that the mean curvature κ of the interface is equal to $\Delta d/(N-1)$. Thus the equation (2.11) is equivalent to (1.6).

3. Liouville type theorems for eternal solutions of a parabolic system

Before starting the proof of the main theorem, we present Liouville type theorems for eternal solutions of reaction-diffusion systems. These are extensions of similar Liouville type results of Berestycki and Hamel [5, 6] to systems of reaction-diffusion equations. Though what we need in the proof of the main theorem is only a special case of such Liouville type results, we state them in a rather general setting since we think those results are important in their own right.

3.1. **Statement of Liouville type theorems (homogeneous case).** Let us first state a result on a reaction-diffusion system of the form:

$$\begin{cases} u_t=D_1\Delta u+f_1(u,v), & x\in\mathbb{R}^N,\ t\in\mathbb{R},\\ v_t=D_2\Delta v+f_2(u,v), & x\in\mathbb{R}^N,\ t\in\mathbb{R},\end{cases} \tag{3.1}$$

where D_1, D_2 are positive constants and f_1,f_2 are smooth functions such that (3.1) is a *competition-diffusion system*, that is, it holds that

$$f_{1,v}=\partial_v f_1<0,\ \ f_{2,u}<0\ \text{ in }\ (p_1^-,p_1^+)\times(p_2^+,p_2^-). \tag{3.2}$$

Furthermore, we assume that $F=(f_1,f_2)$ has two linearly stable equilibria

$$p^+=(p_1^+,p_2^+),\ p^-=(p_1^-,p_2^-)\ \ (p_1^-<p_1^+,\ p_2^+<p_2^-),$$

that is, for some constants $\lambda_\pm>0$ and vectors $\varphi^\pm={}^t(\varphi_1^\pm,\varphi_2^\pm)$ $(\varphi_2^\pm<0<\varphi_1^\pm)$,

$$F(p^\pm)=(0,0),\quad DF(p^\pm)\varphi^\pm=-\lambda_\pm\varphi^\pm, \tag{3.3}$$

where

$$DF(p^\pm)=\begin{pmatrix} f_{1,u}(p^\pm) & f_{1,v}(p^\pm)\\ f_{2,u}(p^\pm) & f_{2,v}(p^\pm)\end{pmatrix}.$$

We also assume:

$$
\text{(A)} \qquad \begin{cases} \text{(3.1) has a traveling wave solution} \\ (u(x,t), v(x,t)) = (\phi_1(n\cdot x - ct), \phi_2(n\cdot x - ct)) \\ \text{with a direction } n \in \mathbb{R}^N \ (|n| = 1) \ \text{ and a speed } c \in \mathbb{R} \text{ satisfying} \\ (\phi_1(\mp\infty), \phi_2(\mp\infty)) = p^\pm \ \text{ and } \phi_1' < 0, \ \phi_2' > 0 \text{ in } \mathbb{R}. \end{cases}
$$

Remark 3.1. The assumption (A) means that the system (3.1) has a planar wave solution whose direction and speed are n and c, respectively.

Theorem 3.2 (Liouville type theorem for a competition-diffusion system)**.** *Assume* (A), (3.3) *and* (3.2). *Let* $(u(x,t), v(x,t))$ $(x \in \mathbb{R}^N, \ t \in \mathbb{R})$ *be a solution of* (3.1) *which satisfies that there are a unit vector n, some constants $c \in \mathbb{R}$, $a < b$ such that, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$
(3.4) \qquad \begin{cases} \phi_1(n\cdot x - ct - a) \le u(x,t) \le \phi_1(n\cdot x - ct - b), \\ \phi_2(n\cdot x - ct - b) \le v(x,t) \le \phi_2(n\cdot x - ct - a), \end{cases}
$$

where (ϕ_1, ϕ_2) *is a function satisfying* (A) *with the speed c. Then there exists $\theta_0 \in (a,b)$ such that, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$
(u(x,t), v(x,t)) = (\phi_1(n\cdot x - ct - \theta_0), \phi_2(n\cdot x - ct - \theta_0)).
$$

Roughly speaking, Theorem 3.2 means that eternal solutions that are sandwiched between two planar wave solutions are precisely planar waves. The following Theorems 3.3, 3.5 and 3.7 have similar meaning.

Let us reformulate Theorem 3.2 in more general settings. As one easily sees, a two-species competition system can be converted to a two-species cooperation-diffusion system by the change of variables $(u,v) \mapsto (u,-v)$. Therefore it suffices to state the results for m-species cooperation-diffusion systems.

First, let us define order relations in $\mathbb{R}^k$ and in $X = C(\mathbb{R}^l : \mathbb{R}^k)$ as follows:

$$
\begin{aligned}
&(u_1, u_2, \cdots, u_k) \preceq (v_1, v_2, \cdots, u_k) \ \text{ if } \ u_i \le v_i, \quad (i = 1, 2, \cdots, k), \\
&u := (u_1, u_2, \cdots, u_k) \prec v := (v_1, v_2, \cdots, v_k) \ \text{ if } \ u \preceq v \ \text{ and } \ u \ne v, \\
&(u_1, u_2, \cdots, u_k) \ll (v_1, v_2, \cdots, v_k) \ \text{ if } \ u_i < v_i, \quad (i = 1, 2, \cdots, k),
\end{aligned}
$$

$$
\begin{aligned}
&u \preceq v \ \text{ if } \ u(x) \preceq v(x) \ \text{ for all } \ x \in \mathbb{R}^l, \\
&u \prec v \ \text{ if } \ u \preceq v \ \text{ and } \ u \ne v, \\
&u \ll v \ \text{ if } \ u(x) \ll v(x) \ \text{ for all } \ x \in \mathbb{R}^l.
\end{aligned}
$$

Now we consider a reaction-diffusion system of the form:

$$
(3.5) \qquad \begin{cases} u_{1,t} = D_1 \Delta u_1 + f_1(u_1, u_2, \cdots, u_m), \quad x \in \mathbb{R}^N, \ t \in \mathbb{R}, \\ u_{2,t} = D_2 \Delta u_2 + f_2(u_1, u_2, \cdots, u_m), \quad x \in \mathbb{R}^N, \ t \in \mathbb{R}, \\ \qquad \vdots \\ u_{m,t} = D_m \Delta u_m + f_m(u_1, u_2, \cdots, u_m), \quad x \in \mathbb{R}^N, \ t \in \mathbb{R}, \end{cases}
$$

where D_l $(l = 1, 2, \cdots, m)$ are positive constants and $f_1, f_2, \cdots, f_m$ are smooth functions such that (3.5) is a *cooperation-diffusion system*, that is, it holds that

$$
(3.6) \quad f_{k,u_l} = \partial_{u_l} f_k \ge 0 \ \ (k \ne l) \ \text{ in } \ (p^-, p^+) := (p_1^-, p_1^+) \times (p_2^-, p_2^+) \times \cdots \times (p_m^-, p_m^+),
$$

$$DF(u) \text{ is an irreducible matrix for each } u \in (p^-, p^+), \tag{3.7}$$

where

$$DF(p^\pm) = \begin{pmatrix} f_{1,u_1}(p^\pm) & f_{1,u_2}(p^\pm) & \cdots & f_{1,u_m}(p^\pm) \\ f_{2,u_1}(p^\pm) & f_{2,u_2}(p^\pm) & \cdots & f_{2,u_m}(p^\pm) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,u_1}(p^\pm) & f_{m,u_2}(p^\pm) & \cdots & f_{m,u_m}(p^\pm) \end{pmatrix}.$$

We say that an $m \times m$ matrix $A = (a^{kl})$ is *reducible* if there is $\emptyset \neq \Lambda \subsetneq \{1, 2, \cdots, m\}$ such that

$$a^{kl} = 0 \quad \text{for} \quad k \in \Lambda, \ l \notin \Lambda.$$

We say that an $m \times m$ matrix A is *irreducible* if A is not reducible.

Furthermore, we assume that $F = (f_1, f_2, \cdots, f_m)$ has two linearly stable equilibria

$$p^+ = (p_1^+, p_2^+, \cdots, p_m^+) \gg p^- = (p_1^-, p_2^-, \cdots, p_m^-),$$

that is, for some constants $\lambda_\pm > 0$ and unit vectors

$$\varphi^\pm = {}^t(\varphi_1^\pm, \varphi_2^\pm, \cdots, \varphi_m^\pm) \gg {}^t(0, 0, \cdots, 0),$$

$$F(p^\pm) = (0, 0, \cdots, 0), \quad DF(p^\pm)\varphi^\pm = -\lambda_\pm \varphi^\pm. \tag{3.8}$$

We also assume:

$$\text{(A')} \qquad \begin{cases} \text{(3.5) has a traveling wave solution } u(x,t) = \phi(n \cdot x - ct) \\ \text{with a direction } n \in \mathbb{R}^N \ (|n| = 1) \ \text{ and a speed } c \in \mathbb{R} \\ \text{satisfying } \phi(\mp\infty) = p^\pm \text{ and } \phi' \ll (0, 0, \cdots, 0) \text{ in } \mathbb{R}. \end{cases}$$

Theorem 3.3 (Liouville type theorem for a cooperation-diffusion system)**.** *Assume* (A′), (3.8) *and* (3.6). *Let* $u(x,t)$ $(x \in \mathbb{R}^N, \ t \in \mathbb{R})$ *be a solution of* (3.5) *which satisfies that there are a unit vector* n, *some constants* $c \in \mathbb{R}$, $a < b$ *such that, for all* $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

$$\phi(n \cdot x - ct - a) \preceq u(x,t) \preceq \phi(n \cdot x - ct - b), \tag{3.9}$$

where ϕ *is a function satisfying* (A′) *with the speed* c. *Then there exists a function* $\widetilde{\phi}$ *satisfying* (A′) *such that* $u(x,t) = \widetilde{\phi}(n \cdot x - ct)$. *If, in addition, assume* (3.7), *then there exists* $\theta_0 \in (a,b)$ *such that*

$$u(x,t) = \phi(n \cdot x - ct - \theta_0) \quad \text{for all} \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

3.2. **Statement of Liouville type theorems (inhomogeneous case).** The Liouville type theorem in the previous subsection can be extended to more general systems. First we consider the following time periodic systems:

$$\begin{cases} u_{l,t} = \displaystyle\sum_{i,j=1}^N D_l^{ij}(t) u_{l,x_i x_j} + q_l(t) \cdot \nabla u_l + f_l(t, u_1, \cdots, u_m) \\ \qquad \text{for} \quad x \in \mathbb{R}^N, \ t \in \mathbb{R} \quad (l = 1, 2, \cdots, m), \end{cases} \tag{3.10}$$

where

$$D_l^{ij}(t) \quad (i,j = 1, 2, \cdots, N), \quad q_l(t) \in \mathbb{R}^N \quad (l = 1, 2, \cdots, m),$$
$$F(t, u_1, \cdots, u_m) = (f_1, f_2, \cdots, f_m)$$

are Hölder continuous in t, smooth in $(u_1, u_2, \cdots, u_m)$ and $f_1, f_2, \cdots, f_m$ satisfy

$$f_{k,u_l}(t,u) = \partial_{u_l} f_k \geq 0 \quad (k \neq l) \quad \text{for each} \quad u \in (p^-(t), p^+(t)), \ t \in \mathbb{R}, \tag{3.11}$$

$$DF(t,u) \text{ is an irreducible matrix for each } u \in (p^-(t), p^+(t)),\ t \in \mathbb{R}, \tag{3.12}$$

where

$$DF(t,p^{\pm}) = \begin{pmatrix} f_{1,u_1}(t,p^{\pm}) & f_{1,u_2}(t,p^{\pm}) & \cdots & f_{1,u_m}(t,p^{\pm}) \\ f_{2,u_1}(t,p^{\pm}) & f_{2,u_2}(t,p^{\pm}) & \cdots & f_{2,u_m}(t,p^{\pm}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,u_1}(t,p^{\pm}) & f_{m,u_2}(t,p^{\pm}) & \cdots & f_{m,u_m}(t,p^{\pm}). \end{pmatrix}.$$

We assume that there are $\alpha_2 \geq \alpha_1 > 0$, $T > 0$ such that

$$\alpha_1 |\xi|^2 \leq \sum_{i,j=1}^{N} D_l^{ij}(t)\xi_i\xi_j \leq \alpha_2 |\xi|^2 \quad \text{for} \quad t \in \mathbb{R},\ \xi \in \mathbb{R}^N \ (l = 1,2,\cdots,m), \tag{3.13}$$

$$\begin{aligned} &D_l^{ij}(t+T) = D_l^{ij}(t),\ q_l(t+T) = q_l(t),\ f_l(t+T,\cdot) = f_l(t,\cdot) \\ &\text{for} \quad t \in \mathbb{R} \quad (i,j = 1,2,\cdots,N;\ l = 1,2,\cdots,m) \end{aligned} \tag{3.14}$$

and there are two smooth functions

$$p^+(t) = (p_1^+(t), p_2^+(t), \cdots, p_m^+(t)) \gg p^-(t) = (p_1^-(t), p_2^-(t), \cdots, p_m^-(t))$$

such that, for some constants $\lambda_{\pm} > 0$ and vector valued functions

$$\varphi^{\pm} = {}^t(\varphi_1^{\pm}, \varphi_2^{\pm}, \cdots, \varphi_m^{\pm}) \gg {}^t(0,0,\cdots,0),$$

$$\begin{aligned} &\frac{dp^{\pm}}{dt} - F(t,p^{\pm}) = (0,0,\ldots,0),\quad \frac{d\varphi^{\pm}}{dt} - DF(t,p^{\pm})\varphi^{\pm} = \lambda_{\pm}\varphi^{\pm}, \\ &p^{\pm}(\cdot + T) = p^{\pm}(\cdot),\ \varphi^{\pm}(\cdot + T) = \varphi^{\pm}(\cdot). \end{aligned} \tag{3.15}$$

We also assume:

$$\text{(A1)} \qquad \begin{cases} \text{(3.10) has a pulsating traveling wave solution} \\ u(x,t) = \phi(n \cdot x - ct, t) \text{ with a direction } n \in \mathbb{R}^N \text{ and} \\ \text{a speed } c \in \mathbb{R} \text{ satisfying } \phi(\mp\infty, t) = p^{\pm}(t) \text{ and} \\ \phi(z, t+T) = \phi(z,t),\ \phi_z(z,t) \ll (0,0,\cdots,0) \text{ for } z \in \mathbb{R},\ t \in \mathbb{R}. \end{cases}$$

Remark 3.4. As shown in Bao and Wang [3], the following time-periodic Lotka–Volterra competition-diffusion system

$$\begin{cases} u_t = d_1(t)u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), \\ v_t = d_2(t)v_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v), \end{cases} \quad x \in \mathbb{R},\ t \in \mathbb{R}$$

satisfies (A1) under the assumption stated below (after the change of variables $(u,v) \mapsto (u,-v)$):

- $d_i(t),\ r_i(t),\ a_i(t),\ b_i(t) \in C^{\frac{\theta}{2}}(\mathbb{R})$ $(i = 1,2; \theta \in (0,1))$ are T-periodic functions satisfying $d_i(t) > 0,\ a_i(t) > 0,\ b_i(t) > 0$ for any $t \in [0,T]$, $\overline{r_i} = \frac{1}{T}\int_0^T r_i(t)dt > 0$ and

$$\overline{r_1} < \min_t \Big(\frac{b_1(t)}{b_2(t)}\Big)\overline{r_2},\quad \overline{r_2} < \min_t \Big(\frac{a_2(t)}{a_1(t)}\Big)\overline{r_1},$$

$$\overline{r_1} + \overline{r_2} > \max_t \Big(\frac{a_2(t)}{a_1(t)}\Big)\overline{r_1},\quad \overline{r_1} + \overline{r_2} > \max_t \Big(\frac{b_1(t)}{b_2(t)}\Big).$$

Theorem 3.5 (Liouville type theorem for t-periodic system)**.** *Assume* (A1), (3.13), (3.14), (3.15) *and* (3.11)*. Let u be a solution of* (3.10) *which satisfies that there are a unit vector n, some constants $c \in \mathbb{R}$, $a < b$ such that, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$\phi(n \cdot x - ct - a, t) \preceq u(x,t) \preceq \phi(n \cdot x - ct - b, t), \tag{3.16}$$

where ϕ is a function satisfying (T) *with the speed c. Then there exists a function $\widetilde{\phi}$ satisfying* (A1) *such that $u(x,t) = \widetilde{\phi}(n \cdot x - ct, t)$. If, in addition, assume* (3.12)*, then there exists $\theta_0 \in (a,b)$ such that*

$$u(x,t) = \phi(n \cdot x - ct - \theta_0, t) \quad \text{for all} \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Next we state the theorem for spatially periodic systems:

$$\begin{cases} u_{l,t} = \displaystyle\sum_{i,j=1}^{N} D_l^{ij}(x) u_{l,x_i x_j} + q_l(x) \cdot \nabla u_l + f_l(x, u_1, \cdots, u_m) \\ \qquad \text{for} \;\; x \in \mathbb{R}^N, \; t \in \mathbb{R} \;\; (l = 1, 2, \cdots, m), \end{cases} \tag{3.17}$$

where

$$\begin{aligned} & D_l^{ij}(x) \;\; (i,j = 1,2,\cdots,N), \quad q_l(x) \in \mathbb{R}^N \;\; (l = 1,2,\cdots,m), \\ & F(x, u_1, \cdots, u_m) = (f_1, f_2, \cdots, f_m) \end{aligned}$$

are Hölder continuous in x, smooth in $(u_1, u_2, \cdots, u_m)$ and $f_1, f_2, \cdots, f_m$ satisfy

$$f_{k,u_l}(x,u) = \partial_{u_l} f_k \geq 0 \;\; (k \neq l) \;\; \text{for each} \;\; u \in (p^-(x), p^+(x)), \; x \in \mathbb{R}^N, \tag{3.18}$$

$$DF(x,u) \text{ is an irreducible matrix for each } u \in (p^-(x), p^+(x)), \; x \in \mathbb{R}^N, \tag{3.19}$$

where

$$DF(x, p^\pm) = \begin{pmatrix} f_{1,u_1}(x,p^\pm) & f_{1,u_2}(x,p^\pm) & \cdots & f_{1,u_m}(x,p^\pm) \\ f_{2,u_1}(x,p^\pm) & f_{2,u_2}(x,p^\pm) & \cdots & f_{2,u_m}(x,p^\pm) \\ \vdots & \vdots & \ddots & \vdots \\ f_{m,u_1}(x,p^\pm) & f_{m,u_2}(x,p^\pm) & \cdots & f_{m,u_m}(x,p^\pm) \end{pmatrix}.$$

We assume that there are $\alpha_2 \geq \alpha_1 > 0$, $L_j > 0$ $(j = 1, 2, \cdots, N)$ such that

$$\alpha_1 |\xi|^2 \leq \sum_{i,j=1}^{N} D_l^{ij}(x) \xi_i \xi_j \leq \alpha_2 |\xi|^2 \;\; \text{for} \;\; x \in \mathbb{R}^N, \; \xi \in \mathbb{R}^N \;\; (l = 1, 2, \cdots, m), \tag{3.20}$$

$$\begin{aligned} & D_l^{ij}(x+k) = D_l^{ij}(x), \; q_l(x+k) = q_l(x), \; f_l(x+k, \cdot) = f_l(x, \cdot) \\ & \text{for} \;\; x \in \mathbb{R}^N, \; k \in \mathbb{L} := L_1\mathbb{Z} \times L_2\mathbb{Z} \times \cdots \times L_N\mathbb{Z} \\ & (i,j = 1,2,\cdots,N; \; l = 1,2,\cdots,m), \end{aligned} \tag{3.21}$$

and there are two smooth functions

$$p^+(x) = (p_1^+(x), p_2^+(x), \cdots, p_m^+(x)) \gg p^-(x) = (p_1^-(x), p_2^-(x), \cdots, p_m^-(x))$$

such that, for some constants $\lambda_\pm > 0$ and vector valued functions

$$\varphi^\pm = {}^t(\varphi_1^\pm, \varphi_2^\pm, \cdots, \varphi_m^\pm) \gg {}^t(0, 0, \cdots, 0),$$

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{N} D_l^{ij}(x)p^{\pm}_{l,x_ix_j} + q_l(x)\cdot\nabla p_l^{\pm} + f_l(x,p^{\pm}) = 0, \\
\displaystyle\sum_{i,j=1}^{N} D_l^{ij}(x)\varphi^{\pm}_{l,x_ix_j} + q_l(x)\cdot\nabla \varphi_l^{\pm} + \sum_{j=1}^{m} f_{l,u_j}(x,p^{\pm})\varphi_j^{\pm} = -\lambda_{\pm}\varphi_l^{\pm}, \\
\qquad \text{for} \ \ x\in\mathbb{R}^N \quad (l=1,2,\cdots,m), \\
p^{\pm}(\cdot+k) = p^{\pm}(\cdot), \ \varphi^{\pm}(\cdot+k)=\varphi^{\pm}(\cdot) \ \text{ for } \ k\in\mathbb{L}.
\end{cases} \tag{3.22}
$$

We also assume:

$$
\text{(A2)} \qquad \begin{cases}
\text{(3.17) has a solution } u(x,t) \text{ such that} \\
\text{for a constant } c\neq 0 \text{ and a unit vector } n\in\mathbb{R}^N, \\
u(x-k,t) = u(x,t+k\cdot n/c), \ \displaystyle\lim_{k\in\mathbb{L},k\cdot n\to\pm\infty} u(x+k,t)\to p^{\mp}(x), \\
cu_t(x,t) \gg (0,0,\cdots,0) \ \text{ for } t\in\mathbb{R}, \ x\in\mathbb{R}^N \text{ and } k\in\mathbb{L}.
\end{cases}
$$

Remark 3.6. In the case where $m = 1$, namely, a scalar bistable equation with spatially periodic coefficients on $\mathbb{R}^N$

$$
u_t - \mathrm{div}\,(A(x)\nabla u) = F(x,u), \quad x\in\mathbb{R}^N, \ t\in\mathbb{R}, \tag{3.23}
$$

Ducrot [8] shows that (A2) is satisfies if and only if there exists no stationary front in the direction n under the following assumption:

- $A : \mathbb{T}^N := \mathbb{R}^N/\mathbb{Z}^N \to \mathcal{S}_N$ is a symmetric matrix valued function of the class $C^{1+\gamma}$ for some $\gamma\in(0,1)$ and satisfies (3.20). F is of the class C^γ in x uniformly with respect to $u\in\mathbb{R}$, the partial derivative F_u is continuous on $\mathbb{T}^N\times\mathbb{R}$. Moreover the equation

$$
\begin{cases}
u_t - \mathrm{div}\,(A(x)\nabla u) = F(x,u), \quad x\in\mathbb{T}^N, \ t>0, \\
u(x,0) = u_0(x), \quad x\in\mathbb{T}^N
\end{cases}
$$

  has two stable stationary states $\psi^- < \psi^+$ with $\psi^{\pm}\in C^2(\mathbb{T}^N)$ and there is no stable stationary state between ψ^+ and ψ^-.

Fang and Zhao [10] give sufficient conditions for (A2) in a more abstract framework.

Theorem 3.7 (Liouville type theorem for x-periodic system)**.** *Assume* (A2)*,* (3.20)*,* (3.21)*,* (3.22) *and* (3.18)*. Let u be a solution of* (3.17) *and v be a solution as in* (A2) *with a speed $c\neq 0$ and a unit vector $n\in\mathbb{R}^N$ which satisfy for some constants $a,b\in\mathbb{R}$ and for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$,*

$$
v(x,t+a) \preceq u(x,t) \preceq v(x,t+b). \tag{3.24}
$$

Then u satisfies (A2) *with the speed $c\neq 0$ and the unit vector $n\in\mathbb{R}^N$. If, in addition, assume* (3.19)*, then there exists θ_0 between a and b such that, for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$,*

$$
u(x,t) = v(x,t+\theta_0).
$$

4. Proof of the main theorem

As we mentioned before, the proof of Theorem 1.3 is based on a rescaling argument and the following two statements (Lemmas 4.1 and 4.2).

Lemma 4.1 ([11]). *Let $(u^\varepsilon, v^\varepsilon)$ be the solution of* (1.1). *Under the assumptions in section 1, there are $C > 0$, $A_i > 0$ $(i = 1, 2, 3)$ and $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0]$, $(x, t) \in \overline{\Omega} \times [Ct^\varepsilon, T]$,*

$$U_0\Big(\frac{d(x,t) + \varepsilon A_1}{\varepsilon}, x\Big) - \varepsilon A_2 \le u^\varepsilon(x,t) \le U_0\Big(\frac{d(x,t) - \varepsilon A_1}{\varepsilon}, x\Big) + \varepsilon A_2,$$
$$V_0\Big(\frac{d(x,t) - \varepsilon A_1}{\varepsilon}, x\Big) - \varepsilon A_3 \le v^\varepsilon(x,t) \le V_0\Big(\frac{d(x,t) + \varepsilon A_1}{\varepsilon}, x\Big) + \varepsilon A_3.$$

In the next lemma, we consider the following system.

$$\text{(4.1)} \qquad \begin{cases} u_t = D_1 \Delta u + (R_1 - a_1 u - b_1 v)u, & x \in \mathbb{R}^N,\ t \in \mathbb{R}, \\ v_t = D_2 \Delta v + (R_2 - a_2 u - b_2 v)v, & x \in \mathbb{R}^N,\ t \in \mathbb{R}, \end{cases}$$

where D_i, R_i, a_i, b_i $(i = 1, 2)$ are positive constants.

Lemma 4.2 (Liouville type theorem). *Suppose that* Assumption 1 *and* (1.2) *hold. Let $u(x,t)$, $v(x,t)$ $(x \in \mathbb{R}^N,\ t \in \mathbb{R})$ be a solution of* (4.1) *satisfying, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$\text{(4.2)} \qquad \begin{cases} \phi(n \cdot x - a) \le u(x,t) \le \phi(n \cdot x - b), \\ \psi(n \cdot x - b) \le v(x,t) \le \psi(n \cdot x - a), \end{cases}$$

where n is a unit vector, $a < b$ are some constants and (ϕ, ψ) is a solution of (1.4). *Then there is a $\theta_0 \in (a, b)$ such that, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,*

$$u(x,t) = \phi(n \cdot x - \theta_0), \quad v(x,t) = \psi(n \cdot x - \theta_0).$$

This lemma is a special case of Theorem 3.2. In fact, (3.2) is obviously satisfied. (1.2) and Assumption 1 imply (3.3) and (A), respectively.

Remark 4.3. From Lemma 4.1, the following holds. (See Theorem 2 in [11].)

Theorem 4.4. *Let $C > 0$, $\varepsilon_0 > 0$ be constants in Lemma 4.1. Then there is a constant $\widetilde{C} > 0$ such that*

$$d_{\mathcal{H}}(\Gamma^\varepsilon(t), \Gamma(t)) < \widetilde{C}\varepsilon \quad \text{for} \quad t \in [Ct^\varepsilon, T],\ \varepsilon \in (0, \varepsilon_0],$$

where $d_{\mathcal{H}}$ denotes the Hausdorff distance between compact sets.

4.1. **Proof of statement (ii).**

Poof of (*ii*) *of Theorem 1.3.* Fix $\mu > 1$, $T_1 \in (T, T_{\max})$ and let C be the constant in Lemma 4.1. To obtain a contradiction, suppose that (*ii*) does not hold. Then there exist $\eta > 0$, $\varepsilon_j > 0$, $(x_j, t_j) \in \overline{\Omega} \times [\mu C t^{\varepsilon_j}, T]$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and for all $j \in \mathbb{N}$,

$$\begin{cases} \Big| u^{\varepsilon_j}(x_j, t_j) - U_0\Big(\frac{d^{\varepsilon_j}(x_j, t_j)}{\varepsilon_j}, x_j\Big)\Big| \ge \eta \quad \text{or} \\ \Big| v^{\varepsilon_j}(x_j, t_j) - V_0\Big(\frac{d^{\varepsilon_j}(x_j, t_j)}{\varepsilon_j}, x_j\Big)\Big| \ge \eta. \end{cases}$$

By extracting a subsequence, it holds that

$$\text{(4.3)} \qquad \Big| u^{\varepsilon_j}(x_j, t_j) - U_0\Big(\frac{d^{\varepsilon_j}(x_j, t_j)}{\varepsilon_j}, x_j\Big)\Big| \ge \eta \quad \text{for all} \quad j \in \mathbb{N} \quad \text{or}$$

$$\text{(4.4)} \qquad \Big| v^{\varepsilon_j}(x_j, t_j) - V_0\Big(\frac{d^{\varepsilon_j}(x_j, t_j)}{\varepsilon_j}, x_j\Big)\Big| \ge \eta \quad \text{for all} \quad j \in \mathbb{N}.$$

Since it is irrelevant in the later argument whether (4.3) holds or (4.4) holds, we may assume that (4.3) holds. By the same reason, we may assume

$$
\begin{aligned}
&x_j \in \Omega^{\varepsilon_j}_{out}(t_j) \cup \Gamma^{\varepsilon_j}(t_j) \quad \text{for all} \quad j \in \mathbb{N}, \quad \text{that is,} \\
&(u^{\varepsilon_j}(x_j,t_j), v^{\varepsilon_j}(x_j,t_j)) \in \Delta_2 \cup S \quad \text{for all} \quad j \in \mathbb{N}.
\end{aligned} \tag{4.5}
$$

Then it holds that

$$
\operatorname{dist}(x_j, \Gamma^{\varepsilon_j}(t_j)) = O(\varepsilon_j), \quad \operatorname{dist}(x_j, \Gamma(t_j)) = O(\varepsilon_j) \quad \text{as} \quad j \to \infty. \tag{4.6}
$$

In fact, if this is not true, then, by Theorem 4.4 and extracting a subsequence, it holds that

$$
\begin{aligned}
&\Big|\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j}\Big| = \frac{\operatorname{dist}(x_j,\Gamma^{\varepsilon_j}(t_j))}{\varepsilon_j} \to \infty \quad \text{as} \quad j \to \infty, \\
&\Big|\frac{d(x_j,t_j)}{\varepsilon_j}\Big| = \frac{\operatorname{dist}(x_j,\Gamma(t_j))}{\varepsilon_j} \to \infty \quad \text{as} \quad j \to \infty, \\
&d^{\varepsilon_j}(x_j,t_j) d(x_j,t_j) > 0 \quad \text{for all} \quad j \in \mathbb{N}.
\end{aligned}
$$

By Lemma 4.1,

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \Big\{ U_0\Big(\frac{d(x_j,t_j)+\varepsilon_j A_1}{\varepsilon_j}, x_j\Big) - \varepsilon_j A_2 - U_0\Big(\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j}, x_j\Big)\Big\} \\
&\le \lim_{j\to\infty} \Big\{ u^{\varepsilon_j}(x_j,t_j) - U_0\Big(\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j}, x_j\Big)\Big\} \\
&\le \lim_{j\to\infty} \Big\{ U_0\Big(\frac{d(x_j,t_j)-\varepsilon_j A_1}{\varepsilon_j}, x_j\Big) + \varepsilon_j A_2 - U_0\Big(\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j}, x_j\Big)\Big\} = 0
\end{aligned}
$$

and this contradicts (4.3). Hence (4.6) holds.

Let $y_j \in \Gamma^{\varepsilon_j}(t_j)$ be a point such that $|y_j - x_j| = d^{\varepsilon_j}(x_j,t_j)$ and let $p_j = p(x_j,t_j)$ be the image of x_j of the projection onto $\Gamma(t_j)$ for each $j \in \mathbb{N}$. Then it is easy to see that the following hold.

$$
(u^{\varepsilon_j}(y_j,t_j), v^{\varepsilon_j}(y_j,t_j)) \in S, \tag{4.7}
$$

$$
d^{\varepsilon_j}(x_j,t_j) = |x_j - y_j|, \tag{4.8}
$$

$$
(u^{\varepsilon_j}(x,t_j), v^{\varepsilon_j}(x,t_j)) \in \Delta_2 \cup S \quad \text{if} \quad |x - x_j| < |y_j - x_j|, \tag{4.9}
$$

$$
x_j - p_j \perp \Gamma(t_j) \quad \text{at} \quad p_j \in \Gamma(t_j), \tag{4.10}
$$

$$
|x_j - p_j| = O(\varepsilon_j), \ |x_j - y_j| = O(\varepsilon_j) \quad \text{as} \quad j \to \infty. \tag{4.11}
$$

We now rescale the solution $(u^{\varepsilon_j}, v^{\varepsilon_j})$ around (p_j,t_j) and define

$$
\begin{cases}
w^j_1(z,\tau) := u^{\varepsilon_j}(p_j + \varepsilon_j \mathcal{R}_j z, t_j + \varepsilon_j^2 \tau), \\
w^j_2(z,\tau) := v^{\varepsilon_j}(p_j + \varepsilon_j \mathcal{R}_j z, t_j + \varepsilon_j^2 \tau),
\end{cases} \tag{4.12}
$$

where $\mathcal{R}_j$ is a matrix in $SO(\mathbb{R}^N)$ that rotates z_N axis onto the outward normal at $p_j \in \Gamma(t_j)$. Since $\underset{0\le t\le T_1}{\cup} \Gamma(t)$ is separated from $\partial\Omega$ by some positive distance, there is a $C_0 > 0$ such that (w^j_1, w^j_2) is defined at least on the box

$$
B_j := \Big\{ (z,\tau) \in \mathbb{R}^N \times \mathbb{R} \,\Big|\, |z| < \frac{C_0}{\varepsilon_j}, \ -(\mu-1)C|\log\varepsilon_j| \le \tau \le \frac{T_1 - T}{\varepsilon_j^2} \Big\}.
$$

Since $(u^\varepsilon, v^\varepsilon)$ satisfies (1.1), we can see that (w_1^j, w_2^j) satisfies

$$\begin{cases} w_{1,\tau}^j = D_1\tilde{k}_j(z)\Delta w_1^j + \varepsilon_j q_1^j(z)\cdot\nabla w_1^j + \tilde{h}_j(z)f(w_1^j, w_2^j), \\ w_{2,\tau}^j = D_2\tilde{k}_j(z)\Delta w_2^j + \varepsilon_j q_2^j(z)\cdot\nabla w_2^j + \tilde{h}_j(z)g(w_1^j, w_2^j) \end{cases} \text{ in } B_j,$$

where

$$\tilde{k}_j(z) = k(p_j + \varepsilon_j\mathcal{R}_j z),\ \tilde{h}_j(z) = h(p_j + \varepsilon_j\mathcal{R}_j z),$$
$$q_i^j(z) = D_i\nabla k(p_j + \varepsilon_j\mathcal{R}_j z)\quad (i = 1, 2).$$

Thus from (4.11), Lemma 4.1, compactness of $\overline{\Omega}$ and standard parabolic estimates, up to extraction of subsequence, x_j and p_j converge to a point $x_* \in \Omega$, (w_1^j, w_2^j) converges to (w_1, w_2) locally uniformly in $\mathbb{R}^N \times \mathbb{R} = \underset{j\geq 1}{\cup} B_j$ as $j\to\infty$ and

$$(w_1(z,\tau), w_2(z,\tau)) \text{ and } (U_0(z_N, x_*), V_0(z_N, x_*))$$

satisfy

$$\begin{cases} U_0(z_N + A_1, x_*) \leq w_1(z,\tau) \leq U_0(z_N - A_1, x_*), \\ V_0(z_N - A_1, x_*) \leq w_2(z,\tau) \leq V_0(z_N + A_1, x_*) \end{cases}$$

and the following system

$$\begin{cases} u_{1,\tau} = D_1 k\Delta u_1 + hf(u_1, u_2), \\ u_{2,\tau} = D_2 k\Delta u_2 + hg(u_1, u_2) \end{cases} \text{ in } \mathbb{R}^N \times \mathbb{R},$$

where $k = k(x_*)$, $h = h(x_*)$. By Lemma 4.2, there is a $\theta_0 \in \mathbb{R}$ such that, for all $(z,\tau) \in \mathbb{R}^N \times \mathbb{R}$,

$$(w_1(z,\tau), w_2(z,\tau)) = (U_0(z_N - \theta_0, x_*), V_0(z_N - \theta_0, x_*)). \tag{4.13}$$

Define

$$z_j := \frac{1}{\varepsilon_j}\mathcal{R}_j^{-1}(x_j - p_j),\quad \tilde{z}_j := \frac{1}{\varepsilon_j}\mathcal{R}_j^{-1}(y_j - p_j)\quad (j\in\mathbb{N}).$$

From (4.11), up to extraction of subsequence, they converge:

$$\lim_{j\to\infty} z_j = z_* = (0,\cdots,0,z_{*,N}),\quad \lim_{j\to\infty}\tilde{z}_j = \tilde{z}_* = (\tilde{z}_{*,1},\cdots,\tilde{z}_{*,N}).$$

By (4.7) and (4.9),

$$\begin{aligned} &(w_1(\tilde{z}_*,0), w_2(\tilde{z}_*,0)) \in S, \\ &(w_1(z,0), w_2(z,0)) \in \Delta_2 \cup S \text{ for } z \text{ with } |z - z_*| \leq |\tilde{z}_* - z_*|. \end{aligned} \tag{4.14}$$

By (4.13),

$$\{z \mid (w_1(z,0), w_2(z,0)) \in S\} = \{z \mid z_N = \theta_0\}\ (=: H).$$

By (4.13) and (4.14),

$$z_* = \tilde{z}_* \text{ or } \partial B_{|\tilde{z}_* - z_*|}(z_*) \text{ and } H \text{ intersect at } \tilde{z}_*.$$

Thus

$$\tilde{z}_* = (0,\cdots,0,\tilde{z}_{*,N}) = (0,\cdots,0,\theta_0).$$

By $(w_1(z_*,0), w_2(z_*,0)) \in \Delta_2 \cup S$ and (4.13),

$$z_{*,N} \geq \theta_0.$$

On the other hand, $d^{\varepsilon_j}(x_j,t_j) = |y_j - x_j|$ implies

$$\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j} = |z_j - \tilde{z}_j| \to |z_* - \tilde{z}_*| = z_{*,N} - \theta_0 \text{ as } j\to\infty.$$

Hence, by (4.3) and (4.13),

$$\begin{aligned} 0 &= |w_1(z_*,0) - U_0(z_{*,N} - \theta_0, x_*)| \\ &= \lim_{j\to\infty} \Big| u^{\varepsilon_j}(x_j,t_j) - U_0\Big(\frac{d^{\varepsilon_j}(x_j,t_j)}{\varepsilon_j}, x_j\Big)\Big| \geq \eta > 0. \end{aligned}$$

This contradiction proves that (ii) of Theorem 1.3 holds. □

4.2. **Proof of statements (i) and (iii).**

Proof of (i), (iii) of Theorem 1.3. First we prove that there is a constant $c_1 > 0$ such that for all $x \in \mathcal{N}_{\tilde{C}\varepsilon}(\Gamma(t))$, $t \in [\mu C t^\varepsilon, T]$ and $\varepsilon \in (0, \varepsilon_0]$,

$$\begin{aligned} &H_u(u^\varepsilon(x,t), v^\varepsilon(x,t))\, n(p(x,t),t) \cdot \nabla u^\varepsilon(x,t) \\ &\quad + H_v(u^\varepsilon(x,t), v^\varepsilon(x,t))\, n(p(x,t),t) \cdot \nabla v^\varepsilon(x,t) \geq \frac{c_1}{\varepsilon}, \end{aligned} \tag{4.15}$$

$$\lim_{\varepsilon\to 0} \sup_{t\in[\mu C t^\varepsilon, T]} \{\varepsilon \|\nabla_{\Gamma(t)} u^\varepsilon(\cdot,t)\|_{L^\infty(\mathcal{N}_{\tilde{C}\varepsilon}(\Gamma(t)))} + \varepsilon \|\nabla_{\Gamma(t)} v^\varepsilon(\cdot,t)\|_{L^\infty(\mathcal{N}_{\tilde{C}\varepsilon}(\Gamma(t)))}\} = 0, \tag{4.16}$$

where C, $\tilde{C}$ and ε_0 are constants in Lemma 4.1 and in Theorem 4.4, respectively and $H(u,v)$ is a function in Remark 1.1, $n(p,t)$ is the outward unit normal vector to $\Gamma(t)$ at $p \in \Gamma(t)$, $p(x,t)$ is the image of x of the projection onto $\Gamma(t)$ and

$$\mathcal{N}_{\tilde{C}\varepsilon}(\Gamma(t)) := \{x \mid \mathrm{dist}(x, \Gamma(t)) < \tilde{C}\varepsilon\}.$$

If (4.15) is not true, then there exist $\varepsilon_j > 0$, $t_j \in [\mu C t^{\varepsilon_j}, T]$ and $x_j \in \mathcal{N}_{\tilde{C}\varepsilon_j}(\Gamma(t_j))$ such that

$$\begin{aligned} &\lim_{j\to\infty} \varepsilon_j \{H_u(u^{\varepsilon_j}(x_j,t_j), v^{\varepsilon_j}(x_j,t_j))\, n(p(x_j,t_j),t_j) \cdot \nabla u^{\varepsilon_j}(x_j,t_j) \\ &\quad + H_v(u^{\varepsilon_j}(x_j,t_j), v^{\varepsilon_j}(x_j,t_j))\, n(p(x_j,t_j),t_j) \cdot \nabla v^{\varepsilon_j}(x_j,t_j)\} = 0. \end{aligned} \tag{4.17}$$

By the same rescaling argument as in the proof of the statement (ii), the rescaled function $(w_1^j(z,\tau), w_2^j(z,\tau))$ converges to $(U_0(z_N - \theta_0, x_*), V_0(z_N - \theta_0, x_*))$ in $C^{2,1}_{loc}(\mathbb{R}^N \times \mathbb{R})$ as $j \to \infty$ and

$$\begin{aligned} &H_u(U_0(-\theta_0, x_*), V_0(-\theta_0, x_*))\, U_0'(-\theta_0, x_*) \\ &\quad + H_v(U_0(-\theta_0, x_*), V_0(-\theta_0, x_*)) V_0'(-\theta_0, x_*) \\ &= \lim_{j\to\infty} \{H_u(w_1^j(0,0), w_2^j(0,0))\, n(p(x_j,t_j),t_j) \cdot \nabla w_1^j(0,0) \\ &\quad + H_v(w_1^j(0,0), w_2^j(0,0))\, n(p(x_j,t_j),t_j) \cdot \nabla w_2^j(0,0)\} = 0. \end{aligned}$$

This contradicts (1.9) in Remark 1.1 and this contradiction implies that (4.15) holds. The proof of (4.16) is similar to that of (4.15) and we omit it.

By (1.9), Theorem 4.4, (4.15) and the implicit function theorem, there is a smooth function $\eta^\varepsilon(\cdot,t)$ defined on $\Gamma(t)$ for each $t \in [\mu C t^\varepsilon, T]$ such that

$$H(u^\varepsilon(x + \eta^\varepsilon(x,t) n(x,t), t), v^\varepsilon(x + \eta^\varepsilon(x,t) n(x,t), t)) = 0, \tag{4.18}$$

$$H(u^\varepsilon(y,t), v^\varepsilon(y,t)) = 0 \Leftrightarrow \exists x \in \Gamma(t),\ \text{s.t.},\ y = x + \eta^\varepsilon(x,t) n(x,t), \tag{4.19}$$

$$\nabla_{\Gamma(t)} \eta^\varepsilon(x,t) = -\frac{H_u \nabla_{\Gamma(t)} u^\varepsilon(p^\varepsilon,t) + H_v \nabla_{\Gamma(t)} v^\varepsilon(p^\varepsilon,t)}{H_u\, n^\varepsilon \cdot \nabla u^\varepsilon(p^\varepsilon,t) + H_v\, n^\varepsilon \cdot \nabla v^\varepsilon(p^\varepsilon,t)} \quad \text{for all } x \in \Gamma(t),\ t \in [\mu C t^\varepsilon, T], \tag{4.20}$$

where $\nabla_{\Gamma(t)}$ denotes the gradient on $\Gamma(t)$, $n^\varepsilon = n(p^\varepsilon, t)$,

$$p^\varepsilon = p^\varepsilon(x,t) = x + \eta^\varepsilon(x,t)n(x,t),$$
$$H_u = H_u(u^\varepsilon(p^\varepsilon,t), v^\varepsilon(p^\varepsilon,t)),\ H_v = H_v(u^\varepsilon(p^\varepsilon,t), v^\varepsilon(p^\varepsilon,t)).$$

From (4.18) and (4.19), it holds that $\Gamma^\varepsilon(t)$ is expressed as the graph of the function $\eta^\varepsilon(\cdot,t)$ on $\Gamma(t)$ for each $t\in[\mu C t^\varepsilon, T]$. By (4.15), (4.16) and (4.20)

$$|\nabla_{\Gamma(t)}\eta^\varepsilon(x,t)| = O(\varepsilon|\nabla_{\Gamma(t)}u^\varepsilon(p^\varepsilon,t)| + \varepsilon|\nabla_{\Gamma(t)}v^\varepsilon(p^\varepsilon,t)|) \to 0$$
$$\text{as } \ \varepsilon\to 0 \ \text{ uniformly for } \ x\in\Gamma(t),\ t\in[\mu C t^\varepsilon, T].$$

This completes the proof of statement (i). Statement (iii) immediately follows from (ii) and

$$\sup_{x\in\Omega,\, t\in[\mu C t^\varepsilon, T]} |d^\varepsilon(x,t) - d(x,t)| \le \tilde{C}\epsilon \ \text{ for all } \ \varepsilon\in(0,\varepsilon_0].$$

□

4.3. **Proof of Lemmas 4.1 and 4.2.**

Proof of Lemma 4.1. In Definition 6.1 in [11], they construct a lower solution $(\hat{u}_\varepsilon^-, \hat{v}_\varepsilon^-)$ and a upper solution $(\hat{u}_\varepsilon^+, \hat{v}_\varepsilon^+)$ of (1.1). By (7.1) in [11] and their construction, it is easy to see that there are $C>0$, $A_i>0$ $(i=1,2,3)$ and $\varepsilon_0>0$ such that, for any $\varepsilon\in(0,\varepsilon_0]$, $(x,t)\in\overline{\Omega}\times[Ct^\varepsilon, T]$,

$$\begin{aligned}
&U_0\Big(\frac{d(x,t)+\varepsilon A_1}{\varepsilon}, x\Big) - \varepsilon A_2\\
&\qquad \le \hat{u}_\varepsilon^-(x,t) \le u^\varepsilon(x,t) \le \hat{u}_\varepsilon^+(x,t)\\
&\qquad\qquad \le U_0\Big(\frac{d(x,t)-\varepsilon A_1}{\varepsilon}, x\Big) + \varepsilon A_2,\\
&V_0\Big(\frac{d(x,t)-\varepsilon A_1}{\varepsilon}, x\Big) - \varepsilon A_3\\
&\qquad \le \hat{v}_\varepsilon^+(x,t) \le v^\varepsilon(x,t) \le \hat{v}_\varepsilon^-(x,t)\\
&\qquad\qquad \le V_0\Big(\frac{d(x,t)+\varepsilon A_1}{\varepsilon}, x\Big) + \varepsilon A_3.
\end{aligned}$$

This completes the proof. □

Proof of Lemma 4.2. This lemma is an easy consequence of Theorem 3.2. In fact, it is obvious that (3.2) holds. (3.3) and (A) follow from (1.2) and Assumption 1, respectively. □

5. Proof of the Liouville type theorems

5.1. **Proof of Theorems 3.2 and 3.3.**

Proof of Theorem 3.2. Let us put

$$(u_1,u_2) = (u,-v),\ f_1(u_1,u_2) = f(u_1,-u_2),\ f_2(u_1,u_2) = -g(u_1,-u_2).$$

Then Theorem 3.2 is an easy consequence of Theorem 3.3 and we omit the detail of the proof. □

Next proposition plays a key rule to prove the uniqueness of the traveling wave solution up to shifts in time.

Proposition 5.1 (strong comparison principle)**.** *Assume* (3.8) *and* (3.6). *Let* $u(x,t)$, $v(x,t)$ *be solutions of* (3.5) *such that*

$$p^- \preceq u,\, v \preceq p^+, \quad u(\cdot,0) \preceq v(\cdot,0)$$

Then $u(\cdot,t) \preceq v(\cdot,t)$ *for any* $t \geq 0$. *If, in addition, assume* (3.7) *and* $u(\cdot,0) \prec v(\cdot,0)$, *then* $u(\cdot,t) \ll v(\cdot,t)$ *for any* $t > 0$.

Proof. First we prove

$$u(\cdot,0) \ll v(\cdot,0) \Rightarrow u(\cdot,t) \ll v(\cdot,t) \;\text{ for all }\; t \geq 0. \tag{5.1}$$

If (5.1) does not hold, then

$$t_0 := \sup\{t' > 0 \mid u(\cdot,t) \ll v(\cdot,t) \;\text{ for all }\; t \in [0,t']\} \in (0,\infty)$$

and

$$\begin{aligned} &u(\cdot,t) \ll v(\cdot,t) \;\text{ for }\; t \in [0,t_0), \\ &u(\cdot,t_0) \preceq v(\cdot,t_0),\; u(\cdot,t_0) \not\ll v(\cdot,t_0). \end{aligned}$$

Hence $\zeta(x,t) := v(x,t) - u(x,t)$ satisfies

$$\begin{aligned} &\zeta(\cdot,t) \gg (0,0,\cdots,0) \;\text{ for }\; t \in [0,t_0), \quad \zeta(\cdot,t_0) \succeq (0,0,\cdots,0), \\ &\exists l_0 \in \{1,2,\cdots,m\},\, x_0 \in \mathbb{R}^N,\; \zeta_{l_0}(x_0,t_0) = 0. \end{aligned}$$

By (3.6), for $(x,t) \in \mathbb{R}^N \times (0,t_0]$,

$$\begin{aligned} \zeta_{l_0,t} - D_{l_0}\Delta\zeta_{l_0} &= f_{l_0}(v) - f_{l_0}(u) \\ &\geq f_{l_0}(\cdots,u_{l_0-1},v_{l_0},u_{l_0+1},\cdots) - f_{l_0}(u) \\ &\geq -M\zeta_{l_0} \quad (M := \sup_{w\in[p^-,p^+]}|DF(w)|). \end{aligned} \tag{5.2}$$

By strong maximum principle for a parabolic equation,

$$\zeta_{l_0}(\cdot,t) = 0 \;\text{ for }\; t \in [0,t_0].$$

This contradicts $\zeta(\cdot,t) \gg (0,0,\cdots,0)$ for $t \in [0,t_0)$. Hence (5.1) holds. We take some smooth functions $u^j(\cdot,0)$, $v^j(\cdot,0)$ such that

$$\begin{aligned} &\lim_{j\to\infty} u^j(\cdot,0) = u(\cdot,0),\; \lim_{j\to\infty} v^j(\cdot,0) = v(\cdot,0), \\ &p^- \ll u^j(\cdot,0) \ll v^j(\cdot,0) \ll p^+ \;\text{ for }\; j \in \mathbb{N}. \end{aligned}$$

and let u^j, v^j be solutions of (3.5) with initial data $u^j(\cdot,0)$, $v^j(\cdot,0)$, respectively. Then by (5.1),

$$u^j(\cdot,t) \ll v^j(\cdot,t) \;\text{ for }\; t \geq 0.$$

Taking limits of both sides of this inequality as $j \to \infty$, by continuously dependence of solutions of a parabolic system on initial data, we get

$$u(\cdot,t) \preceq v(\cdot,t) \;\text{ for }\; t \geq 0.$$

Now we assume (3.7) and prove

$$u(\cdot,0) \prec v(\cdot,0) \Rightarrow u(\cdot,t) \ll v(\cdot,t) \;\text{ for }\; t > 0.$$

If this is not true, then there are $l_0 \in \{1,2,\cdots,m\}$ and $(x_0,t_0) \in \mathbb{R}^N \times (0,\infty)$ such that $u_{l_0}(x_0,t_0) = v_{l_0}(x_0,t_0)$. $\zeta_{l_0} := u_{l_0} - v_{l_0}$ satisfies (5.2), $\zeta_{l_0} \geq 0$ and $\zeta_{l_0}(x_0,t_0) = 0$. By strong maximum principle, $\zeta_{l_0} = 0$ for $(x,t) \in \mathbb{R}^N \times [0,t_0]$. Put

$$\Lambda := \{l \in \{1,2,\cdots,m\} \mid \zeta_l := u_l - v_l \not> 0 \;\text{ on }\; \mathbb{R}^N \times \{t_0\}\}$$

Then by $u(\cdot,0) \prec v(\cdot,0)$,

$$\emptyset \neq \Lambda \subsetneq \{1,2,\cdots,m\}$$

and

$$\begin{cases} \zeta_l(\cdot,t) = 0 \ \text{ in } \ \mathbb{R}^N \times [0,t_0] \ \text{ for } \ l \in \Lambda, \\ v_l(\cdot,t_0) - u_l(\cdot,t_0) = \zeta_l(\cdot,t_0) > 0 \ \text{ in } \ \mathbb{R}^N \ \text{ for } \ l \notin \Lambda. \end{cases}$$

Then

$$f_{l,u_j}(v(\cdot,t_0)) = 0 \ \text{ for } \ l \in \Lambda, \ j \notin \Lambda. \tag{5.3}$$

If (5.3) is not true, then

$$f_{l_0,u_{j_0}}(v(x_0,t_0)) > 0 \ \text{ for some } \ x_0 \in \mathbb{R}^N \ \text{ and } \ l_0 \in \Lambda, \ j_0 \notin \Lambda$$

and hence, by $\zeta_{l_0} = 0$ in $\mathbb{R}^N \times [0,t_0]$ and $v_{j_0}(x_0,t_0) > u_{j_0}(x_0,t_0)$,

$$\begin{aligned} 0 = \zeta_{l_0,t} - D_{l_0}\Delta\zeta_{l_0} &= f_{l_0}(v) - f_{l_0}(u) \\ &> f_{l_0}(\cdots, v_{j_0-1}, u_{j_0}, v_{j_0+1}, \cdots) - f_{l_0}(u) \\ &\geq f_{l_0}(\cdots, u_{l_0-1}, v_{l_0}, u_{l_0+1}, \cdots) - f_{l_0}(u) \\ &\geq -M\zeta_{l_0} = 0 \ \text{ at } \ (x,t) = (x_0,t_0). \end{aligned}$$

This is a contradiction and (5.3) holds. However (5.3) implies that $DF(v(x,t_0))$ is reducible for each $x \in \mathbb{R}^N$ and contradicts the assumption (3.7). The proof is completed. □

Next lemma completes the proof of the last part of Theorem 3.3. We give the proof later.

Lemma 5.2. *Assume* (3.8), (3.6) *and* (3.7). *Let* $\phi(n\cdot x - ct), \widetilde{\phi}(n \cdot x - ct)$ *be functions satisfying* (A′) *with a direction* $n \in \mathbb{R}^N$ *and a speed* c *and for some constants* $a, b \in \mathbb{R}$,

$$\widetilde{\phi}(\cdot - a) \preceq \phi(\cdot) \preceq \widetilde{\phi}(\cdot - b). \tag{5.4}$$

Then $\phi(\cdot) = \widetilde{\phi}(\cdot - \theta_0)$ *for some* $\theta_0 \in \mathbb{R}$.

Proof of Theorem 3.3. Take

$$0 < \delta < \min\left\{ \frac{p_l^+ - p_l^-}{\max\{\varphi_l^-, \varphi_l^+\}} \mid l = 1,2,\cdots,m \right\}$$

such that

$$\delta < \frac{\min\{\lambda_+\varphi_l^+, \, \lambda_-\varphi_l^- \mid l = 1,2,\cdots,m\}}{\max\left\{1, \sup\limits_{u\in[p^-,p^+]} |D^2F(u)|\right\}}, \tag{5.5}$$

where $[p^-,p^+] := \{u \in \mathbb{R}^m \mid p_l^- \leq u_l \leq p_l^+, \ \ l = 1,2,\cdots,m\}$ and

$$|D^2F| := \sqrt{\sum_{l,j,k=1}^m f^2_{l,u_ju_k}}.$$

Then

$$\begin{aligned} F(w) \preceq &F(w - \varepsilon\varphi^+) \ \text{ for any } \ w \in \mathbb{R}^m \ \text{ with} \\ &p^+ \succeq w \succeq p^+ - \frac{\delta}{2}\varphi^+ \ \text{ and for any } \ \varepsilon \in [0, \delta/2], \end{aligned} \tag{5.6}$$

$$(5.7)\qquad \begin{aligned} F(w) \succeq &F(w+\varepsilon\varphi^-) \quad \text{for any} \quad w\in\mathbb{R}^m \quad \text{with} \\ &p^- \preceq w \preceq p^- + \frac{\delta}{2}\varphi^- \quad \text{and for any} \quad \varepsilon\in[0,\delta/2]. \end{aligned}$$

In fact, for any $w \in \mathbb{R}^m$ with $p^+ \succeq w \succeq p^+ - \frac{\delta}{2}\varphi^+$ and $\varepsilon \in [0,\delta/2]$, by using (3.8),

$$\begin{aligned} F(w) &= F(w-\varepsilon\varphi^+) + F(w) - F(w-\varepsilon\varphi^+) - \varepsilon DF(p^+)\varphi^+ - \varepsilon\lambda_+\varphi^+ \\ &= F(w-\varepsilon\varphi^+) + \varepsilon\int_0^1 \{DF(w-s\epsilon\varphi^+) - DF(p^+)\}\varphi^+ ds - \varepsilon\lambda_+\varphi^+. \end{aligned}$$

By (5.5), $\varepsilon \in [0,\delta/2]$ and ${}^t(0,0,\cdots,0) \succeq w - p^+ \succeq -(\delta/2)\varphi^+$,

$$\begin{aligned} \Big|\int_0^1 \{DF(w-s\epsilon\varphi^+) - DF(p^+)\}\varphi^+ ds\Big| &\le (\varepsilon+\delta/2)\sup_{u\in[p^-,p^+]}|D^2F(u)| \\ &\le \delta \sup_{u\in[p^-,p^+]}|D^2F(u)| \le \lambda_+\varphi_l^+ \quad \text{for} \quad l=1,2,\cdots,m. \end{aligned}$$

$$\therefore \epsilon\int_0^1 \{DF(w-s\epsilon\varphi^+) - DF(p^+)\}\varphi^+ ds \preceq \varepsilon\lambda_+\varphi^+.$$

Thus (5.6) holds. The proof of (5.7) is similar and we omit it.

By (3.6),

$$(5.8)\qquad \begin{aligned} &f_l(v) - f_l(u) \ge -M\,(v_l - u_l) \quad (l=1,2,\cdots,m) \\ &\quad \text{for any} \quad u,\, v\in[p^-,p^+] \quad \text{with} \quad u\preceq v \qquad (M := \max_{w\in[p^-,p^+]}|DF(w)|). \end{aligned}$$

Taking appropriate Cartesian coordinates, we may assume that the solution $u(x,t)$ of (3.5) satisfies

$$(5.9)\qquad \phi(x_1 - ct - a) \preceq u(x,t) \preceq \phi(x_1 - ct - b)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Fix $(\rho,\tau) \in \mathbb{R}^{N-1}\times\mathbb{R}$ arbitrary. For $\sigma \in \mathbb{R}$, put

$$w^\sigma(x,t) := u(x_1 + c\tau + \sigma, x' + \rho, t+\tau)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. By (5.9) and monotonicity of $\phi(z)$, for any $\sigma \ge b-a$,

$$w^\sigma(x,t) \preceq \phi(x_1 + \sigma - ct - b) \preceq \phi(x_1 - ct - a) \preceq u(x,t)$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Define

$$\sigma^* := \inf\{\sigma \mid w^{\sigma'} \preceq u \ \text{ holds for all } \ \sigma' \ge \sigma\}.$$

Then clearly $\sigma^* \le b-a$.

Now we prove $\sigma^* \le 0$ by contradiction. Suppose $\sigma^* > 0$. By (5.9), monotonicity of $\phi(z)$ and $\phi(\pm\infty) = p^\mp$, there is $C > b-a$ such that

$$(5.10)\qquad \begin{cases} p^- \preceq u(x_1,x',t) \preceq p^- + \dfrac{\delta}{2}\varphi^- \\ \qquad\qquad \text{for all} \quad x_1 - ct \ge C, \ (x',t)\in\mathbb{R}^{N-1}\times\mathbb{R}, \\ p^+ \succeq u(x_1,x',t) \succeq p^+ - \dfrac{\delta}{2}\varphi^+ \\ \qquad\qquad \text{for all} \quad x_1 - ct \le -C, \ (x',t)\in\mathbb{R}^{N-1}\times\mathbb{R}, \end{cases}$$

where

$$0 < \delta < \min\Big\{\frac{p_l^+ - p_l^-}{\max\{\varphi_l^+,\varphi_l^-\}} \mid l=1,2,\cdots,m\Big\}$$

is a constant for which (5.5) holds.

If $(0,0,\cdots,0) \not\ll \inf\{u - w^{\sigma^*} \mid |x_1 - ct| \leq 2C,\ (x',t) \in \mathbb{R}^{N-1} \times \mathbb{R}\}$, then there exist $l_0 \in \{1,2,\cdots,m\}$, $x_{1,\infty} \in [-2C,2C]$ and

$$x_{1,n} - ct_n \in [-2C,2C],\ (x'_n,t_n) \in \mathbb{R}^{N-1} \times \mathbb{R} \quad (n = 1,2,3,\cdots)$$

such that

$$u_{l_0}(x_{1,n},x'_n,t_n) - w_{l_0}^{\sigma^*}(x_{1,n},x'_n,t_n) \to 0, \quad x_{1,n} - ct_n \to x_{1,\infty} \quad \text{as} \quad n \to \infty.$$

From standard parabolic estimates, up to extraction of subsequence, the function $u^n(x,t) := u(x_1 + ct_n, x' + x'_n, t + t_n)$ converges locally uniformly to a solution U of (3.5) such that

$$\begin{aligned} z(x,t) &:= U(x,t) - W^{\sigma^*}(x,t) \\ &:= U(x,t) - U(x_1 + c\tau + \sigma^*, x' + \rho, t + \tau) \succeq (0,0,\cdots,0), \\ z_{l_0}(x_{1,\infty}&,0,0) = 0 \end{aligned}$$

and by (5.8) and $W^{\sigma^*} \preceq U$,

$$z_{l_0,t} - D_{l_0}\Delta z_{l_0} = f_{l_0}(U) - f_{l_0}(W^{\sigma^*}) \geq -Mz_{l_0}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. By strong maximum principle, for all $(x,t) \in \mathbb{R}^N \times (-\infty,0]$,

$$\begin{aligned} U_{l_0}(x,t) - U_{l_0}(x_1 + c\tau + \sigma^*, x' + \rho, t + \tau) &= U_{l_0}(x,t) - W_{l_0}^{\sigma^*}(x,t) \\ &= z_{l_0}(x,t) = 0. \end{aligned}$$

If $\tau > 0$, then, by

$$\begin{aligned} &\sigma^* > 0,\ \phi(-\infty) = p^+,\ \phi(+\infty) = p^- \quad \text{and} \\ &\phi(x_1 - ct - a) \preceq U(x,t) \preceq \phi(x_1 - ct - b), \end{aligned}$$

$$\begin{aligned} U_{l_0}(x,0) &= U_{l_0}(x_1 - c\tau - \sigma^*, x' - \rho, -\tau) \\ &= \cdots = U_{l_0}(x_1 - cn\tau - n\sigma^*, x' - n\rho, -n\tau) \stackrel{n\to\infty}{\longrightarrow} p_{l_0}^+ \end{aligned}$$

and this contradicts

$$U(x,0) \preceq \phi(x_1 - b) \ll p^+.$$

If $\tau \leq 0$, then

$$\begin{aligned} U_{l_0}(x,0) &= U_{l_0}(x_1 + c\tau + \sigma^*, x' + \rho, \tau) \\ &= \cdots = U_{l_0}(x_1 + cn\tau + n\sigma^*, x' + n\rho, n\tau) \stackrel{n\to\infty}{\longrightarrow} p_{l_0}^- \end{aligned}$$

and this contradicts

$$p^- \ll \phi(x_1 - a) \preceq U(x,0).$$

Then it follows that

$$(0,0,\cdots,0) \ll \inf\{u - w^{\sigma^*} \mid (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |x_1 - ct| \leq 2C\}.$$

Hence, by uniformly continuity of u, there is an $\eta_0 \in (0,\sigma^*)$ such that, for any $\eta \in [0,\eta_0]$,

$$(5.11) \qquad (0,0,\cdots,0) \ll \inf\{u - w^{\sigma^*-\eta} \mid\ (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |x_1 - ct| \leq 2C\}.$$

By $u \succeq w^{\sigma^*}$, $\varphi^{\pm} \gg {}^t(0,0,\cdots,0)$ and uniformly continuity of u, there is an $\eta_1 \in (0,\eta_0]$ such that, for any $\eta \in [0,\eta_1]$,

$$u + \frac{\delta}{2}\varphi^{\pm} \succeq w^{\sigma^*} + \frac{\delta}{2}\varphi^{\pm} \succeq w^{\sigma^*-\eta}.$$

Put $S_\pm := \{(x,t)\in\mathbb{R}^N\times\mathbb{R} \mid \pm(x_1-ct)\le -2C\}$ and

$$\varepsilon_\pm := \inf\{\varepsilon>0 \mid u+\varepsilon\varphi^\pm \succeq w^{\sigma^*-\eta} \text{ on } S_\pm\}\ (\in[0,\delta/2]).$$

We prove $\varepsilon_\pm=0$ by contradiction. Suppose $\varepsilon_\pm>0$. Then, by

$$u+\varepsilon_\pm\varphi^\pm - w^{\sigma^*-\eta} \succeq \epsilon_\pm\varphi^\pm \gg (0,0,\cdots,0) \ \text{ on } \ \partial S_\pm$$

and

$$u(\mp\infty,x',t)+\varepsilon_\pm\varphi^\pm - w^{\sigma^*-\eta}(\mp\infty,x',t) = \varepsilon_\pm\varphi^\pm \gg (0,0,\cdots,0),$$

for each $\mu\in\{+,-\}$, there exist $l_\mu\in\{1,2\}$, $\mu x^\mu_{1,\infty}\in(-\infty,-2C)$ and

$$(x^\mu_n,t^\mu_n)=(x^\mu_{1,n},x^{\mu\prime}_n,t^\mu_n)\in S_\mu \quad (n=1,2,3,\cdots)$$

such that

$$u_{l_\mu}(x^\mu_n,t^\mu_n)+\varepsilon_\mu\varphi^\mu - w^{\sigma^*-\eta}_{l_\mu}(x^\mu_n,t^\mu_n)\to 0, \quad x^\mu_{1,n}-ct^\mu_n\to x^\mu_{1,\infty} \ \text{ as } \ n\to\infty.$$

From standard parabolic estimates, up to extraction of subsequence, the functions $u^{\mu,n}(x,t):=u(x_1+ct^\mu_n,x'+x^{\mu\prime}_n,t+t^\mu_n)$ converge locally uniformly to solutions U^μ of (3.5) such that, for $(x,t)\in S_\mu$,

$$\begin{aligned}
z^\mu(x,t) &:= U^\mu(x,t)+\varepsilon_\mu\varphi^\mu - W^{\mu,\sigma^*-\eta}(x,t)\\
&:= U^\mu(x,t)+\varepsilon_\mu\varphi^\mu - U^\mu(x_1+c\tau+\sigma^*-\eta,x'+\rho,t+\tau)\\
&\succeq (0,0,\cdots,0),\\
z^\mu_{l_\mu}(x^\mu_{1,\infty},0,0)&=0, \quad z^\mu\succeq\epsilon_\mu\varphi^\mu\gg(0,0,\cdots,0) \ \text{ on } \ \partial S_\mu
\end{aligned}$$

and, by (5.10),

$$\begin{aligned}
p^+ \succeq W^{+,\sigma^*-\eta} &\succeq p^+ - \frac{\delta}{2}\varphi^+ \ \text{ on } \ S_+,\\
p^- \preceq U^- &\preceq p^- + \frac{\delta}{2}\varphi^- \ \text{ on } \ S_-
\end{aligned}$$

and hence, by (3.6), (5.6), (5.7), (5.8) and $\varepsilon_\pm\in[0,\delta/2]$,

$$\begin{aligned}
z^+_{l_+,t}-D_{l_+}\Delta z^+_{l_+} &= f_{l_+}(U^+)-f_{l_+}(W^{+,\sigma^*-\eta})\\
&\ge f_{l_+}(U^+)-f_{l_+}(W^{+,\sigma^*-\eta}-\varepsilon_+\varphi^+)\ge -Mz^+_{l_+} \ \text{ on } \ S_+,\\
z^-_{l_-,t}-D_{l_-}\Delta z^-_{l_-} &= f_{l_-}(U^-)-f_{l_-}(W^{-,\sigma^*-\eta})\\
&\ge f_{l_-}(U^-+\varepsilon_-\varphi^-)-f_{l_-}(W^{-,\sigma^*-\eta})\ge -Mz^-_{l_-} \ \text{ on } \ S_-.
\end{aligned}$$

By strong maximum principle,

$$z^\pm_{l_\pm}(x,t)=0 \ \text{ for all } \ (x,t)\in S_\pm\cap(\mathbb{R}^N\times(-\infty,0])$$

and this contradicts $z^\pm\succeq\epsilon_\pm\varphi^\pm\gg(0,0,\cdots,0)$ on $\partial S_\pm$. Thus $\varepsilon_\pm=0$ and hence

$$u\succeq w^{\sigma^*-\eta} \ \text{ on } \ S_\pm \ \text{ for any } \ \eta\in[0,\eta_1].$$

Therefore, by (5.11), it holds that $u\succeq w^{\sigma^*-\eta}$ for any $\eta\in[0,\eta_1]$. This contradicts the minimality of σ^*. Thus $\sigma^*\le 0$ and hence

$$u(x,t)\succeq w^0(x,t)=u(x_1+c\tau,x'+\rho,t+\tau).$$

Since $(\rho,T)\in\mathbb{R}^{N-1}\times\mathbb{R}$ is arbitrary, there is a function $\widetilde{\phi}$ such that

$$u(x,t)=\widetilde{\phi}(x_1-ct) \ \text{ with } \ \widetilde{\phi}(-\infty)=p^+, \ \widetilde{\phi}(+\infty)=p^-.$$

Moreover $\widetilde{\phi}' \preceq (0,0,\cdots,0)$ since $\widetilde{\phi}(x_1 - ct) \succeq \widetilde{\phi}(x_1 - ct + \sigma)$ for all $\sigma > 0$. By strong maximum principle and $\widetilde{\phi}(-\infty) = p^+ \gg p^- = \widetilde{\phi}(+\infty)$,

$$\widetilde{\phi}' \ll (0,0,\cdots,0).$$

If, in addition, assume (3.7), then, by Lemma 5.2,

$$\widetilde{\phi}(\cdot) = \phi(\cdot - \theta_0).$$

Then $\theta_0 \in (a,b)$ follows from $\phi(\cdot - a) \preceq \phi(\cdot - \theta_0) \preceq \phi(\cdot - b)$ and monotonicity of ϕ. □

Proof of Lemma 5.2. Define

$$\tau_0 := \inf\{\tau' > 0 \mid \exists \tau \in \mathbb{R},\ \widetilde{\phi}(\cdot - \tau) \preceq \phi(\cdot) \preceq \widetilde{\phi}(\cdot - \tau - \tau')\}\ (\in [0, b-a])$$

and we prove $\tau_0 = 0$ by contradiction. Suppose $\tau_0 > 0$. Then there are τ_j', $\tau_j \in \mathbb{R}$ such that $\tau_j' \to \tau_0$ as $j \to \infty$,

$$\widetilde{\phi}(\cdot - \tau_j) \preceq \phi(\cdot) \preceq \widetilde{\phi}(\cdot - \tau_j - \tau_j') \quad \text{for} \quad j = 1, 2, \cdots.$$

By (5.4) and monotonicity of $\widetilde{\phi}$,

$$\tau_j \in [a,b].$$

Hence, by extracting a subsequence, we may assume that τ_j converges to a τ_*. Then

$$\widetilde{\phi}(\cdot - \tau_*) \preceq \phi(\cdot) \preceq \widetilde{\phi}(\cdot - \tau_* - \tau_0).$$

Let us take $\delta > 0$, $\varepsilon > 0$, $C > \max\{|a|, |b|\}$ as in the proof of Theorem 3.3. By Proposition 5.1,

$$\widetilde{\phi}(n \cdot x - ct - \tau_*) \ll \phi(n \cdot x - ct)\ ((x,t) \in \mathbb{R}^N \times \mathbb{R}).$$

Hence

$$(0,0,\cdots,0) \ll \inf\{\phi(s) - \widetilde{\phi}(s - \tau_*) \mid -2C \le s \le 2C\}. \tag{5.12}$$

As in the proof of Theorem 3.3, for any sufficiently small $\eta > 0$,

$$\widetilde{\phi}(\cdot - \tau_* - \eta) \preceq \phi(\cdot).$$

This implies

$$\tau_0 := \inf\{\tau' > 0 \mid \exists \tau \in \mathbb{R},\ \widetilde{\phi}(\cdot - \tau) \preceq \phi(\cdot) \preceq \widetilde{\phi}(\cdot - \tau - \tau')\} \le \tau_0 - \eta < \tau_0.$$

This is contradiction and τ_0 is equal to 0. Therefore there is a $\theta_0 \in \mathbb{R}$ such that $\widetilde{\phi}(\cdot - \theta_0) = \phi(\cdot)$. □

5.2. **Outline of the proof of Theorems 3.5 and 3.7.** The proof of the following two propositions is same as that of Proposition 5.1 and we omit the proof.

Proposition 5.3 (strong comparison principle)**.** *Assume* (3.13)*,* (3.14)*,* (3.15) *and* (3.11)*. Let $u(x,t)$, $v(x,t)$ be solutions of* (3.10) *such that*

$$p^- \preceq u,\, v \preceq p^+, \quad u(\cdot,0) \preceq v(\cdot,0).$$

Then $u(\cdot,t) \preceq v(\cdot,t)$ for any $t \ge 0$. If, in addition, assume (3.12) *and $u(\cdot,0) \prec v(\cdot,0)$, then $u(\cdot,t) \ll v(\cdot,t)$ for any $t > 0$.*

Proposition 5.4 (strong comparison principle)**.** *Assume* (3.20)*,* (3.21)*,* (3.22) *and* (3.18)*. Let $u(x,t)$, $v(x,t)$ be solutions of* (3.17) *such that*

$$p^- \preceq u,\ v \preceq p^+, \quad u(\cdot,0) \preceq v(\cdot,0).$$

Then $u(\cdot,t) \preceq v(\cdot,t)$ for any $t \geq 0$. If, in addition, assume (3.19) *and $u(\cdot,0) \prec v(\cdot,0)$, then $u(\cdot,t) \ll v(\cdot,t)$ for any $t > 0$.*

The following two lemmas play key rules to prove the last parts of Theorems 3.5 and 3.7, respectively.

Lemma 5.5. *Assume* (3.13)*,* (3.14)*,* (3.15)*,* (3.11) *and* (3.12)*. Let*

$$\phi(z,t),\ \widetilde{\phi}(z,t) \quad (z = n \cdot x - ct)$$

be functions satisfying (A1) *with a direction $n \in \mathbb{R}^N$ and a speed c and for some constants $a, b \in \mathbb{R}$ and for all $z \in \mathbb{R},\ t \in \mathbb{R}$,*

$$\widetilde{\phi}(z-a,t) \preceq \phi(z,t) \preceq \widetilde{\phi}(z-b,t). \tag{5.13}$$

Then $\phi(z,t) \equiv \widetilde{\phi}(z-\theta_0,t)$ for some $\theta_0 \in \mathbb{R}$.

Lemma 5.6. *Assume* (3.22)*,* (3.18) *and* (3.19)*. Let*

$$u(x,t),\ v(x,t) \quad ((x,t) \in \mathbb{R}^N \times \mathbb{R})$$

be functions satisfying (A2) *with a direction $n \in \mathbb{R}^N$ and a speed $c \neq 0$ and for some constants $a, b \in \mathbb{R}$ and for all $x \in \mathbb{R}^N,\ t \in \mathbb{R}$,*

$$u(x,t+a) \preceq v(x,t) \preceq u(x,t+b). \tag{5.14}$$

Then $v(x,t) \equiv u(x,t-\theta_0)$ for some $\theta_0 \in \mathbb{R}$.

Outline of the proof of Theorem 3.5. Take

$$0 < \delta < \min\Big\{ \frac{p_l^+(t) - p_l^-(t)}{\max\{\varphi_l^+(t), \varphi_l^-(t)\}} \mid t \in \mathbb{R},\ l = 1, 2, \cdots, m \Big\}$$

such that

$$\delta < \min\Big\{ \frac{\lambda_+ \varphi_l^+(t)}{a_+(t)}, \frac{\lambda_- \varphi_l^-(t)}{a_-(t)} \mid t \in \mathbb{R},\ l = 1, 2, \cdots, m \Big\}, \tag{5.15}$$

where $a_\pm(t) := \max\Big\{1, \sup\limits_{w \in [p^-(t), p^+(t)]} (|D^2 F(t,w)||\varphi^\pm(t)|^2)\Big\}$,

$$|D^2 F| := \sqrt{\sum_{l=1}^m \sum_{i,j=1}^N f_{l,u_i u_j}^2}, \quad |\varphi|^2 := \sum_{l=1}^m \varphi_l^2.$$

Then, by the same calculation as in the proof of Theorem 3.3,

$$\begin{cases} F(t,w) - \varepsilon DF(t,p^+(t))\varphi^+(t) - \varepsilon\lambda_+\varphi^+(t) \preceq F(t, w - \varepsilon\varphi^+(t)) \\ \text{for any } \ w \in \mathbb{R}^m,\ t \in \mathbb{R} \ \text{ with } \ p^+(t) \succeq w \succeq p^+(t) - \varepsilon\varphi^+(t) \ \text{ and} \\ \text{for any } \ \varepsilon \in [0, \delta/2], \end{cases} \tag{5.16}$$

$$\begin{cases} F(t,w) + \varepsilon DF(t,p^-(t))\varphi^-(t) + \varepsilon\lambda_-\varphi^-(t) \succeq F(t, w + \varepsilon\varphi^-(t)) \\ \text{for any } \ w \in \mathbb{R}^m,\ t \in \mathbb{R} \ \text{ with } \ p^-(t) \preceq w \preceq p^-(t) + \varepsilon\varphi^-(t) \ \text{ and} \\ \text{for any } \ \varepsilon \in [0, \delta/2], \end{cases} \tag{5.17}$$

We also take $C > b - a$ such that

$$(5.18)\qquad \begin{cases} p^-(t) \preceq u(x,t) \preceq p^-(t) + \dfrac{\delta}{2}\varphi^-(t) \\ \qquad\qquad \text{for all } \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ \text{ with } \ n \cdot x - ct \geq C, \\ p^+(t) \succeq u(x,t) \succeq p^+(t) - \dfrac{\delta}{2}\varphi^+(t) \\ \qquad\qquad \text{for all } \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ \text{ with } \ n \cdot x - ct \leq -C. \end{cases}$$

For any $(\rho, \tau) \in \mathbb{R}^N \times T\mathbb{Z}$ with $n \cdot \rho - c\tau = 0$, an argument similar to that in the proof of Theorem 3.3 shows that, for any $\sigma \geq 0$,

$$w^\sigma(x,t) := u(x + \rho + \sigma n, t + \tau) \preceq u(x,t) \ \text{ for all } \ x \in \mathbb{R}^N, \ t \in \mathbb{R}.$$

This implies that $u(x,t) = \widetilde{\phi}(z,t)$ $(z = n \cdot x - ct)$ for a function $\widetilde{\phi}$ which satisfies

$$\widetilde{\phi}(z, t+T) \equiv \widetilde{\phi}(z,t), \ \widetilde{\phi}_z \gg (0, 0, \cdots, 0), \ \widetilde{\phi}(\pm\infty, \cdot) = p^{\mp}(\cdot).$$

Moreover (3.12) and Lemma 5.5 imply $\widetilde{\phi}(z,t) \equiv \phi(z - \theta_0, t)$. □

Outline of the proof of Theorem 3.7. We assume that the speed c is positive since the sign of the speed is irrelevant in the later argument. Take

$$0 < \delta < \min\Big\{ \frac{p_l^+(x) - p_l^-(x)}{\max\{\varphi_l^+(x), \varphi_l^-(x)\}} \mid x \in \mathbb{R}^N, \ l = 1, 2, \cdots, m \Big\}$$

such that

$$(5.19)\qquad \delta < \min\Big\{ \frac{\lambda_+\varphi_l^+(x)}{a_+(x)}, \frac{\lambda_-\varphi_l^-(x)}{a_-(x)} \mid x \in \mathbb{R}^N, \ l = 1, 2, \cdots, m \Big\},$$

where $a_\pm(x) := \max\Big\{1, \sup_{w \in [p^-(x), p^+(x)]} (|D^2F(x,w)||\varphi^\pm(x)|^2)\Big\}$,

$$|D^2F| := \sqrt{\sum_{l=1}^m \sum_{i,j=1}^N f_{l,u_i u_j}^2}, \quad |\varphi|^2 := \sum_{l=1}^m \varphi^2.$$

Then, by the same calculation as in the proof of Theorem 3.3,

$$(5.20)\qquad \begin{cases} F(x,w) - \varepsilon DF(x, p^+(x))\varphi^+(x) - \varepsilon\lambda_+\varphi^+(x) \preceq F(x, w - \varepsilon\varphi^+(x)) \\ \text{for any } \ w \in \mathbb{R}^m, \ x \in \mathbb{R}^N \ \text{ with } \ p^+(x) \succeq w \succeq p^+(x) - \varepsilon\varphi^+(x) \ \text{ and} \\ \text{for any } \ \varepsilon \in [0, \delta/2], \end{cases}$$

$$(5.21)\qquad \begin{cases} F(x,w) + \varepsilon DF(x, p^-(x))\varphi^-(x) + \varepsilon\lambda_-\varphi^-(x) \succeq F(x, w + \varepsilon\varphi^-(x)) \\ \text{for any } \ w \in \mathbb{R}^m, \ x \in \mathbb{R}^N \ \text{ with } \ p^-(x) \preceq w \preceq p^-(x) + \varepsilon\varphi^-(x) \ \text{ and} \\ \text{for any } \ \varepsilon \in [0, \delta/2], \end{cases}$$

We also take $C > b - a$ such that

$$(5.22)\qquad \begin{cases} p^-(x) \preceq u(x,t) \preceq p^-(x) + \dfrac{\delta}{2}\varphi^-(x) \\ \qquad\qquad \text{for all } \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ \text{ with } \ n \cdot x - ct \geq C, \\ p^+(x) \succeq u(x,t) \succeq p^+(x) - \dfrac{\delta}{2}\varphi^+(x) \\ \qquad\qquad \text{for all } \ (x,t) \in \mathbb{R}^N \times \mathbb{R} \ \text{ with } \ n \cdot x - ct \leq -C. \end{cases}$$

For any $(\rho, \tau) \in \mathbb{L} \times \mathbb{R}$ with $n \cdot \rho - c\tau = 0$, an argument similar to that in the proof of Theorem 3.3 shows that, for any $\sigma \geq 0$,

$$w^\sigma(x,t) := u(x+\rho, t+\tau-\sigma) \preceq u(x,t) \quad \text{for all} \quad x \in \mathbb{R}^N, \ y \in \mathbb{R}.$$

This implies that, for $\rho \in \mathbb{L}$, $t \in \mathbb{R}$,

$$u(x+\rho, t+n\cdot\rho/c) \equiv u(x,t), \ u_t \succeq (0,0,\cdots,0).$$

By (3.24) and $\lim_{k\in\mathbb{L}, n\cdot k \to \pm\infty} v(\cdot + k, t) = p^{\mp}(\cdot)$,

$$\lim_{k\in\mathbb{L}, n\cdot k \to \pm\infty} u(\cdot + k, t) = p^{\mp}(\cdot). \tag{5.23}$$

From (5.23), maximum principle, $u_t \succeq (0,0,\cdots,0)$ and

$$u_{l,tt} \geq \sum_{i,j=1}^N D_l^{ij}(x) u_{l,tx_ix_j} + q_l(x)\cdot\nabla u_{l,t} + f_{l,u_l}(x,u_1,\cdots,u_m)u_{l,t}$$
$$\text{for} \quad x \in \mathbb{R}^N, \ t \in \mathbb{R} \quad (l = 1,2,\cdots,m),$$

it holds that $u_t \gg (0,0,\cdots,0)$. Therefore u is a solution which satisfies (A2). Moreover, if, in addition, assume (3.19), then Lemma 5.6 implies $u(x,t) \equiv v(x,t+\theta_0)$. □

Proof of Lemmas 5.5 and 5.6. The proof of Lemmas 5.5 and 5.6 is based on Propositions 5.3, 5.4 and an argument similar to that in the proof of Lemma 5.2. We give the proof of Lemma 5.6 only. The proof of Lemma 5.5 is easier and omitted. We only consider the case that the peed c is positive since the sign of the speed is irrelevant in the later argument.

Define

$$\tau_0 := \{\tau' \mid \exists\tau \in \mathbb{R}, \, u(x,t+\tau) \preceq v(x,t) \preceq u(x,t+\tau+\tau')$$
$$((x,t) \in \mathbb{R}^N \times \mathbb{R})\} \ (\in [0, b-a])$$

and we prove $\tau_0 = 0$ by contradiction. Suppose $\tau_0 > 0$. Then there are τ_j', $\tau_j \in \mathbb{R}$ such that $\tau_j' \to \tau_0$ as $j \to \infty$,

$$u(x,t+\tau_j) \preceq v(x,t) \preceq u(x,t+\tau_j+\tau_j') \quad \text{for} \quad (x,t) \in \mathbb{R}^N \times \mathbb{R} \quad (j = 1,2,\cdots).$$

By (5.14) and monotonicity of u, v with respect to t,

$$\tau_j \in [a,b].$$

Hence, by extracting a subsequence, we may assume that τ_j converges to a τ_* as $j \to \infty$. Then

$$u(x,t+\tau_*) \preceq v(x,t) \preceq u(x,t+\tau_*+\tau_0) \quad ((x,t) \in \mathbb{R}^N \times \mathbb{R}).$$

Let us take $\delta > 0$, $\varepsilon > 0$, $C > \max\{|a|,|b|\}$ as in the proof of Theorem 3.7. By Proposition 5.4,

$$u(x,t+\tau_*) \ll v(x,t) \quad ((x,t) \in \mathbb{R}^N \times \mathbb{R})$$

and hence

$$(0,0,\cdots,0) \ll \inf\{v(x,t) - u(x,t+\tau_*) \mid (x,t) \in \mathbb{R}^N \times \mathbb{R}, \ |n\cdot x - ct| \leq 2C\}. \tag{5.24}$$

If (5.24) is not true, then there are $l_0 \in \{1,2,\cdots,m\}$, $x_j \in \mathbb{R}^N$, $t_j \in \mathbb{R}$ such that

$$|n\cdot x_j - ct_j| \leq 2C \quad (j \in \mathbb{N}) \quad \text{and} \quad \lim_{j\to\infty}\{v_{l_0}(x_j,t_j) - u_{l_0}(x_j,t_j+\tau_*)\} = 0.$$

Let $k_j \in \mathbb{L}\,(= L_1\mathbb{Z} \times L_2\mathbb{Z} \times \cdots \times L_N\mathbb{Z})$ satisfy

$$x_j \in k_j + [0, L_1) \times [0, L_2) \times \cdots \times [0, L_N) \quad (j \in \mathbb{N}).$$

Then $x_j - k_j$, $t_j - n \cdot k_j/c$ are bounded uniformly for $j \in \mathbb{N}$. Hence, by extracting a subsequence, we may assume that there are $x_* \in \mathbb{R}^N$ and $t_* \in \mathbb{R}$ such that $|x_* \cdot n - ct_*| \leq 2C$, $x_j - k_j \to x_*$, $t_j - n \cdot k_j/c \to t_*$ as $j \to \infty$. Thus

$$\begin{aligned} 0 &= \lim_{j\to\infty} \{v_{l_0}(x_j, t_j) - u_{l_0}(x_j, t_j + \tau_*)\} \\ &= \lim_{j\to\infty} \{v_{l_0}(x_j - k_j, t_j - n \cdot k_j/c) - u_{l_0}(x_j - k_j, t_j - n \cdot k_j/c + \tau_*)\} \\ &= v_{l_0}(x_*, t_*) - u_{l_0}(x_*, t_* + \tau_*). \end{aligned}$$

This contradicts $u(x, t + \tau_*) \ll v(x, t) \quad ((x, t) \in \mathbb{R}^N \times \mathbb{R})$ and (5.24) holds. By an argument similar to that in the proof of Theorem 3.3, for any sufficiently small $\eta > 0$, $u(x_*, t_* + \tau_* + \eta) \preceq v(x_*, t_*)$. Thus

$$\tau_0 = \inf\{\tau' \mid \exists \tau \in \mathbb{R},\ u(\cdot, \cdot + \tau) \preceq v(\cdot, \cdot) \preceq u(\cdot, \cdot + \tau + \tau')\} \leq \tau_0 - \eta < \tau_0.$$

This is contradiction and $\tau_0 = 0$ is proved. This completes the proof. $\square$

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