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Reliability estimation of rare events for stochastic dynamic systems excited by stationary stochastic processes

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Abstract: In the field of reliability analysis of stochastic dynamic systems, the identification of rare parameter configurations that lead to failure is one aspect of special interest. Often the identification of rare events that lead to a structures failure and the estimation of the probability of this failure is computationally very costly or even unfeasible. When regarding the simulation of structures critical combinations could arise because of uncertainties in material parameters, environmental parameters, as well as loading parameters, which should be considered as random or in intervals when modelling safety-relevant systems. This work focuses on the Probability Density Evolution Method, introduced by Jie Li and Jian-Bing Chen in the advent of the century, and its application to highly non-linear dynamic systems subjected to stationary Gaussian stochastic processes. The major interest lies in establishing a robust estimator for the system's probability of failure and reliability towards a performance-based first-passage criterion. Due to the non-linearity and the high dimension of random distributions included in the model, most usual approaches are computationally very costly and still only yield results that come with a high variance or can only be assessed in large intervals. With the Probability Density Evolution Method, a direct deterministic description of the outputs Joint Probability Density Function is available and therefore can be utilised for reliability assessment. On the way of combining the Probability Density Evolution Method with advanced Monte Carlo Simulation techniques to improve the outputs Joint PDF, this work shall help to understand the establishment of a reliability estimator on the PDEM PDF in the first place. For this reason, a suitable numerical example has been set up. On this model, several different scenarios have been established that include different sampling techniques. A comparison between Monte Carlo simulation, quasi Monte Carlo simulation techniques and the PDEM approach has been made. The results of the probability of failure and the evolution of the system reliability were observed for all examples.

Keywords: Stochastic Dynamics, Probability of Failure, Reliability, Monte Carlo Simulation, Probability Density Evolution Method, Stochastic Processes.

1. Introduction

Stochastic dynamic systems describe the movement of particles or larger continua by the well established equations of motion subjected to influences that are quantified by any sort of random variable. Based on the posed problem, the systems physical or random dimension a description can either be done by a single differential equation or also common in engineering analysis for larger structural relations by a combination of multidimensional differential equations e.g. Finite Element systems. In civil engineering examples could be the analysis of large structures under natural influences such as earthquake or wind loadings but also vibration patterns of machinery. This unfolds that for many purposes substantial studies of stochastic dynamic systems are of great interest. In reliability engineering, the analysis of stochastic dynamic systems can yield information on critical events that might happen in the future or during the lifeline of a product. Special interest is caught by certain system behaviours which lead to a failure that affects the system's stability or serviceability. A practical example, which this paper is working towards, is the effect of earthquakes on large structures. Seismic ground motion affecting a structure lead to extreme stresses and displacements which can have an direct impact on the structures serviceability. These examples inherit the issue that it is not possible to exactly predict an earthquake. However, it is feasible to describe certain characteristics of an earthquake and use these to establish a stochastic description of different earthquake signals with similar baseline characteristics. The random nature of the earthquake's occurrence poses the question of how specific structures should be designed to be considered safe. This is especially of interest in areas with higher seismic activity and for non-customary structures such as high rise buildings, dams, oil platforms or bridges.

The challenge for civil engineers is not only to design the structure and compare different designs towards a safety criterion or earthquake code but the challenge also lies in the difficulty of how to describe the stochastic part of the dynamic system and how to analyse it. Most commonly used nowadays is the Monte Carlo Method (MC). By changing parameters due to pre-defined probability distributions, it is a versatile tool to deterministically explore the random behaviour in any system. In principle, for MC there is no boundary for the systems complexity and dimension (random or physical). Nonetheless, practically MC is suffering from the so-called curse of dimensionality. That means for a high random dimension dependent on the complexity of the system (is it a large FE Model or a single analytical equation) a certain number of the random dimension is reached where the computational effort is ultimately too high to evaluate the system sufficiently. Towards serviceability that is formulated using a very small tolerable probability of failure, a large amount of samples is needed. Especially for the estimation of small failure probabilities a large amount of samples is usually necessary to estimate a robust value.

Therefore in this work, a different approach has been chosen to overcome the curse of dimensionality by keeping the needed samples low. As already mentioned MC is a deterministic solver that requires a high number of realisations of every random distribution inside the system to carry out several simulations or calculations to get a sufficiently high number of outcomes that statistically are significant.

In contrast to MC the Probability Density Evolution Method (PDEM) (Li and J.-B. Chen 2010, Ang 2017) offers an approximation scheme that describes for a certain output quantity of interest the whole stochastic behaviour by generating a multivariate Probability Density Function (PDF). Utilising the principle of probability preservation, PDEM captures the input random variables influence by once running a quasi MC simulation only in random space. The information gathered from this quasi-MC run is then further processes by selecting points in the multi-dimensional random space between all input quantities which are describing the main influential behaviour on the system. This is done by generating a set that has a minimized generalized discrepancy to cover the random space as broad as possible. J.-B. Chen et al. 2016

The next step is how to estimate the probability of failure governed by the simulation that is carried out. In this work, only the probability of failure defined by a double-sided first passage reliability statement is analysed. This means for each system configuration the output quantity is analysed in the face of a determined critical value, also known as the system's capacity. Once this capacity is exceeded this output counts as a failed system. The number of failed systems is then divided by all simulated system outputs, this yields an estimator for the probability of failure. However, due to the PDEM's description of the output in the probability space by approximating a multivariate joint PDF the reliability statement can also be calculated utilising this outcome. Due to the utilised approximation scheme, many stability considerations must be made, otherwise, the approximation is not valid. In previous work the goal was to overcome the issue, that in the representative point set not a single realisation of the PDEM's point set is lying in the failure domain. This is realized by using the advanced MC simulation technique Subset sampling. But to quantify the effect of the representative point set on the PDEM's joint PDF output, first a robust estimator for the probability of failure must be established, which is the goal of this work.

2. Reliability Analysis for Stochastic Dynamic Systems

The probability of how long a system/product remains in service can be measured by the reliability. Here a distinction between reliability in time and system reliability can be made. Whereas reliability in time sets the probability of an unaltered serviceability in dependence of time the system reliability tries to neglect the time factor and estimate a factor of the systems reliability in general. For both approaches it is possible that the system is composed of a number of components that each distinctively have different characteristics. In this work the system is composed of a one dimensional stochastic dynamic equation of motion with a double-sided first-passage failure criterion. This criterion is imposed on a one dimensional time dependant stochastic dynamic system. For each realisation of the stochastic system the first exceedance in time of the double-sided first-passage problem is counted. For a number of realisations, e.g. Monte Carlo samples, these exceedances can be counted over time.

2.1. Probability of failure

$\boldsymbol{X}(\boldsymbol{\theta}, t)$ shall be the response of a mechanical system subjected to specific input probability distributions $\boldsymbol{\theta}$ with corresponding random dimension n_{Θ}. A sample realisation of a specific distribution type Θ is denoted by θ. Throughout this work a dynamic system with fixed parameters but with a stochastic excitation is considered to mimic the basic relations between artificial earthquake inputs and mechanical system behaviour. The systems reliability regarding a first-passage problem can be described as $Rel = Pr\{\boldsymbol{X}(\boldsymbol{\theta}, t) \in \Omega_s, t \in (0, T]\}$ the domain Ω_s is considered to be the safe domain of the responses range. As long as the response $\boldsymbol{X}(\boldsymbol{\theta}, t)$ is not exceeding some specific boundaries during the defined time interval $(0, T]$ the system is considered safe. Hence once $\boldsymbol{X}(\boldsymbol{\theta}, t)$ exceeds this boundary the system fails. The domain Ω_s can be of any shape and described in any dimension. Simple and common appearances of the boundaries are a one-sided boundary or a double-sided boundary. For a double-sided boundary problem, the probability of a failure in a defined time T can be stated as the event

$$p_f = Pr\{X(t) \in \Omega_f \,|\, t \in (0, T])\} \tag{1}$$

here C is the limit for the systems response, that define the double-sided boundary. The failure domain Ω_f ca be stated with properties $\Omega_f = \Omega_s^c, \Omega_f \cap \Omega_s = \emptyset, \Omega_f \cup \Omega_s = \Omega_r$. Please note that Ω_r is the response domain related to the systems output $\boldsymbol{X}$ and $\Omega_{\boldsymbol{\Theta}}$ the random domain. The probability of failure for any arbitrary multivariate probability distribution function $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is then calculated by following integral over the whole failure domain Ω_f

$$p_f = \int_{\Omega_f} f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}, \tag{2}$$

since the exact failure domain is usually unknown, by using the indicator function $\mathcal{I}(\boldsymbol{\theta})$ with $\mathcal{I}(\boldsymbol{\theta}) = 1$ if $\boldsymbol{\theta} \in \Omega_f$ and 0 otherwise, the above integral can be expanded on the whole domain and rewritten to

$$p_f = \int_{-\infty}^{+\infty} \mathcal{I}(\boldsymbol{\theta}) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta} = E[\mathcal{I}(\boldsymbol{\theta})]. \tag{3}$$

2.2. Monte Carlo simulation

A versatile approach to estimate the probability of failure is the Monte Carlo simulation, which basically utilizes a deterministic analysis of a certain number of samples of e.g. a double-sided first passage problem. For this purpose to each output $\boldsymbol{X}(\boldsymbol{\theta}, t)$ a certain performance $g(\boldsymbol{\theta}, t)$ can be evaluated using following relation

$$g(\boldsymbol{\theta}, t) = C - |X(t)|, \tag{4}$$

to evaluate if a sample is inside Ω_f above expression can be set in the relation if $g(\boldsymbol{\theta}, t) \leq 0 \rightarrow \boldsymbol{\theta} \in \Omega_f \rightarrow \mathcal{I}(\boldsymbol{\theta}) = 1$ and 0 otherwise. This offers a relation to evaluate the indicator function introduced in eq. (3). The deterministic estimation of p_f shall be expressed by following equation

$$\hat{p}_f = \frac{1}{N_{MC}} \sum_{i=1}^{N_{MC}} \mathcal{I}(\theta_{i,n_\theta}) \tag{5}$$

where N_{MC} is the number of Monte Carlo samples used. For each $\boldsymbol{\theta}_i$ a deterministic analysis of a given mechanical system with n_θ random inputs is carried out. Note that this only includes the random inputs used to generate the stochastic excitation of the system. Later for comparison the variance $\mathrm{Var}[\hat{p}_f]$ and the corresponding coefficient of variation ν_{p_f} are introduced as

$$\mathrm{Var}[\hat{p}_f] = \frac{\hat{p}_f - \hat{p}_f^2}{N_{MC}}, \quad \nu_{p_f} = \frac{\sqrt{\mathrm{Var}[\hat{p}_f]}}{\hat{p}_f}. \tag{6}$$

The accurate estimation of eq. (5) requires a very large amount of samples N_{MC}. In general it is desirable to achieve a small ν_{p_f}. Additionally to above considerations in Zio 2014 an estimator for the required minimum number of samples is described to be

$$\hat{N}_{MC} = \frac{1 - p_f}{p_f \nu_{p_f}^2} \tag{7}$$

for $p_f = \nu_{p_f} = 0.01$ a minimum number of $\hat{N}_{MC} = 9.9 \cdot 10^5$ is required to achieve the desired ν_{p_f}.

3. Probability Density Evolution Method

After MC the target $\boldsymbol{X}(\boldsymbol{\theta}, t)$ are deterministic timelines. For N_{MC} realisations of $\boldsymbol{\theta}$ this means the whole mechanical system must be evaluated for each sample. As aforementioned this is computationally expensive due to the necessary high number of samples. The Probability Density Evolution Method (PDEM) allows to approximate the joint Probability Density Function (PDF) for the target value. The derivation and mathematical as well as physical evolution of the PDEM can be seen in Li and J.-B. Chen 2010. In this work just a brief overview of the relation is given. Further information about the in this work used approximation scheme can be found in J.-B. Chen and Li 2010, an alternative approximation using a Finite Element approach is revisited in Papadopoulos and Kalogeris 2016.

For an arbitrary equation of motion a generalized multi dimensional stochastic dynamic system with applied random force can be stated as

$$\boldsymbol{M}(\boldsymbol{\Theta})\ddot{\boldsymbol{X}}(t) + \boldsymbol{C}(\boldsymbol{\Theta})\dot{\boldsymbol{X}}(t) + \boldsymbol{K}(\boldsymbol{\Theta})\boldsymbol{X}(t) = \boldsymbol{F}(\boldsymbol{\Theta}, t), \tag{8}$$

in which $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \dots, \Theta_{n_\theta})$ are the random parameters of the physical properties in the system with random dimension n_θ, described by a known joint PDF. $\boldsymbol{X}(t), \dot{\boldsymbol{X}}(t) \text{and} \ddot{\boldsymbol{X}}(t)$ are respectively the displacements, velocities and accelerations of the system. $\boldsymbol{M}(\boldsymbol{\Theta})$, $\boldsymbol{C}(\boldsymbol{\Theta})$ and $\boldsymbol{K}(\boldsymbol{\Theta})$ are the $n \times n$ mass, damping and stiffness matrices with physical dimension n. $\boldsymbol{F}(\boldsymbol{\Theta}, t)$ is the external force which can be time dependant. Influences of random distributions could appear in the systems material parameters as well as in the external applied force. eq. (8) must be solvable in a deterministic procedure, this implies that all distributions $\boldsymbol{\Theta}$ must be known.

Under the condition that the random distributions $\boldsymbol{\Theta}$ as input distributions do not change and that no additional randomness in time is induced in the system, that means the systems randomness is subjected to the the Principle of Probability Preservation (P3) given as $Pr\{\boldsymbol{X}(\boldsymbol{\theta}, t) \in \Omega_X \times \Omega_{\boldsymbol{\Theta}}, t \in (0, T]\} = const.$, following several derivations as in e.g. Li et al. 2012, the Generalized Probability Density Evolution Equation (GDEE) for a multi dimensional mechanical system can be stated to be

$$\frac{\partial p_{\mathbf{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}, t)}{\partial t} + \dot{\boldsymbol{X}}(\boldsymbol{\theta}, t)\frac{\partial p_{\boldsymbol{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}, t)}{\partial \boldsymbol{x}} = 0, \tag{9}$$

where $p_{\mathbf{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}, t)$ is the joint PDF of $\boldsymbol{X}$ and $\boldsymbol{\Theta}$. $\dot{\boldsymbol{X}}(\boldsymbol{\theta}, t)$ is the fist order derivative of the targeted response $\boldsymbol{X}(t)$. In contrast to the purely deterministic output of MC now a total probabilistic output is available.

3.1. Point selection

Prior to solving eq. (9) a deterministic analysis of eq. (8) needs to be carried out. Alike MC the PDEM uses a set of samples to assesses the system behaviour. However, to achieve this with a benefit, PDEM is choosing a sample that utilises quasi Monte Carlo sample features to cover the probability space as broad as possible and only realises samples around which the highest probability mass is centred. For this purpose the sample set is firstly generated and then a certain number of desired points are chosen to suffice a minimized generalized F-discrepancy, as in Zhou et al. 2019 and in J.-B. Chen and Zhang 2013 or on the basis og the GL_2 discrepancy as in J.-B. Chen et al. 2016. The generated sample set shall be called the representative point set $\{\boldsymbol{\theta} = \theta_{1,n_\theta}, \theta_{2,n_\theta}, \dots, \theta_{q,n_\theta}\}$ with q as the number of desired samples, which is considerately smaller than $\hat{N}_{MC}$. A drawback in computation is that to generate the representative Point Set (PS) when using e.g. a Sobol sequence as quasi Monte Carlo simulation procedure, at least 10^6 samples need to be generate in order to select a low discrepancy point set. However, these samples are not evaluated in the system but are generated due to the know distributions.

3.2. Solving procedure

1. Generate the representative point set $\{\boldsymbol{\theta}_{q,n_{\boldsymbol{\Theta}}}\}$. Each sample within the point set has an assigned probability given to be $P_q = \int_{V_q} p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) d\boldsymbol{\theta}$, in which V_q refers to the space covered by this sample due to the quasi MC sample and choosing of points in the procedure of minimizing the discrepancy. The assigned probability can be seen as a weighting factor for each sample in the representative point set and suffices $\sum_{i=1}^{q} P_i = 1$. The assigned probability weights the starting condition for the approximation scheme as following $p_{\boldsymbol{X\Theta}}(z, \boldsymbol{\theta}_q, t_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) P_q$, where δ is the Kronecker delta.

2. Carry out a deterministic analysis of the mechanical system eq. (8) using $\{\boldsymbol{\theta}_{q,n_{\boldsymbol{\Theta}}}\}$. The output is given to be $\boldsymbol{X}(\boldsymbol{\theta}, t)$. Calculate $\dot{\boldsymbol{X}}(\boldsymbol{\theta}, t)$ by using e.g. a central difference scheme.

3. For given initial condition and above values solve eq. (9) for each point $i = 1, 2, \dots, q$ in the representative point set to obtain the joint PDF of $\mathbf{X}$ and $\boldsymbol{\Theta}$ $p_{\mathbf{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}_i, t)$ using a initial value solving procedure, e.g. a finite difference scheme as in J.-B. Chen and Li 2010.

4. To obtain the joint PDF for the output value calculate

the summation of all previously obtained PDF values, i.e.: $p_{\boldsymbol{X}}(\boldsymbol{x}, t) = \sum_{i=1}^{q} p_{\boldsymbol{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}_i, t)$.

3.3. Approximation scheme

The approximated joint PDF $p_{\mathbf{X\Theta}}(\boldsymbol{x}, \boldsymbol{\theta}, t)$ is highly dependant on the chosen approximation scheme. The stability of the approximation scheme on the other hand is of course dependant on the analysed system. Since the external force in the following example is highly non-linear, the stability of the approximation scheme is difficult to achieve. The difficulty arises from the trade-off between accuracy and stability. However, since the goal is to capture the probability of failure for rare random events the accuracy is of utmost importance. In this work a closer investigation of the approximation schemes parameters in the face of the reliability statement has been carried out.
The hybrid Lax-Wendroff approximation scheme used as in Li and J.-B. Chen 2010 and J.-B. Chen and Li 2010 to approximate the joint PDF for each point θ_i in the representative point set the scheme is given to be

$$p_j^{(k+1)} = p_j^{(k)} - \frac{1}{2}(\lambda a - |\lambda a|)\Delta p_{j+\frac{1}{2}}^{(k)} - \frac{1}{2}(\lambda a - |\lambda a|)\Delta p_{j-\frac{1}{2}}^{(k)} - \frac{1}{2}(|\lambda a| - \lambda^2 a^2)(\psi_{j+\frac{1}{2}}\Delta p_{j+\frac{1}{2}}^{(k)} - \psi_{j-\frac{1}{2}}\Delta p_{j-\frac{1}{2}}^{(k)}). \tag{10}$$

in which $p_j^{(k)}$ is the approximated value for a certain point in the representative point set for $p_{\mathbf{X\Theta}}(x_j, \boldsymbol{\theta}_i, t_k)$. At this point the discretisation of time and space domain are introduced. That means $t_k = k \cdot \Delta\hat{t}$ and $x_j = j \cdot \Delta x$. For the sake of brevity the parameter a is actually the approximated speed $\dot{x}^{(k)}$ from the input of eq. (9). For each time step k the Courant-Friedrich-Lewy condition towards $\lambda = \frac{\Delta\hat{t}}{\Delta x}$ should be checked: $|\lambda \cdot a| \leq 1$. Note at this point that not only the space grid with Δx can be chose in the approximation scheme but also the time grid $\Delta\hat{t}$ can be chosen to suffice the CFL condition. This might lead to a finer discretisation of the time domain $\Delta\hat{t}$ than the original simulation time domain Δt. To achieve a higher stability in the case of discontinuities following consecutive gradients, forward and backward on the space grid, are calculated and shall serve as discontinuity indicators

$$r_{j+\frac{1}{2}}^{+} = \frac{p_{j+2}^{(k)} - p_{j+1}^{(k)}}{p_{j+1}^{(k)} - p_j^{(k)}}, r_{j+\frac{1}{2}}^{-} = \frac{p_{j+2}^{(k)} - p_{j+1}^{(k)}}{p_{j+1}^{(k)} - p_j^{(k)}}$$
$$r_{j-\frac{1}{2}}^{+} = \frac{p_{j+1}^{(k)} - p_j^{(k)}}{p_j^{(k)} - p_{j-1}^{(k)}}, r_{j-\frac{1}{2}}^{-} = \frac{p_{j-1}^{(k)} - p_{j-2}^{(k)}}{p_j^{(k)} - p_{j-1}^{(k)}} \tag{11}$$

Supposing that for example $r_{j+\frac{1}{2}}^{+} = 1$ this means the two points in x forward the grid are on a straight line. If the values in above equations get large the points are widely spread from one and another which indicates an abrupt change of the curve. As proposed in Li and J.-B. Chen 2010 a superbee (aka. Roe-Sweby) flux limiter based on following gradient control equation $\psi_0(r) = max(0, min(2r, 1), min(r, 2)$ is used which leads to following specific flux limiters based on the probability ratios utilised in eq. (11)

$$\psi_{j+\frac{1}{2}}(r_{j+\frac{1}{2}}^{+}, r_{j+\frac{1}{2}}^{-}) = u(-a)\psi_0(r_{j+\frac{1}{2}}^{+}) + u(a)\psi_0(r_{j+\frac{1}{2}}^{-})$$
$$\psi_{j-\frac{1}{2}}(r_{j-\frac{1}{2}}^{+}, r_{j-\frac{1}{2}}^{-}) = u(-a)\psi_0(r_{j-\frac{1}{2}}^{+}) + u(a)\psi_0(r_{j-\frac{1}{2}}^{-}), \tag{12}$$

which are case sensitive dependant mainly on the speed. In all above equations $u(\cdot)$ is the Heaviside step funtion given to $u(x) = 1$, if $x \geq 0$ and otherwise $u(x) = 0$.

3.4. Reliability in PDEM

The P3 states that no additional randomness than the initial input random distributions is induced in the system over time. It is possible to use this statement in combination with the fact that once a failure criterion has been reached and the lets say first-passage boundary condition has been violated i.e. the probability trajectory has left Ω_s and entered Ω_f, this probability trajectory must not return to the safe space because the systems state has already been in a state of failure. These trajectories now contribute to the failure probability. Within the approximation scheme of PDEM it is possible to introduce a absorbing boundary condition on the joint PDF that realizes the aforementioned considerations

$$p_{X\boldsymbol{\Theta}}(x, \boldsymbol{\theta}, t) = 0 \quad \text{for} \quad x \in \Omega_f \tag{13}$$

For the specific description of the failure probability a sort of violation of P3 is used but by not inducing new randomness but by reducing the probability mass for those events that already lead to a failure. This ensures that these events can not contribute to the safe domain, as in regulation with the definition of the probability of failure of a first-passage problem. The boundary "absorbs" the probability mass that is reaching the pre-defined capacity. The by absorbing boundary conditions obtained probability density is called the remaining probability density and denoted by $\breve{p}_X(x, t)$ for the resulting systems response PDF and $\breve{p}_X(x, \boldsymbol{\theta}, t)$ for the input joint sample PDF. The relationship with the pre defined restrictions from above are given as

$$\breve{p}_X(x, t) = \int_{\Omega_{\boldsymbol{\Theta}}} \breve{p}_{X\boldsymbol{\Theta}}(x, \boldsymbol{\theta}, t) d\boldsymbol{\theta} \tag{14}$$

For a certain threshold C as already introduced in section 2.2 the systems reliability can be calculated by

$$Rel(t) = \int_{-C}^{C} \breve{p}_X(x, t)dx = \int_{-\infty}^{\infty} \breve{p}_X(x, t)dx \tag{15}$$

from which the probability of failure can be calculated as $p_f(t) = 1 - Rel(t)$.
At this point it is very important to note that in the representative point set in most cases after the quasi MC simulation and point selection procedure no sample in the failure region is present. However, due to the single realisations and the approximation of the overall joint PDF the PDF tails still reach the failure region.

4. Numerical Example

With the above considerations about the estimation of a probability of failure as well as the PDEM several Monte Carlo simulations (MCS) and PDEM simulations are carried out to investigate the effect of different grid parameters for the approximation scheme on the estimation of the probability of failure for a highly non-linear mechanical system with a high random dimension.

4.1. Model

A simple Single Degree-of-Freedom (SDOF) linear oscillator in the form

$$m\ddot{\boldsymbol{x}}(t) + c\dot{\boldsymbol{x}}(t) + k\boldsymbol{x}(t) = \boldsymbol{F}(\boldsymbol{\theta}, t), \tag{16}$$

with mass $m = 50$ kg, spring constant $k = 1922$ N/m and damping coefficient $c = 37.2$ kg/s is introduced. $\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}$ are displacements, velocities and accelerations of the system. The forcing term on the right hand side $\boldsymbol{F}(\boldsymbol{\theta}, t)$ is modelled by a stochastic process using the Spectral Representation Method (SRM) formulated in Shinozuka and Deodatis 1991. To generate a stationary stochastic process following model is used

$$\boldsymbol{F}(\theta, t) = \sum_{n=0}^{N-1} \sqrt{4S_X(\omega_n)\Delta\omega}\cos(\omega_n t + \varphi_n(\theta)), \tag{17}$$

where $\omega_n = n\Delta\omega, \quad n = 0, 1, 2, \ldots, N-1$ and $\Delta\omega = \frac{\omega_u}{N}$. In theory to achieve a stationary process the series is summed over $N \to \infty$ and $\varphi_n(\theta)$ as $n = n_\Theta$ uniform distributed phase angles between $[0, 2\pi]$. For $S_X(\omega)$ the following Power Spectral Density (PSD)

$$S_X(\omega) = \frac{1 + a - \omega^2}{(\omega_p^2 - \omega) + (2\zeta\omega\omega_p)^2}, \tag{18}$$

is used with $a = 5$ and $\zeta = 0.25$ as shape parameters and $\omega_p = 10$ rad/s as peak frequency. Since it is a Gaussian stationary process, the overall mean of the signal should be zero. The above differential equation of the mechanical system is solved using an explicit Runge-Kutta. The capacity within the performance function C is chosen to be 0.0055 m. The demand D is the maximum absolute displacement of the system

$$g(\boldsymbol{\theta}) = 0.0055 - \max_{t\in[0,T_E]} \left[|\boldsymbol{x}(\boldsymbol{\theta}, t)|\right]. \tag{19}$$

This equation states a double-sided boundary condition. All other system parameters, as well as the performance functions capacity is chosen in order to generate a relatively small probability of failure which is $p_f < 0.01$. The random dimension n_θ describing the distinct distribution functions to generate the random phase angles corresponds to the simulation time. This relationship was chosen to be $n_\varphi \geq \lceil \frac{T_E \omega_u}{2\pi} \rceil$ with $\omega_u = 25$ rad/s is the so called cut-off frequency and T_E is the total simulation time given to be 49 s. This results in $n_\varphi = 195$ uniform distributed phase angles.

5. Results

In following table 1 specific results for the system in eq. (16) with the properties explained in previous section are presented.

The method refers to the sample realisation technique and output evaluation method. MC means crude Monte Carlo simulation on the system in eq. (16). PS means the evaluation of the representative point set (PS) generated on the way to calculate the joint PDF for PDEM. Note that for the generation of the samples in PS a quasi Monte Carlo Sobol sequence with 10^6 samples was generated, from these set 185 samples have ben chosen according to minimizing the discrepancy of the set. The values of PDEM are derived from the multivariate joint PDF. N_s is the number of samples used for this method. $\sqrt{\mathrm{Var}[x]}$ is the mean standard deviation over time of the systems response x. $\mathrm{E}[x]$ is the mean over time of the mean displacements in x. $\max[|x|]$ is the overall maximum over all samples and the whole time domain. ν_{p_f} is the coefficient of variation calculated according to eq. (6) and p_f likewise, according to the considerations in section 2.1. In fig. 1 the reliability for the above given MC and PDEM scenarios is plotted over time. Additional another MC simulation with a lesser number of samples is used to compare, and to indicate the overall range of the reliability curves. The figure fig. 2 depicts one realisation of the multivariate joint PDF yielded by PDEM. A brighter colour in the contour plot means a higher probability, the highest probability is centred around zero. A darker colour indicates a lower PDF value and therefore lower probability. The critical values of the double-sided boundary condition are shown as thick lines at the top and at the bottom of the graph.

Table 1. Results for different simulation methods and number of samples N_s

Method	MC	PS	PDEM
N_s	10^6	185	as PS
$\sqrt{\mathrm{Var}[x]}$ (m)	0.0012	0.0013	0.0014
$\mathrm{E}[x]$ (m)	$2.1940 \cdot 10^{-7}$	$-7.3904 \cdot 10^{-7}$	$-3.8215 \cdot 10^{-6}$
$\max[\|x\|]$ (m)	0.0075	0.0056	-
ν_{p_f}	0.0103	0.9973	-
p_f (%)	0.0095	0.0054	0.0104

6. Discussion

There are several issues here, first in table 1 it can be seen that the number of chosen points for the PS is extremely small in comparison with the MC simulation, due to the quasi MC sampling the probability of failure is not zero but the coefficient of variation is very high. The PDEM result is based on this PS but approximating the PDF according to the PDEM scheme presented in this work. This leads to an overestimation of the probability of failure, calculated by the PDF with absorbing boundary conditions. Second, when regarding this result closer as in fig. 1 the overall systems behaviour described by the PDEM is different. This is because of the induced absorbing boundary condition. For most of the time the PDF does not reach the critical value but it seems whenever it does, a higher amount of the probability trajectory becomes absorbed. Therefore some abrupt changes in the reliability curve for the PDEM result can be observed. How-

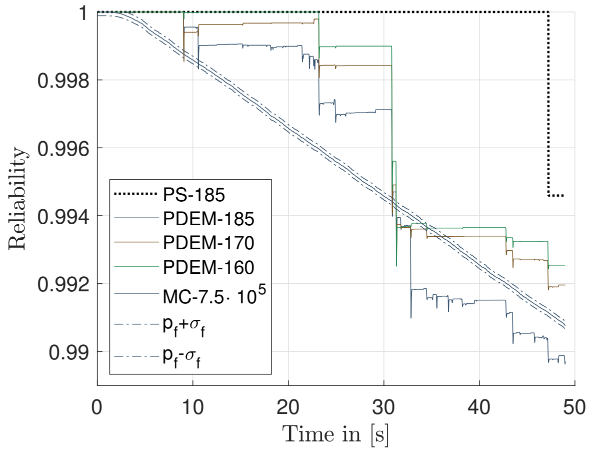

Figure 1. Three different reliability curves corresponding to the simulations presented in table 1.

ever interestingly, the PDEM result is approximated with the weights according to the PS that inhibits just a single sample that fails and still it simulates more frequent failures due to absorbed probability trajectories of the PDF. Nonetheless, the approximation performance is not quite good since all PDEM estimations are lying outside of the one sigma interval of the high MC result. Additionally, some numerical oscillations can be observed, since after the reliability is lowering, it immediately increases in the next time step afterwards. The solving procedure for PDEM seems to be not calibrated totally correct. This may be caused by choosing a unsuitable time and/or space grid. From my observations the CFL condition alone is not sufficient to ensure a robust calculation. But considering the extremely small number of samples and the needed computation time, further calibration of the solving procedure to gain a robust estimator should be carried out. For future considerations the random dimension of the mechanical system should be achieved. Either by decreasing the overall simulation time, choosing a different example or using other methods than the SRM. It is planned to extend the reliability estimation also on non-stationary processes. Suitable for this extension might be the Stochastic Harmonic Function representation introduced by J. Chen et al. 2013 which can reduce the total number of used RVs by defining the PSDs omega partition as random and the Super Asperity Model by Nozu 2014, which relies on earthquake measurements and observations to reduce the number of RVs for generation of signals. All in all, for this presented numerical example the system parameters as well as PDEM approximation parameters definitely must be explored further to achieve a robust estimator for the reliability and the probability of failure.

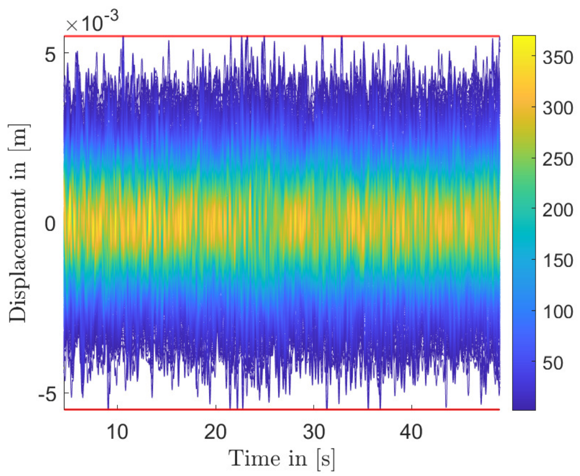

Figure 2. Joint PDF values of eq. (14) for the PDEM result, above and below, the line in red, the critical value.

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