Studies on the asymptotic invariants of cohomology groups and the positivity in complex geometry

(コホモロジー群の漸近的な不変量と複素幾何学における正値性の研究)

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Preface

The purpose of this thesis is to reveal relations among the asymptotic behavior of the cohomology group for high tensor powers of a line bundle on a compact complex manifold, the geometry of the positivity (amplitude) and the positivity of the curvature of a line bundle.

Let X be a compact complex manifold of dimension n. For the study of the geometric structure of X, it is fundamental to consider a line bundle L on X and various properties of L. Many important properties (for example, the amplitude, the Kodaira dimension and so on) of L can be described by the asymptotic behavior of the cohomology group $H^i(X, \mathcal{O}_X(L^m))$ as $m \to \infty$. The cohomology group $H^i(X, \mathcal{O}_X(L^m))$ depends on the holomorphic structure of L. However, the asymptotic behavior as $m \to \infty$ is determined only by the differential structure (the first Chern class) of L. The first Chern class of L can be regarded as the space of (Chern) curvatures of L. Therefore it is natural to investigate relations between the asymptotic behavior of the cohomology group and the curvature of a line bundle.

In Chapter 2, we study relations between the q-amplitude and the curvature of a line bundle. The q-amplitude is an asymptotic vanishing property of the higher cohomology group whose degree is strictly larger than q. Strictly speaking, a line bundle L on X is called q-ample, if for any coherent sheaf $\mathcal{F}$ on X there exists an integer m_0 such that $H^i\big(X, \mathcal{F} \otimes \mathcal{O}_X(L^m)\big) = 0$ for $i > q$, $m \geq m_0$. The q-amplitude of a line bundle depends only on the first Chern class of the line bundle. Therefore it is natural to ask what is a characterization of a q-ample line bundle in terms of the curvature.

For this problem, we consider the Andreotti-Grauert vanishing theorem. The Andreotti-Grauert vanishing theorem in the classical theory of several complex variables can be generalized to an asymptotic cohomology vanishing theorem of a line bundle on a compact complex manifold. This generalized Andreotti-Grauert vanishing theorem asserts that a q-positive line bundle is always q-ample. Here a line bundle L on X is called q-positive, if L admits a smooth metric whose Chern curvature has at least $(n-q)$-positive eigenvalues everywhere on X. A 0-positive line bundle is a positive line bundle in the usual sense. Further, it follows from the Serre vanishing theorem that, a line bundle is ample in the usual sense of algebraic geometry if and only if the line bundle is 0-ample. Thanks to the Kodaira embedding theorem, we know that a positive line bundle coincides with an ample line bundle. In particular, the converse implication of the Andreotti-Grauert theorem holds when q is zero.

Therefore it is of interest to know whether the converse implication of the Andreotti-Grauert theorem holds. That is to say, is a q-ample line bundle L always q-positive ? This problem was first posed by Demailly-Peternell-Schneider. In Chapter 2, we investigate this problem and give affirmative answers in the following cases.

(i) The problem is affirmatively solved on an arbitrary compact complex manifold if L is semi-ample.

(ii) The problem is affirmatively solved on a smooth projective surface without any assumptions on L.

These results give a characterization of a q-ample line bundle in terms of the curvature. In the proof of result (ii), we give a numerical characterization of a $(n-1)$-ample line bundle. From this numerical characterization, we construct a metric with $(n-1)$-positive curvature by using a solution of a global equation (Monge-Ampère equation). On a surface, the asymptotic behavior of the cohomology group can be described by the curvature, thanks to result (ii) and the theorem of Demailly on the holomorphic Morse inequality. In this meaning, result (ii) is an important result on a surface.

The Griffiths conjecture says that, any ample vector bundle would admit a hermitian metric with Griffiths positive curvature. It would be a generalization of the Kodaira embedding theorem. The Griffiths conjecture has similarity to the converse of the Andreotti-Grauert theorem. On the other hand, Ottem recently constructed a counterexample to the converse of the Andreotti-Grauert theorem on a higher dimensional manifold, by investigating the topology of the zero locus of a section of a q-positive line bundle, in the spirit of the Lefschetz hyperplane theorem for an ample line bundle. In Chapter 3, we study the Lefschetz type theorem for a vector bundle in this view point.

Specifically, we compare the homotopy groups of a compact complex manifold X with those of the zero locus S of a section of an ample vector bundle E on X. Then it is conjectured that the relative homotopy group $\pi_i(X, S)$ vanishes for $i \leq n-r$. Here r is the rank of E. We affirmatively solve this problem under the assumption that the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r-1)$-positive. The proof is based on the Morse theory. This assumption is satisfied when E is globally generated. Therefore this result can be seen as generalizations of the classical results. Moreover, we prove that the natural map $\pi_1(S) \to \pi_1(X)$ between fundamental groups is surjective when S has the expected dimension. For the proof of this result, we consider the cohomological dimension (cohomological completeness) of the complement $X \setminus S$. Then we construct suitable sections which separate an étale covering. For this purpose, we solve the $\overline{\partial}$-equation with the L^2-estimate. This idea is based on the technique of Napier-Ramachandran. We estimate the number of such sections by the dimension of the cohomology of the formal scheme. Further, we apply the duality theorem on the formal scheme.

In Chapter 4, we give a characterization of the amplitude of a vector bundle on a curve. As an application, we give a partial result for the Hartshorne conjecture. The Hartshorne conjecture says that some positive multiple of a smooth subvariety with ample normal bundle would move (as a cycle) in a large algebraic family. Chapter 4 is based on a joint work with Ottem.

In Chapter 5, we investigate the restricted volume of a big line bundle along a subvariety. We give an analytic description of the restricted volume in terms of the curvature currents associated to singular metrics. Further, we give a relation between the existence of a Zariski decomposition and the behavior of the restricted volume of a big line bundle.

In Chapter 6, we consider the extendability of a singular metric of a line bundle from a subvariety to an ambient space. We prove that the extendability of singular metrics with positive curvature currents of a line bundle implies the amplitude of the line bundle. It gives an ampleness criterion of a line bundle with the extendability of singular metrics.

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1
Preliminaries

In this chapter, we collect the basic notations, definitions and results that are often used in the subsequent chapters.

1.1 Notations and conventions

Throughout this thesis, we denote by X a compact complex manifold. Sometimes we may assume that X is Kähler or projective. Further, we denote by L a (holomorphic) line bundle on X. For simplicity, we denote by L^k, the k-th tensor product $L^{\otimes k}$ of L, except Chapter 6. In Chapter 6, we use additive notation for the tensor product of a line bundle. We use the words "divisors", "line bundles" and "invertible sheaves" interchangeably.

1.2 Currents

Let X be a compact Kähler manifold. Since X is a Kähler manifold, $H^{p,p}(X, \mathbb{C})$ is identified with the quotient of the space of d-closed (p,p)-currents modulo the dd^c-exact currents. For our purpose, the case of $p = 1$ is important. We say that a function φ is a potential function of a $(1,1)$-current T if $T = dd^c\varphi$. Notice that a (local) potential function is uniquely determined modulo the pluriharmonic functions. If T is d-closed, we can locally take a potential function of T. A d-closed $(1,1)$-current is said to have analytic (resp. algebraic) singularities (along the subschema $V(\mathcal{I})$ defined by an ideal sheaf $\mathcal{I}$), if its potential function φ can be locally written as

$$\varphi = \frac{c}{2}\log(|f_1|^2 + ... + |f_k|^2) + v$$

for some $c \in \mathbb{R}_{>0}$ (resp. $c \in \mathbb{Q}_{>0}$), where $f_1, \dots, f_k$ are local generators of $\mathcal{I}$ and v is a smooth function. Then $V(\mathcal{I})$ is called the singular locus of the current.

1.3 Pull-backs of $(1,1)$-currents

Let us confirm the definition of the pull-back of a d-closed $(1,1)$-current by a holomorphic map. Let T be a d-closed $(1,1)$-current on X and let $f : Z \to X$ a holomorphic map from a complex manifold Z to X. Assume that the image of Z by f is not contained in the polar set of a potential function of T. Then we can define the pull-back of T by f as follows: Since T is d-closed, we can locally take a potential function φ of T. Then the pullback of T is (locally) defined to be $f^*T := dd^c f^*\varphi$. It determines a global d-closed $(1,1)$-current on Z since $dd^c f^*\varphi$ does not depend on the choice of a local potential function φ. In particular, we can restrict a d-closed $(1,1)$-current to a submanifold if the submanifold is not contained in the polar set of its potential function. Notice that the pull-back f^*T is also positive if T is positive.

1.4 On the line bundles and transcendental classes

In this section, we recall the definition of an ample (nef, big, pseudo-effective) line bundle.

Let us recall that an class lies in $H^{1,1}(X, \mathbb{Z})$ if and only if the class is the first Chern class of some line bundle. The Néron-Severi space $NS_{\mathbb{R}}(X)$ is the $\mathbb{R}$-vector space defined by $NS_{\mathbb{R}}(X) := H^{1,1}(X, \mathbb{Z}) \otimes \mathbb{R}$.

If some positive multiple L^m of L has a holomorphic section, L is called $\mathbb{Q}$-*effective*. When L is $\mathbb{Q}$-effective, the complete linear system of L^m gives the rational map $\Phi_{|L^m|} : X \to \mathbb{P}(|L^m|)$. When the rational map induced by a sufficiently large multiple of L is a birational map, the line bundle L is called *big*. The pseudo-effective cone is defined by the closure of the cone generated by positive linear combinations of big line bundles. If the first Chern class of L lies in the pseudo-effective cone, L is called *psuedo-effective*. A line bundle L is *semi-ample*, if its holomorphic global sections of some positive multiple of L have no common zero set. When L is semi-ample, the the complete linear system of a sufficiently large multiple of L induces the holomorphic map to the projective space. If the holomorphic map give an embedding to the projective space, L is called *ample*. If the first Chern class of L lies in the closure of the cone generated by positive linear combinations of ample line bundles, L is called *numerically effective* (*nef* in short).

Now we consider a transcendental class α in $H^{1,1}(X,\mathbb{R})$. The Néron-Severi space is contained in $H^{1,1}(X,\mathbb{R})$ however $NS_{\mathbb{R}}(X) \subsetneqq H^{1,1}(X,\mathbb{R})$ in general. We recall the definition of a Kähler (nef, big, pseudo-effective) class. Certainly, when α is the first Chern class of L, L is ample (resp. nef, big, psuedo-effective) if and only if α is Kähler (resp. nef, big, pseudo-effective).

Definition 1.4.1. Let α be a class in $H^{1,1}(X,\mathbb{R})$. Fix a hermitian form ω on X.
(1) α is said to be a *Kähler class* if there exists a smooth positive form in α.
(2) α is said to be a *nef class* if for each $\varepsilon > 0$ there exists a smooth form θ_ε in α such that $\theta_\varepsilon \geq -\varepsilon\omega$.
(3) α is said to be a *big class* if there exists a Kähler current in α.
(4) α is said to be a *pseudo-effective class* if there exists a positive current in α.

1.5 On the ample vector bundles

In this section, we confirm the definition of an ample vector bundle. Let E be a vector bundle of rank r on X. Then we consider the projective space bundle $\mathbb{P}(E)$ of E, which is a complex manifold of dimension $n+r-1$. In this thesis, we denote by $\mathbb{P}(E)$ the projective space bundle associated to E in the sense of Grothendieck. That is,

$$\mathbb{P}(E) = \mathbf{Proj}\Big(\bigoplus_{k=0}^{\infty} \mathrm{Sym}^k(E)\Big).$$

By the construction, we have the morphism $\mathbb{P}(E) \to X$ and the invertible sheaf $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$. We denote by $\mathcal{O}_{\mathbb{P}(E)}(-1)$, the dual line bundle of $\mathcal{O}_{\mathbb{P}(E)}(1)$. A fibre of $\mathbb{P}(E) \to X$ is the space of 1-dimensional quotients of E. Further the restriction to a fibre of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is equal to the hyperplane bundle on the projective space (the space of 1-dimensional quotients of E).

Definition 1.5.1. (1) A vector bundle E is called *ample* if the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is an ample line bundle on $\mathbb{P}(E)$.
(2) A vector bundle E is called *Griffiths-positive* if there exists a hermitian metric on E such that the Chern curvature is Griffith-positive.

We can easily see that a Griffiths positive vector bundle is ample. However, we do not know whether an ample vector bundle is always Griffiths positive. It is so-called the Griffiths conjecture.

1.6 Asymptotic invariants of base loci

In this section, we collect the definitions and properties of the augmented base locus and the restricted base locus of a divisor (line bundle). See Definition 1.2, 1.12 in [ELMNP09] for more details.

Definition 1.6.1. Let L be an $\mathbb{R}$-divisor (line bundle) on X.
(1) When L is a $\mathbb{Q}$-divisor, *the stable base locus* $\mathbb{B}(L)$ of L is defined by

$$\mathbb{B}(L) := \bigcap_{k} \mathrm{Bs}(|kL|)$$

where k runs through all positive integers such that kL is a $\mathbb{Z}$-divisor.
(2) *The augmented base locus* $\mathbb{B}_{+}(L)$ of L is defined by

$$\mathbb{B}_{+}(L) := \bigcap_{L \equiv A+E} \mathrm{Supp}(E)$$

where the intersection is taken over all decomposition $L \equiv A + E$, A and E are $\mathbb{R}$-divisors such that A is ample and E is effective.
(3) *The restricted base locus* $\mathbb{B}_{-}(L)$ of L is defined by

$$\mathbb{B}_{-}(L) := \bigcup_{A} \mathbb{B}(L + A)$$

where the union is taken over all ample $\mathbb{R}$-divisors A such that $L + A$ is a $\mathbb{Q}$-divisor.

Let us recall the definitions of the non-Kähler locus and the non-nef locus of a class $\alpha \in H^{1,1}(X, \mathbb{R})$ (see definition 3.3, 3.17 in [Bou04]).

Definition 1.6.2. (1) Assume α is a big class (that is, it possesses a Kähler current). Then *the non-Kähler locus* $E_{nK}(\alpha)$ of α is defined to be

$$E_{nK}(\alpha) := \bigcap_{T\in\alpha} E_{+}(T)$$

where T ranges among the Kähler currents in α. Here $E_{+}(T)$ denotes $\{x \in X \mid \nu(T, x) > 0\}$.

(2) Assume α is a pseudo-effective class (that is, it possesses a positive current). Then *the non-nef locus* $E_{nn}(\alpha)$ of α is defined to be

$$E_{nn}(\alpha) := \{x \in X \mid \nu(\alpha, x) > 0.\}.$$

Here $\nu(\alpha, x)$ is $\sup_{\varepsilon>0} \nu(T_{\min,\varepsilon}, x)$, where $T_{\min,\varepsilon}$ is a current with minimal singularities in $\alpha + \varepsilon\{\omega\}$ and $\{\omega\}$ is the class of a Kähler form ω on X.

In this thesis, we need the following properties of these base loci. We state the properties of the non-Kähler (non-nef) locus without the proofs (but we give the references). Note that the non-Kähler (resp. non-nef) locus of α coincides with the augmented (resp. restricted) base locus of L when α is the first Chern class of some line bundle L. Thus, the augmented (restricted) base locus also satisfies the following properties.

Proposition 1.6.3. (1) ([ELMNP09, Section 5]). *Given a class $\alpha \in H^{1,1}(X, \mathbb{R})$, we have $E_{nK}(\beta) \subset E_{nK}(\alpha)$ for every class β in a sufficiently small open neighborhood of $\alpha \in H^{1,1}(X, \mathbb{R})$.*
(2) ([Bou04, Theorem 3.17]). *If α is big, there is a Kähler current S in α with analytic singularities such that $E_+(S) = E_{nK}(\alpha)$.*
(3) ([Bou04, Proposition 3.6]). *If α is big, we have*

$$E_{nn}(\alpha) = \{x \in X \mid \nu(T_{\min}, x) > 0.\}$$

where $T_{\min}$ is a current with minimal singularities in α.

Precisely speaking, property (1) was proved only for the augmented base locus in [ELMNP09, Section 5]. However, we shall give the proof for the non-Kähler locus in the proof of Proposition 5.4.10.

1.7 Numerical characterizations of pseudo-effective line bundles

In this section, we give a numerical characterization of the pseudo-effective line bundles, which was established in [BDPP04]. In this section, we assume that X is projective.

Definition 1.7.1. ([BDPP04, Definition 1.3]). A curve C on X is called *a strongly movable curve* if

$$C = \mu_*(A_1 \cap \cdots \cap A_{n-1})$$

for suitable very ample divisors A_i on $\tilde{X}$, where $\mu : \tilde{X} \to X$ is a birational morphism.

The following theorem gives a numerical characterization of the pseudo-effective line bundles. We shall apply this theorem in Chapter 6, 4.

Theorem 1.7.2. ([BDPP04, 0.2, 1.5 Theorem]). *The following conditions are equivalent.*

(1) *A line bundle L is pseudo-effective.*

(2) *The degree of L on C is semi-positive for every strongly movable curve C.*

2

Asymptotic cohomology vanishing and a converse to the Andreotti-Grauert theorem

2.1 Introduction

In complex geometry, the concept of positivity plays an important role. In particular, a positive line bundle is fundamental and important in the theory of several complex variables and algebraic geometry. For that reason, a positive line bundle has been characterized in various ways. For example, some positive multiple gives an embedding to the projective space (geometric characterization), all higher cohomology groups of some positive multiple are zero (cohomological characterization), and the intersection number with any subvariety is positive (numerical characterization). The purpose of this chapter is to generalize these characterizations to a q-positive line bundle.

Throughout this chapter, let X be a compact complex manifold of dimension n, L a line bundle on X and q an integer with $0 \leq q \leq n-1$. Sometimes we may assume that X is Kähler or projective.

In this chapter, we study relations between the q-positivity and the cohomological q-amplitude of a line bundle. The fundamental relations are discussed in [DPS96]. Further, Küronya and Totaro investigated the cohomological q-amplitude of a line bundle in terms of algebraic geometry (see [Kür10], [Tot10]). We consider a q-ample line bundle in terms of complex geometry. Let us recall the definition of a q-positive (cohomologically q-ample) line bundle.

Definition 2.1.1. (1) A holomorphic line bundle L on X is called *q-positive*, if there exists a (smooth) hermitian metric h whose Chern curvature $\sqrt{-1}\Theta_h(L)$

has at least $(n-q)$ positive eigenvalues at any point on X as a $(1,1)$-form.

(2) A holomorphic line bundle L on X is called *cohomologically q-ample*, if for any coherent sheaf $\mathcal{F}$ on X there exists a positive integer $m_0 = m_0(\mathcal{F}) > 0$ such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(L^m)) = 0 \quad \text{for } i > q,\ m \geq m_0.$$

Andreotti and Grauert proved that a q-positive line bundle is always q-ample. (see [AG62, Théorème 14], [DPS96, Proposition 2.1]). It is of interest to know whether the converse implication of the Andreotti-Grauert theorem holds. In this chapter, we mainly discuss the following problem.

Problem 2.1.2. ([DPS96, Problem 2.2]). *Does the converse implication of the Andreotti-Grauert theorem hold ? That is to say, is a q-ample line bundle always q-positive ?*

This problem was first posed by Demailly, Peternell and Schneider in [DPS96]. Precisely speaking, they consider a uniformly q-ample line bundle. However, Totaro showed that the uniform q-amplitude is the same concept as the cohomological q-amplitude (see [Tot10, Theorem 6.2]). It is a natural question. However it has been an open problem for a long time, except the case of $q = 0$.

A 0-positive line bundle is a positive line bundle in the usual sense. Further, it follows from the Serre vanishing theorem that a cohomologically 0-ample line bundle is ample in the usual sense of algebraic geometry. Thanks to the Kodaira embedding theorem, we know that a positive line bundle coincides with an ample line bundle. Therefore Problem 2.1.2 is affirmatively solved in the case of $q = 0$.

In Section 2.2, we study this problem when X is a smooth projective surface. The main result of this section is an affirmative answer for Problem 2.1.2 on a surface (Theorem 2.1.3). For the proof of Theorem 2.1.3, we establish Theorem 2.2.1. Theorem 2.2.1 also leads to Corollary 2.2.6, which can be seen as the generalization of [FO09, Theorem 1]. Theorem 2.2.1 is proved by using a solution of the Monge-Ampère equation.

Theorem 2.1.3. *On a smooth projective surface X, the converse of the Andreotti-Grauert theorem holds. That is, the following conditions are equivalent.*

(A) *L is cohomologically q-ample.*

(B) *L is q-positive.*

In his paper [Dem10-B], Demailly proved the converse of the holomorphic Morse inequality under various situations. These results can be seen as a "partial" converse of the Andreotti-Grauert theorem. The original part of this chapter is to give an "exact" converse (see [Dem10-B] or Section 2.5 for the precise difference). By combining Theorem 2.1.3 and the result of [Dem10-B], the asymptotic behavior of the higher cohomology on a surface can be interpreted in terms of the curvature.

In Section 2.3, the various characterizations of the q-positivity of a semi-ample line bundle are given on an arbitrary compact complex manifold. A line bundle L is called *semi-ample*, if its holomorphic global sections of some positive multiple of L have no common zero set. A semi-ample line bundle induces a holomorphic map to the projective space. (See [Laz] for more details on a semi-ample line bundle.)

Theorem 2.3.1 gives a relation between the fibre dimension of a holomorphic map and the q-positivity. When the map is the holomorphic map associated to a sufficiently large multiple of a semi-ample line bundle L, condition (B) in Theorem 2.3.1 is equivalent to the cohomological q-amplitude of L (see [Som78-B, Proposition 1.7]). It leads to the following theorem:

Theorem 2.1.4. *Let L be a semi-ample line bundle on a compact complex manifold X. Then the following conditions (A), (B) and (C) are equivalent.*

(A) *L is q-positive.*

(B) *The semi-ample fibration of L has fibre dimensions at most q.*

(C) *L is cohomologically q-ample.*

Further if X is projective, the conditions above are equivalent to condition (D).

(D) *For every subvariety Z with $\dim Z > q$, there exists a curve C on Z such that the degree of L on C is positive.*

Condition (B) (resp.(C), (D)) gives a geometric (resp. cohomological, numerical) characterization of a q-positive line bundle. In particular, the converse of the Andreotti-Grauert theorem holds for a semi-ample line bundle on an arbitrary compact complex manifold.

In Section 2.4, we consider the Zariski-Fujita type theorem (Theorem 2.4.1) in order to investigate the q-positivity of a big line bundle. In particular, we know that the converse of the Andreotti-Grauert theorem for a big line bundle is reduced to the case of varieties of smaller dimension (the non-ample locus).

2.2 Monge-Ampère equations and $(n-1)$-positivity

This section is devoted to prove Theorem 2.2.1 and its corollaries. Throughout this section, let L be a line bundle on a compact Kähler manifold X and ω a Kähler form on X. First we give the proof of Theorem 2.2.1.

Theorem 2.2.1. *Let L be a line bundle on a compact Kähler manifold X and ω a Kähler form on X. Assume that the intersection number $\left(L\cdot\{\omega\}^{n-1}\right)$ is positive. Here $\{\omega\}$ denotes the cohomology class in $H^{1,1}(X,\mathbb{R})$ which is defined by a d-closed $(1,1)$-form ω.*

Then L is $(n-1)$-positive. That is, there exists a smooth hermitian metric h whose Chern curvature $\sqrt{-1}\Theta_h(L)$ has at least 1 positive eigenvalue at every point on X.

Proof. The main idea of the proof is to use a solution of the Monge-Ampère equation. In order to solve the Monge-Ampère equation, we make use of the following Calabi-Yau type theorem. It is a deep result which was proved as a special case in [Yau78]. Roughly speaking, it says that the product of the eigenvalues of a Kähler form (which represents a given Kähler class) can be controlled.

Theorem 2.2.2. ([Yau78]). *Let M be a compact Kähler manifold of dimension n and $\widetilde{\omega}$ a Kähler form on M. For a positive smooth (n,n)-form $F>0$ with $\int_M F=\int_M \widetilde{\omega}^n$, there exists a function $\varphi\in C^\infty(M,\mathbb{R})$ with the following properties :*

(1) $(\widetilde{\omega}+dd^c\varphi)^n=F$ *at every point on* M

(2) $(\widetilde{\omega}+dd^c\varphi)$ *is a Kähler form on* M.

Fix a smooth hermitian metric h of L. Then the Chern curvature $\sqrt{-1}\Theta_h(L)$ represents the first Chern class of L. We want to construct a real-valued smooth function φ on X such that $\sqrt{-1}\Theta_h(L)+dd^c\varphi$ is $(n-1)$-positive (that is, the $(1,1)$-form has at least 1 positive eigenvalue everywhere). If we obtain a function φ with the condition above, we can easily see that L is $(n-1)$-positive. In fact, the Chern curvature associated to the metric defined by $he^{-2\varphi}$ is equal to $\sqrt{-1}\Theta_h(L)+dd^c\varphi$. Therefore it is sufficient to construct a function φ with the condition above. To construct such function, we make use of Theorem 2.2.2.

Since ω is a positive form, the $(1,1)$-form $\sqrt{-1}\Theta_h(L) + k\omega$ is a Kähler form on X for a sufficiently large constant $k > 0$. Now we consider the following Monge-Ampère equation:

$$\begin{aligned}\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)^n &= D_k\big(k\omega\big)^n,\\ \big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big) &> 0.\end{aligned}$$

Here D_k is a positive constant which depends on k. In order to solve this equation, we need to define D_k by

$$D_k := \frac{\int_X \big(\sqrt{-1}\Theta_h(L) + k\omega\big)^n}{\int_X \big(k\omega\big)^n}.$$

When D_k is defined as above, we know that there exists a solution of the equation, thanks to Theorem 2.2.2. In fact, by applying Theorem 2.2.2 to a Kähler form

$$\widetilde{\omega} := \big(\sqrt{-1}\Theta_h(L) + k\omega\big)$$

and a smooth (n,n)-form $F := D_k\big(k\omega\big)^n$, we can obtain a solution. Notice that the equality $\int_X \widetilde{\omega}^n = \int_X F$ holds from the definition of D_k.

Now we shall show that the constant D_k is greater than 1 for a sufficiently large $k > 0$. We use the assumption of the theorem only for this argument. By the definition of D_k we obtain

$$\begin{aligned}D_k &= \frac{\int_X \big(\sqrt{-1}\Theta_h(L) + k\omega\big)^n}{\int_X \big(k\omega\big)^n}\\ &= \frac{\big(L + k\{\omega\}\big)^n}{\big(k\{\omega\}\big)^n}\\ &= \frac{\big(L^n\big) + kn\big(L^{n-1}\cdot\{\omega\}\big) + \cdots + k^{n-1}n\big(L\cdot\{\omega\}^{n-1}\big)}{k^n(\{\omega\}^n)} + 1.\end{aligned}$$

The molecule in the right hand is a polynomial of degree $(n-1)$ with respect to k. Further, the coefficient of the highest degree term is equal to $n(L\cdot\{\omega\}^{n-1})$. It is positive by the assumption. Therefore the first term in the right hand is greater than 0 for a sufficiently large $k > 0$. Hence D_k is greater than 1.

Finally we show that $\sqrt{-1}\Theta_h(L) + dd^c\varphi$ has at least 1 positive eigenvalue at every point on X. Here $\lambda_1(x) \geq \cdots \geq \lambda_n(x)$ denote the eigenvalues of

$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)$$

with respect to $k\omega$ at $x \in X$. Then the function λ_i for $i = 1, 2, \ldots, n$ is well-defined as a function on X. Further λ_i for $i = 1, 2, \ldots, n$ is continuous, since the j-th symmetric function in $\lambda_1, \ldots, \lambda_n$ is smooth. Since φ is a solution of the Monge-Ampère equation, the functions λ_i satisfy the following equality everywhere:

$$\lambda_1(x)\lambda_2(x)\cdots\lambda_n(x) = D_k > 1, \qquad \text{for any } x \in X.$$

In addition, $\lambda_n(x)$ is positive for any point $x \in X$ since

$$\big(\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi\big)$$

is a Kähler $(1,1)$-from. Thus we know $\lambda_1(x) > 1$ at every point on X since D_k is greater than 1.

Now the eigenvalues of

$$\sqrt{-1}\Theta_h(L) + dd^c\varphi = (\sqrt{-1}\Theta_h(L) + k\omega + dd^c\varphi) - k\omega$$

are $(\lambda_1 - 1), \ldots, (\lambda_n - 1)$ since all eigenvalues of $k\omega$ are 1. Thus $\big(\sqrt{-1}\Theta_h(L) + dd^c\varphi\big)$ has 1 positive eigenvalue $(\lambda_1 - 1)$ everywhere. It completes the proof. $\square$

On the rest of this section, we give the proofs of Theorem 2.2.3 and Corollary 2.2.6.

Theorem 2.2.3. (=Theorem 2.1.3). *On a smooth projective surface X, the converse of the Andreotti-Grauert theorem holds. That is, the following conditions are equivalent.*

(A) *L is cohomologically q-ample.*

(B) *L is q-positive.*

In the statement of Theorem 2.2.3, it follows that condition (B) leads to (A) from the Andreotti-Grauert theorem. The converse is an open problem and a main subject in this chapter. Theorem 2.2.3 claims the converse is affirmative on a smooth projective surface.

For the proof of Theorem 2.2.3, we shall prepare Lemma 2.2.4 and Lemma 2.2.5. These lemmata may be known facts. However we give proofs for readers' convenience. Lemma 2.2.4 can be proved even if X has singularities. However for our purpose, we need only the case where X is smooth.

Lemma 2.2.4. *Let L be a line bundle on a smooth projective variety X. Then the following conditions are equivalent.*

(1) *The dual line bundle L^{-1} is not pseudo-effective.*

(2) *L is cohomologically $(n-1)$-ample.*

Proof. This theorem follows from the Serre duality theorem.

First we confirm that condition (2) implies (1). For a contradiction, we assume that L^{-1} is psuedo-effective. Then we can take an ample line bundle A on X such that $A \otimes L^{-m}$ has a non-zero section for every positive integer $m > 0$. Note that A does not depend on m. For the coherent sheaf $\mathcal{O}_X(K_X \otimes A^{-1})$, we can take a positive integer m such that

$$h^n\big(X, \mathcal{O}_X(K_X \otimes A^{-1} \otimes L^m)\big) = 0$$

from condition (2). Here K_X denotes the canonical bundle on X. The Serre duality theorem asserts that $h^0\big(X, \mathcal{O}_X(A \otimes L^{-m})\big) = 0$. This is a contradiction to the choice of A.

Conversely we show that condition (1) implies (2). Fix an ample line bundle B on X. For a given coherent sheaf $\mathcal{F}$ on X, we can take the following resolution of $\mathcal{F}$ by choosing an integer $k > 0$:

$$0 \longrightarrow \mathcal{G} \longrightarrow \oplus_{i=1}^{N} \mathcal{O}_X(B^{-k}) \longrightarrow \mathcal{F} \longrightarrow 0.$$

In fact, $\mathcal{F} \otimes \mathcal{O}_X(B^k)$ is globally generated for a sufficient large k since B is ample. Therefore we have a surjective map $\oplus_{i=1}^{N} \mathcal{O}_X(B^{-k}) \longrightarrow \mathcal{F}$ as a sheaf morphism. We define $\mathcal{G}$ by the kernel of its map.

Thus it is sufficient to show that there is a positive integer m_0 such that

$$h^n\big(X, \mathcal{O}_X(B^{-k} \otimes L^m)\big) = 0 \quad \text{for } m \geq m_0$$

In fact, the long exact sequence yields $h^n\big(X, \mathcal{F} \otimes \mathcal{O}_X(L^m)\big) = 0$ for $m \geq m_0$. It means that L is cohomologically $(n-1)$-ample.

Since L^{-1} is not psuedo-effective, there is a sufficiently large integer m_0 such that $K_X^{-1} \otimes B^k \otimes L^{-m}$ is not psuedo-effective for $m \geq m_0$. It implies

$$h^0\big(X, \mathcal{O}_X(K_X^{-1} \otimes B^k \otimes L^{-m})\big) = 0.$$

Again by using the Serre duality theorem, we have

$$h^n(X, \mathcal{O}_X(B^{-k} \otimes L^m)) = 0$$

for $m \geq m_0$. □

Lemma 2.2.5. *Let L be a line bundle on a smooth projective variety X. Then the following conditions are equivalent.*

(2) *L is cohomologically $(n-1)$-ample.*

(3) *There exists a strongly movable curve C on X such that the degree of L on C is positive.*

Proof. The deep result proved in [BDPP04] yields Lemma 2.2.5. It follows from [BDPP04, Theorem2.2] that the cone of pseudo-effective line bundles is the dual cone of strongly movable curves. That is, a line bundle is pseudo-effective if and only if the degree of the line bundle on every strongly movable curve is semi-positive. From Lemma 2.2.4, L is cohomological $(n-1)$-ample if and only if L^{-1} is not psuedo-effective. Therefore then there exists a strongly movable curve C such that

$$(L^{-1} \cdot C) < 0.$$

It completes the proof. □

By applying Lemma 2.2.5 and Theorem 2.2.1, we shall complete the proof of Theorem 2.2.3.

Proof of Theorem 2.2.3. When X is a projective surface, the closure of the cone of strongly movable curves coincides with the closure of the cone of ample line bundles (that is, the nef cone). Indeed, the dual cone of pseudo-effective line bundles equals to the closure of the cone of ample line bundles. Therefore, when L is cohomologically $(n-1)$-ample, we have an ample line bundle H with $(L \cdot H) > 0$ by Lemma 2.2.5.

Since L is ample, the first Chern class of L contains a Kähler form ω. Since the intersection number $(L \cdot H)$ equals to $(L \cdot \{\omega\})$, the line bundle L satisfies the assumption in Theorem 2.2.1. Therefore if follows that L is 1-positive from Theorem 2.2.1. □

At the end of this section, we prove Corollary 2.2.6, which can be seen as the generalization of [FO09, Theorem 1] to a psuedo-effective line bundle. In [FO09], in order to show that an effective line bundle is $(n-1)$-positive, Fuse and Ohsawa apply $(n-1)$-convexity of a non-compact complex manifold. We make use of the Monge-Ampère equation instead of $(n-1)$-convexity of a non-compact complex manifold.

Corollary 2.2.6. *Let L be a pseudo-effective line bundle on a compact Kähler manifold X. Assume that the first Chern class of L is not trivial.*

Then L is $(n-1)$-positive.

A line bundle is called *pseudo-effective* if there exists a singular hermitian metric h whose curvature current $\sqrt{-1}\Theta_h(L)$ is positive on X as a $(1,1)$-current. A pseudo-effective line bundle (which is not numerically trivial) is cohomologically $(n-1)$-ample (see Lemma 2.2.4). Therefore a pseudo-effective line bundle should be $(n-1)$-positive if the converse of the Andreotti-Grauert theorem holds. Corollary 2.2.6 claims that it is affirmative on a compact Kähler manifold.

Proof. Under the assumption of Corollary 2.2.6, we show that the line bundle L satisfies the assumption in Theorem 2.2.1. In fact, it follows from the following lemma. Hence the intersection number $\left(L \cdot \{\omega\}^{n-1}\right)$ must be positive. It follows that L is $(n-1)$-positive from Theorem 2.2.1. $\square$

Finally we prove Lemma 2.2.7.

Lemma 2.2.7. *Let L be a pseudo-effective line bundle whose first Chern class $c_1(L)$ is not zero. Then the intersection number $(L \cdot \{\omega\}^{n-1})$ is positive for any Kähler form ω on X.*

Proof. We take an arbitrary smooth $(n-1, n-1)$-form γ on X. Since the $(n-1, n-1)$-form ω^{n-1} is strictly positive, there exists a large constant $C > 0$ such that

$$-C\omega^{n-1} \leq \gamma \leq C\omega^{n-1}.$$

Here we implicitly used the compactness of X. Since L is pseudo-effective, we can take a singular metric h on L such that the curvature current $\sqrt{-1}\Theta(L)$ is positive. It gives the following inequality:

$$-C\int_X \sqrt{-1}\Theta(L) \wedge \omega^{n-1} \leq \int_X \sqrt{-1}\Theta(L) \wedge \gamma \leq C\int_X \sqrt{-1}\Theta(L) \wedge \omega^{n-1}.$$

The $(1,1)$-current $\sqrt{-1}\Theta(L)$ represents the first Chern class of L. Thus the integral in the right and left hands

$$\int_X \sqrt{-1}\Theta(L) \wedge \omega^{n-1}$$

agrees with the intersection number $(L \cdot \{\omega\}^{n-1})$. If the intersection number is zero,

$$\int_X \sqrt{-1}\Theta(L) \wedge \gamma = 0$$

for any smooth $(n-1, n-1)$-form γ from the inequality above. It means that the $(1,1)$-current $\sqrt{-1}\Theta(L)$ is a zero current. This is a contradiction to the assumption that $c_1(L)$ is not zero. Hence the intersection number $(L \cdot \{\omega\}^{n-1})$ is positive for any Kähler form ω on X. $\square$

2.3 Fiber dimensions and q-positivity

The main purpose of this section is to give the proof of Theorem 2.1.4. For this purpose, we first consider Theorem 2.3.1.

Theorem 2.3.1. *Let $\Phi : X \longrightarrow Y$ be a holomorphic map (possibly not surjective) from X to a compact complex manifold Y. Then the following conditions are equivalent.*

(A) *Fix a Hermitian form ω (that is, a positive definite $(1,1)$-form) on Y. Then there exists a function $\varphi \in C^\infty(X, \mathbb{R})$ such that the $(1,1)$-form $\Phi^*\omega + dd^c\varphi$ is q-positive (that is, the form has at least $(n-q)$ positive eigenvalues at any point on X as a $(1,1)$-form).*

(B) *The map Φ has fibre dimensions at most q.*

Throughout this section, let $\Phi : X \longrightarrow Y$ be a holomorphic map from X to a compact complex manifold Y. Fix a hermitian form ω on Y. Set $\widetilde{\omega} := \Phi^*\omega$, which is a semi-positive $(1,1)$-form on X.

First we show that condition (A) implies (B). For a contradiction, we assume that there is a fibre F of the map Φ with $\dim F > q$. Then by condition (A), X allows a smooth function φ such that $\widetilde{\omega} + dd^c\varphi$ is q-positive. Since F is a fibre, the restriction to F of $\widetilde{\omega} = \Phi^*\omega$ is equal to zero. It implies that the restriction to F of $dd^c\varphi$ is q-positive. Even if F has singularities, we can define the q-positivity (see Definition 2.4.2) Thus, it follows from $\dim F > q$, that there is a smooth function $\varphi|_F$ on F whose Levi form has at least 1 positive eigenvalue.

Since F is compact, the function $\varphi|_F$ has the maximum value. Suppose that $p \in F$ attains the maximum value of $\varphi|_F$. Then the Levi-form of $\varphi|_F$ at p has no positive eigenvalues. It is a contradiction. Hence condition (A) leads to (B).

On the rest of this section, we shall show that condition (B) implies (A). From now on, we assume that the dimension of every fibre of the map Φ is at most q. Then we want to construct a function φ with condition (A). For this purpose, we define the degenerated locus of a hermitian form by the pull-back of the map Φ as follows:

Definition 2.3.2. The *degenerated locus* by the pull-back of Φ is defined to be

$$B_q := \{\, p \in X \mid \Phi^*\omega \text{ has at least } (q+1) \text{ zero eigenvalues at } p.\}.$$

Since $\widetilde{\omega} = \Phi^*\omega$ is a semi-positive form, $\widetilde{\omega}$ is q-positive outside B_q. Therefore if B_q is empty, condition (A) is satisfied by taking $\varphi := 0$. It is sufficient to consider the case where B_q is not empty. The following lemma asserts that the degenerated locus is locally the zero set of finite holomorphic functions.

Proposition 2.3.3. *The degenerated locus B_q is a closed analytic set on X.*

Proof. First we show that B_q is a closed set in X. Fix a hermitian form $\overline{\omega}$ on X. We denote by $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, the eigenvalues of $\widetilde{\omega}$ with respect to $\overline{\omega}$. The j-th symmetric function in $\lambda_1, \ldots, \lambda_n$ is smooth since $\widetilde{\omega}$ and $\overline{\omega}$ is smooth forms. Therefore λ_i for $0 \leq i \leq n$ is a (well-defined) continuous function on X. Now p is contained in B_q if and only if $\lambda_{n-q}(p) = 0$. Thus the degenerated locus is closed since λ_{n-q} is a continuous function.

It remains to show that B_q is the zero set of finite holomorphic functions. Now we take a local coordinate $(w_1, \ldots, w_m)$ on Y. Here m denotes the dimension of Y. Then the degenerate locus of ω coincides with the degenerate locus of $\sum_{i=1}^m dd^c|w_i|^2$. In fact, it follows since we have

$$C\sum_{i=1}^{m} dd^c|w_i|^2 \geq \omega \geq (1/C)\sum_{i=1}^{m} dd^c|w_i|^2$$

for a sufficiently large constant $C > 0$.

The holomorphic map Φ can be locally written as

$$(z_1, \ldots, z_n) \longmapsto (f_1(z), \ldots, f_m(z))$$

for some holomorphic functions f_i. Here $(z_1, \ldots, z_n)$ denotes a local coordinate on X. Then B_q is equal to the locus where the hermitian form $dd^c\sum_{i=1}^m dd^c|f_i(z)|^2$ has at least $(q+1)$ zero eigenvalues. In general, a semi-positive hermitian form has at least $(q+1)$ zero eigenvalues if and only if j-th symmetric function in the eigenvalues is zero for $n-q \leq j \leq n$. By a simple computation, j-th symmetric function σ_j in $\lambda_1, \ldots, \lambda_n$ is described as follows:

$$\begin{aligned}\sigma_j &= \sum_{0 \leq i_1 < \cdots < i_j \leq n} \det\Big(\big\langle \mathbb{G}_{i_a}, \mathbb{G}_{i_b}\big\rangle\Big)_{a,b=1,\ldots,n} \\ &= \Bigg| \sum_{0 \leq i_1 < \cdots < i_j \leq n} \ \sum_{0 \leq \alpha_1 < \cdots < \alpha_j \leq m} \det\Big(g_{i_b}^{\alpha_a}\Big)_{a,b=1,\ldots,j}\Bigg|^2.\end{aligned}$$

Here g_i^α denotes the differential $\partial f_\alpha / \partial z_i$, $\mathbb{G}_i$ a vector $(g_i^1, \ldots, g_i^m)$, and $\langle \cdot, \cdot \rangle$ the standard hermitian metric on $\mathbb{C}^m$. Therefore the set defined by $\sigma_j = 0$ $(n-q \leq j \leq n)$ is the zero set of finite holomorphic functions. $\square$

Thanks to Proposition 2.3.3, we can consider the dimension of B_q. When the dimension of B_q is less than or equal to q, we can easily see condition (A) in Theorem 2.3.1 from Lemma 2.3.8. When the dimension is greater than q, we factor B_q to subvarieties of smaller dimension. For this purpose we need Lemma 2.3.5. The assumption on fibre dimensions is used in the proof of this lemma. Later we need to treat an analytic set which may not be closed. For that reason, Lemma 2.3.5 is formulated for an analytic set (possibly not closed).

Definition 2.3.4. A subset V in X is called *an analytic set* if at every point p in V there exist a small neighborhood of p and finite holomorphic functions on the neighborhood such that V is the zero set of these functions.

Note that an analytic set does "not" mean a closed analytic set in this chapter. For example, the set $\{1/n \in \mathbb{C} \mid n \in \mathbb{N}\}$ is an analytic set, but not a closed analytic set.

Lemma 2.3.5. *Let W be an irreducible analytic set (possibly not closed, singular) on X. Assume that the dimension of W is greater than q.*

Then the degenerate locus D defined by

$$D := \{p \in W_{\text{reg}} \mid \text{The restriction } \widetilde{\omega}|_{W_{\text{reg}}} \text{ has at least } (q{+}1) \text{ zero eigenvalues at } p\}$$

is a closed analytic set on W_{reg} and properly contained in W_{reg}. Here W_{reg} denotes the regular locus of W.

Proof. We have already proved that D is a closed analytic set on W_{reg} in the proof of Proposition 2.3.3. It remains to show that D is a properly contained subset in W_{reg}. For a contradiction, we assume $D = W_{\text{reg}}$.

We take a point p in W_{reg} such that $\Phi|_W : W \longrightarrow Y$ is a smooth morphism at p, and a fibre F of Φ containing p. Further we take an open ball U in W with a local coordinate $(z_1, \ldots, z_r)$ centered at p. We may assume that the first coordinate $(z_1, \ldots, .z_s)$ also becomes a local coordinate on F_{reg}. Here r (resp. s) denotes the dimension of W (resp. F).

Now we consider the restriction f of Φ which is defined by

$$f := \Phi|_{F^{\perp}} : F^{\perp} \longrightarrow Y, \text{ where } F^{\perp} = \{ (0, \ldots, 0, z^{s+1}, \ldots, z^r) \in U. \}.$$

Then the fibre of $\Phi(p)$ by f is a zero dimensional analytic set. It implies the Jacobian of f is not identically zero on some neighborhood of p. Hence the holomorphic map f gives a local isomorphism at some point. This means the restriction of $\widetilde{\omega}$ to $F^{\perp}$ has $(n - s)$ positive eigenvalues. Note that s is less than or equal to q. This is a contradiction to $W_{\text{reg}} = D$. $\square$

Lemma 2.3.5 leads to Proposition 2.3.6. Later we shall apply this proposition to B_q. The set B_q is a closed analytic set. However we formulate this proposition for an analytic set to prove Proposition 2.3.6 with induction on the dimension. Remark that "dimension " in Proposition 2.3.6 does not necessarily mean the pure dimension.

Proposition 2.3.6. *Let V be an analytic set of dimension k (possibly, not closed, not irreducible, singular). Then there exist sets D_ℓ $(0 \leq \ell \leq k-1)$ with following properties:*

(1) *D_ℓ is an analytic set on X.*

(2) $D_k := V \supseteq D_{k-1} \supseteq \cdots \supseteq D_1 \supseteq D_0$.

(3) $\dim D_\ell = \ell$ *for* $\ell = 0, 1, 2, \ldots, k$.

(4) *$D_\ell \setminus D_{\ell-1}$ for $\ell = 1, 2, \ldots, k$ is a disjoint union of non-singular analytic sets.*

(5) *For an irreducible component W of $D_\ell \setminus D_{\ell-1}$ with $\dim W > q$, the $(1,1)$-form $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues.*

Proof. We prove this proposition by induction on the dimension $k = \dim V$. When k is zero, we set $D_0 = V$. Then the properties in Proposition 2.3.6 hold. From now on, we assume that k is greater than zero.

We consider the decomposition $V = V_{\text{reg}} \cup V_{\text{sing}}$. Here V_{reg} (resp. V_{sing}) denotes the regular locus (resp. the singular locus) of V. Note that this decomposition is a disjoint union. Since the dimension of V_{sing} is smaller than k, we obtain $\widetilde{D}_\ell$ $(0 \leq \ell \leq \dim V_{\text{sing}})$ with the properties in Proposition 2.3.6 by the induction hypothesis.

Let $V_{\text{reg}} = \bigcup_{i \in I} V_i$ be the irreducible decomposition of V_{reg}. For an irreducible component V_i, we set $D^i_{\dim V_i} := V_i$ if the dimension of V_i is less than or equal to q. Otherwise, we investigate the degenerated locus D^i of V_i which is defined by

$$D^i := \{p \in (V_i)_{\text{reg}} \mid \widetilde{\omega}|_{(V_i)_{\text{reg}}} \text{ has at least } (q+1) \text{ zero eigenvalues at } p. \}.$$

It follows that D^i is an analytic set and properly contained in V_i from Lemma 2.3.5. In particular, the dimension of D^i is smaller than k. Therefore by applying the induction hypothesis to D^i, we obtain D^i_ℓ $(0 \leq \ell \leq \dim V_i)$ with the properties in Proposition 2.3.6.

For each ℓ, we define the set D_ℓ to be the union of $\widetilde{D}_\ell$ and D^i_ℓ $(i \in I)$. Then we can easily see that D_ℓ has the properties in Proposition 2.3.6. In fact, it follows since $\widetilde{D}_\ell$ and D^i_ℓ satisfy the properties in Proposition 2.3.6 and D_ℓ is a disjoint union of them. $\square$

For the proof of Theorem 2.3.1, we apply Proposition 2.3.6 to B_q. Then we obtain D_ℓ with the properties in Proposition 2.3.6. By using these properties, we shall construct a function φ whose Levi-form has $(n - \dim W)$ positive eigenvalues in the normal direction of an irreducible component W of $D_\ell \setminus D_{\ell-1}$. On the other hand, the restriction $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues from property (5) if the dimension of W is greater than q. Thus if there is such function φ, $\widetilde{\omega} + dd^c\varphi$ has at least $(n - q)$ positive eigenvalues everywhere. To construct such function, we prepare Proposition 2.3.7, 2.3.9. The proofs of these Propositions are similar to the proof of [Dem90, Theorem 4] (cf. [Siu77]).

Proposition 2.3.7. *For $\ell = 0, 1, \ldots, k$, there exists a function $\varphi_\ell \in C^2(X, \mathbb{R})$ with the following properties :*

(1) *Let W be an irreducible component of $D_\ell \setminus D_{\ell-1}$. Then the Levi form $dd^c\varphi_\ell$ has $(n - \dim W)$ positive eigenvalues in the normal direction of W.*

(2) *The Levi form $dd^c\varphi_\ell$ is semi-positive at every point in $\overline{D_\ell}$.*

Here $\overline{D_\ell}$ denotes the closure of D_ℓ in X. Note $\overline{D_\ell}$ may not be a closed analytic set. For example, the closure of $\{(x, y) \in \mathbb{C}^2 \mid x = e^y\}$ in the 2-dimensional projective space is not a closed analytic set. In order to show Proposition 2.3.7, we prepare the following lemma. If D_ℓ for $\ell = 0, 1, \ldots, k$ is a closed analytic set, the statement of this lemma is same as that of Proposition 2.3.7.

Lemma 2.3.8. *Let ℓ be an integer with $0 \leq \ell \leq k$. For every open neighborhood U of $\overline{D_\ell} \setminus D_\ell$, there exists a function $\varphi_U \in C^\infty(X, \mathbb{R})$ with following properties :*

(1) *Let W be an irreducible component of $D_\ell \setminus D_{\ell-1}$. Then the Levi form $dd^c\varphi_U$ has $(n - \dim W)$ positive eigenvalues in the normal direction at every point in $W \setminus U$.*

(2) *The Levi form $dd^c\varphi_U$ is semi-positive at every point in $\overline{D_\ell}$.*

Proof. For a given U, we take an open covering of X by open balls U_j $(j = 1, 2, \ldots, N)$. Further, we can assume the following properties for this covering.

(*a*) The members U_j $(j = 1, 2, \ldots, s)$ cover $D_\ell \setminus D_{\ell-1}$.

(*b*) The members U_j $(j = 1, 2, \ldots, s)$ are contained in U.

(*c*) Every U_k which intersects with U_j $(j = 1, 2, \ldots, s)$ is also contained in U.

We denote by $\mathcal{I}_{B_q}$, the ideal sheaf associated to a closed analytic set B_q. For every $j = 1, 2, \ldots, s$, we take holomorphic functions $\{f_{j,i}\}_{i=1}^{N_j}$ on U_j such that these functions generate the ideal sheaf $\mathcal{I}_{B_q}$. Further, for every $j = s+1, \ldots, N$, we take holomorphic functions $\{f_{j,i}\}_{i=1}^{N_j}$ on U_j such that these functions generate the ideal sheaf $\mathcal{I}_{D_\ell}$. Note D_ℓ is a closed analytic set on U_j $(j = s+1, \ldots, N)$. Therefore we can define the ideal sheaf $\mathcal{I}_{D_\ell}$ and take its generators.

Let $\{\rho_j\}_{j=1}^{N}$ be a partition of unity associated to the covering of X. Now we define φ_U to be

$$\varphi_U := \sum_{j=1}^{N} \rho_j \Bigl(\sum_{i=1}^{N_j} |f_{j,i}|^2 \Bigr).$$

Then φ_U satisfies properties (1), (2). In fact, an easy computation yields

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\varphi_U = \sum_{j=1}^{N}\sum_{i=1}^{N_j} \Bigl\{ &|f_{j,i}|^2 \sqrt{-1}\partial\overline{\partial}\rho_j + \sqrt{-1}\overline{f_{j,i}}\partial f_{j,i} \wedge \overline{\partial}\rho_j \\ &+ \sqrt{-1} f_{j,i}\partial\rho_j \wedge \overline{\partial} f_{j,i} + \sqrt{-1}\rho_j \partial f_{j,i} \wedge \overline{\partial f_{j,i}} \Bigr\}.\end{aligned}$$

By the definition, $f_{j,i}$ is identically zero on $\overline{D_\ell}$. (Notice that $\overline{D_\ell}$ is contained in B_q.) Therefore the first three terms are zero on $\overline{D_\ell}$. Further the last term is clearly semi-positive. Therefore property (2) is satisfied. For every point p in $W \setminus U$, we can take j_0 such that U_{j_0} does not intersect with $\overline{D_\ell} \setminus D_{\ell-1}$ and $\rho_{j_0}(p) \neq 0$ by the choice of a covering. Hence the last term has property (2) since $\partial f_{j_0,i} \wedge \overline{\partial f_{j_0,i}}$ has $(n - \dim W)$ positive eigenvalues at p in the normal direction of W. $\square$

Before the proof of Proposition 2.3.7, we recall the definition of a C^2-norm $\| \cdot \|_{C^2}$ on $C^2(X, \mathbb{R})$. We take an open covering of X by open balls U_j $(j =, 1, 2, \ldots, N)$ with a differential coordinate $(x_1^j, \ldots, x_{2n}^j)$. Let V_j be a relatively compact set in U_j such that $\{V_j\}_{j=1}^{N}$ is also an open covering of X. Then the C^2-norm $|\cdot|_{C^2}$ with respect to the open covering is defined to be

$$|f|_{C^2} := \sum_{j=1}^{N}\sum_{\alpha,\beta=1}^{2n,2n} \sup_{p\in\overline{V_j}} \left| \frac{\partial^2 f}{\partial x_\alpha^j \partial x_\beta^j}(p) \right| + \sum_{j=1}^{N}\sum_{\alpha=1}^{k} \sup_{p\in\overline{V_j}} \left| \frac{\partial f}{\partial x_\alpha^j}(p) \right| + \sup_{p\in X} |f(p)|$$

for every $f \in C^2(X, \mathbb{R})$. Certainly the norm depends on the choice of an open covering. However the topology induced by these norms is unique. For our purpose, we fix the norm. Let us begin the proof of Proposition 2.3.7.

Proof of Proposition 2.3.7. Choose a family of open neighborhoods $\{U_i\}_{i=1}^{\infty}$ of $\overline{D_\ell} \setminus D_{\ell-1}$ such that $\bigcap_{i=1}^{\infty} U_i = \overline{D_\ell} \setminus D_{\ell-1}$. For each positive integer i, we can take a function $\varphi_{U_i} \in C^\infty(X, \mathbb{R})$ with the properties in Lemma 2.3.8. We set

$$A_i = \| \varphi_{U_i} \|_{C^2} .$$

Here $\| \cdot \|_{C^2}$ denotes the fixed C^2-norm.

Now we define a function φ_ℓ on X to be

$$\varphi_\ell := \sum_{i=1}^{\infty} \frac{1}{2^i A_i} \varphi_{U_i}.$$

By the definition of A_i, the sum in the definition uniformly converges with respect to the C^2-norm. Hence we obtain

$$dd^c \varphi_\ell = \sum_{i=1}^{\infty} \frac{1}{2^i A_i} dd^c \varphi_{U_i}.$$

Properties (1), (2) in Lemma 2.3.8 and the choice of U_i lead to the properties in Proposition 2.3.7.

$\square$

Proposition 2.3.9. *For every integer $\ell = 0, 1, \ldots, k$, there exists a function $\widetilde{\varphi}_\ell \in C^2(X, \mathbb{R})$ with property $(*)$.*

$(*)$ *Let m be an integer with $0 \leq m \leq \ell$ and W an irreducible component of $D_m \setminus D_{m-1}$. Then the Levi-form $dd^c \widetilde{\varphi}_\ell$ has $(n - \dim W)$ positive eigenvalues in the normal direction of W.*

Before the proof of Proposition 2.3.9, we confirm that Proposition 2.3.9 and Proposition 2.3.6 complete the proof of Theorem 2.3.1. That is, there is a smooth function φ on X such that the $(1,1)$-form $\widetilde{\omega} + dd^c \varphi$ is q-positive.

First we obtain $\{D_\ell\}_{\ell=0}^{k}$ with properties in Proposition 2.3.6 by applying Proposition 2.3.6 to B_q. Now we take $\widetilde{\varphi_k}$ with property $(*)$ in Proposition 2.3.9. Recall k is the dimension of B_q. Then we shall show that $\widetilde{\omega} + \varepsilon dd^c \widetilde{\varphi}_k$ is q-positive for a sufficiently small $\varepsilon > 0$.

Now $\widetilde{\omega}$ is q-positive at $x \notin B_q$. Hence when x is not contained in B_q, the form $\widetilde{\omega} + \varepsilon dd^c \widetilde{\varphi}_k$ is q-positive for a small $\varepsilon > 0$. When x is contained in B_q, there is a positive integer ℓ such that $x \in D_\ell \setminus D_{\ell-1}$. (Otherwise x is contained in D_0. Then the Levi-form of $\widetilde{\varphi}_k$ has n positive eigenvalues at x.) For an irreducible component W of $D_\ell \setminus D_{\ell-1}$ containing x, the $(1,1)$-form $\widetilde{\omega}|_W$ has $(\dim W - q)$ positive eigenvalues along W. On the other hand, the

Levi-form of $\widetilde{\varphi}_k$ has $(n - \dim W)$ positive eigenvalues in the normal direction of W (property $(*)$ in Proposition 2.3.9). Thus $\widetilde{\omega} + \varepsilon dd^c \widetilde{\varphi}_k$ has $(n - q)$ positive eigenvalues. The function $\widetilde{\varphi}_k$ may not be smooth. However we can approximate it with smooth functions without the loss of the q-positivity since it is C^2-function (for instance, see [Ric68]). It completes the proof of Theorem 2.3.1.

Proof of Proposition 2.3.9. We prove Proposition 2.3.9 by induction on ℓ. When ℓ is zero, the claim is obvious. Thus we assume ℓ is greater than 0. By the induction hypothesis, we obtain a smooth function $\widetilde{\varphi}_{\ell-1}$ with property $(*)$. Further we take φ_ℓ with the properties in Proposition 2.3.7. We define a function $\widetilde{\varphi}_\ell$ to be $\varphi_\ell + \varepsilon \widetilde{\varphi}_{\ell-1}$. Then the function satisfies property $(*)$ for a sufficiently small $\varepsilon > 0$.

□

Finally we see that it follows Theorem 2.1.4 from Theorem 2.3.1.

Theorem 2.3.10. (=Theorem 2.1.4). *Let L be a semi-ample line bundle on a compact complex manifold X. Then the following conditions (A), (B) and (C) are equivalent.*

(A) *L is q-positive.*

(B) *The semi-ample fibration of L has fibre dimensions at most q.*

(C) *L is cohomologically q-ample.*

Further if X is projective, the conditions above are equivalent to condition (D).

(D) *For every subvariety Z with $\dim Z > q$, there exists a curve C on Z such that the degree of L on C is positive.*

Proof. We can easily confirm the equivalence between condition (B) and (C) from the standard argument of the spectral sequence (see [Som78-B, Proposition 1.7]).

The equivalence between condition (A) and (B) is directly followed from Theorem 2.3.1. Indeed, we apply Theorem 2.3.1 to the semi-ample fibration $\Phi := \Phi_{|L^m|} : X \longrightarrow \mathbb{P}^{N_m}$ and $\omega := \omega_{FS}$, where ω_{FS} is the Fubini-Study form. Then $\Phi^*\omega_{FS} + dd^c\varphi$ is q-positive for some function φ by Theorem 2.3.1. The form represents the Chern class of L^m. Therefore condition (B) implies (A). Conversely, if L is q-positive, it is cohomologically q-ample by the Andreotti-Grauert theorem. Therefore the fibre dimension of the semi-ample fibration must be at most q.

It remains to show the equivalence between condition (B) and (D). In this step, we use the assumption that X is projective. Assume that the fibre dimension of the semi-ample fibration is at most q. Then for any subvariety Z with $\dim Z > q$, we can take a curve on Z which is not contracted by Φ. Then the degree of L on the curve is positive by the projection formula. Conversely, if there exists a fibre F with $\dim F > q$, the degree on every curve in F is zero. □

2.4 Zariski-Fujita type theorems of big line bundles

In this section, we prove Theorem 2.4.1. Theorem 2.4.1 says that, a big line bundle is q-positive if and only if the restriction to the non-ample locus of the line bundle is q-positive. See [ELMNP06] or [Bou04, Section 3.5] for the definition and properties of the non-ample locus. (Sometimes, the non-ample locus is called the augmented base locus or the non-Kähler locus.)

Theorem 2.4.1. *Let L be a big line bundle on a smooth projective variety X. Assume that the restriction of L to the non-ample locus $\mathbb{B}_+(L)$ is q-positive. Then L is q-positive on X.*

Recall that 0-positive is positive in the usual sense (that is, ample). Hence Theorem 2.4.1 claims that L is ample on X if the restriction to the non-ample locus of L is ample. It can be seen as the parallel of the the Zariski-Fujita theorem (see [Zar89] and [Fuj83] for the Zariski-Fujita theorem).

Throughout this section, we denote by X a compact Kähler manifold and by L a line bundle on X. Moreover fix a smooth hermitian metric h of L. The Chern curvature $\sqrt{-1}\Theta_h(L)$ associated to h represents the first Chern class $c_1(L)$.

In this section, we treat a closed analytic set which may have the singularities on X. For this purpose, we extend the concept of q-positivity concept from a manifold to an analytic space. Note the following definition does not depend on the choice of a hermitian metric h of L.

Definition 2.4.2. Let V be a closed analytic set on X. The restriction $L|_V$ to V of L is *q-positive* if there exists a real-valued continuous function φ on V with the following condition:

For every point p on V, there exist a neighborhood U of p on X and a C^2-function $\widetilde{\varphi}$ on U such that $\widetilde{\varphi}|_{V\cap U} = \varphi$ and the $(1,1)$-form $\sqrt{-1}\Theta_h(L) + dd^c\widetilde{\varphi}$ has at least $(n-q)$-positive eigenvalues on U.

For the proof of Theorem 2.4.1, we prepare the following lemma.

Lemma 2.4.3. *Let V be a closed analytic set (possibly not irreducible) on X. If the restriction $L|_V$ to V of L is q-positive, then X allows a function $\varphi_V \in C^\infty(X, \mathbb{R})$ on X such that $\sqrt{-1}\Theta_h(L) + dd^c\varphi_V$ has at least $(n-q)$ positive eigenvalues on some neighborhood of V.*

Proof. We take a smooth extension $\widetilde{\varphi}$ to X of the potential function in Definition 2.4.2. Let $V = \bigcup_{i\in I} V_i$ be the irreducible decomposition. From the construction of $\widetilde{\varphi}$, the restriction to $(V_i)_{\mathrm{reg}}$ of $\sqrt{-1}\Theta_h(L) + dd^c\widetilde{\varphi}$ has at least $(\dim V_i - q)$ positive eigenvalues. Then we can revise the positivity in the normal direction of V with the same argument as Proposition 2.3.9. It leads to Lemma 2.4.3. $\square$

Proof of Theorem 2.4.1.
By the property of the non-ample locus (see [Bou04, Theorem 3.17]), there exists a d-closed current T on X with following properties:

(1) T represents the first Chern class of L.

(2) T has analytic singularities along the non-ample locus of L.

(3) For some hermitian form ω on X, the inequality $T \geq \omega$ holds as a $(1,1)$-current.

From property (1), we obtain an L^1-function φ_s on X with $T = \sqrt{-1}\Theta_h(L) + dd^c\varphi_s$. On the other hand, by applying Lemma 2.4.3 to the non-ample locus, we can obtain a function $\varphi_{\mathbb{B}_+} \in C^\infty(X, \mathbb{R})$ such that $\sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$ is q-positive on some neighborhood U on the non-ample locus.

Then we shall see that φ_s and $\varphi_{\mathbb{B}}$ can be glued. For a real number $C > 0$, we define the function ψ_C to be $\psi_C := \max\{\varphi_{\mathbb{B}_+} - C, \varphi_s\}$. For a large $C > 0$, $\varphi_{\mathbb{B}} - C$ is smaller than φ_s outside some neighborhood U_C of the non-ample locus. By taking a sufficiently large $C > 0$, we may assume that U_C is relatively compact in U.

On the other hand, the function φ_s has a pole along the non-ample locus. That is, $\varphi_s(x) = -\infty$ for any point x on the non-ample locus. Hence there exists a neighborhood V_C of the non-ample locus such that φ_s is smaller than $\varphi_{\mathbb{B}_+} - C$ even if C is large. We may assume V_C is relatively compact in U_C.

Outside U_C, the $(1,1)$-form

$$\sqrt{-1}\Theta_h(L) + dd^c\psi_C = \sqrt{-1}\Theta_h(L) + dd^c\varphi_s$$

has n positive eigenvalues. On the other hand, inside V_C the $(1,1)$-form

$$\sqrt{-1}\Theta_h(L) + dd^c\psi_C = \sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$$

has $(n-q)$-positive eigenvalues. In order to investigate the positivity on $\overline{U_C} \setminus V_C$, we apply Lemma 2.4.4.

Lemma 2.4.4. *Let γ be a smooth d-closed $(1,1)$-form on X and a function φ_i (for $i = 1, 2$) be a γ-psh function on X. Then the function $\max(\varphi_1, \varphi_2)$ is also a γ-psh function on X. (See Section 6.1 for the definition of a γ-psh function.)*

Proof. First we remark γ-plurisubharmonicity is a local property. We can locally take a smooth potential function of γ since γ is a d-closed $(1,1)$-form. Thus we can locally write $\gamma = dd^c\psi$ for some function ψ. By the assumption, $dd^c(\psi + \varphi_i)$ is a positive current. Therefore the Levi form of

$$\max(\psi + \varphi_1, \psi + \varphi_2) = \psi + \max(\varphi_1, \varphi_2)$$

is also a positive current. It means that $\gamma + dd^c \max(\varphi_1, \varphi_2) \geq 0$. Upper semi-continuity of functions is preserved. Hence the function $\max(\varphi_1, \varphi_2)$ is a γ-psh function. $\square$

Since U_C is relatively compact in U, $\sqrt{-1}\Theta_h(L) + dd^c\varphi_{\mathbb{B}_+}$ is q-positive on $\overline{U_C}$. Certainly $\sqrt{-1}\Theta_h(L) + dd^c\varphi_s$ is q-positive (0-positive) on $\overline{U_C} \setminus V_C$. Therefore it follows from the lemma above that, $\sqrt{-1}\Theta_h(L) + dd^c\psi_C$ is q-positive on $\overline{U_C} \setminus V_C$. The function ψ_C may not be smooth. However we can approximate it with smooth functions without the loss of the q-positivity since ψ_C is continuous. Therefore L is q-positive on X. $\square$

When the dimension of the non-ample locus is smaller q, the assumption in Theorem 2.4.1 is automatically satisfied. Thus the following corollary holds.

Corollary 2.4.5. *Assume the dimension of the non-ample locus of L is less than or equal to q. Then L is q-positive.*

Under the assumption in Corollary 2.4.5, L is cohomologically q-ample (see [Kür10]). Corollary 2.4.5 claims that the q-positivity has the same property.

2.5 On the holomorphic Morse inequalities

In his paper [Dem10-B], Demailly proved the converse of the holomorphic Morse inequality on a surface. This result has the similarity to the converse of the Andreotti-Grauert theorem. In this section, we explain the difference between his result and the result (Theorem 2.1.3) in this chapter. First we recall the holomorphic Morse inequality which is closely related with the Andreotti-Grauert vanishing theorem.

Definition 2.5.1. Let L be a line bundle on a compact complex manifold X. Then the asymptotic q-cohomology of L is defined to be

$$\hat{h}^i(L) := \limsup_{m\to\infty} \frac{n!}{m^n} h^i(X, \mathcal{O}_X(L^m))$$

In his paper [Dem85], Demailly gave a relation between the dimension of the asymptotic cohomology of a line bundle and certain Monge-Ampère integrals of the curvature. It is so-called Demailly's holomorphic Morse inequality. For simplicity, we assume that X is projective.

Theorem 2.5.2. ([Dem85]). *For every holomorphic line bundle L on a projective manifold X, we have the (weak) Morse inequality*

$$\hat{h}^i(L) \leq \inf_{h:\text{hermitian metric on } L} \int_{X(h,i).} (\sqrt{-1}\Theta_h(L))^n(-1)^i,$$

where h runs through smooth hermitian metrics on L, and $X(h,i)$ is the set defined by

$$X(h,i) := \left\{x \in X \mid \sqrt{-1}\Theta_h(L) \text{ has a signature } (n-i,i) \text{ at } x.\right\}.$$

The holomorphic Morse inequality would be seen as an asymptotic version of the Andreotti-Grauert vanishing theorem. In his paper [Dem10-A], Demailly conjectured that the inequality would actually be an equality. The conjecture has the similarity to Problem 2.1.2. In [Dem10-B], he showed the converse of the holomorphic Morse inequality holds in the following case:

(1) The case where X is projective surface.

(2) The case where X is an arbitrary projective manifold and $i = 0$.

Result (2) can be seen as a "partial" converse of the Andreotti-Grauert theorem. However, Result (2) seems not to lead to Theorem 2.1.3.

2.6 Examples

Thanks to Theorem 2.1.3, a 1-ample line bundle is always 1-positive on a smooth projective surface. However, in general, it is difficult to construct a concrete metric whose curvature is 1-positive. Thus it seems to be worth collecting examples which can be explicitly computed. This subsection is devoted to give such examples. For simplicity, we use additive notation for line bundles in this section.

Example 2.6.1. Let X be the product of two 1-dimensional projective spaces. Denote by $p_i : X \to \mathbb{P}^1$, the i-th projection ($i = 1, 2$). Then a line bundle L on X can be written as

$$L_{(a,b)} = a\ p_1^*\mathcal{O}_{\mathbb{P}^1}(1) + b\ p_2^*\mathcal{O}_{\mathbb{P}^1}(1)$$

with integers a, b. Here $\mathcal{O}_{\mathbb{P}^1}(1)$ is the hyperplane bundle on $\mathbb{P}^1$. From a simple computation (or Corollary 2.2.5), L (which parametrized by integers a, b) is 1-ample if and only if $a > 0$ or $b > 0$. Then a metric on L which is induced by the pullback of suitable multiple of the Fubini-Study metric has a 1-positive curvature.

Example 2.6.2. Let E be an elliptic curve. We set $X := E \times E$ with projections $p_i : X \to E$ ($i = 1, 2$). We consider line bundles

$$F_1 := p_1^*(\mathcal{O}_E(p)), \qquad F_2 := p_2^*(\mathcal{O}_E(p)), \qquad \Gamma := \mathcal{O}_X(\Delta),$$

where p is a point on E and $\Delta \subset X = E \times E$ is the diagonal divisor. It is known that an arbitrary line bundle on X can be written as a linear combination of F_1, F_2 and Γ. Thanks to Proposition 2.2.5, L is 1-ample if and only if $-L$ is not pseudo-effective. Since the automorphism group of X (which is a connected algebraic group) acts transitively on X, the pseudo-effective cone corresponds with the nef cones. Thus, the line bundle L is 1-ample if and only if $(L^2) < 0$ or $(L \cdot A) > 0$, where A is an ample line bundle such as $A := F_1 + F_2 + \Gamma$. The intersection numbers among F_1, F_2 and Γ can be computed as follows:

$$\begin{aligned}(\Gamma \cdot F_1) = (\Gamma \cdot F_2) = (F_1 \cdot F_2) = 1,\\ (\Gamma^2) = (F_1^2) = (F_2^2) = 0.\end{aligned}$$

By the argument above, we have the following proposition.

Proposition 2.6.3. *A line bundle $L = aF_1 + bF_2 + c\Gamma$ is 1-ample if and only if*

$$\begin{gathered} a + b + c > 0 \quad \textit{or} \\ ab + bc + ca < 0. \end{gathered}$$

Now we construct a metric on L whose curvature is 1-positive under the condition above on a, b and c. Denote by h, a hermitian metric on $\mathcal{O}_E(p)$ such that the pull-back of the Chern curvature by the universal covering $\mathbb{C} \to E$ is equal to $du \wedge d\bar{u}$. Here u is a (standard) coordinate on $\mathbb{C}$. On the other hand, we can construct a metric k on Γ whose Chern curvature can be written as:

$$\begin{aligned}\Theta_k(\Gamma) &= dd^c|z-w|^2 \\ &= dz \wedge d\bar{z} + dw \wedge d\bar{w} - dz \wedge d\bar{w} - dw \wedge d\bar{z}.\end{aligned}$$

Here (z, w) is a local coordinate on X which is induced by the universal covering $\mathbb{C}^2 \to X$. Then the Chern curvature of

$$L = aF_1 + bF_2 + c\Gamma$$

associated to a metric $p_1^*(h^{\otimes a}) \otimes p_2^*(h^{\otimes b}) \otimes k^{\otimes c}$ is

$$(a+c)dz \wedge d\bar{z} + (b+c)dw \wedge d\bar{w} - cdz \wedge d\bar{w} - cdw \wedge d\bar{z}.$$

Eigenvalues of the curvature are solutions of the equation

$$\det \begin{pmatrix} (a+c)-x & -c \\ -c & (b+c)-x \end{pmatrix} = 0.$$

The a necessary and sufficient condition that the equation has at least 1-positive solution is

$$\begin{aligned} a+b+2c > 0 \ \text{ or} \\ ab+bc+ca < 0. \end{aligned}$$

It is easy to see that this condition is equivalent to the condition in Proposition 2.6.3.

2.7 Counterexamples to Problem 2.1.3

In this section, we study Ottem's counterexample to Problem 2.1.3. Further we investigate the converse implication of Andreotti-Grauert vanishing theorem on a non-compact manifold.

By the (classical) Andreotti-Grauert vanishing theorem, a q-complete complex space is always cohomologically q-complete. Let us confirm the definitions. Let M be a non-compact, irreducible and reduced analytic space of dimension n and q an integer with $0 \leq q \leq (n-1)$.

Definition 2.7.1. (1) M is called *q-complete*, if there exists a (smooth) exhaustive function $\varphi \in C^\infty(M, \mathbb{R})$ whose Levi-form $\sqrt{-1}\partial\overline{\partial}\varphi$ has at least $(n-q)$ positive eigenvalues at any point on M as a $(1,1)$-form.

(2) M is called *cohomologically q-complete*, if for any coherent sheaf $\mathcal{F}$ on M,

$$H^i(M, \mathcal{F}) = 0 \quad \text{for } i > q.$$

It is natural to ask whether the converse implication holds. It is a non-compact version of Problem 2.1.3.

Problem 2.7.2. *If M is cohomologically q-complete, is M q-complete ?*

In their paper [ES80], Eastwood and Suria proved that the problem above is affirmatively solved, if M is a domain with a smooth boundary in a Stein manifold. Another proof is given for a domain with a smooth boundary in $\mathbb{C}^n$ in [Wat94].

It is well-known that any non-compact complex space of dimension n is cohomologically $(n-1)$-compete. If Problem 2.7.2 is affirmative, any non-compact complex space of dimension n should be $(n-1)$-compete. In the case where complex space is non-singular, Greene and Wu proved $(n-1)$-completeness of non-compact analytic space in [GW75]. In the case where complex space has singularities, that is proved by Ohsawa (see [Ohs84]).

In this section, we show that the observation for Ottem's example gives a counterexample to Problem 2.1.3. See [Ott11, Section 10] for the example. The essential devotion of the counterexample is due to Ottem.

Proposition 2.7.3. *For a pair (n, q) of positive integers with $n/2 - 1 < q < n - 2$, there exists a complex manifold M of dimension n such that M is cohomologically q-complete, but not q-complete. In particular, Problem 2.7.2 is negative in general.*

Proof. We give the proof only in the case where $(n, q) = (4, 1)$. (A slight change in the proof gives the proof of other cases.)

We consider a smooth Enriques surface S in the projective space $\mathbb{P}^4$. Then we shall show that the complement $\mathbb{P}^4 \setminus S$ is cohomologically 1-complete, but not 1-complete. We denote by M, the complement $\mathbb{P}^4 \setminus S$. Since S is an Enriques surface, the fundamental group $\pi_1(S)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore, we have $H^1(S, \mathbb{Q}) = 0$, (which is isomorphic to $H^1(\mathbb{P}^4, \mathbb{Q})$). Thus we can conclude that M is cohomologically 1-complete from [Ogu73, Theorem 4.4].

It remains to show that M is not 1-complete. We assume that M is 1-complete for a contradiction. By the definition, there exists an exhaustive function $\varphi \in C^\infty(M, \mathbb{R})$ such that $\sqrt{-1}\partial\bar{\partial}\varphi$ has at least 3 positive eigenvalues at any point on M. We can assume that $\varphi \geq 0$ since φ is exhaustive. Now we consider a function f on $\mathbb{P}^4$ which is defined to be

$$f := \begin{cases} 1/\varphi & \text{if } x \notin S, \\ 0 & \text{others.} \end{cases}$$

A simple computation implies that the critical points of f on M are equal to that of φ. Therefore we have

$$\sqrt{-1}\partial\bar{\partial}f = \frac{-\sqrt{-1}\partial\bar{\partial}\varphi}{\varphi^2}$$

at the critical points of φ on M. It implies that the index (the number of negative eigenvalues of the Hessian) at the critical points is greater than or equal to 3. Note that the index of the Hessian of a smooth function is equal to the number of the negative eigenvalues of the Levi-form. Therefore by applying the standard Morse theory, for an arbitrary number $\delta > 0$ we have that, X is obtained from W_δ by successively attaching cells of dimension ≥ 3. Here W_δ is $f^{-1}([0, \delta])$. In particular, we have the isomorphism $\pi_i(W_\delta, S) \cong \pi_i(\mathbb{P}^4, S)$ for $i = 0, 1, 2$ for any $\delta > 0$. Since we triangulate $\mathbb{P}^4$ with S as a subcomplex, we can take a neighborhood U of S which deformation retracts onto S. Since φ is an exhaustive function, f is continuous. Thus, W_δ is contained in U for a sufficiently small $\delta > 0$, since f has a positive valued on M. Then we have the following commutative diagram

$$\begin{tikzcd} & \pi_i(\mathbb{P}^4, S) & \\ \pi_i(W_\delta, S) \arrow[ur, "\cong"] \arrow[rr] & & \pi_i(U, S). \arrow[ul] \end{tikzcd}$$

The diagonal map on the left is an isomorphism for $i = 0, 1, 2$. Since U can retracts onto S, we have $\pi_i(U, S) = 0$ for any i. Therefore we obtain $\pi_i(\mathbb{P}^4, S) = 0$ for $i = 0, 1, 2$.

By the argument above,we have that, if M is 1-complete, then the map $\pi_1(S) \to \pi_1(\mathbb{P}^4)$ (which is induced by the inclusion map) should be an isomorphism. However, since $\mathbb{P}^4$ is simply connected, it is a contradiction. $\square$

3
Weak Lefschetz theorems and the topology of zero loci of ample vector bundles

3.1 Introduction

Topology and the concept of positivity in algebraic geometry have been nourishing each other for a long time. For example, there are the Lefschetz hyperplane theorem, Connected theorem of Fulton and Hansen, and Barth-Larsen theorem and so on. In particular, the Lefschetz hyperplane theorem has been developed for various purposes. The theorem asserts that the homotopy groups of a smooth projective variety X can be compared with those of the zero locus of a section of an ample line bundle on X. In this chapter, we investigate the homotopy groups of the zero locus of a section of an ample *vector bundle* on X, in the sprit of the Lefschetz hyperplane theorem. In this direction, Sommese gave the following celebrated result:

Theorem 3.1.1. ([Som78-B, Proposition 1.16], cf. [Laz, Theorem 7.1.1]). *Let E be an ample vector bundle of rank r on a smooth projective variety X of dimension n. Consider a holomorphic section $s \in H^0(X, \mathcal{O}_X(E))$ and the zero locus defined to be*

$$S := s^{-1}(0) := \{x \in X \mid s(x) = 0 \in E_x.\}.$$

Then

$$H_i(X, S : \mathbb{Z}) = 0 \qquad \text{for } i \leq n - r.$$

It is natural and of interest to ask whether the relative homotopy groups $\pi_i(X, S)$ vanish in the same setting of this theorem. In this chapter, we mainly study the following problem.

Problem 3.1.2. *In the same assumption in Theorem 3.1.1, does the relative homotopy group $\pi_i(X, S)$ vanish for $i \leq n - r$?*

In his paper [Oko87], Okonek proved that Problem 3.1.2 is affirmative if E is ample, globally generated and S has the expected codimension r. Further, Lazarsfeld affirmatively solved Problem 3.1.2 without assuming $\mathrm{codim}_X S = r$, when $E \otimes B^{-1}$ is globally generated where B is an ample and globally generated line bundle. (See [Laz83, Theorem 3.5].) The following theorem claims that Problem 3.1.2 is affirmatively solved under the weaker assumption than them.

Theorem 3.1.3. *Let E be a vector bundle of rank r on a smooth projective variety X of dimension n. Consider a section $s \in H^0(X, \mathcal{O}_X(E))$ and the zero locus S of s. Assume that the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r + k - 1)$-positive on $\mathbb{P}(E^*)$.*
Then

$$\pi_i(X, S) = 0 \qquad \text{for } i \leq n - r - k.$$

In particular, the map (which is induced by the inclusion map $j : S \hookrightarrow X$) $j_* : \pi_i(S) \longrightarrow \pi_i(X)$ is isomorphic for $i < n - r - k$, and surjective for $i = n - r - k$.

Here $\mathbb{P}(E^*)$ is the projective space bundle associated to the dual vector bundle E^* of E, and $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is the tautological line bundle on $\mathbb{P}(E^*)$.

If E is ample and globally generated, $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r-1)$-positive, thanks to [Som78-A, Proposition 1.3] (cf. [DPS96, Proposition 2.8]). Remark that the theorem above holds without assuming $\mathrm{codim}_X S = r$. Therefore this theorem can be seen as the generalization of [Oko87] and [Laz83, Theorem 3.5].

Problem 3.1.2 is closely related with the Griffiths conjecture for a vector bundle. The Griffiths conjecture says that any ample vector bundle would be Griffiths positive. (A Griffiths positive vector bundle is always ample.) From a simple computation, we can easily see that $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r - 1)$-positive when E is Griffiths positive. (For instance, see [DPS96, Proposition 2.8].) Thus if the Griffiths conjecture is affirmative, Problem 3.1.2 should be affirmative under only the assumption that E is ample.

Problem 3.1.2 is also related with the converse implication of the Andreotti-Grauert vanishing theorem. The Andreotti-Grauert theorem asserts that any q-positive line bundle is always q-ample. (See [AG62, Théorèm 14], [DPS96, Proposition 2.1].) In [DPS96], Demailly, Peternell and Schneider asked whether the converse implication of the Andreotti-Grauert theorem holds. It is a natural question. However, it has been an open problem for a long time except the case of $q = 0$.

E is ample if and only if $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r-1)$-ample on $\mathbb{P}(E^*)$. It follows from the standard argument on the spectral sequences of the direct image sheaves. For instance, see [DPS96, Example 2.7] or [Ott11, Proposition 4.1]. On the other hand, when E is Griffiths positive, $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r-1)$-positive. Theorem 3.1.3 says that, if the converse of the Andreotti-Grauert theorem for $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ holds, Problem 3.1.2 is affirmative.

The converse of the Andreotti-Grauert theorem for a semi-ample line bundle holds on any compact manifold. Moreover, the converse holds on a smooth projective surface without any assumption on a line bundle. (See Chapter2.) However Ottem recently constructed a counterexample to the converse of the Andreotti-Grauert theorem, by investigating the Lefschetz hyperplane type theorem. (See [Ott11] for the precise argument.) If there similarly exists a counterexample to Problem 3.1.2, then the example would be a counterexample to the Griffiths conjecture. In this sense, Problem 3.1.2 is interesting.

$$\begin{tikzcd}
E : \text{ample} \arrow[r, leftrightarrow, "\text{equivalent}"] \arrow[d, shift left, "\text{open}"] & \mathcal{O}_{\mathbb{P}(E^*)}(-1) : (r-1)\text{-ample} \arrow[d, shift left, "\text{open}"] \\
E : \text{Griffiths positive} \arrow[r, "\text{hold}"'] \arrow[u, shift left, "\text{hold}"] & \mathcal{O}_{\mathbb{P}(E^*)}(-1) : (r-1)\text{-positive.} \arrow[u, shift left, "\text{Andreotti-Grauert}"]
\end{tikzcd}$$

Problem 3.1.2 seems to be a hard problem. However we can give a partial result (Corollary 3.1.4) on Problem 3.1.2, in terms of the weak Lefschetz theorem.

Corollary 3.1.4. *Let E be an ample vector bundle of rank r on a smooth projective variety X of dimension n, and R a nowhere dense analytic set on X. We consider a section $s \in H^0(X, \mathcal{O}_X(E))$ and the zero locus S of s. Assume S has the expected codimension r (that is, $\mathrm{codim}_X(S) = r$) and $r < n$. Then the map $j_* : \pi_1(S \backslash R) \longrightarrow \pi_1(X \backslash R)$ is surjective. In particular, when $R = \phi$, the map $j_* : \pi_1(S) \longrightarrow \pi_1(X)$ is surjective.*

Corollary 3.1.4 follows from Theorem 3.3.6. The proof of Theorem 3.3.6 is based on the method of [NR98] and its variation. We mainly use the L^2 $\overline{\partial}$-method and the theory of a formal scheme. In [NR98], Napier and Ramachandran constructed holomorphic sections of an ample line bundle which separate sheets of an étale covering, by using the L^2 $\overline{\partial}$-method. We slightly generalize its construction. The generalization and the computation of cohomology groups of a formal scheme lead to Theorem 3.3.6.

This chapter is organized in the following way: In Section 3.2, we prove Theorem 3.1.3. Section 3.3 is devoted to give the proof of Theorem 3.3.6 and Corollary 3.1.4. In Section 3.4, we give Theorem 3.4.1 as an application of the slight variation of the Napier-Ramachandran's method established in Section 3.3. In Section 3.5, we collect the known facts on the topology in complex geometry for readers' convenient.

3.2 The Lefschetz type theorems and the Morse theory

3.2.1 Preliminaries

In the proof of Theorem 3.1.3, we shall use the standard Morse theory. In this subsection, we collect materials on the Morse theory for the proof of the theorem.

Let Φ be a real-valued smooth function on a compact C^∞-manifold M of (real) dimension k. A point $m \in M$ is called a *critical point* if $d\Phi = 0$ at m. In terms of a local coordinate $(x_1, \ldots, x_k)$, the (real) Hessian of Φ at $m \in M$ is defined to be

$$\left(\frac{\partial^2}{\partial x_i \partial x_j}\Phi(m)\right)_{1\leq i,\ j\leq k},$$

which is a symmetric quadric form on the tangent space T_mM at m. When m is a critical point of Φ, the Hessian does not depend on the choice of a local coordinate. The number of negative eigenvalues of the Hessian at a critical point is called the *index* at the critical point. A function on M is called *non-degenerate*, if the function has only non-degenerate manifold. See [Bot59-A, Definition 3.1] for the definition of a non-degenerate manifold. (In this chapter, we do not use the precise definition.) The following theorem says that a non-degenerate function allows us to reconstruct the manifold as a CW complex.

Theorem 3.2.1. ([Bot59-B, Theorem 3]). *Let Φ be a real-valued smooth function on a compact C^∞-manifold M. Assume that Φ is non-degenerate. Let λ be the infimum of the indices of Φ at the critical points. Then for an arbitrary real number a, M is obtained from W_a by successively attaching cells of dimension $\geq \lambda$. In particular, we have*

$$\pi_i(X, W_a) = 0 \qquad \text{for } i \leq \lambda - 1.$$

Here W_a is $\{m \in M \mid \Phi(m) < a\}$.

In the proof of Theorem 3.1.3, we construct a non-degenerate function from a suitable perturbation. For this purpose, we apply the following approximation theorem. We take an open covering of M by open balls U_j ($j =, 1, 2, \ldots, N$) with a coordinate $(x_1^j, \ldots, x_k^j)$. Let V_j be a relatively compact set in U_j such that $\{V_j\}_{j=1}^N$ is also an open covering of X. Then the C^2-norm $|\cdot|_{C^2}$ with respect to the open covering is defined to be

$$|f|_{C^2} := \sum_{j=1}^{N} \sum_{\alpha,\beta=1}^{k,k} \sup_{p\in\overline{V_j}} \left| \frac{\partial^2 f}{\partial x_\alpha^j \partial x_\beta^j}(p) \right| + \sum_{j=1}^{N} \sum_{\alpha=1}^{k} \sup_{p\in\overline{V_j}} \left| \frac{\partial f}{\partial x_\alpha^j}(p) \right| + \sup_{p\in X} |f(p)|$$

for every function $f \in C^2(X, \mathbb{R})$. A C^2-function is called $(\varepsilon, 2)$-*small* when the C^2-norm of the function is smaller than ε.

Theorem 3.2.2. ([Whi55]). *Let Φ be a real-valued smooth function on a compact C^∞-manifold M. Then for an arbitrary positive number $\varepsilon > 0$, there exists an $(\varepsilon, 2)$-small function η_ε such that $\Phi + \eta_\varepsilon$ is non-degenerate on M.*

3.2.2 Proof of Theorem 3.1.3

This subsection is devoted to give the proof of Theorem 3.1.3. For the proof, we shall construct a non-degenerate function with a suitable index, from a hermitian metric whose Chern curvature satisfies a partial positivity condition. Then we apply the standard Morse theory.

Let us begin to prove Theorem 3.1.3. Since $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r + k - 1)$-positive on $\mathbb{P}(E^*)$, there exists a smooth hermitian metric h such that the Chern curvature associated to h has at least $(n - k)$ positive eigenvalues at any point on $\mathbb{P}(E^*)$. The metric h can be regarded as the function F_h on the total space of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ defined by

$$F_h : \mathcal{O}_{\mathbb{P}(E^*)}(-1) \longrightarrow \mathbb{R}, \quad t \longmapsto |t|_h^2.$$

Here $|t|_h$ is the norm of $t \in \mathcal{O}_{\mathbb{P}(E^*)}(-1)$ with respect to h. Since the total space of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is equal to the blow-up of the total space of E along the image of the zero section, we have the isomorphism

$$\pi : \mathcal{O}_{\mathbb{P}(E^*)}(-1) \setminus \mathbb{P}(E^*) \xrightarrow{\cong} E \setminus X. \tag{3.1}$$

By this isomorphism, the functions on $\mathcal{O}_{\mathbb{P}(E^*)}(-1)\setminus\mathbb{P}(E^*)$ can be identified with the functions on $E\setminus X$. In particular, F_h can be regarded as the function on $E \setminus X$. A given section s of E can be seen as a holomorphic map from

X to the total space of E. We denote by Φ_h, the pull-back of F_h by the holomorphic map $s : X \setminus S \to E \setminus X$. Then Φ_h can be written as

$$\Phi_h : X \setminus S \to \mathbb{R}, \qquad x \longmapsto |\pi^{-1}(s(x))|_h^2.$$

The function Φ_h satisfies the following properties:

Claim 3.2.3. *The function $\Phi_h : X \setminus S \to \mathbb{R}$ satisfies the following properties:*
(1) *Φ_h is a positive-valued smooth function on $X \setminus S$.*
(2) *The Levi-form $\sqrt{-1}\partial\overline{\partial}\Phi_h$ has at least $(n-r-k+1)$ negative eigenvalues at a critical point of Φ_h.*

Proof. The first property is obvious by the definition of Φ_h. It is sufficient to (locally) confirm the second property. In the proof, we denote by $[s(x)] \in \mathbb{P}(E^*)$, the line through a point $s(x) \in E_x$. Recall that $\mathbb{P}(E^*) \to X$ has the space of the lines in E_x as the fibre of $x \in X$. Thus we have the natural projection $E \setminus X \to \mathbb{P}(E^*)$.

Fix an arbitrary point $p \in X \setminus S$. Let W be a small open neighborhood of p and $\overline{W}$ the image of $s(W)$ by $E \setminus X \to \mathbb{P}(E^*)$. Now we take a local frame t of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)|_{\overline{W}}$ adapted at $[s(p)]$ with respect to the restriction $h|_W$ of h. (For simplicity, we denote by the same notation h, the restriction $h|_W$.) That is, t is a non-vanishing (holomorphic) map from $\overline{W}$ to $\mathcal{O}_{\mathbb{P}(E^*)}(-1)|_{\overline{W}}$ such that

$$|t([s(x)])|_h^2 = 1 \ \text{ and } \ d(|t([s(x)])|_h^2) = 0 \quad \text{at } [s(p)].$$

Notice that W is isomorphic to $\overline{W}$. Therefore $\pi^{-1} \circ s$ can be seen as a non-vanishing section of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on $\overline{W}$. Since t is also non-vanishing on $\overline{W}$, there exists a non-vanishing holomorphic function g on W such that $\pi^{-1}(s(x)) = g([s(x)])t([s(x)])$ on W. Then by the definition of Φ_h, we have

$$\Phi_h(x) = |g([s(x)])|^2 |t([s(x)])|_h^2 \quad \text{on } W.$$

Let us compute the Levi-form of Φ_h. A simple computation yields

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\Phi_h =& \sqrt{-1}|t|_h^2 \partial g \wedge \overline{\partial g} - \sqrt{-1} g \overline{\partial g} \wedge \partial|t|_h^2 \\ &+ \sqrt{-1}\overline{g}\partial g \wedge \overline{\partial}|t|_h^2 + |g|^2 \sqrt{-1}\partial\overline{\partial}|t|_h^2.\end{aligned}$$

The second and third term are equal to zero at $[s(p)]$, since t is adapted at $[s(p)]$. Moreover it is easy to see that the first term is also equal to zero at

$[s(p)]$ if p is a critical point of Φ_h. In fact, we have

$$\begin{aligned} d\Phi_h(x) &= |t|_h^2 d|g|^2 + |g|^2 d|t|_h^2 \\ &= |t|_h^2 d|g|^2 \qquad\qquad \text{at } [s(p)]. \end{aligned}$$

The second equality follows since t is adapted at $[s(p)]$. Thus if p is a critical point of Φ_h, then $dg = 0$ at $[s(p)]$. From the argument above, we have

$$\sqrt{-1}\partial\bar{\partial}\Phi_h(x) = |g|^2 \sqrt{-1}\partial\bar{\partial}|t|_h^2$$

if p is a critical point of Φ_h.

For the proof of the claim, we need to compute the number of negative eigenvalues of the Levi form $\sqrt{-1}\partial\bar{\partial}|t|_h^2$. Since t is a frame adapted at $[s(p)]$, we have

$$\sqrt{-1}\partial\bar{\partial}\log|t([s(x)])|_h^2 = \sqrt{-1}\partial\bar{\partial}|t([s(x)])|_h^2 \qquad \text{at } [s(p)].$$

By the definition of the Chern curvature, $\sqrt{-1}\Theta_h = -\sqrt{-1}\partial\bar{\partial}\log|\tilde{t}|_h$ for any frame $\tilde{t}$ of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$. Therefore $-\sqrt{-1}\partial\bar{\partial}\log|\tilde{t}|_h^2$ has at least $(n-k)$ positive eigenvalues. Since $\overline{W}$ is a (local) submanifold of codimension $(r-1)$ on $\mathbb{P}(E^*)$, the restriction of $-\sqrt{-1}\partial\bar{\partial}\log|\tilde{t}|_h^2$ to $\overline{W}$ has at least $(n-r-k+1)$ positive eigenvalues. Therefore $\sqrt{-1}\partial\bar{\partial}|t([s(x)])|_h^2$ has at least $(n-r-k+1)$ negative eigenvalues at $[s(p)]$. It completes the proof of the claim. □

We shall construct a non-degenerate function with a suitable index by using this claim. First we choose sufficiently small neighborhoods $U,\ V \subset X$ of S such that V is relatively compact in U. Further we choose a smooth function ρ_V such that ρ_V is identically 0 on V and identically 1 on $X \backslash U$. Then the function $\rho_V\Phi_h$ is a smooth function on X. By applying the approximation theorem (Theorem 3.2.2) to $\rho_V\Phi_h$, for an arbitrary positive number ε, we obtain an $(\varepsilon, 2)$-small function η_ε such that $\rho_V\Phi_h + \eta_\varepsilon$ is non-degenerate on X.

Claim 3.2.4. *For a sufficiently small $\varepsilon > 0$, the Levi-form of $\rho_V\Phi_h + \eta_\varepsilon$ has at least $(n-r-k+1)$ negative eigenvalues at a critical point of $\rho_V\Phi_h + \eta_\varepsilon$ on $X \setminus U$.*

Proof. Remark that $\rho_V\Phi_h = \Phi_h$ on $X \setminus U$, since ρ_V is identically 1 on $X \setminus U$. Thus, it follows the Levi-form of $\rho_V\Phi_h + \eta_\varepsilon$ has at least $(n-r-k+1)$ negative eigenvalues at a critical point of Φ_h on $X \setminus U$ for a sufficiently small ε from Claim 3.2.3. Since this is an open condition, there exists an open neighborhood W of the critical points of Φ_h such that the Levi-form of $\rho_V \Phi_h + \eta_\varepsilon$ still has at least $(n - r - k + 1)$ negative eigenvalues on W. It is easy to see that the critical points of $\rho_V \Phi_h + \eta_\varepsilon$ is contained in W for a sufficiently small $\varepsilon > 0$. In fact, if p is not a critical point of Φ_h, then p is not also a critical point of $\rho_V \Phi_h + \eta_\varepsilon$ for a small ε. $\square$

Now we take a positive number δ with

$$0 < \delta < \inf\{\ \rho_V(x)\Phi_h(x) + \eta_\varepsilon(x) \mid x \in X \setminus V\}.$$

Then

$$W_\delta = \{x \in X \bigm| \rho_V(x)\Phi_h(x) + \eta_\varepsilon(x) < \delta\}$$

is contained in U. Since $\rho_V \Phi_h + \eta_\varepsilon$ is non-degenerate, we can apply the standard Morse theory (Theorem 3.2.1). (Note that the index of the Hessian of a smooth function is equal to the number of the negative eigenvalues of its Levi-form.) Then X is obtained from W_δ by successively attaching cells of dimension $\geq n - r - k + 1$. In particular, the natural map $\pi_i(W_\delta, S) \to \pi_i(X, S)$ is an isomorphism for $i \leq n - r - k$. On the other hand, since W_δ is contained in U, we have the following commutative diagram

$$\begin{tikzcd}
 & \pi_i(X,S) & \\
\pi_i(W_\delta,S) \arrow[ur, "\cong"] \arrow[rr] & & \pi_i(U,S). \arrow[ul]
\end{tikzcd}$$

The diagonal map on the left is an isomorphism for $i \leq n - r - k$. Now we can choose U which retracts onto S, since X can be triangulated with S as a subcomplex. Then $\pi_i(U, S) = 0$ for any i. Therefore we obtain $\pi_i(X, S) = 0$ for $i \leq n - r - k$.

3.3 The weak Lefschetz type theorems

3.3.1 On the key estimates

The methods of the proofs of Theorem 3.3.6, 3.4.1 are based on the technics established in [NR98]. (Corollary 3.1.4 follows from Theorem 3.3.6.) In their paper [NR98], Napier and Ramachandran showed that the image of the fundamental group of a submanifold with ample normal bundle is of finite index. The strategy of the proof may be divided into two steps. In first step, they consider a suitable covering space and construct L^2-sections of an ample line bundle, which separate sheets of the covering. In second step, they study the finite dimensionality of the space of sections on the formal scheme, which gives a bound on the index.

For our purpose, we need to construct L^2-sections of a big line bundle. The following proposition is a slight generalization of [NR98, Theorem 2.2]. The proof is based on a standard application of the $\overline{\partial}$-method with L^2-estimates (which is established in [Dem82]), and is parallel to the proof of [NR98, Theorem 2.2]. For the readers' convenience, here we give details of the proof.

Proposition 3.3.1. (cf. [NR98, Theorem 2.2]). *Let X be a projective variety of dimension n (not necessarily smooth), U an open set such that U does not intersect with the singular locus X_{sing} of X, and R a nowhere dense analytic subset on X such that $R \supset X_{\mathrm{sing}}$. We denote by M the complement $X \setminus R$. Let $\pi : \widetilde{M} \longrightarrow M$ be an étale covering of degree d $(1 \leq d \leq \infty)$. Fix a big line bundle L on X and a hermitian line bundle (F, k) on X. Then there exist positive numbers c_0 and ν_0 (depending on X, L and F) with the following properties :*

(1) *For any $\nu \geq \nu_0$,*

$$c_0 \nu^n d \leq \dim H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big).$$

(2) *For any $\nu \geq \nu_0$,*

$$c_0 \nu^n (d-1) \leq \dim H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big) - \dim H^0\big(X, \mathcal{O}_X(E_\nu \otimes \Omega^n_X)\big).$$

Here $K_{\widetilde{M}}$ is the canonical bundle on $\widetilde{M}$, E_ν is $L^\nu \otimes F$, and Ω^n_X is the dualizing sheaf of X. See [NR98, Lemma 1.2] for the definition of the dualizing sheaf. Moreover

$$H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big)$$

is the space of holomorphic n-forms valued in $\pi^* E_\nu$ with L^2-condition $(*)$ on $U \cap R$.

$(*)$ For a point $r \in U \cap R$, there exist a neighborhood V_r of r and trivializations of L and F on V_r such that

$$\int_{\overline{V_r}} \sqrt{-1}^{n^2} s \wedge \overline{s} < \infty$$

for any connected component $\overline{V_r}$ of $\pi^{-1}(V_r)$. Here we regard s as a holomorphic n-form on $\pi^{-1}(V_r)$ under the fixed trivialization of $\pi^* E_\nu$.

Remark 3.3.2. A hermitian line bundle (F, k) does not necessarily have semi-positive curvature. In [NR98, Theorem 2.2], (F, k) is assumed to have semi-positive curvature. However, we need to remove this assumption for our purpose.

Proof. Since L is a big line bundle, some positive multiple L^{m_0} of L can be written as the tensor product of an ample line bundle A and an effective line bundle F. Now we take a smooth metric h_A of A whose curvature is a Kähler form on the regular locus X_{reg} of X. Further we take a singular metric h_F of F obtained from a section of E. Then the curvature current associated to h_F is (semi-)positive as a $(1,1)$-current on X_{reg}. We denote by ω, the Kähler form $\frac{1}{m_0}\sqrt{-1}\Theta_{h_A}(A)$ on X_{reg}. Then the curvature current $\sqrt{-1}\Theta_h(L)$ associated to h satisfies $\sqrt{-1}\Theta_h(L) \geq \omega > 0$, since the curvature current of F is semi-positive. Here h is the metric on L which is defined by $(h_A \otimes h_E)^{1/m_0}$.

Now we fix a point p in M such that h is smooth on a neighborhood of p. Further we take a local coordinate $(z_1, \ldots, z_n)$ centered at p, and a function φ_p on X with the following properties:

(a) The support of φ_p is contained in a (small) neighborhood of p.

(b) $\varphi_p = \log(|z_1|^2 + \cdots + |z_n|^2)$ on a smaller neighborhood of p.

(c) φ_p is smooth except p.

Since $\sqrt{-1}\Theta_h(L)$ is strictly positive, there exists a sufficiently large integer a_0 with the following properties on X_{reg}:

$$\frac{a_0}{2}\omega + \sqrt{-1}\Theta_k(F) \geq \omega,$$
$$a_0\sqrt{-1}\Theta_h(L) + \sqrt{-1}\partial\overline{\partial}\varphi_p \geq \frac{a_0}{2}\omega.$$

The regular locus is not compact. However, we can take a_0 with the properties above since φ_p has a compact support. We fix local trivializations of L and F on a neighborhood of p. Since π is an étale covering, the coordinate $(z_1, \ldots, z_n)$ also becomes a local coordinate on a neighborhood of a point in the fibre $\pi^{-1}(p)$ of p. This coordinate gives the local trivialization of $K_{\widetilde{M}}$. Under these trivializations, a n-form s valued in $\pi^* E_\nu$ can be seen as a function on a neighborhood of $\pi^{-1}(p)$. Thus for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, the differential $(\partial^{|\alpha|}/\partial z^\alpha)s$ can be defined on a neighborhood of $\pi^{-1}(p)$.

Fix an integer ν_0 with $\nu_0 > n a_0$. Then we construct holomorphic n-forms valued in $\pi^* E_\nu$ for any $\nu \geq \nu_0$ by solving $\overline{\partial}$-equations with L^2-estimates. From now on, we denote by $|\alpha|$ the norm of α which is defined by $|\alpha| = \sum_{i=1}^n \alpha_i$.

Claim 3.3.3. *For $\nu \geq \nu_0$, a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \leq \nu/a_0 - n$ and a point q in $\pi^{-1}(p)$, there exists a holomorphic n-form $s_{\alpha,q}$ valued in $\pi^* E_\nu$ with the following properties:*

(1) *For any multi-index β with $|\beta| \leq \nu/a_0 - n$ and a point r in $\pi^{-1}(p)$,*

$$\frac{\partial^{|\beta|}}{\partial z^\beta} s_{\alpha,q}(r) = \begin{cases} 1 & \text{if } r = q \text{ and } \beta = \alpha, \\ 0 & \text{others.} \end{cases}$$

(2)

$$\int_{\widetilde{M}} \left| s_{\alpha,q} \right|^2_{\pi^*(h^\nu \otimes k), \pi^*(\omega)} \exp\left(- \frac{\nu}{a_0} \pi^* \varphi_p \right) \left(\pi^* \omega \right)^n < \infty.$$

Proof of the claim. First we take a smooth n-form u valued in $\pi^* E_\nu$ with the following properties:

(a) For a multi-index β and $r \in \pi^{-1}(p)$,

$$\frac{\partial^{|\beta|}}{\partial z^\beta} u_{\alpha,q}(r) = \begin{cases} 1 & \text{if } r = q \text{ and } \beta = \alpha, \\ 0 & \text{others.} \end{cases}$$

(b) The support of u is contained in a (small) neighborhood of q.

(c) u is holomorphic on a smaller neighborhood of q.

Now we consider a $\overline{\partial}$-equation $\overline{\partial} \eta = \overline{\partial} u$. In order to obtain a solution of the equation by applying [Dem82, THÉORÈM 4.1], we need to compute the curvature of $\pi^* E_\nu$ and confirm the L^2-boundedness of $\overline{\partial} u$ with respect to the suitable metric.

From the choice of a_0 and ν_0, we have

$$\begin{aligned} &\sqrt{-1} \Theta_{\pi^*(h^\nu \otimes k) \exp(-\frac{\nu}{a_0} \pi^* \varphi_p)} (\pi^* E_\nu) \\ &= \nu \sqrt{-1} \Theta_{\pi^* h^\nu} (\pi^* L) + \sqrt{-1} \Theta_{\pi^* k} (\pi^* F) + \frac{\nu}{a_0} \sqrt{-1} \partial \overline{\partial} \pi^* \varphi_p \\ &\geq \frac{\nu}{2} \pi^* \omega + \sqrt{-1} \Theta_{\pi^* k} (\pi^* F) \geq \pi^* \omega, \end{aligned}$$

for any $\nu \geq \nu_0$. Thus it is sufficient to see the L^2-boundedness of $\overline{\partial} u$. The support of $\overline{\partial} u$ does not contain the pole of $\pi^* \varphi_p$ since u is holomorphic on a neighborhood of q. Further $\pi^* h$ is smooth at q by the choice of p. Therefore we obtain

$$\int_{\widetilde{M}} |\overline{\partial} u|^2_{\pi^*(h^\nu \otimes k), \pi^*(\omega)} \exp\left(- \frac{\nu}{a_0} \pi^* \varphi_p \right) \left(\pi^* \omega \right)^n < \infty.$$

Since X is projective, $M = X \setminus R$ admits a complete Kähler metric (see [Dem82]). Therefore the étale covering $\widetilde{M}$ also admits a complete Kähler metric. (Remark that $\pi^*\omega$ is not complete.) Thus by applying [Dem82, THÉORÈM 4.1], we obtain a n-form η valued in π^*E_ν such that

$$\overline{\partial}\eta = \overline{\partial}u,$$
$$\int_{\widetilde{M}} |\eta|^2_{\pi^*(h^\nu\otimes k),\pi^*(\omega)} \exp\big(-\frac{\nu}{a_0}\pi^*\varphi_p\big)\big(\pi^*\omega\big)^n < \infty.$$

Then $s_{\alpha,q} = u - \eta$ is a holomorphic n-form valued in π^*E_ν with L^2-condition (2) in the claim. The L^2-condition and the choice of φ_p yield η vanishes at any point in $\pi^{-1}(p)$ to an order $(\nu/a_0) - n$. Thus $s_{\alpha,q}$ satisfies condition (1) in the claim by the choice of u. □

We shall see that $s_{\alpha,q}$ satisfies condition $(*)$ in Proposition 3.3.1. Note the function $\exp(-\varphi_p)$ is bounded below. Fix trivializations of L and F on a neighborhood V_r of $r \in U \cap R$. Then under these trivializations, h is also bounded below function on V_r, since the curvature associated to h is positive. Here we implicitly used the known fact that an almost plurisubharmonic is always upper semi-continuous. Thus Claim 3.3.3 yields

$$\int_{\pi^{-1}(V_r)} |s_{\alpha,q}|_{\pi^*\omega}(\pi^*\omega)^n < \infty.$$

Here we regarded $s_{\alpha,q}$ as a holomorphic n-form on $\pi^{-1}(V_r)$. Recall

$$|s_{\alpha,q}|_{\widetilde{\omega}} \frac{\widetilde{\omega}^n}{n!} = \sqrt{-1}^{n^2} s_{\alpha,q} \wedge \overline{s_{\alpha,q}}$$

for any hermitian form $\widetilde{\omega}$. Therefore $s_{\alpha,q}$ satisfies condition $(*)$.

Finally we show estimates (1), (2) in Proposition 3.3.1. Let V_ν be the $\mathbb{C}$-vector space which is spaned by $\{s_{\alpha,q}\}_{\alpha,q}$ where α runs through multi-indices with $|\alpha| \leq (\nu/a_0) - n$ and q runs through the fibre $\pi^{-1}(p)$ of p. Since $s_{\alpha,q}$ satisfies condition $(*)$, we have

$$V_\nu \subset H^0_{L^2(U\cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^*E_\nu \otimes K_{\widetilde{M}})\big).$$

We can easily show that $\{s_{\alpha,q}\}_{\alpha,q}$ is linearly independent from property (1) in Claim 3.3.3. Thus the dimension of V_ν is equal to the number of pairs of multi-index α with $|\alpha| \leq (\nu/a_0) - n$ and $q \in \pi^{-1}(p)$. It implies that the dimension is greater than or equal to $c_0\nu^n d$ for some number c_0 depending only on a_0 and n. Therefore conclusion (1) in Proposition 3.3.1 holds.

It remains to show conclusion (2). The pull-back $\pi^* s$ of a holomorphic n-form s valued in E_ν on X satisfies condition $(*)$ since s is globally defined on X. Thus we have

$$\pi^* H^0\big(X, \mathcal{O}_X(E_\nu \otimes \Omega_X^n)\big) \subset H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big).$$

Fix a point q_0 in the fibre of p. Let W_ν be the $\mathbb{C}$-vector space which is spaned by $\{s_{\alpha,q}\}_{\alpha, q \neq q_0}$. It follows any section in W_ν is orthogonal to $\pi^* H^0\big(X, \mathcal{O}_X(E_\nu \otimes \Omega_X^n)\big)$ from property (1) in Claim 3.3.3. Further the dimension of W_ν is greater than or equal to $c_0 \nu^n (d-1)$. Therefore we obtain estimate (2). □

3.3.2 On the formal schemes

In the proofs of Theorem 3.3.6, 3.4.1, we consider a formal scheme (a formal completion). In this subsection, we explain notations of a formal scheme and give the proof of Proposition 3.3.4. This proposition and its proof are often applied.

Let X be an irreducible analytic space (not necessarily compact) and Y an analytic subspace on X and let $\mathcal{I}$ be the ideal sheaf associated to Y. Note that Y is not necessarily irreducible and reduced. Then we denote by Y_k, the k-th infinitesimal neighborhood of Y which is defined by $\mathcal{I}^k$. That is, Y_k is the analytic space with structure sheaf $\mathcal{O}_{Y_k} := \mathcal{O}_X / \mathcal{I}^k$. The ringed space defined by

$$(\widehat{X}, \mathcal{O}_{\widehat{X}}) = (Y, \varprojlim_k \mathcal{O}_X / \mathcal{I}^k)$$

is called the *formal completion* with respect to $\mathcal{I}$. For a given coherent sheaf $\mathcal{F}$ on X, the formal completion $\widehat{\mathcal{F}}$ with respect to Y is defined to be

$$\widehat{\mathcal{F}} := \varprojlim_k \mathcal{F} \otimes \mathcal{O}_X / \mathcal{I}^k,$$

which is a $\mathcal{O}_{\widehat{X}}$-module sheaf. The cohomology groups of the formal completion $\widehat{\mathcal{F}}$ may be infinite dimensional space even if Y is compact. However under a suitable condition on Y, the cohomology groups are finite dimensional.

Proposition 3.3.4. *Let X be an irreducible analytic space (not necessarily compact), Y a compact analytic subspace on X, $\mathcal{I}$ the ideal sheaf associated to Y and E a locally free sheaf on X. Assume that there exists a positive integer k_0 such that*

$$H^0(X, \mathcal{O}_X(E) \otimes \mathcal{I}^k / \mathcal{I}^{k+1}) = 0 \quad \textit{for } k \geq k_0.$$

Then $H^0(\widehat{X}, \widehat{\mathcal{O}_X(E)})$ *injects into* $H^0(Y, \mathcal{O}_{Y_{k_0}}(E))$. *Moreover for a sufficiently large* k *we have*

$$H^0(\widehat{X}, \widehat{\mathcal{O}_X(E)}) = H^0(X, \mathcal{O}_{Y_k}(E)).$$

Remark 3.3.5. When Y is a locally complete intersection, we have the isomorphism $\mathcal{I}^k/\mathcal{I}^{k+1} \cong S^k N^*_{Y/X}$ as a $\mathcal{O}_X$-module. Then, if the normal bundle $N_{Y/X}$ satisfies some positivity conditions (such as ample), the assumption in Proposition 3.3.4 is satisfied (for instance see [Laz, Proposition 6.3.14]).

Proof. For every positive integer k, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \otimes \mathcal{I}^k/\mathcal{I}^{k+1} \longrightarrow \mathcal{O}_{Y_{k+1}}(E) \longrightarrow \mathcal{O}_{Y_k}(E) \longrightarrow 0.$$

By the assumption, the map

$$H^0(X, \mathcal{O}_{Y_{k+1}}(E)) \to H^0(X, \mathcal{O}_{Y_k}(E))$$

is injective for $k \geq k_0$. It implies that the map

$$H^0(X, \mathcal{O}_{Y_k}(E)) \to H^0(X, \mathcal{O}_{Y_{k_0}}(E))$$

is injective for $k \geq k_0$. Therefore we have

$$H^0(\widehat{X}, \widehat{\mathcal{O}_X(E)}) = \varprojlim_k H^0(X, \mathcal{O}_{Y_k}(E)) \subset H^0(X, \mathcal{O}_{Y_{k_0}}(E)).$$

The first equality follows from the general fact on the formal scheme (see [Har77, Proposition 9.2]).

Finally we prove the latter conclusion. By the argument above, we have already known that the dimension of the cohomology groups

$$\{H^0(X, \mathcal{O}_{Y_k}(E))\}_{k \geq k_0}$$

is decreasing for $k \geq k_0$. The dimension of the cohomology groups above is finite, since Y is compact. Therefore the cohomology groups must be isomorphic for a sufficiently large k. It implies

$$H^0(\widehat{X}, \widehat{\mathcal{O}_X(E)}) = \varprojlim_k H^0(Y, \mathcal{O}_{Y_k}(E)) = H^0(Y, \mathcal{O}_{Y_k}(E)).$$

□

3.3.3 Proof of Theorem 3.3.6

In this subsection, we prove the following theorem. Corollary 3.1.4 follows from this theorem.

Theorem 3.3.6. *Let Y be a connected analytic subset on a smooth projective variety X of dimension n, U a connected open neighborhood of Y, and R a nowhere dense analytic subset on X. Assume that the cohomological dimension of the complement $X \setminus Y$ is smaller than or equal to $(n-2)$. Then the map $j_* : \pi_1(U \setminus R) \longrightarrow \pi_1(X \setminus R)$ is surjective. In particular, if $R = \phi$, the map $j_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is surjective.*

Before the proof, we confirm the definition of the cohomological dimension.

Definition 3.3.7. Let Y be an analytic subset on a projective variety X. The (algebraic) *cohomological dimension* of $X \setminus Y$ is the smallest integer q such that $H^i(X \setminus Y, \mathcal{F}) = 0$ for any $i > q$ and any coherent sheaf $\mathcal{F}$ on X. We denote by $\mathrm{cd}(X \setminus Y)$ the cohomological dimension of $X \setminus Y$.

Let us begin to prove Theorem 3.3.6. We denote by M the complement $X \setminus R$ and by d the index of the image of $\pi_1(U \setminus R)$ by j_* in $\pi_1(X \setminus R)$. Let $\pi : \widetilde{M} \to M$ be an étale covering with $\pi_*(\pi_1(\widetilde{M})) = j_*(\pi_1(U \setminus R))$. From the construction, the degree of π is equal to d. Since U is connected and smooth, $U \setminus R$ is also connected. Thus, there exists a holomorphic map $\widetilde{j}$ with the following commutative diagram:

$$\begin{tikzcd} & \widetilde{M} \arrow[dr, "\pi"] & \\ U \setminus R \arrow[ur, "\widetilde{j}"] \arrow[rr, "j"'] & & M. \end{tikzcd}$$

Fix a big (or ample) line bundle L on X. By applying Proposition 3.3.1, we obtain positive numbers c_0 and ν_0 such that

$$c_0\nu^n(d-1) \leq \dim H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big) - \dim H^0\big(X, \mathcal{O}_X(E_\nu \otimes K_X)\big)$$

for any $\nu \geq \nu_0$.

Lemma 3.3.8. *In the situation above, we have*

$$\dim H^0_{L^2(U \cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})\big) \leq \dim H^0(\widehat{U}, \mathcal{O}_U(\widehat{E_\nu \otimes K_U})).$$

Proof. If s is a section in $H^0_{L^2(U\cap R)}(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}}))$, the pull-back $\widetilde{j}^*(s)$ is a holomorphic n-form valued in E_ν on $U \setminus R$. By the L^2-condition on $U \cap R$ (condition $(*)$ in Proposition 3.3.1), $\widetilde{j}^*(s)$ is L^2-bounded on a neighborhood of a point in $U \cap R$. Therefore $\widetilde{j}^*(s)$ can be extended to the holomorphic section on U by the Riemann extension theorem. It implies that

$$\dim H^0_{L^2(U\cap R)}(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^* E_\nu \otimes K_{\widetilde{M}})) \leq \dim H^0(U, \mathcal{O}_U(E_\nu \otimes K_U)).$$

Now $E_\nu \otimes K_U$ is a locally free sheaf on a smooth U and Y is connected. Thus the map

$$H^0(U, \mathcal{O}_U(E_\nu) \otimes K_U) \to H^0(\widehat{U}, \mathcal{O}_U(\widehat{E_\nu \otimes K_U}))$$

is injective. It follows from the general fact on a formal completion. (For instance, see [BS76, Proposition VI.2.7].) □

From [Har77, Proposition 9.2], we have

$$H^0(\widehat{U}, \mathcal{O}_U(\widehat{E_\nu \otimes K_U}) = \varprojlim_k H^0(U, \mathcal{O}_{Y_k}(E_\nu \otimes K_U)),$$

$$H^0(\widehat{X}, \mathcal{O}_X(\widehat{E_\nu \otimes K_X}) = \varprojlim_k H^0(X, \mathcal{O}_{Y_k}(E_\nu \otimes K_X)).$$

Since the supports of the sheaves which appear in the right hands is contained in U, the left hands coincide. These arguments and Lemma 3.3.8 yield

$$\begin{aligned} c_0 \nu^n (d-1) \leq \dim & H^0(\widehat{X}, \mathcal{O}_X(\widehat{E_\nu \otimes K_X})) \\ & - \dim H^0(X, \mathcal{O}_X(E_\nu \otimes K_X)). \end{aligned} \tag{3.2}$$

From now on, we estimate the right hand (3.2) by using the formal duality theorem.

Claim 3.3.9. *The right hand of* (3.2) *is estimated from above by the dimension of* $H^{n-1}(X \setminus Y, \mathcal{O}_X(E_\nu^*))$.

It follows $d = 1$ from this claim. In fact, $H^{n-1}(X \setminus Y, \mathcal{O}_X(E_\nu^*))$ is zero since $\mathrm{cd}(X \setminus Y) \leq n-2$. By inequality (3.2), we obtain $d = 1$. Thus it sufficient to see the claim above for the proof of Theorem 3.3.6.

Proof. In the proof, we use the theory of a local cohomology. See [Har68] for the local cohomology. Consider the long exact sequence of a local cohomology

$$\begin{aligned} \cdots \to & H^{n-i}_Y(X, \mathcal{O}_X(E_\nu^*)) \to H^{n-i}(X, \mathcal{O}_X(E_\nu^*)) \\ \to & H^{n-i}(X \setminus Y, \mathcal{O}_X(E_\nu^*)) \to H^{n-i+1}_Y(X, \mathcal{O}_X(E_\nu^*)) \\ \to & H^{n-i+1}(X, \mathcal{O}_X(E_\nu^*)) \to H^{n-i+1}(X \setminus Y, \mathcal{O}_{X\setminus Y}(E_\nu^*)) \to \cdots . \end{aligned}$$

Let us investigate the local cohomology $H^{n-i+1}_Y(X, \mathcal{O}_X(E^*_\nu))$. Now we have

$$\begin{aligned} H^{n-i+1}_Y(X, \mathcal{O}_X(E^*_\nu)) &= \varinjlim_k \operatorname{Ext}^{n-i+1}(\mathcal{O}_{Y_k}, \mathcal{O}_X(E^*_\nu)) \\ &= \varinjlim_k \operatorname{Ext}^{n-i+1}(\mathcal{O}_{Y_k}(E_\nu \otimes K_X), \mathcal{O}_X(K_X)). \end{aligned}$$

The Serre duality theorem on X implies

$$\operatorname{Ext}^{n-i+1}(\mathcal{O}_{Y_k}(E_\nu \otimes K_X), \mathcal{O}_X(K_X))$$

is equal to the dual space of

$$H^{i-1}(X, \mathcal{O}_{Y_k}(E_\nu \otimes K_X)).$$

Note that we can apply the Serre duality theorem since X is smooth. The inverse limit of the cohomology above is equal to the cohomology of the formal completion with respect to Y. Therefore we obtain

$$H^{n-i+1}_Y(X, \mathcal{O}_X(E^*_\nu)) = H^{i-1}(\widehat{X}, \mathcal{O}_X(\widehat{E_\nu \otimes} K_X))^*.$$

On the other hand, by using the Serre duality theorem on X again, we have

$$H^{n-i+1}(X, \mathcal{O}_X(E^*_\nu)) = H^{i-1}(X, \mathcal{O}_X(E_\nu \otimes K_X))^*.$$

Thus, the long exact sequence corresponds with

$$\begin{aligned} \cdots \to & H^{n-i}(X \setminus Y, \mathcal{O}_{X\setminus Y}(E^*_\nu)) \to H^{i-1}(\widehat{X}, \mathcal{O}_X(\widehat{E_\nu \otimes K_X}))^* \\ \to & H^{i-1}(X, \mathcal{O}_X(E_\nu \otimes K_X))^* \to \cdots . \end{aligned}$$

The conclusion follows from taking the dual spaces of this exact sequence. □

3.3.4 Proof of Corollary 3.1.4

In this subsection, we give the proof of Corollary 3.1.4. By Theorem 3.1.1, we know that S is connected under the assumption that E is ample and $r \leq n$. It is sufficient to prove $\operatorname{cd}(X \setminus S) \leq n-2$. A section $s \in H^0(X, \mathcal{O}_X(E))$ induces the morphism $s^* : \mathcal{O}_X(E^*) \to \mathcal{O}_X$. This morphism is a surjective map to $\mathcal{I}_S \subset \mathcal{O}_X$ since the codimension of S is equal to the expected codimension r. By taking symmetric powers and **Proj**, we have the embedding

$$i : \mathbf{Proj}\Big(\bigoplus_{k=0}^{\infty} \mathcal{I}_S^k\Big) \hookrightarrow \mathbf{Proj}\Big(\bigoplus_{k=0}^{\infty} \operatorname{Sym}^k(E^*)\Big).$$

The left hand is equal to the blow-up $\mathrm{Bl}_S(X)$ of X along S and the right hand is equal to $\mathbb{P}(E^*)$. Then we can easily check that $\mathcal{O}_{\mathrm{Bl}_S(X)}(F) = i^*\mathcal{O}_{\mathbb{P}(E^*)}(-1)$, where F is the exceptional divisor of $\mathrm{Bl}_S(X) \to X$.

Since E is ample, $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(r-1)$-ample (see [DPS96, Example 2.7]). It follows this line bundle is $(n-2)$-ample from $r < n$. Thus, the restriction $\mathcal{O}_{\mathrm{Bl}_S(X)}(F)$ is also $(n-2)$-ample. Note the restriction to a subvariety of a q-ample line bundle is also q-ample. In general, the cohomological dimension of the complement of the zero locus of a q-ample line bundle is smaller than or equal to q. (For instance, see [Ott11, Proposition 5.1].) Thus, we have $\mathrm{cd}(\mathrm{Bl}_S(X) \setminus F) \leq (n-2)$. Since $\mathrm{Bl}_S(X) \setminus F$ is isomorphic to $X \setminus S$, we obtain $\mathrm{cd}(X \setminus S) \leq n-2$. Thus the conclusion of Corollary 3.1.4 follows from Theorem 3.3.6.

Remark 3.3.10. (1) The same conclusion in Corollary 3.1.4 holds under the weaker assumption that $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is $(n-2)$-ample by the proof.

(2) If $X \setminus S$ is q-complete, then it follows $\pi_i(X, S) = 0$ for $i \leq n-q-1$ from the same argument in Section 3.2.2. Here (non-compact) complex manifold is called *q-complete* if the manifold admits a (smooth) exhaustive function whose Levi form has at least $(n-q)$ positive eigenvalues at any point.

(3) If a complex manifold Z is q-complete, then $\mathrm{cd}(Z) \leq q$. (It follows from the Andreotti-Grauert vanishing theorem.) But in general the converse is failed even if Z is a quasi-projective manifold. Therefore Theorem 3.3.6 is worth.

3.4 The weak Lefschetz type theorems for effective divisors

3.4.1 Applications of the key estimates

The aim of this section is to give the proof of the following theorem. This theorem have been already proved in [Nor83, Section 2]. It nevertheless seems to be worth to display another proof with the Napier-Ramachandran's method.

Theorem 3.4.1. *Let Y be a connected effective divisor on a projective variety X of dimension n (not necessarily smooth), U a connected open neighborhood of Supp(Y) such that U does not intersect with the singular locus X_{sing} of X, and R a nowhere dense analytic subset on X such that $R \supset X_{\mathrm{sing}}$. Assume that the Kodaira dimension of Y is larger than or equal to 2. Then the map*

$j_* : \pi_1(U \setminus R) \longrightarrow \pi_1(X \setminus R)$ is surjective. In particular, when X is normal and R is the singular locus X_{sing}, the map $j_* : \pi_1(Y) \longrightarrow \pi_1(X)$ is surjective.

Remark 3.4.2. The latter conclusion of this theorem follows from Corollary 3.5.4.

If the normal bundle $N_{Y/X}$ is ample, the image of $j_* : \pi_1(U \setminus R) \longrightarrow \pi_1(X \setminus R)$ is of finite index in $\pi_1(X \setminus R)$ by the result of [NR98]. However the normal bundle $N_{Y/X}$ may not be ample, even if the Kodaira dimension of Y is larger than or equal to 2. Therefore we do not know whether the image of the map is of finite index in $\pi_1(X \setminus R)$. In oder to overcome the difficulty, we need to establish the slight generalization (Proposition 3.3.1) of [NR98, Theorem 2.2].

The proof have been divided into 3 steps. In first step, we prove the theorem when Y is nef and big divisor. In this step, we apply Proposition 3.3.1. For the proof of the general case, we use the induction on the dimension of X. In second step, we consider the case where X is a projective surface. In third step , we use the induction hypothesis on the dimension for the proof of the general case. Throughout this section, we use the same notations in the proof of Theorem 3.3.6. For example we denote by d, the index

$$d = [\pi_1(X \setminus R) : j_*\pi_1(U \setminus R)].$$

Notice that our purpose is to prove $d = 1$.

3.4.2 The case where Y is nef and big

In this step, we assume that Y is a connected nef and big divisor. Then we shall prove $d = 1$.

By applying (2) in Proposition 3.3.1 to a big line bundle L and a line bundle F on X, we obtain positive integers c_0 and ν_0 (depending on X, L and F) such that

$$\begin{aligned} c_0\nu^n(d-1) \leq \dim & H^0_{L^2(U\cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^*E_\nu \otimes K_{\widetilde{M}})\big) \\ & - \dim H^0\big(X, \mathcal{O}_X(E_\nu \otimes \Omega^n_X)\big) \end{aligned}$$

for any $\nu \geq \nu_0$. Now X may have singularities, however, a neighborhood of Y is smooth. Thus Lemma 3.3.8 yields

$$\dim H^0_{L^2(U\cap R)}\big(\widetilde{M}, \mathcal{O}_{\widetilde{M}}(\pi^*E_\nu \otimes K_{\widetilde{M}})\big) \leq \dim H^0(\widehat{U}, \mathcal{O}_U(\widehat{E_\nu \otimes K_U})).$$

These estimates above hold for an arbitrary big line bundle L. In order to estimate the right hand, we take the big line bundle $\mathcal{O}_X(Y)$ associated to the big divisor Y as L. Then we can prove the following claim since Y is nef.

Claim 3.4.3. *Put $L := \mathcal{O}_X(Y)$. Then there exists a sufficient negative line bundle F (which is independent of k, ν) such that*

$$H^0(Y, \mathcal{O}_Y(E_\nu) \otimes \mathcal{I}^k/\mathcal{I}^{k+1}) = 0 \qquad \text{for } k \geq \nu.$$

Here $\mathcal{I}$ is the ideal sheaf associated to Y.

Proof of the claim. Since Y is an effective divisor on a smooth variety U, the ideal sheaf $\mathcal{I}$ is a locally complete intersection. Therefore we have the isomorphism $(\mathcal{I}^k/\mathcal{I}^{k+1})^* \cong \mathcal{O}_Y(kY)$. It yields

$$H^0(Y, \mathcal{O}_Y(E_\nu) \otimes \mathcal{I}^k/\mathcal{I}^{k+1}) = H^0(Y, \mathcal{O}_Y((\nu - k)Y) \otimes \mathcal{O}_Y(F \otimes \Omega_X^n)). \quad (3.3)$$

Here we used $E_\nu = L^\nu \otimes F = \mathcal{O}_X(\nu Y) \otimes F$. We shall show the right hand is equal to zero under a suitable choice of F.

Since Y is a nef divisor, $\mathcal{O}_Y(Y)$ is nef. That is, $\mathcal{O}_{Y_i}(Y)$ is nef for an irreducible component Y_i of Y. It implies $\mathcal{O}_Y((k-\nu)Y)$ is a nef line bundle for $k \geq \nu$. Therefore for the proof, it is sufficient to show that there exists a negative line bundle F such that $H^0(Y, \mathcal{O}_Y(-P) \otimes \mathcal{O}_Y(F \otimes \Omega_X^n)) = 0$ for any nef line P on Y.

If Y is irreducible and reduced, it is easy to see the existence of such negative line bundle F. If fact, if we take a line bundle F such that $F \otimes \Omega_X^n$ is negative on Y, then $\mathcal{O}_Y(-P) \otimes F \otimes \Omega_X^n$ is also negative for any nef divisor P. It implies the right hand in (3.3) is equal to zero.

In general, Y is not irreducible and reduced. For the precise proof, we need the following argument: Let $Y = \sum_{i=1}^N a_i Y_i$ be the irreducible decomposition. A section in the right hand in (3.3) can be seen as the family of sections

$$s_i \in H^0(Y_i, \mathcal{O}_{a_i Y_i}((\nu - k)Y) \otimes \mathcal{O}_{a_i Y_i}(F \otimes \Omega_X^n))$$

satisfying suitable gluing conditions on $Y_i \cap Y_j$. Therefore it is sufficient to show the existence of F such that $H^0(Y_i, \mathcal{O}_{a_i Y_i}(-P) \otimes \mathcal{O}_{a_i Y_i}(F \otimes \Omega_X^n)) = 0$ for any nef divisor P. Since Y_i is irreducible and reduced, we can take a line bundle F such that $H^0(Y_i, \mathcal{O}_{Y_i}(-kY - P) \otimes \mathcal{O}_{Y_i}(F \otimes \Omega_X^n)) = 0$ for any $i = 1, \ldots, N$ and k with $a_i \geq k \geq 0$. Then from the same argument as Lemma 3.3.4, we can easily show $H^0(Y_i, \mathcal{O}_{a_i Y_i}(-P) \otimes \mathcal{O}_{a_i Y_i}(F \otimes \Omega_X^n)) = 0$ for any nef divisor P. The choice of F does not depend on nef line bundles P. It competes the proof.

□

By this claim and Proposition 3.3.4, we have

$$\begin{aligned} \dim H^0(\widehat{U}, \mathcal{O}_U(\widehat{E_\nu \otimes K_U})) &\leq \dim H^0(Y, \mathcal{O}_{Y_\nu}(E_\nu \otimes K_U)) \\ &= \dim H^0(Y, \mathcal{O}_{Y_\nu}(E_\nu \otimes \Omega_X^n)). \end{aligned}$$

It follows the last equality since the support of $\mathcal{O}_{Y_\nu}(E_\nu)$ is contained in U. Then we consider the following exact sequence:

$$0 \longrightarrow \mathcal{I}^\nu \otimes \mathcal{O}_X(E_\nu \otimes \Omega_X^n) \longrightarrow \mathcal{O}_X(E_\nu \otimes \Omega_X^n) \longrightarrow \mathcal{O}_{Y_\nu}(E_\nu \otimes \Omega_X^n) \longrightarrow 0.$$

The long exact sequence (induced by this sequence) and the inequalities above yield

$$c_0\nu^n(d-1) \leq \dim H^1(X, \mathcal{I}^\nu \otimes \mathcal{O}_X(E_\nu \otimes \Omega_X^n))$$

for any $\nu \geq \nu_0$. It follows

$$\dim H^1(X, \mathcal{I}^\nu \otimes \mathcal{O}_X(E_\nu)) = \dim H^1(X, \mathcal{O}_X(F \otimes \Omega_X^n))$$

from $\mathcal{I} = \mathcal{O}_X(-Y)$ and $L = \mathcal{O}_X(Y)$. By this argument, we obtain

$$c_0\nu^n(d-1) \leq \dim H^1(X, \mathcal{O}_X(F \otimes \Omega_X^n)) \quad \text{for any} \quad \nu \geq \nu_0.$$

The right hand of the inequality does not depend on ν. Letting $\nu \to \infty$, we conclude that $c_0(d-1)$ must be zero. Hence $d = 1$ when Y is a connected nef and big divisor.

Remark 3.4.4. If X is Cohen-Macaulay, the right-hand is equal to zero for a sufficiently negative F. However we do not suppose that X is Cohen-Macaulay.

3.4.3 The case where X is a surface

In this section, we consider the case where X is a surface (and Y is not necessarily nef). Then, by the assumption of Theorem 3.4.1, Y is a big divisor. If Y is nef, it follows $d = 1$ from Section 3.4.2. However Y may not be nef even if Y is a big divisor. To overcome the difficulty, we use Lemma 3.4.5. This lemma says that Y can be assumed to be a nef and big divisor by changing the coefficients of irreducible components of Y.

Lemma 3.4.5. *Let Y be an effective divisor on a projective surface X and $Y = \sum_{i=1}^N a_i Y_i$ the irreducible decomposition of Y. Assume that Y is a big divisor, its support is connected and does not intersect with X_{sing}. Then there exist positive integers b_i $(1 \leq i \leq N)$ such that the effective divisor $\sum_{i=1}^N b_i Y_i$ is nef and big.*

Remark 3.4.6. This proposition does not hold without assuming that Y is connected. In fact, we consider the one point blow-up $\pi : \mathrm{Bl}_p(\mathbb{P}^2) \to \mathbb{P}^2$ of $\mathbb{P}^2$

and $Y = H+F$, where H is the pull-back of a line on $\mathbb{P}^2$ which does not pass through p and F is the exceptional divisor. Then Y is a big divisor. However even if we change the coefficients of H and F, the intersection number with F is negative (that is, not nef).

Proof. First we prove the following claim:

Claim 3.4.7. *If there exists a connected nef and big $\mathbb{Q}$-divisor $\widetilde{Y}$ such that $0 \le \widetilde{Y} \le Y$, then the conclusion holds in Lemma 3.4.5.*

Proof of the claim. Let $\{\widetilde{Y_\ell}\}_{\ell \in \mathbb{Z}_{\ge 0}}$ be the family of $\mathbb{Q}$-divisors defined as follows: $\widetilde{Y_0} := \phi$, $\widetilde{Y_1} := \widetilde{Y}$ and $\widetilde{Y_\ell} :=$ the sum of those irreducible components of Y which do not intersect with $\widetilde{Y_{\ell-2}}$ but intersect with $\widetilde{Y_{\ell-1}}$ for $\ell \ge 2$. Put $m := \sup\{\ell \mid \widetilde{Y_\ell} \ne \phi\}$. Note that every irreducible component Y_i of Y is contained some $\mathbb{Q}$-divisor $\widetilde{Y_{\ell_i}}$. It follows since Y is connected. We give the proof by the induction on m.

When $m = 1$, the support of $\widetilde{Y}$ coincides with that of Y. Therefore, by changing the coefficients of the irreducible components of Y, we may assume that $Y = \widetilde{Y}$. Then Y may not be a $\mathbb{Z}$-divisor. However, some positive multiple $m_0\widetilde{Y}$ is a $\mathbb{Z}$-divisor since $Y = \widetilde{Y}$ is a $\mathbb{Q}$-divisor. On the other hand, $m_0\widetilde{Y}$ is nef and big by the assumption. It leads to the conclusion in the claim.

From now on, we consider the case where $m > 1$. Since every irreducible component E of $\widetilde{Y_2}$ intersects with $\widetilde{Y_1}$, the intersection number $(E \cdot \widetilde{Y_1})$ is positive. Therefore there exists a sufficiently large integer a such that $(E \cdot (\widetilde{Y_1} + a\widetilde{Y_2})) > 0$ for every irreducible component E of $\widetilde{Y_2}$. Then it is easy to see that $\widetilde{Y_1} + a\widetilde{Y_2}$ is a connected nef and big divisor. In fact, let E be an irreducible divisor (curve) on X. If E is not contained in the support of $\widetilde{Y_1} + a\widetilde{Y_2}$, the intersection number $(E \cdot (\widetilde{Y_1} + a\widetilde{Y_2}))$ is semi-positive. If E is contained in the support of $\widetilde{Y_2}$, the intersection number is positive by the choice of a. Further, when E is contained in the support of $\widetilde{Y_1}$, the intersection number is also positive since $\widetilde{Y_1}$ is nef. Since $\widetilde{Y_1}$ is big, $\widetilde{Y_1} + a\widetilde{Y_2}$ is also big.

Putting $\widetilde{Z_1} := \widetilde{Y_1} + a\widetilde{Y_2}$, we construct $\{\widetilde{Z}\}_\ell$ by the same construction as $\{\widetilde{Y_\ell}\}_\ell$. Then by the construction, we have $\widetilde{Z_1} = \widetilde{Y_1} + a\widetilde{Y_2}$ and $\widetilde{Z_\ell} = \widetilde{Y_{\ell-1}}$ for $\ell \ge 2$. Thus $\sup\{\ell \mid \widetilde{Z_\ell} \ne \phi\} = m - 1$. By the induction hypothesis on m, we obtain the conclusion.

□

Finally, we see the existence of $\widetilde{Y}$ in the claim by using the Zariski decomposition of Y. Recall that any big divisor admits a Zariski decomposition on a smooth projective surface. Now X may have singularities. However, since Y does not intersection with X_{sing}, we can assume X is smooth without changing Y, by taking a suitable resolution of singularities of X.

Let $Y = P + N$ be the Zariski decomposition of Y. Note that P is a nef and big $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor such that $0 \leq P, N \leq Y$. We consider the decomposition $P = \sum_i P_i$ suct that the support of P_i is disjoint each other. Since P is a nef and big divisor, the self-intersection number $(P^2) = \sum_i (P_i^2)$ is positive. Thus, the self-intersection number $(P_{i_0}^2)$ must be positive for some i_0. Since P is a nef divisor, P_{i_0} is also a nef divisor. In fact, for every irreducible component E of P_i, $(E \cdot P_i) = (E \cdot P) \geq 0$. Here the first equality follows since P_i is disjoint each other. We knew that P_{i_0} is a connected nef and big divisor such that $0 \leq P_{i_0} \leq P$. It completes the proof of the lemma. □

By Lemma 3.4.5, we may assume that Y is a nef and big divisor by changing the coefficients of the irreducible components of Y. Recall that d is the index of the image of $j_* : \pi_1(U \setminus R) \longrightarrow \pi_1(X \setminus R)$. Therefore d is invariant even if we change the coefficients of the irreducible components of Y. When Y is a connected nef and big divisor, it follows $d = 1$ from Section 3.4.2. In this step, we proved that the theorem holds on a surface.

3.4.4 General cases

In this subsection, we prove the general case under the induction hypothesis. Let X be a projective variety of dimension n. We may assume $n > 2$. By the induction hypothesis, Theorem 3.4.1 can be assumed to hold when the dimension of a projective variety is smaller than n. First we fix an embedding of X to the projective space $\mathbb{P}^N$.

Proposition 3.4.8. *Let $X \subset \mathbb{P}^N$ be an embedded projective variety of dimension $n > 2$, U an open set in X which does not intersect with X_{sing} and Y a connected divisor whose Kodaira dimension ≥ 2. Then a general hyperplane H in $\mathbb{P}^N$ satisfies the following properties:*

(1) *$X \cap H$ is an irreducible projective variety of dimension $(n-1)$.*

(2) *$U \cap H$ is smooth (that is, does not intersection with $(X \cap H)_{\text{sing}}$).*

(3) *The Kodaira dimension $\kappa_{X \cap H}(Y \cap H) \geq 2$.*

(4) *$Y \cap H$ is a connected divisor.*

Proof. By applying the Bertini type theorem, conditions (1) and (2) are satisfied for a general hyperplane H. We prove condition (3). The rational map $\Phi := \Phi_{|m_0 Y|} : X \cdots \to \mathbb{P}(|m_0 Y|)$ associated to the complete linear system $|m_0 Y|$ has the generic rank ≥ 2. From the definition of the Kodaira dimension, the tangent linear map $d\Phi_x : T_x X \to T_{\Phi(x)}\mathbb{P}^N$ has rank ≥ 2 at a general point $x \in X$. Therefore a general hyperplane H passes through a general point and the restriction $d\Phi_x|_{T(X\cap H)}$ has rank ≥ 2. Therefore the restriction of $\Phi_{|m_0 Y|}$ to a general hyperplane H has still rank ≥ 2. It means $\kappa_{X\cap H}(Y \cap H) \geq 2$.

Finally we check condition (4). Let H be a hyperplane in $\mathbb{P}^N$. Consider the following exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(-qH) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y/\mathcal{I}^q_{Y\cap H} = \mathcal{O}_{q(Y\cap H)} \longrightarrow 0,$$

where q is a positive integer and $\mathcal{I}_{Y\cap H}$ is the ideal sheaf associated to the effective divisor $Y \cap H$ on Y. Now since Y is a divisor on a smooth variety U, Y is a locally complete intersection and equidimensional. Therefore Y is Cohen-Macaulay (see [Har77, Remark 7.61.]). It implies that $H^1(Y, \mathcal{O}_Y(-qH)) = 0$ for a sufficiently large q. Hence we have the surjective map

$$H^0(Y, \mathcal{O}_Y) \longrightarrow H^0(Y \cap H, \mathcal{O}_Y/\mathcal{I}^q_{Y\cap H}) = H^0(Y \cap H, \mathcal{O}_{(Y\cap H)_q}).$$

Now $\dim H^0(Y, \mathcal{O}_Y) = 1$ since Y is connected. If $Y \cap H$ is not connected, $\dim H^0(Y \cap H, \mathcal{O}_{(Y\cap H)_q})$ is greater than or equal to 2. It is a contradiction to the subjectivity of the map. Therefore $Y \cap H$ must be connected. $\square$

Take a hyperplane H in $\mathbb{P}^N$ with the conditions in Proposition 3.4.8. Then we consider the following commutative diagram:

$$\begin{array}{ccc} X \setminus R & \longleftarrow & (X \cap H) \setminus R \\ \uparrow & & \uparrow \\ U \setminus R & \longleftarrow & \widetilde{U}, \end{array}$$

where $\widetilde{U} \subset (X \cap H)$ is a connected open neighborhood of $(Y \cap H)$. By Proposition 3.4.8, $Y \cap U$ is connected. However we do not know whether $U \cap H$ is connected. Therefore we need to take such connected neighborhood $\widetilde{U}$. Then we have

$$\begin{array}{ccc} \pi_1(X \setminus R) & \longleftarrow & \pi_1((X \cap H) \setminus R) \\ \uparrow & & \uparrow \\ \pi_1(U \setminus R) & \longleftarrow & \pi_1(\widetilde{U}). \end{array}$$

By the induction hypothesis, the vertical map on the right is surjective. If the horizontal map on the above is surjective, then the vertical map on the left is also surjective. It completes the proof of Theorem 3.4.1.

It remains to show that the horizontal map on the above is surjective. Note that $X \cap H$ is an ample divisor on X. If $X \cap H$ does not intersect with X_{sing}, it follows this map is surjective in Section 3.4.2. (However $X \cap H$ may intersect with X_{sing}.) Hence we have already proved Theorem 3.4.1 when X is smooth.

The general case can be reduced to the smooth case as follows: Take a resolution of singularities $p : \widetilde{X} \to X$ of X. Then the pull-back p^*H is a nef and big divisor since H is an ample divisor. (Note the pull-back can be defined since H is a Cartier divisor.) Now $p : \widetilde{X} \to X$ is not a fibre connected when X is not a normal variety. Therefore $p^{-1}(U)$ and p^*Y may not be connected. For this reason, we need to take a new connected nef and big divisor and its connected open neighborhood. Let $p^*Y = \sum_i^N \overline{Y}_i$ be a decomposition such that the effective divisors $\overline{Y}_i$ is disjoint each other. Now p^*Y is nef and big, the self-inersection number

$$((p^*Y)^n) = \sum_i^N (\overline{Y}_i^n)$$

is positive. Therefore $(\overline{Y}_{i_0}^n)$ is positive for some i_0. Since p^*Y is nef, $\overline{Y}_i$ is also nef. In fact, for an irreducible curve C contained in $\overline{Y}_i$, the intersection number $(C \cdot \overline{Y}_i) = (C \cdot p^*Y)$ is semi-positive. Now we take a connected open neighborhood $\overline{U}$ such that $Y_{i_0} \subset \overline{U}$ and $\overline{U} \subset p^{-1}(U)$. Then the map $\pi_1(\overline{U} \setminus p^{-1}(R)) \to \pi_1(\widetilde{X} \setminus p^{-1}(R))$ is surjective from Section 3.4.2. Note that $\overline{U}$ is a sufficiently small neighborhood of $\overline{Y_{i_0}}$. Hence the horizontal map on the below is surjective in the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(X \setminus R) & \longleftarrow & \pi_1((X \cap H) \setminus R) \\
\uparrow & & \uparrow \\
\pi_1(\widetilde{X} \setminus p^{-1}(R)) & \longleftarrow & \pi_1(\overline{Y_{i_0}} \setminus p^{-1}(R)).
\end{array}$$

In order to show that the horizontal map on the above is surjective, it is sufficient to check that the vertical map on the left is surjective. Now there exists an analytic subspace Z such that p is an isomorphism on $\widetilde{X} \setminus Z$, since p is a birational morphism. Thus, the map

$$\pi_1(\widetilde{X} \setminus (p^{-1}(R) \cup Z)) \xrightarrow{p_*} \pi_1(X \setminus (R \cup p(Z)))$$

is an isomorphism. Moreover the map

$$\pi_1(X \setminus (R \cup p(Z))) \to \pi_1(X \setminus R)$$

is also surjective since $X \setminus R$ is smooth and $p(Z)$ is an analytic set. Here we used Corollary 3.5.4. Therefore the following commutative diagram

$$\begin{array}{ccc}
\pi_1(X \setminus R) & \longleftarrow & \pi_1(X \setminus (R \cup p(Z))) \\
\uparrow & & \uparrow \\
\pi_1(\widetilde{X} \setminus p^{-1}(R)) & \longleftarrow & \pi_1(\widetilde{X} \setminus (p^{-1}(R) \cup Z)).
\end{array}$$

implies the vertical map on the left is surjective.

3.5 Examples

In this section, we discuss on Theorem 3.5.3. It may be a well-known fact. However for the readers' convenience, here we give the sketch of the proof. First we confirm the definition.

Definition 3.5.1. Let X be an irreducible analytic space (not necessarily smooth) and Z an analytic subspace on X. Then X is called *topologically unibranch along* Z, if for an arbitrary point $p \in Z$ and an open neighborhood U of p, there exists a open neighborhood V such that $V \subset U$ and $(V \setminus (V \cap Z))$ is connected.

Lemma 3.5.2. *Let X be an (irreducible) normal variety. Then X is topologically unibranch along any analytic subspace Z on X.*

Proof. Take a point p on Z and an open neighborhood U of p. Then we can easily show that $(V \setminus (V \cap Z))$ is connected for any connected open neighborhood V of p, by the extension theorems of holomorphic functions. In fact, we assume that $(V \setminus (V \cap Z))$ is a disjoint union of V_0 and V_1 for a contradiction. Then we consider the function F defined by $F \equiv 0$ on V_0 and $F \equiv 1$ on V_1. Since F is a bounded holomorphic function on $(V \setminus (V \cap Z))$, F can be extend to $V \setminus (V \cap X_{sing})$ by the Riemann extension theorem. Further F can be actually extended to V since X is a normal variety. It is a contradiction. □

Theorem 3.5.3. *Let Z be an analytic subspace on an (irreducible) analytic space X. Assume that X is topologically unibranch along Z. Then the map $j_* : \pi_1(X \setminus Z) \to \pi_1(X)$ is surjective.*

Proof. Let $\pi : \widetilde{X} \to X$ be the universal covering of X. Since π is a local biholomorphic, there exists an open covering $\{V_i\}_{i=1}^{\infty}$ of $\widetilde{X}$ such that $V_i \setminus (V_i \cap \pi^{-1}(Z))$ is connected. Note that $V_i \setminus (V_i \cap \pi^{-1}(Z))$ is path-connected. First we show that $\pi^{-1}(X \setminus Z)$ is connected. Take arbitrary points p and q in $\pi^{-1}(X \setminus Z)$. Since $\widetilde{X}$ is path-connected, there exists a path $\gamma : [0,1] \to \widetilde{X}$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Since the image of γ is compact, the image of γ is covered by finite members. Therefore there exist finite members $\{V_i\}_{i=1}^{N}$ such that

(1) The image of γ is covered by $\{V_i\}_{i=1}^{N}$.
(2) V_1 contains p and V_N contains q.
(3) V_i intersections with V_{i+1} for $i = 1, 2, \ldots, (N-1)$.

Let p_i be a point in $V_i \cap V_{i+1}$ for $i = 2, 3, \ldots, (N-2)$. Since $V_i \setminus (V_i \cap \pi^{-1}(Z))$ is path-connected, p_i and p_{i+1} can be connected by a path in $V_{i+1} \setminus (V_{i+1} \cap \pi^{-1}(Z))$ for $i = 2, 3, \ldots, (N-2)$. By the same reason, p and p_2 (q and p_{N-2}) can be connected by a path in $V_1 \setminus (V_1 \cap \pi^{-1}Z)$. By connecting these pathes, we obtain a path in $\pi^{-1}(X \setminus Z)$ connecting p and q. Thus, $\pi^{-1}(X \setminus Z)$ is connected.

Let $\overline{\pi} : \overline{X} \to X$ be an étale covering with $\pi_*(\pi_1(\overline{X})) = j_*\pi_1(X \setminus Z)$. Then we have

$$\begin{tikzcd} & \widetilde{X} \arrow[dl] \arrow[dr, "\pi"] & \\ \overline{X} \arrow[rr, "\overline{\pi}"'] & & X. \end{tikzcd}$$

Since $\pi^{-1}(X \setminus Z)$ is connected, $\overline{\pi}^{-1}(X \setminus Z)$ is also connected. Now $\overline{\pi}$ has a section on $X \setminus Z$ by the construction. Therefore the restriction $\overline{\pi}|_{\overline{\pi}^{-1}(X \setminus Z)} : \overline{\pi}^{-1}(X \setminus Z) \to (X \setminus Z)$ has a section. That is, there exists a holomorphic map $s : (X \setminus Z) \to \overline{\pi}^{-1}(X \setminus Z)$ such that $\overline{\pi} \circ s = id$. Therefore $\overline{\pi}_* : \pi_1(\overline{\pi}^{-1}(X \setminus Z)) \to \pi_1((X \setminus Z))$ is surjective. Thus j_* is surjective. $\square$

As a corollary of Theorem 3.5.3, we obtain the following.

Corollary 3.5.4. *Let X be an (irreducible) normal variety.*
(1) *Then the map $\pi_1(X_{\text{reg}}) \to \pi_1(X)$ is surjective.*
(2) *Let Z be an analytic subspace on a smooth variety X. Then the map $\pi_1(X \setminus Z) \to \pi_1(X)$ is surjective.*

Remark 3.5.5. In the setting of Corollary 3.5.4, if X is a smooth variety and the codimension of Z is greater than or equal to two, the map is an isomorphism. In fact, it follows from the real codimension ≥ 3, by using the van-Kampen theorem. However, the map is necessarily not an isomorphism when X has singularities. See Example 3.5.6.

Example 3.5.6. (Landman's example). (cf. [Laz, Example 3.1.33]).

This example implies that (1) in Corollary 3.5.4 does not necessarily hold when X is not normal (even if $\mathrm{codim}_X X_{\mathrm{sing}} \geq 2$).

Let Y be a smooth projective variety and distinct points p, q on X. Taking a sufficiently large projective embedding $Y \subset \mathbb{P}^N$, a general point on the secant line through p, q does not intersect with any other secant lines of Y. We denote by X the image by the projection from a general point of the line and by o the image of p, q. Then X has singularities only at o and $X \setminus \{o\}$ is an isomorphic to $Y \setminus \{p, q\}$. Note that X is not topologically unibranch along o. Then we have

$$\pi_1(X \setminus o) \xrightarrow{\cong} \pi_1(Y \setminus \{p, q\}) \xrightarrow{\cong} \pi_1(Y)$$

by the remark above. On the other hand, $\pi_1(X)$ is an isomorphic to $\pi_1(Y) \times \mathbb{Z}$. It follows from the van-Kampen theorem. In fact, take an open neighborhood U_1 in X such that

(1) U_1 is a neighborhood of the image $\bar{\gamma}$ of a path γ.
(2) U_1 is homotopic to a circle S^1.

Here γ is a path in Y connecting p and q and $\bar{\gamma}$ is the image of γ by the projection. On the other hand, U_2 defined by $U_2 := X \setminus \bar{\gamma}$ is connected and, homotopic to $Y \setminus \gamma$. Therefore we have

$$\pi_1(U_2) \xrightarrow{\cong} \pi_1(Y \setminus \gamma) \xrightarrow{\cong} \pi_1(Y).$$

Here we use $\mathrm{codim}_{\mathbb{R},Y}\gamma \geq 3$. Now the intersection $U_1 \cap U_2$ is equal to $U_1 \setminus \bar{\gamma}$. Therefore $U_1 \cap U_2$ is homotopic to $S^{2n-1} \times [0, 1)$. Here n is the (complex) dimension of Y. It impies $\pi_1(U_1 \cap U_2) = \{1\}$. By the van-Kampen theorem, $\pi_1(X)$ is an isomorphic to $\pi_1(Y) \times \mathbb{Z}$.

4

On the ample vector bundles on curves

4.1 Introduction

In algebraic geometry, there are various geometric results on the amplitude of the normal bundle of a subvariety. In this chapter, we study the Hartshorne conjecture on the normal bundle, which was first posed in his paper [Har70].

Conjecture 4.1.1. ([Har70, Conjecture 4.4]). *Let S be a smooth subvariety on a smooth projective variety X. Assume that the normal bundle of S in X is ample. Then some positive multiple of S would move as a cycle in a large family.*

It is easy to see that this conjecture is affirmative when the codimension of S in X is one. In their paper [FL82], Fulton and Lazarsfeld constructed a counterexample for the conjecture as follows: Let E be an ample vector bundle on a smooth projective variety $S = \mathbb{P}^2$. Then we consider the projective closure $\mathbb{P}(\mathcal{O}_S \oplus E^*)$ of E, which we denote by X. We identify S with the zero-section $\mathbb{P}(\mathcal{O}_S)$ of $X = \mathbb{P}(\mathcal{O}_S \oplus E^*)$. Then the normal bundle of S in X is equal to E. When E is Gieseker's ample vector bundle on $S = \mathbb{P}^2$, this construction gives a counterexample for the Hartshorne conjecture. (See [Gie71] for Gieseker's ample vector bundle.)

For that reason, the conjecture is negative in general. However the conjecture still remains open when S is a curve. In this chapter, we give a characterization of the amplitude of a vector bundle on a curve. As an application, we give a partial answer of the Hartshorne conjecture for a curve (Theorem 4.1.2).

Theorem 4.1.2. *Let E be an ample vector bundle on a projective curve C. Here we identify C with the zero-section $\mathbb{P}(\mathcal{O}_C)$ in $\mathbb{P}(\mathcal{O}_C \oplus E^*)$. Then there* *exist a positive integer k_0 and an algebraic family $\{C_t\}_{t\in T}$ of 1-cycles with irreducible T such that*

(1) $C_{t_0} = k_0 C$ *for some* $t_0 \in T$,

(2) $\mathbb{P}(\mathcal{O}_C \oplus E^*) = \bigcup_{t\in T} C_t$.

This theorem assures that the counterexample of Fulton and Lazarsfeld can never be modified into a counterexample for a curve. When the rank of E is 2, this theorem have been proved in [BPS90]. Therefore Theorem 4.1.2 can be seen as the generalization of the result of [BPS90]. Barlet, Peternell and Schneider proved Proposition 4.1.3 when the rank of E is 2 by using some geometric facts concerning the ruled surface. We shall generalize this proposition to a vector bundle of an arbitrary rank, by applying the properties of a q-ample line bundle and a numerical characterization of the pseudo-effective line bundles (which was established in [BDPP04]). The method of this chapter seems to be essentially different from that of [BPS90].

Proposition 4.1.3. *Let E be an ample vector bundle on a smooth projective curve C. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on the projective space bundle $\mathbb{P}(E^*)$ associated to the dual vector bundle E^* of E. Then for an arbitrary point e on $\mathbb{P}(E^*)$, there exists an irreducible curve D_e on $\mathbb{P}(E^*)$ such that*

(1) *D_e passes through $e \in \mathbb{P}(E^*)$,*

(2) *The restriction of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ to D_e is an ample line bundle on D_e.*

The main result of [CF90] implies Theorem 4.1.2. However the methods in this chapter give a direct generalization of [BPS90] and do not need the result of [CF90]. In the proof of the main result of [CF90], Campana and Flenner constructed a curve with the properties in Proposition 4.1.3 when E is a semi-stable and ample vector bundle. (See [CF90, Corollary (8) and Lemma (9)].) Further for the main result, they used the Harder-Narasimhan filtration of a vector bundle by semi-stables subbundles. Proposition 4.1.3 may give another proof of the main result of [CF90] without the Harder-Narasimhan filtration and semi-stability of vector bundles.

4.2 Characterizations of the amplitude of vector bundles on curves

The main aim of this section is to prove Theorem 4.2.1 and Proposition 4.1.3. First we consider Theorem 4.2.1, which gives a characterization of the amplitude of a vector bundle on a curve.

Theorem 4.2.1. *Let E be a vector bundle on a smooth projective curve C. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on the projective space bundle $\mathbb{P}(E^*)$. Then the following conditions are equivalent.*

(*a*) *E is ample.*

(*b*) *There exists a strongly movable curve D on $\mathbb{P}(E^*)$ such that the degree of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on D is positive.*

In this section, we denote by X, the projective space bundle $\mathbb{P}(E^*)$ associated to the dual vector bundle E^* of E and by L, the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on X for simplicity.

For the proof of Theorem 4.2.1, we consider the following proposition. Theorem 4.2.1 follows from this proposition and Proposition 2.2.5.

Proposition 4.2.2. *The following conditions are equivalent.*

(*a*) *E is ample.*

(*c*) *$L = \mathcal{O}_{\mathbb{P}(E^*)}(-1)$ is cohomologically $(r-1)$-ample on $X = \mathbb{P}(E^*)$.*

Proof. The following lemma asserts that it is sufficient to see the cohomology groups vanish when $\mathcal{F}$ is a negative multiple of a fixed ample line bundle. It may be known fact but we give the complete proof for readers' convenience (cf. [Ott11, Lemma 2.1]).

Lemma 4.2.3. *Let M be a line bundle on a projective variety Y. Fix an ample line bundle A on Y. Then M is cohomologically q-ample if and only if for each $k \geq 0$ exists a positive integer $m_0 = m_0(k) > 0$ such that*

$$H^i(Y, A^{-k} \otimes M^m) = 0 \quad \text{for } i > q,\ m \geq m_0.$$

Proof. Let $\mathcal{F}$ be a coherent sheaf on Y. Then for a sufficiently large k_0, the sheaf $\mathcal{F} \otimes A^{k_0}$ is generated by its global sections. Thus, there is a surjective map $\mathcal{E} \to \mathcal{F}$ where $\mathcal{E}$ is a sum of line bundles of the form A^{-r_i}. Here r_i is a positive integer for each i. Let $\mathcal{G}$ be the kernel of the map $\mathcal{E} \to \mathcal{F}$. Then we consider the exact sequence

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0.$$

It follows that $H^i(Y, L^m \otimes \mathcal{E}) = 0$ for $i > q$ and $m \gg 0$ from the assumption and the form of $\mathcal{E}$. Thus, for a sufficiently large m, we have the isomorphism $H^{i+1}(Y, L^m \otimes \mathcal{G}) \cong H^i(Y, L^m \otimes \mathcal{F})$ for $i > q$. Then we can easily see that L is cohomologically q-ample by the induction on q. □

First we show that condition (a) implies (c) in Theorem 4.2.1. That is, for any coherent sheaf $\mathcal{F}$ on X there exists a large integer m_0 (which depends on $\mathcal{F}$) such that

$$H^r(X, L^m \otimes \mathcal{F}) = 0 \quad \text{for} \quad m \geq m_0.$$

It follows from a formula on direct image sheaves of line bundles on the projective space bundle and a simple computation of the spectral sequences.

Let $X = \mathbb{P}(E^*) \xrightarrow{\pi} C$ be the natural projection to C. We fix an ample line bundle B on C such that $\pi^*B \otimes L^{-1}$ is ample on X. Note that we can take such line bundle since the restriction of L^{-1} to a fibre of π is ample. We denote by A, the ample line bundle $\pi^*B \otimes L^{-1}$ on X. Thanks to Lemma 4.2.3, it is sufficient to show that for each $k \geq 0$, $H^i(X, L^m \otimes A^{-k}) = 0$ for $m \gg 0$ and $i > q$.

By a formula for higher direct images of line bundles, we have

$$\pi_*(L^{-m}) = \mathrm{Sym}^m(E^*), \qquad R^{r-1}\pi_*(L^{m+r}) = \mathrm{Sym}^m(E) \otimes \det E$$

for a positive integer m and all other direct image sheaves vanish (see [Laz, Appendix A]). By using the Leray spectral sequence, we obtain

$$H^i(C, \mathrm{Sym}^m(E) \otimes \mathcal{F}) = H^{r+i-1}(X, L^{m+r} \otimes \pi^*(\mathcal{F} \otimes \det E^*))$$

for a coherent sheaf $\mathcal{F}$ on C and a positive integer m. For any k, by taking $\mathcal{F} = B^{-k} \otimes \det E$, we obtain

$$\begin{aligned} H^i(C, \mathrm{Sym}^m(E) \otimes B^{-k} \otimes \det E) &= H^{r+i-1}(X, L^{m+r} \otimes \pi^* B^{-k}) \\ &= H^{r+i-1}(X, L^{m+r-k} \otimes A^{-k}). \end{aligned}$$

The left hand is equal to zero for $i > 0$ and a sufficiently large m, since E is an ample vector bundle on C. Thus, for any integer k there is a large integer $m_0 = m(k)$ such that $H^r(X, L^m \otimes A^{-k}) = 0$ for $m \geq m_0$. Therefore it follows L is an $(r-1)$ line bundle on X from Lemma 4.2.3.

Conversely, we assume that L is cohomologically $(r-1)$-ample. Then by the same argument, for any coherent sheaf $\mathcal{F}$ on C, there is an integer m_0 such that $H^1(C, \mathrm{Sym}^m(E) \otimes \mathcal{F}) = 0$ for $m \geq m_0$. It implies that E is ample. □

At the end of this section, we give the proof of Proposition 4.1.3 by applying Proposition 4.2.1.

Proposition 4.2.4. (=Proposition 4.1.3). *Let E be an ample vector bundle on a smooth projective curve C. Consider the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ on the projective space bundle $\mathbb{P}(E^*)$ associated to the dual vector bundle E^* of E. Then for an arbitrary point e on $\mathbb{P}(E^*)$, there exists an irreducible curve D_e on $\mathbb{P}(E^*)$ such that*

(1) *D_e passes through $e \in \mathbb{P}(E^*)$,*

(2) *The restriction of $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ to D_e is an ample line bundle on D_e.*

Proof. By the definition of a strongly movable curve, we obtain a birational morphism $\mu : \widetilde{X} \to X$ and very ample divisors A_i for $i = 1, 2, \ldots, r-1$ such that $(\mu^* L \cdot A_1 \cdots A_{r-1}) > 0$.

Let e be a point in X. We can take a point $\tilde{e}$ on $\widetilde{X}$ with $\mu(\tilde{e}) = e$ since $\mu : \widetilde{X} \to X$ is surjective. Then there exist a smooth curve $\widetilde{C}$ on $\widetilde{X}$ with the following properties:

(1) $\widetilde{C}$ passes through $\tilde{e}$.

(2) The intersection number $(\mu^* L \cdot \widetilde{C})$ is positive.

In fact, a general member of $|A_i|_{\tilde{e}}$ is irreducible and smooth, since A_i is very ample (see [Zha09, Theorem 1.3]). Here $|A_i|_{\tilde{e}}$ is the linear system passing through $\tilde{e}$ in the complete linear system of A_i. Thus by taking a complete intersection of general members of $|A_i|_{\tilde{e}}$, we can obtain a curve $\widetilde{C}$ with the properties above.

Let D_e be the push-forward of $\widetilde{C}$. Notice that D_e is not a point by the construction of $\widetilde{C}$. Since $\widetilde{C}$ passes through $\tilde{e}$ and $\mu(\tilde{e}) = e$, the push-forward D_e passes through e. Further, we have $(L \cdot D_e) > 0$ by the projection formula and property (2). It completes the proof of Proposition 4.1.3. □

4.3 Proof of Theorem 4.1.2

In this subsection, we give the proof of Theorem 4.1.2 by applying Proposition 4.1.3. Before the proof, we prepare the following lemma.

Lemma 4.3.1. *Let S be a m-cycle on a compact Kähler manifold Y. Assume that there exist analytic families $\{S^i_t\}_{t \in T_i}$ of m-cycles with irreducible T_i with the following properties:*

(1) *For each i, there is $t_i \in T_i$ such that $S^i_{t_i}$ is equal to some positive multiple $k_i S$ of S.*

(2) *$\bigcup_{i \in I} \bigcup_{t \in T_i} S^i_t$ contains some open set on Y.*

Then there exists an analytic family $\{S_t\}_{t\in T}$ *of* m*-cycles with irreducible* T *with the following properties:*

(1) S_{t_0} *is equal to some positive multiple* kS *of* S *for some* $t_0 \in T$.

(2) $\bigcup_{t\in T} S_t = Y$.

Proof. Since Y is a compact Kähler manifold, the Barlet space of m-cycles has at most countable irreducible components. Further, each irreducible component is compact (see [Lie78]). Remark that in general components of the Barlet space may not be compact even if Y is compact. Let $\{Z_\ell\}_{\ell=1}^{\infty}$ be the irreducible components of the Barlet space of m-cycles which contain some positive multiple of S. Then for each $i \in I$ there is ℓ_i such that Z_{ℓ_i} contains T_i. Thus we have

$$\bigcup_{i\in I}\bigcup_{t\in T_i} S_t^i \subset \bigcup_{\ell=1}^{\infty}\bigcup_{t\in Z_\ell} S_t^\ell.$$

Now $\bigcup_{t\in Z_\ell} S_t^\ell$ is a closed analytic subset in Y since Z_ℓ is compact. In fact, we consider the graph of the analytic family $\{S_t^\ell\}_{t\in Z_\ell}$. The graph

$$\{(y,t) \in Y \times Z_\ell \mid y \in S_t^\ell\}.$$

is a closed analytic set in $Y \times Z_\ell$. Further, the first projection $Y \times Z_\ell \xrightarrow{p_1} Y$ is a proper map since Z_ℓ is compact. Therefore the image of the graph by the projection is a closed analytic set in Y by the proper mapping theorem.

By property (2), the countable union of closed analytic sets $\bigcup_{t\in Z_\ell} S_t^\ell$ contains an open set on Y. Thus there is an integer ℓ_0 such that $\bigcup_{t\in Z_{\ell_0}} S_t^{\ell_0}$ contains the open set. Since $\bigcup_{t\in Z_{\ell_0}} S_t^{\ell_0}$ is a closed analytic set and Y is connected, we have $\bigcup_{t\in Z_{\ell_0}} S_t^{\ell_0} = Y$. It follows that the analytic family $\{S_t^{\ell_0}\}_{t\in Z_{\ell_0}}$ satisfies the conclusion of the lemma from the choice of Z_{ℓ_0}. □

Let us prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Take a normalization $\nu : \overline{C} \to C$ of C. Since ν is a finite morphism and E is ample on C, the pull-back ν^*E is also ample on $\overline{C}$. We assume that there is an algebraic family $\{\overline{C}_t\}_{t\in T}$ of 1-cycles on $\overline{C}$ with the conditions in Theorem 4.1.2. Then the push-forward $\{\nu_*\overline{C}_t\}_{t\in T}$ is an algebraic family of 1-cycles on C since ν is a finite map (see [Bar75], [Bar80-B]). Futher, this family satisfies the conditions in Theorem 4.1.2 by the construction. Thus, we may assume that C is smooth curve.

We take an arbitrary point $e \neq 0_c$ in the fibre E_c of $E \to C$ at $c \in C$. Now $\pi : X = \mathbb{P}(E^*) \to C$ has the space of lines of E_c as the fibre of $c \in C$.

Thus we can identify e with the corresponding point of X, which we denote by the same notation e. By applying Proposition 4.1.3, for this $e \in X$, we take an irreducible curve D_e with the following properties:

(1) D_e passes through $e \in X$.

(2) The restriction of L to D_e is ample on D_e.

From property (2), there exist a positive integer k and a (holomorphic) section $s_e \in H^0(D_e, L^k|_{D_e})$ such that the value of s_e at e is not zero. We consider the image of multi-section induced by s_e and want to construct an analytic family of 1-cycles from the homotheties. For this purpose, we consider the following graph:

$$\big\{(f, \alpha) \in L|_{D_e} \times \mathbb{C} \;\big|\; f^k = \alpha \cdot s_e\big\}.$$

This graph induced the analytic family $\{\widetilde{C}^e_\alpha\}_{\alpha\in\mathbb{C}}$ on $L|_{D_e}$ with the following properties:

(1) $\widetilde{C}^e_0$ is equal to kD_e as a cycle.

(2) The fibre of $L|_{D_e}$ at e is covered by $\bigcup_{\alpha\in\mathbb{C}} \widetilde{C}^e_\alpha$.

Now we have the natural injection $0 \to L \to \pi^*E$ and the natural projection $\pi^*E \to E$. Then the composition $L \to E$ is equal to the blow-up of the total space of E along the image of the zero-section. Thus by taking the push-forward of the cycle above, we have an analytic family on $\{C^e_\alpha\}_{\alpha\in\mathbb{C}}$ of 1-cycles on E with the following properties:

(1) C^e_0 is equal to kC as a cycle.

(2) The line passing through e in E_c is covered by $\bigcup_{\alpha\in\mathbb{C}} C^e_\alpha$.

By varying $e \in X$, we obtain analytic families $\{C^e_\alpha\}_{\alpha\in\mathbb{C}}$ such that $\{C^e_\alpha\}$ contains some positive multiple of C and

$$E = \bigcup_{e\in X}\bigcup_{\alpha\in\mathbb{C}} C^e_\alpha.$$

Since E is a (Zariski) open set in $X = \mathbb{P}(\mathcal{O}_C \oplus E^*)$, Lemma 4.3.1 yields the conclusion of Theorem 4.1.2.

□

In the proof of Theorem 4.1.2, we have proved the following corollary.

Corollary 4.3.2. *Let E be an ample vector bundle of rank r on a projective curve C. Then there exists a positive integer k_0 and an algebraic family $\{C_t\}_{t\in T}$ of 1-cycles with irreducible T such that*

(1) *$C_{t_0} = k_0C$ for some $t_0 \in T$,*

(2) *$E = \bigcup_{t\in T} C_t$.*

5 Restricted volumes and divisorial Zariski decompositions

5.1 Introduction

Throughout this chapter, let X be a smooth projective variety of dimension n, D a (big) divisor on X and V an irreducible subvariety of dimension d on X, unless otherwise mentioned. Then the restricted volume of D along V is defined to be

$$\mathrm{vol}_{X|V}(D) := \limsup_{k\to\infty} \frac{\dim H^0(X|V, \mathcal{O}_X(kD))}{k^d/d!}.$$

Here we denote by $H^0(X|V, \mathcal{O}_X(kD)) \subseteq H^0(V, \mathcal{O}_V(kD))$ the space of global sections of $\mathcal{O}_V(kD)$ on V that can be extended to X. Roughly speaking, the restricted volume measures the number of sections of $\mathcal{O}_V(kD)$ which can be extended to X. The notion of the restricted volume first appeared in [Tsu06]. The restricted volume has many applications in various situations (see [HM06], [Tak06]). The properties of the restricted volume are studied in [ELMNP09], [BFJ09] and so on.

On the other hand, it is an important problem to determine when D admits a Zariski decomposition. Here a decomposition $D = P + N$ is said to be a Zariski decomposition, if P is a nef $\mathbb{R}$-divisor and N is an effective $\mathbb{R}$-divisor such that the following map is an isomorphism for any positive integer $k > 0$:

$$H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \longrightarrow H^0(X, \mathcal{O}_X(kD)).$$

This map is the natural map induced by the section e_k, where e_k is the standard section of the effective divisor $\lceil kN \rceil$. Here $\lfloor G \rfloor$ (resp. $\lceil G \rceil$) denotes the divisor defined by the round-downs (resp. the round-ups) of the coefficients of an $\mathbb{R}$-divisor G.

When D is an ample divisor, the restricted volume $\mathrm{vol}_{X|V}(D)$ of D along V is equal to the self-intersection number $(D^d \cdot V)$ of D on V. Therefore the restricted volume $\mathrm{vol}_{X|V}(D)$ along V depends only on the first Chern class (the numerical class) of D when D is ample. In general, the restricted volume has the same property if V is not contained in the augmented base locus $\mathbb{B}_+(D)$ of D (see [ELMNP09, Theorem A]). The augmented base locus of D is a subvariety on X which measures how far D is from ample divisors (see [ELMNP06, Section 1] for the precise definition and the properties).

It is natural to ask whether the restricted volume $\mathrm{vol}_{X|V}(D)$ of D depends only on the numerical class of V. In [BFJ09], the question is affirmatively answered when the codimension of V is one. In this chapter, we give a necessary and sufficient condition for D, that the restricted volume $\mathrm{vol}_{X|V}(D)$ of D depends only on the numerical class of V. The condition is related to the existence of a Zariski decomposition of D as follows:

Theorem 5.1.1. *Let D be a big divisor on a smooth projective variety X. Then the following conditions are equivalent.*

(1) *D admits a Zariski decomposition.*

(2) $\mathrm{vol}_{X|V}(D) = \mathrm{vol}_{X|V'}(D)$ *holds for any pair of subvarieties V and V' on X such that $V \equiv V'$ and $V, V' \not\subseteq \mathbb{B}_+(D)$.*

(3) $\mathrm{vol}_{X|C}(D) = \mathrm{vol}_{X|C'}(D)$ *holds for any pair of curves C and C' on X such that $C \equiv C'$ and $C, C' \not\subseteq \mathbb{B}_+(D)$.*

Remark 5.1.2. It is sufficient for the proof of Theorem 5.1.1 to show that condition (1) (resp. (3)) implies condition (2) (resp. (1)) since condition (2) clearly leads to condition (3).

When subvarieties V and V' are numerically equivalent, we write $V \equiv V'$. Condition (2) means that the restricted volume $\mathrm{vol}_{X|V}(D)$ of D depends only on the numerical class of V. Theorem 5.1.1 implies that the restricted volumes along some numerically equivalent subvarieties are different when D does not admit a Zariski decomposition.

When V is the ambient space X, the restricted volume of D is equal to the usual volume $\mathrm{vol}_X(D)$ of D. The usual volume has been studied by several authors. The general theory is presented in details in [Laz]. In his paper [Bou02], Boucksom gave an analytic description of the usual volume with positive curvature currents which represent the first Chern class of D, by using a result of Fujita on the approximation of Zariski decompositions and the singular holomorphic Morse inequalities. In other words, Boucksom expressed the usual volume of D in terms of the first Chern class of D.

The restricted volume $\mathrm{vol}_{X|V}(D)$ along V depends only on the first Chern class $c_1(D)$ of D if V is not contained in the augmented base locus $\mathbb{B}_+(D)$ of D. Then Boucksom's description for the usual volume can be generalized to the restricted volume as follows:

Theorem 5.1.3. *Let D be a big divisor on a smooth projective variety X. Assume that V is not contained in the augmented base locus $\mathbb{B}_+(D)$ of D. Then the restricted volume of D along V satisfies the following equality:*

$$\mathrm{vol}_{X|V}(D) = \sup_{T \in c_1(D)} \int_{V_{\mathrm{reg}}} (T|_{V_{\mathrm{reg}}})^d_{\mathrm{ac}}$$

where T ranges among positive $(1,1)$-currents with analytic singularities in $c_1(D)$ whose singular loci do not contain V.

Here we denote by $T|_{V_{\mathrm{reg}}}$ the restriction of T to the regular locus V_{reg} of V and by $(T|_{V_{\mathrm{reg}}})_{\mathrm{ac}}$ the absolutely continuous part of $T|_{V_{\mathrm{reg}}}$ (see Section 5.2.2 for the precise definition). Theorem 5.1.3 enables us to define the restricted volume of a transcendental class on a compact Kähler manifold in natural way.

Definition 5.1.4. Let W be an irreducible analytic subset of dimension d on a compact Kähler manifold M and let α a class in $H^{1,1}(M, \mathbb{R})$. Assume that W is not contained in the non-Kähler locus $E_{nK}(\alpha)$ of α. Then *the restricted volume* of α along W is defined to be

$$\mathrm{vol}_{M|W}(\alpha) := \sup_{T \in \alpha} \int_{W_{\mathrm{reg}}} (T|_{W_{\mathrm{reg}}})^d_{\mathrm{ac}}$$

where T ranges among positive $(1,1)$-currents with analytic singularities in α whose singular loci do not contain W.

Here the non-Kähler locus is an analytic counterpart of the augmented base locus (see [Bou04, Definition 3.14] for the precise definition of the non-Kähler locus). When α is the first Chern class of some divisor D, the non-Kähler locus $E_{nK}(\alpha)$ coincides with the augmented base locus $\mathbb{B}_+(D)$. For this extended definition, the properties of the usual restricted volume hold. For example, the continuity, log concavity, Fujita's approximations and so on (see Section 5.4.2). Moreover, an analogue of Theorem 5.1.1 holds for the extended definition as follows. The proof gives another proof of Theorem 5.1.1 by using analytic methods (see Section 5.4.3).

Theorem 5.1.5. *Let α be a big class in $H^{1,1}(X, \mathbb{R})$ on a smooth projective variety X. Then the following conditions are equivalent.*

(1) *α admits a Zariski decomposition.*

(2) $\mathrm{vol}_{X|V}(\alpha) = \mathrm{vol}_{X|V'}(\alpha)$ *holds for any pair of subvarieties V and V' on X such that $V \equiv V'$ and $V, V' \not\subseteq E_{nK}(\alpha)$.*

(3) $\mathrm{vol}_{X|C}(\alpha) = \mathrm{vol}_{X|C'}(\alpha)$ *holds for any pair of curves C and C' on X such that $C \equiv C'$ and $C, C' \not\subseteq E_{nK}(\alpha)$.*

Here we say that a big class admits a Zariski decomposition if the positive part of its divisorial Zariski decomposition is nef (see Section 5.2.4). When α is the first Chern class of some divisor D (that is, α is contained in the Néron-Severi space), a Zariski decomposition of α coincides with that of D. However, a class α is not necessarily contained in the Néron-Severi space of X even if X is projective. Therefore Theorem 5.1.5 is essentially stronger statement than Theorem 5.1.1.

5.2 Preliminaries

In this section, we prepare for the proofs. The propositions in this section may be known facts. However we give comments or references for the readers' convenience. Throughout this section, M denotes a compact Kähler manifold of dimension n.

5.2.1 Multiplier ideal sheaves and Skoda's lemma

In this chapter, we often use the description of the restricted volume with the multiplier ideal sheaf which was proved in [ELMNP09]. We denote by $\mathcal{I}(T)$ the multiplier ideal sheaf associated to a d-closed $(1,1)$-current T. That is, $\mathcal{I}(T)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-2\varphi}$ is locally integrable, where φ is a local potential function of T. (Note that this definition does not depend on the choice of a local potential function.) See [DEL00], [Dem] for more details. Skoda's Lemma gives a relation between the Lelong number of T and the multiplier ideal sheaf $\mathcal{I}(T)$. Here the Lelong number $\nu(T, x)$ of an almost positive $(1,1)$-current $T = dd^c\varphi$ at x is defined by $\nu(T, x) := \liminf_{z \to x} \dfrac{\varphi(z)}{\log|z - x|}$ where z is a local coordinate centered at x.

Lemma 5.2.1. ([Sko72]). *Let φ be a potential function of an almost positive current T.*

(a) *If* $\nu(T,x) < 1$, *then* $e^{-2\varphi}$ *is integrable in a neighborhood of* x. *In other words, the stalk* $\mathcal{I}(T)_x$ *of* $\mathcal{I}(T)$ *at* x *is equal to the stalk* $\mathcal{O}_{M,x}$ *of the structure sheaf of* M *at* x.

(b) *If* $\nu(T,x) \geq n+s$ *for some positive integer* s, *then* $e^{-2\varphi} \geq C|z-x|^{-2n-2s}$ *in a neighborhood of* x. *In particular, we have* $\mathcal{I}(T)_x \subseteq \mathfrak{m}_{M,x}^{s+1}$, *where* $\mathfrak{m}_{M,x}$ *is the maximal ideal of* $\mathcal{O}_{M,x}$.

5.2.2 Lebesgue decompositions

A positive current T can be locally regarded as a $(1,1)$-form with measure coefficients. Thus it admits the Lebesgue decomposition into the absolutely continuous part and the singular part with respect to the Lebesgue measure. Therefore we obtain the decomposition $T = T_{\mathrm{ac}} + T_{\mathrm{sing}}$, where T_{ac} (resp. T_{sing}) is the absolutely continuous part (resp. the singular part) of T. This decomposition is globally determined thanks to the uniqueness of the Lebesgue decomposition. Now T_{ac} is considered as a $(1,1)$-form with L^1_{loc}-function coefficients. Thus we can define the product T_{ac}^k of T_{ac} almost everywhere. We have $T_{\mathrm{ac}} \geq \gamma$ if $T \geq \gamma$ for some smooth $(1,1)$-form γ. In particular, the absolutely continuous part T_{ac} is positive if T is a positive $(1,1)$-current (see [Bou02, Section 2.3] for more details).

5.2.3 Approximations of currents

Let $T = \theta + dd^c\varphi$ be a $(1,1)$-current in a class $\alpha \in H^{1,1}(M,\mathbb{R})$, where θ is a smooth $(1,1)$-form in α and φ is an L^1-function on M. We assume that $T \geq \gamma$ holds for a smooth form γ. Fix a Kähler form ω on M. Then we can approximate T by smooth forms in the following sense:

Theorem 5.2.2. ([Dem82, THÉORÈME 9.1]). *There exists a decreasing sequence of smooth functions* φ_k *converging to* φ *such that if we set* $T_k = \theta + dd^c\varphi_k \in \alpha$, *we have*

(a) $T_k \longrightarrow T$ *weakly and* $T_k \longrightarrow T_{\mathrm{ac}}$ *almost everywhere on* M.

(b) $T_k \geq \gamma - C\lambda_k\omega$, *where* C *is a positive constant depending only on* (M,ω), *and* $\{\lambda_k\}_{k=1}^{\infty}$ *is a decreasing sequence of continuous functions such that* $\lambda_k(x) \searrow \nu(T,x)$ *for all* $x \in M$.

Roughly speaking, Theorem 5.2.2 says that it is possible to smooth a given current T insides the class α, but only with the loss of positivity controlled by the Lelong numbers of T. By the proof of Theorem 5.2.2 in [Dem82], we may add the following property to Theorem 5.2.2. (Recall that T_k is obtained from T by convolution with a regularized kernel.)

(c) *If T is smooth on a given open set U of M, then T_k converges to T in $C^\infty(U)$.*

The following theorem asserts that it is possible to approximate a given current with currents with analytic singularities. There is a loss of positivity but it is arbitrary small.

Theorem 5.2.3. ([Dem92], [Bou02, Theorem 2.4]). *There exists a sequence of functions φ_k with analytic singularities converging to φ such that if we set $T_k = \theta + dd^c\varphi_k \in \alpha$, we have*

(a′) *$T_k \longrightarrow T$ weakly and $T_{k,\mathrm{ac}} \longrightarrow T_{\mathrm{ac}}$ almost everywhere.*

(b′) *$T_k \geq \gamma - \varepsilon_k\omega$, where ε_k is a positive number converging to zero.*

(c′) *The Lelong number $\nu(T_k, x)$ increases to $\nu(T, x)$ uniformly with respect to $x \in M$.*

In the proof of [Bou02, Theorem 2.4], the convergence $T_{k,\mathrm{ac}} \longrightarrow T_{\mathrm{ac}}$ in (a′) was obtained from only property (a) in Theorem 5.2.2. Therefore we may add the following property (d′) thanks to property (c).

(d′) *If T is smooth on a given open set U of M, then $T_{k,\mathrm{ac}}$ converges to T_{ac} in $C^\infty(U)$.*

It yields the following corollary.

Corollary 5.2.4. *Let W be an irreducible analytic subset on M. Assume that $T|_{W_{\mathrm{reg}}}$ is smooth except some analytic set on W_{reg}. Then T_k in Theorem 5.2.3 satisfies the following property:*

$$\left(T_k|_{W_{\mathrm{reg}}}\right)_{\mathrm{ac}} \longrightarrow \left(T|_{W_{\mathrm{reg}}}\right)_{\mathrm{ac}} \qquad \textit{almost everywhere on } W_{\mathrm{reg}}.$$

5.2.4 Divisorial Zariski decompositions

In this subsection, we confirm the definition of the divisorial Zariski decomposition of a class. The divisorial Zariski decomposition of a big divisor coincides with its σ-decomposition. The divisorial Zariski decomposition is studied in [Bou04] and the σ-decomposition is studied in [Nak].

Let α be a pseudo-effective class in $H^{1,1}(M, \mathbb{R})$. Then the effective $\mathbb{R}$-divisor N is defined to be

$$N := \sum_{F:\text{prime div}} \nu(\alpha, F)F.$$

Here $\nu(\alpha, F)$ denotes the Lelong number along a prime divisor F which is defined by $\inf_{x\in F} \nu(\alpha, x)$. The class $\{N\}$ of N is called the negative part of the divisorial Zariski decomposition of α. The class P defined by $P := \alpha - \{N\}$ is called the positive part. Then the decomposition $\alpha = P + \{N\}$ is said to be the divisorial Zariski decomposition of α. In general, the positive part P is nef in codimension one (that is, the codimension of its non-nef locus is strictly larger than one). We say that α admits a Zariski decomposition if the positive part P is nef. If α is the first Chern class of a big divisor, this definition coincides with that of the divisor (which was described in Section 5.1). For example, if M is surface, any big class admits a Zariski decomposition (see [Bou04, section 4]). By the construction of N, positive currents in α and positive currents in P are identified by the correspondence $T \in \alpha \longmapsto T - [N] \in P$.

5.3 Restricted volumes and Zariski decompositions

5.3.1 Positive parts and restricted volumes

The main aim in this section is to prove Theorem 5.1.1. Throughout this section, let D be a big divisor on a smooth projective variety X of dimension n. Then we consider the divisorial Zariski decomposition $D = P + N$ of D. We first establish Proposition 5.3.1 for the proof of Theorem 5.1.1. This proposition asserts that the restricted volume of D can be computed with the positive part P.

Proposition 5.3.1. *Let W be an irreducible subvariety of dimension d on X. Assume that W is not contained in the augmented base locus $\mathbb{B}_+(D)$ of D. Then the equality $\mathrm{vol}_{X|W}(D) = \mathrm{vol}_{X|W}(P)$ holds.*

Remark 5.3.2. In general, P is an $\mathbb{R}$-divisor. Then $\mathrm{vol}_{X|W}(P)$ can be defined by the limit of the restricted volumes of $\mathbb{Q}$-divisors which converge to P in the Néron-Severi space. Thanks to the continuity of the restricted volume (see [ELMNP09, Theorem 5.2]), $\mathrm{vol}_{X|W}(P)$ does not depend on the choice of $\mathbb{Q}$-divisors which converge to P.

Proof. Since D is a big divisor, there is an effective $\mathbb{Q}$-divisor which is $\mathbb{Q}$-linearly equivalent to D. Therefore we may assume that D is effective. (Recall that the restricted volume has the homogeneity.) Moreover we may assume that the support of D does not contain W, since W is not contained in $\mathbb{B}_+(D)$. In particular, W is not contained in the support of N nor in that of P.

Since $D = P + N$ is a divisorial Zariski decomposition, there exists the natural isomorphism $H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \cong H^0(X, \mathcal{O}_X(kD))$ induced by the section e_k for a positive integer $k > 0$, where e_k is the standard section of the effective divisor $\lceil kN \rceil$ (see [Bou04, Theorem 5.5] or [Nak]). Then we consider the following commutative diagram:

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) & \xrightarrow{\cdot e_k} & H^0(X, \mathcal{O}_X(kD)) \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ H^0(W, \mathcal{O}_W(\lfloor kP \rfloor)) & \xrightarrow{\cdot e_k|_W} & H^0(W, \mathcal{O}_W(kD)), \end{array}$$

where f and g are the restriction maps. The diagram induces the map $\mathrm{Im}(f) \xrightarrow{\cdot e_k|_W} \mathrm{Im}(g)$. This map is surjective since the horizontal map above in the diagram is an isomorphism. Now $e_k|_W$ is a nonzero section since W is not contained in the support of N. It implies that the map below in the diagram is injective. Thus, the map $\mathrm{Im}(f) \to \mathrm{Im}(g)$ is an isomorphism. It yields

$$\mathrm{vol}_{X|W}(D) = \limsup_{k \to \infty} \frac{h^0(X|W, \mathcal{O}_X(\lfloor kP \rfloor))}{k^d/d!}. \tag{5.1}$$

When P is a $\mathbb{Q}$-divisor, Proposition 5.3.1 follows from this equality and the homogeneity of the restricted volume. However, we need the following argument when P is an $\mathbb{R}$-divisor.

Let $P = \sum_i a_i D_i$ be the irreducible decomposition of P. Note that a_i is positive for any i since P is effective. We want to approximate the $\mathbb{R}$-divisor P with suitable $\mathbb{Q}$-divisors. For this purpose, we define a $\mathbb{Q}$-divisor P_ℓ by $P_\ell := \ell^{-1}\lfloor \ell P \rfloor$. Then, from the definition of the round-down, we obtain $\lfloor \ell P \rfloor \leq \ell P \leq \lfloor \ell P \rfloor + F$ for any positive integer ℓ, where F is the effective divisor defined by $F := \sum_i D_i$. These inequalities imply that P_ℓ converges to P in the Néron-Severi space. For a sufficiently large ℓ, $\mathbb{B}_+(P_\ell)$ does not contain W (see Proposition 1.6.3 (1)). Therefore we have

$$\mathrm{vol}_{X|W}(P) = \lim_{\ell \to \infty} \mathrm{vol}_{X|W}(P_\ell)$$

from the continuity of the restricted volume.

Now we prove the inequality $\mathrm{vol}_{X|W}(D) \geq \mathrm{vol}_{X|W}(P_\ell)$ for any ℓ in order to show the inequality $\mathrm{vol}_{X|W}(D) \geq \mathrm{vol}_{X|W}(P)$. By the homogeneity of the restricted volume and equality (5.1), we obtain the following equalities:

$$\operatorname{vol}_{X|W}(D) = \limsup_{k\to\infty} \frac{h^0(X|W, \mathcal{O}_X(\lfloor kP \rfloor))}{k^d/d!} = \limsup_{k\to\infty} \frac{h^0(X|W, \mathcal{O}_X(\lfloor \ell k P \rfloor))}{\ell^d k^d/d!},$$

$$\operatorname{vol}_{X|W}(P_\ell) = \limsup_{k\to\infty} \frac{h^0(X|W, \mathcal{O}_X(k\lfloor \ell P \rfloor))}{\ell^d k^d/d!}.$$

Note that $\lfloor \ell k P \rfloor - k\lfloor \ell P \rfloor$ is an effective divisor and its support is contained in the support of D. Since W is not contained in the support of D, we have

$$\frac{h^0(X|W, \mathcal{O}_X(\lfloor \ell k P \rfloor))}{\ell^d k^d/d!} \geq \frac{h^0(X|W, \mathcal{O}_X(k\lfloor \ell P \rfloor))}{\ell^d k^d/d!}.$$

It implies the inequality $\operatorname{vol}_{X|W}(D) \geq \operatorname{vol}_{X|W}(P_\ell)$ for any ℓ. Therefore we obtain $\operatorname{vol}_{X|W}(D) \geq \operatorname{vol}_{X|W}(P)$.

Finally we show the converse inequality $\operatorname{vol}_{X|W}(D) \leq \operatorname{vol}_{X|W}(P)$. For this purpose, we shall estimate $\lfloor \ell k P \rfloor - k\lfloor \ell P \rfloor$ from above. By a simple computation, we obtain

$$\begin{aligned} \lfloor \ell k P \rfloor - k\lfloor \ell P \rfloor &= \sum_i \left(\lfloor \ell k a_i \rfloor - k\lfloor \ell a_i \rfloor\right) D_i \\ &\leq \sum_i \left(k(\ell a_i - \lfloor \ell a_i \rfloor)\right) D_i \\ &\leq \sum_i k D_i = kF. \end{aligned}$$

Since the support of F does not contain W, the inequality above yields

$$\begin{aligned} \operatorname{vol}_{X|W}(D) &= \limsup_{k\to\infty} \frac{h^0(X|W, \mathcal{O}_X(\lfloor \ell k P \rfloor))}{\ell^d k^d/d!} \\ &\leq \limsup_{k\to\infty} \frac{h^0\big(X|W, \mathcal{O}_X\big(k(\lfloor \ell P \rfloor + F)\big)\big)}{\ell^d k^d/d!} \\ &= \frac{1}{\ell^d} \operatorname{vol}_{X|W}(\lfloor \ell P \rfloor + F). \end{aligned}$$

Now we have $\operatorname{vol}_{X|W}(\lfloor \ell P \rfloor + F)\ell^{-d} = \operatorname{vol}_{X|W}(P_\ell + \ell^{-1}F)$ from the homogeneity of the restricted volume. Further, $P_\ell + \ell^{-1}F$ converges to P in the Néron-Severi space when ℓ tends to infinity. The continuity of the restricted volume implies that $\operatorname{vol}_{X|W}(\lfloor \ell P \rfloor + F)\ell^{-d}$ converges to $\operatorname{vol}_{X|W}(P)$. Hence we obtain the converse inequality $\operatorname{vol}_{X|W}(D) \leq \operatorname{vol}_{X|W}(P)$. □

Corollary 5.3.3. *Let W be an irreducible subvariety of dimension d on X Assume that W is not contained in $\mathbb{B}_+(D)$. If D admits a Zariski decomposition (that is, the positive part P of its divisorial Zariski decomposition is nef), then the equality* $\mathrm{vol}_{X|W}(D) = (W \cdot P^d)$ *holds.*

Proof. By Proposition 5.3.1, we have $\mathrm{vol}_{X|W}(D) = \mathrm{vol}_{X|W}(P)$. Since P is nef, there exist ample $\mathbb{Q}$-divisors A_k which converge to P in the Néron-Severi space. Since A_k is ample, the restricted volume $\mathrm{vol}_{X|W}(A_k)$ of A_k along W is equal to the self-intersection number $(W \cdot A_k^d)$ on W. By the continuity of the restricted volume and the self-intersection number, we obtain

$$\begin{aligned}\mathrm{vol}_{X|W}(D) &= \mathrm{vol}_{X|W}(P)\\ &= \lim_{k\to\infty} \mathrm{vol}_{X|W}(A_k)\\ &= \lim_{k\to\infty} (W \cdot A_k^d) = (W \cdot P^d).\end{aligned}$$

□

5.3.2 Proof of Theorem 5.1.1

This subsection is devoted to complete the proof of Theorem 5.1.1. First, we shall see that condition (1) implies condition (2). We assume that D admits a Zariski decomposition $D = P + N$ (that is, the positive part P of its divisorial Zariski decomposition is nef). Take subvarieties V and V' on X such that $V \equiv V'$ and $V, V' \not\subseteq \mathbb{B}_+(D)$. Then the restricted volumes of D can be computed by the self-intersection number of the positive part P from Corollary 5.3.3. That is, $\mathrm{vol}_{X|V}(D) = (V \cdot P^d)$ and $\mathrm{vol}_{X|V'}(D) = (V' \cdot P^d)$ hold. Since V and V' are numerically equivalent, $(V \cdot P^d)$ coincides with $(V' \cdot P^d)$. Hence the equality $\mathrm{vol}_{X|V}(D) = \mathrm{vol}_{X|V'}(D)$ holds.

We shall show that condition (3) implies condition (1). Let $D = P + N$ be a divisorial Zariski decomposition of a big divisor D. We assume that P is not nef for a contradiction. Since P is not nef, the restricted base locus $\mathbb{B}_-(P)$ of P is not empty. From this condition, we want to construct curves C, C' such that the restricted volume $\mathrm{vol}_{X|C}(P)$ along C is different from the restricted volume $\mathrm{vol}_{X|C'}(P)$ along C'.

For a construction of such curves, we take a very ample divisor A on X and a point x_0 in $\mathbb{B}_-(P)$. Then there are smooth curves C and C' with the following properties:

(1) C and C' are not contained in the augmented base locus $\mathbb{B}_+(D)$.
(2) C passes through $x_0 \in \mathbb{B}_-(P)$.

(3) C' does not intersect with the restricted base locus $\mathbb{B}_{-}(P)$.

(4) C and C' are complete intersections of members of the complete linear system of A.

We can easily see that there exist such curves: A general member of $|A|_{x_0}$ is irreducible and smooth, where $|A|_{x_0}$ is the linear system passing through x_0 in the complete linear system $|A|$ of A (see [Zha09, Theorem 2.5]). Then by taking a complete intersection of general members of $|A|_{x_0}$, we can take a curve C with properties (1), (2), (4). Now we construct a curve C' with properties (1), (3), (4). By the construction of the divisorial Zariski decomposition, the restricted base locus $\mathbb{B}_{-}(P)$ of the positive part P is the countable union of subvarieties of codimension ≥ 2. Thus the codimension of the intersection of $\mathbb{B}_{-}(P)$ and H is greater than or equal to 3 for a "very" general member H of $|A|$. It implies that a curve which is a complete intersection of very general members of $|A|$ does not intersect with $\mathbb{B}_{-}(P)$.

Now C and C' are numerically equivalent since C and C' are complete intersections of members of the same complete linear system. Thus, it follows the equality $\mathrm{vol}_{X|C}(D) = \mathrm{vol}_{X|C'}(D)$ from condition (3) in Theorem 5.1.1. By Proposition 5.3.1, we have the equality $\mathrm{vol}_{X|C}(P) = \mathrm{vol}_{X|C'}(P)$. It is sufficient for a contradiction to prove the following lemma. In fact, $(C \cdot P)$ is equal to $(C' \cdot P)$ since C and C' are numerically equivalent. Therefore the following lemma implies $\mathrm{vol}_{X|C}(D) < \mathrm{vol}_{X|C'}(D)$. It is a contradiction.

Lemma 5.3.4. *In the situation above, the followings hold.*

(A) $\mathrm{vol}_{X|C'}(P) = (C' \cdot P)$, (B) $\mathrm{vol}_{X|C}(P) < (C \cdot P)$.

Proof. First we show equality (A). In general, P is an $\mathbb{R}$-divisor. Thus we should approximate P with $\mathbb{Q}$-divisors. We take ample $\mathbb{R}$-divisors A_k with the following properties:

(i) $P + A_k$ is a $\mathbb{Q}$-divisor for a positive integer $k > 0$.

(ii) $P + A_k$ converges to P in the Néron-Severi space.

Since A_k is ample, $\mathbb{B}_{+}(P + A_k) \subseteq \mathbb{B}_{+}(P)$ for any k. Thus, it follows that C' is not contained in $\mathbb{B}_{+}(P + A_k)$ from property (1). We take a positive integer a_k such that $a_k(P + A_k)$ is a $\mathbb{Z}$-divisor. Then the homogeneity and the description of the restricted volume with the asymptotic multiplier ideal sheaf (which was proved in [ELMNP09, Theorem 2.13]) yields

$$\mathrm{vol}_{X|C'}(P+A_k) = \limsup_{\ell \to \infty} \frac{1}{\ell a_k} h^0\big(C', \mathcal{O}_{C'}\left(\ell a_k(P+A_k)\right) \otimes \mathcal{J}\left(\|\ell a_k(P+A_k)\|\right)|_{C'}\big).$$

Here $\mathcal{J}(\|L\|)$ denotes the asymptotic multiplier ideal sheaf associated to a divisor L (see [DEL00] for the definition). Further $\mathcal{I}|_V$ denotes the ideal $\mathcal{I} \cdot \mathcal{O}_V$ for a ideal $\mathcal{I}$ and a subvariety V on X.

We shall investigate the asymptotic multiplier ideal sheaf $\mathcal{J}(\|\ell a_k(P + A_k)\|)$ along C'. Let $T_{\min,k}$ be a current with minimal singularities in the first Chern class of $a_k(P + A_k)$. Then the restricted base locus of $a_k(P + A_k)$ is equals to the set $\{x \in X \mid \nu(T_{\min,k}, x) > 0\}$ by Proposition 1.6.3 (3). On the other hand, the restricted base locus of $P + A_k$ is contained in the restricted base locus of P, since A_k is an ample divisor. Further, C' does not intersect with the restricted base locus of P from property (3). Therefore the Lelong number of $T_{\min,k}$ at $x \in C'$ is zero. Thus we have $\mathcal{J}(\ell T_{\min,k})|_{C'} = \mathcal{O}_{C'}$ for every $\ell > 0$ by Skoda's Lemma. Thus, from Theorem 5.4.3 (which is proved in Section 5.4), we have

$$\mathrm{vol}_{X|C'}(P + A_k) = \limsup_{\ell \to \infty} \frac{h^0(C', \mathcal{O}_{C'}(\ell a_k(P + A_k))}{\ell a_k}.$$

Since C' is not contained in $\mathbb{B}_+(P)$, $(P + A_k)$ is an ample divisor on C'. By the Riemann-Roch formula, we obtain $\mathrm{vol}_{X|C'}(P + A_k) = \big((P + A_k) \cdot C'\big)$. It follows $\mathrm{vol}_{X|C'}(P) = (P \cdot C')$ from the continuity of the restricted volume.

Finally we show inequality (B). Consider the following commutative diagram:

$$\begin{array}{ccc}
H^0(X, \mathcal{O}_X(\lfloor kP \rfloor) \otimes \mathcal{J}(\|\lfloor kP \rfloor\|)) & \longrightarrow & H^0(C, \mathcal{O}_C(\lfloor kP \rfloor) \otimes \mathcal{J}(\|\lfloor kP \rfloor\|)|_C) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) & \longrightarrow & H^0(C, \mathcal{O}_C(\lfloor kP \rfloor)).
\end{array}$$

The vertical map on the left hand is an isomorphism (see [DEL00, Theorem 1.8]). Thus the vertical map on the right hand is surjective onto the image of the horizontal map. It yields

$$\limsup_{k \to \infty} \frac{h^0(X|C, \mathcal{O}_X(\lfloor kP \rfloor))}{k} \leq \limsup_{k \to \infty} \frac{h^0(C, \mathcal{O}_C(\lfloor kP \rfloor) \otimes \mathcal{J}(\|\lfloor kP \rfloor\|)|_C)}{k}.$$

We have already proved that the left hand is equals to $\mathrm{vol}_{X|C}(P)$ in the proof of Proposition 5.3.1. For the proof of inequality (B), we need to estimate the right hand from above. For this purpose, we shall investigate the asymptotic multiplier ideal sheaf $\mathcal{J}(\|\lfloor kP \rfloor\|)$ along C.

Take a positive current S_k in the first Chern class of $\lfloor kP \rfloor$ such that $\mathcal{J}(S_k) = \mathcal{J}(\|\lfloor kP \rfloor\|)$. Let $P = \sum a_i D_i$ be an irreducible decomposition of P

and let F the divisor which is defined by $F := \sum_i D_i$. Then $kP - \lfloor kP \rfloor \leq F$ for any positive integer k. Thus we obtain

$$\nu(kT_{\min}, x) \leq \nu(S_k, x) + \nu([F], x)$$

by the definition of a current with minimal singularities. Here $[F]$ denotes the positive current defined by the effective divisor F and $T_{\min}$ denotes a current with minimal singular in $c_1(P)$. Since $\nu(T_{\min}, x_0)$ is positive by property (2), we can take a positive rational number p/q which is smaller than $\nu(T_{\min}, x_0)$. For simplicity, we put $c := \nu([F], x_0)$. Then we have $kp - c < \nu(S_{kq}, x_0)$. Skoda's Lemma implies $\mathcal{J}(\| \lfloor kP \rfloor \|)_{x_0} \subseteq \mathfrak{m}_{X,x_0}^{kp-c-n+1}$, where $\mathfrak{m}_{X,x_0}$ is the maximal ideal in $\mathcal{O}_{X,x_0}$. Thus we obtain

$$\begin{aligned}
\limsup_{k\to\infty} & \frac{h^0\big(C, \mathcal{O}_C(\lfloor kP \rfloor) \otimes \mathcal{J}(\| \lfloor kP \rfloor \|)|_C\big)}{k} \\
&\leq \limsup_{k\to\infty} \frac{h^0\big(C, \mathcal{O}_C(\lfloor kqP \rfloor) \otimes \mathfrak{m}_{X,x_0}^{kp-c-n+1}|_C\big)}{kq} \\
&\leq \limsup_{k\to\infty} \frac{h^0\big(C, \mathcal{O}_C(\lfloor kqP \rfloor) \otimes \mathfrak{m}_{C,x_0}^{kp-c-n+1}\big)}{kq} \\
&= \limsup_{k\to\infty} \frac{h^0\big(C, \mathcal{O}_C(\lfloor kqP \rfloor - (kp-c-n+1)[x_0])\big)}{kq},
\end{aligned}$$

where $[x_0]$ is the divisor on C defined by x_0. Now $(\lfloor qkP \rfloor - (kp-c-n+1)[x_0])$ is nef on C. Thus the dimension of its first cohomology group converges to zero when k tends to infinity. By using the Riemann-Roch formula again, we obtain

$$\begin{aligned}
\mathrm{vol}_{X|C}(P) &\leq \limsup_{k\to\infty} \frac{h^0\big(C, \mathcal{O}_C(\lfloor kqP \rfloor - (kp-c-n+1)[x_0])\big)}{kq} \\
&= \limsup_{k\to\infty} \frac{\big(C \cdot (\lfloor kqP \rfloor - (kp-c-n+1)[x_0])\big)}{kq} \\
&\leq (C \cdot P) - \frac{p}{q} < (C \cdot P).
\end{aligned}$$

□

In the proof of Lemma 5.3.4, we have already proved the following corollary.

Corollary 5.3.5. *Let C be a smooth curve on X. Assume that C is not contained in $\mathbb{B}_+(D)$. Then we have*

$$\mathrm{vol}_{X|C}(D) \leq (C \cdot D) - \sum_{x \in C \cap \mathbb{B}_-(D)} \nu(T_{\min}, x).$$

5.4 An analytic description of restricted volumes with positive curvature currents

5.4.1 Proof of Theorem 5.1.3

The main aim of this subsection is to prove Theorem 5.1.3. Before the proof of Theorem 5.1.3, we need to show that the integral in Theorem 5.1.3 is always finite.

Proposition 5.4.1. *Let W be an irreducible analytic subset of dimension d on a compact Kähler manifold M and T a positive d-closed $(1,1)$-current on M. Assume that the polar set of a potential function of T does not contain W. Then the integral $\int_{W_{\mathrm{reg}}} (T|_{W_{\mathrm{reg}}})^d_{\mathrm{ac}}$ is finite.*

Proof. In [Bou02, Lemma 2.11], it has been proved that $\int_W S^d_{\mathrm{ac}}$ is finite for a positive d-closed current S on W when W is non-singular. Since T is a positive d-closed current on M, the restriction $T|_{W_{\mathrm{reg}}}$ is also a positive d-closed current. Thus, Proposition 5.4.1 holds when W is non-singular. It is enough to consider the case where W has singularities. Then we take an embedded resolution $\mu : \widetilde{W} \subseteq \widetilde{M} \longrightarrow W \subseteq M$ of $W \subseteq M$. That is, μ is a modification from a compact complex manifold $\widetilde{M}$ to M and its restriction to W gives a resolution of singularities of W. Then the following lemma assures that Proposition 5.4.1 holds even if W has singularities. In fact, we have

$$\int_{\widetilde{W}} ((\mu^* T)|_{\widetilde{W}})^d_{\mathrm{ac}} = \int_{W_{\mathrm{reg}}} (T|_{W_{\mathrm{reg}}})^d_{\mathrm{ac}}$$

by this lemma. The left hand is finite since $\widetilde{W}$ is non-singular. Thus the right hand is also finite. $\square$

Lemma 5.4.2. *Let $\mu : \widetilde{W} \subseteq \widetilde{M} \longrightarrow W \subseteq M$ be an embedded resolution of $W \subseteq M$. In the assumption of Proposition 5.4.1, we have*

$$\int_{W_{\mathrm{reg}}} (T|_{W_{\mathrm{reg}}})^d_{\mathrm{ac}} = \int_{\widetilde{W}} ((\mu^* T)|_{\widetilde{W}})^d_{\mathrm{ac}}.$$

Proof. The map $\widetilde{W} \xrightarrow{\mu} W$ is a modification. Therefore $(\mu^*T)|_{\widetilde{W}}$ is identified with $T|_{W_{\mathrm{reg}}}$ by μ on some Zariski open set. Now $\big((\mu^*T)|_{\widetilde{W}}\big)_{\mathrm{ac}}$ and $\big(T|_{W_{\mathrm{reg}}}\big)_{\mathrm{ac}}$ are $(1,1)$-forms with L^1-functions as coefficients. Since a Zariski closed set is of measure zero with respect to the Lebesgue measure, we obtain $\int_{W_{\mathrm{reg}}} (T|_{W_{\mathrm{reg}}})^d_{\mathrm{ac}} = \int_{\widetilde{W}} \big((\mu^*T)|_{\widetilde{W}}\big)^d_{\mathrm{ac}}$. □

The rest of this subsection is devoted to prove Theorem 5.1.3.

Proof of Theorem 5.1.3.

(Step1) In this step, we prove the inequality $\geq$ in Theorem 5.1.3 by using the singular holomorphic Morse inequalities (see [Bon98]) and Proposition 5.4.3. Proposition 5.4.3 is proved at the end of this subsection. Let T be a positive d-closed $(1,1)$-current with analytic singularities in the first Chern class $c_1(D)$ whose singular locus does not contain V.

First, we consider the case where V is non-singular. Then we obtain the following inequality:

$$\begin{aligned}\mathrm{vol}_{X|V}(D) &= \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT_{\min})|_V\big)}{k^d/d!}\\ &\geq \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT)|_V\big)}{k^d/d!}\\ &\geq \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT|_V)\big)}{k^d/d!}.\end{aligned}$$

Here $T_{\min}$ denotes a current with minimal singularities in $c_1(D)$. The first equality follows from Proposition 5.4.3 and the second inequality follows from the restriction formula (see [DEL00, Corollary 1.3] for the restriction formula). By using the singular holomorphic Morse inequality, we have

$$\begin{aligned}\mathrm{vol}_{X|V}(D) &\geq \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT|_V)\big)}{k^d/d!}\\ &\geq \int_V (T|_V)^d_{\mathrm{ac}}.\end{aligned}$$

Therefore the inequality $\geq$ in Theorem 5.1.3 holds when V is non-singular.

Now we consider the case where V has singularities. Then we take an embedded resolution $\mu : \widetilde{V} \subseteq \widetilde{X} \longrightarrow V \subseteq X$. The augmented base locus of the pull back μ^*D of D does not contain $\widetilde{V}$, since $\mu : \widetilde{V} \longrightarrow V$ is a modification. By applying the singular holomorphic Morse inequality and restriction formula to μ^*D, $\widetilde{V}$ and μ^*T again, we obtain

$$\mathrm{vol}_{\widetilde{X}|\widetilde{V}}(\mu^*D) \geq \int_{\widetilde{V}} ((\mu^*T)\ |_{\widetilde{V}})^d_{\mathrm{ac}}.$$

By Lemma 5.4.2, we obtain $\int_V (T|_{V_{\rm reg}})^d_{\rm ac} = \int_{\widetilde{V}}((\mu^*T)|_{\widetilde{V}})^d_{\rm ac}$. On the other hand, we have $\mathrm{vol}_{X|V}(D) = \mathrm{vol}_{\widetilde{X}|\widetilde{V}}(\mu^*D)$ from [ELMNP09, Lemma 6.7]. Thus the inequality $\geq$ in Theorem 5.1.3 holds even if V has singularities.

(Step2) In this step, we shall show the converse inequality $\leq$ by applying Fujita's approximation theorem for the restricted volume (which is proved in [ELMNP09, Proposition 2.11]). By applying [ELMNP09, Proposition 2.11], for an arbitrary number $\varepsilon > 0$, we can find a modification $\pi_\varepsilon : X_\varepsilon \longrightarrow X$ and the expression ${\pi_\varepsilon}^*D = A_\varepsilon + E_\varepsilon$ such that $({A_\varepsilon}^d \cdot V_\varepsilon) \geq \mathrm{vol}_{X|V}(D) - \varepsilon$. Here A_ε is an ample $\mathbb{Q}$-divisor and E_ε is an effective $\mathbb{Q}$-divisor whose support does not contain the strict transformation V_ε of V.

Let ω_ε be a Kähler form on X_ε in the first Chern class of A_ε. Since the support of E_ε does not contain V_ε, we may restrict $[E_\varepsilon]$ to V_ε, where $[E_\varepsilon]$ denotes the current defined by the effective divisor E_ε. Then we obtain

$$\begin{aligned}
({A_\varepsilon}^d \cdot V_\varepsilon) &= \int_{V_\varepsilon} (\omega_\varepsilon|_{V_\varepsilon})^d \\
&= \int_{V_\varepsilon} \big((\omega_\varepsilon + [E_\varepsilon])|_{V_\varepsilon}\big)^d_{\rm ac} \\
&= \int_{V_{\rm reg}} \big\{\big(\pi_{\varepsilon *}(\omega_\varepsilon + [E_\varepsilon])\big)|_{V_{\rm reg}}\big\}^d_{\rm ac}.
\end{aligned}$$

The third equality follows from the same argument as the proof of Lemma 5.4.2.

Since π_ε is a modification, its push-forward $\pi_{\varepsilon *}(\omega_\varepsilon + [E_\varepsilon])$ is a positive current in the Chern class $c_1(D)$. However the push-forward may not have analytic singularities. For the proof, we need to approximate the push-forward by positive currents with analytic singularities. When we approximate a given current by Theorem 5.2.3, the approximation sequence may lose positivity. Now $(\omega_\varepsilon + [E_\varepsilon])$ is a Kähler current but the push-forward may not be a Kähler current. For that reason, we need to consider a Kähler current before we apply Theorem 5.2.3.

For simplicity, we put $T_\varepsilon := \pi_{\varepsilon *}(\omega_\varepsilon + [E_\varepsilon])$. Since V is not contained in the augmented base locus $\mathbb{B}_+(D)$, there is a Kähler current S in $c_1(D)$ with analytic singularities whose singular locus does not contain V. By Fatou's Lemma, we obtain

$$
\begin{aligned}
\mathrm{vol}_{X|V}(D) - \varepsilon &\leq ({A_\varepsilon}^d \cdot V_\varepsilon) \\
&\leq \int_{V_{\mathrm{reg}}} \liminf_{\delta \to 0} \left\{(1-\delta)(T_\varepsilon|_{V_{\mathrm{reg}}}) + \delta(S|_{V_{\mathrm{reg}}})\right\}_{\mathrm{ac}}^d \\
&\leq \liminf_{\delta \to 0} \int_{V_{\mathrm{reg}}} \left\{(1-\delta)(T_\varepsilon|_{V_{\mathrm{reg}}}) + \delta(S|_{V_{\mathrm{reg}}})\right\}_{\mathrm{ac}}^d.
\end{aligned}
$$

Hence there exists a sufficiently small number $\delta_0 > 0$ with the following inequality:

$$
\mathrm{vol}_{X|V}(D) - 2\varepsilon \leq \int_{V_{\mathrm{reg}}} \{(1-\delta_0)(T_\varepsilon|_{V_{\mathrm{reg}}}) + \delta_0(S|_V)\}_{\mathrm{ac}}^d.
$$

Note that $(1-\delta_0)T_\varepsilon + \delta_0 S$ is a Kähler current in $c_1(D)$. By applying the approximation theorem (Theorem 5.2.3 and Corollary 5.2.4) to $(1-\delta_0)T_\varepsilon + \delta_0 S$, we can find positive currents U_k in $c_1(D)$ with the following properties.

(1) U_k has analytic singularities for every integer k.

(2) $(U_k|_{V_{\mathrm{reg}}})_{\mathrm{ac}} \longrightarrow \left\{(1-\delta_0)T_\varepsilon|_{V_{\mathrm{reg}}} + \delta_0 S|_{V_{\mathrm{reg}}}\right\}_{\mathrm{ac}}$ almost everywhere on V_{reg}

(3) U_k is a positive current for a sufficiently large k.

Fatou's Lemma and property (2) imply

$$
\begin{aligned}
\mathrm{vol}_{X|V}(D) - 2\varepsilon &\leq \int_{V_{\mathrm{reg}}} \left\{(1-\delta_0)(T_\varepsilon|_{V_{\mathrm{reg}}})_{\mathrm{ac}} + \delta_0(S|_{V_{\mathrm{reg}}})_{\mathrm{ac}}\right\}^d \\
&= \int_{V_{\mathrm{reg}}} \liminf_{k \to \infty} (U_k|_{V_{\mathrm{reg}}})_{\mathrm{ac}}^d \\
&\leq \liminf_{k \to \infty} \int_{V_{\mathrm{reg}}} (U_k|_{V_{\mathrm{reg}}})_{\mathrm{ac}}^d.
\end{aligned}
$$

Therefore we have

$$
\mathrm{vol}_{X|V}(D) - 3\varepsilon \leq \int_{V_{\mathrm{reg}}} (U_{k_0}|_{V_{\mathrm{reg}}})_{\mathrm{ac}}^d
$$

for a sufficiently large k_0. Now ε is an arbitrary positive number and U_{k_0} is a positive current with analytic singularities in the Chern class $c_1(D)$. It completes the proof of Theorem 5.1.3.

□

At the end of this subsection, we prove the following proposition, which is a variation of [ELMNP09, Theorem 2.13].

Proposition 5.4.3. *Let V be an irreducible subvariety of dimension d on X. Assume that V is not contained in $\mathbb{B}_+(D)$. Then the following equality holds.*

$$\mathrm{vol}_{X|V}(D) = \limsup_{k\to\infty} \frac{h^0(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT_{\min})|_V)}{k^d/d!},$$

where $T_{\min}$ is a current with minimal singularities in $c_1(D)$.

Proof. By [ELMNP09, Theorem 2.13], we know

$$\mathrm{vol}_{X|V}(D) = \limsup_{k\to\infty} \frac{h^0(V, \mathcal{O}(kD)\otimes \mathcal{J}(\|kD\|)|_V)}{k^d/d!}.$$

In order to prove Proposition 5.4.3, we should compare the multiplier ideal sheaf $\mathcal{I}(kT_{\min})$ with the asymptotic multiplier ideal sheaf $\mathcal{J}(\|kD\|)$. By the definition of a current with minimal singularities, we have $\mathcal{J}(\|kD\|) \subseteq \mathcal{I}(kT_{\min})$ for all positive integer k. Hence it is sufficient to prove the inequality $\geq$ in Proposition 5.4.3. For this purpose, we establish the following lemma.

Lemma 5.4.4. *Let D be a big divisor on a smooth projective variety X. There is an effective divisor E (independent of k) with the following properties:*

(i) *The support of E does not contain V.*

(ii) *$\mathcal{I}(kT_{\min})\cdot\mathcal{O}_X(-E) \subseteq \mathcal{J}(\|kD\|)$ for a sufficiently large k.*

Proof. This proof is essentially based on the argument in [DEL00, Theorem 1.11]. Fix a very ample divisor A on X. For an arbitrary point $x \in X$, there exists a zero-dimensional complete intersection P_x of the complete linear system $|A|$ containing x. The Ohsawa-Takegoshi-Manivel L^2-extension theorem asserts that, there exists an ample divisor B (which depends only on A) such that for any divisor F and a singular hermitian metric h on F with the positive curvature current T_h, the following restriction map is surjective (see [OT87], [Man93]):

$$H^0(X, \mathcal{O}_X(F+B)\otimes\mathcal{I}(T_h)) \longrightarrow H^0(P_x, \mathcal{O}_{P_x}(F+B)\otimes\mathcal{I}(T_h|_{P_x})).$$

Moreover, the Ohsawa-Takegoshi-Manivel L^2-extension theorem claims that, for a section on P_x, the extended section satisfies an L^2-estimate with a constant which is independent of F. Further the L^2-estimate depends only on A.

Since D is big and V is not contained in the augmented base locus of D, we can take $E \in |k_0 D - B|$ with property (i) by choosing a sufficiently large k_0. We apply the Ohsawa-Takegoshi-Manivel L^2-extension theorem to $F_k := (k-k_0)D + E$ equipped with a singular hermitian metric $h_{\min}^{\otimes k-k_0} \otimes h_E$. Here $h_{\min}$ denotes a singular hermitian metric with minimal singularities and h_E denotes a singular hermitian metric defined by the standard section of the effective divisor E. Then for a sufficiently large k and a given point $x \in X$, we obtain the global section s_x of $F_k + B \sim kD$ with the following estimates:

$$\int_X |s_x|^2_{h_{\min}^{\otimes k-k_0} \otimes h_E \otimes h_B} \omega^n \leq C \qquad \textit{and} \qquad |s_x(x)|^2_{h_{\min}^{\otimes k-k_0} \otimes h_E \otimes h_B} = 1,$$

where C is a constant depending only A and h_B is a smooth hermitian metric on B with the positive curvature. Here ω is a Kähler form on X. From the second equality, we obtain

$$|s_x(x)|^2 e^{-2(k-k_0)\varphi_{\min} - 2\varphi_E - 2\varphi_B} = 1,$$

where $\varphi_{\min}$, φ_E, φ_B is the weight of the hermitian metric $h_{\min}$,h_E,h_B respectively. Since φ_B is a smooth function and X is compact, there is a constant C' such that

$$\varphi_{\min} + \frac{1}{k-k_0}\varphi_E \leq \frac{1}{k-k_0} \log |s_x(x)| + C'.$$

The evaluation map $H^0(X, \mathcal{O}_X(kD)) \longrightarrow \mathbb{C}$ is a bounded operator on the Hilbert space $H^0(X, \mathcal{O}_X(kD))$ with the L^2-norm. Moreover the operation norm is equal to the Bergman kernel

$$\sum_{j=1}^{N_k} |f_j(x)|_{h_{\min}^{\otimes k-k_0} \otimes h_E \otimes h_B}$$

where $\{f_j\}_{j=1}^{N_k}$ is an orthonormal basis of $H^0(X, \mathcal{O}_X(kD))$. Therefore there is a constant C'' such that

$$\log |s_x(x)| \leq \log \sum_{j=1}^{N_k} |f_j| + C''.$$

These inequalities imply that the function $\frac{1}{k-k_0}\log\sum_j |f_j|$ has less singularities than $\varphi_{\min} + \frac{1}{k-k_0}\varphi_E$. By the definition of the asymptotic multiplier ideal sheaf, we obtain property (ii). □

We shall complete the proof of Proposition 5.4.3 by using Lemma 5.4.4. From property (i) in the previous lemma, we may consider the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_V(kD - E) \longrightarrow \mathcal{O}_V(kD) \longrightarrow \mathcal{O}_{V\cap E}(kD) \longrightarrow 0.$$

Since the dimension of the intersection $V \cap E$ is smaller than $\dim V = d$, we have

$$\limsup_{k\to\infty} \frac{h^0(V\cap E, \mathcal{O}_{V\cap E}(kD))}{k^d/d!} = 0.$$

Hence we obtain

$$\limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT_{\min})|_V\big)}{k^d/d!} \le \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD-E)\otimes \mathcal{I}(kT_{\min})|_V\big)}{k^d/d!}.$$

On the other hand, E satisfies property (ii) in Lemma 5.4.4. It implies

$$\limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD-E)\otimes \mathcal{I}(kT_{\min})|_V\big)}{k^d/d!} \le \limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{J}(\|kD\|)|_V\big)}{k^d/d!}.$$

These inequalities assert

$$\limsup_{k\to\infty} \frac{h^0\big(V, \mathcal{O}_V(kD)\otimes \mathcal{I}(kT_{\min})|_V\big)}{k^d/d!} \le \mathrm{vol}_{X|V}(D).$$

□

5.4.2 Properties of restricted volumes

Theorem 5.1.3 enables us to define the restricted volume for a big class on a compact Kähler manifold (see Definition 5.1.4). In this subsection, we study the properties of the restricted volume of a class on a compact Kähler manifold. Throughout this subsection, we denote by M a compact Kähler manifold and by W an irreducible analytic subset on M of dimension d and by α a big class in $H^{1,1}(M,\mathbb{R})$.

Proposition 5.4.5. *Assume that α is a nef class and W is not contained in the non-Kähler locus $E_{nK}(\alpha)$ of α. Then the restricted volume $\mathrm{vol}_{M|W}(\alpha)$ is equal to the self-intersection number $(\alpha^d \cdot W)$ on W.*

Proof. When W is non-singular, this proposition is proved by using the same argument as [Bou02, Theorem 4.1]. By using Lemma 5.4.2, we can give the proof even if W has singularities. □

The following proposition is the generalization of Proposition 5.3.1 to a class on a compact Kähler manifold. The proof gives another proof of Proposition 5.3.1 without the approximation of the positive part P by $\mathbb{Q}$-divisors.

Proposition 5.4.6. *Let $\alpha = P + \{N\}$ be the divisorial Zariski decomposition of α. Assume that W is not contained in $E_{nK}(\alpha)$. Then W is not contained in $E_{nK}(P)$ and the equality $\mathrm{vol}_{M|W}(\alpha) = \mathrm{vol}_{M|W}(P)$ holds.*

Proof. The proposition is based on the following fact. Positive currents in α and positive currents in P are identified by the correspondence $T \longmapsto T - [N]$. First we show the following claim.

Claim 5.4.7. *We have $E_{nK}(\alpha) = E_{nK}(P)$.*

Proof. For a point $x \notin E_{nK}(\alpha)$, there is a Kähler current T in α with analytic singularities such that T is smooth at x. Note $T - [N]$ is a Kähler current since T is a Kähler current in α. In fact, $T \geq \omega$ for some Kähler form ω. Then the negative part of the Siu decomposition of $T - \omega$ still contains $[N]$. It yields $T - [N] \geq \omega$. Therefore $T - [N]$ is a Kähler current in P. We can easily see that the support of N is contained in $E_{nK}(\alpha)$. Since x is not contained in the support of N, the Kähler current $T - [N]$ is smooth at x. Thus x is not contained in $E_{nK}(P)$.

Conversely we take a point $x \notin E_{nK}(P)$. Then there is a Kähler current S in P such that S is smooth at x. We may assume that $S \geq \omega$. We shall show that x is not contained in the support of N. To prove this, we consider the surjective map:

$$\{\text{smooth real } d\text{-closed } (1,1)\text{-form}\} \longrightarrow H^{1,1}(M, \mathbb{R}), \quad \theta \mapsto \{\theta\}.$$

We regard the space of smooth real d-closed $(1,1)$-forms as the topological space with the Fréchet topology. For a smooth $(n-1, n-1)$-form γ, the integral $\int_M \theta_k \wedge \gamma$ tends to $\int_M \theta \wedge \gamma$ if θ_k converges to θ in the Frécht topology. Hence it follows that the above map $\theta \longmapsto \{\theta\}$ is continuous from the duality theorem. Thus the map is an open map from the open mapping theorem.

Since the map is an open map, for a positive number ε, there is a sufficiently small $\delta > 0$ such that $\delta c_1(N)$ contains a smooth form η with

$-\varepsilon\omega \leq \eta \leq \varepsilon\omega$. Since S is a Kähler current, $S+\eta+(1-\delta)[N]$ is still a positive current for a sufficiently small ε. Further the current belongs to the class α. Now the Lelong number of $S+\eta+(1-\delta)[N]$ at x is equal to the Lelong number of $(1-\delta)[N]$ since S and η are smooth at x. If $\nu([N],x)$ is positive, it is a contradiction to the construction of N. (Recall $N=\sum \nu(T_{\min}, E)E$, where $T_{\min}$ is a current with minimal singularities.) Thus x is not contained in the support of N. It implies that the Kähler current $S+[N]$ is smooth at x. Hence x is not contained in $E_{nK}(\alpha)$. □

Finally, we prove $\mathrm{vol}_{M|W}(\alpha)=\mathrm{vol}_{M|W}(P)$. Note that we can define the restricted volume of P thanks to the claim above. Since the support of the current $[N]|_{W_{\mathrm{reg}}}$ is contained in $N\cap W$, the absolutely continuous part of $[N]|_{W_{\mathrm{reg}}}$ is zero. It implies that $[N]|_{W_{\mathrm{reg}}}$ does not affect the integration on W. Therefore it follows Proposition 5.4.6 from the correspondence between positive currents in α and in P. □

The following theorem says that Fujita's approximation theorem for the restricted volume of a class holds. It leads to the continuity of the restricted volume.

Theorem 5.4.8. *The restricted volume of a class α along W can be approximated by self-intersection numbers of semi-positive classes. That is, the following equality holds.*

$$\mathrm{vol}_{M|W}(\alpha)=\sup_{\pi^*T=B+[E]}\left(\{B\}^d\cdot\widetilde{W}\right),$$

where the supremum is taken over all resolutions $\pi:\widetilde{M}\longrightarrow M$ of positive currents $T\in\alpha$ with analytic singularities such that π is an isomorphism at a generic point of W and $\widetilde{W}\not\subseteq \mathrm{Supp}(E)$. (Here $\widetilde{W}$ denotes the strict transformation of W.)

Proof. Let T be a positive current with analytic singularities in the class α whose singular locus does not contain W. Then we take a modification μ such that $\mu^*T=B+[E]$ and μ is an isomorphism at a generic point on W.

Lemma 5.4.2 yields

$$\begin{aligned}\int_{W_{\mathrm{reg}}} T|_{W\mathrm{reg}} &= \int_{\widetilde{W}} (\mu^* T|_{\widetilde{W}})^d_{\mathrm{ac}} \\ &= \int_{\widetilde{W}} ((B+[E])|_{\widetilde{W}})^d_{\mathrm{ac}} \\ &= \int_{\widetilde{W}} B^d = (\{B\}^d \cdot \widetilde{W}).\end{aligned}$$

Therefore we obtain $\mathrm{vol}_{M|W}(\alpha) = \sup_{\pi^*T=B+[E]} \left(\{B\}^d \cdot \widetilde{W}\right)$ from the definition of the restricted volume of α along W. $\square$

In order to show the continuity of the restricted volume, we consider the "domain" of the restricted volume for a given analytic subset W on M. Further, we prove the convexity of the domain and log concavity of the restricted volume.

Definition 5.4.9. For an irreducible analytic subset W on M, *the domain of the restricted volume* is defined to be $\mathrm{Big}^W(M) := \{\beta \in H^{1,1}(M,\mathbb{R}) \mid W \not\subseteq E_{nK}(\beta)\}$.

Proposition 5.4.10. (1) *$\mathrm{Big}^W(M)$ is an open convex set in $H^{1,1}(M,\mathbb{R})$.*
(2) *For $\beta_1, \beta_2 \in \mathrm{Big}^W(M)$, we have*

$$\mathrm{vol}_{M|W}(\beta_1+\beta_2)^{1/d} \geq \mathrm{vol}_{M|W}(\beta_1)^{1/d} + \mathrm{vol}_{M|W}(\beta_2)^{1/d}.$$

Proof. (1) The convexity is easily proved from $E_{nK}(\beta+\beta') \subseteq E_{nK}(\beta') \bigcup E_{nK}(\beta)$ and $E_{nK}(\beta) = E_{nK}(k\beta)$ for $k > 0$. For a given class β, we can see $E_{nK}(\beta') \subseteq E_{nK}(\beta)$ for every class β' in a suitable open neighborhood of β by using the argument in Lemma 5.4.7. It asserts the domain is an open set.

(2) In his paper [Bou02], Boucksom showed the log concavity for the volume of a transcendental class. Hence it follows the log concavity of the restricted volumes of nef classes from Proposition 5.4.5. By using Proposition 5.4.8, we can conclude that the restricted volume has the log concavity on $\mathrm{Big}^W(M)$. $\square$

Corollary 5.4.11. *The following map is continuous.*

$$\mathrm{vol}_{M|W}(\cdot) : \mathrm{Big}^W(M) \longrightarrow \mathbb{R}, \qquad \beta \longmapsto \mathrm{vol}_{M|W}(\beta)$$

Proof. It is known fact that a concave function on an open convex set in $\mathbb{R}^N$ is continuous. Therefore Corollary 5.4.11 follows from Proposition 5.4.10. $\square$

5.4.3 Proof of Theorem 5.1.5

In this subsection, we prove Theorem 5.1.5 by using the analytic description of the restricted volume with currents. It gives another proof of Theorem 5.1.1. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class on a smooth projective variety X and $\alpha = P + \{N\}$ the divisorial Zariski decomposition. We have $E_{nK}(\alpha) = E_{nK}(P)$ by Lemma 5.4.7. Hence we can consider the restricted volume of P along V.

The strategy of the proof is essentially same as Theorem 5.1.1. From Proposition 5.4.6, we have $\mathrm{vol}_{X|V}(\alpha) = \mathrm{vol}_{X|V}(P)$ for an irreducible subvariety V on X such that $V \not\subseteq E_{nK}(\alpha)$. Moreover, Proposition 5.4.5 says that $\mathrm{vol}_{X|V}(P) = (V \cdot P^d)$ holds if P is nef. Hence when α admits a Zariski decomposition, the restricted volumes along any cohomologus subvarieties coincide.

Let us show that condition (3) implies condition (1). We assume the non-nef locus $E_{nn}(P)$ is not empty for a contradiction and fix a very ample divisor A on X. Then there are smooth curves C and C' with the following properties:

(1) C' does not intersect with the non-nef locus $E_{nn}(P)$.
(2) C and C' are not contained in the non-Kähler locus $E_{nK}(\alpha)$.
(3) C intersects with the non-nef locus $E_{nn}(P)$ at $x_0 \in X$.
(4) C and C' are complete intersections of members of the complete linear system of A.

Then we prove the following lemma for a contradiction.

Lemma 5.4.12. *In the situation above, the followings hold.*

$$(A) \quad \mathrm{vol}_{X|C'}(\alpha) = (C' \cdot P), \qquad (B) \quad \mathrm{vol}_{X|C}(\alpha) < (C \cdot P).$$

Proof. From the definition of the restricted volume of P and Proposition 5.4.6, we obtain

$$\mathrm{vol}_{X|C}(\alpha) = \mathrm{vol}_{X|C}(P) = \sup_{T \in P} \int_C (T|_C)_{\mathrm{ac}}.$$

Here T runs through positive currents with analytic singularities in the class P whose singular loci do not contain C. Now $T|_C$ is also a positive current with analytic singularities. In general, the Siu decomposition coincides with the Lebesgue decomposition for a d-closed positive current with analytic singularities. Therefore we have $(T|_C)_{\mathrm{ac}} = T|_C - \sum_{x \in C} \nu(T|_C, x)[x]$. On the other hand, we have

$$\int_C T|_C = (C \cdot P).$$

In fact, we can easily see

$$\int_C T|_C = (C \cdot P) + \int_C dd^c\varphi|_C,$$

where φ is an L^1-function on X such that $T = \theta + dd^c\varphi$. Here θ denotes a smooth $(1,1)$-form in P. By applying the approximation theorem (Theorem 5.2.2) to $\varphi|_C$, we obtain smooth functions φ_k on C such that $dd^c\varphi_k$ weakly converges to $dd^c\varphi|_C$. Thus $\int_C dd^c\varphi_k$ tends to $\int_C dd^c\varphi|_C$. On the other hand, $\int_C dd^c\varphi_k$ is equal to zero for every k from Stokes's theorem. (Note that $dd^c\varphi_k$ is smooth.)

Hence we obtain

$$\begin{aligned}\mathrm{vol}_{X|C}(\alpha) &= \sup_{T \in c_1(P)} \{(C \cdot P) - \sum_{x \in C} \nu(T|_C, x)\} \\ &= (C \cdot P) - \inf_{T \in P} \sum_{x \in C} \nu(T|_C, x).\end{aligned}$$

In general, the Lelong number of the restriction of a current is more than or equal to the Lelong number of the current. Further, $\nu(T_{\min}, x) \leq \nu(T, x)$ holds from the definition of a current with minimal singularities. Therefore we obtain

$$\mathrm{vol}_{X|C}(\alpha) \leq (C \cdot P) - \sum_{x \in C} \nu(T_{\min}, x).$$

The curve C intersects with the non-nef locus $E_{nn}(P)$ at x_0 from property (3). Hence $\nu(T_{\min}, x_0)$ is positive. It implies $\mathrm{vol}_{X|C}(\alpha) \leq (C \cdot P) - \nu(T_{\min}, x_0) < (C \cdot P)$. Here $T_{\min}$ is a current with minimal singularities in P. Therefore inequality (B) holds.

Finally we shall prove equality (A). By the first half argument, we have $\mathrm{vol}_{X|C'}(\alpha) \leq (C' \cdot P)$. To prove the converse inequality, we take a Kähler current S with analytic singularities in α. We may assume $S \geq \omega$, where ω is a Kähler form on X. By applying the approximation theorem (Theorem 5.2.3) to a current $T_{\min}$ with minimal singularities in P, we obtain positive currents T_k with analytic singularities with the following properties.

(b′) $T_k \geq -\varepsilon_k\omega$ and ε_k converges to zero.

(c′) The Lelong number $\nu(T_k, x)$ increases to $\nu(T_{\min}, x)$ for every point $x \in X$.

For every positive number δ, there is $k(\delta)$ such that $(1-\delta)T_{k(\delta)} + \delta S$ is a positive current. Since $(1-\delta)T_{k(\delta)} + \delta S$ is a positive current with analytic singularities, the inequality

$$\mathrm{vol}_{X|C'}(\alpha) \geq \int_{C'} \left(((1-\delta)T_{k(\delta)} + \delta S)|_{C'}\right)_{\mathrm{ac}}$$

holds by the definition of the restricted volume. The Lelong number of T_k at every point in C is zero by property (3). It implies T_k is smooth on C. Thus we obtain

$$\mathrm{vol}_{X|C'}(\alpha) \geq (1-\delta)(C' \cdot P) - \delta \int_{C'} (S|_{C'})_{\mathrm{ac}}.$$

for every δ. When δ tends to zero, we obtain $\mathrm{vol}_{X|C'}(\alpha) \geq (C' \cdot P)$. □

6

An ampleness criterion with the extendability of singular positive metrics

6.1 Introduction

Throughout this chapter, let us denote by X a smooth projective variety of dimension n, by L a line bundle on X. In the theory of several complex variables and algebraic geometry, it is fundamental to consider a singular metric on L whose Chern curvature is a positive $(1,1)$-current. A singular metric on L with positive curvature current corresponds to a θ-plurisubharmonic function, where θ is a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$ of the line bundle L. (For simplicity of notation, we will abbreviate "θ-plurisubharmonic" to "θ-psh".) Here a function $\varphi : X \longrightarrow [-\infty, \infty)$ is called a *θ-psh function*, if φ is upper semi-continuous on X and the Levi form $\theta + dd^c\varphi$ is a positive $(1,1)$-current. We will denote by $\mathrm{Psh}(X, \theta)$ the set of θ-psh functions on X. That is, $\mathrm{Psh}(X, \theta)$ is the set

$$\{\varphi : X \longrightarrow [-\infty, \infty) \mid \varphi \text{ is upper semi-continuous and } \theta + dd^c\varphi \geq 0. \}.$$

It is of interest to know when a $\theta|_V$-psh function on a (closed) subvariety $V \subseteq X$ can be extended to a global θ-psh function on X. Coman, Guedj and Zeriahi proved that a $\theta|_V$-psh function on any subvariety V can be extended a global θ-psh function on X, if L is an ample line bundle (see [CGZ10, Theorem B]). Note that a $\theta|_V$-psh function can be defined even if V has singularities (see [CGZ10, Section 2] for the precise definition).

Theorem 6.1.1. ([CGZ10, Theorem B]). *Let L be an ample line bundle on a smooth projective variety X and let θ be a Kähler form representing $c_1(L)$.*

Then for any subvariety $V \subseteq X$, any $\theta|_V$-psh function on V extends to a θ-psh function on X.

The following theorem asserts that the converse implication of Theorem 6.1.1 holds, which is a main result in this chapter. Theorem 6.1.2 gives an ampleness criterion by the extendability of singular metrics (θ-psh functions).

Theorem 6.1.2. *Let L be a pseudo-effective line bundle whose first Chern class $c_1(L)$ is not zero. Assume that L satisfies the following property : For any subvariety V and any $\theta|_V$-psh function $\varphi \in \mathrm{Psh}(V, \theta|_V)$, there exists a global θ-psh function $\widetilde{\varphi} \in \mathrm{Psh}(X, \theta)$ such that $\widetilde{\varphi}|_V = \varphi$. Here θ is a smooth d-closed $(1,1)$-form representing $c_1(L)$. Then L is an ample line bundle.*

A line bundle L is called *pseudo-effective* if $\mathrm{Psh}(X, \theta)$ is not empty. We can easily check that the definition of pseudo-effective line bundles does not depend on the choice of $\theta \in c_1(L)$. In the proof of Theorem 6.1.2, we consider only the case when V is a strongly movable curve (see Section 6.3). Thus for an ampleness criterion, it is sufficient to check the extendability from a strongly movable curve.

It is important to emphasize that even if a given $\theta|_V$-psh function φ is smooth at some point on V, the extended function $\widetilde{\varphi}$ may not be smooth at the point. The fact seems to make the proof of Theorem 6.1.2 difficult.

Remark that the assumption that the first Chern class $c_1(L)$ is not zero is necessary. Indeed, when the first Chern class $c_1(L)$ is zero and θ is equal to zero as a $(1,1)$-form, a θ-psh function is always constant, from the maximum principle of psh functions. Hence any $\theta|_V$-psh function can be extended. However L is not an ample line bundle. In other words, a line bundle which satisfies the extendability condition in Theorem 6.1.2 must be ample or numerically trivial (that is, $c_1(L)$ is zero).

This chapter is organized in the following way: In Section 6.2, we collect materials to prove Theorem 6.1.2. Section 6.3 is devoted to give the proof of Theorem 6.1.2. In Section 6.4, there are two examples which give ideas for the proof of Theorem 6.1.2.

In this chapter, we use additive notation for tensor product of a line bundles.

6.2 Preliminaries

In this section, we collect materials for the proof of Theorem 6.1.2. The propositions in this section may be known facts. However we give comments or references for the readers' convenience.

Theorem 6.2.1 ([Zha09, Theorem 1.3]). *Let X be a smooth projective variety of dimension at least two and p, q be points on X. Fix an ample line bundle B on X. Then there exists a smooth curve C with the following properties:*

(1) *C is a complete intersection of the complete linear system $|mB|$ for some $m > 0$.*

(2) *C contains points p and q.*

Proof. Take an embedding of X into the projective space $\mathbb{P}^N$. Then two points p, q are always in general position in $\mathbb{P}^N$ (see [Zha09, Definition 1.1] for the definition). [Zha09, Theorem 1.3] asserts that a general member of $|mB|_{p,q}$ is irreducible and smooth, where $|mB|_{p,q}$ is a linear system in $|mB|$ passing through p and q. Then by taking a complete intersection of general members of $|mB|_{p,q}$, we can construct a curve with the properties above. $\square$

Lemma 6.2.2. *Let C be an irreducible curve on X and p be a non-singular point on C. Assume that the intersection number $(L \cdot C)$ is positive (that is, the restriction $L|_C$ is ample). Then there exists a function φ on C with the following properties:*

(1) $\varphi \in \mathrm{Psh}(C, \theta|_C)$

(2) *The function φ has pole at p (that is, $\varphi(p) = -\infty$) and is smooth except p.*

Proof. By the assumption, $L|_C$ is ample on C. Therefore we can obtain a smooth strictly $\theta|_C$-psh function φ_1 on C. Even if C has singularities, we can obtain such function. In fact, there exists an integer $m_0 > 0$ such that the complete linear system of $m_0 L|_C$ gives an embedding of C to the projective space $\mathbb{P}^N$, since $L|_C$ is ample. Now there exists a smooth strictly θ_0-psh function ψ on $\mathbb{P}^N$. Here θ_0 is a $(1,1)$-form representing the first Chern class of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$. Since the restriction to C of $\mathcal{O}_{\mathbb{P}^N}(1)$ is equal to $m_0 L|_C$, the function $(1/m_0)\psi|_C$ gives a smooth strictly $\theta|_C$-psh function on C.

Let z be a local coordinate on C centered at p. We define a function φ_2 on C to be $\varphi_2 := \rho \log |z|^2$, where ρ is a smooth function on C whose support is contained in some neighborhood of p. Then φ_2 has a pole only at p. Further φ_2 is an almost psh function (that is, there exists a smooth $(1,1)$-form γ such that $dd^c\varphi_2 \geq \gamma$). Then a function φ which is defined to be $\varphi = (1-\varepsilon)\varphi_1 + \varepsilon\varphi_2$ satisfies the condition above for a sufficiently small $\varepsilon > 0$.

In fact, property (1) follows from the strictly positivity of the Levi-form of φ_1. The function φ has a pole only at p thanks to φ_2. $\square$

Lemma 2.2.7, 6.2.2 say that there exists many $\theta|_C$-psh functions on a complete intersection of very ample divisors.

For the proof of Theorem 6.1.2, it is important to obtain strict positivity from extended θ-psh functions. The main idea for the purpose is to use the volume of a line bundle and its expression formula in terms of current integration, which is proved in [Bou02].

Definition 6.2.3. Let M be a line bundle on a projective variety Y of dimension d. Then the volume of M on Y is defined to be

$$\mathrm{vol}_Y(\mathrm{M}) = \limsup_{k\to\infty} \frac{\dim H^0(Y, \mathcal{O}_Y(kM))}{k^d/d!}.$$

The volume asymptotically measures the number of global holomorphic sections. The volume of a line bundle can be defined for a $\mathbb{Q}$-line bundle and, depends only on the numerical class (the first Chern class) of the line bundle. Moreover the volume is a continuous function on the $\mathbb{Q}$-vector space $N^1(Y)_{\mathbb{Q}}$ of the numerical equivalent classes of $\mathbb{Q}$-line bundles. (See [Laz, Proposition 2.2.35, 2.2.41] for the precise statement.) The properties above are used in the proof of Theorem 6.1.2.

The following proposition gives a relation between the volume and curvature currents of a line bundle. It is proved by using singular Morse inequalities (which is proved in [Bon98]) and approximations of θ-psh functions (see [Bou02] for details).

Proposition 6.2.4. ([Bou02, Proposition 3.1]). *Let M be a pseudo-effective line bundle on a smooth projective variety Y of dimension d and η a smooth $(1,1)$-form which represents the first Chern class $c_1(M)$ of M. Then for any η-psh function φ on Y, we have*

$$\mathrm{vol}_Y(\mathrm{M}) \geq \int_Y \left(\eta + dd^c\varphi\right)^d_{\mathrm{ac}}.$$

Here $\left(\eta + dd^c\varphi\right)_{\mathrm{ac}}$ means the absolutely continuous part of a positive current $\left(\eta + dd^c\varphi\right)$ by the Lebesgue decomposition. (See Section 5.2.2.) We use only property that when φ is smooth on an open set, the equality

$$\left(\eta + dd^c\varphi\right)_{\mathrm{ac}} = \left(\eta + dd^c\varphi\right)$$

holds on the open set.

Actually, the inequality above would be an equality by taking supremum of the right-hand side over all η-psh functions (see [Bou02, Theorem 1.2]). It is generalized to the restricted volume along a subvariety (cf. Theorem 5.1.3). These expressions of the volume and restricted volume with current integrations give an example, which show us that there exists a θ-psh function on some subvariety which can not be extended to a global θ-psh function even if L is a big line bundle (see Section 6.4.2).

In Section 6.3, we need to approximate a given θ-psh function by almost psh functions with mild singularities. For the purpose, we use Theorem 6.2.5. Theorem 6.2.5 says that it is possible to approximate a given almost psh function with the same singularities as a logarithm of a sum of squares of holomorphic functions without a large loss of positivity of the Levi form.

Theorem 6.2.5. ([Dem, Theorem 13.12]). *Let φ be an almost psh function on a compact complex manifold X such that $dd^c\varphi > \gamma$ for some continuous $(1,1)$-form γ. Fix a hermitian form ω on X. Then there exists a sequence of almost psh functions φ_k and a decreasing sequence $\delta_k > 0$ converging to 0 with the following properties:*

(A) $\varphi(x) < \varphi_k(x) \leq \sup_{|\zeta - x| < r} \varphi(\zeta) + C\Big(\dfrac{|\log r|}{k} + r + \delta_k\Big)$
with respect to coordinate open sets covering X.

(B) *φ_k has the same singularities as a logarithm of a sum of squares of holomorphic functions. In particular, φ_k is smooth except the polar set of φ.*

(C) $dd^c\varphi_k \geq \gamma - \delta_k\omega$.

6.3 Proof of Theorem 6.1.2

In this section, we give the proof of Theorem 6.1.2. Let L be a line bundle with the assumption of Theorem 6.1.2 and θ be a smooth d-closed $(1,1)$-form which represents the first Chern class $c_1(L)$. According to the Nakai-Moishezon-Kleiman criterion (cf. [Laz, Theorem 1.2.23]), in order to show that L is ample, it is enough to see the self-intersection number $(L^d \cdot V)$ along V is positive for any irreducible subvariety V.

For this purpose, we first show the following proposition which implies that the self-intersection number along an irreducible curve is always positive.

Proposition 6.3.1. *Let L be a line bundle with the assumption in Theorem 6.1.2 and V be an (irreducible) subvariety on X. Then*

(1) *The restriction $L|_V$ to V is pseudo-effective.*

(2) *The restriction $L|_V$ to V is not numerically trivial.*

Remark 6.3.2. When V is non-singular, property (2) means that the first Chern class $c_1(L|_V)$ is not zero.

Proof. First we take two different points p, q on $V_{\rm reg}$. Here $V_{\rm reg}$ means the regular locus of V. Then we can take a smooth curve C on X such that C contains p, q by Theorem 6.2.1. By the construction, the curve C is a complete intersection of the linear system of some very ample line bundles. It follows that the intersection number $(L \cdot C)$ along C is positive from Lemma 2.2.7. Lemma 6.2.2 asserts that there exists a function $\varphi \in \mathrm{Psh}(C, \theta|_C)$ such that φ has a pole at p and is smooth at q. The $\theta|_C$-function φ on C can be extended to a global θ-function on X by the assumption of the extendability. The extended function to X does not have a pole at $q \in V$. It means that the restriction to V of the function is well-defined (that is, the function is not identically $-\infty$ on V). We denote by $\widetilde{\varphi}$ the restriction to V of the function. The function $\widetilde{\varphi}$ gives an element in $\mathrm{Psh}(V, \theta|_V)$. Hence $L|_V$ is pseudo-effective.

From now on, we show that $L|_V$ is not numerically trivial. For a contradiction, we assume that $L|_V$ is numerically trivial. First we consider the case where V is non-singular. Then there exists a function on V such that

$$\theta|_V + dd^c\widetilde{\varphi} = dd^c\psi.$$

from the $\partial\bar{\partial}$-Lemma, since $L|_V$ is numerically trivial (that is, first Chern class $c_1(L|_V)$ is zero). Since the function $\widetilde{\varphi}$ is a $\theta|_V$-psh, ψ is a psh function on V. It follows that ψ is actually a constant by the maximum principle of psh functions. Therefore $\theta|_V + dd^c\widetilde{\varphi}$ is a zero current. We know that θ-pluriharmonic functions are always smooth. Hence the function $\widetilde{\varphi}$ is smooth on V. However $\widetilde{\varphi}$ has a pole at p by the construction. This is a contradiction.

We need to consider the case where V has singularities. Then we take an embedded resolution

$$\mu : \widetilde{V} \subseteq \widetilde{X} \longrightarrow V \subseteq X$$

of $V \subseteq X$. That is, $\mu : \widetilde{X} \longrightarrow X$ is a birational morphism and the restriction of μ to $\widetilde{V}$ gives a resolution of singularities of V. Since p is contained in the regular locus of V, μ is an isomorphism on some neighborhood of p. Further the pull-back $(\mu^*L)|_{\widetilde{V}}$ is also numerically trivial since $L|_V$ is numerically trivial. The same argument asserts that any function in $\mathrm{Psh}(\widetilde{\mathrm{V}}, (\mu^*\theta)|_{\widetilde{\mathrm{V}}})$ is always smooth. Note that the pull-back $\mu^*\widetilde{\varphi}$ is a $(\mu^*\theta)|_{\widetilde{V}}$-psh function on V. It shows that the pull-back $\mu^*\widetilde{\varphi}$ is smooth on $\widetilde{V}$. The function $\widetilde{\varphi}$ is also smooth at p since μ is an isomorphism on some neighborhood of p. However $\widetilde{\varphi}$ has a pole at p by the construction. This is a contradiction. Thus $L|_V$ is not numerically trivial even if V has singularities. $\square$

Corollary 6.3.3. *Let L be a line bundle with assumption in Theorem 6.1.2. Then the intersection number $(L \cdot C)$ is positive for any irreducible curve C on X.*

Proof. Any pseudo-effective line bundle on a curve which is not numerically trivial is always ample. Thus, the corollary follows from Proposition 6.3.1. $\square$

In oder to show $(L^d \cdot V) > 0$ for any subvariety, we need only consider the case where the dimension of V is larger than or equal to two from the corollary above. Moreover the corollary above asserts that L is a nef line bundle on X. It is well-known that the volume of $L|_V$ is equal to the self-intersection number $(L^d \cdot V)$ along V for a nef line bundle. (see [Laz, Section 2.2 C]). That is, the equality holds

$$\mathrm{vol}_V(L) = (L^d \cdot V)$$

for any irreducible subvariety V of dimension d. (Note the restriction of a nef line bundle is also nef.) Therefore for the proof of Theorem 6.1.2, it is enough to show that the volume $\mathrm{vol}_V(L)$ is always positive for any irreducible subvariety V of dimension $d \geq 2$. From now on, we will show that the volume $\mathrm{vol}_V(L)$ is positive for a subvariety V of dimension $d \geq 2$, by using Proposition 6.2.4.

We first consider the case where V is non-singular. Even if V has singularities, the same argument can be justified by taking an embedded resolution of $V \subseteq X$. We argue the case at the end of this section.

Fix a point p on V. Let $(z_1, z_2, \ldots, z_d)$ be a local coordinate centered at p. We consider an open ball B, which is defined by

$$B := \big\{(z_1, z_2, \ldots, z_d) \bigm| |z|^2 < 1\big\}.$$

Since $dd^c|z|^2$ is a strictly positive $(1,1)$-form on B, there exists a large positive number A such that

$$Add^c|z|^2 + \theta|_V > 0 \quad \text{on } B. \tag{6.1}$$

For every point y on the boundary ∂B of B, we can take a curve C_y on V such that C_y contains p and y. We can take such curve from Theorem

6.2.1 and the assumption that the dimension of V is larger than or equal to two. By Lemma 2.2.7 and property (1) in Theorem 6.2.1, the restriction of L to C_y is ample. Therefore there exists a function φ_y on C_y with following properties:

$$\varphi_y \in \mathrm{Psh}(C_y, \theta|_{C_y}), \tag{6.2}$$
$$\varphi_y(p) = -\infty, \tag{6.3}$$
$$\varphi_y(y) = 0, \tag{6.4}$$

Indeed, we can take a function $\varphi_y \in \mathrm{Psh}(C_y, \theta|_{C_y})$ such that φ_y has a pole at p and φ_y is smooth at y by Lemma 6.2.2. After replacing φ_y by $\varphi_y - \varphi_y(y)$, the function satisfies property (6.4). Note that the function is a $\theta|_{C_y}$-psh function even if we replace φ_y by $\varphi_y - \varphi_y(y)$.

Now the function φ_y on C_y can be extended to a $\theta|_V$-psh function $\widetilde{\varphi}_y$ on V by the assumption of Theorem 6.1.2. In fact, we can extend φ_y to a global θ-psh function on X by the assumption in Theorem 6.1.2. From property (6.4), the extended function does not have pole at y. Thus we can restrict the function to V. The function gives the extension of φ_y to V, which we denote by $\widetilde{\varphi}_y$. Then the function $\widetilde{\varphi}_y$ satisfies the following properties:

$$\widetilde{\varphi}_y \in \mathrm{Psh}(V, \theta|_V), \tag{6.5}$$
$$\widetilde{\varphi}_y(p) = -\infty, \tag{6.6}$$
$$\widetilde{\varphi}_y(y) = 0. \tag{6.7}$$

In the following step, we approximate the function $\widetilde{\varphi}_y$ with the same singularities as a logarithm of a sum of squares of holomorphic functions. If the extended function $\widetilde{\varphi}_y$ is continuous on some neighborhood of y, this step is not necessary. However the function $\widetilde{\varphi}_y$ may not be continuous at y even if φ_y is smooth at y on C_y. Thus the following step seems to be necessary in general.

Lemma 6.3.4. *Fix a hermitian form ω on X. For every positive number ε and a point $y \in \partial B$, there exist a neighborhood U_y of y which is independent of ε and an almost psh $\widetilde{\varphi}_{y,\varepsilon}$ with following properties:*

$$\theta|_V + dd^c\widetilde{\varphi}_{y,\varepsilon} \geq -\varepsilon\omega, \tag{6.8}$$
$$\widetilde{\varphi}_{y,\varepsilon}(y) > 0 \textit{ and } \widetilde{\varphi}_{y,\varepsilon} \textit{ is smooth on some neighborhood of } y, \tag{6.9}$$
$$-A > \widetilde{\varphi}_{y,\varepsilon} \quad \text{on } U_y. \tag{6.10}$$

Proof. By applying Theorem 6.2.5 to $\varphi = \widetilde{\varphi}_y$ and $\gamma = -\theta|_V$, we obtain almost psh functions $\{\widetilde{\varphi}_{y,k}\}_{k=1}^{\infty}$ with the properties in Theorem 6.2.5. For a given ε, by taking a sufficiently large $k = k(\varepsilon, y)$, property (6.8) holds from property (C).

From the left side inequality of property (A) in Theorem 6.2.5 and (6.7), we can easily check property (6.9) for every positive integer k. In fact property (B) implies that if $\widetilde{\varphi}_{y,k}$ does not have a pole at a point, $\widetilde{\varphi}_{y,k}$ is smooth at the point. In particular, $\widetilde{\varphi}_{y,k}$ is smooth on some neighborhood of y. In order to show the existence of U_y with property (6.10), we estimate the right hand inequality of property (A). We can easily show that there exists a sufficiently small $r_1 > 0$ which does not depend on ε such that

$$0 < C(\frac{|\log r_1|}{k} + r_1 + \delta_k) < A \quad \text{for any } k \geq \lceil \frac{1}{r_1} \rceil. \tag{6.11}$$

Here $\lceil \cdot \rceil$ means round up of a real number. Indeed, for any $k \geq \lceil \frac{1}{r_1} \rceil$, we have

$$\frac{|\log r_1|}{k} + r_1 + \delta_k \leq r_1 |\log r_1| + r_1 + \delta_k.$$

Now C depends on the choice of coordinate open sets covering V. However C is independent of ε. (We may assume that the coordinate open set $\big(B, (z_1, z_2, \ldots, z_d)\big)$ is a member of coordinate open sets covering V.) Therefore inequality (6.11) holds for a sufficiently small r_1 which is independent of ε.

On the other hand, $\widetilde{\varphi}_y$ has a pole at p by (6.6). Thus we have

$$\sup_{|z - z(p)| < r_2} \widetilde{\varphi}_y(z) < -2A$$

for a sufficiently small $r_2 > 0$. Here we used upper semi-continuity of $\widetilde{\varphi}_y$. Then we define U_y to be

$$U_y := \big\{ z \in B \ \big| \ |z - z(p)| < r_3 \big\},$$

where r_3 is $\min\{r_1, r_2\}$. Then the right hand of property (A) in Theorem 6.2.5 is strictly smaller than A for any $k \geq \lceil \frac{1}{r_3} \rceil$. We emphasize that r_1 and r_2 do not depend on ε. Therefore we obtain U with property (6.10). $\square$

By using these functions, we construct an almost psh function whose value at p is smaller than values on the boundary ∂B of B. From property (6.9), there exists a neighborhood W_y of y such that

$$\widetilde{\varphi}_{y,\varepsilon} > 0 \quad \text{on } W_y.$$

Since ∂B is a compact set, we can cover ∂B by finite members $\{W_{y_i}\}_{i=1}^{N}$. Now we define a function Φ_ε to be

$$\Phi_\varepsilon := \max_{i=1,\dots,N} \{\widetilde{\varphi}_{y_i,\varepsilon}\}.$$

Lemma 6.3.5. *Then the function Φ_ε satisfies the following properties:*

$$\theta|_V + dd^c\Phi_\varepsilon \geq -\varepsilon\omega, \tag{6.12}$$

$$\Phi_\varepsilon > 0 \text{ on some neighborhood } V_\varepsilon \text{ of } \partial B, \tag{6.13}$$

$$-A > \Phi_\varepsilon \text{ on some neighborhood } U \text{ of } p. \tag{6.14}$$

Proof. The property (6.12) follows from Lemma 2.4.4 and property (6.8). The property (6.13) is clear by the definition of Φ_ε and property (6.9). If a neighborhood U is defined to be $U := \cap_{i=1}^{N} U_{y_i}$, property (6.14) holds from property (6.10). Here U_{y_i} is a neighborhood of p with the property (6.10) in Lemma 6.3.4. □

Remark 6.3.6. We can assume that a neighborhood U in property (6.14) does not depend on ε. It follows from the definition of U in the proof of Lemma 6.3.5. The fact is essentially important in the estimation of the volume $\mathrm{vol}_V(L)$ with current integrations.

We want to construct a almost psh function whose Levi-form is strictly positive on some neighborhood of p. The integral of the Levi-form would imply that the volume $\mathrm{vol}_V(L)$ is positive. For this purpose, we define a new function Ψ_ε on V as follows:

$$\Psi_\varepsilon := \begin{cases} \Phi_\varepsilon & \text{on } V \setminus B \\ \max\{\Phi_\varepsilon, A|z|^2 - A\} & \text{on } B. \end{cases} \tag{6.15}$$

Then the function Ψ_ε satisfies the following properties:

Lemma 6.3.7. *The function Ψ_ε satisfies the following properties:*

$$\theta|_V + dd^c\Psi_\varepsilon \geq -\varepsilon\omega, \tag{6.16}$$

$$\Psi_\varepsilon = A|z|^2 - A \text{ on } U, \tag{6.17}$$

where U is a neighborhood of p which is independent of ε.

Proof. From property (6.14), we have $\Phi_\varepsilon < -A$ for some neighborhood U of p which is independent of ε. Therefore the inequality

$$\Phi_\varepsilon < -A \leq A|z|^2 - A$$

holds on U. Thus property (6.17) holds.

Further by the choice of A, the $(1,1)$-form $Add^c|z|^2+\theta|_V$ is strictly positive on the neighborhood B of p. In particular,

$$Add^c|z|^2 + \theta|_V \geq -\varepsilon\omega$$

holds. Hence we have

$$\theta|_V + dd^c \max\left\{\Phi_\varepsilon, A|z|^2 - A\right\} \geq -\varepsilon\omega \quad \text{on } B$$

from Lemma 2.4.4 and property (6.12).

On the other hand, we obtain

$$\max\left\{\Phi_\varepsilon, A|z|^2 - A\right\} = \Phi_\varepsilon$$

on some neighborhood of ∂B from property (6.13). Therefore the function Ψ_ε satisfies

$$\theta|_V + dd^c\Psi_\varepsilon = \theta|_V + dd^c\Phi_\varepsilon \geq -\varepsilon\omega$$

on the neighborhood of ∂B from property (6.12). Thus property (6.16) holds on X. $\square$

Finally, we estimate the volume $\mathrm{vol}_V(L)$ of L with current integrations for the computation of the intersection number $(L^d \cdot V)$. The function Ψ_ε is a $(\theta|_V + \varepsilon\omega)$-psh function by property (6.16). Here ω can be assumed to be a Kähler form which represents the first Chern class $c_1(B)$ of B, where B is an ample line bundle on V. The d-closed $(1,1)$-form $(\theta|_V + \varepsilon\omega)$ represents the first Chern class $c_1(L) + \varepsilon c_1(B)$. Thus by Proposition 6.2.4, we have

$$\mathrm{vol}_V\big(L + \varepsilon B\big) \geq \int_V \big(\theta|_V + \varepsilon\omega + dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}}.$$

Since $\big(\theta|_V + \varepsilon\omega + dd^c\Psi_\varepsilon\big)$ is a positive current, the absolute continuous part is (semi)-positive. It shows

$$\mathrm{vol}_V\big(L + \varepsilon B\big) \geq \int_U \big(\theta|_V + \varepsilon\omega + dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}},$$

where U is a neighborhood of p which satisfies the properties in Lemma 6.3.7. If we let ε tend to zero, the left hand of the inequality above converges to $\mathrm{vol}_V(L)$ from the continuity of the volume. Thus we have

$$\begin{aligned}
\mathrm{vol}_V\big(L\big) &\geq \liminf_{\varepsilon\to 0} \int_U \big(\theta|_V + \varepsilon\omega + dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}} \\
&\geq \int_U \liminf_{\varepsilon\to 0} \big(\theta|_V + \varepsilon\omega + dd^c\Psi_\varepsilon\big)^d_{\mathrm{ac}} \\
&= \int_U \big(\theta|_V + dd^c A|z|^2\big)^d_{\mathrm{ac}}.
\end{aligned}$$

The second inequality follows form Fatou's lemma. Here we used the fact that U does not shrink even if ε goes to 0, since U is independent of ε, The equality follows from property (6.17). By the choice of A (see (6.1)), the right hand of the inequality above

$$\int_U \left(\theta|_V + dd^c A|z|^2\right)^d_{\mathrm{ac}} = \int_U \left(\theta|_V + dd^c A|z|^2\right)^d$$

is positive. Hence we proved the volume $\mathrm{vol}_V(L)$ is positive for a non-singular subvariety V.

When V has singularities, we take an embedded resolution $\mu : \widetilde{V} \subseteq \widetilde{X} \longrightarrow V \subseteq X$. Then we can show that $\mathrm{vol}_{\widetilde{V}}(\mu^* L) > 0$ by the same argument as above. Note that we used only the following property on the line bundle L in the argument above.

$(*)$ For every point $y \in \partial B$, there exists a $\theta|_V$-psh function $\widetilde{\varphi}_y$ such that $\widetilde{\varphi}_y(p) = -\infty$ and $\widetilde{\varphi}_y(y) = 0$.

We can easily show that property $(*)$ holds for the pull-back $\mu^* L$ of L as follows: We first choose a point p on $\widetilde{V}$ such that μ is an isomorphism on a neighborhood B of p. For every point $y \in \partial B$, we consider a curve C_y on $\widetilde{V}$ which contains the points p and y. Since μ is an isomorphism on B, the push-forward $\mu(C_y)$ is not a point. Therefore it follows that the intersection number $\big(L \cdot \mu(C_y)\big)$ is positive from Proposition 6.3.1. Lemma 6.2.2 implies that there exists a $\theta|_{\mu(C_y)}$-psh function φ_y such that $\varphi_y\big(\mu(p)\big) = -\infty$ and $\varphi_y(\mu(y)) = 0$. By the assumption of Theorem 6.1.2 on L, we can extend φ_y to a global θ-psh function $\widetilde{\varphi}_y$ on X. Then the pull-back $\mu^*\widetilde{\varphi}_y$ of $\widetilde{\varphi}_y$ satisfies property $(*)$ which we want to obtain. By the same argument as above, we obtain $\mathrm{vol}_{\widetilde{V}}(\mu^* L) > 0$. Since the restriction $\mu|_{\widetilde{V}}$ a birational morphism from $\widetilde{V}$ to V, $\mathrm{vol}_V(L) = \mathrm{vol}_{\widetilde{V}}(\mu^* L)$ (see [Laz, Proposition 2.2.43]). Hence we proved the volume $\mathrm{vol}_V(L) > 0$ for any subvariety V, even if V has singularities.

Since L is a nef line bundle, the volume of L on V coincides with the intersection number $(L^d \cdot V)$. Therefore L is an ample line bundle by the Nakai-Moishezon-Kleiman criterion.

6.4 Examples

Example 6.4.1. (This example shows us that there exists a θ-psh function on some subvariety which can not extended to a global θ-psh function even if L is semi-ample and big.)

Let $\pi : X := \mathrm{Bl}_{\mathrm{p}}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$ be a blow-up along a point $p \in \mathbb{P}^2$ and L the pull-back of the hyperplane bundle by π. Then L is a semi-ample and big. (However L is not ample.) We denote by θ the pull-back of the Fubini-Study form on $\mathbb{P}^2$. Note that θ represents the first Chern class $c_1(L)$ of L. By the definition of θ, the restriction of θ to E is zero $(1,1)$-form, where E is the exceptional divisor. Therefore any $\theta|_E$-psh on E is constant by the maximum principle of psh functions. It says that a global θ-psh function has the same value along E.

Now we denote by C an irreducible curve on X which intersects E with at least two points. Then C is not contractive by π. Therefore the degree of L on C is positive by the projection formula (that is, $L|_C$ is ample on C). It implies that there exist many $\theta|_C$-psh functions on C. In particular, there exists a $\theta|_C$-psh function which has different values at intersection points with E. Indeed, we can obtain such function by using Lemma 6.2.2. Such function can not extend to a global θ-psh function.

Example 6.4.2. (Relations between the restricted volume of a line bundle and the extendability of θ-psh functions.)

Recall the definition of the restricted volume of a line bundle. The restricted volume of L along a subvariety V is defined to be

$$\mathrm{vol}_{X|V}(L) = \limsup_{k\to\infty} \frac{\dim H^0\big(X|V, \mathcal{O}(kL)\big)}{k^d/d!},$$

where

$$H^0\big(X|V, \mathcal{O}(kL)\big) = \mathrm{Im}\Big(H^0\big(X, \mathcal{O}_X(kL)\big) \longrightarrow H^0\big(V, \mathcal{O}_V(kL)\big)\Big).$$

The restricted volume measures the number of sections of $\mathcal{O}_V(kL)$ which can be extended to X. In Chapter 5, restricted volumes can be expressed with current integrations as follows (see Chapter 5 for details).

Theorem 6.4.3. (=Theorem 5.1.3). Assume that V is not contained in the augmented base locus $\mathbb{B}_+(L)$. Then the restricted volume of L along V satisfies the following equality :

$$\mathrm{vol}_{X|V}(L) = \sup_{\varphi\in \mathrm{Psh}(X,\theta)} \int_{V_{\mathrm{reg}}} (\theta|_{V_{\mathrm{reg}}} + dd^c\varphi|_{V_{\mathrm{reg}}})^d_{\mathrm{ac}},$$

for φ ranging among θ-psh functions on X with analytic singularities whose singular locus does not contain V.

The right hand integral measures Monge-Ampère products of $\theta|_V$-psh functions on V which can be extended to X. On the other hand, the volume of the line bundle $L|_V$ can also be expressed with current integrations (see Proposition 6.2.4 and [Bou02]). If any $\theta|_V$-psh function can be extended to a global θ-psh function, the restricted volume along V and the volume on V coincides. However there exists an example such that they are different even if V is not contained in the augmented base locus.

For example, when X is a surface, a big line bundle L admits a Zariski decomposition. That is, there exist nef $\mathbb{Q}$-divisor P and effective $\mathbb{Q}$-divisor N such that

$$H^0(X, \mathcal{O}_X(\lfloor kP \rfloor)) \longrightarrow H^0(X, \mathcal{O}_X(kL))$$

is an isomorphism. The map is multipling the section e_k, where e_k is the standard section of the effective divisor $\lceil kN \rceil$. Here $\lfloor G \rfloor$ (resp. $\lceil G \rceil$) denotes round down (resp. round up) of an $\mathbb{R}$-divisor G. Let V be an irreducible curve which is not contained in the augmented base locus $\mathbb{B}_+(L)$. Then $L|_V$ is an ample line bundle. Further, the restricted volume along V is computed by the self-intersection number of the positive part P along V when L admits a Zariski decomposition (see Proposition 5.3.1). That is,

$$\begin{aligned}
&\mathrm{vol}_{X|V}(L) = (P \cdot V), \\
&\mathrm{vol}_V(L) = (L \cdot V) = (P \cdot V) + (N \cdot V).
\end{aligned}$$

Therefore, the volume and restricted volume may be different value unless $(N \cdot V)$ is not equal to zero. When L is not nef (that is, N is non-zero divisor) and V is an ample divisor, $(N \cdot V)$ is not zero.

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