Topology, symplectic geometry and complex geometry of solvmanifolds –From nilpotent to solvable–
(和訳: 可解多様体のトポロジー、シンプレクティック幾何学および複素幾何学 –冪零から可解へ–)

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Introduction

Nilmanifolds are important objects for symplectic geometry and complex geometry. Thurston suggested 4-dimensional nilmanifolds as the first examples of complex and symplectic manifolds admitting no Kähler structure ([46]). Until now, many mathematicians gave many results in complex geometry and symplectic geometry on nilmanifolds. We introduce some well-known theorems.

Facts 1. *Let G be a simply connected nilpotent Lie group with a lattice (cocompact discrete subgroup) Γ and $\mathfrak{g}$ be a Lie algebra (the space of left-invariant complex structure). Consider the DGA $\bigwedge \mathfrak{g}^*$ of the left-invariant differential on G/Γ. Then:*

(1)The inclusion $\bigwedge \mathfrak{g}^* \subset A^*(G/\Gamma)$ induces a cohomology isomorphism

$$H^*(\mathfrak{g}) \cong H^*(G/\Gamma).$$

Thus the DGA $\bigwedge \mathfrak{g}^*$ is Sullivan's minimal model of $A^*(G/\Gamma)$.(Nomizu [32])

(2)Suppose G/Γ is symplectic. G/Γ satisfies the hard Lefschetz property if and only if G/Γ is torus.(Benson-Gordon [5])

(3) G/Γ is formal in the sense of Sullivan if and only if G/Γ is torus.(Hasegawa [20])

(4)If G/Γ is cohomologically symplectic, then G/Γ is really symplectic. (Corollary of Nomizu's theorem)

(5) Suppose G admits a left-invariant complex structure. Consider the Dolbeault complex $A^{*,*}(G/\Gamma)$ and its sub-DBA $\bigwedge^{*,*} \mathfrak{g}^*_{\mathbb{C}}$ of left-invariant $\mathbb{C}$-valued differential forms. Under some conditions, the inclusion $\bigwedge^{*,*} \mathfrak{g}^*_{\mathbb{C}} \subset A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism

$$H^{*,*}(\mathfrak{g}_{\mathbb{C}}) \cong H^{*,*}(G/\Gamma).$$

(Sakane [42], Cordero-Fernández-Gray-Ugarte [10] and Console-Fino [9])

Enlarging the class of nilmanifolds to solvmanifolds, geometry of solvmanifolds are more complicated than geometry of nilmanifolds. In this thesis we consider how we extend the above facts to geometry of solvmanifolds or how geometry of solvmanifolds and geometry of nilmanifolds are different.

Summary of main results

(Chapter 1) We consider the space of differential forms on the solvmanifold G/Γ with values in certain flat bundle so that this space has a structure of a differential graded algebra(DGA). We construct Sullivan's minimal model of this DGA. This result is an extension of Nomizu's theorem for ordinary coefficients in the nilpotent case. By using this result, we refine Hasegawa's result of formality of nilmanifolds and Benson-Gordon's result of hard Lefschetz properties of nilmanifolds.

(Chapter 2) We consider aspherical manifolds with torsion-free virtually polycyclic fundamental groups, constructed by Baues. We prove that if those manifolds are cohomologically symplectic, then they are symplectic. As a corollary we show that cohomologically symplectic solvmanifolds are symplectic.

(Chapter 3) We consider semi-direct products $\mathbb{C}^n \ltimes_\phi N$ of Lie groups with lattices Γ such that N are nilpotent Lie groups with left-invariant complex structures. We compute the Dolbeault cohomology of direct sums of holomorphic line bundles over G/Γ by using the Dolbeaut cohomology of the Lie algebras of the direct product $\mathbb{C}^n \times N$. As a corollary of this computation, we can compute the Dolbeault cohomology $H^{p,q}(G/\Gamma)$ of G/Γ by using a finite dimensional cochain complexes. Computing some examples, we observe that the Dolbeault cohomology varies for choices of lattices Γ.

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Chapter 1

Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems

1.1 The Purpose of this chapter

The main purpose of this part is to compute the de Rham cohomology of solvmanifolds with values in local coefficients associated to some diagonal representations by using of the invariant forms and the unipotent hulls. The computations are natural extensions of Nomizu's computations of untwisted de Rham cohomology of nilmanifolds by the invariant forms in [32]. The computations give natural extensions of Hasegawa's result of formality of nilmanifolds ([20]) and Benson and Gordon's result of hard Lefschetz properties of nilmanifolds ([6]).

First we explain the central tools of this paper called the unipotent hulls and algebraic hulls. Let G be a a simply connected solvable Lie group, there exists a unique algebraic group $\mathbf{H}_G$ called the algebraic hull of G with an injection $\psi : G \to \mathbf{H}_G$ so that:
(1) $\psi(G)$ is Zariski-dense in $\mathbf{H}_G$.
(2) The centralizer $Z_{\mathbf{H}_G}(\mathbf{U}(\mathbf{H}_G))$ of $\mathbf{U}(\mathbf{H}_G)$ is contained in $\mathbf{U}(\mathbf{H}_G)$.
(3) $\dim \mathbf{U}(\mathbf{H}_G) = \dim G$.
where we denote $\mathbf{U}(\mathbf{H})$ the unipotent radical of an algebraic group $\mathbf{H}$. We denote $\mathbf{U}_G = \mathbf{U}(\mathbf{H}_G)$ and call it the unipotent hull of G.

We consider Hain's DGAs in [19] which are expected to be effective techniques for studying de Rham homotopy theory of non-nilpotent spaces. Let M be a C^∞-manifold and $\rho:\pi_1(M,x)\to(\mathbb{C}^*)^n$ a representation and $\mathbf{T}$ the Zariski-closure of $\rho(\pi_1(M,x))$ in $(\mathbb{C}^*)^n$. Let $\{V_\alpha\}$ be the set of one-dimensional representations for all characters α of $\mathbf{T}$ and (E_α, D_α) be a rank one flat bundle with the monodromy $\alpha\circ\rho$ and $A^*(M,E_\alpha)$ the space of E_ρ-valued C^∞-differential forms. Denote $A^*(M,\mathcal{O}_\rho)=\bigoplus_\alpha A^*(M,E_\alpha)$ and $D=\bigoplus_\alpha D_\alpha$. Then $(A^*(M,\mathcal{O}_\rho),D)$ is a cohomologically connected(i.e. the 0-th cohomology is isomorphic to the ground field) DGA. In this paper we construct Sullivan's minimal model([45]) of such DGAs on solvmanifolds.

On simply connected solvable Lie groups, we consider DGAs of left-invariant differential forms with local systems which are analogues of Hain's DGA's. Suppose G is a simply connected solvable Lie group and $\mathfrak{g}$ is the Lie algebra of G. Consider the adjoint representation $\mathrm{Ad}:G\to \mathrm{Aut}(\mathfrak{g})$ and its derivation $\mathrm{ad}:\mathfrak{g}\to D(\mathfrak{g})$ where $D(\mathfrak{g})$ be the Lie algebra of the derivations of $\mathfrak{g}$. We construct representations of $\mathfrak{g}$ and G as following.

Construction 1.1.1. *Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. There exists a subvector space (not necessarily Lie algebra) V of $\mathfrak{g}$ so that $\mathfrak{g}=V\oplus\mathfrak{n}$ as the direct sum of vector spaces and for any $A,B\in V$ $(\mathrm{ad}_A)_s(B)=0$ where $(ad_A)_s$ is the semi-simple part of ad_A (see [14, Proposition III.1.1]). We define the map $\mathrm{ad}_s:\mathfrak{g}\to D(\mathfrak{g})$ as $\mathrm{ad}_{sA+X}=(\mathrm{ad}_A)_s$ for $A\in V$ and $X\in\mathfrak{n}$. Then we have $[\mathrm{ad}_s(\mathfrak{g}),\mathrm{ad}_s(\mathfrak{g})]=0$ and ad_s is linear (see [14, Proposition III.1.1]). Since we have $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{n}$, the map $\mathrm{ad}_s:\mathfrak{g}\to D(\mathfrak{g})$ is a representation and the image $\mathrm{ad}_s(\mathfrak{g})$ is abelian and consists of semi-simple elements. We denote by $\mathrm{Ad}_s:G\to\mathrm{Aut}(\mathfrak{g})$ the extension of ad_s. Then $\mathrm{Ad}_s(G)$ is diagonalizable.*

Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Then $\mathbf{T}$ is diagonalizable. Let $\{V_\alpha\}$ be the set of one-dimensional representations for all characters α of $\mathbf{T}$. We consider V_α the representation of $\mathfrak{g}$ which is the derivation of $\alpha\circ\mathrm{Ad}_s$. Then we have the cochain complex of Lie algebra $(\bigwedge\mathfrak{g}^*_\mathbb{C}\otimes V_\alpha, d_\alpha)$. Denote $A^*(\mathfrak{g}_\mathbb{C},\mathrm{ad}_s)=\bigoplus_\alpha\bigwedge\mathfrak{g}^*_\mathbb{C}\otimes V_\alpha$ and $d=\bigoplus_\alpha d_\alpha$. Then $(A^*(\mathfrak{g}_\mathbb{C},\mathrm{ad}_s),d)$ is a cohomologically connected DGA. In this paper we compute the cohomology of this DGA by the unipotent hull $\mathbf{U}_G$ of G. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}_G$ and $\bigwedge\mathfrak{u}^*$ be the cochain complex of the dual space $\mathfrak{u}^*$ of $\mathfrak{u}$. We prove the following theorem.

Theorem 1.1.1 (Theorem 1.5.4). *We have a quasi-isomorphism (i.e. a morphism which induces a cohomology isomorphism) of DGAs*

$$\bigwedge\mathfrak{u}^*\to A^*(\mathfrak{g}_\mathbb{C},\mathrm{ad}_s).$$

Thus $\bigwedge\mathfrak{u}^*$ is Sullivan's minimal model of $A^*(\mathfrak{g}_\mathbb{C},\mathrm{ad}_s)$.

Suppose G has a lattice Γ i.e. a cocompact discrete subgroup of G. We call a compact homogeneous space G/Γ a solvmanifold. We have $\pi_1(G/\Gamma) \cong \Gamma$. For the restriction of the semi-simple part of the adjoint representation $\mathrm{Ad}_{s|_\Gamma}$ on Γ, we consider Hain's DGA $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$. By using Theorem 1.1.1, we prove:

Theorem 1.1.2 (Corollary 1.7.5). *Let G be a simply connected solvable Lie group with a lattice Γ and $\mathbf{U}_G$ be the unipotent hull of G. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}_G$. Then we have a quasi-isomorphism*

$$\bigwedge \mathfrak{u}^* \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}}).$$

Thus $\bigwedge \mathfrak{u}^*$ is Sullivan's minimal model of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$.

If G is nilpotent, the adjoint operator Ad is a unipotent representation and hence $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}}) = A^*_{\mathbb{C}}(G/\Gamma)$ and $A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s) = \bigwedge \mathfrak{g}_{\mathbb{C}} = \bigwedge \mathfrak{u}^*$. In this case, Theorem 1.1.2 reduce to the classical theorem proved by Nomizu in [32]. Moreover this result gives more progressed computations of untwisted de Rham cohomology of solvmanifolds than the results of Mostow and Hattori (see Corollary 1.7.4 and Section 1.10).

We call a DGA A formal if there exists a finite diagram of morphisms

$$A \to C_1 \leftarrow C_2 \cdots \leftarrow H^*(A)$$

such that all morphisms are quasi-isomorphisms and we call manifolds M formal if the de Rham complex $A^*(M)$ is formal. In [20] Hasegawa showed that formal nilmanifolds are tori. By the results of this paper, we have a natural extension of Hasegawa's theorem for solvmanifolds.

Theorem 1.1.3 (Theorem 1.8.2). *Let G be a simply connected solvable Lie group. Then the following conditions are equivalent:*
(A) The DGA $A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s)$ is formal.
(B) $\mathbf{U}_G$ is abelian.
(C) $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.
Moreover suppose G has a lattice Γ. Then the above three conditions are equivalent to the following condition:
(D) $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$ is formal.

In [6] Benson and Gordon showed that symplectic nilmanifolds with the hard Lefschetz properties are tori. We can also have an extension of Benson and Gordon's theorem.

Theorem 1.1.4 (Theorem 1.8.4). *Let G be a simply connected solvable Lie group. Suppose $\dim G = 2n$ and G has an G-invariant symplectic form ω. Then the following conditions are equivalent:*
(A)

$$[\omega]^{n-i}\wedge : H^{i}(A^{*}(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_{s})) \to H^{2n-i}(A^{*}(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_{s}))$$

is an isomorphic for any $i \leq n$.
(B) $\mathbf{U}_G$ is abelian.
(C) $G = \mathbb{R}^{n} \ltimes_{\phi} \mathbb{R}^{m}$ such that the action $\phi : \mathbb{R}^{n} \to \mathrm{Aut}(\mathbb{R}^{m})$ is semi-simple.

Suppose G has a lattice Γ and G/Γ has a symplectic form(not necessarily G-invariant) ω. Then the conditions (B) and (C) are equivalent to the following condition:
(D)

$$[\omega]^{n-i}\wedge : H^{i}(A^{*}(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})) \to H^{2n-i}(A^{*}(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$$

is an isomorphism for any $i \leq n$

Remark 1.1.1. *As a representation in an algebraic group, Ad_s is independent of the choice of a subvector space V in Construction 1.1.1 (see Lemma 1.2.5). By this, the structures of DGAs $A^{*}(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_{s})$ and $A^{*}(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ are independent of the choice of a subvector space V.*

Finally we consider relations with Kähler geometries. We review studies of Kähler structures on solvmanifolds briefly. See [5] and [22] for more details. In [7] Benson and Gordon conjectured that for a completely solvable simply connected Lie group G with a lattice Γ, G/Γ has a Kähler metric if and only if G/Γ is a torus. In [21] Hasegawa studied Kähler structures on some classes of solvmanifolds which are not only completely solvable type and suggested a generalized version of Benson-Gordon's conjecture: A compact solvmanifold can have a Kähler structure if and only if it is a finite quotient of a complex torus that is a holomorphic fiber bundle over a complex torus with fiber a complex torus. In [1] Arapura showed Benson-Gordon's conjecture and also showed that the fundamental group of a Kähler solvmanifold is virtually abelian by the result in [2]. In [1] a proof of Hasegawa's conjecture was also written but we notice that this proof contains a gap and Hasegawa complement in [22]. We also notice that Baues and Cortés showed a more generalized version of Benson-Gordon's conjecture for aspherical manifolds with polycyclic fundamental groups in [5].

By the theory of Higgs bundle studied by Simpson, we have a twisted analogues of formality(see [13]) and the hard Lefschetz properties of compact Kähler manifolds. We have:

Theorem 1.1.5 (Special case of Thoerem 1.4.1). *Suppose M is a compact Kähler manifold with a Kähler form ω and $\rho : \pi_1(M) \to (\mathbb{C}^*)^n$ is a representation. Then the following conditions hold:*
(A) *(formality) The DGA $A^*(M, \mathcal{O}_\rho)$ is formal.*
(B) *(hard Lefschetz) For any $0 \leq i \leq n$ the linear operator*

$$[\omega]^{n-i}\wedge : H^i(A^*(M, \mathcal{O}_\rho)) \to H^{2n-i}(A^*(M, \mathcal{O}_\rho))$$

is an isomorphism where $\dim_{\mathbb{R}} M = 2n$.

Now by Theorem 1.1.5 formality and hard Lefschetz property of DGA $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$ are criteria for G/Γ to has a Kähler metric. We will see such conditions are stronger than untwisted formality and hard Lefschetz property.

Remark 1.1.2. *There exist examples of solvmanifolds G/Γ which satisfy formality and the hard Lefschetz property of the untwisted de Rham complex $A^*(G/\Gamma,)$ but do not satisfy formality and the hard Lefschetz property of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$.*

However we will see these criteria can not classify Kähler solvmanifolds completely.

Remark 1.1.3. *There exist examples of non-Kähler solvmanifolds which satisfy formality and the hard Lefschetz property of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$.*

1.2 Preliminaries on algebraic hulls

Let G be a discrete group (resp. a Lie group). We call a map $\rho : G \to GL_n(\mathbb{C})$ a representation, if ρ is a homomorphism of groups (resp. Lie groups).

1.2.1 Algebraic groups

In this paper an algebraic group means an affine algebraic variety $\mathbf{G}$ over $\mathbb{C}$ with a group structure such that the multiplication and inverse are morphisms of varieties. All algebraic groups in this paper arise as Zariski-closed subgroups of $GL_n(\mathbb{C})$. Let k be a subfield of $\mathbb{C}$. We call $\mathbf{G}$ k-algebraic if $\mathbf{G}$ is defined by polynomials with coefficient in k. We denote $\mathbf{G}(k)$ the k-points of $\mathbf{G}$. We say that an algebraic group is diagonalizable if it is isomorphic to a closed subgroup of $(\mathbb{C}^*)^n$ for some n.

1.2.2 Algebraic hulls

A group Γ is polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. For a polycyclic group Γ, we denote $\operatorname{rank}\Gamma = \sum_{i=1}^{i=k} \operatorname{rank}\Gamma_{i-1}/\Gamma_i$. Let G be a simply connected solvable Lie group and Γ be a lattice in G. Then Γ is torsion-free polycyclic and $\dim G = \operatorname{rank}\Gamma$ (see [40, Proposition 3.7]). Let $\rho : G \to GL_n(\mathbb{C})$, for $g \in G$ be a representation. Let $\mathbf{G}$ and $\mathbf{G}'$ be the Zariski-closures of $\rho(G)$ and $\rho(\Gamma)$ in $GL_n(\mathbb{C})$. Then we have $\mathbf{U}(\mathbf{G}) = \mathbf{U}(\mathbf{G}')$ (see [40, Theorem 3.2]).

We review the algebraic hulls.

Proposition 1.2.1. ([40, Proposition 4.40]) *Let G be a simply connected solvable Lie group (resp. torsion-free polycyclic group). Then there exists a unique $\mathbb{R}$-algebraic group $\mathbf{H}_G$ with an injective group homomorphism $\psi : G \to \mathbf{H}_G(\mathbb{R})$ so that:*
(1) $\psi(G)$ is Zariski-dense in $\mathbf{H}_\mathbf{G}$.
(2) $Z_{\mathbf{H}_G}(\mathbf{U}(\mathbf{H}_G)) \subset \mathbf{U}(\mathbf{H}_G)$.
(3) $\dim \mathbf{U}(\mathbf{H}_G) = \dim G$ (resp. $\operatorname{rank} G$).
Such $\mathbf{H}_G$ is called the algebraic hull of G.

We denote $\mathbf{U}_G = \mathbf{U}(\mathbf{H}_G)$ and call $\mathbf{U}_G$ the unipotent hull of G.

1.2.3 Direct constructions of algebraic hulls

Let $\mathfrak{g}$ be a solvable Lie algebra, and $\mathfrak{n} = \{X \in \mathfrak{g} | \mathrm{ad}_X \text{ is nilpotent}\}$. $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$ and called the nilradical of $\mathfrak{g}$. Then we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$. Consider the adjoint representation $\mathrm{ad} : \mathfrak{g} \to D(\mathfrak{g})$ and the representation $\mathrm{ad}_s : \mathfrak{g} \to D(\mathfrak{g})$ as Construction 1.1.1.

Let $\bar{\mathfrak{g}} = \operatorname{Im} \mathrm{ad}_s \ltimes \mathfrak{g}$ and

$$\bar{\mathfrak{n}} = \{X - \mathrm{ad}_{sX} \in \bar{\mathfrak{g}} | X \in \mathfrak{g}\}.$$

Then we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \bar{\mathfrak{n}}$ and $\bar{\mathfrak{n}}$ is the nilradical of $\bar{\mathfrak{g}}$ (see [14]). Hence we have $\bar{\mathfrak{g}} = \operatorname{Im} \mathrm{ad}_s \ltimes \bar{\mathfrak{n}}$.

Lemma 1.2.2. *Suppose $\mathfrak{g} = \mathbb{R}^k \ltimes_\phi \mathfrak{n}$ such that ϕ is a semi-simple action and $\mathfrak{n}$ is nilpotent. Then $\bar{\mathfrak{n}} = \mathbb{R}^k \oplus \mathfrak{n}$.*

Proof. By the assumption, for $X+Y \in \mathbb{R}^k \ltimes_\phi \mathfrak{n}$, we have $\mathrm{ad}_{sX+Y} = \mathrm{ad}_X$. Hence we have

$$[X_1+Y_1-\mathrm{ad}_{sX_1+Y_1}, X_2+Y_2-\mathrm{ad}_{sX_2+Y_2}] = [X_2, Y_2]$$

for $X_1+Y_1, X_2+Y_2 \in \mathbb{R}^k \ltimes_\phi \mathfrak{n}$. Hence the lemma follows. □

Let G be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of G. Let N be the subgroup of G which corresponds to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. We consider the exponential map $\exp : \mathfrak{g} \to G$. In general exp is not a diffeomorphism. But we have the useful property of exp as following.

Lemma 1.2.3. ([12, Lemma 3.3]) *Let V be a subvector space (not necessarily Lie algebra) V of $\mathfrak{g}$ so that $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of vector spaces. We define the map $F : \mathfrak{g} = V \oplus \mathfrak{n} \to G$ as $F(A+X) = \exp(A)\exp(X)$ for $A \in V$, $X \in \mathfrak{n}$. Then F is a diffeomorphism and we have the commutative diagram*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r] & G/N \cong \mathbb{R}^k \arrow[r] & 1 \\ 0 \arrow[r] & \mathfrak{n} \arrow[r] \arrow[u, "\exp"] & \mathfrak{g} \arrow[r] \arrow[u, "F"] & \mathfrak{g}/\mathfrak{n} \cong \mathbb{R}^k \arrow[r] \arrow[u, "\exp=\mathrm{id}_{\mathbb{R}^k}"'] & 0 \end{tikzcd}$$

where $\dim G/N = k$.

By this Lemma, for $A \in V$, $X \in \mathfrak{n}$, the extension $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g}_\mathbb{C})$ is given by

$$\mathrm{Ad}_s(\exp(A)\exp(X)) = \exp((\mathrm{ad}_A)_s) = (\exp(\mathrm{ad}_A))_s$$

and we have $\mathrm{Ad}_s(G) = \{(\exp(\mathrm{ad}_A))_s \in \mathrm{Aut}(\mathfrak{g}_\mathbb{C}) | A \in V\}$. Let $\bar{G} = \mathrm{Ad}_s(G) \ltimes G$. Then the Lie algebra of $\bar{G}$ is $\bar{\mathfrak{g}}$. For the nilradical $\bar{N}$ of $\bar{G}$, by the spritting $\bar{\mathfrak{g}} = \mathrm{Im}\,\mathrm{ad}_s \ltimes \bar{\mathfrak{n}}$ we have $\bar{G} = \mathrm{Ad}_s(G) \ltimes \bar{N}$ such that we can regard $\mathrm{Ad}_s(G) \subset \mathrm{Aut}(\bar{\mathrm{N}})$ and $\mathrm{Ad}_s(G)$ consists of semi-simple automorphisms of $\bar{N}$. By the construction of $\bar{\mathfrak{n}}$ we have $\bar{G} = G \cdot \bar{N}$.

A simply connected nilpotent Lie group is considered as the real points of a unipotent $\mathbb{R}$-algebraic group (see [36, p. 43]) by the exponential map. We have the unipotent $\mathbb{R}$-algebraic group $\bar{\mathbf{N}}$ with $\bar{\mathbf{N}}(\mathbb{R}) = \bar{N}$. We identify $\mathrm{Aut}_{\mathrm{a}}(\bar{\mathbf{N}})$ with $\mathrm{Aut}(\mathfrak{n}_\mathbb{C})$ and $\mathrm{Aut}_{\mathrm{a}}(\bar{\mathbf{N}})$ has the $\mathbb{R}$-algebraic group structure with $\mathrm{Aut}_a(\bar{\mathbf{N}})(\mathbb{R}) = \mathrm{Aut}(\bar{N})$. So we have the $\mathbb{R}$-algebraic group $\mathrm{Aut}_{\mathrm{a}}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}$. By $\mathrm{Ad}_s(G) \ltimes G = \mathrm{Ad}_s(G) \ltimes \bar{N}$, we have the injection $I : G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N} = \mathrm{Aut}_{\mathrm{a}}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}(\mathbb{R})$. Let $\mathbf{G}$ be the Zariski-closure of $I(G)$ in $\mathrm{Aut}_{\mathrm{a}}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}$.

Proposition 1.2.4. *Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}\,\bar{\mathbf{N}}$. Then we have $\mathbf{G} = \mathbf{T} \ltimes \bar{\mathbf{N}}$ and $\mathbf{G}$ is the algebraic hull of G with the unipotent hull $\mathbf{U}_G = \bar{\mathbf{N}}$. Hence the Lie algebra of unipotent hull $\mathbf{U}_G$ of G is*

$$\bar{\mathfrak{n}}_{\mathbb{C}} = \{X - \mathrm{ad}_{sX} \in \bar{\mathfrak{g}}_{\mathbb{C}} | X \in \mathfrak{g}_{\mathbb{C}}\}.$$

Proof. The algebraic group $\mathbf{T} \ltimes \bar{\mathbf{N}}$ is the Zariski-closure of $\mathrm{Ad}_s(G) \ltimes \bar{N}$ in $\mathrm{Aut}(\bar{N}) \ltimes \bar{N}$. By $\mathrm{Ad}_s(G) \cdot I(G) = \mathrm{Ad}_s(G) \ltimes \bar{N}$, we have $\mathbf{T} \cdot \mathbf{G} = \mathbf{T} \ltimes \bar{\mathbf{N}}$. Since $\mathbf{T}$ is a diagonalizable algebraic group, we have $\bar{\mathbf{N}} \subset \mathbf{G}$. Otherwise since $\mathbf{G} \subset \mathbf{T} \ltimes \bar{\mathbf{N}}$ is a connected solvable algebraic group, we have $\mathbf{U}(\mathbf{G}) = \bar{\mathbf{N}} \cap \mathbf{G} = \bar{\mathbf{N}}$. Since we have $\mathrm{Ad}_s(G) \ltimes \bar{N} = G \cdot \bar{N}$, $\mathbf{G}$ is identified with the Zariski-closure of $\mathrm{Ad}_s(G) \ltimes \bar{N}$. Hence we have $\mathbf{G} = \mathbf{T} \ltimes \bar{\mathbf{N}}$. By $\dim G = \dim \bar{N}$, we can easily check that $\mathbf{T} \ltimes \bar{\mathbf{N}}$ is the algebraic hull of G. $\square$

By this proposition the Zariski-closure $\mathbf{T}$ of $\mathrm{Ad}_s(G)$ is a maximal torus of the algebraic hull of G. By the uniqueness of the algebraic hull (see [40, Lemma 4.41]), we have:

Lemma 1.2.5. *Let $\mathbf{H}_G$ be the algebraic hull of G and $q : \mathbf{H}_G \to \mathbf{H}_G/\mathbf{U}_G$ the quotient map. Then for any injection $\psi : G \to \mathbf{H}_G(\mathbb{R})$ as in Proposition 1.2.1, there exists an isomorphism $\varphi : \mathbf{H}_G/\mathbf{U}_G \to \mathbf{T}$ such that the diagram*

$$\begin{tikzcd}
\mathbf{H}_G/\mathbf{U}_G \arrow[r, "\varphi"] & \mathbf{T} \\
G \arrow[u, "q\circ\psi"] \arrow[ur, "\mathrm{Ad}_s"'] &
\end{tikzcd}$$

commutes.

Lemma 1.2.6. *Let $H_G = \mathbf{H}_G(\mathbb{R})$ be the real points of the algebraic hull of G. Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ and $T = \mathbf{T}(\mathbb{R})$ its real points. Then we have a semi-direct product*

$$H_G = T \ltimes G.$$

Proof. By $\mathrm{Im}(\mathrm{ad}_s) \ltimes \bar{\mathfrak{n}} = \mathrm{Im}(\mathrm{ad}_s) \ltimes \mathfrak{g}$, we have $\mathrm{Ad}_s(G) \ltimes \bar{N} = \mathrm{Ad}_s(G) \ltimes I(G)$. Hence the lemma follows from Proposition 1.2.4. $\square$

Proposition 1.2.7. ([26]) *Let G be a simply connected solvable Lie group. Then $\mathbf{U}_G$ is abelian if and only if $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.*

Proof. Suppose $\mathbf{U}_G$ is abelian. Then by proposition 1.2.4, the Lie algebra $\bar{\mathfrak{n}}$ is abelian. By $\mathfrak{n} \subset \bar{\mathfrak{n}}$, the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is abelian. By $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$, $\mathfrak{g}$ is two-step solvable. We consider the lower central series $\mathfrak{g}^i$ as $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$ for $i \geq 1$. We denote $\mathfrak{n}' = \cap_{i=0}^{\infty} \mathfrak{g}^i$. Then by [11, Lemma 4.1], we have $\mathfrak{g} = \mathfrak{g}/\mathfrak{n}' \ltimes \mathfrak{n}'$. For this decomposition, the subspace $\{X - \mathrm{ad}_{sX} | X \in \mathfrak{g}/\mathfrak{n}'\} \subset \bar{\mathfrak{n}}$ is a Lie subalgebra of $\bar{\mathfrak{n}}$. Since $\mathfrak{g}/\mathfrak{n}'$ is a nilpotent subalgebra of $\mathfrak{g}$, this space is identified with $\mathfrak{g}/\mathfrak{n}'$. Thus since $\bar{\mathfrak{n}}$ is abelian, $\mathfrak{g}/\mathfrak{n}'$ is also abelian. We show that the action of $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{n}$ is semi-simple. Suppose for some $X \in \mathfrak{g}/\mathfrak{n}'$ ad_X on $\mathfrak{n}$ is not semi-simple . Then $\mathrm{ad}_X - \mathrm{ad}_{sX}$ on $\mathfrak{n}$ is not trivial. Since we have $\bar{\mathfrak{n}} = \{X - \mathrm{ad}_{sX} | X \in \mathfrak{g}\}$, we have $[\bar{\mathfrak{n}}, \mathfrak{n}] \neq \{0\}$. This contradicts $\bar{\mathfrak{n}}$ is abelian. Hence the action of $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{n}$ is semi-simple. Hence the first half of the proposition follows. The converse follows from Lemma 1.2.2. □

1.3 Left-invariant forms and the cohomology of solvmanifolds

Let G be a simply connected solvable Lie group, $\mathfrak{g}$ the Lie algebra of G and $\rho : G \to GL(V_\rho)$ a representation on a $\mathbb{C}$-vector space V_ρ. We consider the cochain complex $\bigwedge \mathfrak{g}^*$ with the derivation d which is the dual to the Lie bracket of $\mathfrak{g}$. Then $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho$ is a cochain complex with the derivation $d_\rho = d + \rho_*$ where ρ_* is the derivation of ρ and consider $\rho_* \in \mathfrak{g}^*_{\mathbb{C}} \otimes \mathfrak{gl}(V_\rho)$. We can consider the cochain complex $(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho, d_\rho)$ the twisted G-invariant differential forms on G. Consider the cochain complex $A^*_{\mathbb{C}}(G) \otimes V_\rho$ with the derivation d such that

$$d(\omega \otimes v) = (d\omega) \otimes v \quad \omega \in A^*_{\mathbb{C}}(G), \quad v \in V_\rho.$$

By the left action of G (given by $(g \cdot f)(x) = f(g^{-1}x)$, $f \in C^\infty(G)$, $g \in G$) and ρ, we have the action of G on $A^*_{\mathbb{C}}(G) \otimes V_\rho$. Denote $(A^*_{\mathbb{C}}(G) \otimes V_\rho)^G$ the G-invariant elements of $A^*_{\mathbb{C}}(G) \otimes V_\rho$. Then we have an isomorphism

$$(A^*_{\mathbb{C}}(G) \otimes V_\rho)^G \cong \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho.$$

Suppose G has a lattice Γ. Since $\pi_1(G/\Gamma) = \Gamma$, we have a flat vector bundle $E_{\rho_{|\Gamma}}$ with flat connection $D_{\rho_{|\Gamma}}$ on G/Γ whose monodromy is $\rho_{|\Gamma}$. Let $A^*(G/\Gamma, E_{\rho_{|\Gamma}})$ be the cochain complex of $E_{\rho_{|\Gamma}}$-valued differential forms with the derivation $D_{\rho_{|\Gamma}}$. Consider the cochain complex $A^*_{\mathbb{C}}(G) \otimes V_\rho$ with derivation d such that

$$d(\omega \otimes v) = (d\omega) \otimes v \quad \omega \in A^*_{\mathbb{C}}(G), \quad v \in V_\rho.$$

Then we have the G-action on $A^*_{\mathbb{C}}(G) \otimes V_\rho$ and denote $(A^*_{\mathbb{C}}(G) \otimes V_\rho)^\Gamma$ the subcomplex of Γ-invariant elements of $A^*_{\mathbb{C}}(G) \otimes V_\rho$. We have the isomorphism $(A^*_{\mathbb{C}}(G) \otimes V_\rho)^\Gamma \cong A^*(G/\Gamma, E_{\rho_{|\Gamma}})$. Thus we have

$$\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho \cong (A^*_{\mathbb{C}}(G) \otimes V_\rho)^G \subset (A^*_{\mathbb{C}}(G) \otimes V_\rho)^\Gamma \cong A^*(G/\Gamma, E_{\rho_{|\Gamma}})$$

and we have the inclusion $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho \to A^*(G/\Gamma, E_{\rho_{|\Gamma}})$.

We call a representation ρ Γ-admissible if for the representation $\rho \oplus \mathrm{Ad} : G \to GL_n(\mathbb{C}) \times \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$, $(\rho \oplus \mathrm{Ad})(G)$ and $(\rho \oplus \mathrm{Ad})(\Gamma)$ have the same Zariski-closure in $GL_n(\mathbb{C}) \times \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$.

Theorem 1.3.1. ([28],[40, Theorem 7.26]) *If ρ is Γ-admissible, then the inclusion*

$$\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\rho \to A^*(G/\Gamma, E_{\rho_{|\Gamma}})$$

induces a cohomology isomorphism.

Proposition 1.3.2. *Let G be a simply connected solvable Lie group with a lattice Γ. We suppose $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We consider the diagonalizable representation $\mathrm{Ad}_s : G \to \mathrm{Aut}(G)$. Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ and α be a character of $\mathbf{T}$. Then $\alpha \circ \mathrm{Ad}_s$ is Γ-admissible.*

Proof. Let $\mathbf{G}$ be the Zariski-closure of $\mathrm{Ad}(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We first show that $\mathbf{T}$ is a maximal torus of $\mathbf{G}$. For the direct sum $\mathfrak{g} = V \oplus \mathfrak{n}$ as Construction 1.1.1, the map $F : V \oplus \mathfrak{n} \to G$ defined by $F(A+X) = \exp(A)\exp(X)$ for $A \in V$, $X \in \mathfrak{n}$ is a diffeomorphism (see [12, Lemaa 3.3]). For $A \in V$, we consider the Jordan decomposition $\mathrm{Ad}(\exp(A)) = \exp((\mathrm{ad}_A)_s)\exp((\mathrm{ad}_A)_n)$. Then we have $\exp((\mathrm{ad}_A)_s), \exp((\mathrm{ad}_A)_n) \in \mathbf{G}$. For $X \in \mathfrak{n}$, we have $\exp(\mathrm{ad}_X) \in \mathbf{U}(\mathbf{G})$. Hence we have $\mathrm{Ad}(G) \subset \mathbf{T}\mathbf{U}(\mathbf{G}) \subset \mathbf{G}$. Since $\mathbf{G}$ is Zariski-closure of $\mathrm{Ad}(G)$, $\mathbf{G} = \mathbf{T}\mathbf{U}(\mathbf{G})$. Thus $\mathbf{T}$ is a maximal torus of $\mathbf{G}$.

We take a spritting $\mathbf{G} = \mathbf{T} \ltimes \mathbf{U}(\mathbf{G})$. We consider the algebraic group

$$\mathbf{G}' = \{(\alpha(t), (t,u)) \in \mathbb{C}^* \times \mathbf{G} | (t,u) \in \mathbf{T} \ltimes \mathbf{U}(\mathbf{G})\}.$$

Then we have

$$\begin{aligned}&(\alpha \circ \mathrm{Ad}_s \oplus \mathrm{Ad})(G)\\ &\quad = \{(\alpha(\exp((\mathrm{ad}_A)_s)), \exp(\mathrm{ad}_A)\exp(\mathrm{ad}_X)) | A + X \in V \oplus \mathfrak{n}\}\\ &\quad \subset \mathbf{G}'.\end{aligned}$$

Since $\mathbf{G}$ is Zariski-closure of $\mathrm{Ad}(G)$, $(\alpha \circ \mathrm{Ad}_s \oplus \mathrm{Ad})(G)$ is Zariski-dense in $\mathbf{G}'$. Since $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure, $(\alpha \circ \mathrm{Ad}_s \oplus \mathrm{Ad})(G)$ and $(\alpha \circ \mathrm{Ad}_s \oplus \mathrm{Ad})(\Gamma)$ have the same Zariski-closure $\mathbf{G}'$. □

1.4 Hain's DGAs

1.4.1 Constructions

Let M be a C^∞-manifold, $\mathbf{S}$ be a reductive algebraic group and $\rho : \pi_1(M,x) \to \mathbf{S}$ be a representation. We assume the image of ρ is Zariski-dense in $\mathbf{S}$. Let $\{V_\alpha\}$ be the set of irreducible representations of $\mathbf{S}$ and (E_α, D_α) be a flat bundle with the monodromy $\alpha \circ \rho$ and $A^*(M, E_\alpha)$ the space of E_ρ-valued C^∞-differential forms. Then we have an algebra isomorphism of $\bigoplus_\alpha V_\alpha \otimes V_\alpha^*$ and the coordinate ring $\mathbb{C}[\mathbf{S}]$ of $\mathbf{S}$ (see [19, Section 3]). Denote

$$A^*(M, \mathcal{O}_\rho) = \bigoplus_\alpha A^*(M, E_\alpha) \otimes V_\alpha^*$$

and $D = \bigoplus_\alpha D_\alpha$. Then by the wedge product, $(A(M, \mathcal{O}_\rho)), D)$ is a cohomologically connected DGA with coefficients in $\mathbb{C}$.

Suppose $\mathbf{S}$ is a diagonal algebraic group. Then $\{V_\alpha\}$ is the set of one-dimensional representations for all algebraic characters α of $\mathbf{T}$ and (E_α, D_α) are rank one flat bundles with the monodromy $\alpha \circ \rho$. In this case for characters α and β, we have the wedge product $A^*(M, E_\alpha) \otimes A^*(M, E_\beta) \to A^*(M, E_{\alpha\beta})$ and $D_{\alpha\beta}(\psi_\alpha \wedge \psi_\beta) = D_\alpha\psi_\alpha \wedge \psi_\beta + (-1)^p \psi_\alpha \wedge D_\beta \psi_\beta$ for $\psi_\alpha \in A^p(M, E_\alpha)$, $\psi_\beta \in A^q(M, E_\beta)$ (see [30] for details in this case).

1.4.2 Formality and the hard Lefschetz properties of compact Kähler manifolds

In this subsection we will prove the following theorem by theories of Higgs bundles studied by Simpson.

Theorem 1.4.1. *Let M be a compact Kähler manifold with a Kähler form ω and $\rho : \pi_1(M) \to \mathbf{S}$ a representation to a reductive algebraic group $\mathbf{S}$ with the Zariski-dense imaga. Then the following conditions hold:*
(A) *(formality) The DGA $A^*(M, \mathcal{O}_\rho))$ is formal.*
(B) *(hard Lefschetz) For any $0 \le i \le n$ the linear operator*

$$[\omega]^i \wedge : H^{n-i}(A^*(M, \mathcal{O}_\rho))) \to H^{n+i}(A^*(M, \mathcal{O}_\rho)))$$

is an isomorphism where $\dim_{\mathbb{R}} M = 2n$.

Let M be a compact Kähler manifold and E a holomorphic vector bundle on M with the Dolbeault operator $\bar\partial$. For a $\mathrm{End}(E)$-valued holomorphic form θ, we denote $D'' = \bar\partial + \theta$. We call (E, D'') a Higgs bundle if it satisfies the Leibniz rule: $D''(ae) = \bar\partial(a)e + (-1)^p D''(e)$ for $a \in A^p(M)$, $e \in A^0(E)$ and the integrability: $(D'')^2 = 0$. Let h be a Hermitian metric on E. For a Higgs bundle $(E, D'' = \bar{\partial} + \theta)$. We define $D'_h = \partial_h + \bar{\theta}_h$ as follows: ∂_h is the unique operator which satisfies

$$h(\bar{\partial}e, f) + h(e, \partial_h f) = \bar{\partial}h(e, f)$$

and $\bar{\theta}_h$ is defined by $(\theta e, f) = (e, \bar{\theta}_h f)$. Let $D_h = D'_h + D''$. Then D_h is a connection. We call a Higgs bundle (E, D'', h) with a metric harmonic if D_h is flat i.e. $(D_h)^2 = 0$.

For two Higgs bundles (E, D''), (F, D'') with metric h_E, h_F, the tensor product $(E \otimes F, D'' \otimes 1 + 1 \otimes D'')$ is an also Higgs bundle and $h_E \otimes h_F$ gives the connection $D_{h_E \otimes h_F} = D_{h_E} \otimes 1 + 1 \otimes D_{h_F}$ on $E \otimes F$. If (E, D'', h_E) and (F, D'', h_F) are harmonic, $(E \otimes F, D'' \otimes 1 + 1 \otimes D'')$ is also a harmonic Higgs bundle with the flat connection $D_{h_E} \otimes 1 + 1 \otimes D_{h_F}$.

Theorem 1.4.2. ([44, Theorem 1]) *Let (E, D) be a flat bundle on M whose monodromy is semi-simple. Then D is given by a harmonic Higgs bundle (E, D'', h) that is $D = D_h$.*

Theorem 1.4.3. ([44, Lemma 2.2]) *Let (E, D'', h) be a harmonic Higgs bundle with the flat connection $D = D' + D''$. Then the inclusion*

$$(\operatorname{Ker} D', D'') \to (A^*(E), D)$$

and the quotient

$$(\operatorname{Ker} D', D'') \to (H_{D'}(A^*(E)), D'') = (H^*_D(A^*(E)), 0)$$

induce the cohomology isomorphisms.

Theorem 1.4.4. ([44, Lemma 2.6]) *Let (E, D'', h) be a harmonic Higgs bundle with the flat connection $D = D' + D''$. Then for any $0 \leq i \leq n$ the linear operator*

$$[\omega]^{n-i} \wedge : H^i_D(A^*(E_\rho)) \to H^{2n-i}_D(A^*(E_\rho))$$

is an isomorphism.

Proof of Theorem 1.4.1.
By Theorem 1.4.2 and 1.4.4, the condition (B) holds. By Theorem 1.4.2, for $(A^*(E_\alpha), D_\alpha)$, we have $D_\alpha = D'_\alpha + D''_\alpha$ such that D''_α is a harmonic Higgs bundle. Denote $D' = \bigoplus_\alpha D'_\alpha$ and $D'' = \bigoplus_\alpha D''_\alpha$. Then by properties of Higgs bundle, $(\operatorname{Ker} D', D'')$ is a DGA, and the maps

$$(\operatorname{Ker} D', D'') \to (A^*(M, \mathcal{O}_\rho)), D)$$

and

$$(\operatorname{Ker} D', D'') \to (H^*_D(A^*(M, \mathcal{O}_\rho)), 0)$$

are DGA homomorphisms, thus quasi-isomorphisms by Theorem 1.4.3. Hence the condition (A) holds. $\square$

1.5 Minimal models of invariant forms on solvable Lie groups with local systems

Let G be a simply connected solvable Lie group and $\mathfrak{g}$ the Lie algebra of G. Consider the diagonal representation Ad_s as in Section 1 and the derivation ad_s of Ad_s. For some basis $\{X_1, \dots X_n\}$ of $\mathfrak{g}_{\mathbb{C}}$, Ad_s is represented by diagonal matrices. Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. Let $\{V_\alpha\}$ be the set of one-dimensional representations for all characters α of $\mathbf{T}$. We consider V_α the representation of $\mathfrak{g}$ which is the derivation of $\alpha \circ \mathrm{Ad}_s$. Then we have the cochain complex of Lie algebra $(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha, d_\alpha)$. Denote $d = \bigoplus_\alpha d_\alpha$. Then $(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha, d)$ is a cohomologically connected DGA with coefficients in $\mathbb{C}$ as the last section. By $\mathrm{Ad}_s(G) \subset \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ we have $\mathbf{T} \subset \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ and hence we have the action of $\mathbf{T}$ on $\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$. Denote $(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)^{\mathbf{T}}$ the sub-DGA of $\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ which consists of the $\mathbf{T}$-invariant elements of $\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$.

Lemma 1.5.1. *We have an isomorphism*

$$H^*((\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)^{\mathbf{T}}) \cong H^*(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha).$$

Proof. We show that the action of $\mathrm{Ad}_s(G) \subset \mathbf{T}$ on the cohomology $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$ is trivial. Consider the direct sum $\mathfrak{g} = V \oplus \mathfrak{n}$ as Construction 1.1.1. Then we have $\mathrm{Ad}_s(G) = \mathrm{Ad}_s(\exp(V))$ by Lemma 1.2.3. For $A \in V$, the action $\mathrm{Ad}_s(\exp(A))$ on the cochain complex $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ is a semi-simple part of the action of $\exp(A)$ on $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ via $\mathrm{Ad} \otimes \alpha \circ \mathrm{Ad}_s$. Since the action of G on the cohomology $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$ via $\mathrm{Ad} \otimes \alpha \circ \mathrm{Ad}_s$ is the extension of the Lie derivation on $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$, this G-action on $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$ is trivial. Hence for $A \in V$ the action of $\mathrm{Ad}_s(\exp(A)) = (\exp(\mathrm{ad}_A))_s$ on the cohomology $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$ is trivial.

Since $\mathbf{T}$ is the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ and the action of $\mathbf{T}$ on $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ is algebraic, the action of $\mathbf{T}$ on $H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)$ is also trivial. Since the action of $\mathbf{T}$ on $\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha$ is diagonalizable, we have an isomorphism

$$H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha) \cong H^*(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)^{\mathbf{T}} \cong H^*((\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_\alpha)^{\mathbf{T}}).$$

Hence we have the lemma. □

Consider the unipotent hull $\mathbf{U}_G$ of G. Let $\mathfrak{u}$ be the $\mathbb{C}$-Lie algebra of $\mathbf{U}_G$ and $\mathfrak{u}^*$ the $\mathbb{C}$-dual space. We consider the DGA $\bigwedge \mathfrak{u}^*$ with coefficients in $\mathbb{C}$.

Lemma 1.5.2. *We have an isomorphism of DGA*

$$\bigwedge \mathfrak{u}^* \cong (\bigoplus_\alpha \bigwedge \mathfrak{g}^*_\mathbb{C} \otimes V_\alpha)^{\mathbf{T}}.$$

Proof. Let $\{x_1, \dots, x_n\}$ be the dual of a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ such that Ad_s is represented by diagonal matrices. We define characters α_i as $t \cdot X_i = \alpha_i(t) X_i$ for $t \in \mathbf{T}$. Then we have $t \cdot x_i = \alpha_i^{-1}(t) x_i$. Hence the vector space $(\bigoplus_\alpha \bigwedge^1 \mathfrak{g}^*_\mathbb{C} \otimes V_\alpha)^{\mathbf{T}}$ is spanned by $\{x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n}\}$ where $V_{\alpha_i} \ni v_{\alpha_i} \neq 0$. For

$$\omega = \sum_{i_1,\dots i_p,\alpha} a_{i_1,\dots i_p,\alpha} x_{i_1} \wedge \cdots \wedge x_{i_p} v_\alpha \in (\bigoplus_\alpha \bigwedge^p \mathfrak{g}^*_\mathbb{C} \otimes V_\alpha)^{\mathbf{T}},$$

since any $x_{i_1} \wedge \cdots \wedge x_{i_p} v_\alpha$ is an eigenvector of the action of $\mathbf{T}$, if $a_{i_1,\dots i_p,\alpha} \neq 0$ then $x_{i_1} \wedge \cdots \wedge x_{i_p} v_\alpha$ is also a $\mathbf{T}$-invariant element. Since we have

$$t \cdot x_{i_1} \wedge \cdots \wedge x_{i_p} = \alpha_{i_1}^{-1}(t) \dots \alpha_{i_p}^{-1}(t) x_{i_1} \wedge \cdots \wedge x_{i_p}$$

for $t \in \mathbf{T}$, we have

$$x_{i_1} \wedge \cdots \wedge x_{i_p} \otimes v_\alpha = x_{i_1} v_{\alpha_{i_1}} \wedge \cdots \wedge x_{i_p} v_{\alpha_{i_p}}.$$

Thus the DGA $(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_\mathbb{C} \otimes V_\alpha)^{\mathbf{T}}$ is generated by $\{x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n}\}$. Consider the Maurer-Cartan equations

$$dx_k = -\sum_{ij} c^k_{ij} x_i \wedge x_j$$

and denote $\mathrm{ad}_{sX_i}(X_j) = a_{ij} X_j$. Since $\mathrm{Ad}_{sg}(X_k) = \alpha_i(\mathrm{Ad}_{sg}) X_k$ for $g \in G$, we have $dv_{\alpha_k} = \sum_{i=1}^n \mathrm{ad}_{sX_i}(X_k) x_i v_{\alpha_k} = \sum_{i=1}^n a_{ik} x_i v_{\alpha_k}$. Then we have

$$d_{\alpha_k}(x_k \otimes v_{\alpha_k}) = -\sum_{ij} (c^k_{ij} x_i \wedge x_j \otimes v_{\alpha_k} - a_{ik} x_i \wedge x_k \otimes v_{\alpha_k}).$$

Hence the DGA $(\bigoplus_\alpha \bigwedge \mathfrak{g}^*_\mathbb{C} \otimes V_\alpha)^{\mathbf{T}}$ is isomorphic to a free DGA generated degree 1 elements $\{y_1, \dots y_n\}$ such that

$$d(y_k) = -\sum_{ij} (c^k_{ij} y_i \wedge y_j - a_{ik} y_i \wedge y_k).$$

Let $\mathfrak{h}$ be the Lie algebra which is the dual of the free DGA $(\bigoplus_\alpha \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha)^{\mathbf{T}}$ and $\{Y_1, \dots, Y_n\}$ the dual basis of $\{y_1, \dots y_n\}$. It is sufficient to show $\mathfrak{h} \cong \mathfrak{u}$. Then the bracket of $\mathfrak{h}$ is given by

$$[Y_i, Y_j] = \sum_k c_{ij}^k Y_k - a_{ij} Y_j + a_{ji} Y_i.$$

Otherwise by Section 2.3, we have $\mathfrak{u} \cong \{X - \mathrm{ad}_{sX} | X \in \mathfrak{g}_{\mathbb{C}}\} \subset D(\mathfrak{g}_{\mathbb{C}}) \ltimes \mathfrak{g}_{\mathbb{C}}$. For the basis $\{X_1 - \mathrm{ad}_{sX_1}, \dots, X_n - \mathrm{ad}_{sX_n}\}$ of $\mathfrak{u}$, we have

$$[X_i - \mathrm{ad}_{sX_i}, X_j - \mathrm{ad}_{sX_j}] = \sum_k c_{ij}^k X_k - a_{ij} X_j + a_{ji} X_i.$$

By $[\mathfrak{g}; \mathfrak{g}] \subset \mathfrak{n}$, we have $[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{n}_{\mathbb{C}}$ where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$. By this we have

$$\sum_k c_{ij}^k X_k - a_{ij} X_j + a_{ji} X_i \in \mathfrak{n}_{\mathbb{C}},$$

and hence we have

$$ad_{s \sum_k c_{ij}^k X_k - a_{ij} X_j + a_{ji} X_i} = 0.$$

This gives

$$\begin{aligned}[X_i - \mathrm{ad}_{sX_i}, X_j - \mathrm{ad}_{sX_j}] \\ = \sum_k c_{ij}^k (X_k - \mathrm{ad}_{sX_k}) - a_{ij}(X_j - \mathrm{ad}_{sX_j}) + a_{ji}(X_i - \mathrm{ad}_{sX_i}).\end{aligned}$$

This gives an isomorphism $\mathfrak{h} \cong \mathfrak{u}$. Hence the lemma follows. □

Since $\mathrm{Ad}_s(G)$ is Zariski-dense in $\mathbf{T}$, $\mathrm{Ad}_s(G)$-invariant elements are also $\mathbf{T}$-invariant. In particular we have the following lemma.

Lemma 1.5.3. *Let $T = \mathbf{T}(\mathbb{R})$ be the real points of $\mathbf{T}$. Then we have*

$$(\bigoplus_\alpha \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha)^T \cong (\bigoplus_\alpha \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha)^{\mathbf{T}} \cong \bigwedge \mathfrak{u}^*.$$

Later we use this lemma.

Denote $A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s) = \bigoplus_\alpha \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha$. By lemma 1.5.1 and 1.5.2 we have:

Theorem 1.5.4. *We have a quasi-isomorphism of DGAs*

$$\bigwedge \mathfrak{u}^* \to A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s).$$

Thus $\bigwedge \mathfrak{u}^*$ is the minimal model of $A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s)$.

1.6 Cohomology of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$

Consider the two DGA $A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s)$ and $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$. For any character α of an algebraic group $\mathbf{T}$ which is the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$. We have the inclusion

$$\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha \cong (A_{\mathbb{C}}^*(G) \otimes V_\alpha)^G \subset (A_{\mathbb{C}}^*(G) \otimes V_\alpha)^\Gamma \cong A^*(G/\Gamma, E_{\alpha \circ \mathrm{Ad}_{s|\Gamma}}) .$$

Thus we have the morphism of DGAs

$$\phi : A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s) \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}).$$

Proposition 1.6.1. *The morphism* $\phi : A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s) \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ *is injective and the induced map*

$$\phi^* : H^*(A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s)) \to H^*(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$$

is also injective.

Proof. Since G has a lattice Γ, G is unimodular(see [40, Remark 1.9]). Choose a Haar measure $d\mu$ such that the volume of G/Γ is 1. We define a map $\varphi_\alpha : (A_{\mathbb{C}}^*(G) \otimes V_\alpha)^\Gamma \to \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha$ as

$$\varphi_\alpha(\omega \otimes v_\alpha)(X_1, \dots X_p) = \int_{G/\Gamma} \frac{\omega_x}{\alpha(x)}(X_1, \dots X_p) d\mu \cdot v_\alpha$$

for $\omega \otimes v_\alpha \in (A_{\mathbb{C}}^p(G) \otimes V_\alpha)^\Gamma$, $X_1, \dots, X_p \in \mathfrak{g}_C$. Then each φ_α is a morphism of cochain complexes and we have $\varphi_\alpha \circ \phi_{|\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha} = \mathrm{id}_{|\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha}$ (see [40, Remark 7.30]). Thus the restriction

$$\phi^* : H^*(\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha) \to H^*(A^*(G/\Gamma, E_\alpha))$$

is injective. By this it is sufficient to show that two distinct characters α, β with $\alpha \circ \mathrm{Ad}_{s|\Gamma} = \beta \circ \mathrm{Ad}_{s|\Gamma}$ satisfy $\varphi_\beta \circ \phi_{|\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha} = 0$. For $\omega \otimes v_\alpha \in \bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha$, we have

$$\varphi_\beta \circ \phi_{|\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_\alpha}(\omega \otimes v_\alpha) = \int_{G/\Gamma} \frac{\alpha(x)}{\beta(x)} \omega_x(X_1, \dots X_p) d\mu \cdot v_\alpha.$$

Since $\omega \in \bigwedge \mathfrak{g}_{\mathbb{C}}^*$, $\omega_x(X_1, \dots X_p)$ is constant on G/Γ. Let $\lambda = \frac{\beta}{\alpha} d(\frac{\alpha}{\beta})$. Then λ is a G-invariant form. Choose $\eta \in \bigwedge \mathfrak{g}_{\mathbb{C}}^*$ such that $\lambda \wedge \eta = d\mu$. Then we have

$$d\left(\frac{\alpha}{\beta}\eta\right) = \frac{\alpha}{\beta}\lambda \wedge \eta = \frac{\alpha}{\beta} d\mu.$$

By $\alpha \circ \mathrm{Ad}_{s|_\Gamma} = \beta \circ \mathrm{Ad}_{s|_\Gamma}$, $\frac{\alpha}{\beta}\eta$ is Γ-invariant and we can consider $\frac{\alpha}{\beta}\eta$ a differential form on G/Γ. Hence by Stokes' theorem, we have

$$\int_{G/\Gamma} \frac{\alpha(x)}{\beta(x)} \omega_x(X_1, \dots X_p) d\mu = \omega(X_1, \dots X_p) \int_{G/\Gamma} \frac{\alpha(x)}{\beta(x)} d\mu$$
$$= \omega(X_1, \dots X_p) \int_{G/\Gamma} d\left(\frac{\alpha}{\beta}\eta\right) = 0.$$

This prove the proposition. □

Corollary 1.6.2. *Let G be a simply connected solvable Lie group with a lattice Γ. We suppose $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Gamma)$ have the same Zariski-closure in $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Then we have an isomorphism*

$$H^*(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})) \cong H^*(A^*(\mathfrak{g}_\mathbb{C}, \mathrm{ad}_s)).$$

Proof. Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$. For any 1-dimensional representation V_α of $\mathbf{T}$ given by a character α of $\mathbf{T}$, we consider a flat bundle E_α on G/Γ given by the representation $\alpha \circ \mathrm{Ad}_s$ and the two cochain complex $A^*(G/\Gamma, E_\alpha)$ and $\bigwedge \mathfrak{g}_\mathbb{C}^* \otimes V_\alpha$ as above. Then since $\alpha \circ \mathrm{Ad}_s$ is Γ-admissible, by Theorem 1.3.1 we have an isomorphism

$$H^*(\bigwedge \mathfrak{g}^* \otimes V_\alpha) \cong H^*(A^*(G/\Gamma, E_\alpha)).$$

By the definitions of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$ and $A^*(\mathfrak{g}_\mathbb{C}, \mathrm{ad}_s)$ the corollary follows. □

1.7 Extensions

In this Section we extend Corollary 1.6.2 to the case of general sovmanifolds. To do this we consider infra-solvmanifolds which are generalizations of solvmanifolds.

1.7.1 Infra-solvmanifold

Let G be a simply connected solvable Lie group. We consider the affine transformation group $\mathrm{Aut}(G) \ltimes G$ and the projection $p : \mathrm{Aut}(G) \ltimes G \to \mathrm{Aut}(G)$. Let $\Gamma \subset \mathrm{Aut}(G) \ltimes G$ be a discrete subgroup such that $p(\Gamma)$ is contained in a compact subgroup of $\mathrm{Aut}(G)$ and the quotient G/Γ is compact. We call G/Γ an infra-solvmanifold.

Theorem 1.7.1. *[4, Theorem 1.5] For two infra-solvmanifolds G_1/Γ_1 and G_2/Γ_2, if Γ_1 is isomorphic to Γ_2, then G_1/Γ_1 is diffeomorphic to G_2/Γ_2.*

1.7.2 Extensions for infra-solvmanifolds

Let Γ be a torsion-free polycyclic group. and $\mathbf{H}_\Gamma$ be the algebraic hull. Then there exists a finite index normal subgroup Δ of Γ and a simply connected solvable subgroup G of $\mathbf{H}_\Gamma$ such that Δ is a lattice of G, and G and Δ have the same Zariski-closure in $\mathbf{H}_\Gamma$ (see [4, Proposition 2.9]). Since the Zariski-closure of Δ in $\mathbf{H}_\Gamma$ is finite index normal subgroup of $\mathbf{H}_\Gamma$, this group is the algebraic hull $\mathbf{H}_\Delta$ of Δ by the properties in Proposition 1.2.1. By $\operatorname{rank}\Gamma = \dim G$, $\mathbf{H}_\Delta$ is also the algebraic hull $\mathbf{H}_G$ of G. Hence we have the commutative diagram

$$\begin{tikzcd} G \arrow[r] & \mathbf{H}_\Delta(=\mathbf{H}_G) \arrow[r] & \mathbf{H}_\Gamma . \\ \Delta \arrow[u] \arrow[ur] \arrow[r] & \Gamma \arrow[ur] & \end{tikzcd}$$

Since Δ is a finite index normal subgroup of Γ, by this diagram $\mathbf{H}_\Delta$ is a finite index normal subgroup of $\mathbf{H}_\Gamma$. We suppose $\mathbf{H}_\Gamma/\mathbf{U}_\Gamma$ is diagonalizable. Let $\mathbf{T}$ and $\mathbf{T}'$ be maximal daiagonalizable subgroups of $\mathbf{H}_\Gamma$ and $\mathbf{H}_\Delta$. Then we have decompositions $\mathbf{H}_\Gamma = \mathbf{T} \ltimes \mathbf{U}_\Gamma$, $\mathbf{H}_\Delta = \mathbf{T}' \ltimes \mathbf{U}_\Gamma$. Since $\mathbf{T}/\mathbf{T}' = \mathbf{H}_\Gamma/\mathbf{H}_\Delta$ is a finite group, we have a finite subgroup $\mathbf{T}''$ of $\mathbf{T}$ such that $\mathbf{T} = \mathbf{T}''\mathbf{T}'$(see [8, Proposition 8.7]).

Lemma 1.7.2. $\mathbf{H}_\Gamma = \Gamma\mathbf{H}_\Delta$.

Proof. Consider the quotient $q : \mathbf{H}_\Gamma \to \mathbf{H}_\Gamma/\mathbf{H}_\Delta$. Since Γ is Zariski-dense in $\mathbf{H}_\Gamma$, $q(\Gamma)$ is Zariski-dense in $\mathbf{H}_\Gamma/\mathbf{H}_\Delta$. Since $\mathbf{H}_\Gamma/\mathbf{H}_\Delta$ is a finite group, $q(\Gamma) = \mathbf{H}_\Gamma/\mathbf{H}_\Delta$. Thus we have

$$\Gamma\mathbf{H}_\Delta = \Gamma\mathbf{T}' \ltimes \mathbf{U}_\Gamma = \mathbf{T}''\mathbf{T}' \ltimes \mathbf{U}_\Gamma = \mathbf{H}_\Gamma.$$

□

Let $H_\Gamma = \mathbf{H}_\Gamma(\mathbb{R})$, $T' = \mathbf{T}'(\mathbb{R})$ and $T'' = \mathbf{T}''(\mathbb{R})$. Then by Lemma 1.2.6 and $H_G = H_\Delta$, we have $H_\Gamma = T'T'' \ltimes G$. Hence we have $\Gamma \subset H_\Gamma \subset \operatorname{Aut}(G) \ltimes G$. Since Δ is a lattice of G and a finite index normal subgroup of Γ, Γ is a discrete subgroup of $\operatorname{Aut}(G) \ltimes G$ and G/Γ is compact and hence an infra-sovmanifold.

Theorem 1.7.3. *Let Γ be a torsion-free polycyclic group and $\Gamma \to \mathbf{H}_\Gamma$ be the algebraic hull of Γ. Suppose $\mathbf{H}_\Gamma/\mathbf{U}_\Gamma$ is diagonalizable. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}_\Gamma$. Let ρ be the composition*

$$\Gamma \to \mathbf{H}_\Gamma \to \mathbf{H}_\Gamma/\mathbf{U}_\Gamma.$$

Then we have a quasi-isomorphism

$$\bigwedge \mathfrak{u}^* \to A^*(G/\Gamma, \mathcal{O}_\rho).$$

Proof. In this proof for a DGA A with a group G-action, we denote $(A)^G$ the sub DGA which consists of G-invariant elements of A. Consider decompositions $\mathbf{H}_\Gamma = \mathbf{T} \ltimes \mathbf{U}_\Gamma$, $\mathbf{H}_\Delta = \mathbf{T}' \ltimes \mathbf{U}_\Gamma$ as above. Let $\{V_\alpha\}$ be the set of 1-dimensional representations of $\mathbf{T}$ for all characters α of $\mathbf{T}$. Consider the DGA $\bigoplus_\alpha A^*(G) \otimes V_\alpha$ with the derivation d given by

$$d(\omega \otimes v_\alpha) = (d\omega) \otimes v_\alpha \qquad \omega \in A^*(G), \quad v_\alpha \in V_\alpha$$

and the products given by

$$(\omega_1 \otimes v_\alpha) \wedge (\omega_2 \otimes v_\beta) = (\omega_1 \wedge \omega_2) \otimes (v_\alpha \otimes v_\beta).$$

Then by the definition, we have

$$A^*(G/\Gamma, \mathcal{O}_\rho) = (\bigoplus_\alpha A^*(G) \otimes V_\alpha)^\Gamma.$$

Let $\{V_{\alpha'}\}$ and $\{V_{\alpha''}\}$ be the sets of 1-dimensional representations of $\mathbf{T}'$ and $\mathbf{T}''$ for all characters α' of $\mathbf{T}'$ and α'' of $\mathbf{T}''$. By $\mathbf{T} = \mathbf{T}'\mathbf{T}''$, we have $\{V_\alpha\} = \{V_{\alpha'} \otimes V_{\alpha''}\}$. Then we have

$$H^*(A^*(G/\Gamma, \mathcal{O}_\rho)) = H^*\left(\left(\bigoplus_{\alpha',\alpha''} A^*(G) \otimes (V_{\alpha'} \otimes V_{\alpha''})\right)^\Gamma\right).$$

Since Δ is a finite index normal subgroup of Γ, we have

$$\begin{aligned} &H^*\left(\left(\bigoplus_{\alpha',\alpha''} A^*(G) \otimes (V_{\alpha'} \otimes V_{\alpha''})\right)^\Gamma\right) \\ &\qquad \cong H^*\left(\left(\left(\bigoplus_{\alpha',\alpha''} A^*(G) \otimes (V_{\alpha'} \otimes V_{\alpha''})\right)^\Delta\right)^{\Gamma/\Delta}\right). \end{aligned}$$

Since $\Delta \subset \mathbf{T}' \ltimes \mathbf{U}_\Gamma$, for a character α'' of $\mathbf{T}''$ Δ acts trivially on $V_{\alpha''}$. Hence we have

$$H^*\left(\left(\bigoplus_{\alpha',\alpha''} A^*(G)\otimes (V_{\alpha'}\otimes V_{\alpha''})\right)^{\Delta}\right)^{\Gamma/\Delta}$$
$$= H^*\left(\bigoplus_{\alpha''}\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{\Delta}\otimes V_{\alpha''}\right)^{\Gamma/\Delta}.$$

Since Δ is a lattice of G and we assume that $\mathrm{Ad}(G)$ and $\mathrm{Ad}(\Delta)$ have the same Zariski-closure, by Corollary 1.6.2 we have

$$H^*\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{\Delta}\right)\cong H^*\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{G}\right).$$

By Lemma 1.5.1 and 1.5.3, we have

$$H^*\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{G}\right) \cong H^*\left(\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{G}\right)^{T'}\right).$$

Hence we have

$$H^*\left(\bigoplus_{\alpha''}\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{\Delta}\otimes V_{\alpha''}\right)^{\Gamma/\Delta}$$
$$\cong H^*\left(\bigoplus_{\alpha''}\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{G}\right)^{T'}\otimes V_{\alpha''}\right)^{\Gamma/\Delta}.$$

Since $H_\Delta = T' \ltimes G$, we have

$$H^*\left(\bigoplus_{\alpha''}\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^G\right)^{T'} \otimes V_{\alpha''}\right)^{\Gamma/\Delta}$$

$$= H^*\left(\bigoplus_{\alpha''}\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^{H_\Delta} \otimes V_{\alpha''}\right)^{\Gamma/\Delta}$$

$$\cong H^*\left(\left(\bigoplus_{\alpha''}\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\otimes V_{\alpha''}\right)^{\Gamma H_\Delta}\right).$$

By Lemma 1.7.2, we have

$$H^*\left(\left(\bigoplus_{\alpha''}\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\otimes V_{\alpha''}\right)^{\Gamma H_\Delta}\right)$$

$$= H^*\left(\left(\bigoplus_{\alpha''}\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\otimes V_{\alpha''}\right)^{H_\Gamma}\right).$$

Since $H_\Gamma = T'T'' \ltimes G$, as above we have

$$H^*\left(\left(\bigoplus_{\alpha''}\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\otimes V_{\alpha''}\right)^{H_\Gamma}\right)$$

$$= H^*\left(\left(\bigoplus_{\alpha''}\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^G\right)^{T'} \otimes V_{\alpha''}\right)^{T''}\right).$$

Thus it is sufficient to show that the DGA

$$\left(\bigoplus_{\alpha''}\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^G\right)^{T'} \otimes V_{\alpha''}\right)^{T''}$$

is isomorphic to $\bigwedge \mathfrak{u}^*$. By Lemma 1.5.3 we have

$$\left(\bigoplus_{\alpha''}\left(\left(\bigoplus_{\alpha'} A^*(G)\otimes V_{\alpha'}\right)^G\right)^{T'} \otimes V_{\alpha''}\right)^{T''} \cong \Big(\bigoplus_{\alpha''}\bigwedge \mathfrak{u}^*\otimes V_{\alpha''}\Big)^{T''}.$$

Now let $\bigwedge \mathfrak{u}^* = \bigoplus_\beta A_{\beta''}$ be the weight decomposition of T'' for characters β'' of $\mathbf{T}''$. Then we have

$$(\bigoplus_{\alpha''} \bigwedge \mathfrak{u}^* \otimes V_{\alpha''})^{T''} = (\bigoplus_{\beta''} \bigoplus_{\alpha''} A_{\beta''} \otimes V_{\alpha''})^{T''} = \bigoplus_{\alpha''} A_{(\alpha'')^{-1}} \otimes V_{\alpha''}.$$

It is easily seen that

$$\bigoplus_{\alpha''} A_{(\alpha'')^{-1}} \otimes V_{\alpha''} \cong \bigwedge \mathfrak{u}^*.$$

Hence the theorem follows. □

Obviously a solvmanifold G/Γ is a infra-solvmanifold with polycyclic fundamental group Γ. Since $\mathbf{T}$ is the Zariski-closure of $\mathrm{Ad}_s(\Gamma)$ and diagonalizable, we have:

Corollary 1.7.4. *Let G be a simply connected solvable Lie group with a lattice Γ and $\mathbf{U}_G$ be the unipotent hull of G. Let $\mathfrak{u}$ be the Lie algebra of $\mathbf{U}_G$. Then we have a quasi-isomorphism*

$$\bigwedge \mathfrak{u}^* \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}}).$$

Thus $\bigwedge \mathfrak{u}^*$ is the minimal model of $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$.

Consider the injection $\phi : A^*(\mathfrak{g}_\mathbb{C}, \mathrm{ad}_s) \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$. By Theorem 1.5.4, Proposition 1.6.1 and above Corollary, $\phi : A^*(\mathfrak{g}_\mathbb{C}, \mathrm{ad}_s) \to A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})$ is a quasi-isomorphism. Hence we have:

Corollary 1.7.5. *Let G be a simply connected solvable Lie group with a lattice Γ. Then we have an isomorphism*

$$H^*(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|_\Gamma}})) \cong H^*(A^*(\mathfrak{g}_\mathbb{C}, \mathrm{ad}_s)).$$

We can apply this corollary to computations of the untwisted de Rham cohomology of solvmanifolds by invariant forms. We have an extension of Mostow's theorem(=Theorem 1.3.1) for the untwisted cohomology.

Corollary 1.7.6. *Let G be a simply connected solvable Lie group with a lattice Γ. Let $\mathbf{T}$ be the Zariski-closure of $\mathrm{Ad}_s(G)$ in $\mathrm{Aut}(\mathfrak{g}_\mathbb{C})$. Denote A_Γ a set of characters of $\mathbf{T}$ such that for $\alpha \in A_\Gamma$ the restriction of $\alpha \circ \mathrm{Ad}_s$ on*

Γ is trivial. Consider the sub-DGA $\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha$ of $A^*(\mathfrak{g}_{\mathbb{C}},\mathrm{ad}_s)$. Then we have a quasi-isomorphisms

$$\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}\to\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\to A_{\mathbb{C}}^*(G/\Gamma).$$

Moreover the DGA $\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}$ is a sub-DGA of $\bigwedge\mathfrak{u}^*$.

Proof. Since we can consider $A_{\mathbb{C}}^*(G/\Gamma)=A^*(G/\Gamma,E_{\mathbf{1}})$ for the trivial character $\mathbf{1}$, $A_{\mathbb{C}}^*(G/\Gamma)$ is a sub-DGA of $A^*(G/\Gamma,\mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$. Then we have

$$\phi^{-1}(A_{\mathbb{C}}^*(G/\Gamma))=\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha.$$

Since we define $A^*(G/\Gamma,\mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})=\bigoplus A^*(G/\Gamma,E_{\alpha\circ\mathrm{Ad}_{s|\Gamma}})$ as a direct sum of cochain complexes and $\phi: A^*(\mathfrak{g}_{\mathbb{C}},\mathrm{ad}_s)\to A^*(G/\Gamma,\mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ is a quasi-isomorphism by Corollary 1.7.5, the restriction $\phi:\phi^{-1}(A_{\mathbb{C}}^*(G/\Gamma))\to A_{\mathbb{C}}^*(G/\Gamma)$ is also a quasi-isomorphism. By Lemma 1.5.1, the inclusion

$$\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}\to\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha$$

is a quasi-isomorphism. By Lemma 1.5.2, $\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}$ is a sub-DGA of $\bigwedge\mathfrak{u}^*$. Hence the corollary follows. □

1.8 Formality and hard Lefschetz properties

In [20], Hasegawa proved the following theorem.

Theorem 1.8.1. ([20]) *Consider a DGA $\bigwedge\mathfrak{n}^*$ which is the dual of a nilpotent Lie algebra $\mathfrak{n}$. Then $\bigwedge\mathfrak{n}^*$ is formal if and only if $\mathfrak{n}$ is abelian.*

By Hasegawa's theorem, Theorem 1.5.4, Proposition 1.2.7 and Corollary 1.7.4, we have the following theorem.

Theorem 1.8.2. *Let G be a simply connected solvable Lie group. Then the following conditions are equivalent:*
(A) The DGA $A^*(\mathfrak{g}_{\mathbb{C}},\mathrm{ad}_s)$ is formal
(B) $\mathbf{U}_G$ is abelian.

(C) $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple. Moreover suppose* G *has a lattice* Γ. *Then the above three conditions are equivalent to the following condition:*
(D) $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ *is formal.*

In [6], Benson and Gordon proved:

Theorem 1.8.3. ([6], see also [17, Section 4.6.4]) *Consider a DGA* $\bigwedge \mathfrak{n}^*$ *which is the cochain complex of the dual of a nilpotent Lie algebra* $\mathfrak{n}$. *Suppose we have* $[\omega] \in H^2(\bigwedge \mathfrak{n}^*)$ *such that* $[\omega]^n \neq 0$ *where* $2n = \dim \mathfrak{n}$. *Then for any* $0 \leq i \leq n$ *the linear operator*

$$[\omega]^{n-i} \wedge : H^i(\bigwedge \mathfrak{n}^*) \to H^{2n-i}(\bigwedge \mathfrak{n}^*)$$

is an isomorphism if and only if $\mathfrak{n}$ *is abelian.*

By this theorem, we have:

Theorem 1.8.4. *Let* G *be a simply connected solvable Lie group. Suppose* $\dim G = 2n$ *and* G *has an* G*-invariant symplectic form* ω. *Then the following conditions equivalent:*
(A)

$$[\omega]^{n-i} \wedge : H^i(A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s)) \to H^{2n-i}(A^*(\mathfrak{g}_{\mathbb{C}}, \mathrm{ad}_s))$$

is an isomorphic for any $i \leq n$.
(B) $\mathbf{U}_G$ *is abelian.*
(C) $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*

Suppose G *has a lattice* Γ *and* G/Γ *has a symplectic form(not necessarily* G*-invariant)* ω. *Then the conditions (B) and (C) are equivalent to the following condition:*
(D)

$$[\omega]^{n-i} \wedge : H^i(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})) \to H^{2n-i}(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$$

is an isomorphism for any $i \leq n$

For infra-solvmanifolds, by Theorem 1.7.3 and Proposition 1.2.7 we have:

Theorem 1.8.5. *Let* M *be a infra-solvmanifold with the torsion-free polycyclic fundamental group* Γ *and* $\Gamma \to \mathbf{H}_\Gamma$ *be the algebraic hull of* Γ. *Suppose* $\mathbf{H}_\Gamma/\mathbf{U}_\Gamma$ *is diagonalizable. Let* ρ *be the composition*

$$\Gamma \to \mathbf{H}_\Gamma \to \mathbf{H}_\Gamma/\mathbf{U}_\Gamma.$$

Then following conditions are equivalent:
(A) $A^*(M, \mathcal{O}_\rho)$ *is formal.*
(B) $\mathbf{U}_\Gamma$ *is abelian.*
(C) M *is finitely covered by a solvmanifold* G/Γ *such that* $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *with a semi-simple action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *and* Γ *is a lattice of* G.
If $\dim M = 2n$ *and* M *has a symplectic form* ω, *the conditions (A), (B) and (C) are equivalent to the following condition:*
(D)

$$[\omega]^{n-i}\wedge : H^i(A^*(M, \mathcal{O}_\rho)) \to H^{2n-i}(A^*(M, \mathcal{O}_\rho))$$

is an isomorphism for any $i \leq n$.

1.9 Examples and remarks

Let G be a simply connected solvable Lie group with a lattice Γ. Suppose $\mathbf{U}_G$ is abelian. In [26] the author showed that G/Γ is formal and if G/Γ has a symplectic form, then G/Γ is hard Lefschetz. But the converses of these results are not true. See the following examples.

Example 1. *([43])*
We consider a 8-dimensional solvable Lie group $G = G_1 \times \mathbb{R}$ *such that :* G_1 *is the matrix group as*

$$\left\{ \begin{pmatrix} e^{a_1 t} & 0 & 0 & 0 & 0 & e^{-a_3 t}x_2 & z_1 \\ 0 & e^{a_2 t} & 0 & e^{-a_1 t}x_3 & 0 & 0 & z_2 \\ 0 & 0 & e^{a_3 t} & 0 & e^{-a_2 t}x_1 & 0 & z_3 \\ 0 & 0 & 0 & e^{-a_1 t} & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & e^{-a_2 t} & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & e^{-a_3 t} & x_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t, x_i, y_i \in \mathbb{R} \right\},$$

where a_1, a_2, a_3 *are distinct real numbers such that* $a_1 + a_2 + a_3 = 0$.
Let $\mathfrak{g}$ *be the Lie algebra of* G *and* $\mathfrak{g}^*$ *the dual of* $\mathfrak{g}$. *The cochain complex* $(\bigwedge \mathfrak{g}^*, d)$ *is generated by a basis* $\{\alpha, \beta, \zeta_i, \eta_i\}$ *of* $\mathfrak{g}^*$ *such that:*

$$d\alpha = 0,\ d\beta = 0,$$

$$d\zeta_i = a_i \alpha \wedge \zeta_i,$$

$$d\eta_1 = -a_1 \alpha \wedge \eta_1 - \zeta_2 \wedge \zeta_3,$$

$$d\eta_2 = -a_2 \alpha \wedge \eta_2 - \zeta_3 \wedge \zeta_1,$$

$$d\eta_3 = -a_3\alpha \wedge \eta_3 - \zeta_1 \wedge \zeta_2.$$

In [43] Sawai showed that for some a_1, a_2, a_3, G has a lattice Γ and G/Γ satisfies formality and has a G-invariant symplectic form

$$\omega = \alpha \wedge \beta + p(\zeta_1 \wedge \eta_1 - \zeta_2 \wedge \eta_2) + q(-\zeta_2 \wedge \eta_2 + \zeta_3 \wedge \eta_3)$$

satisfying the hard Lefschetz property where $pq \neq 0$ and $p + q \neq 0$. We have

$$\mathrm{Ad}_s(G) = \left\{ \begin{pmatrix} e^{a_1 t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_2 t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_3 t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-a_1 t} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-a_2 t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-a_3 t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Let $\mathbf{T}$ is the Zariski closure of $\mathrm{Ad}_s(G)$. Then for some characters $\alpha_1, \alpha_2, \alpha_3$ of $\mathbf{T}$, the cochain complexes $(\bigwedge \mathfrak{g}^* \otimes V_{\alpha_i}, d_{\alpha_i})$ are given by:

$$d_{\alpha_i}(v_{\alpha_i}) = -a_i \alpha \otimes v_{\alpha_i}$$

for $v_{\alpha_i} \in V_{\alpha_i}$.

We have

$$d_{\alpha_2}(\zeta_2 \otimes v_{\alpha_2}) = a_2\alpha \wedge \zeta_2 \otimes v_{\alpha_2} + \zeta_2 \wedge a_2\alpha \otimes v_{\alpha_2} = 0,$$

$$d_{\alpha_3}(\zeta_3 \otimes v_{\alpha_3}) = a_3\alpha \wedge \zeta_3 \otimes v_{\alpha_3} + \zeta_3 \wedge a_3\alpha \otimes v_{\alpha_3} = 0,$$

$$\begin{aligned} & d_{\alpha_2\alpha_3}(\eta_1 \otimes v_{\alpha_2\alpha_3}) \\ & \quad = -(a_1 + a_2 + a_3)\alpha \wedge \eta_1 \otimes v_{\alpha_2\alpha_3} - \zeta_2 \wedge \zeta_3 \otimes v_{\alpha_2\alpha_3} \\ & \qquad = -\zeta_2 \wedge \zeta_3 \otimes v_{\alpha_2\alpha_3}. \end{aligned}$$

Hence in $H^2(\bigwedge \mathfrak{g}^*_{\mathbb{C}} \otimes V_{\alpha_2\alpha_3})$,

$$[\zeta_2 \otimes v_{\alpha_2}] \cdot [\zeta_3 \otimes v_{\alpha_3}] = 0$$

and we have the Massey triple product

$$\langle [\zeta_2 \otimes v_{\alpha_2}], [\zeta_3 \otimes v_{\alpha_3}], [\zeta_3 \otimes v_{\alpha_3}] \rangle = [\eta_1 \wedge \zeta_3 \otimes v_{\alpha_2\alpha_3^2}]$$

in the quotient of

$$H^2(\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_{\alpha_2\alpha_3^2})$$

by

$$([\zeta_2 \otimes v_{\alpha_2}] \cdot H^1(\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_{\alpha_3^2}) + [\zeta_3 \otimes v_{\alpha_3}] \cdot H^1(\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_{\alpha_2\alpha_3})).$$

This Massey product is not zero. Hence the DGA $\bigoplus_\alpha \bigwedge \mathfrak{g}^* \otimes V_\alpha$ *has a non-zero Massey product and it is not formal.*

Remark 1.9.1. *In [30], Narkawicz gave examples of complements* X *of hyperplane arrangements which are formal but for some diagonal representations of* $\pi_1(X,x)$ *the DGA* $A^*(X,\mathcal{O}_\rho)$ *is non-formal.*

We have $d_{\alpha_1}(\zeta_1 \otimes v_{\alpha_1}) = 0$ *and the cohomology class* $[\zeta_1 \otimes v_{\alpha_1}] \in H^1(\bigwedge \mathfrak{g}_{\mathbb{C}}^* \otimes V_{\alpha_1})$ *is not zero. We have*

$$\begin{aligned}\omega^3 = -6p(q+p)\alpha \wedge \beta \wedge \zeta_1 \wedge \eta_1 \wedge \zeta_2 \wedge \eta_2 \\ - 6(p+q)q\alpha \wedge \beta \wedge \zeta_2 \wedge \eta_2 \wedge \zeta_3 \wedge \eta_3 \\ + 6pq\alpha \wedge \beta \wedge \zeta_1 \wedge \eta_1 \wedge \zeta_3 \wedge \eta_3 \\ - 6pq(p+q)\zeta_1 \wedge \eta_1 \wedge \zeta_2 \wedge \eta_2 \wedge \zeta_3 \wedge \eta_3,\end{aligned}$$

and

$$\omega^3 \wedge \zeta_1 \otimes v_{\alpha_1} = -6(p+q)q\alpha \wedge \beta \wedge \zeta_1 \wedge \zeta_2 \wedge \eta_2 \wedge \zeta_3 \wedge \eta_3 \otimes v_{\alpha_1}.$$

Otherwise we have

$$d_{\alpha_1}(\alpha \wedge \beta \wedge \zeta_1 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \otimes v_{\alpha_1}) = -\alpha \wedge \beta \wedge \zeta_1 \wedge \zeta_2 \wedge \eta_2 \wedge \zeta_3 \wedge \eta_3 \otimes v_{\alpha_1}.$$

Hence $[\omega]^3 \wedge ([\zeta_1 \otimes v_{\alpha_1}]) = 0$ *and the operator* $[\omega]^3\wedge$ *is not injective.*

Theorem 1.9.1. *For* G/Γ, *the DGA* $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$ *is not formal and the linear operator*

$$[\omega]^3\wedge : H^1(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})) \to H^7(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$$

is not an isomorphism. Thus $\mathbf{U}_G$ *is not abelian. In particular* G/Γ *is not Kähler.*

As above examples, comparing with untwisted versions, formality and the hard Lefschetz properties of the DGA $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ are useful criteria for formal and hard Lefschetz solvmanifolds to be not Kähler. But we have a non-Kähler symplectic solvmanifold such that $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ is formal and hard Lefschetz. In [1] Arapura showed that for a simply connected solvable Lie group G with a lattice Γ if a solvmanifold G/Γ admits a Kähler structure then Γ is virtually abelian. In [3] it was proved that a lattice of a simply connected solvable Lie group G is virtually nilpotent if and only if G is type (I) i.e. for any $g \in G$ the all eigenvalues of Ad_g have absolute value 1. Thus by Theorem 1.8.2 and 1.8.4, we have:

Corollary 1.9.2. *Let $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple. Suppose G is not type (I) and has a lattice Γ. Then $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ is formal but G/Γ has no Kähler structure. If G/Γ has a symplectic form ω, then the operator*

$$[\omega]^{n-i}\wedge : H^i(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})) \to H^{2n-i}(A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}}))$$

is an isomorphism for any $i \leq n$ where $\dim G = 2n$.

We give complex examples.

Example 2. *([29])*
Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then G has an invariant symplectic form. In [29], it was shown that G has a lattice Γ. Thus G/Γ is a non-Kähler complex solvmanifold but $A^*(G/\Gamma, \mathcal{O}_{\mathrm{Ad}_{s|\Gamma}})$ is formal and hard Lefschetz.

1.10 On isomorphism $H^*(G/\Gamma, \mathbb{C}) \cong H^*(\mathfrak{g}_\mathbb{C})$

Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. We give new criteria for the isomorphism $H^*(G/\Gamma, \mathbb{C}) \cong H^*(\mathfrak{g}_\mathbb{C})$ to hold by using Corollary 1.7.4. Take a basis $X_1, \dots, X_n$ of $\mathfrak{g}_\mathbb{C}$ such that Ad_s is represented by diagonal matrices as $\mathrm{Ad}_{sg} = diag(\alpha_1(g), \dots, \alpha_n(g))$. For $\{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ write $\alpha_{i_1 \dots i_p}$ as the product of characters $\alpha_{i_1}, \dots, \alpha_{i_p}$.

Corollary 1.10.1. *Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. Suppose (G, Γ) satisfies the following condition :*
$(C_{G,\Gamma})$: For any $\{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ if the character $\alpha_{i_1 \dots i_p}$ is non-trivial then the restriction of $\alpha_{i_1 \dots i_p|\Gamma}$ on Γ is also non-trivial.
Then an isomorphism $H^*(G/\Gamma, \mathbb{C}) \cong H^*(\mathfrak{g}_\mathbb{C})$ holds.

Proof. Let $x_1,\dots,x_n$ be a basis of $\mathfrak{g}_{\mathbb{C}}^*$ which is dual to $X_1,\dots,X_n$. Consider the DGA $\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}$ as Corollary 1.7.4. By $\mathrm{Ad}_{sg}^*\cdot x_i=\alpha_i(g)^{-1}x_i$ we have

$$\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}$$
$$=\left\langle x_{i_1}\wedge\cdots\wedge x_{i_p}\otimes v_{\alpha_{i_1\dots i_p}}\;\middle|\;\begin{array}{c}1\le i_1<i_2<\cdots<i_p\le n,\\ \text{the restriction of } \alpha_{\mathrm{i}_1\dots\mathrm{i}_p} \text{ on } \Gamma \text{ is trivial}\end{array}\right\rangle$$

as the proof of Lemma 1.5.2. Suppose (G,Γ) satisfies the condition $(C_{G,\Gamma})$. Then we have

$$\left(\bigoplus_{\alpha\in A_\Gamma}\bigwedge\mathfrak{g}_{\mathbb{C}}^*\otimes V_\alpha\right)^{\mathbf{T}}=\left(\bigwedge\mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}}.$$

Hence by Corollary 1.7.4, we have an isomorphism

$$H^*\left(\left(\bigwedge\mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}}\right)\cong H^*(G/\Gamma,\mathbb{C}).$$

This implies that the inclusion $\bigwedge(\mathfrak{g}_{\mathbb{C}})^*\subset A_{\mathbb{C}}^*(G/\Gamma)$ induces an isomorphism

$$H^*(\mathfrak{g}_{\mathbb{C}})\cong H^*(G/\Gamma,\mathbb{C}).$$

□

Remark 1.10.1. *We have examples such that we can apply of this corollary but can not use Mostow's theorem(=Theorem 1.3.1).*

Example 3. *Let* $G=\mathbb{R}\ltimes_\phi\mathbb{R}^2$ *with* $\phi(t)=\begin{pmatrix}\cos\pi t & -\sin\pi t\\ \sin\pi t & \cos\pi t\end{pmatrix}$. *Then* G *has a lattice* $\Gamma=\mathbb{Z}\ltimes\mathbb{Z}^2$. *In this case* G *is not completely solvable and* (G,Γ) *does not satisfies the Mostow's condition. But diagonalization of* Ad_s *is given by* $\mathrm{Ad}_s(t,x,y)=diag(1,e^{\pi t\sqrt{-1}},e^{-\pi t\sqrt{-1}})$ *and hence* (G,Γ) *satisfies the condition* $(C_{,G,\Gamma})$. *Thus we have an isomorphism* $H^*(\mathfrak{g}_{\mathbb{C}})\cong H^*(G/\Gamma,\mathbb{C})$.

For a character α of G, if the restriction of α on Γ is trivial, then the image $\alpha(G)=\alpha(G/\Gamma)$ is compact and hence α is a unitary character. Hence the above corollary reduce to the following corollary.

Corollary 1.10.2. *Let G be a simply connected solvable Lie group with a lattice Γ and $\mathfrak{g}$ be the Lie algebra of G. Suppose G satisfies the following condition :*
(D_G): For each $\{i_1,\dots,i_p\}\subset\{1,\dots,n\}$ the character $\alpha_{i_1\dots i_p}$ is not a non-trivial unitary character.
Then an isomorphism $H^*(G/\Gamma,\mathbb{C})\cong H^*(\mathfrak{g}_{\mathbb{C}})$ holds.

Since the condition (D_G) does not concern with Γ, this corollary is more useful than the above corollary. Clearly a completely solvable Lie group satisfies the condition (D_G). Hence this corollary is a generalization of Hattori's result in [24].

Example 4. *Let $G=\mathbb{R}^s\ltimes_\phi(\mathbb{R}^s\times\mathbb{C})$ such that*

$$\phi(t_1,\dots,t_s)=\begin{pmatrix} e^{t_1} & 0 & \dots & 0 & 0\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & e^{t_s} & 0 & 0\\ 0 & \dots & 0 & e^{-\frac{1}{2}(t_1+\dots t_s)}\cos\varphi & -e^{-\frac{1}{2}(t_1+\dots t_s)}\sin\varphi\\ 0 & \dots & 0 & e^{-\frac{1}{2}(t_1+\dots t_s)}\sin\varphi & e^{-\frac{1}{2}(t_1+\dots t_s)}\cos\varphi \end{pmatrix},$$

where $\varphi=c_1t_1+\dots+c_st_s$. Then a diagonalization of Ad_s is given by

$$\mathrm{Ad}_s=diag(e^{t_1},\dots,e^{t_s},e^{-\frac{1}{2}(t_1+\dots t_s)+\varphi\sqrt{-1}},e^{-\frac{1}{2}(t_1+\dots t_s)-\varphi\sqrt{-1}},1,\dots,1).$$

By this, G satisfies the condition (D_G) for any.

Proposition 1.10.3. *For any lattice Γ, we have $b_p(G/\Gamma)=b_{2s+2-p}(G/\Gamma)={}_sC_p$ for $1\le p\le s$ and $b_{s+1}(G/\Gamma)=0$.*

Proof. For a coordinate $(t_1,\dots t_s,x_1,\dots x_s,z)\in\mathbb{R}^s\ltimes_\phi(\mathbb{R}^s\times\mathbb{C})$, the cochin complex $\bigwedge\mathfrak{g}_{\mathbb{C}}^*$ is generated by

$$\{dt_1,\dots,dt_s,e^{-t_1}dx_1,\dots,e^{-t_s}dx_s,e^{\frac{1}{2}(t_1+\dots t_s)-\varphi\sqrt{-1}}dz,e^{\frac{1}{2}(t_1+\dots t_s)+\varphi\sqrt{-1}}d\bar{z}\}.$$

Since G satisfies the condition $(D_{p,G})$, we have an isomorphism

$$H^p(G/\Gamma,\mathbb{C})\cong H^*\left(\left(\bigwedge^p\mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}}\right).$$

We have

$$\left(\bigwedge^p\mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}}=\langle dt_{i_1}\wedge\dots\wedge dt_{i_p}|1\le i_1<\dots<i_p\le s\rangle$$

for $1 \leq p \leq s$ and $\left(\bigwedge^{s+1} \mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}} = 0$. Since the restriction of the derivation on $(\bigwedge^p \mathfrak{g}_{\mathbb{C}}^*)^{\mathbf{T}}$ is 0 for $1 \leq p \leq s+1$, we have

$$H^*\left(\left(\bigwedge^p \mathfrak{g}_{\mathbb{C}}^*\right)^{\mathbf{T}}\right) \cong \langle dt_{i_1} \wedge \cdots \wedge dt_{i_p} | 1 \leq i_1 < \cdots < i_p \leq s \rangle.$$

By the Poincaré duality, we have the proposition. □

We can construct a lattice of G by using of number theory. Let K be a finite extension field of $\mathbb{Q}$ with the degree $s+2 (s>0)$. Suppose K admits embeddings $\sigma_1, \dots \sigma_s, \sigma_{s+1}, \sigma_{s+2}$ into $\mathbb{C}$ such that $\sigma_1, \dots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \sigma_{s+2}$ are complex ones satisfying $\sigma_{s+1} = \bar{\sigma}_{s+2}$. We can choose K admitting such embeddings(see [34]). Denote $\mathcal{O}_K$ the ring of algebraic integers of K, $\mathcal{O}_K^*$ the group of units in $\mathcal{O}_K$ and

$$\mathcal{O}_K^{*+} = \{a \in \mathcal{O}_K^* : \sigma_i > 0 \text{ for all } 1 \leq i \leq s\}.$$

Define $\sigma : \mathcal{O}_K \to \mathbb{R}^s \times \mathbb{C}$ by

$$\sigma(a) = (\sigma_1(a), \dots, \sigma_s(a), \sigma_{s+1}(a))$$

for $a \in \mathcal{O}_K$. Then the image $\sigma(\mathcal{O}_K)$ is a lattice in $\mathbb{R}^s \times \mathbb{C}$. We denote

$$\sigma(a) \cdot \sigma(b)$$
$$= (\sigma_1(a)\sigma_1(b), \dots, \sigma_s(a)\sigma_s(b), \sigma_{s+1}(a)\sigma_{s+1}(b), \dots, \sigma_{s+t}(a)\sigma_{s+t}(b))$$

for $a, b \in \mathcal{O}_K$. Define $l : \mathcal{O}_K^{*+} \to \mathbb{R}^{s+1}$ by

$$l(a) = (\log|\sigma_1(a)|, \dots, \log|\sigma_s(a)|, 2\log|\sigma_{s+1}(a)|)$$

for $a \in \mathcal{O}_K^{*+}$. Then by Dirichlet's units theorem, $l(\mathcal{O}_K^{*+})$ is a lattice in the vector space $L = \{x \in \mathbb{R}^{s+1} | \sum_{i=1}^{s+1} x_i = 0\}$. By this we have a geometrical representation of the semi-direct product $l(\mathcal{O}_K^{*+}) \ltimes_\phi \sigma(\mathcal{O}_K)$ with

$$\phi(t_1, \dots, t_{s+1})(\sigma(a)) = \sigma(l^{-1}(t_1, \dots, t_{s+1})) \cdot \sigma(a)$$

for $(t_1, \dots, t_{s+1}) \in l(\mathcal{O}_K^{*+})$. Since $l(\mathcal{O}_K^{*+})$ and $\sigma(\mathcal{O}_K)$ are lattices of L and $\mathbb{R}^s \times \mathbb{C}$ respectively, we have a extension $\bar{\phi} : L \to \mathrm{Aut}(\mathbb{R}^s \times \mathbb{C})$ of ϕ and $l(\mathcal{O}_K^{*+}) \ltimes_\phi \sigma(\mathcal{O}_K)$ can be seen as a lattice of $L \ltimes_{\bar{\phi}} (\mathbb{R}^s \times \mathbb{C})$. Since we have $\phi(t_1, \dots t_{s+1}) = diag(e^{t_1}, \dots, e^{t_s}, \sigma_{s+1}(l^{-1}(t_1, \dots, t_{s+1})))$ and σ_{s+1} is a complex embedding of K, for some $c_1, \dots, c_s \in \mathbb{R}$, the Lie group $L \ltimes_{\bar{\phi}} (\mathbb{R}^s \times \mathbb{C})$ is identified with the Lie group G as above.

Remark 1.10.2. *In [34], for each s Oeljeklaus and Toma constructed a LCK (locally conformal Kähler) structure on the manifold $G/l(\mathcal{O}_K^{*+}) \ltimes_\phi \sigma(\mathcal{O}_K)$ and showed that for $s = 2$ this LCK manifold is a counter example of Vaisman's conjecture(i.e. Every compact LCK manifold has odd odd Betti number). By the above proposition, for $s = 2m$ the Betti number $b_p = b_{4m+2-p} = {}_{2m}C_p$ is even for odd number $1 \leq p < 2m$. Hence for any even s, $G/l(\mathcal{O}_K^{*+}) \ltimes_\phi \sigma(\mathcal{O}_K)$ is also a counter example of Vaisman's conjecture.*

Chapter 2

Cohomologically symplectic solvmanifolds are symplectic

2.1 The Purpose of this chapter

A $2n$-dimensional compact manifold M is called cohomologically symplectic (c-symplectic) if we have $\omega \in H^2(M, \mathbb{R})$ such that $\omega^n \neq 0$. A compact symplectic manifold is c-symplectic but the converse is not true in general. For example $\mathbb{C}P^2 \# \mathbb{C}P^2$ is c-symplectic but not symplectic. But for some class of manifolds these two conditions are equivalent. For examples, nilmanifolds i.e. compact homogeneous spaces of nilpotent simply connected Lie group. In [32], for a nilpotent simply connected Lie group G with a cocompact discrete subgroup Γ (such subgroup is called a lattice), Nomizu showed that the De Rham cohomology $H^*(G/\Gamma, \mathbb{R})$ of G/Γ is isomorphic to the cohomology $H^*(\mathfrak{g})$ of the Lie algebra of G. By the application of Nomizu's theorem, if G/Γ is c-symplectic then G/Γ is symplectic (see [17, p.191]). Every nilmanifold can be represented by such G/Γ (see [27]).

Consider Solvmanifolds i.e. compact homogeneous spaces of solvable simply connected Lie groups. Let G be a solvable simply connected Lie group with a lattice Γ. We assume that for any $g \in G$ the all eigenvalues of the adjoint operator Ad_g are real. With this assumption, in [24] Hattori extended Nomizu's theorem. By Hattori's theorem, for such case, without difficulty, we can similarly show that if G/Γ is c-symplectic, then G/Γ is symplectic. But the isomorphism $H^*(G/\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g})$ fails to hold for general solvable Lie groups, and not all solvmanifolds can be represented by G/Γ. Thus it is a considerable problem whether every c-symplectic solvmanifold is symplectic.

Let Γ be a torsion-free virtually polycyclic group. In [4] Baues constructed the compact aspherical manifold M_Γ with $\pi_1(M_\Gamma) = \Gamma$. Baues proved that every infra-solvmanifold (see [4] for the definition) is diffeomorphic to M_Γ. In particular the class of such aspherical manifolds contains the class of solvmanifolds. We prove that if M_Γ is c-symplectic then M_Γ is symplectic. In other words, for a torsion-free virtually polycyclic group Γ with $2n = \operatorname{rank}\Gamma$, if there exists $\omega \in H^2(\Gamma, \mathbb{R})$ such that $\omega^n \neq 0$ then we have a symplectic aspherical manifold with the fundamental group Γ.

2.2 Aspherical manifolds with torsion-free virtually polycyclic fundamental groups

Definition 2.2.1. *A group Γ is polycyclic if it admits a sequence*

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. We denote $\operatorname{rank}\Gamma = \sum_{i=1}^{i=k} \operatorname{rank}\Gamma_{i-1}/\Gamma_i$.

Proposition 2.2.2. ([40, Proposition 3.10]) *The fundamental group of a solvmanifold is torsion-free polycyclic.*

Let k be a subfield of $\mathbb{C}$. Let Γ be a torsion-free virtually polycyclic group. For a finite index polycyclic subgroup $\Delta \subset \Gamma$, we denote $\operatorname{rank}\Gamma = \operatorname{rank}\Delta$.

Definition 2.2.3. *We call a k-algebraic group $\mathbf{H}_\Gamma$ a k-algebraic hull of Γ if there exists an injective group homomorphism $\psi : \Gamma \to \mathbf{H}_\Gamma(k)$ and $\mathbf{H}_\Gamma$ satisfies the following conditions:*
(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_\Gamma$.
(2) $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma)$ where $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma))$ is the centralizer of $\mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \operatorname{rank}\Gamma$.

Theorem 2.2.4. ([4, Theorem A.1]) *There exists a k-algebraic hull of Γ and a k-algebraic hull of Γ is unique up to k-algebraic group isomorphism.*

Let Γ be a torsion-free virtually polycyclic group and $\mathbf{H}_\Gamma$ the $\mathbb{Q}$-algebraic hull of Γ. Denote $H_\Gamma = \mathbf{H}_\Gamma(\mathbb{R})$. Let U_Γ be the unipotent radical of H_Γ and T a maximal reductive subgroup. Then H_Γ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$. Let $\mathfrak{u}$ be the Lie algebra of U_Γ. Since the exponential map $\exp : \mathfrak{u} \longrightarrow U_\Gamma$ is a diffeomorphism, U_Γ is diffeomorphic to $\mathbb{R}^n$ such that $n = \operatorname{rank}\Gamma$. For the semi-direct product $H_\Gamma = T \ltimes U_\Gamma$, we denote $\phi : T \to \mathrm{Aut}(U_\Gamma)$ the action of T on U_Γ. Then we have the homomorphism $\alpha : H_\Gamma \longrightarrow \mathrm{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that $\alpha(t, u) = (\phi(t), u)$ for $(t, u) \in T \ltimes U_\Gamma$. By the property (2) in Definition 2.2.3, ϕ is injective and hence α is injective.

In [4] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma)\backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call M_Γ a standard Γ-manifold.

Theorem 2.2.5. ([4, Theorem 1.2, 1.4]) *A standard Γ-manifold is unique up to diffeomorphism. A solvmanifold with the fundamental group Γ is diffeomorphic to the standard Γ-manifold M_Γ.*

Let $A^*(M_\Gamma)$ be the de Rham complex of M_Γ. Then $A^*(M_\Gamma)$ is the set of the Γ-invariant differential forms $A^*(U_\Gamma)^\Gamma$ on U_Γ. Let $(\bigwedge \mathfrak{u}^*)^T$ be the left-invariant forms on U_Γ which are fixed by T. Since $\Gamma \subset H_\Gamma = U_\Gamma \cdot T$, we have the inclusion

$$(\bigwedge \mathfrak{u}^*)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

Theorem 2.2.6. ([4, Theorem 1.8]) *This inclusion induces an isomorphism on cohomology.*

By the application of the above facts, we prove the main theorem of this paper.

Theorem 2.2.7. *Suppose M_Γ is c-symplectic. Then M_Γ admits a symplectic structure. In particular cohomologically symplectic solvmanifolds are symplectic.*

Proof. Since we have the isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge \mathfrak{u}^*)^T)$, we have $\omega \in (\bigwedge^2 \mathfrak{u}^*)^T$ such that $0 \neq [\omega]^n \in H^{2n}((\bigwedge \mathfrak{u}^*)^T)$. This gives $0 \neq \omega^n \in (\bigwedge \mathfrak{u}^*)^T$ and hence $0 \neq \omega^n \in \bigwedge \mathfrak{u}^*$. Since ω^n is a non-zero invariant $2n$-form on U_Γ, we have $(\omega^n)_p \neq 0$ for any $p \in U_\Gamma$. Hence by the inclusion $(\bigwedge \mathfrak{u}^*)^T \subset A^*(U_\Gamma)^T = A^*(M_\Gamma)$, we have $(\omega^n)_{\Gamma p} \neq 0$ for any $\Gamma p \in \Gamma\backslash U_\Gamma = M_\Gamma$. This implies that ω is a symplectic form on M_Γ. Hence we have the theorem. □

2.3 Remarks

Let $G = \mathbb{R} \ltimes_\phi U_3(\mathbb{C})$ such that

$$\phi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{i\pi t} \cdot x & z \\ 0 & 1 & e^{-i\pi t} \cdot y \\ 0 & 0 & 1 \end{pmatrix},$$

and $D = \mathbb{Z} \ltimes_{\phi} D'$ with

$$D' = \left\{ \begin{pmatrix} 1 & x_1 + ix_2 & z_1 + iz_2 \\ 0 & 1 & y_1 + iy_2 \\ 0 & 0 & 1 \end{pmatrix} : x_1, y_2, z_2 \in \mathbb{Z},\ x_2, y_1, z_1 \in \mathbb{R} \right\}.$$

Then D is not discrete and G/D is compact. We have $D/D_0 \cong \mathbb{Z} \ltimes_{\varphi} U_3(\mathbb{Z})$ such that

$$\varphi(t) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (-1)^t x & z \\ 0 & 1 & (-1)^{-t} y \\ 0 & 0 & 1 \end{pmatrix},$$

where D_0 is the identity component of D. Denote $\Gamma = D/D_0$. We have the algebraic hull $H_\Gamma = \{\pm 1\} \ltimes_{\psi} (U_3(\mathbb{R}) \times \mathbb{R})$ such that

$$\psi(-1) \cdot \left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, t \right) = \left(\begin{pmatrix} 1 & -x & z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, t \right).$$

The dual of the Lie algebra $\mathfrak{u}$ of $U_3(\mathbb{R}) \times \mathbb{R}$ is given by $\mathfrak{u}^* = \langle \alpha, \beta, \gamma, \delta \rangle$ such that the differential is given by

$$d\alpha = d\beta = d\delta = 0,$$

$$d\gamma = -\alpha \wedge \beta,$$

and the action of $\{\pm 1\}$ is given by

$$(-1) \cdot \alpha = -\alpha,\ (-1) \cdot \beta = -\beta,$$

$$(-1) \cdot \gamma = \gamma,\ (-1) \cdot \delta = \delta.$$

Then we have a diffeomorphism $M_\Gamma \cong G/D$ and an isomorphism $H^*(M_\Gamma, \mathbb{R}) \cong H^*((\bigwedge \mathfrak{u}^*)^{\{\pm 1\}})$. By simple computations, $H^2((\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}) = 0$ and hence the solvmanifold G/D is not symplectic.

Remark 2.3.1. *The proof of the Theorem 2.2.7 contains a proof of the following proposition.*

Proposition 2.3.1. *If M_Γ admits a symplectic structure, then U_Γ has an invariant symplectic form.*

Otherwise for the above example, $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$ has an invariant symplectic form but M_Γ is not symplectic. Thus the converse of this proposition is not true. If Γ is nilpotent, then T is trivial and any invariant symplectic form on U_Γ induces the symplectic form on M_Γ. Hence for nilmanifolds the converse of Proposition 2.3.1 is true.

Remark 2.3.2. *Γ is a finite extension of a lattice of $U_\Gamma = U_3(\mathbb{R}) \times \mathbb{R}$. Hence M_Γ is finitely covered by a Kodaira-Thurston manifold (see [46], [17, p.192]). M_Γ is an example of a non-symplectic manifold finitely covered by a symplectic manifold.*

Let $H = G \times \mathbb{R}$. Then the dual of the Lie algebra $\mathfrak{h}$ of H is given by $\mathfrak{h}^* = \langle \sigma, \tau, \zeta_1, \zeta_2, \eta_1, \eta_2, \theta_1, \theta_2 \rangle$ such that the differential is given by

$$d\sigma = d\tau = 0,$$

$$d\zeta_1 = \tau \wedge \zeta_2,\ d\zeta_2 = -\tau \wedge \zeta_1,$$

$$d\eta_1 = \tau \wedge \eta_2,\ d\eta_2 = -\tau \wedge \eta_1,$$

$$d\theta_1 = -\zeta_1 \wedge \eta_1 + \zeta_2 \wedge \eta_2,\ d\theta_2 = -\zeta_1 \wedge \eta_2 - \zeta_2 \wedge \eta_1.$$

By simple computations, any closed invariant 2-form $\omega \in \bigwedge^2 \mathfrak{h}^*$ satisfies $\omega^4 = 0$. Hence H has no invariant symplectic form. Otherwise we have a lattice $\Delta = 2\mathbb{Z} \ltimes U_3(\mathbb{Z} + i\mathbb{Z}) \times \mathbb{Z}$ which is also a lattice of $\mathbb{R}^2 \times U_3(\mathbb{C})$. Thus H/Δ is diffeomorphic to a direct product of a 2-dimensional torus and an Iwasawa manifold (see [18]). Since an Iwasawa manifold is symplectic (see [18]), H/Δ is also symplectic. By this example we can say:

Remark 2.3.3. *For a simply connected nilpotent Lie group G with lattice Γ, if the nilmanifold G/Γ is symplectic then G has an invariant symplectic form. But suppose G is solvable we have an example of a symplectic solvmanifold G/Γ such that G has no invariant symplectic form.*

Chapter 3

Techniques of computations of Dolbeault cohomology of solvmanifolds

3.1 The purpose of this chapter

Let G be a simply connected solvable Lie group and $\mathfrak{g}$ the Lie algebra of G. We assume that G admits a lattice Γ and a left-invariant complex structure J. We consider the Dolbeault cohomology $H^{*,*}_{\bar\partial}(G/\Gamma)$ of the complex solvmanifold G/Γ. We also consider the cohomology $H^{*,*}_{\bar\partial}(\mathfrak{g})$ of the differential bigraded algebra (shortly DBA) $\bigwedge^{*,*}\mathfrak{g}^*$ of the complex valued left-invariant differential forms with the operator $\bar\partial$.

The purpose of this part is to prove that one can compute the Dolbeault cohomology of certain class of solvmanifolds G/Γ by using finite dimensional DBAs. In this part we consider a Lie group G as in the following assumption.

Assumption 3.1.1. *G is the semi-direct product $\mathbb{C}^n \ltimes_\phi N$ so that:*
(1) N is a simply connected nilpotent Lie group with a left-invariant complex structure J.
Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of $\mathbb{C}^n$ and N respectively.
(2) For any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of (N, J).
(3) ϕ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ of N.
(4) G has a lattice Γ. (Then Γ can be written by $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ such that Γ' and Γ'' are lattices of $\mathbb{C}^n$ and N respectively and for any $t \in \Gamma'$ the action $\phi(t)$ preserves Γ''.)

(5) The inclusion $\bigwedge^{*,*}\mathfrak{n}^* \subset A^{*,*}(N/\Gamma'')$ induces an isomorphism

$$H^{*,*}_{\bar\partial}(\mathfrak{n}) \cong H^{*,*}_{\bar\partial}(N/\Gamma'').$$

Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a character (i.e. a representation on 1-dimensional vector space $\mathbb{C}_\alpha$) of $\mathbb{C}^n$. By the projection $\mathbb{C}^n \ltimes_\phi N \to \mathbb{C}^n$, we regard α as a character of G as in Assumption 3.1.1. We consider the holomorphic line bundle $L_\alpha = (G \times \mathbb{C}_\alpha)/\Gamma$ and the Dolbeault complex $(A^{*,*}(G/\Gamma, L_\alpha), \bar\partial)$ with values in the line bundle L_α. Let $\mathcal{L}$ be the set of isomorphism classes of line bundles over G/Γ given by characters of $\mathbb{C}^n$. We consider the direct sum

$$\bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

of Dolbeault complexes. Then by the wedge products and the tensor products, the direct sum $\bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$ is a DBA.

Theorem 3.1.2. *There exists a subDBA $A^{*,*}$ of*

$$\bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

such that we have a DBA isomorphism $\iota : \bigwedge^{*,*}(\mathfrak{a} \oplus \mathfrak{n})^* \cong A^{*,*}$ and the inclusion

$$\Phi : A^{*,*} \to \bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

induces a cohomology isomorphism.

See Section 3 for the construction of $A^{*,*}$.

Corollary 3.1.3. *We have the finite dimensional subDBA $B^{*,*} = \Phi^{-1}(A^{*,*}(G/\Gamma))$ of $A^{*,*}(G/\Gamma)$ such that the inclusion $\Phi : B^{*,*} \to A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism*

$$H^{*,*}_{\bar\partial}(B^{*,*}) \cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

See Corollary 3.4.2 for the construction of $B^{*,*}$.

Remark 3.1.1. *Let N be a simply connected nilpotent Lie group with a lattice Γ'' and a left-invariant complex structure J. Like Nomizu's theorem ([32]) for the de Rham cohomology of nilmanifolds, it is known that an isomorphism $H^{*,*}_{\bar\partial}(N/\Gamma'') \cong H^{*,*}_{\bar\partial}(\mathfrak{n})$ holds if (N, J, Γ'') meet one of the following conditions:*

(N) The complex manifold N/Γ'' has the structure of an iterated principal holomorphic torus bundle ([10]).

(Q) J is a small deformation of a rational complex structure i.e. for the rational structure $\mathfrak{n}_{\mathbb{Q}} \subset \mathfrak{n}$ of the Lie algebra $\mathfrak{n}$ induced by a lattice Γ'' (see [40, Section 2]) we have $J(\mathfrak{n}_{\mathbb{Q}}) \subset \mathfrak{n}_{\mathbb{Q}}$ ([9]).

(C) (N, J) is a complex Lie group ([42]).

By using Corollary 3.1.3, we actually compute the Dolbeault cohomology of some examples in Section 5. Unlike nilmanifolds, we observe that in many cases the Dolbeault cohomology of solvmanifolds can not be completely computed by using only Lie algebras. Moreover we give examples of non-Kähler complex solvmanifolds with the Hodge symmetry.

Remark 3.1.2. *If N has a nilpotent complex structure (see [10]), then $(\bigwedge^{*,*}(\mathfrak{a} \oplus \mathfrak{n})^*, \bar\partial)$ is the minimal model of the DBA $\bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$ (see [31]).*

3.2 Holomorphic line bundles over complex tori

Lemma 3.2.1. *Let Γ be a finitely generated free abelian group and $\alpha : \Gamma \to \mathbb{C}^*$ a character of Γ. If the character α is non-trivial, then we have $H^*(\Gamma, \mathbb{C}_\alpha) = 0$.*

Proof. First we assume $\Gamma \cong \mathbb{Z}$. Then we have

$$H^0(\mathbb{Z}, \mathbb{C}_\alpha) = \{m \in \mathbb{C}_\alpha | \alpha(g)m = m, \text{ for all } g \in \mathbb{Z}\} = 0.$$

Like the de Rham cohomology of S^1, by the Poincaré duality we have

$$H^1(\mathbb{Z}, \mathbb{C}_\alpha) \cong H^0(\mathbb{Z}, \mathbb{C}_{\alpha^{-1}})^* = 0,$$

and obviously $H^p(\mathbb{Z}, \mathbb{C}_\alpha) = 0$ for $p \geq 2$. Hence the lemma holds in this case. In general case, we consider a decomposition $\Gamma = A \oplus B$ such that A is a rank 1 subgroup and the restriction of α on A is also non-trivial. Then we have the Hochshild-Serre spectral sequence E_r such that

$$E_2^{p,q} = H^p(\Gamma/A, H^q(A, \mathbb{C}_\alpha))$$

and this converges to $H^{p+q}(\Gamma, \mathbb{C}_\alpha)$. Since $H^q(A, \mathbb{C}_\alpha) = 0$ for any q, we have $E_2 = 0$ and hence the lemma follows. □

We consider a complex vector space $\mathbb{C}^n$ with a lattice Γ. Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a C^∞-character of $\mathbb{C}^n$. We have the holomorphic line bundle $L_\alpha = (\mathbb{C}^n \times \mathbb{C}_\alpha)/\Gamma$ over the complex torus $\mathbb{C}^n/\Gamma$. We define the equivalence relation on the space of C^∞-characters of $\mathbb{C}^n$ such that $\alpha \sim \beta$ if $\alpha\beta^{-1}$ is holomorphic.

Lemma 3.2.2. *Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a C^∞-character of $\mathbb{C}^n$. There exists a unique unitary character β such that $\alpha \sim \beta$.*

Proof. For a coordinate $(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) \in \mathbb{C}^n$, a character α is written as

$$\alpha(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) = \exp(\sum_{i=1}^{n}(a_i x_i + b_i y_i + \sqrt{-1}(c_i x_i + d_i y_i)))$$

for some $a_i, b_i, c_i, d_i \in \mathbb{R}^n$. Let α' be the holomorphic character defined by

$$\alpha'(x_1+\sqrt{-1}y_1, \dots, x_n+\sqrt{-1}y_n) = \exp(\sum_{i=1}^{n}(-a_i(x_i+\sqrt{-1}y_i)+\sqrt{-1}b_i(x_i+\sqrt{-1}y_i)).$$

Then the character $\beta = \alpha\alpha'$ is unitary. If a unitary character is holomorphic, then it is trivial. Hence such β is unique. □

Lemma 3.2.3. ([38]) *Let $\beta : \mathbb{C}^n \to \mathbb{C}^*$ be a unitary C^∞-character of $\mathbb{C}^n$. Then the holomorphic line bundle L_β is trivial if and only if the restriction of β on Γ is trivial.*

Proposition 3.2.4. *Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a C^∞-character of $\mathbb{C}^n$. If L_α is a non-trivial holomorphic line bundle, then the Dolbeault cohomology $H^{*,*}_{\bar\partial}(\mathbb{C}^n/\Gamma, L_\alpha)$ with values in the line bundle L_α is trivial.*

Proof. Let β be the unitary character such that $\alpha \sim \beta$ as in Lemma 3.2.2. Then we have $L_\alpha \cong L_\beta$. Let D be the flat connection on L_β induced by β. We have the decomposition $D = \partial + \bar\partial$ so that $\bar\partial$ is the Dolbeault operator on L_β. Since β is unitary, we have a Hermitian metric on L_β such that for a Kähler metric on $\mathbb{C}^n/\Gamma$ we have the standard identity of the Laplacians of D and $\bar\partial$ (see [?, Section 7]). Hence we have an isomorphism $H^*_{\bar\partial}(\mathbb{C}^n/\Gamma, L_\beta) \cong H^*_D(\mathbb{C}^n/\Gamma, L_\beta)$. If L_β is non-trivial, then we have $H^*_D(\mathbb{C}^n/\Gamma, L_\beta) = H^*(\Gamma, \mathbb{C}_\beta) = 0$ by Lemma 3.2.1. Hence the proposition follows. □

3.3 The construction of $A^{*,*}$

Let G be a Lie group as in Assumption 3.1.1. Consider the decomposition $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n}^{1,0} \oplus \mathfrak{n}^{0,1}$. By the condition (2), this decomposition is a direct sum of $\mathbb{C}^n$-modules. By the condition (3) we have a basis $Y_1, \dots, Y_m$ of $\mathfrak{n}^{1,0}$ such that the action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi(t) = \mathrm{diag}(\alpha_1(t), \dots, \alpha_m(t))$. Since Y_j is a left-invariant vector field on N, the vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes_\phi N$ is left-invariant. Hence we have a basis $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$ of $\mathfrak{g}^{1,0}$. Let $x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m$ be the basis of $\bigwedge^{1,0} \mathfrak{g}^*$ which is dual to $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$. Then we have

$$\bigwedge^{p,q} \mathfrak{g}^* = \bigwedge^{p} \langle x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m \rangle \otimes \bigwedge^{q} \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \dots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle.$$

Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a character of $\mathbb{C}^n$. Let $(A^{*,*}(G) \otimes \mathbb{C}_\alpha)^\Gamma$ be the space of $\mathbb{C}_\alpha$-valued Γ-invariant differential forms on G. Then we can identify the Dolbeault complex $A^{*,*}(G/\Gamma, L_\alpha)$ with $(A^{*,*}(G) \otimes \mathbb{C}_\alpha)^\Gamma$. Hence for $\omega \in \bigwedge^{*,*} \mathfrak{g}^*$ and $v_\alpha \in \mathbb{C}_\alpha$, we have

$$\omega \otimes (\alpha^{-1} v_\alpha) \in A^{*,*}(G/\Gamma, L_\alpha).$$

Let $\mathcal{L}$ be the set as in Introduction. By Section 2, we can regard $\mathcal{L}$ as the set of isomorphism classes of line bundles over G/Γ given by unitary characters of $\mathbb{C}^n$. We consider the DBA $\bigoplus_{L_\alpha \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\alpha)$. We define the DBA $A^{*,*}$ to prove Theorem 3.1.2.

Definition 3.3.1. *Let $x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m$ be the basis of $\bigwedge^{1,0} \mathfrak{g}^*$ as above. By Lemma 3.2.2, we have the unitary character β_j such that $\alpha_j \sim \beta_j$. We consider the holomorphic line bundles $L_{\beta_j^{-1}}$ over G/Γ. By $A^{*,*}(G/\Gamma, L_{\beta_j^{-1}}) = (A^{*,*}(G) \otimes \mathbb{C}_{\beta_j^{-1}})^\Gamma$, for $\mathbb{C}_{\beta_j^{-1}} \ni v_{\beta_j^{-1}} \neq 0$ we consider*

$$\alpha_j^{-1} y_j \otimes (\beta_j v_{\beta_j^{-1}}) \in A^{*,*}(G/\Gamma, L_{\beta_j^{-1}}).$$

Let $A^{*,*}$ be the subDBA of $\bigoplus_{L_\alpha \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\alpha)$ defined by

$$A^{p,q} = \bigwedge^{p} \langle x_1, \dots, x_n, \alpha_1^{-1} y_1 \otimes (\beta_1 v_{\beta_1^{-1}}), \dots, \alpha_m^{-1} y_m \otimes (\beta_m v_{\beta_m^{-1}}) \rangle$$
$$\bigotimes \bigwedge^{q} \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1 \otimes (\gamma_1 v_{\gamma_1^{-1}}), \dots, \bar{\alpha}_m^{-1} \bar{y}_m \otimes (\gamma_m v_{\gamma_m^{-1}}) \rangle.$$

Lemma 3.3.2. *Let $\iota : \bigwedge^{*,*}(\mathfrak{a}\oplus\mathfrak{n})^* \to A^{*,*}$ be the algebra homomorphism defined by*

$$\iota(x_i) = x_i,$$
$$\iota(\alpha_j^{-1}y_j) = \alpha_j^{-1}y_j \otimes \beta_j v_{\beta_j^{-1}}.$$

Then we have a DBA isomorphism

$$\iota : (\bigwedge^{*,*}(\mathfrak{a}\oplus\mathfrak{n})^*, \bar\partial) \cong (A^{*,*}, \bar\partial).$$

Proof. Since $\alpha_j^{-1}\beta_j$ is holomorphic, we have

$$\bar\partial(\alpha_j^{-1}y_j \otimes \beta_j v_{\beta_j^{-1}}) = \alpha_j^{-1}(\bar\partial y_j) \otimes \beta_j v_{\beta_j^{-1}}.$$

This implies $\bar\partial \circ \iota = \iota \circ \bar\partial$. Hence the lemma follows. □

Let g be the left-invariant Hermitian metric on G defined by

$$g = x_1\bar x_1 + \cdots + x_n\bar x_n + \alpha_1^{-1}\bar\alpha_1^{-1}y_1\bar y_1 + \cdots + +\alpha_m^{-1}\bar\alpha_m^{-1}y_m\bar y_m.$$

Let $\beta : \mathbb{C}^n \to \mathbb{C}^*$ be a unitary C^∞-character of $\mathbb{C}^n$. Take $\mathbb{C}_\beta \ni v_\beta \neq 0$. Then $\beta^{-1}v_\beta$ is a C^∞-frame of the line bundle $L_\beta = (G\times\mathbb{C}_\beta)/\Gamma$. We define the Hermitian metric h_β on L_β such that $h_\beta(\beta^{-1}v_\beta, \beta^{-1}v_\beta) = 1$. Let $\bar *_{g\otimes h_\beta} : A^{p,q}(G/\Gamma, L_\beta) \to A^{n+m-p,n+m-q}(G/\Gamma, L_\beta^*)$ be the $\mathbb{C}$-anti-linear Hodge star operator of $g\otimes h_\beta$ on $A^{*,*}(G/\Gamma, L_\beta)$ and let

$$\bar\delta = \bar *_{g\otimes h_\beta} \circ \bar\partial \circ \bar *_{g\otimes h_\beta}, \quad \Box_{g\otimes h_\beta} = \bar\partial\bar\delta + \bar\delta\bar\partial$$

and

$$\mathcal{H}^{p,q}(G/\Gamma, L_\beta) = \{\omega \in A^{*,*}(G/\Gamma, L_\beta) | \Box_{g\otimes h_\beta}\omega = 0\}.$$

We consider the $\bar\partial$-Laplace operator $\oplus\Box_{g\otimes h_\beta}$ on the direct sum $\bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$.

We consider the basis $x_1,\dots,x_n,y_1,\dots,y_m$ of $\bigwedge^{1,0}(\mathfrak{a}\oplus\mathfrak{n})^*$. Let g' be the Hermitian metric on $\mathfrak{a}\oplus\mathfrak{n}$ defined by

$$g' = x_1\bar x_1 + \cdots + x_n\bar x_n + y_1\bar y_1 + \cdots + y_m\bar y_m.$$

Let $\bar *_{g'} : \bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^* \to \bigwedge^{n+m-p,n+m-q}(\mathfrak{a}\oplus\mathfrak{n})^*$ be the $\mathbb{C}$-anti-linear Hodge star operator of g' on $\bigwedge^{*,*}(\mathfrak{a}\oplus\mathfrak{n})^*$ and let

$$\bar\delta = \bar *_{g'} \circ \bar\partial \circ \bar *_{g'}, \quad \Box_{g'} = \bar\partial\bar\delta + \bar\delta\bar\partial$$

and

$$\mathcal{H}^{p,q}(\mathfrak{a}\oplus\mathfrak{n}) = \{\omega \in \bigwedge^{*,*}(\mathfrak{a}\oplus\mathfrak{n})^* | \Box_{g'}\omega = 0\}.$$

Lemma 3.3.3. *We consider the isomorphism* $\iota : \bigwedge^{*,*}(\mathfrak{a}\oplus\mathfrak{n})^* \cong A^{*,*}$ *as in Lemma 3.3.2. Then we have*

$$\iota \circ \Box_{g'} = (\oplus \Box_{g\otimes h_\alpha}) \circ \iota.$$

Proof. Let $\oplus\bar{*}_{g\otimes h_\alpha}$ be the Hodge star operator on $\bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$. It is sufficient to show

$$\iota \circ \bar{*}_{g'} = (\oplus\bar{*}_{g\otimes h_\alpha}) \circ \iota.$$

For a multi-index $I = \{i_1, \dots, i_r\}$, we write $x_I = x_{i_1}\wedge\cdots\wedge x_{i_r}$, $y_I = y_{i_1}\wedge\cdots\wedge y_{i_r}$, $\alpha_I = \alpha_{i_1}\cdots\alpha_{i_r}$ and $\beta_I = \beta_{i_1}\cdots\beta_{i_r}$. For multi-indices $I, K \subset \{1,\dots,n\}$ and $J, L \subset \{1,\dots,m\}$, we have

$$\bar{*}_{g'}(x_I \wedge y_J \wedge \bar{x}_K \wedge \bar{y}_L) = \epsilon x_{I'} \wedge y_{J'} \wedge \bar{x}_{K'} \wedge \bar{y}_{L'}$$

where I', J', K' and L' are complements and ϵ is the sign of a permutation. We also have

$$\begin{aligned}\oplus\, \bar{*}_{g\otimes h_\alpha}&(x_I \wedge \alpha_J^{-1} y_J \wedge \bar{x}_K \wedge \bar{\alpha}_L^{-1}\bar{y}_L \otimes \beta_J\gamma_L v_{\beta_J^{-1}\gamma_L^{-1}}) \\ &= \epsilon x_{I'} \wedge \alpha_{J'}^{-1} y_{J'} \wedge \bar{x}_{K'} \wedge \bar{\alpha}_{L'}^{-1}\bar{y}_{L'} \otimes \beta_J^{-1}\gamma_L^{-1} v_{\beta_J\gamma_L}.\end{aligned}$$

Hence we only need to show

$$\beta_J^{-1}\gamma_L^{-1} = \beta_{J'}\gamma_{L'}.$$

Since a Lie group with a lattice is unimodular (see [40, Remark 1.9]), the action ϕ on $\mathfrak{n}$ is represented by unimodular matrices. Hence we have $\alpha_J\bar{\alpha}_L\alpha_{J'}\bar{\alpha}_{L'} = 1$. This implies $\beta_J^{-1}\gamma_L^{-1} = \beta_{J'}\gamma_{L'}$. Hence the lemma follows. □

Corollary 3.3.4. *The inclusion*

$$\Phi : A^{*,*} \to \bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

induces an injection

$$H^{p,q}_{\bar\partial}(\mathfrak{a}\oplus\mathfrak{n}) \cong H^{p,q}(A^{*,*}) \to H^{p,q}_{\bar\partial}(\bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)).$$

Proof. We have isomorphisms $\mathcal{H}^{p,q}(G/\Gamma, L_\beta) \cong H^{p,q}_{\bar\partial}(G/\Gamma, L_\beta)$ and $\mathcal{H}^{p,q}(\mathfrak{a}\oplus\mathfrak{n}) \cong H^{p,q}_{\bar\partial}(\mathfrak{a}\oplus\mathfrak{n})$ (see [41]). By Lemma 3.3.3, we have

$$\iota(\mathcal{H}^{p,q}(\mathfrak{a}\oplus\mathfrak{n})) \subset \bigoplus_{L_\beta\in\mathcal{L}} \mathcal{H}^{p,q}(G/\Gamma, L_\beta).$$

Hence the corollary follows. □

3.4 Proof of the main theorem

Proposition 3.4.1. *Let G be a Lie group as in Assumption 3.1.1. G/Γ is a holomorphic fiber bundle over a torus with a nilmanifold as a fiber,*

$$N/\Gamma'' \to G/\Gamma \to \mathbb{C}^n/\Gamma'$$

such that the structure group of this fibration is discrete.

Proof. Consider the covering $\mathbb{C}^n \times (N/\Gamma'') \to G/\Gamma$ such that the covering transformation is the action of Γ' on $\mathbb{C}^n \times (N/\Gamma'')$ given by $g \cdot (a,b) = (a+g, \phi(g)b)$. Hence we have the fiber bundle $G/\Gamma \to \mathbb{C}^n/\Gamma'$ with the fiber N/Γ'' and the discrete structure group $\phi(\Gamma') \subset \mathrm{Aut}(N)$. Since $\phi(g)$ is a holomorphic automorphism, this fiber bundle is holomorphic. □

Proof of Theorem 3.1.2. For $L_\beta \in \mathcal{L}$, by Borel's results in [25, Appendix 2], we have the spectral sequence (E_r, d_r) of the filtration of $A^{p,q}(G/\Gamma, L_\beta)$ induced by the holomorphic fiber bundle $p : G/\Gamma \to \mathbb{C}^n/\Gamma'$ as in Proposition 3.4.1 such that:
(1)E_r is 4-graded, by the fiber-degree, the base-degree and the type. Let ${}^{p,q}E_r^{s,t}$ be the subspace of elements of E_r of type (p,q), fiber-degree s and base-degree t. We have ${}^{p,q}E_r^{s,t} = 0$ if $p+q = s+t$ or if one of p,q,s,t is negative.
(2) If $p+q = s+t$, then we have

$${}^{p,q}E_2^{s,t} \cong \sum_{i\ge 0} H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma', L_\beta \otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma''))$$

where $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic fiber bundle $\bigcup_{b\in\mathbb{C}^n/\Gamma'} H_{\bar\partial}^{p,q}(p^{-1}(b))$.
(3)The spectral sequence converges to $H_{\bar\partial}(G/\Gamma, L_\beta)$.

By the assumption $H_{\bar\partial}^{*,*}(\mathfrak{n}) \cong H_{\bar\partial}^{*,*}(N/\Gamma'')$, the fiber bundle $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic vector bundle with the fiber $H_{\bar\partial}^{p-i,q-s+i}(\mathfrak{n})$ induced by the action ϕ of Γ on $H_{\bar\partial}^{p-i,q-s+i}(\mathfrak{n})$. Since the action ϕ on $\mathfrak{n}$ is semi-simple, the action of $\mathbb{C}^n$ on $H_{\bar\partial}^{p-i,q-s+i}(\mathfrak{n})$ induced by ϕ is diagonalizable. The fiber bundle splits as $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'') = \oplus L_{\delta_j}$ for some $L_{\delta_j} \in \mathcal{L}$. Hence we have

$$H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma', L_\beta \otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma'')) = H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma', \bigoplus_{\delta_j} L_\beta \otimes L_{\delta_j}).$$

By Proposition 3.2.4, we have $H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma', L_\beta \otimes L_{\delta_j}) \cong H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma')$ if $L_\beta \otimes L_{\delta_j}$ is trivial and $H_{\bar\partial}^{i,i-s}(\mathbb{C}^n/\Gamma', L_\beta \otimes L_{\delta_j})) = 0$ if $L_\beta \otimes L_{\delta_j}$ is non-trivial.

Hence we have

$$H^{i,i-s}_{\bar{\partial}}(\mathbb{C}^n/\Gamma', \bigoplus_{L_\beta\in\mathcal{L}} L_\beta\otimes\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')) \cong H^{i,i-s}_{\bar{\partial}}(\mathbb{C}^n/\Gamma')\otimes H^{p-i,q-s+i}_{\bar{\partial}}(\mathfrak{n}).$$

For the direct sum $\bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$, we consider this spectral sequence E_r. Then we have

$$\begin{aligned}{}^{p,q}E_2^{s,t} &\cong \sum_{i\geq 0} H^{i,i-s}_{\bar{\partial}}(\mathbb{C}^n/\Gamma', \bigoplus_{L_\beta\in\mathcal{L}} L_\beta\otimes\mathbf{H}^{p-i,q-s+i}(N/\Gamma''))\\ &\cong \sum_{i\geq 0} H^{i,i-s}_{\bar{\partial}}(\mathbb{C}^n/\Gamma')\otimes H^{p-i,q-s+i}_{\bar{\partial}}(\mathfrak{n})\end{aligned}$$

This implies an isomorphism $E_2 \cong \bigoplus_{p,q} H^{p,q}_{\bar{\partial}}(\mathfrak{a}\oplus\mathfrak{n})$. On the other hand, by Corollary 3.3.4, we have an injection

$$H^{p,q}_{\bar{\partial}}(\mathfrak{a}\oplus\mathfrak{n}) \to H^{p,q}_{\bar{\partial}}(\bigoplus_{L_\beta\in\mathcal{L}} A^{*,*}(G/\Gamma, L_\beta) \cong E_\infty.$$

Hence the spectral sequence degenerates at E_2 and the theorem follows. □

Corollary 3.4.2. *Let $B^{*,*}\subset A^{*,*}(G/\Gamma)$ be the subDBA of $A^{*,*}(G/\Gamma)$ given by*

$$B^{p,q} = \left\langle x_I\wedge\alpha_J^{-1}\beta_J y_J\wedge\bar{x}_K\wedge\bar{\alpha}_L^{-1}\gamma_L\bar{y}_L \,\middle|\, \begin{array}{c}|I|+|K|=p,\ |J|+|L|=q\\ \text{the restriction of } \beta_J\gamma_L \text{ on } \Gamma \text{ is trivial}\end{array}\right\rangle.$$

Then the inclusion $B^{*,*}\subset A^{*,*}(G/\Gamma)$ induces a cohomology isomorphism

$$H^{*,*}_{\bar{\partial}}(B^{*,*}) \cong H^{*,*}_{\bar{\partial}}(G/\Gamma).$$

Proof. By Lemma 3.2.3,

$$\Phi(x_I\wedge\alpha_J^{-1}y_J\wedge\bar{x}_K\wedge\bar{\alpha}_L^{-1}\bar{y}_L\otimes\beta_J\gamma_L v_{\beta_J^{-1}\gamma_L^{-1}}) \in A^{*,*}(G/\Gamma)$$

if and only if the restriction of $\beta_J\gamma_L$ on Γ is trivial. Hence we have $\Phi^{-1}(A^{*,*}(G/\Gamma)) = B^{*,*}$. □

Remark 3.4.1. *Suppose $\phi:\mathbb{C}^n\to \mathrm{Aut}(\mathfrak{n}^{1,0})$ is a holomorphic map. Since each α_j is holomorphic, β_j is trivial. Hence we have $B^{p,0} = \bigwedge^{p,0}\mathfrak{g}^*$. Moreover if N is a complex Lie group, then $G=\mathbb{C}^n\ltimes_\phi N$ is also a complex Lie group and any element of $B^{1,0}=\mathfrak{g}^{1,0}$ is holomorphic and hence $\bar{\partial}B^{p,0}=0$. Hence we have an isomorphism*

$$H^{p,q}(G/\Gamma) \cong \bigwedge^p \mathfrak{g}^{1,0}\otimes H^q_{\bar{\partial}}(B^{0,q}).$$

Remark 3.4.2. *We suppose the following condition:*
($\star$) For multi-indices J, L, if the restriction of $\beta_J\gamma_L$ on Γ is trivial, then $\beta_J\gamma_L$ itself is trivial.
Then we have $B^{*,*} \subset \bigwedge^{*,*}\mathfrak{g}^*$ and hence we have an isomorphism

$$H^{*,*}_{\bar\partial}(\mathfrak{g}) \cong H^{*,*}_{\bar\partial}(G/\Gamma).$$

3.5 Examples

3.5.1 Example 1

Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that $\phi(x+\sqrt{-1}y) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then for some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugate to an element of $SL(2,\mathbb{Z})$. Hence for any $0 \neq b \in \mathbb{R}$ we have a lattice $\Gamma = (a\mathbb{Z} + b\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$. Then for a coordinate $(z_1 = x+\sqrt{-1}y, z_2, z_3) \in \mathbb{C} \ltimes_\phi \mathbb{C}^2$ we have

$$\bigwedge^{p,q}\mathfrak{g}^* = \bigwedge^{p,q}\langle dz_1,\, e^{-x}dz_2, e^x dz_3\rangle \otimes \langle d\bar z_1,\, e^{-x}d\bar z_2,\, e^x d\bar z_3\rangle.$$

Since we have $e^x \sim e^{-\sqrt{-1}y}$, the subDBA

$$A^{*,*} \subset \bigoplus_{L_\beta \in \mathcal{L}} A^{*,*}(G/\Gamma, L_\beta)$$

as in Definition 3.3.1 is given by

$$A^{p,q} = \bigwedge^{p,q}\langle dz_1,\, e^{-x}dz_2 \otimes e^{-\sqrt{-1}y}v_{e^{\sqrt{-1}y}},\, e^x dz_3 \otimes e^{\sqrt{-1}y}v_{e^{-\sqrt{-1}y}}\rangle$$
$$\bigotimes\langle d\bar z_1,\, e^{-x}d\bar z_2 \otimes e^{-\sqrt{-1}y}v_{e^{\sqrt{-1}y}},\, e^x d\bar z_3 \otimes e^{\sqrt{-1}y}v_{e^{-\sqrt{-1}y}}\rangle.$$

$B^{p,q} \subset A^{p,q}(G/\Gamma)$ varies for a choice of $b \in \mathbb{R}$ as the following.
(A) If $b = 2n\pi$ for $n \in \mathbb{Z}$, then we have:

$$B^{p,q} = \bigwedge^{p,q}\langle dz_1,\, e^{-x-\sqrt{-1}y}dz_2, e^{x+\sqrt{-1}y}dz_3\rangle \otimes \langle d\bar z_1,\, e^{-x-\sqrt{-1}y}d\bar z_2, e^{x+\sqrt{-1}y}d\bar z_3\rangle.$$

(B) If $b = (2n-1)\pi$ for $n \in \mathbb{Z}$, then we have:

$$B^{1,0} = \langle dz_1\rangle,\; B^{0,1} = \langle d\bar z_1\rangle,$$
$$B^{2,0} = \langle dz_2 \wedge dz_3\rangle,\; B^{0,2} = \langle d\bar z_2 \wedge d\bar z_3\rangle,$$

$$B^{1,1} = \langle dz_1 \wedge d\bar{z}_1,\ e^{-2x-2\sqrt{-1}y} dz_2 \wedge d\bar{z}_2,\ e^{2x+2\sqrt{-1}y} dz_3 \wedge d\bar{z}_3,\ dz_2 \wedge d\bar{z}_3,\ dz_3 \wedge d\bar{z}_2 \rangle,$$

$$B^{3,0} = \langle dz_1 \wedge dz_2 \wedge dz_3 \rangle,$$

$$B^{2,1} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1,\ e^{-2x-2\sqrt{-1}y} dz_1 \wedge dz_2 \wedge d\bar{z}_2,\ e^{2x+2\sqrt{-1}y} dz_1 \wedge dz_3 \wedge d\bar{z}_3,\ dz_1 \wedge dz_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_2 \rangle,$$

$$B^{1,2} = \langle dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3,\ e^{-2x-2\sqrt{-1}y} dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2,\ e^{2x+2\sqrt{-1}y} dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_3,\ dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1,\ dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_1 \rangle,$$

$$B^{0,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$

$$B^{3,1} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \rangle,\ B^{1,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \rangle,$$

$$B^{2,2} = \langle dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3,\ e^{-2x-2\sqrt{-1}y} dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2,\ e^{2x+2\sqrt{-1}y} dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_3,\ dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \rangle,$$

$$B^{3,2} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle, B^{2,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$

$$B^{3,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.$$

(C) If $b \neq n\pi$ for any $n \in \mathbb{Z}$, then we have:

$$B^{1,0} = \langle dz_1 \rangle,\ B^{0,1} = \langle d\bar{z}_1 \rangle,$$

$$B^{2,0} = \langle dz_2 \wedge dz_3 \rangle,\ B^{0,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$

$$B^{1,1} = \langle dz_1 \wedge d\bar{z}_1, dz_2 \wedge d\bar{z}_3, dz_3 \wedge d\bar{z}_2 \rangle,$$

$$B^{3,0} = \langle dz_1 \wedge dz_2 \wedge dz_3 \rangle, B^{2,1} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1,\ dz_1 \wedge dz_2 \wedge d\bar{z}_3,\ dz_1 \wedge dz_3 \wedge d\bar{z}_2 \rangle,$$

$$B^{1,2} = \langle dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3,\ dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1, dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_1 \rangle, B^{0,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$

$$B^{3,1} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \rangle, B^{1,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \rangle,$$

$$B^{2,2} = \langle dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3, dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \rangle,$$

$$B^{3,2} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle, B^{2,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$

$$B^{3,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.$$

By Corollary 3.4.2, for each case we have an isomorphism $H^{p,q}_{\bar{\partial}}(G/\Gamma) \cong B^{p,q}$. Moreover considering the left-invariant Hermitian metric $g = dz_1 d\bar{z}_1 + e^{-2x} dz_2 d\bar{z}_2 + e^{2x} dz_3 d\bar{z}_3$, we have $\mathcal{H}^{p,q}(G/\Gamma) \cong B^{p,q}$.

Remark 3.5.1. *In the case (A), the Dolbeault cohomology $H^{*,*}_{\bar\partial}(G/\Gamma)$ is isomorphic to the Dolbeault cohomology of complex 3-torus. But G/Γ is not homeomorphic to a complex 3-torus. Moreover considering the metric g, the space of the harmonic forms does not satisfy Hodge symmetry (i.e. $\bar{\mathcal{H}}^{p,q}(G/\Gamma) \neq \mathcal{H}^{q,p}(G/\Gamma)$).*

Remark 3.5.2. *By Hattori's result in [24], we have an isomorphism $H^*(G/\Gamma) \cong H^*(\mathfrak{g})$ of the de Rham cohomology of G/Γ and the Lie algebra cohomology. Hence considering the space $\mathcal{H}^k_d(\mathfrak{g})$ of left-invariant d-harmonic forms of the left-invariant Hermitian metric g, we have $\mathcal{H}^k_d(\mathfrak{g}) \cong \mathcal{H}^k_d(G/\Gamma)$. By simple computations, in the case (C) we have the Hodge decomposition $\mathcal{H}^k_d(G/\Gamma) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(G/\Gamma)$. Hence G/Γ has cohomological properties (for example the Frölicher spectral sequence degenerates at E_1) of compact Kähler manifolds. But by Arapura's result (solving Benson-Gordon's conjecture) in [1], G/Γ admits no Kähler structure.*

Remark 3.5.3. *In the case (C), an isomorphism $H^{*,*}_{\bar\partial}(\mathfrak{g}) \cong H^{*,*}_{\bar\partial}(G/\Gamma)$ holds. But in the other cases, this isomorphism does not hold.*

3.5.2 Example 2

Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(x+\sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}.$$

Then we have $a+\sqrt{-1}b, c+\sqrt{-1}d \in \mathbb{C}$ such that $\mathbb{Z}(a+\sqrt{-1}b)+\mathbb{Z}(c+\sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a+\sqrt{-1}b)$ and $\phi(c+\sqrt{-1}d)$ are conjugate to elements of $SL(4,\mathbb{Z})$ where we regard $SL(2,\mathbb{C}) \subset SL(4,\mathbb{R})$ (see [23]). Hence we have a lattice $\Gamma = (\mathbb{Z}(a+\sqrt{-1}b)+\mathbb{Z}(c+\sqrt{-1}d)) \ltimes_\phi \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$. For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$, we have

$$\bigwedge^{p,q} \mathfrak{g}^* = \bigwedge^{p,q} \langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \otimes \langle d\bar z_1, e^{-\bar z_1}d\bar z_2, e^{\bar z_1}d\bar z_3\rangle.$$

We have

$$A^{p,q}$$
$$= \bigwedge^{p,q}\langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \bigotimes \langle d\bar z_1, e^{-\bar z_1}d\bar z_2 \otimes e^{-2\sqrt{-1}y_1} v_{e^{2\sqrt{-1}y_1}}, e^{\bar z_1}d\bar z_3 \otimes e^{2\sqrt{-1}y_1} v_{e^{-2\sqrt{-1}y_1}}\rangle$$

for $z_1 = x_1 + \sqrt{-1}y_1$.

If $b, d \in \pi\mathbb{Z}$, then we have

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge^{p,q} \langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \otimes \langle d\bar{z}_1, e^{-z_1}d\bar{z}_2, e^{z_1}d\bar{z}_3\rangle.$$

If $b \notin \pi\mathbb{Z}$ or $c \notin \pi\mathbb{Z}$, then we have

$$B^{0,1} = \langle d\bar{z}_1\rangle,\ B^{0,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3\rangle,\ B^{0,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\rangle$$

and

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge^{p} \langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \otimes B^{0,q}.$$

Bibliography

[1] D. Arapura, *Kähler solvmanifolds*, Int. Math. Res. Not. **3** (2004), 131–137, MR2038772, Zbl 1091.53047.

[2] D. Arapura, M. Nori, *Solvable fundamental groups of algebraic varieties and Kähler manifolds*, Compositio Math. **116** (1999), no. 2, 173–188, MR1686777, Zbl 0971.14020.

[3] L. Auslander, *An exposition of the structure of solvmanifolds*, I. Algebraic theory. Bull. Amer. Math. Soc. **79** (1973), no. 2, 227–261, MR0486307, Zbl 0265.22016.

[4] O. Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology **43** (2004), no. 4, 903–924, MR2061212, Zbl 1059.57022.

[5] O. Baues, V. Cortés, *Aspherical Kähler manifolds with solvable fundamental group*, Geom. Dedicata **122** (2006), 215–229, MR2295551, Zbl 1128.53043.

[6] C. Benson, and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), no. 4, 513–518, MR0976592, Zbl 0672.53036.

[7] C. Benson, C. S. Gordon, *Kähler structures on compact solvmanifolds*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 971–980. MR0993739, Zbl 0689.53036.

[8] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012, Zbl 0726.20030.

[9] S. Console, A. Fino, Dolbeault cohomology of compact nilmanifolds. Transform. Groups **6** (2001), no. 2, 111–124.

[10] L. A. Cordero, M. Fernández, A. Gray, L. Ugarte, Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology. Trans. Amer. Math. Soc. **352** (2000), no. 12, 5405–5433.

[11] D. Burde, K. Dekimpe, K. Vercammen, *Complete LR-structures on solvable Lie algebras*, J. Group Theory 13 (2010), no. 5, 703–719, MR2720199, Zbl pre05825397.

[12] K. Dekimpe, *Solvable Lie algebras, Lie groups and polynomial structures*, Compositio Math. **121** (2000), no. 2, 183-204, MR1757881, Zbl 1002.22003.

[13] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274, MR0382702, Zbl 0312.55011.

[14] N. Dungey, A. F. M. ter Elst, D. W. Robinson, *Analysis on Lie Groups with Polynomial Growth*, Progress in Mathematics, 214. Birkhäuser Boston (2003). MR2000440, Zbl 1041.43003.

[15] J. Dixmier, Cohomologie des algébres de Lie nilpotentes. Acta Sci. Math. Szeged **16** (1955), 246–250.

[16] S. Dragomir, L. Ornea, Locally conformal Kähler geometry. Progress in Mathematics, 155. Birkhäuser Boston, Inc., Boston, MA, 1998.

[17] Y. Félix, J. Oprea and D. Tanré, *Algebraic Models in Geometry*, Oxford Graduate Texts in Mathematics 17, Oxford University Press 2008, MR2403898, Zbl 1149.53002.

[18] M. Fernandez, A. Gray, The Iwasawa manifold. Differential geometry, Peniscola 1985, 157–159, Lecture Notes in Math., 1209, Springer, Berlin, 1986.

[19] R. M. Hain, *The Hodge de Rham theory of relative Malcev completion*, Ann. Sci. Ecole Norm. Sup. (4) **31** (1998), no. 1, 47–92, MR1604294, Zbl 0911.14008.

[20] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71, MR0946638, Zbl 0691.53040.

[21] K. Hasegawa, *A class of compact Kählerian solvmanifolds and a general conjecture*, Geom. Dedicata **78** (1999), no. 3, 253–258, MR1725375, Zbl 0940.53037.

[22] K. Hasegawa, *A note on compact solvmanifolds with Kähler structures*, Osaka J. Math. **43** (2006), no. 1, 131–135, MR2222405, Zbl 1105.32017.

[23] K. Hasegawa, Small deformations and non-left-invariant complex structures on six-dimensional compact solvmanifolds. Differential Geom. Appl. **28** (2010), no. 2, 220–227.

[24] A. Hattori, *Spectral sequence in the de Rham cohomology of fibre bundles*, J. Fac. Sci. Univ. Tokyo Sect. I **8** 1960 289–331 (1960), MR0124918, Zbl 0099.18003.

[25] F. Hirzebruch, Topological Methods in Algebraic Geometry, third enlarged ed., Springer-Verlag, 1966.

[26] H. Kasuya, *Formality and hard Lefschetz properties of aspherical manifolds*, http://arxiv.org/abs/0910.1175. to appear in Osaka J. Math.

[27] A. Malcev, On a class of homogeneous spaces. (Russian) Izvestiya Akad. Nauk. SSSR. Ser. Mat. **13**, (1949). 9–32.

[28] G. D. Mostow, *Cohomology of topological groups and solvmanifolds* Ann. of Math. (2) **73** (1961), 20–48, MR0125179, Zbl 0103.26501.

[29] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Differential Geometry **10** (1975), 85–112, MR0393580, Zbl 0297.32019.

[30] A. Narkawicz, *Cohomology Jumping Loci and Relative Malcev Completion*, Thesis (Ph.D.)–Duke University. 2007. MR2711033.

[31] J. Neisendorfer, L. Taylor, Dolbeault homotopy theory. Trans. Amer. Math. Soc. **245** (1978), 183–210.

[32] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups*, Ann. of Math. (2) **59**, (1954). 531–538, MR0064057, Zbl 0058.02202.

[33] A. L. Onishchik, E. B. Vinberg (Eds.), *Lie groups and Lie algebras. II. Discrete subgroups of Lie groups and cohomologies of Lie groups and Lie algebras*, Encyclopaedia of Mathematical Sciences, 21. Springer-Verlag, Berlin, (2000), MR1756406, Zbl 0932.00011.

[34] K. Oeljeklaus, M. Toma, *Non-Kähler compact complex manifolds associated to number fields*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 161–171, MR2141693, Zbl 1071.32017.

[35] L. Ornea, M. Pilca, Remarks on the product of harmonic forms, **250** (2011), Pacific J. Math. No. 2, 353–363.

[36] L. Ornea, M. Verbitsky, Morse-Novikov cohomology of locally conformally Kähler manifolds. J. Geom. Phys. **59** (2009), no. 3, 295–305.

[37] M. Parton, V. Vuletescu, Examples of non-trivial rank in locally conformal Kähler geometry, Math. Z. (2010), DOI 10.1007/s00209-010-0791-5.

[38] A. Polishchuk, Abelian Varieties, Theta Functions and the Fourier Transform. Cambridge University Press 2002.

[39] J. Oprea, A. Tralle, Symplectic manifolds with no Kähler structure. Lecture Notes in Math. 1661, Springer.

[40] M. S. Raghunathan, *Discrete subgroups of Lie Groups*, Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68, MR0507234, Zbl 0254.22005.

[41] S. Rollenske, Lie-algebra Dolbeault-cohomology and small deformations of nilmanifolds. J. Lond. Math. Soc. (2) **79** (2009), no. 2, 346–362.

[42] Y. Sakane, On compact complex parallelisable solvmanifolds. Osaka J. Math. **13** (1976), no. 1, 187–212.

[43] H. Sawai, *A construction of lattices on certain solvable Lie groups*, Topology Appl. **154** (2007), no. 18, 3125–3134, MR2364640, Zbl 1138.53065.

[44] C. T. Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. No. **75** (1992), 5–95, MR1179076, Zbl 0814.32003.

[45] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. **47** (1977), 269–331 (1978), MR0646078, Zbl 0374.57002.

[46] W. P. Thurston, Some simple examples of symplectic manifolds. Proc. Amer. Math. Soc. **55** (1976), no. 2, 467–468.

[47] T. Yamada, A pseudo-Kähler structure on a nontoral compact complex parallelizable solvmanifold. Geom. Dedicata **112** (2005), 115–122.