Symmetrized Max-Plus Algebra and Ultradiscrete sine-Gordon Equation

(対称Max-Plus代数と超離散sine-Gordon方程式)

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Contents

Chapter 1

Introduction

1.1 Problem

Ultradiscrete integrable systems are integrable systems where independent variables take values in $\mathbb{Z}$, and dependent variables in the max-plus algebra $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. Among them is the famous box-ball system [17], represented by the equation

$$U_n^{t+1} = \min\left[1 - U_n^t, \sum_{k=-\infty}^{n-1} U_k^t - \sum_{k=-\infty}^{n-1} U_k^{t+1}\right]. \tag{1.1}$$

Defining $S_n^t = \sum_{k=n}^{\infty} \sum_{l=-\infty}^{t} U_k^l$, we obtain

$$S_{n+1}^{t+1} + S_n^{t-1} = \max\left[S_{n+1}^{t-1} + S_n^{t+1} - 1, S_n^t + S_{n+1}^t\right], \tag{1.2}$$

which is *ultradiscretization* of the discrete KdV equation

$$(1+\delta)\sigma_{n+1}^{t+1}\sigma_n^{t-1} = \delta\sigma_{n+1}^{t-1}\sigma_n^{t+1} + \sigma_n^t\sigma_{n+1}^t. \tag{1.3}$$

Ultradiscretization [18] is a systematic procedure to obtain ultradiscrete systems from discrete systems. The fundamental formula of the procedure is

$$\lim_{\epsilon \to +0} \epsilon \log\left(e^{A/\epsilon} + e^{B/\epsilon}\right) = \max[A, B], \tag{1.4a}$$

$$\lim_{\epsilon \to +0} \epsilon \log\left(e^{A/s} \cdot e^{B/s}\right) = A + B. \tag{1.4b}$$

This may be understood as transformation of addition into max operation and of multiplication into addition. Setting $\delta = e^{-1/\epsilon}$ in (1.3) and applying ultradiscretization, we obtain (1.2).

The problem is, however, that ultradiscretization cannot be applied to subtraction, which is of course contained by many discrete integrable systems. This is because the equation

$$\max[x, a] = b \tag{1.5}$$

for x has no solution in $\mathbb{R}_{\max}$ when $a > b$.

1.2 Contents of the thesis

We focus on the symmetrized max-plus algebra [16, 1], denoted by $\mathrm{u}\mathbb{R}$ in this thesis, in order to solve the problem. This algebra is an extension of $\mathbb{R}_{\max}$ and looks natural in the sense it traces the construction of $\mathbb{Z}$ from $\mathbb{N}^2$. Linear algebra over $\mathrm{u}\mathbb{R}$ is also possible [16, 1], and ultradiscretization with $\mathrm{u}\mathbb{R}$ is presented in [3]. These theories of $\mathrm{u}\mathbb{R}$ are mainly developed in the field of discrete event systems and seems little known to the field of integrable systems. Several other attempts [15, 6, 19, 9] have also been made to solve the problem, and ultradiscretization with parity variables [13] is of special interest. We compare this method and ultradiscretization with $\mathrm{u}\mathbb{R}$.

The discrete sine-Gordon equation [5, 2]

$$(1-\delta)\tau_l^m\tau_{l+1}^{m+1} = \tau_{l+1}^m\tau_l^{m+1} - \delta\sigma_{l+1}^m\sigma_l^{m+1}, \tag{1.6a}$$

$$(1-\delta)\sigma_l^m\sigma_{l+1}^{m+1} = \sigma_{l+1}^m\sigma_l^{m+1} - \delta\tau_{l+1}^m\tau_l^{m+1} \tag{1.6b}$$

has not been ultradiscretized until recent years because soliton solutions include subtraction or even complex numbers. The first attempt is made by [7, 8] where a τ_l^m-only trilinear equation is exploited to exclude subtraction. Here we propose another method to ultradiscretize the sine-Gordon equation which utilizes $\mathrm{u}\mathbb{R}$. Both the equation and the solutions are ultradiscretized keeping subtraction and complex numbers in a highly direct fashion.

Noncommutative integrable systems have been drawing more interest in the last two decades. It is difficult to point out the first appearance of such systems, but the noncommutative KdV equation is already mentioned in [11]. The first discrete noncommutative integrable system is probably the noncommutative discrete KP equation [14, 10]. Along this line, we propose the noncommutative discrete sine-Gordon equation, explore relations to other integrable systems, and construct multisoliton solutions by the Darboux transformation. Moreover, we also propose the noncommutative ultradiscrete sine-Gordon equation and explicitly derive 1-soliton and 2-soliton solutions by ultradiscretization with $\mathrm{u}\mathbb{R}$. As a result, we have a complete set of commutative and noncommutative versions of the continuous, discrete, and ultradiscrete sine-Gordon equations.

The rest of the thesis is organized as follows.

In Section 2.1, the construction and properties of $\mathrm{u}\mathbb{R}$ and matrices over $\mathrm{u}\mathbb{R}$ are reviewed in some detail. Ultradiscrete complex numbers are also introduced. In Section 2.2, ultradiscretization with these algebras is reviewed. Ultradiscretization with $\mathrm{u}\mathbb{R}$ and that with parity variables are compared.

In Section 3.1, the discrete sine-Gordon equation and 1-soliton and 2-soliton solutions are reviewed. Special solutions such as the traveling-wave and kink-antikink solutions are explicitly presented. In Section 3.2, the ultradiscrete sine-Gordon equation is proposed and the solutions are obtained. Because of ultradiscretization with u$\mathbb{R}$, correspondence between the discrete and ultradiscrete systems are direct, which is also supported by figures.

In Section 4.1, the noncommutative discrete sine-Gordon equation is proposed. Relation to other integrable systems including the noncommutative discrete KP equation is explained, and multisoliton solutions are constructed by repeating the Darboux transformation. In Section 4.2, the noncommutative ultradiscrete sine-Gordon equation is proposed and 1-soliton and 2-soliton solutions are derived. Also figures of solutions for both equations are displayed.

In Chapter 5, concluding remarks are presented.

Chapter 2

Symmetrized max-plus algebra and ultradiscretization

In this chapter, we review the construction and properties of the symmetrized max-plus algebra $\mathrm{u}\mathbb{R}$ and matrices over $\mathrm{u}\mathbb{R}$ in some detail [16, 1]. Ultradiscrete complex numbers are also introduced. Ultradiscretization with $\mathrm{u}\mathbb{R}$ [3] is then reviewed. Most of the results are already known, but we dare explain them because they are important and seem little known to the community of integrable systems. Lastly, we compare ultradiscretization with $\mathrm{u}\mathbb{R}$ and that with parity variables, using two simple examples.

2.1 Symmetrized max-plus algebra

2.1.1 Dioids

A semiring is a set R with addition $\oplus$ and multiplication $\otimes$ such that

- $\oplus$ is associative and commutative with null element 0,
- $\otimes$ is associative with unit element 1,
- $\otimes$ is distributive over $\oplus$,
- and 0 is absorbing, that is, $0 \otimes x = x \otimes 0 = 0$ for any $x \in R$.

An idempotent semiring, also called a dioid, D is a semiring where addition is idempotent, that is, $x \oplus x = x$ for any $x \in D$. A subset C of D is called a subdioid if

- C includes 0 and 1,
- and is closed under $\oplus$ and $\otimes$.

A dioid is called commutative if multiplication is commutative.

Most of the time, $\otimes$ is omitted in expressions for brevity. $\bigoplus$ and $\bigotimes$ are used like

$$\bigoplus_{i=1}^{n} x_i = x_1 \oplus x_2 \oplus \cdots \oplus x_n, \quad \bigotimes_{i=1}^{n} x_i = x_1 \otimes x_2 \otimes \cdots \otimes x_n.$$

Multiplicative inverse of x, if exists, is denoted by x^{-1}, and powers of x by

$$x^n = \begin{cases} \bigotimes_{i=1}^{n} x & (n > 0), \\ 1 & (n = 0), \\ \left(x^{-1}\right)^{-n} & (n < 0). \end{cases}$$

In commutative case, fractions like $x/y = xy^{-1}$ are also used.

2.1.2 Pair of the max-plus algebra

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$. $\mathbb{R}_{\max}$ has the obvious total order. Define $\oplus$ and $\otimes$ by

$$x \oplus y = \max(x, y), \quad x \otimes y = x + y \tag{2.1}$$

for $x, y \in \mathbb{R}_{\max}$. With these operations, $\mathbb{R}_{\max}$ becomes a commutative dioid called the max-plus algebra. The null element is $-\infty$ and the unit element is 0. We extend $\oplus$ and $\otimes$ over $\mathbb{R}^2_{\max}$ by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus y_1, x_2 \oplus y_2), \tag{2.2}$$

$$(x_1, x_2) \otimes (y_1, y_2) = (x_1 y_1 \oplus x_2 y_2, x_1 y_2 \oplus x_2 y_1). \tag{2.3}$$

Then $\mathbb{R}^2_{\max}$ is a commutative dioid with null element $(-\infty, -\infty)$ and unit element $(0, -\infty)$. $\mathbb{R}_{\max}$ is embedded into $\mathbb{R}^2_{\max}$ by $x \longmapsto (x, -\infty)$.

Define minus sign $\ominus$ by

$$\ominus(x_1, x_2) = (x_2, x_1) \tag{2.4}$$

for $x = (x_1, x_2) \in \mathbb{R}^2_{\max}$. We have

$$\ominus(\ominus x) = x, \quad \ominus(x \oplus y) = (\ominus x) \oplus (\ominus y), \quad \ominus(xy) = (\ominus x) y \tag{2.5}$$

and therefore write $x \ominus y$ for $x \oplus (\ominus y)$, which is regarded as subtraction.

Define absolute value $|\,|_\oplus : \mathbb{R}^2_{\max} \to \mathbb{R}_{\max}$ by

$$|(x_1, x_2)|_\oplus = x_1 \oplus x_2. \tag{2.6}$$

We have

$$|x|_\oplus = |\ominus x|_\oplus, \quad |x \oplus y|_\oplus = |x|_\oplus \oplus |y|_\oplus, \quad |xy|_\oplus = |x|_\oplus |y|_\oplus. \tag{2.7}$$

Define balance operator $^\bullet$ by

$$(x_1, x_2)^\bullet = (x_1, x_2) \ominus (x_1, x_2) = (x_1 \oplus x_2, x_1 \oplus x_2). \tag{2.8}$$

We have

$$x^\bullet = (\ominus x)^\bullet, \quad (x^\bullet)^\bullet = x^\bullet, \quad (x \oplus y)^\bullet = x^\bullet \oplus y^\bullet, \quad x(y^\bullet) = (xy)^\bullet. \tag{2.9}$$

2.1.3 Symmetrized max-plus algebra

It is natural to consider balance relation ∇ defined by

$$(x_1, x_2) \ \nabla \ (y_1, y_2) \iff x_1 \oplus y_2 = x_2 \oplus y_1. \tag{2.10}$$

∇ is reflexive and symmetric, but *not* transitive: if $x_1 > x_2$, then $(x_1, x_2) \ \nabla \ (x_1, x_1)$ and $(x_1, x_1) \ \nabla \ (x_2, x_1)$ hold, but $(x_1, x_2) \ \nabla \ (x_2, x_1)$ does not hold since $x_1 \oplus x_1 = x_1 \neq x_2 = x_2 \oplus x_2$.

Therefore, we introduce another relation $\mathcal{R}$ defined by

$$(x_1, x_2) \ \mathcal{R} \ (y_1, y_2) \iff \begin{cases} (x_1, x_2) \ \nabla \ (y_1, y_2) & (\text{when } x_1 \neq x_2 \text{ and } y_1 \neq y_2), \\ (x_1, x_2) = (y_1, y_2) & (\text{otherwise}). \end{cases} \tag{2.11}$$

$\mathcal{R}$ is an equivalence relation compatible with the operations $\oplus$, $\otimes$, $\ominus$, $|\,|_\oplus$, $^\bullet$, and the relation ∇. Thus, we can define the quotient structure

$$\mathrm{u}\mathbb{R} = \mathbb{R}^2_{\max} / \mathcal{R}\,. \tag{2.12}$$

This is called the symmetrized max-plus algebra [16, 1]. Usually this is denoted by $\mathbb{S}$, but we use $\mathrm{u}\mathbb{R}$ to imply it is somehow a whole set of *ultradiscrete real numbers*. We will also introduce $\mathrm{u}\mathbb{Z}$, $\mathrm{u}\mathbb{C}$ later.

Proposition 2.1. We have three kinds of equivalence classes:

$$\begin{aligned} \overline{(x, -\infty)} &= \{(x, t) : t \in \mathbb{R}_{\max} \text{ and } x > t\}, \\ \overline{(-\infty, x)} &= \{(t, x) : t \in \mathbb{R}_{\max} \text{ and } t < x\}, \\ \overline{(x, x)} &= \{(x, x)\}. \end{aligned}$$

Proof. For any $(x_1, x_2) \in \mathbb{R}^2_{\max}$, only one of the three conditions $x_1 > x_2$, $x_1 < x_2$, and $x_1 = x_2$ hold. In the first case, we have

$$(x_1, x_2) \ \nabla \ (x_1, -\infty) \implies (x_1, x_2) \ \mathcal{R} \ (x_1, -\infty)$$

and thus $(x_1, x_2) \in \overline{(x_1, -\infty)}$. The second case is similar, and the third case is trivial. ∎

$\mathbb{R}_{\max}$ is embedded into $\mathrm{u}\mathbb{R}$ by $x \longmapsto \overline{(x, -\infty)}$. Define

$$\ominus\mathbb{R}_{\max} = \left\{\overline{(-\infty, x)} : x \in \mathbb{R}_{\max}\right\}, \quad \mathbb{R}^{\bullet}_{\max} = \left\{\overline{(x, x)} : x \in \mathbb{R}_{\max}\right\}. \tag{2.13}$$

Then $\mathrm{u}\mathbb{R}$ has a decomposition

$$\mathrm{u}\mathbb{R} = \mathbb{R}_{\max} \cup \ominus\mathbb{R}_{\max} \cup \mathbb{R}^{\bullet}_{\max}, \tag{2.14}$$

and $\overline{(-\infty, -\infty)}$ is the only element which belongs to any two of the three sets. Thus, we simply write x for $\overline{(x, -\infty)}$, $\ominus x$ for $\overline{(-\infty, x)}$, and $x^{\bullet}$ for $\overline{(x, x)}$.

Example.

$$(-3) \oplus (-2) = -2, \quad 3 \ominus 2 = 3, \quad 2 \ominus 3 = \ominus 3, \quad 2 \ominus 2 = 2^{\bullet},$$
$$2 \otimes 3 = 5, \quad \ominus(-2) \otimes (-3) = \ominus(-5), \quad 2 \otimes 3^{\bullet} = 5^{\bullet}.$$

Define sign function $\operatorname{sgn} x$ by

$$\operatorname{sgn} x = \begin{cases} 0 & (x \in \mathbb{R}), \\ 0^{\bullet} & (x \in \mathbb{R}^{\bullet}_{\max}), \\ \ominus 0 & (x \in \ominus\mathbb{R}). \end{cases} \tag{2.15}$$

$x \in \mathrm{u}\mathbb{R}$ is said to be positive if $\operatorname{sgn} x = 0$, negative if $\operatorname{sgn} x = \ominus 0$, and balanced if $\operatorname{sgn} x = 0^{\bullet}$.

Define $\mathrm{u}\mathbb{R}^{\vee} = \mathbb{R}_{\max} \cup \ominus\mathbb{R}_{\max}$. $x \in \mathrm{u}\mathbb{R}$ is said to be signed if $x \in \mathrm{u}\mathbb{R}^{\vee}$. It is somewhat confusing that $-\infty$ is a signed element, but this is a minor problem.

Proposition 2.2. Let $\mathrm{u}\mathbb{R}^{\otimes}$ denote the whole set of invertible elements in $\mathrm{u}\mathbb{R}$. Then,

$$\mathrm{u}\mathbb{R}^{\otimes} = \mathrm{u}\mathbb{R}^{\vee} \setminus \{-\infty\} = \mathrm{u}\mathbb{R} \setminus \mathbb{R}^{\bullet}_{\max}. \tag{2.16}$$

Proof. For any $x \in \mathbb{R}$, we have $x \otimes (-x) = (\ominus x) \otimes (\ominus(-x)) = 0$. And for any $x \in \mathbb{R}_{\max}$ and $y \in \mathrm{u}\mathbb{R}$, we have $(x^{\bullet})y = (xy)^{\bullet} \neq 0$. ∎

Define $\mathrm{u}\mathbb{Z}, \mathrm{u}\mathbb{Z}^{\vee} \subset \mathrm{u}\mathbb{R}$ by

$$\mathrm{u}\mathbb{Z} = \{-\infty\} \cup \mathbb{Z} \cup \ominus\mathbb{Z} \cup \mathbb{Z}^{\bullet}, \quad \mathrm{u}\mathbb{Z}^{\vee} = \mathrm{u}\mathbb{Z} \cap \mathrm{u}\mathbb{R}^{\vee} \tag{2.17}$$

with obvious notations. $\mathrm{u}\mathbb{Z}$ is a subdioid of $\mathrm{u}\mathbb{R}$ and can be regarded as a whole set of *ultradiscrete integers*. $x \in \mathrm{u}\mathbb{Z}$ is said to be even if $|x|_{\oplus}$ is even, odd if $|x|_{\oplus}$ is odd. We do not define whether $-\infty$ is even or odd. We have of course

$$\mathrm{u}\mathbb{Z}^{\otimes} = \mathrm{u}\mathbb{Z}^{\vee} \setminus \{-\infty\}. \tag{2.18}$$

2.1.4 Properties of balance relation

We make much use of ∇, rather than $\mathcal{R}$, since members of $\mathbb{R}^{\bullet}_{\max}$ can be regarded as a kind of null elements by virtue of the following proposition.

Proposition 2.3. For any $x \in \mathrm{u}\mathbb{R}$,

$$x \nabla -\infty \iff x \in \mathbb{R}^{\bullet}_{\max}. \tag{2.19}$$

Proof. Assume $x = \overline{(x_1, x_2)}$. Then,

$$x \nabla -\infty \iff x_1 \oplus -\infty = x_2 \oplus -\infty \iff x \in \mathbb{R}^{\bullet}_{\max}. \qquad \blacksquare$$

Proposition 2.4. For any $x \in \mathrm{u}\mathbb{R}$ and $t \in \mathbb{R}_{\max}$,

$$x \nabla t^{\bullet} \text{ and } x \notin \mathbb{R}^{\bullet}_{\max} \iff |x|_{\oplus} \leqslant t. \tag{2.20}$$

Proof. Assume $x = \overline{(x_1, x_2)}$. Then,

$$\begin{aligned} x \nabla t^{\bullet} \text{ and } x \notin \mathbb{R}^{\bullet}_{\max} &\iff x_1 \oplus t = x_2 \oplus t \text{ and } x_1 \neq x_2 \\ &\iff x_1 \leqslant t \text{ and } x_2 \leqslant t \\ &\iff |x|_{\oplus} \leqslant t. \end{aligned} \qquad \blacksquare$$

Proposition 2.5. For any $x, y \in \mathrm{u}\mathbb{R}$, we have

$$x \nabla y \iff x \ominus y \nabla -\infty. \tag{2.21}$$

Proof. Assume $x = \overline{(x_1, x_2)}$, $y = \overline{(y_1, y_2)}$. Then,

$$\begin{aligned} x \nabla y &\iff x_1 \oplus y_2 = x_2 \oplus y_1 \\ &\iff (x_1 \oplus y_2) \oplus -\infty = (x_2 \oplus y_1) \oplus -\infty \\ &\iff x \ominus y \nabla -\infty. \end{aligned} \qquad \blacksquare$$

Proposition 2.6. For any $x, y, z, w \in \mathrm{u}\mathbb{R}$, we have

$$x \nabla y \text{ and } z \nabla w \implies x \oplus z \nabla y \oplus w, \tag{2.22}$$
$$x \nabla y \implies xz \nabla yz. \tag{2.23}$$

Proof.

$$\begin{aligned} x \nabla y \text{ and } z \nabla w &\iff x \ominus y \nabla -\infty \text{ and } z \ominus w \nabla -\infty \\ &\implies x \ominus y \oplus z \ominus w \nabla -\infty \\ &\iff x \oplus z \nabla y \oplus w \end{aligned}$$

and

$$\begin{aligned} x \nabla y &\iff x \ominus y \nabla -\infty \\ &\implies (x \ominus y)z \nabla -\infty \\ &\iff xz \nabla yz. \end{aligned} \qquad \blacksquare$$

Proposition 2.7 (Weak substitution).

$$x \mathrel{\nabla} y,\ cy \mathrel{\nabla} z, \text{ and } y \in \mathrm{u}\mathbb{R}^{\vee} \implies cx \mathrel{\nabla} z. \tag{2.24}$$

Proof. Assume $x = \overline{(x_1, x_2)}, y = \overline{(y_1, y_2)}$, etc. When $y_2 = -\infty$,

$$x_1 = x_2 \oplus y_1, \quad c_1 y_1 \oplus z_2 = c_2 y_1 \oplus z_1.$$

Adding $c_1 x_2 \oplus c_2 x_2$ to the both sides of the second equality, we have

$$\begin{aligned} & c_1 y_1 \oplus c_1 x_2 \oplus c_2 x_2 \oplus z_2 = c_2 y_1 \oplus c_1 x_2 \oplus c_2 x_2 \oplus z_1 \\ & \implies c_1 x_1 \oplus c_2 x_2 \oplus z_2 = c_2 x_1 \oplus c_1 x_2 \oplus z_1 \\ & \implies cx \mathrel{\nabla} z. \end{aligned}$$

Similarly for the case $y_1 = -\infty$. ■

Corollary 2.8 (Weak transitivity).

$$x \mathrel{\nabla} y,\ y \mathrel{\nabla} z, \text{ and } y \in \mathrm{u}\mathbb{R}^{\vee} \implies x \mathrel{\nabla} z. \tag{2.25}$$

Proof. Set $c = 0$ in the previous proposition. ■

Proposition 2.9 (Reduction of balances).

$$x \mathrel{\nabla} y \text{ and } x, y \in \mathrm{u}\mathbb{R}^{\vee} \implies x = y. \tag{2.26}$$

Proof. Assume $x = \overline{(x_1, x_2)}, y = \overline{(y_1, y_2)}$. When $x_2 = -\infty$, we have

$$x_1 \oplus y_2 = y_1 \implies y_2 = -\infty \text{ and } x_1 = y_1.$$

Similarly for the case $x_1 = -\infty$. ■

Proposition 2.10. For any $x, y, z, w \in \mathrm{u}\mathbb{R}$, we have

$$x \mathrel{\nabla} y \text{ and } z \mathrel{\nabla} w \implies xz \mathrel{\nabla} yw. \tag{2.27}$$

Proof. If both of xz and yw are signed, then $x = y$ and $z = w$, which imply $xz \mathrel{\nabla} yw$. If both of xz and yw are balanced, then $xz \mathrel{\nabla} yw$ trivially holds.

For the remaining cases, we can assume xz are signed and yw are balanced without loss of generality. We have

$$xz \mathrel{\nabla} yz, \quad xw \mathrel{\nabla} yw, \quad xz \mathrel{\nabla} xw, \quad yz \mathrel{\nabla} yw.$$

If $yz \in \mathrm{u}\mathbb{R}^{\vee}$ or $xw \in \mathrm{u}\mathbb{R}^{\vee}$, then weak transitivity implies $xz \mathrel{\nabla} yw$. Otherwise, we have $y = s^{\bullet}, w = t^{\bullet}$ for some $s, t \in \mathbb{R}_{\max}$. Then, $|x|_{\oplus} \leqslant s$ and $|z|_{\oplus} \leqslant t$ imply $|xz|_{\oplus} \leqslant st \iff xz \mathrel{\nabla} yw$. ■

2.1.5 Matrices and determinants

Let $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ denote the whole set of $N \times N$ matrices over $\mathrm{u}\mathbb{R}$. Define addition $\oplus$ by

$$(a_{ij}) \oplus (b_{ij}) = (a_{ij} \oplus b_{ij}) \tag{2.28}$$

and multiplication $\otimes$ by

$$(a_{ij}) \otimes (b_{ij}) = (c_{ij}), \quad c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj} \tag{2.29}$$

for any $(a_{ij}), (b_{ij}) \in \mathrm{uMat}(N, \mathrm{u}\mathbb{R})$. Then $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ becomes a dioid, noncommutative when $N > 1$. $\ominus$, $^\bullet$, and ∇ are of course defined by

$$\ominus(a_{ij}) = (\ominus a_{ij}), \quad (a_{ij})^\bullet = (a_{ij}^\bullet), \tag{2.30}$$

$$(a_{ij}) \,\nabla\, (b_{ij}) \iff a_{ij} \,\nabla\, b_{ij} \text{ for any } i, j \tag{2.31}$$

respectively. $(a_{ij}) \in \mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ is said to be signed if all the elements are signed. The whole set of signed elements in $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ is denoted by $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})^\vee$. $\mathrm{u}\mathbb{R}$ is embedded into $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ by

$$x \longmapsto \begin{pmatrix} x & -\infty & \cdots & -\infty \\ -\infty & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\infty \\ -\infty & \cdots & -\infty & x \end{pmatrix}. \tag{2.32}$$

For any permutation $\sigma \in S_N$, define $\mathrm{sgn}(\sigma)$ by

$$\mathrm{sgn}(\sigma) = \begin{cases} 0 & \text{(when } \sigma \text{ is even)}, \\ \ominus 0 & \text{(when } \sigma \text{ is odd)}. \end{cases} \tag{2.33}$$

And define the determinant of a matrix $A = (a_{ij}) \in \mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ by

$$\det A = \bigoplus_\sigma \mathrm{sgn}(\sigma) \bigotimes_i a_{i\sigma(i)}. \tag{2.34}$$

$\det A$ is also denoted by $|A|$ or $|a_{ij}|$. This satisfies some basic properties completely analogous to that of ordinary determinants. Proofs are also quite similar, thus we simply list the properties below.

Proposition 2.11.

$$|{}^tA| = |A| \tag{2.35}$$

where ${}^tA$ denotes transposition of A.

Proposition 2.12.

$$\begin{aligned}&\big|\, v_1 \;\cdots\; \lambda v_j \oplus u \;\cdots\; v_N \,\big| \\ &\quad = \lambda \,\big|\, v_1 \;\cdots\; v_j \;\cdots\; v_N \,\big| \oplus \big|\, v_1 \;\cdots\; u \;\cdots\; v_N \,\big| \end{aligned} \tag{2.36}$$

where $v_j = {}^t(a_{1j}, \dots, a_{Nj})$ and $u = {}^t(u_1, \dots, u_N)$.

Proposition 2.13. For any permutation $\sigma \in S_N$,

$$|a_{i\sigma(j)}| = \mathrm{sgn}(\sigma)|a_{ij}|. \tag{2.37}$$

Corollary 2.14. If $v_j = v_k$ for some $j \neq k$, then

$$\big|\, v_1 \;\cdots\; v_N \,\big| \nabla -\infty. \tag{2.38}$$

Let $\mathrm{cof}_{ij}(A)$ denote the cofactor of a_{ij} in $|A|$, which by definition satisfies

$$|A| = \bigoplus_i a_{ij} \otimes \mathrm{cof}_{ij}(A) \tag{2.39}$$

for any j. Define the adjacent matrix of A by

$$\mathrm{adj}\, A = (b_{ij}), \quad b_{ij} = \mathrm{cof}_{ji}(A). \tag{2.40}$$

Theorem 2.15.

$$A \otimes \mathrm{adj}\, A \;\nabla\; |A|, \quad \mathrm{adj}\, A \otimes A \;\nabla\; |A|. \tag{2.41}$$

Remark. $A \otimes \mathrm{adj}\, A$ and $\mathrm{adj}\, A \otimes A$ not necessarily coincide. For example, we have

$$\mathrm{adj}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \ominus 1 \\ \ominus(-1) & 0 \end{pmatrix}$$

and thus

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \ominus 1 \\ \ominus(-1) & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1^\bullet \\ 0^\bullet & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & \ominus 1 \\ \ominus(-1) & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2^\bullet \\ (-1)^\bullet & 1 \end{pmatrix}.$$

If $|A| \in \mathrm{u}\mathbb{R}^{\otimes}$, define A^{-1} by

$$A^{-1} = |A|^{-1}\, \mathrm{adj}\, A. \tag{2.42}$$

This is not a multiplicative inverse in general, but plays a similar role with regard to ∇. Therefore we use the notation A^{-1}.

2.1.6 Ultradiscrete complex numbers

It is well known that we can construct complex numbers by 2×2 real matrices, using

$$\mathrm{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as the imaginary unit. We try to construct *ultradiscrete complex numbers* in a similar way.

Define $\mathrm{I} \in \mathrm{uMat}(2, \mathrm{u}\mathbb{R})$ by

$$\mathrm{I} = \begin{pmatrix} -\infty & \ominus 0 \\ 0 & -\infty \end{pmatrix}. \tag{2.43}$$

We have

$$\mathrm{I}^2 = \begin{pmatrix} -\infty & \ominus 0 \\ 0 & -\infty \end{pmatrix} \begin{pmatrix} -\infty & \ominus 0 \\ 0 & -\infty \end{pmatrix} = \begin{pmatrix} \ominus 0 & -\infty \\ -\infty & \ominus 0 \end{pmatrix} = \ominus 0. \tag{2.44}$$

Define $\mathrm{u}\mathbb{C} \subset \mathrm{uMat}(2, \mathrm{u}\mathbb{R})$ by

$$\mathrm{u}\mathbb{C} = \{x \oplus y\mathrm{I} \mid x, y \in \mathrm{u}\mathbb{R}\}. \tag{2.45}$$

Proposition 2.16. $\mathrm{u}\mathbb{C}$ is a commutative subdioid of $\mathrm{uMat}(2, \mathrm{u}\mathbb{R})$.

Proof. Obviously $\mathrm{u}\mathbb{C}$ includes $-\infty$ and 0. For any $a \oplus b\mathrm{I}, c \oplus d\mathrm{I} \in \mathrm{u}\mathbb{C}$, we have

$$\begin{aligned} (a \oplus b\mathrm{I}) \oplus (c \oplus d\mathrm{I}) &= (a \oplus c) \oplus (b \oplus d)\mathrm{I} \in \mathrm{u}\mathbb{C}, \\ (a \oplus b\mathrm{I}) \otimes (c \oplus d\mathrm{I}) &= (ac \ominus bd) \oplus (ad \oplus bc)\mathrm{I} \in \mathrm{u}\mathbb{C}. \end{aligned}$$

And $\mathrm{u}\mathbb{C}$ is commutative because I^0 and I^1 are commutative. ∎

When $z \in \mathrm{u}\mathbb{C}$ is expressed as $z = x + y\mathrm{I}$ where $x, y \in \mathrm{u}\mathbb{R}$, we write

$$\mathrm{uRe}\, z = x, \quad \mathrm{uIm} = y. \tag{2.46}$$

The whole set of signed elements of $\mathrm{u}\mathbb{C}$ is denoted by $\mathrm{u}\mathbb{C}^{\vee}$.

If $\det(x \oplus y\mathrm{I}) = x^2 \oplus y^2 \in \mathrm{u}\mathbb{R}^{\otimes}$, we have

$$(x \oplus y\mathrm{I})^{-1} = \frac{x \ominus y\mathrm{I}}{x^2 \oplus y^2}$$

and

$$(x \oplus y\mathrm{I})(x \oplus y\mathrm{I})^{-1} = 0 \oplus \frac{(xy)^{\bullet}}{x^2 \oplus y^2}\mathrm{I} \mathrel{\nabla} 0.$$

2.2 Ultradiscretization with the symmetrized max-plus algebra

2.2.1 Usual ultradiscretization

Let $f(s)$ be a real function. For large s, assume $f(s) > 0$ and define $\tilde{f}(s)$ by

$$f(s) = e^{\tilde{f}(s)s}. \tag{2.47}$$

We write

$$f(s) \xrightarrow{\text{ud}} F, \quad F = \lim_{s\to\infty} \tilde{f}(s) \tag{2.48}$$

if the limit exists. When

$$f(s) \xrightarrow{\text{ud}} F, \quad g(s) \xrightarrow{\text{ud}} G, \tag{2.49}$$

we have

$$f(s) + g(s) \xrightarrow{\text{ud}} \max(F, G), \quad f(s)g(s) \xrightarrow{\text{ud}} F + G. \tag{2.50}$$

This is the fundamental formula for ultradiscretization [18]. We want to drop the requirement for positivity in order to widen applicability of the procedure.

2.2.2 Ultradiscretization with negative numbers

Let $f(s)$ and $g(s)$ be real functions. We say $f(s)$ is asymptotically equivalent to $g(s)$ if there exists a real number s_0 such that $g(s) \neq 0$ for any $s > s_0$ and

$$\lim_{s\to\infty} \frac{f(s)}{g(s)} = 1. \tag{2.51}$$

We also say $f(s)$ is asymptotically equivalent to 0 if there exists a real number s_1 such that $f(s) = 0$ for any $s > s_1$. Asymptotic equivalence is an equivalence relation and denoted by $f(s) \sim g(s)$.

We are interested in asymptotic equivalence to exponential functions. If

$$f(s) \sim \mu_F e^{\tilde{F}s} \quad \left(\mu_F \in \mathbb{R}^\times, \tilde{F} \in \mathbb{R}\right),$$

we write

$$f(s) \xrightarrow{\text{ud}} F, \quad F = S(\mu_F) \otimes \tilde{F} \in \mathrm{u}\mathbb{R}^\vee \tag{2.52}$$

where

$$S(\mu) = \begin{cases} 0 & (\mu > 0), \\ \ominus 0 & (\mu < 0). \end{cases} \tag{2.53}$$

We regard $0 \sim \mu e^{(-\infty)s}$ for some $\mu \in \mathbb{R}^\times$ and $0 \xrightarrow{\text{ud}} -\infty$ as a convention.

It is very important here to notice that μ_F is *not* restricted to positive numbers, unlike the usual ultradiscretization procedure.

Theorem 2.17 (Ultradiscretization of addition). Let $f(s), g_1(s), \ldots, g_n(s)$ be real functions satisfying

$$f(s) = \sum_{k=1}^{n} g_k(s)$$

and

$$f(s) \xrightarrow{\text{ud}} F, \quad g_k(s) \xrightarrow{\text{ud}} G_k.$$

Then,

$$F \nabla \bigoplus_{k=1}^{n} G_k.$$

Proof. Assume $f(s) \sim \mu_F e^{\widetilde{F}s}$ and $g_k(s) \sim \mu_{G_k} e^{\widetilde{G}_k s}$. We can also assume

$$\widetilde{G}_1 \geqslant \widetilde{G}_2 \geqslant \cdots \geqslant \widetilde{G}_n$$

without loss of generality. The case $\widetilde{G}_1 = -\infty$ is trivial, so $\widetilde{G}_1 \neq -\infty$ below.

Let

$$\mu_G = \lim_{s \to \infty} e^{-\widetilde{G}_1 s} \sum_{k=1}^{n} g_k(s).$$

We have $\widetilde{G}_1 = \cdots = \widetilde{G}_j > \widetilde{G}_{j+1}$ for some j $(1 \leqslant j \leqslant n, G_{n+1} = -\infty)$ and thus

$$\mu_G = \sum_{k=1}^{j} \mu_{G_k}.$$

If all of $\mu_{G_1}, \ldots, \mu_{G_j}$ have the same sign, μ_G also has the same sign and

$$F = G_1 \Longrightarrow F \nabla \bigoplus_{k=1}^{n} G_k.$$

Otherwise, we have $G_l = \ominus G_m$ for some l, m $(1 \leqslant l < m \leqslant j)$ and

$$G_1^{\bullet} = \bigoplus_{k=1}^{j} G_k = \bigoplus_{k=1}^{n} G_k.$$

When $\mu_G \neq 0$,

$$\widetilde{F} = \widetilde{G}_1 \Longrightarrow F \nabla \bigoplus_{k=1}^{n} G_k.$$

When $\mu_G = 0$,

$$\widetilde{F} < \widetilde{G}_1 \Longrightarrow F \nabla \bigoplus_{k=1}^{n} G_k. \qquad \blacksquare$$

Remark. $f(s) \xrightarrow{\text{ud}} F$ and $g(s) \xrightarrow{\text{ud}} G$ do not imply $f(s) + g(s) \xrightarrow{\text{ud}} F \oplus G$ because $f(s) + g(s)$ might be no longer asymptotically equivalent to exponential functions. But if $f(s), g(s)$ can be expressed by power series in $\delta = \mu_D e^{\widetilde{D}s}$ where $\widetilde{D} < 0$, this is not a problem because $f(s) + g(s)$ can also be expressed by a power series in δ.

Theorem 2.18 (Ultradiscretization of multiplication). Let $f(s), g(s), h(s)$ be real functions satisfying

$$f(s) = g(s)h(s)$$

and

$$g(s) \xrightarrow{\text{ud}} G, \quad h(s) \xrightarrow{\text{ud}} H.$$

Then,

$$f(s) \xrightarrow{\text{ud}} F = G \otimes H.$$

Proof. Assume $g(s) \sim \mu_G e^{\widetilde{G}s}$ and $h(s) \sim \mu_H e^{\widetilde{H}s}$. Clearly we have

$$f(s) \sim \mu_G \mu_H e^{(\widetilde{G}+\widetilde{H})s}$$

and thus

$$f(s) \xrightarrow{\text{ud}} F = s(\mu_G \mu_H) \otimes \widetilde{G} \otimes \widetilde{H} = G \otimes H. \qquad \blacksquare$$

Corollary 2.19 (Ultradiscretization of polynomials). Let real functions $f(s), g_{kl}(s)$ satisfy

$$f(s) = \sum_{k=1}^{n} \prod_{l=1}^{m_k} g_{kl}(s)$$

and

$$f(s) \xrightarrow{\text{ud}} F, \quad g_{kl}(s) \xrightarrow{\text{ud}} G_{kl}.$$

Then,

$$F \nabla \bigoplus_{k=1}^{n} \bigotimes_{l=1}^{m_k} G_{kl}.$$

Proof. Let $g_k(s) = \prod_{l=1}^{m_k} g_{kl}(s)$. Clearly we have

$$g_k(s) \xrightarrow{\text{ud}} G_k = \bigotimes_{l=1}^{m_k} G_{kl}$$

and thus

$$F \nabla \bigoplus_{k=1}^{n} G_k. \qquad \blacksquare$$

2.2.3 Ultradiscretization of matrices

Consider a matrix-valued function $f(s) = (f_{ij}(s)) : \mathbb{R} \to \mathrm{Mat}(N, \mathbb{R})$. If

$$f_{ij}(s) \xrightarrow{\text{ud}} F_{ij},$$

we write

$$f(s) \xrightarrow{\text{ud}} F = (F_{ij}) \in \mathrm{uMat}(N, \mathrm{u}\mathbb{R}). \tag{2.54}$$

This is a componentwise property; there is no exponential functions of matrices.

Corollary 2.20. Let matrix-valued functions $f(s), g_{kl}(s)$ satisfy

$$f(s) = \sum_{k=1}^{n} \prod_{l=1}^{m_k} g_{kl}(s)$$

and

$$f(s) \xrightarrow{\text{ud}} F, \quad g_{kl}(s) \xrightarrow{\text{ud}} G_{kl}.$$

Then,

$$F \nabla \bigoplus_{k=1}^{n} \bigotimes_{l=1}^{m_k} G_{kl}. \tag{2.55}$$

Proof. Apply Corollary 2.19 componentwisely. ■

2.2.4 Ultradiscretization of complex numbers

Considering 2×2-matrix construction of complex numbers, we have

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{ud}} I = \begin{pmatrix} -\infty & \ominus 0 \\ 0 & -\infty \end{pmatrix}.$$

Let $f(s) = u(s) + v(s)i$ where $u(s), v(s)$ are real functions. If

$$u(s) \xrightarrow{\text{ud}} U, \quad v(s) \xrightarrow{\text{ud}} V,$$

we have of course

$$f(s) \xrightarrow{\text{ud}} F = U \oplus VI \in \mathrm{u}\mathbb{C}. \tag{2.56}$$

Corollary 2.21. Let complex-valued functions $f(s), g_{kl}(s)$ satisfy

$$f(s) = \sum_{k=1}^{n} \prod_{l=1}^{m_k} g_{kl}(s)$$

and

$$f(s) \xrightarrow{\mathrm{ud}} F, \quad g_{kl}(s) \xrightarrow{\mathrm{ud}} G_{kl}.$$

Then,

$$F \nabla \bigoplus_{k=1}^{n} \bigotimes_{l=1}^{m_k} G_{kl}.$$

Proof. This is immediate from Corollary 2.20. ∎

2.2.5 Comparison to ultradiscretization with parity variables

As stated in Chapter 1, several other attempts [15, 6, 19, 9] have been made to extend ultradiscretization so that it can be applied to subtraction. Ultradiscretization with parity variables [13], or p-ultradiscretization, is of special interest.

Let us illustrate p-ultradiscretization with an example from [13]. Consider the simple recurrence formula

$$x_{n+1} = a x_n. \tag{2.57}$$

We want to ultradiscretize this including the signs of a, x_n. First, we separate signs, or *parities*, and absolute values by

$$y_n = |x_n|, \quad \sigma_n = \frac{x_n}{y_n}, \quad b = |a|, \quad \rho = \frac{a}{b}. \tag{2.58}$$

Second, rewrite signs as

$$\sigma_n = h(\sigma_n) - h(-\sigma_n), \quad \sigma = h(\rho) - h(-\rho) \tag{2.59}$$

where h is a step function defined by

$$h(x) = \begin{cases} 1 & (x > 0), \\ 0 & (x < 0). \end{cases} \tag{2.60}$$

Third, rewrite (2.57) without subtraction:

$$\begin{aligned} (h(\sigma_{n+1}) - h(-\sigma_{n+1}))y_{n+1} &= (h(\rho) - h(-\rho))(h(\sigma_n) - h(-\sigma_n))by_n \\ \iff h(\sigma_{n+1})y_{n+1} &+ (h(\rho)h(-\sigma_n) + h(-\rho)h(\sigma_n))by_n \\ &= h(-\sigma_{n+1})y_{n+1} + (h(\rho)h(\sigma_n) + h(-\rho)h(-\sigma_n))by_n. \end{aligned} \tag{2.61}$$

Last, by setting $b = e^{Bs}, y_n = e^{Y_n s}$ and taking the limit $s \to \infty$, we obtain an implicit formula

$$\begin{aligned} &\max[H(\sigma_{n+1}) + Y_{n+1}, \max[H(\rho) + H(-\sigma_n), H(-\rho) + H(\sigma_n)] + B + Y_n] \\ &= \max[H(-\sigma_{n+1}) + Y_{n+1}, \max[H(\rho) + H(\sigma_n), H(-\rho) + H(-\sigma_n)] + B + Y_n] \end{aligned} \tag{2.62}$$

where H is of course defined by

$$H(x) = \begin{cases} 0 & (x > 0), \\ -\infty & (x < 0). \end{cases} \tag{2.63}$$

(2.62) is the p-ultradiscretization of (2.57). In this case, we can obtain the explicit formula

$$\sigma_{n+1} = \rho\sigma_n, \quad Y_{n+1} = B + Y_n \tag{2.64}$$

by enumerating the values of ρ, σ_n.

Clearly p-ultradiscretization and ultradiscretization with $u\mathbb{R}$ have the same philosophy: separate signs or coefficients from absolute values or exponential parts. However, ultradiscretization with $u\mathbb{R}$ is much simpler in the sense that it enables straightforward replacement of $+, -, \times, =$ to $\oplus, \ominus, \otimes, \nabla$, respectively. In the above example, if $a \xrightarrow{\text{ud}} A$ and $x_n \xrightarrow{\text{ud}} X_n$, simply we have

$$X_{n+1} = A \otimes X_n \tag{2.65}$$

by virtue of Theorem 2.18. Since we can write $X_n = (\operatorname{sgn} X_n) \otimes |X_n|_\oplus$, this equation has the same information of (2.64).

We also give an example for addition. Consider the recurrence formula

$$x_{n+1} = x_n + a_n. \tag{2.66}$$

Defining

$$y_n = |x_n|, \quad \sigma_n = \frac{x_n}{y_n}, \quad b_n = |a_n|, \quad \rho_n = \frac{a_n}{b_n} \tag{2.67}$$

and setting $b_n = e^{B_n s}, y_n = e^{Y_n s}$, we obtain the p-ultradiscretization

$$\begin{aligned} &\max[H(\sigma_{n+1}) + Y_{n+1}, H(-\sigma_n) + Y_n, H(-\rho_n) + B_n] \\ &\quad = \max[H(-\sigma_{n+1}) + Y_{n+1}, H(\sigma_n) + Y_n, H(\rho_n) + B_n]. \end{aligned} \tag{2.68}$$

By enumerating the values of σ_n, ρ_n, we have

$$\sigma_{n+1} = \begin{cases} \sigma_n & (\sigma_n = \rho_n \text{ or } Y_n > B_n), \\ \rho_n & (\sigma_n = -\rho_n \text{ and } Y_n < B_n), \\ \text{(indefinite)} & (\sigma_n = -\rho_n \text{ and } Y_n = B_n) \end{cases} \tag{2.69a}$$

and

$$Y_{n+1} = \begin{cases} \max[Y_n, B_n] & (\sigma_n = \rho_n \text{ or } Y_n \neq B_n), \\ \text{(less than or equal to } Y_n) & (\sigma_n = -\rho_n \text{ and } Y_n = B_n). \end{cases} \tag{2.69b}$$

In the case of ultradiscretization with u$\mathbb{R}$, simply we have

$$X_{n+1} \nabla X_n \oplus A_n \tag{2.70}$$

by virtue of Theorem 2.17, when $x_n \xrightarrow{\text{ud}} X_n$, $a_n \xrightarrow{\text{ud}} A_n$. Considering the law of $\oplus$, Proposition 2.4, and Proposition 2.9, (2.70) has the same information of (2.69a), (2.69b).

From the above examples, we guess that the two methods of ultradiscretization do not differ with regard to the information contained by resulting equations. But the simplicity of ultradiscretization with u$\mathbb{R}$ enables us to manipulate complicated expressions more easily. It is somewhat insightful that indefiniteness is unavoidable in both methods.

Chapter 3

Discrete and ultradiscrete sine-Gordon equations

In this chapter, we first review the discrete sine-Gordon equation [5, 2] and several results around it. Explicit calculation of the traveling-wave, kink-antikink, kink-kink, and breather solutions are perhaps presented for the first time. Then, we propose the ultradiscrete sine-Gordon equation. The solutions are obtained in two ways: by calculations completely inside u$\mathbb{R}$, and by ultradiscretization with u$\mathbb{R}$. The correspondence between the discrete and ultradiscrete systems is quite clear. Similarity of profiles of solutions is also visually confirmed by figures.

3.1 Discrete sine-Gordon equation

3.1.1 Discrete sine-Gordon equation

For any function $f = f(l, m)$ over $\mathbb{Z}^2$, define shift operations by

$$f_l = f_l(l, m) = f(l + 1, m), \quad f_m = f_m(l, m) = f(l, m + 1). \tag{3.1a}$$

Inverse operations are denoted by

$$f_{\bar{l}} = f(l - 1, m), \quad f_{\bar{m}} = f(l, m - 1). \tag{3.1b}$$

Let $\tau = \tau(l, m)$, $\sigma = \sigma(l, m)$ be functions $\mathbb{Z}^2 \to \mathbb{C}$. The discrete sine-Gordon equation (dsG) [5, 2] is given by

$$(1 - \delta)\tau\tau_{lm} = \tau_l\tau_m - \delta\sigma_l\sigma_m, \tag{3.2a}$$
$$(1 - \delta)\sigma\sigma_{lm} = \sigma_l\sigma_m - \delta\tau_l\tau_m, \tag{3.2b}$$

where $\delta \in \mathbb{C}^\times$ is a parameter with small absolute value. The vacuum solution

$$\tau = \sigma = 1 \tag{3.3}$$

is the simplest solution, other than the null solution $\tau = \sigma = 0$.

Calculating the cross product of the both sides of (3.2a), (3.2b), we have

$$\tau\tau_{lm}(\sigma_l\sigma_m - \delta\tau_l\tau_m) = \sigma\sigma_{lm}(\tau_l\tau_m - \delta\sigma_l\sigma_m)$$
$$\iff \frac{\tau_{lm}\sigma_m}{\sigma_{lm}\tau_m} - \frac{\tau_l\sigma}{\sigma_l\tau} + \delta\left(\frac{\sigma_m\sigma}{\tau_m\tau} - \frac{\tau_{lm}\tau_l}{\sigma_{lm}\sigma_l}\right) = 0$$

and thus

$$\frac{w_{lm}}{w_m} - \frac{w_l}{w} + \delta\left(\frac{1}{w_m w} - w_{lm}w_l\right) = 0, \tag{3.4}$$

where w is defined by

$$w = \frac{\tau}{\sigma}. \tag{3.5}$$

If we introduce u defined by

$$u = \frac{2}{i}\log w, \tag{3.6}$$

we have

$$e^{i(u_{lm}-u_m)/2} - e^{i(u_l-u)/2} + \delta\left(e^{i(-u_m-u)/2} - e^{i(u_{lm}+u_l)/2}\right) = 0$$
$$\iff \sin\left(\frac{u_{lm} - u_l - u_m + u}{4}\right) = \delta\sin\left(\frac{u_{lm} + u_l + u_m + u}{4}\right). \tag{3.7}$$

Each of (3.4) and (3.7) is also called the discrete sine-Gordon equation.

Assume u is also a function $u(x, y)$ of continuum variables $x, y \in \mathbb{R}$ and has an expansion

$$u(x+r, y+s) = u + (ru_x + su_y) + \frac{1}{2}\left(r^2u_{xx} + 2rsu_{xy} + s^2u_{yy}\right) + \cdots \tag{3.8}$$

where $u_x = \partial u/\partial x$, etc. Connect l, m to x, y via the Miwa transformation

$$u(x, y; l, m) = u(x + la, y + mb) \tag{3.9}$$

where $a, b \in \mathbb{R}^\times$ are parameters. Then we have

$$u_{lm} - u_l - u_m + u = abu_{xy} + (\text{higher-order terms of } a, b).$$

Setting $\delta = ab$ and taking the limit $a, b \to 0$ successively, we obtain the (continuous) sine-Gordon equation

$$u_{xy} = 4\sin u. \tag{3.10}$$

3.1.2 1-soliton and 2-soliton solutions

As a 1-soliton solution, assume

$$\tau = 1 + f_j, \quad \sigma = 1 - f_j, \quad f_j = c_j p_j^l q_j^m \tag{3.11}$$

where $c_j, p_j, q_j \in \mathbb{C}^\times$ are constants. By substitution, we find the dispersion relation

$$(1-\delta)(1+p_j q_j) = (1+\delta)(p_j + q_j) \iff q_j = \frac{(1+\delta)p_j - (1-\delta)}{(1-\delta)p_j - (1+\delta)} \tag{3.12}$$

is a necessary and sufficient condition for (3.11) to become a solution [7]. As a 2-soliton solution, assume

$$\tau = 1 + f_1 + f_2 + \alpha f_1 f_2, \quad \sigma = 1 - f_1 - f_2 + \alpha f_1 f_2 \tag{3.13}$$

where $\alpha \in \mathbb{C}^\times$ is a constant. This time, the pair of the dispersion relation (3.12) and the relation

$$\alpha = -\frac{p_1 - p_2}{1 - p_1 p_2} \frac{q_1 - q_2}{1 - q_1 q_2} = \left(\frac{p_1 - p_2}{1 - p_1 p_2} \right)^2 \tag{3.14}$$

is a necessary and sufficient condition [7].

Figure 3.1 shows the 1-soliton solution $(j = 1)$ with

$$\delta = 0.04, \quad c_1 = -1, \quad p_1 = 2,$$

and the 2-soliton solution with

$$\delta = 0.04, \quad c_1 = c_2 = -2.125, \quad p_1 = q_2 = 2,$$

in the light-cone coordinates

$$(n, t) = \left(\frac{l + m}{2}, \frac{l - m}{2} \right) \iff (l, m) = (n + t, n - t). \tag{3.15}$$

3.1.3 Traveling-wave solution

Replacing c_j by ic_j in the 1-soliton solution, we obtain

$$w = \frac{1 + ic_j (p_j q_j)^n \left(p_j q_j^{-1}\right)^t}{1 - ic_j (p_j q_j)^n \left(p_j q_j^{-1}\right)^t}, \quad u = 4 \arctan \left(c_j (p_j q_j)^n \left(p_j q_j^{-1}\right)^t \right) \tag{3.16}$$

in the light-cone coordinates. This solution corresponds to a so-called traveling-wave solution for the (continuous) sine-Gordon equation (if we restrict $\delta, c, p \in \mathbb{R}^\times$). Figure 3.2 shows the solution with

$$\delta = 0.04, \quad c_1 = 1, \quad p_1 = 2.$$

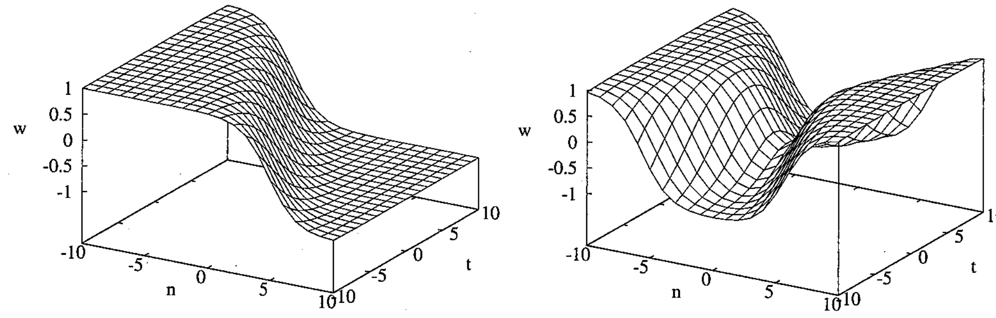

Figure 3.1: 1-soliton solution (left) and 2-soliton solution (right) for dsG.

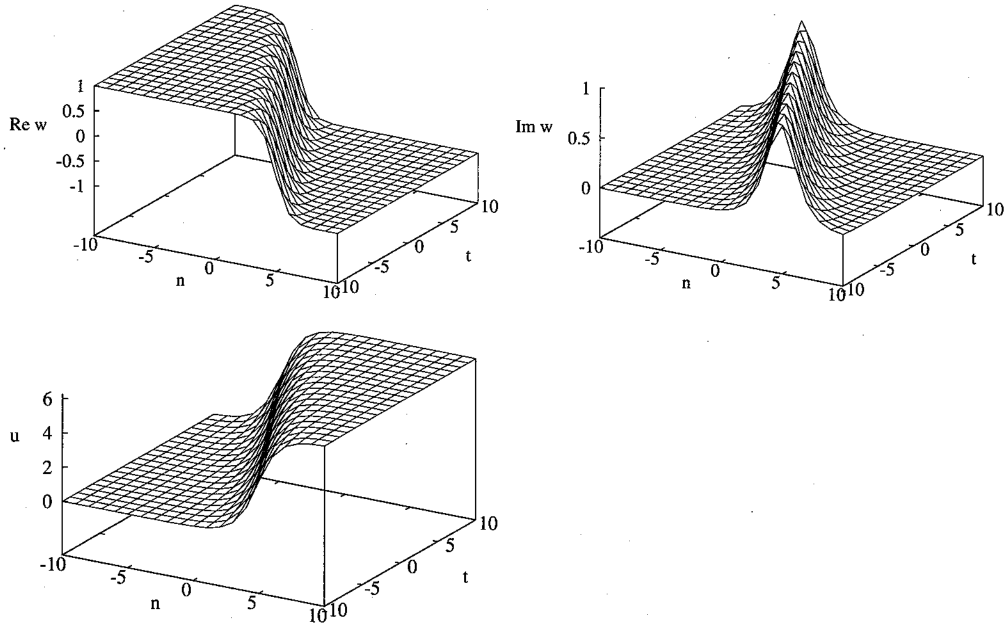

Figure 3.2: Traveling-wave solution for dsG.

3.1.4 Kink-antikink and kink-kink solutions

Set

$$p_1 = q_2 \tag{3.17}$$

in the 2-soliton solution. Then

$$p_1 = \frac{(1+\delta)p_2 - (1-\delta)}{(1-\delta)p_2 - (1+\delta)} \Longleftrightarrow p_2 = \frac{(1+\delta)p_1 - (1-\delta)}{(1-\delta)p_1 - (1+\delta)} \tag{3.18}$$

and thus

$$p_2 = q_1. \tag{3.19}$$

Rewriting in the light-cone coordinates, we have

$$\begin{aligned}
\tau &= 1 + c_1 p_1^{n+t} p_2^{n-t} + c_2 p_2^{n+t} p_1^{n-t} + \alpha c_1 c_2 (p_1 p_2)^{2n} \\
&= (p_1 p_2)^n \left(\alpha c_1 c_2 (p_1 p_2)^n + (p_1 p_2)^{-n} + c_1 \left(p_1 p_2^{-1}\right)^t + c_2 \left(p_1 p_2^{-1}\right)^{-t}\right), \\
\sigma &= (p_1 p_2)^n \left(\alpha c_1 c_2 (p_1 p_2)^n + (p_1 p_2)^{-n} - c_1 \left(p_1 p_2^{-1}\right)^t - c_2 \left(p_1 p_2^{-1}\right)^{-t}\right).
\end{aligned}$$

We set

$$\beta = \pm \frac{p_1 - p_2}{1 - p_1 p_2}, \quad c_1 = -c_2 = i\beta^{-1} \tag{3.20}$$

and define

$$\mathrm{ch}(p,l) = \frac{p^l + p^{-l}}{2}, \quad \mathrm{sh}(p,l) = \frac{p^l - p^{-l}}{2}. \tag{3.21}$$

Then

$$w = \frac{\beta\, \mathrm{ch}(p_1 p_2, n) + i\, \mathrm{sh}\left(p_1 p_2^{-1}, t\right)}{\beta\, \mathrm{ch}(p_1 p_2, n) - i\, \mathrm{sh}\left(p_1 p_2^{-1}, t\right)}, \quad u = 4 \arctan\left(\frac{\mathrm{sh}\left(p_1 p_2^{-1}, t\right)}{\beta\, \mathrm{ch}(p_1 p_2, n)}\right). \tag{3.22}$$

This corresponds to a symmetric kink-antikink solution. Figure 3.3 shows the solution with

$$\delta = 0.04, \quad c_1 = -c_2 = 2.125i, \quad p_1 = 2.$$

Similarly, setting

$$p_1 q_2 = 1 \tag{3.23}$$

gives

$$p_1^{-1} = \frac{(1+\delta)p_2 - (1-\delta)}{(1-\delta)p_2 - (1+\delta)} \Longleftrightarrow p_2^{-1} = \frac{(1+\delta)p_1 - (1-\delta)}{(1-\delta)p_1 - (1+\delta)}$$

and thus

$$p_2 q_1 = 1. \tag{3.24}$$

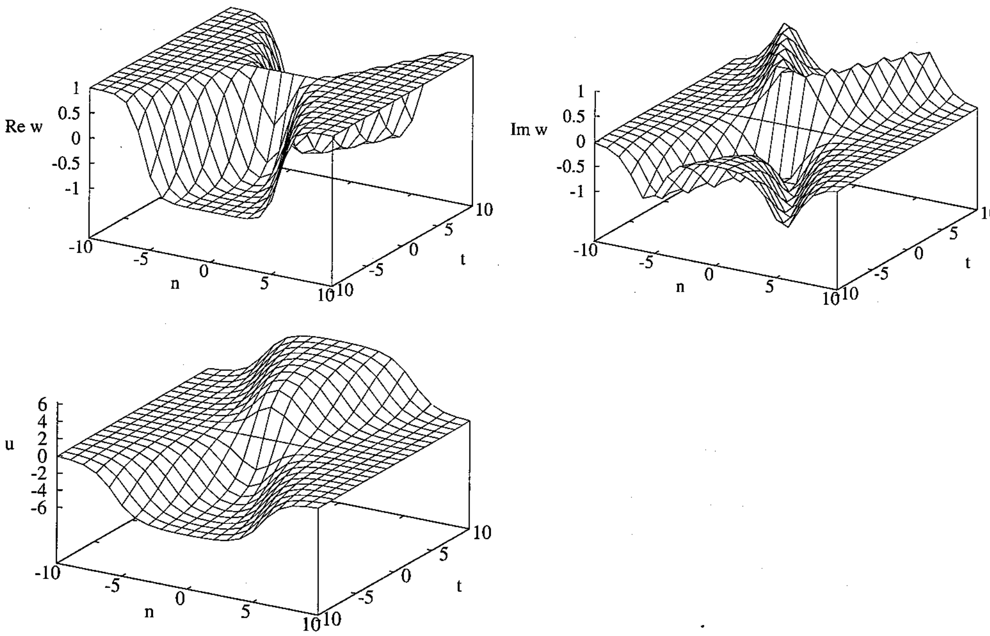

Figure 3.3: Symmetric kink-antikink solution for dsG.

We have

$$\tau = \beta^{-1}(p_1p_2)^t\left(\beta(p_1p_2)^t + \beta(p_1p_2)^{-t} + i\left(p_1p_2^{-1}\right)^n - i\left(p_1p_2^{-1}\right)^{-n}\right),$$
$$\sigma = \beta^{-1}(p_1p_2)^t\left(\beta(p_1p_2)^t + \beta(p_1p_2)^{-t} - i^{-1}\left(p_1p_2^{-1}\right)^n + i\left(p_1p_2^{-1}\right)^{-n}\right)$$

for the same β, c_1, c_2 defined above and

$$w = \frac{\beta\,\mathrm{ch}(p_1p_2,t) + i\,\mathrm{sh}\left(p_1p_2^{-1},n\right)}{\beta\,\mathrm{ch}(p_1p_2,t) - i\,\mathrm{sh}\left(p_1p_2^{-1},n\right)}, \quad u = 4\arctan\left(\frac{\mathrm{sh}\left(p_1p_2^{-1},n\right)}{\beta\,\mathrm{ch}(p_1p_2,t)}\right). \tag{3.25}$$

This corresponds to a symmetric kink-kink solution. Figure 3.4 shows the solution with

$$\delta = 0.04, \quad c_1 = -c_2 = -0.470588i, \quad p_1 = 2.$$

3.1.5 Breather solution

Consider the kink-antikink solution where p_1, p_2 are complex numbers satisfying

$$p_1p_2 \in \mathbb{R}_{>0}, \quad \left|p_1p_2^{-1}\right| = 1. \tag{3.26}$$

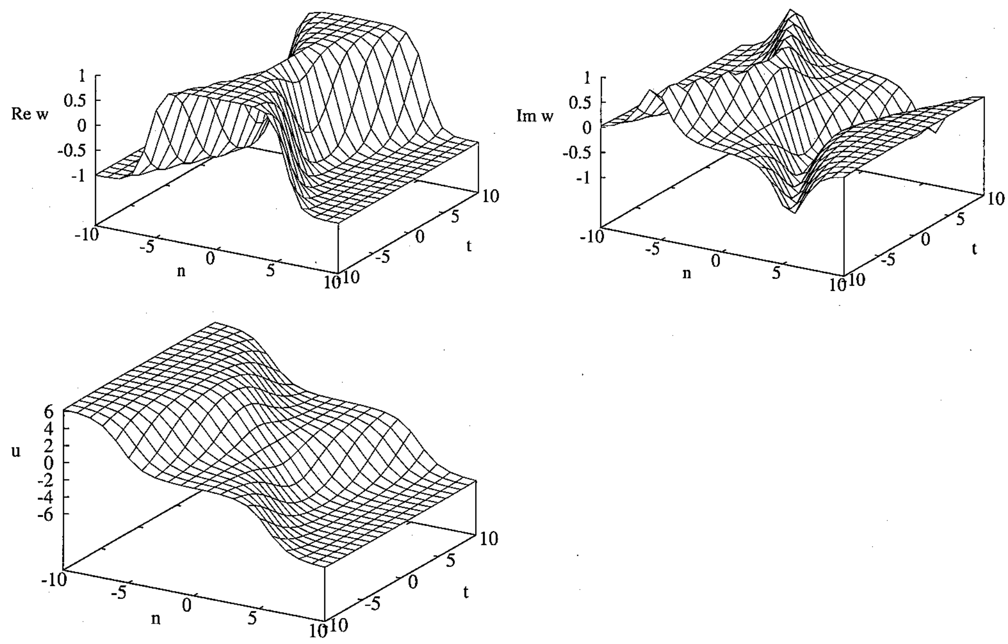

Figure 3.4: Symmetric kink-kink solution for dsG.

Such p_1, p_2 are complex conjugates of each other. If we write

$$p_1 = g + hi, \quad p_2 = g - hi$$

and substitute these into (3.18), we find g, h must satisfy

$$\begin{aligned} &(1-\delta)\left(1+g^2+h^2\right) = 2(1+\delta)g \\ \Longleftrightarrow\ &(1-\delta)g^2 - 2(1+\delta)g + (1-\delta)\left(1+h^2\right) = 0. \end{aligned}$$

As a quadratic equation of g, the condition for the existence of real roots is give by

$$(1+\delta)^2 - (1-\delta)^2\left(1+h^2\right) \geqslant 0 \Longleftrightarrow h^2 \leqslant \left(\frac{1+\delta}{1-\delta}\right)^2 - 1. \tag{3.27}$$

Such a real number h does exist if $\delta \geqslant 0$, and g is given by

$$g = \frac{1+\delta}{1-\delta} \pm \sqrt{\left(\frac{1+\delta}{1-\delta}\right)^2 - (1+h^2)}. \tag{3.28}$$

Rewriting $p_1 = re^{i\theta}, p_2 = re^{-i\theta}$, we have

$$\beta = i\gamma, \quad \gamma = \pm\frac{2r\sin\theta}{1-r^2}, \quad \mathrm{sh}\left(p_1 p_2^{-1}, t\right) = i\sin 2t\theta$$

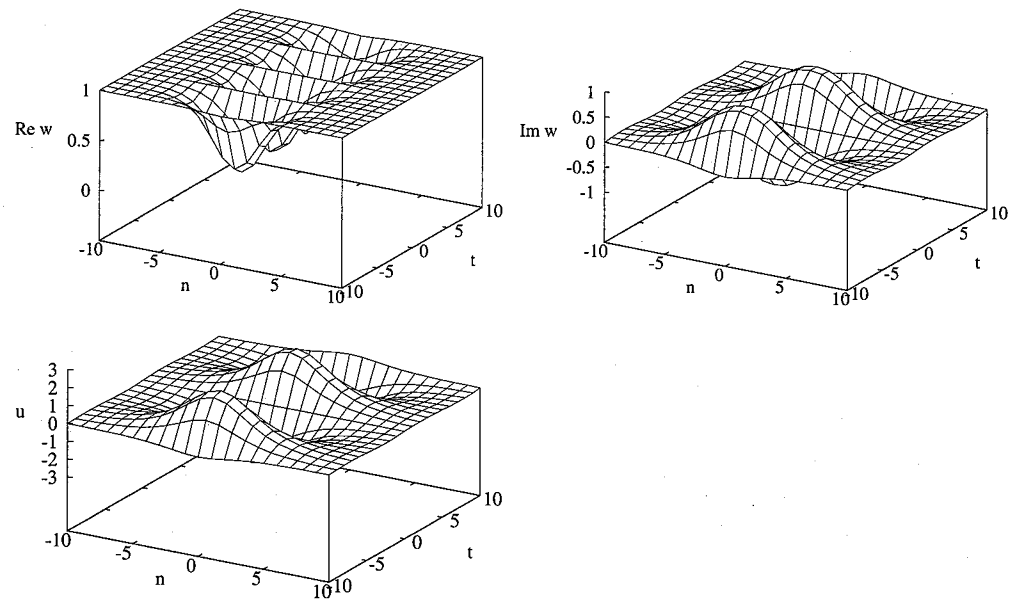

Figure 3.5: Standing breather solution for dsG.

and thus

$$w = \frac{\gamma\,\mathrm{ch}(r^2, n) + i \sin 2t\theta}{\gamma\,\mathrm{ch}(r^2, n) - i \sin 2t\theta}, \quad u = 4 \arctan\left(\frac{\sin 2t\theta}{\gamma\,\mathrm{ch}(r^2, n)}\right). \tag{3.29}$$

This corresponds to a standing breather solution. Figure 3.5 shows the solution with

$$\delta = 0.04, \quad c_1 = -c_2 = 0.75, \quad p_1 = 0.75 + 0.25i.$$

3.2 Ultradiscrete sine-Gordon equation

3.2.1 Ultradiscrete sine-Gordon equation

We perform ultradiscretization of dsG through the parametrization

$$\delta = \mu_D e^{\widetilde{D}s}, \quad \widetilde{D} < 0. \tag{3.30}$$

This can be regarded as an other aspect of continuum limit since $\delta \to 0$ as $s \to \infty$. Assuming $\delta \xrightarrow{\text{ud}} D, \tau \xrightarrow{\text{ud}} T, \sigma \xrightarrow{\text{ud}} S$, we obtain

$$TT_{lm} \oslash T_l T_m \ominus DS_l S_m, \tag{3.31a}$$

$$SS_{lm} \oslash S_l S_m \ominus DT_l T_m. \tag{3.31b}$$

We call the pair (3.31a), (3.31b) the ultradiscrete sine-Gordon equation (udsG). The vacuum solution

$$\mathsf{T} = \mathsf{S} = 0 \tag{3.32}$$

is the simplest solution, other than the null solution $\mathsf{T} = \mathsf{S} = -\infty$. We can also ultradiscretize (3.4) to obtain

$$W_{lm}W_m^{-1} \ominus W_l W^{-1} \oplus \mathsf{D}\left(W_m^{-1}W^{-1} \ominus W_{lm}W_l\right) \nabla -\infty \tag{3.33}$$

where

$$w \xrightarrow{\text{ud}} W \nabla \mathsf{T}\mathsf{S}^{-1}. \tag{3.34}$$

We also call (3.33) the ultradiscrete sine-Gordon equation. Ultradiscretization of (3.7) is unclear.

3.2.2 Deterministic time evolution and class of solutions

It seems sensible to restrict ourselves to the class of signed solutions, that is,

$$\mathsf{T}, \mathsf{S}, W \in \mathrm{u}\mathbb{C}^{\vee} \text{ for any } (l, m) \in \mathbb{Z}^2 \tag{3.35}$$

since it permits basic properties like weak substitution. The null and vacuum solutions are signed solutions.

The problem is that udsG no longer admits time evolution, at least deterministic one, in general, since balance relation is not equality. For example, if we have

$$f(t+1) \nabla \left(\text{expression including } f(t)\right) = 3^{\bullet},$$

we cannot determine $f(t+1)$ from $f(t)$, since this relation is satisfied whenever $|f(t+1)|_{\oplus} \leqslant 3$. Strictly speaking, udsG is not an *equation.*

But in some cases, it actually becomes an equation, or furthermore, a deterministically evolutionary form. Multiplying T^{-1} to (3.31a) and S^{-1} to (3.31b), we have

$$\begin{aligned}
\mathsf{T}^{-1}\mathsf{T}\mathsf{T}_{lm} &\nabla \mathsf{T}^{-1}\left(\mathsf{T}_l\mathsf{T}_m \ominus \mathsf{D}\mathsf{S}_l\mathsf{S}_m\right), \\
\mathsf{S}^{-1}\mathsf{S}\mathsf{S}_{lm} &\nabla \mathsf{S}^{-1}\left(\mathsf{S}_l\mathsf{S}_m \ominus \mathsf{D}\mathsf{T}_l\mathsf{T}_m\right).
\end{aligned}$$

If $\mathsf{T}^{-1}\mathsf{T} = \mathsf{S}^{-1}\mathsf{S} = 0$ and the right hand sides are signed, we obtain

$$\mathsf{T}_{lm} = \mathsf{T}^{-1}\left(\mathsf{T}_l\mathsf{T}_m \ominus \mathsf{D}\mathsf{S}_l\mathsf{S}_m\right), \tag{3.36a}$$

$$\mathsf{S}_{lm} = \mathsf{S}^{-1}\left(\mathsf{S}_l\mathsf{S}_m \ominus \mathsf{D}\mathsf{T}_l\mathsf{T}_m\right) \tag{3.36b}$$

by reduction of balances (Proposition 2.9). We call (3.36a), (3.36b) the deterministically evolutionary form of udsG. If we replace D by $\ominus$D and restrict ranges of

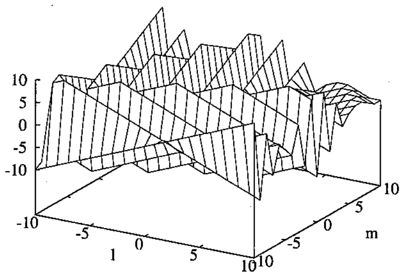

Figure 3.6: Positive ($\mathbb{R}$) evolution of udsG.

D, T, S to $\mathbb{R}$ for example, the assumptions are satisfied, and we obtain the completely ordinary-looking ultradiscrete equation:

$$T_{lm} = \max(T_l + T_m, D + S_l + S_m) - T, \tag{3.37a}$$

$$S_{lm} = \max(S_l + S_m, D + T_l + T_m) - S. \tag{3.37b}$$

Deterministic time evolution is also possible in other settings, which are presented in the following sections.

It might be natural to think we should consider (3.36a), (3.36b), or even (3.37a), (3.37b) only. However, it seems that the former cannot capture the traveling-wave, kink-antikink, and kink-kink solutions. And the latter does not even seem to contain soliton solutions. Therefore, we consider (3.31a), (3.31b) primarily. Perhaps, some indeterminacy is unavoidable (see Section 2.2).

For those who are interested, we show one example of positive ($\mathbb{R}$) time evolution by (3.37a), (3.37b) in Figure 3.6. Initial values are set as

$$T(l, -10) = l, \quad S(-10, m) = m, \quad -10 \leqslant l, m \leqslant 10$$

with $D = -1$. It exhibits alternating (in the sense of $\pm$) nature.

3.2.3 1-soliton solution

Consider a signed solution T, S satisfying

$$T \nabla 0 \oplus F_j, \quad S \nabla 0 \ominus F_j, \quad F_j = C_j P_j^l Q_j^m \tag{3.38}$$

where $C_j \in u\mathbb{C}^{\vee}$ and $P_j, Q_j \in u\mathbb{R}^{\otimes}$. Weakly substituting these into (3.31a), (3.31b), we have

$$0 \oplus P_j Q_j F_j^2 \oplus (0 \oplus P_j Q_j) F_j \nabla 0 \oplus P_j Q_j F_j^2 \oplus (P_j \oplus Q_j) F_j, \tag{3.39a}$$

$$0 \oplus P_j Q_j F_j^2 \ominus (0 \oplus P_j Q_j) F_j \nabla 0 \oplus P_j Q_j F_j^2 \ominus (P_j \oplus Q_j) F_j \tag{3.39b}$$

where $0 \oplus D = 0 \ominus D = 0$ is used. The dispersion relation

$$0 \oplus P_j Q_j \nabla P_j \oplus Q_j \tag{3.40}$$

is a sufficient condition for (3.39a), (3.39b) to hold, since we can construct them by adding and multiplying same numbers to the both sides of (3.40). Rewriting (3.40), we have

$$(P_j \ominus 0) Q_j \nabla (P_j \ominus 0)$$

and thus $P_j = 0$ or $Q_j = 0$. Obviously, (3.38) and (3.40) can be obtained by ultradiscretizing (3.11) and (3.12), respectively, through

$$c_j = \mu_{C_j} e^{\widetilde{C}_j s} \xrightarrow{\text{ud}} C_j, \quad p_j = \mu_{P_j} e^{\widetilde{P}_j s} \xrightarrow{\text{ud}} P_j, \quad q_j \xrightarrow{\text{ud}} Q_j \tag{3.41a}$$

or

$$c_j = \mu_{C_j} e^{\widetilde{C}_j s} \xrightarrow{\text{ud}} C_j, \quad p_j \xrightarrow{\text{ud}} P_j, \quad q_j = \mu_{Q_j} e^{\widetilde{Q}_j s} \xrightarrow{\text{ud}} Q_j. \tag{3.41b}$$

The solution is, however, not completely determined yet, because balance relation is not equality as stated before. So we try to utilize reduction of balances. If $C_j \in \mathrm{u}\mathbb{Z}$ is an odd number and $P_j, Q_j \in \mathrm{u}\mathbb{Z}$ are even numbers, then F_j is always odd and $0 \oplus F_j, 0 \ominus F_j$ can never be balanced since 0 is even. By reduction of balances, we obtain

$$T = 0 \oplus F_j, \quad S = 0 \ominus F_j, \tag{3.42}$$

and W is also immediately determined since S^{-1} is signed. This solution admits deterministic time evolution since

$$|T_l T_m|_\oplus > |D S_l S_m|_\oplus, \quad |S_l S_m|_\oplus > |D T_l T_m|_\oplus$$

and thus

$$T^{-1}(T_l T_m \ominus D S_l S_m) = T^{-1} T_l T_m \in \mathrm{u}\mathbb{R}^\otimes,$$
$$S^{-1}(S_l S_m \ominus D T_l T_m) = S^{-1} S_l S_m \in \mathrm{u}\mathbb{R}^\otimes.$$

Figure 3.7 shows the solution $(j = 1)$ with

$$D = -1, \quad C_1 = \ominus 1, \quad P_1 = 2$$

in the light-cone coordinates (3.15). It is somehow difficult to depict ultradiscrete numbers in figures; here signs and absolute values are displayed separately, and signs are mapped from $\ominus 0, 0^\bullet, \oplus 0$ to $-1, 0, 1$, respectively (balanced elements do not appear in the figure, though). Observe that the form of the 1-soliton solution w for dsG is preserved in the signs. Absolute values are always 0, corresponding to the fact that w asymptotically behaves as ± 1.

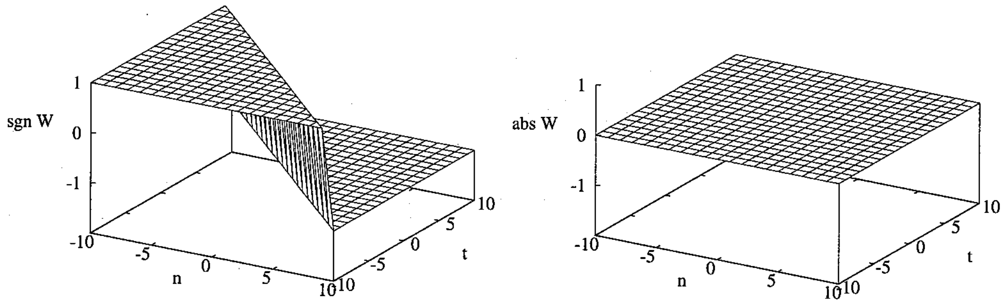

Figure 3.7: Signs (left) and absolute values (right) of 1-soliton solution for udsG.

3.2.4 Traveling-wave solution

If we replace C_j by $C_j I$ and redefine $F_j = C_j P_j^l Q_j^m$ ($C_j \in u\mathbb{R}^{\otimes}$), we obtain

$$T = 0 \oplus F_j I, \quad S = 0 \ominus F_j I \tag{3.43}$$

and

$$W \nabla \frac{(0 \ominus F_j^2) \oplus F_j I}{0 \oplus F_j^2}. \tag{3.44}$$

We choose odd C_j and even P_j, Q_j so that $0 \ominus F_j^2$ is always signed and reduction of balances can be applied. This solution no longer admits deterministic time evolution, but is apparently ultradiscretization of the traveling-wave solution (3.16) for dsG. Figure 3.8 shows the solution ($j = 1$) with

$$D = -1, \quad C_1 = 1, \quad P_1 = 2.$$

The uRe and uIm parts are displayed separately. The profile of the traveling-wave solution for dsG is preserved well.

3.2.5 2-soliton solution

Assume

$$T \nabla 0 \oplus F_1 \oplus F_2 \oplus AF_1F_2, \quad S \nabla 0 \ominus F_1 \ominus F_2 \oplus AF_1F_2 \tag{3.45}$$

where $A \in u\mathbb{R}^{\otimes}$. We also assume

$$P_1 \neq P_2, \quad Q_1 \neq Q_2.$$

By substitution, we find the pair of the dispersion relation(3.40) and the relation

$$A(0 \ominus P_1P_2)(0 \ominus Q_1Q_2) \oplus (P_1 \ominus P_2)(Q_1 \ominus Q_2) \nabla -\infty. \tag{3.46}$$

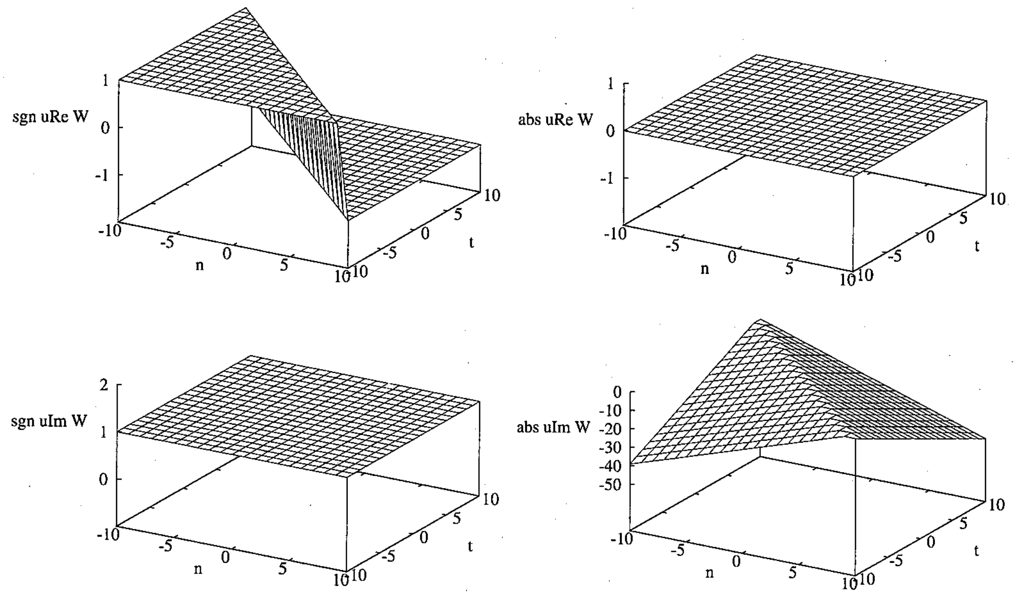

Figure 3.8: Traveling-wave solution for udsG.

is a sufficient condition for (3.45) to become a solution. Obviously, (3.46) is ultradiscretization of (3.14). When $P_1 = P_2 = 0$ or $Q_1 = Q_2 = 0$, any A satisfies (3.46). When $P_1 = Q_2 = 0$, we have

$$A(0 \ominus P_2)(0 \ominus Q_1) \oplus (0 \ominus P_2)(Q_1 \ominus 0) \mathrel{\nabla} -\infty \Longrightarrow A = 0. \tag{3.47}$$

The case $P_2 = Q_1 = 0$ is similar.

We can choose $A, C_j, P_j, Q_j \in \mathrm{u}\mathbb{Z}$ such that $0 \oplus AF_1F_2$ is always positive, even and $F_1 \oplus F_2$ is negative, odd. Then the solution is determined as

$$T = 0 \oplus F_1 \oplus F_2 \oplus AF_1F_2, \quad S = 0 \ominus F_1 \ominus F_2 \oplus AF_1F_2. \tag{3.48}$$

This admits deterministic time evolution, of course. Figure 3.9 shows the solution with

$$D = -1, \quad C_1 = C_2 = \ominus 1, \quad P_1 = Q_2 = 4.$$

3.2.6 Kink-antikink and kink-kink solutions

If we replace C_1 by $C_1 I$, C_2 by $\ominus C_2 I$, and redefine $F_j = C_j P_j^l Q_j^m$ ($C_j \in \mathrm{u}\mathbb{R}^{\otimes}$) in the 2-soliton solution, we obtain

$$T \mathrel{\nabla} 0 \oplus AF_1F_2 \oplus (F_1 \ominus F_2)\,I, \quad S \mathrel{\nabla} 0 \oplus AF_1F_2 \ominus (F_1 \ominus F_2)\,I \tag{3.49}$$

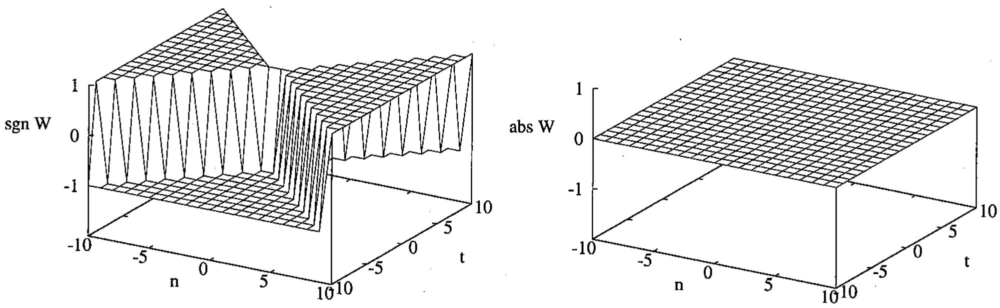

Figure 3.9: 2-soliton solution for udsG.

and

$$W \nabla \frac{\left((0 \oplus AF_1F_2)^2 \ominus (F_1 \ominus F_2)^2\right) \oplus (0 \oplus AF_1F_2)(F_1 \ominus F_2)I}{(0 \oplus AF_1F_2)^2 \oplus (F_1 \ominus F_2)^2}. \tag{3.50}$$

We choose $C_j, P_j, Q_j \in u\mathbb{Z}$ such that

$$|F_1|_{\oplus} \equiv 1, \quad |F_2|_{\oplus} \equiv 3 \pmod 4 \tag{3.51a}$$

or

$$|F_1|_{\oplus} \equiv 3, \quad |F_2|_{\oplus} \equiv 1 \pmod 4). \tag{3.51b}$$

Then $F_1 \ominus F_2$ and $(0 \oplus AF_1F_2)^2 \ominus (F_1 \ominus F_2)^2$ are always signed and balance relations become equalities.

If we set

$$P_1 = Q_2, \quad P_2 = Q_1, \tag{3.52}$$

we have the kink-antikink solution. Similarly, setting

$$P_1 = Q_2^{-1}, \quad P_2 = Q_1^{-1} \tag{3.53}$$

gives the kink-kink solution. These solutions does not admit deterministic time evolution, but are ultradiscretization of (3.22), (3.25) except that they are not symmetric since C_1, C_2 can take different values. Ultradiscretization of the breather solution is unclear, however.

Figure 3.10 shows the kink-antikink solution with

$$D = -1, \quad C_1 = 1, \quad C_2 = -1, \quad P_1 = Q_2 = 4,$$

and Figure 3.11 shows the kink-kink solution with

$$D = -1, \quad C_1 = \ominus 1, \quad C_2 = \ominus(-1), \quad P_1 = Q_2^{-1} = 4.$$

Observe that in the uIm part two waves approach to each other for negative t, collide around $t = 0$, and move away from each other for positive t. In the kink-antikink solution, the two waves have the same sign and *bump up* by collision. In the kink-kink solution, the two have opposite signs and *reflect* by collision.

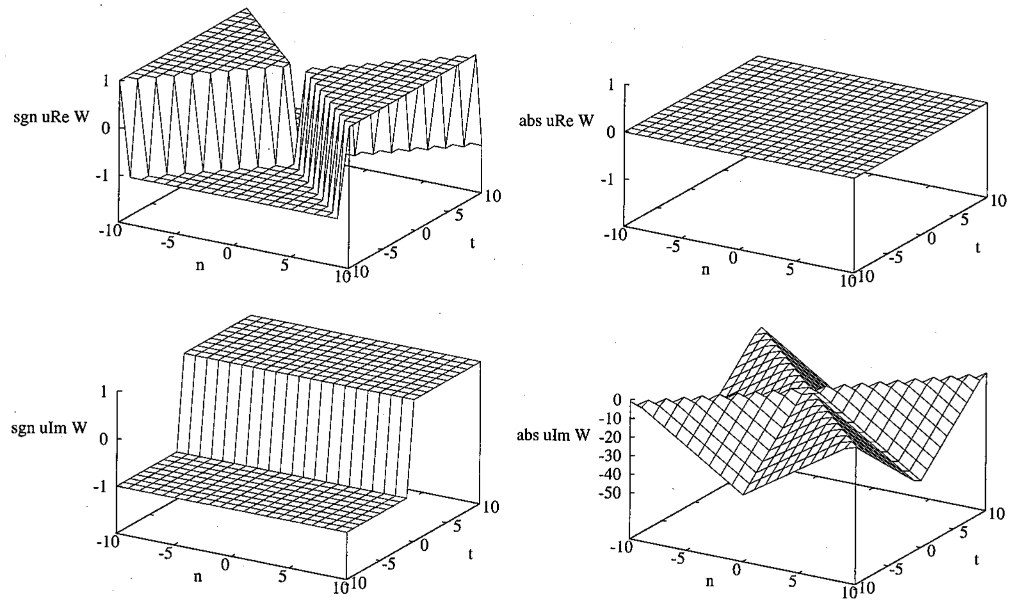

Figure 3.10: Kink-antikink solution for udsG.

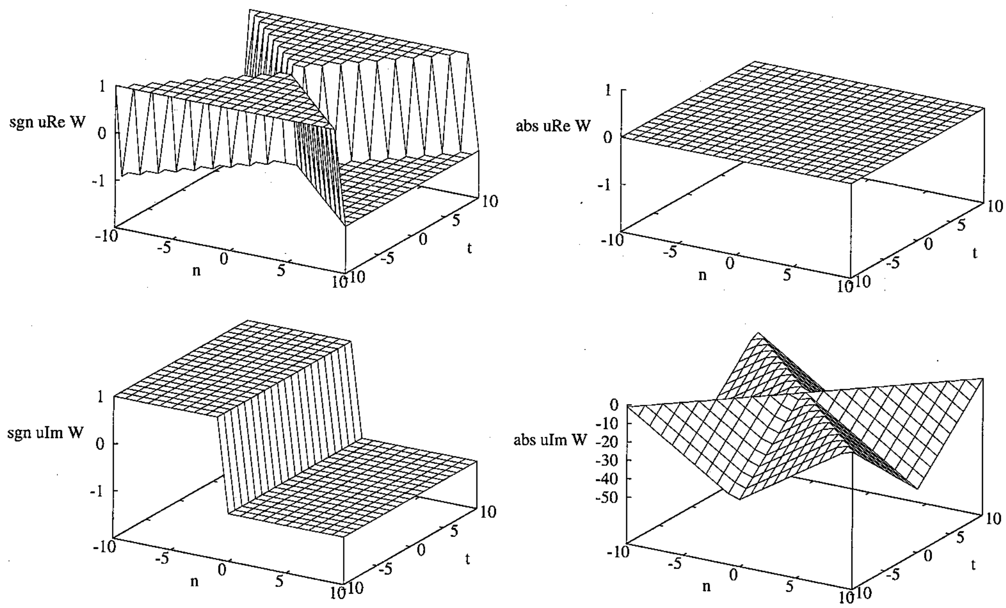

Figure 3.11: Kink-kink solution for udsG.

Chapter 4

Noncommutative discrete and ultradiscrete sine-Gordon equations

In this chapter, we propose the noncommutative discrete sine-Gordon equation as a compatibility condition of a certain linear system. This equation reduces to the commutative version once the underlying algebra turns out to be commutative and one simple reduction condition is applied. Reduction from the noncommutative discrete KP equation [14, 10] also gives the equation, and continuum limit of the equation gives the noncommutative (continuous) sine-Gordon equation, which is already known in a different context [12]. We define the Darboux transformation, which constructs new solutions from old ones, and obtain Casoratian-type solutions by repeating it. Explicitly setting the starting solutions for repetition, we derive so-called multisoliton solutions.

Along the construction of Casoratian-type solutions, quasideterminants [4] are used, which is a noncommutative extension of determinants. The theory needs some space for explanation, but it is not essential to the main story. Therefore, we only briefly explain the definition and some properties of them in Appendix A. For detail, see [4].

We finally propose the noncommutative ultradiscrete sine-Gordon equation. Noncommutative ultradiscrete setting is probably one of the hardest environments for integrable systems to exist, but we manage to obtain 1-soliton and 2-soliton solutions by ultradiscretization with $\mathrm{u}\mathbb{R}$.

Notations are slightly changed in this chapter because of the complexity of expressions we are going to manipulate. Shifts are always indicated after a comma like $f_{,l}$. This is to distinguish indices and shifts. In addition, shift operators T_l, T_m are also used:

$$T_l f = f_{,l} = f(l+1, m), \quad T_m f = f_{,m} = f(l, m+1). \tag{4.1}$$

Do not confuse these with the ultradiscretized τ function of the previous chapter; in the noncommutative setting, τ functions do not seem to exist. We also use superscripts for elements of matrices. For example,

$$w = (w^{\iota\kappa}) = \begin{pmatrix} w^{11} & \cdots & w^{1N} \\ \vdots & \ddots & \vdots \\ w^{N1} & \cdots & w^{NN} \end{pmatrix}.$$

4.1 Noncommutative discrete sine-Gordon equation

4.1.1 Linear system

Let $w = w(l, m), v = v(l, m)$ be functions $\mathbb{Z}^2 \to \mathrm{Mat}(N, \mathbb{C})$ and

$$B_l = \begin{pmatrix} w_{,l}w^{-1} & -a\lambda \\ -a\lambda & v_{,l}v^{-1} \end{pmatrix}, \tag{4.2a}$$

$$B_m = \begin{pmatrix} 1 & -b\lambda^{-1}v_{,m}w^{-1} \\ -b\lambda^{-1}w_{,m}v^{-1} & 1 \end{pmatrix} \tag{4.2b}$$

where $a, b, \lambda \in \mathbb{C}^\times$ are parameters. Consider the linear system

$$T_l \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_l \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \quad T_m \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_m \begin{pmatrix} \phi \\ \psi \end{pmatrix} \tag{4.3}$$

for $\phi, \psi : \mathbb{Z}^2 \to \mathrm{Mat}(N, \mathbb{C})$. Calculating shift operations in two ways, we have

$$T_m T_l \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_{l,m} T_m \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_{l,m} B_m \begin{pmatrix} \phi \\ \psi \end{pmatrix},$$

$$T_l T_m \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_{m,l} T_l \begin{pmatrix} \phi \\ \psi \end{pmatrix} = B_{m,l} B_l \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

These must coincide, so we require the compatibility condition

$$B_{l,m} B_m = B_{m,l} B_l. \tag{4.4}$$

This is equivalent to

$$w_{,lm}w_{,m}^{-1} - w_{,l}w^{-1} + ab\left(v_{,m}w^{-1} - w_{,lm}v_{,l}^{-1}\right) = 0, \tag{4.5a}$$

$$v_{,lm}v_{,m}^{-1} - v_{,l}v^{-1} + ab\left(w_{,m}v^{-1} - v_{,lm}w_{,l}^{-1}\right) = 0. \tag{4.5b}$$

We call the pair (4.5a), (4.5b) the noncommutative discrete sine-Gordon equation (ncdsG). When (w, v) is a solution for ncdsG, ${}^t(\phi\ \psi)$ satisfying (4.3) is called the eigenfunction of (w, v) for eigenvalue λ.

Proposition 4.1. When $N = 1$, the reduction condition

$$wv = 1 \tag{4.6}$$

gives the (commutative) discrete sine-Gordon equation [5, 2]

$$\frac{w_{,lm}}{w_{,m}} - \frac{w_{,l}}{w} + ab\left(\frac{1}{w_{,m}w} - w_{,lm}w_{,l}\right) = 0. \tag{4.7}$$

Proof. Under (4.6), (4.5a) is apparently equivalent to (4.7). Since

$$\begin{aligned}
&(\text{LHS of (4.5a)}) \times \left(w_{,m}w - \frac{1}{w_{,lm}w_{,l}}\right) \\
&= w_{,lm}w + \frac{1}{w_{,lm}w} - w_{,l}w_{,m} - \frac{1}{w_{,l}w_{,m}} \\
&= (\text{LHS of (4.5b)}) \times \left(\frac{1}{w_{,m}w} - w_{,lm}w_{,l}\right),
\end{aligned}$$

(4.5b) is also equivalent to (4.7). ∎

For any w_0 satisfying

$$abw_{0,\bar{l}m} = \frac{w_{0,l\bar{m}}}{ab}, \tag{4.8}$$

ncdsG is solved by $(w, v) = (w_0, abw_{0,\bar{l}m})$, which is not an interesting solution. We consider other types of solutions in the rest of this section.

4.1.2 Reduction from the noncommutative discrete KP equation

Let $w_i = w_i(n_1, n_2, n_3)$ $(i = 1, 2, 3)$ be functions $\mathbb{Z}^3 \to \mathrm{Mat}(N, \mathbb{C})$. The noncommutative discrete KP equation [14, 10] is the set of equations

$$w_{i,j}(c_i - c_j)w_i^{-1} + w_{j,k}(c_j - c_k)w_j^{-1} + w_{k,i}(c_k - c_i)w_k^{-1} = 0 \tag{4.9}$$

for any combination of $i, j, k \in \{1, 2, 3\}$. Here i, j, k can take same values, and shifts are denoted like

$$w_{1,2} = w_1(n_1, n_2 + 1, n_3).$$

$c_i \in \mathbb{C}^\times$ are parameters taking mutually different values.

We perform change of variables in such a way that new $w_i(n_1, n_2, n_3)$ corresponds to old $w_i(n_1 - n_3, n_2, n_3)$. Then we have

$$\begin{aligned}
&w_{1,2}w_1^{-1} + (\delta - 1)w_{2,13}w_2^{-1} - \delta w_{3,1}w_3^{-1} = 0, \\
&w_{1,2}w_1^{-1} = w_{2,1}w_2^{-1}, \quad w_{2,13}w_2^{-1} = w_{3,2}w_3^{-1}, \quad w_{3,1}w_3^{-1} = w_{1,13}w_1^{-1}
\end{aligned}$$

where

$$\delta = \frac{c_1 - c_3}{c_1 - c_2}. \tag{4.10}$$

By imposing the reduction condition

$$w_i(n_1 + 2, n_2, n_3) = w_i(n_1, n_2, n_3), \tag{4.11}$$

we obtain

$$w_{1,2}w_1^{-1} + (\delta - 1)v_{2,3}w_2^{-1} - \delta v_3 w_3^{-1} = 0, \tag{4.12a}$$

$$v_{1,2}v_1^{-1} + (\delta - 1)w_{2,3}v_2^{-1} - \delta w_3 v_3^{-1} = 0, \tag{4.12b}$$

$$w_{1,2}w_1^{-1} = v_2 w_2^{-1}, \quad v_{2,3}w_2^{-1} = w_{3,2}w_3^{-1}, \quad v_3 w_3^{-1} = v_{1,3}w_1^{-1} \tag{4.12c}$$

$$v_{1,2}v_1^{-1} = w_2 v_2^{-1}, \quad w_{2,3}v_2^{-1} = v_{3,2}v_3^{-1}, \quad w_3 v_3^{-1} = w_{1,3}v_1^{-1} \tag{4.12d}$$

where $v_i = w_{i,1}$.

Proposition 4.2. For any w_1, v_1 satisfying (4.12a)–(4.12d),

$$(w, v) = (w_1, v_1) \qquad (l = n_2, m = n_3)$$

solves ncdsG with $ab = \delta$.

Proof. Let us rewrite (4.12a) using only w_1, v_1. From (4.12b)–(4.12d),

$$\begin{aligned}(\delta - 1)v_{2,3}w_2^{-1} &= w_{3,2}v_{3,2}^{-1} \cdot (\delta - 1)v_{3,2}v_3^{-1} \cdot v_3 w_3^{-1} \\ &= w_{1,23}v_{1,2}^{-1} \cdot \left(\delta w_{1,3}v_1^{-1} - v_{1,2}v_1^{-1}\right) \cdot v_1 w_{1,3}^{-1} \\ &= \delta w_{1,23}v_{1,2}^{-1} - w_{1,23}w_{1,3}^{-1}\end{aligned}$$

and thus

$$\begin{aligned}0 &= w_{1,2}w_1^{-1} + (\delta - 1)v_{2,3}w_2^{-1} - \delta v_3 w_3^{-1} \\ &= w_{1,2}w_1^{-1} - w_{1,23}w_{1,3}^{-1} + \delta \left(w_{1,23}v_{1,2}^{-1} - v_{1,3}w_1^{-1}\right).\end{aligned}$$

Similarly,

$$0 = v_{1,2}v_1^{-1} - v_{1,23}v_{1,3}^{-1} + \delta \left(v_{1,23}w_{1,2}^{-1} - w_{1,3}v_1^{-1}\right). \qquad \blacksquare$$

Remark. The solution constructed here seems to be only a part of the whole solutions of ncdsG, since it satisfies extra conditions

$$w_{1,2}w_1^{-1} \cdot v_{1,2}v_1^{-1} = 1, \quad v_{1,3}w_1^{-1} \cdot w_{1,3}v_1^{-1} = 1.$$

4.1.3 Continuum limit

Assume w is also a function $w(x,t)$ of continuum variables $x, t \in \mathbb{R}$ and has an expansion

$$w(x+r, t+s) = w + (rw_x + sw_t) + \frac{1}{2}\left(r^2 w_{xx} + 2rsw_{xt} + s^2 w_{tt}\right) + \cdots \quad (4.13)$$

where $w_x = \partial w/\partial x$, etc. Connect l, m to x, t via the Miwa transformation

$$w(x,t;l,m) = w(x + la, t + mb). \quad (4.14)$$

Assume similarly for $v = v(x,t;l,m)$. Then we have

$$\begin{aligned}
w_{,l} &= w + aw_x + \frac{a^2}{2} w_{xx} + \cdots, \\
w_{,lm} &= w + (aw_x + bw_t) + \frac{1}{2}\left(a^2 w_{xx} + 2abw_{xt} + b^2 w_{tt}\right) + \cdots, \\
w_{,m}^{-1} &= w^{-1} - bw^{-1}w_t w^{-1} \\
&\qquad - \frac{b^2}{2}\left(w^{-1}w_{tt}w^{-1} - 2w^{-1}w_t w^{-1} w_t w^{-1}\right) + \cdots, \\
v_{,l} &= \cdots
\end{aligned}$$

and from (4.5a), (4.5b)

$$\begin{aligned}
0 &= w_{,lm} w_{,m}^{-1} - w_{,l} w^{-1} + ab\left(v_{,m} w^{-1} - w_{,lm} v_{,l}^{-1}\right) \\
&= ab\left(w_{xt}w^{-1} - w_x w^{-1} w_t w^{-1} + vw^{-1} - wv^{-1}\right) + (\text{higher-order terms}), \\
0 &= ab\left(v_{xt}v^{-1} - v_x v^{-1} v_t v^{-1} + wv^{-1} - vw^{-1}\right) + (\text{higher-order terms}).
\end{aligned}$$

Taking the limit $a, b \to 0$ successively, we obtain

$$\begin{aligned}
w_{xt}w^{-1} - w_x w^{-1} w_t w^{-1} + vw^{-1} - wv^{-1} &= 0, \\
v_{xt}v^{-1} - v_x v^{-1} v_t v^{-1} + wv^{-1} - vw^{-1} &= 0.
\end{aligned}$$

Since $-w^{-1}w_t w^{-1} = (w^{-1})_t$, these are transformed into

$$\left(w_x w^{-1}\right)_t = wv^{-1} - vw^{-1}, \quad (4.15a)$$

$$\left(w_x w^{-1} + v_x v^{-1}\right)_t = 0. \quad (4.15b)$$

We call the pair (4.15a), (4.15b) the noncommutative sine-Gordon equation. A quite similar equation with the same name has been derived in a different context [12, (3.10)].

Proposition 4.3. When $N = 1$, the reduction condition

$$wv = 1 \tag{4.16}$$

gives the (commutative) sine-Gordon equation

$$u_{xt} = 4 \sin u, \tag{4.17}$$

where u is defined by

$$u = \frac{2}{i} \log w. \tag{4.18}$$

Proof. Under (4.16), (4.15b) clearly holds. And (4.17) is immediate from (4.15a) since

$$u_{xt} = \frac{2}{i} \frac{w_{xt}w - w_x w_t}{w^2}, \quad \sin u = \frac{w^2 - w^{-2}}{2i}. \qquad \blacksquare$$

4.1.4 Darboux transformation

Let ${}^t(\phi_\lambda\ \psi_\lambda)$, ${}^t(\phi_\mu\ \psi_\mu)$ be eigenfunctions of (w, v) for eigenvalues λ, μ, respectively. Define the Darboux transformation of (w, v), ${}^t(\phi_\lambda\ \psi_\lambda)$ by ${}^t(\phi_\mu\ \psi_\mu)$ as

$$\widetilde{w} = \psi_\mu \phi_\mu^{-1} w, \quad \widetilde{v} = \phi_\mu \psi_\mu^{-1} v, \quad \begin{pmatrix} \widetilde{\phi}_\lambda \\ \widetilde{\psi}_\lambda \end{pmatrix} = K \begin{pmatrix} \phi_\lambda, \\ \psi_\lambda \end{pmatrix} \tag{4.19}$$

where

$$K = \begin{pmatrix} -\mu\psi_\mu\phi_\mu^{-1} & \lambda \\ \lambda & -\mu\phi_\mu\psi_\mu^{-1} \end{pmatrix}. \tag{4.20}$$

Theorem 4.4. $(\widetilde{w}, \widetilde{v})$ is a solution to ncdsG and ${}^t\left(\widetilde{\phi}_\lambda\ \widetilde{\psi}_\lambda\right)$ is an eigenfunction of $(\widetilde{w}, \widetilde{v})$ for eigenvalue λ.

Proof. From the linear system (4.3), we can write

$$w_{,l}w^{-1} = (\phi_{\mu,l} + a\mu\psi_\mu)\phi_\mu^{-1}, \quad v_{,l}v^{-1} = (\psi_{\mu,l} + a\mu\phi_\mu)\psi_\mu^{-1},$$
$$w_{,m}v^{-1} = b^{-1}\mu(\phi_\mu - \phi_{\mu,m})\psi_\mu^{-1}, \quad v_{,m}w^{-1} = b^{-1}\mu(\psi_\mu - \psi_{\mu,m})\phi_\mu^{-1}.$$

Then we have

$$\begin{aligned}
\widetilde{w}_{,l}\widetilde{w}^{-1} &= \psi_{\mu,l}\left(\psi_\mu^{-1} + a\mu\phi_{\mu,l}^{-1}\right)\\
&= v_{,l}v^{-1} + a\mu\left(\psi_{\mu,l}\phi_{\mu,l}^{-1} - \phi_\mu\psi_\mu^{-1}\right),\\
\widetilde{v}_{,l}\widetilde{v}^{-1} &= \phi_{\mu,l}\left(\phi_\mu^{-1} + a\mu\psi_{\mu,l}\right)\\
&= w_{,l}w^{-1} + a\mu\left(\phi_{\mu,l}\psi_{\mu,l}^{-1} - \psi_\mu\phi_\mu^{-1}\right),\\
\widetilde{w}_{,m}\widetilde{v}^{-1} &= b^{-1}\mu\psi_{\mu,m}\left(\phi_{\mu,m}^{-1} - \phi_\mu^{-1}\right)\\
&= v_m w^{-1} + b^{-1}\mu\left(\psi_{\mu,m}\phi_{\mu,m}^{-1} - \psi_\mu\phi_\mu^{-1}\right),\\
\widetilde{v}_{,m}\widetilde{w}^{-1} &= b^{-1}\mu\phi_{\mu,m}\left(\psi_{\mu,m}^{-1} - \psi_\mu^{-1}\right)\\
&= w_{,m}v^{-1} + b^{-1}\mu\left(\phi_{\mu,m}\psi_{\mu,m}^{-1} - \phi_\mu\psi_\mu\right),
\end{aligned}$$

which imply

$$\begin{aligned}
\left(\widetilde{w}_{,l}\widetilde{w}^{-1}\right)_{,m} - \widetilde{w}_{,l}\widetilde{w}^{-1} + ab\left(\widetilde{v}_{,m}\widetilde{w}^{-1} - \left(\widetilde{w}_{,m}\widetilde{v}^{-1}\right)_{,l}\right) = 0,\\
\left(\widetilde{v}_{,l}\widetilde{v}^{-1}\right)_{,m} - \widetilde{v}_{,l}\widetilde{v}^{-1} + ab\left(\widetilde{w}_{,m}\widetilde{v}^{-1} - \left(\widetilde{v}_{,m}\widetilde{w}^{-1}\right)_{,l}\right) = 0.
\end{aligned}$$

Define $\widetilde{B}_l, \widetilde{B}_m$ by

$$\widetilde{B}_l = \begin{pmatrix} \widetilde{w}_{,l}\widetilde{w}^{-1} & -a\lambda \\ -a\lambda & \widetilde{v}_{,l}\widetilde{v}^{-1} \end{pmatrix}, \quad \widetilde{B}_m = \begin{pmatrix} 1 & -b\lambda^{-1}\widetilde{w}_{,m}\widetilde{v}^{-1} \\ -b\lambda^{-1}\widetilde{v}_{,m}\widetilde{w}^{-1} & 1 \end{pmatrix}.$$

Then we have

$$K_{,l}B_l = \widetilde{B}_l K, \quad K_{,m}B_m = \widetilde{B}_m K$$

and thus

$$\begin{aligned}
T_l\left(K\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right) &= K_{,l}B_l\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \widetilde{B}_l\left(K\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right),\\
T_m\left(K\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right) &= K_{,m}B_m\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \widetilde{B}_m\left(K\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right).
\end{aligned} \qquad \blacksquare$$

4.1.5 Casoratian-type solutions

Let λ_k $(k = 1, 2, \ldots)$ be mutually different eigenvalues and ${}^t(\phi_k\ \psi_k)$ be eigenfunctions for λ_k. Define repetition of the Darboux transformation by

$$w^{(n)} = \psi_n^{(n)}\left(\phi_n^{(n)}\right)^{-1} w^{(n-1)}, \tag{4.21a}$$

$$v^{(n)} = \phi_n^{(n)}\left(\psi_n^{(n)}\right)^{-1} v^{(n-1)}, \tag{4.21b}$$

$$\phi_k^{(n+1)} = \lambda_k\psi_k^{(n)} - \lambda_n\psi_n^{(n)}\left(\phi_n^{(n)}\right)^{-1}\phi_k^{(n)}, \tag{4.21c}$$

$$\psi_k^{(n+1)} = \lambda_k\phi_k^{(n)} - \lambda_n\phi_n^{(n)}\left(\psi_n^{(n)}\right)^{-1}\psi_k^{(n)} \tag{4.21d}$$

and

$$w^{(0)} = w, \quad v^{(0)} = v, \quad \phi_k^{(1)} = \phi_k, \quad \psi_k^{(1)} = \psi_k. \tag{4.21e}$$

For notational convenience, we introduce *reduced* shift operator T defined by

$$\mathrm{T} f(\phi_1, \psi_1, \phi_2, \psi_2, \ldots) = f(\lambda_1 \psi_1, \lambda_1 \phi_1, \lambda_2 \psi_2, \lambda_2 \phi_2, \ldots), \tag{4.22}$$

where $f(x_1, x_2, \ldots)$ is any rational function of noncommutative variables x_j. For example, we have

$$\mathrm{T}\phi_k = \lambda_k \psi_k, \quad \mathrm{T}\psi_k = \lambda_k \phi_k, \quad \mathrm{T}^2 \phi_k = \lambda_k^2 \phi_k.$$

Lemma 4.5.

$$\mathrm{T}\phi_k^{(n+1)} = \lambda_k \psi_k^{(n+1)}, \quad \mathrm{T}\psi_k^{(n+1)} = \lambda_k \phi_k^{(n+1)}. \tag{4.23}$$

Proof. By induction. Obviously, $\phi_k^{(n+1)}, \psi_k^{(n+1)}$ are rational functions of ϕ_j, ψ_j. Assume $\mathrm{T}\phi_k^{(n)} = \lambda_k \psi_k^{(n)}, \mathrm{T}\psi_k^{(n)} = \lambda_k \phi_k^{(n)}$ for certain n. Then,

$$\mathrm{T}\phi_k^{(n+1)} = \lambda_k \left(\lambda_k \phi_k^{(n)}\right) - \lambda_n \left(\lambda_n \phi_n^{(n)}\right) \left(\lambda_n \psi_n^{(n)}\right)^{-1} \lambda_k \psi_k^{(n)} = \lambda_k \psi_k^{(n+1)}.$$

Similarly, $\mathrm{T}\psi_k^{(n+1)} = \lambda_k \phi_k^{(n+1)}$. ■

Theorem 4.6.

$$w^{(n)} = \prod_{j=1}^{n} (-\lambda_j^{-1}) \cdot \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n & 1 \\ T\phi_1 & T\phi_2 & \cdots & T\phi_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & T^{n-1}\phi_2 & \cdots & T^{n-1}\phi_n & 0 \\ T^n\phi_1 & T^n\phi_2 & \cdots & T^n\phi_n & \boxed{0} \end{vmatrix} w, \quad (4.24a)$$

$$v^{(n)} = \prod_{j=1}^{n} (-\lambda_j^{-1}) \cdot \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ T\psi_1 & T\psi_2 & \cdots & T\psi_n & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\psi_1 & T^{n-1}\psi_2 & \cdots & T^{n-1}\psi_n & 0 \\ T^n\psi_1 & T^n\psi_2 & \cdots & T^n\psi_n & \boxed{0} \end{vmatrix} v, \quad (4.24b)$$

$$\phi_k^{(n+1)} = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n & \phi_k \\ T\phi_1 & T\phi_2 & \cdots & T\phi_n & T\phi_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & T^{n-1}\phi_2 & \cdots & T^{n-1}\phi_n & T^{n-1}\phi_k \\ T^n\phi_1 & T^n\phi_2 & \cdots & T^n\phi_n & \boxed{T^n\phi_k} \end{vmatrix}, \quad (4.24c)$$

$$\psi_k^{(n+1)} = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & \psi_k \\ T\psi_1 & T\psi_2 & \cdots & T\psi_n & T\psi_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\psi_1 & T^{n-1}\psi_2 & \cdots & T^{n-1}\psi_n & T^{n-1}\psi_k \\ T^n\psi_1 & T^n\psi_2 & \cdots & T^n\psi_n & \boxed{T^n\psi_k} \end{vmatrix}. \quad (4.24d)$$

Here, quasideterminants [4] are used (see Appendix A). When $n = 0$, (4.24a) and (4.24b) read

$$w^{(0)} = 1 \cdot \begin{vmatrix} \boxed{1} \end{vmatrix} w, \quad v^{(0)} = 1 \cdot \begin{vmatrix} \boxed{1} \end{vmatrix} v,$$

respectively.

Proof. By induction. The case $n = 0$ is trivial. Assume $w^{(n-1)}, v^{(n-1)}, \phi_k^{(n)}, \psi_k^{(n)}$

have the above expressions for certain $n > 0$. Then,

$$
\begin{aligned}
w^{(n)} &= \lambda_n^{-1} \left(T\phi_n^{(n)}\right) \left(\phi_n^{(n)}\right)^{-1} w^{(n-1)} \\
&= -\prod_{j=1}^{n} (-\lambda_j^{-1}) \cdot \begin{vmatrix} T\phi_1 & \cdots & T\phi_n \\ \vdots & \ddots & \vdots \\ T^n\phi_1 & \cdots & \boxed{T^n\phi_n} \end{vmatrix} \begin{vmatrix} \phi_1 & \cdots & \phi_n \\ \vdots & \ddots & \vdots \\ T^{n-1}\phi_1 & \cdots & \boxed{T^{n-1}\phi_n} \end{vmatrix}^{-1} \\
&\quad \times \begin{vmatrix} \phi_1 & \cdots & \phi_{n-1} & 1 \\ T\phi_1 & \cdots & T\phi_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & \cdots & T^{n-1}\phi_{n-1} & \boxed{0} \end{vmatrix} w.
\end{aligned}
$$

By the column homological relation (Proposition A.3),

$$
\begin{aligned}
w^{(n)} &= -\prod_{j=1}^{n} (-\lambda_j^{-1}) \cdot \begin{vmatrix} T\phi_1 & \cdots & T\phi_n \\ \vdots & \ddots & \vdots \\ T^n\phi_1 & \cdots & \boxed{T^n\phi_n} \end{vmatrix} \begin{vmatrix} \phi_1 & \cdots & \boxed{\phi_n} \\ \vdots & \ddots & \vdots \\ T^{n-1}\phi_1 & \cdots & T^{n-1}\phi_n \end{vmatrix}^{-1} \\
&\quad \times \begin{vmatrix} \phi_1 & \cdots & \phi_{n-1} & \boxed{1} \\ T\phi_1 & \cdots & T\phi_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & \cdots & T^{n-1}\phi_{n-1} & 0 \end{vmatrix} w,
\end{aligned}
$$

and by Sylvester's identity (Proposition A.4),

$$
w^{(n)} = \prod_{j=1}^{n} (-\lambda_j^{-1}) \cdot \begin{vmatrix} \phi_1 & \cdots & \phi_n & 1 \\ T\phi_1 & \cdots & T\phi_n & 0 \\ \vdots & \ddots & \vdots & \vdots \\ T^n\phi_1 & \cdots & T^n\phi_n & \boxed{0} \end{vmatrix} w.
$$

Similarly for $v^{(n)}$. Next,

$$
\begin{aligned}
\phi_k^{(n+1)} &= T\phi_k^{(n)} - \left(T\phi_n^{(n)}\right)\left(\phi_n^{(n)}\right)^{-1}\phi_k^{(n)} \\
&= \begin{vmatrix} T\phi_1 & \cdots & T\phi_{n-1} & T\phi_k \\ \vdots & \ddots & \vdots & \vdots \\ T^n\phi_1 & \cdots & T^n\phi_{n-1} & \boxed{T^n\phi_k} \end{vmatrix} - \begin{vmatrix} T\phi_1 & \cdots & T\phi_{n-1} & T\phi_n \\ \vdots & \ddots & \vdots & \vdots \\ T^n\phi_1 & \cdots & T^n\phi_{n-1} & \boxed{T^n\phi_n} \end{vmatrix} \\
&\quad \times \begin{vmatrix} \phi_1 & \cdots & \phi_{n-1} & \phi_n \\ T\phi_1 & \cdots & T\phi_{n-1} & T\phi_n \\ \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & \cdots & T^{n-1}\phi_{n-1} & \boxed{T^{n-1}\phi_n} \end{vmatrix}^{-1} \\
&\quad \times \begin{vmatrix} \phi_1 & \cdots & \phi_{n-1} & \phi_k \\ T\phi_1 & \cdots & T\phi_{n-1} & T\phi_k \\ \vdots & \ddots & \vdots & \vdots \\ T^{n-1}\phi_1 & \cdots & T^{n-1}\phi_{n-1} & \boxed{T^{n-1}\phi_k} \end{vmatrix} .
\end{aligned}
$$

By the column homological relation and Sylvester's identity,

$$
\phi_k^{(n+1)} = \begin{vmatrix} \phi_1 & \cdots & \phi_n & \phi_k \\ \vdots & \ddots & \vdots & \vdots \\ T^n\phi_1 & \cdots & T^n\phi_n & \boxed{T^n\phi_k} \end{vmatrix} .
$$

Similarly for $\psi_k^{(n+1)}$. ∎

4.1.6 Multisoliton solutions

The simplest solution for ncdsG is the vacuum solution $(w, v) = (1, 1)$. The linear system of the vacuum solution is

$$
T_l \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 1 & -a\lambda \\ -a\lambda & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{4.25a}
$$

$$
T_m \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 1 & -b\lambda^{-1} \\ -b\lambda^{-1} & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{4.25b}
$$

which has two basic solutions

$$
\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} (1-a\lambda)^l\left(1-b\lambda^{-1}\right)^m \\ (1-a\lambda)^l\left(1-b\lambda^{-1}\right)^m \end{pmatrix}, \begin{pmatrix} (1+a\lambda)^l\left(1+b\lambda^{-1}\right)^m \\ -(1+a\lambda)^l\left(1+b\lambda^{-1}\right)^m \end{pmatrix}. \tag{4.26}
$$

Let λ_k $(k = 1, 2, \ldots)$ be mutually different eigenvalues and define

$$\phi_k = (1 - a\lambda_k)^l \left(1 - b\lambda_k^{-1}\right)^m + (1 + a\lambda_k)^l \left(1 + b\lambda_k^{-1}\right)^m c_k, \tag{4.27a}$$
$$\psi_k = (1 - a\lambda_k)^l \left(1 - b\lambda_k^{-1}\right)^m - (1 + a\lambda_k)^l \left(1 + b\lambda_k^{-1}\right)^m c_k, \tag{4.27b}$$

where $c_k \in \mathrm{Mat}(N, \mathbb{C})$ are parameters introducing noncommutativity. ${}^t(\phi_k\ \psi_k)$ is of course an eigenfunction of the vacuum solution for eigenvalue λ_k. Repeating the Darboux transformation by ${}^t(\phi_k\ \psi_k)$, we can construct multisoliton solutions.

A 1-soliton solution is given by

$$w = \psi_1\phi_1^{-1} = (1 - f_1)(1 + f_1)^{-1}, \tag{4.28a}$$
$$v = \phi_1\psi_1^{-1} = (1 + f_1)(1 - f_1)^{-1}, \tag{4.28b}$$

where f_k is defined by

$$f_k = \left(\frac{1 + a\lambda_k}{1 - a\lambda_k}\right)^l \left(\frac{1 + b\lambda_k^{-1}}{1 - b\lambda_k^{-1}}\right)^m c_k. \tag{4.29}$$

As a concrete example, Figure 4.1 shows the behavior of

$$w = \begin{pmatrix} w^{11} & w^{12} \\ w^{21} & w^{22} \end{pmatrix} \qquad (N = 2)$$

with

$$a = b = 0.2, \quad c_1 = \begin{pmatrix} 2 & -4 \\ 1 & -1.5 \end{pmatrix}, \quad \lambda_1 = \frac{5}{3}$$

in the light-cone coordinates (3.15). A 2-soliton solution is given by

$$\begin{aligned} w &= \left(\lambda_2\phi_2 - \lambda_1\phi_1\psi_1^{-1}\psi_2\right)\left(\lambda_2\psi_2 - \lambda_1\psi_1\phi_1^{-1}\phi_2\right)^{-1}\psi_1\phi_1^{-1} \\ &= \left(\lambda_2\phi_2\psi_2^{-1} - \lambda_1\phi_1\psi_1^{-1}\right)\left(\lambda_2\phi_1\psi_1^{-1} - \lambda_1\phi_2\psi_2^{-1}\right)^{-1} \\ &= \left(\lambda_2(1 + f_2)(1 - f_2)^{-1} - \lambda_1(1 + f_1)(1 - f_1)^{-1}\right) \\ &\qquad \times \left(\lambda_2(1 + f_1)(1 - f_1)^{-1} - \lambda_1(1 + f_2)(1 - f_2)^{-1}\right)^{-1}, \end{aligned} \tag{4.30a}$$
$$\begin{aligned} v &= \left(\lambda_2(1 - f_2)(1 + f_2)^{-1} - \lambda_1(1 - f_1)(1 + f_1)^{-1}\right) \\ &\qquad \times \left(\lambda_2(1 - f_1)(1 + f_1)^{-1} - \lambda_1(1 - f_2)(1 + f_2)^{-1}\right)^{-1}. \end{aligned} \tag{4.30b}$$

Figure 4.2 shows the solution with

$$a = b = 0.2, \quad c_1 = \begin{pmatrix} 2.5 & -0.8 \\ 2 & 1.8 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1.5 & 1.2 \\ -1 & 0.5 \end{pmatrix}, \quad \lambda_1 = \lambda_2^{-1} = \frac{5}{3}.$$

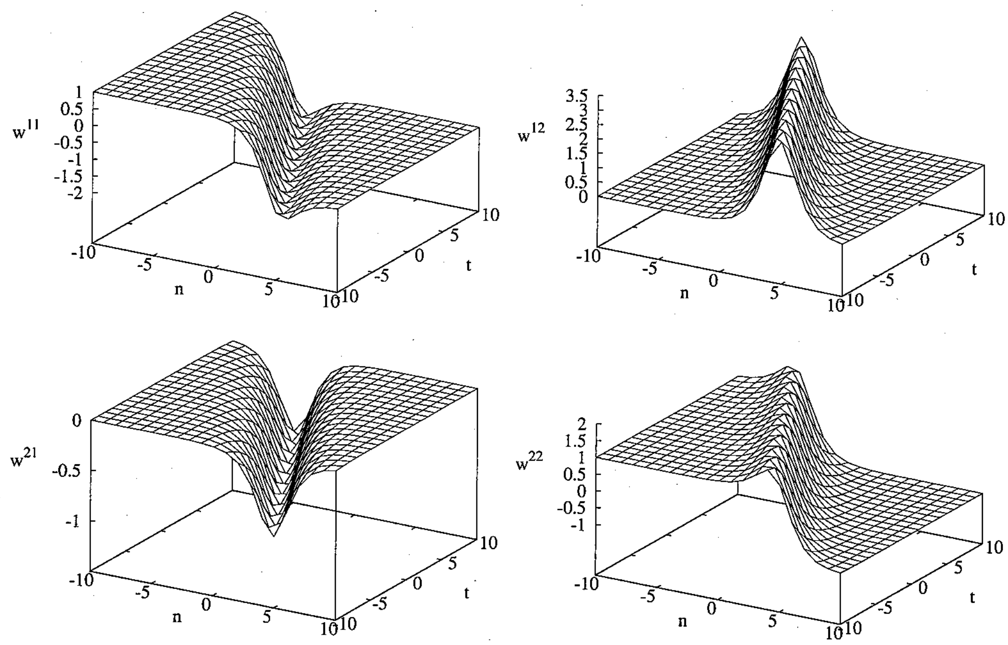

Figure 4.1: 1-soliton solution for ncdsG.

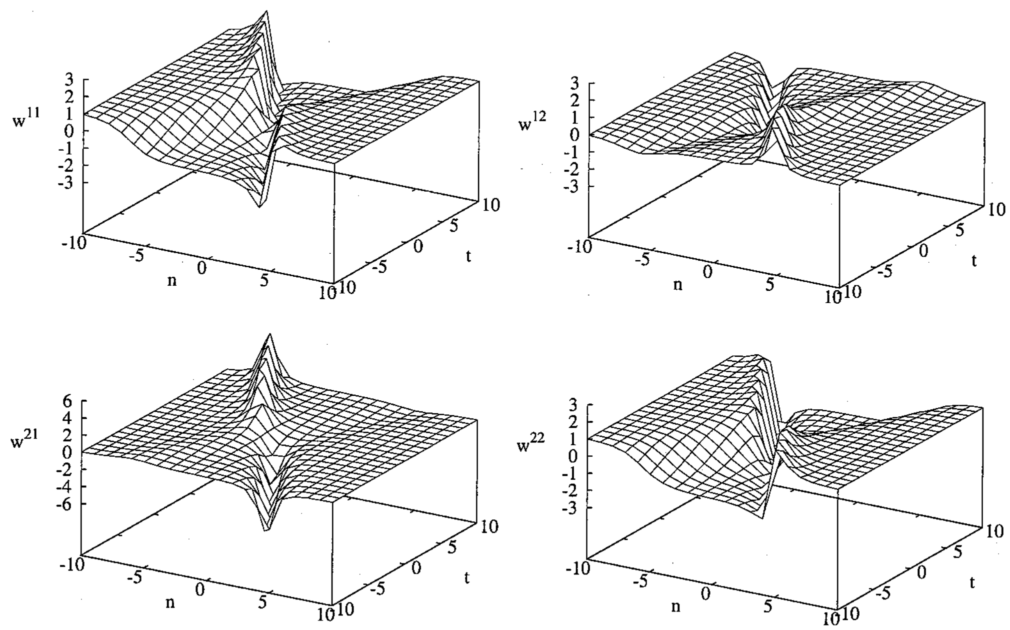

Figure 4.2: 2-soliton solution for ncdsG.

4.2 Noncommutative ultradiscrete sine-Gordon equation

4.2.1 Ultradiscretization

We perform ultradiscretization of ncdsG by the parametrization

$$a = \mu_A e^{\tilde{A}s}, \quad b = \mu_B e^{\tilde{A}s}, \quad \tilde{A}, \tilde{B} < 0. \tag{4.31}$$

Assuming

$$a \xrightarrow{\text{ud}} A, \quad b \xrightarrow{\text{ud}} B, \quad w \xrightarrow{\text{ud}} W, \quad v \xrightarrow{\text{ud}} V, \tag{4.32}$$

we obtain

$$W_{,lm} W_{,m}^{-1} \ominus W_{,l} W^{-1} \oplus AB \left(V_{,m} W^{-1} \ominus W_{,lm} V_{,l}^{-1} \right) \nabla -\infty, \tag{4.33a}$$

$$V_{,lm} V_{,m}^{-1} \ominus V_{,l} V^{-1} \oplus AB \left(W_{,m} V^{-1} \ominus V_{,lm} W_{,l}^{-1} \right) \nabla -\infty. \tag{4.33b}$$

We call the pair (4.33a), (4.33b) the noncommutative ultradiscrete sine-Gordon equation (ncudsG). Because $\mathrm{uMat}(N, \mathrm{u}\mathbb{C})$ can be realized by $\mathrm{uMat}(2N, \mathrm{u}\mathbb{R})$, we use $\mathrm{uMat}(N, \mathrm{u}\mathbb{R})$ as the underlying algebra for simplicity.

4.2.2 1-soliton solution

In order to ultradiscretize solutions for ncdsG, we introduce

$$p_j = \frac{1 + a\lambda_j}{1 - a\lambda_j}, \quad q_j = \frac{1 + b\lambda_j^{-1}}{1 - b\lambda_j^{-1}}. \tag{4.34}$$

These solve the dispersion relation

$$(1 - ab)(1 + p_j q_j) = (1 + ab)(p_j + q_j), \tag{4.35}$$

and any solution of (4.35) is parametrized by λ_j through (4.34) unless $ab = 1$. As in the commutative case, (4.35) is ultradiscretized to

$$0 \oplus P_j Q_j \ \nabla\ P_j \oplus Q_j \tag{4.36}$$

where $p_j \xrightarrow{\text{ud}} P_j, q_j \xrightarrow{\text{ud}} Q_j$.

We can directly discretize the 1-soliton solution (4.28a), (4.28b) to obtain

$$w \xrightarrow{\text{ud}} W \ \nabla\ (0 \ominus F_1)(0 \oplus F_1)^{-1}, \tag{4.37a}$$

$$v \xrightarrow{\text{ud}} V \ \nabla\ (0 \oplus F_1)(0 \ominus F_1)^{-1} \tag{4.37b}$$

where

$$F_j = P_j^l Q_j^m C_j, \quad c_j \xrightarrow{\text{ud}} C_j \in \mathrm{uMat}(N, \mathrm{u}\mathbb{R}). \tag{4.38}$$

This relation is valid, but inadequate to determine W, V in many cases. For simplicity, we assume $N = 2$ hereafter. If we write $W = (W^{lk}), F_j = (F_j^{lk})$, the $(1,2)$-th element of $(0 \ominus F_1)(0 \oplus F_1)^{-1}$ is given by

$$\frac{\ominus F_1^{12}\left((0 \ominus F_1^{11}) \oplus (0 \oplus F_1^{11})\right)}{\det(0 \oplus F_1)} = \frac{\ominus F_1^{12}\left(0 \oplus (F_1^{11})^\bullet\right)}{\det(0 \oplus F_1)},$$

and $\left|F_1^{11}\right|_\oplus$ exceeds 0 for large $\pm l$ or $\pm m$. Then this element is balanced and W^{12} cannot be determined. Therefore, we need more precise expressions to ultradiscretize.

Define

$$g_j = (1 + f_j)(1 - f_j)^{-1}, \quad h_j = (1 - f_j)(1 + f_j)^{-1}. \tag{4.39}$$

Of course, $w = h_1, v = g_1$ is a 1-soliton solution for ncdsG. Writing $f_j = (f_j^{lk})$, we have

$$g_j = \begin{pmatrix} \frac{(1+f_j^{11})(1-f_j^{22})+f_j^{12}f_j^{21}}{\det(1-f_j)} & \frac{2f_j^{12}}{\det(1-f_j)} \\ \frac{2f_j^{21}}{\det(1-f_j)} & \frac{(1+f_j^{22})(1-f_j^{11})+f_j^{21}f_j^{12}}{\det(1-f_j)} \end{pmatrix}, \tag{4.40a}$$

$$h_j = \begin{pmatrix} \frac{(1-f_j^{11})(1+f_j^{22})+f_j^{12}f_j^{21}}{\det(1+f_j)} & \frac{-2f_j^{12}}{\det(1+f_j)} \\ \frac{-2f_j^{21}}{\det(1+f_j)} & \frac{(1-f_j^{22})(1+f_j^{11})+f_j^{21}f_j^{12}}{\det(1+f_j)} \end{pmatrix}. \tag{4.40b}$$

By ultradiscretization, we obtain

$$g_j \xrightarrow{\text{ud}} G_j \nabla \begin{pmatrix} \frac{(0\oplus F_j^{11})(0\ominus F_j^{22})\oplus F_j^{12}F_j^{21}}{\det(0\ominus F_j)} & \frac{F_j^{12}}{\det(0\ominus F_j)} \\ \frac{F_j^{21}}{\det(0\ominus F_j)} & \frac{(0\oplus F_j^{22})(0\ominus F_j^{11})\oplus F_j^{21}F_j^{12}}{\det(0\ominus F_j)} \end{pmatrix}, \tag{4.41a}$$

$$h_j \xrightarrow{\text{ud}} H_j \nabla \begin{pmatrix} \frac{(0\ominus F_j^{11})(0\oplus F_j^{22})\oplus F_j^{12}F_j^{21}}{\det(0\oplus F_j)} & \frac{\ominus F_j^{12}}{\det(0\oplus F_j)} \\ \frac{\ominus F_j^{21}}{\det(0\oplus F_j)} & \frac{(0\ominus F_j^{22})(0\oplus F_j^{11})\oplus F_j^{21}F_j^{12}}{\det(0\oplus F_j)} \end{pmatrix}. \tag{4.41b}$$

We can choose $C_j, P_j, Q_j \in u\mathbb{Z}$ such that $\left(0 \oplus F_j^{11}\right)\left(0 \ominus F_j^{22}\right)$ is always even and $F_j^{12}F_j^{21}$ odd. Then all the elements on the RHS of (4.41a), (4.41b) are signed and G_j, H_j are completely determined. Figure 4.3 shows $W = H_1$ with

$$A = B = -1, \quad C_1 = \begin{pmatrix} \ominus(-7) & -8 \\ \ominus(-5) & \ominus 7 \end{pmatrix}, \quad P_1 = 2, \quad Q_1 = 0.$$

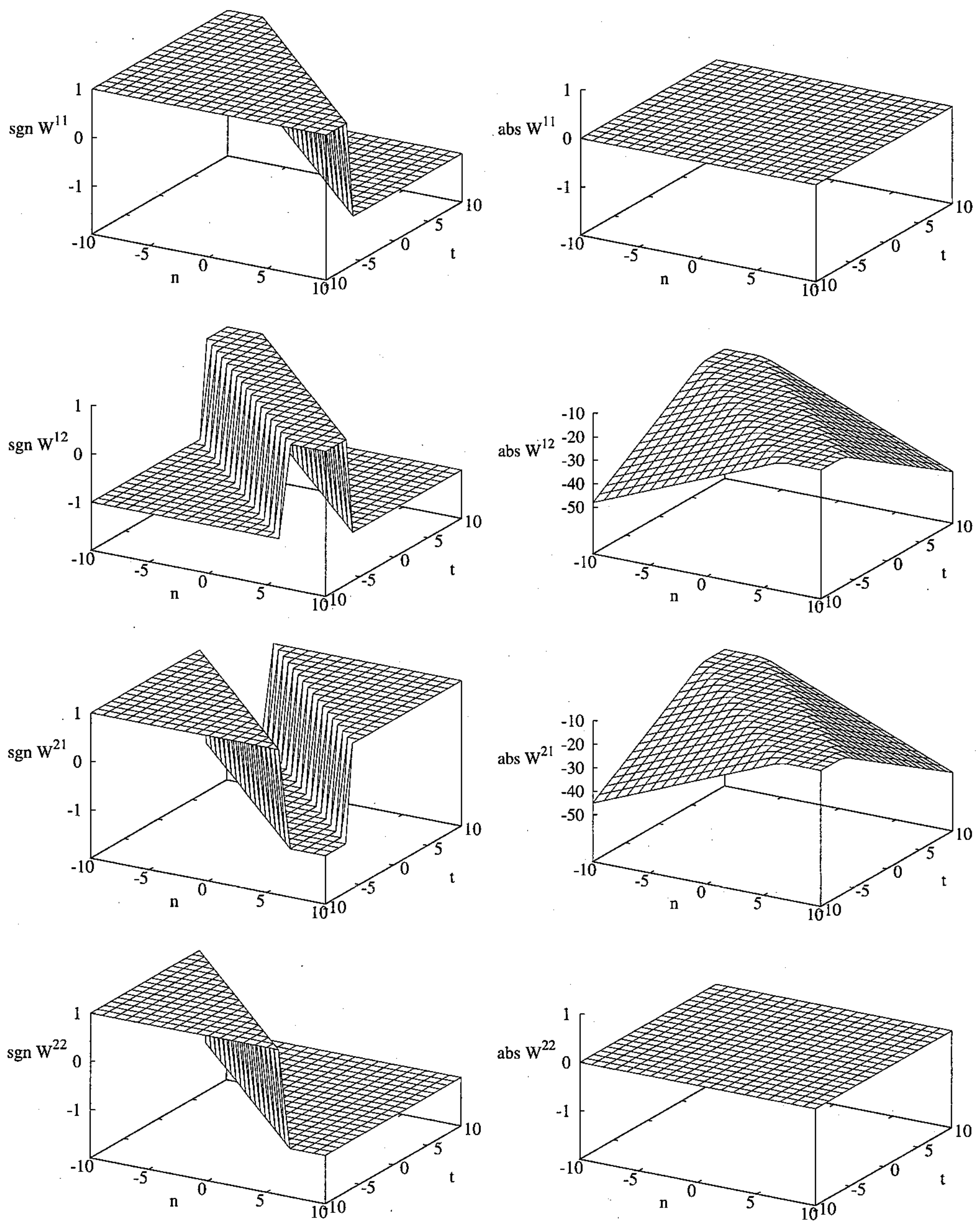

Figure 4.3: 1-soliton solution for ncudsG.

4.2.3 2-soliton solution

Ultradiscretization of (4.30a), (4.30b) gives

$$W \nabla (L_2G_2 \ominus L_1G_1)(L_2G_1 \ominus L_1G_2)^{-1}, \tag{4.42a}$$
$$V \nabla (L_2H_2 \ominus L_1H_1)(L_2H_1 \ominus L_1H_2)^{-1} \tag{4.42b}$$

where

$$\lambda_j \xrightarrow{\text{ud}} L_j. \tag{4.43}$$

In order to determine the value of L_j, we examine the relation

$$\lambda_j = \frac{p_j - 1}{a(p_j + 1)} = \frac{b(q_j + 1)}{q_j - 1}. \tag{4.44}$$

By the dispersion relation (4.36), we have $P_j = 0$ or $Q_j = 0$. When $Q_j = 0$, q_j behaves like a constant with regard to the ultradiscretization parameter s and p_j cannot behave like one. Therefore, we have

$$L_j = \frac{P_j \ominus 0}{A(P_j \oplus 0)} = \begin{cases} A^{-1} & (|P_j|_\oplus > 0), \\ \ominus A^{-1} & (|P_j|_\oplus < 0). \end{cases} \tag{4.45a}$$

Similarly, when $P_j = 0$, we have

$$L_j = \frac{B(Q_j \oplus 0)}{Q_j \ominus 0} = \begin{cases} B & (|Q_j|_\oplus > 0), \\ \ominus B & (|Q_j|_\oplus < 0). \end{cases} \tag{4.45b}$$

If we choose $P_2 = Q_1 = 0$, we have $|L_1|_\oplus > |L_2|_\oplus$ and thus

$$W \nabla G_1G_2^{-1}, \quad V \nabla H_1H_2^{-1}. \tag{4.46a}$$

Similarly, if $P_1 = Q_2 = 0$,

$$W \nabla G_2G_1^{-1}, \quad V \nabla H_2H_1^{-1}. \tag{4.46b}$$

Figure 4.4 shows behavior of $W = G_1G_2^{-1}$ with parameters $A = B = -1$,

$$C_1 = \begin{pmatrix} \ominus 2 & \ominus(-13) \\ \ominus 11 & 15 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -3 & \ominus(-15) \\ \ominus(-7) & \ominus(-13) \end{pmatrix}, \quad P_1 = Q_2 = 4.$$

These are chosen so that every elements involved are signed.

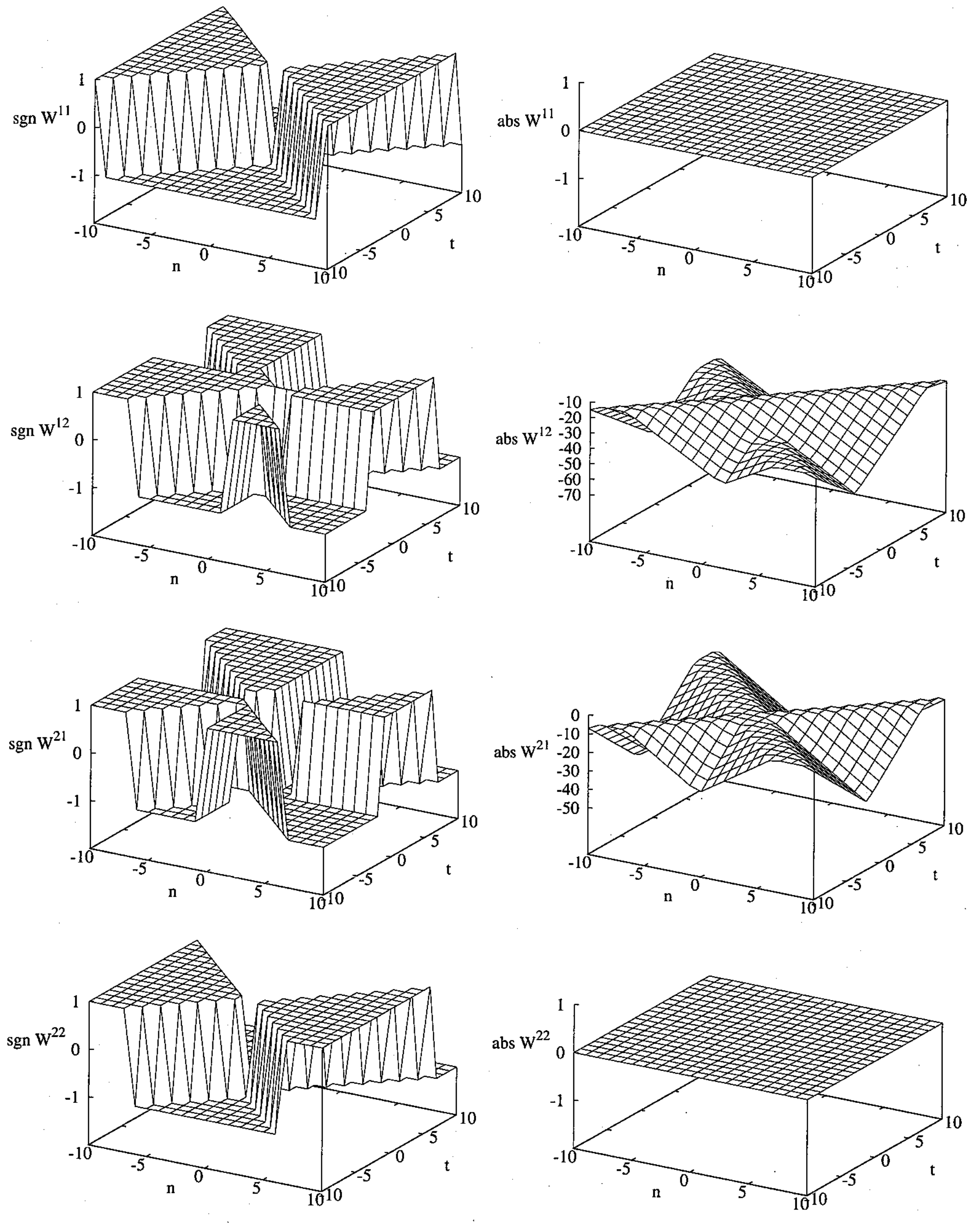

Figure 4.4: 2-soliton solution for ncudsG.

Chapter 5

Conclusion

We have reviewed the construction and properties of the symmetrized max-plus algebra $\mathrm{u}\mathbb{R}$ and related algebras. Ultradiscretization with $\mathrm{u}\mathbb{R}$, which enables ultradiscretization of subtraction, is also reviewed. Comparison to ultradiscretization with parity variables shows the simplicity of ultradiscretization with $\mathrm{u}\mathbb{R}$.

We have proposed the ultradiscrete sine-Gordon equation and constructed signed 1-soliton and 2-soliton solutions utilizing $\mathrm{u}\mathbb{R}$. The traveling-wave, kink-antikink, and kink-kink solutions, which contain ultradiscrete complex numbers, do exist and their correspondence to those for the discrete sine-Gordon equation is quite clear. When the range of solutions are restricted to $\mathrm{u}\mathbb{R}$, even deterministic time evolution is possible.

We have also proposed the noncommutative discrete sine-Gordon equation and revealed its relation to other integrable systems including the noncommutative discrete KP equation. Also, multisoliton solutions are constructed by repetition of the Darboux transformation. And finally, the noncommutative ultradiscrete sine-Gordon equation and its signed 1-soliton and 2-soliton solutions are derived by ultradiscretization with $\mathrm{u}\mathbb{R}$.

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Appendix A

Quasideterminants

Quasideterminants [4] are noncommutative extension of determinants, or, more precisely, determinants divided by cofactors. Here we describe the definition and some properties required for Theorem 4.6. See [4] for more detail.

Let R be a ring and $\mathrm{Mat}(N, R)$ be the whole set of $N \times N$ matrices over R. R is not commutative in general. For any $(a_{ij}), (b_{ij}) \in \mathrm{Mat}(N, R)$, define addition by

$$(a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij}) \tag{A.1}$$

and multiplication by

$$(a_{ij})(b_{ij}) = (c_{ij}), \quad c_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}. \tag{A.2}$$

Ordering of multiplication is important here.

For any $A = (a_{ij}) \in \mathrm{Mat}(N, R)$, define the (p, q)-th quasideterminant $|A|_{pq}$ by

$$|A|_{pq} = a_{pq} - r_p^q \,(A^{pq})^{-1}\, c_q^p, \tag{A.3}$$

where r_p^q is the p-th row of A without the q-th element, c_q^p is the q-th column of A without the p-th element, and A^{pq} is A without the p-th row and the q-th column. $|A|_{pq}$ is also written as

$$|A|_{pq} = \begin{vmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \boxed{a_{pq}} & \vdots \\ a_{N1} & \cdots & a_{NN} \end{vmatrix}. \tag{A.4}$$

For example, we have

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \quad \begin{vmatrix} a_{11} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{vmatrix} = a_{12} - a_{11} a_{21}^{-1} a_{22}$$

and

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - \begin{pmatrix} a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$

$$\begin{aligned} &= a_{11} - a_{12}\left(a_{22} - a_{23}a_{33}^{-1}a_{32}\right)^{-1} a_{21} \\ &\quad - a_{13}\left(a_{23} - a_{22}a_{32}^{-1}a_{33}\right)^{-1} a_{21} \\ &\quad - a_{12}\left(a_{32} - a_{33}a_{23}^{-1}a_{22}\right)^{-1} a_{31} \\ &\quad - a_{13}\left(a_{33} - a_{32}a_{22}^{-1}a_{23}\right)^{-1} a_{31}. \end{aligned}$$

Proposition A.1. If we write $A^{-1} = (b_{ij})$, we have

$$b_{ij} = |A|_{ji}^{-1}. \tag{A.5}$$

Proposition A.2. Quasideterminants are invariant under row and column permutations. (If the row or column contains the *box*, it is moved together.)

Proposition A.3 (Homological relations). For $p_1 \neq p_2, q_1 \neq q_2, i \neq p, j \neq q$, we have the row homological relation

$$|A|_{pq_1} |A^{pq_2}|_{iq_1}^{-1} + |A|_{pq_2} |A^{pq_1}|_{iq_2}^{-1} = 0 \tag{A.6}$$

and the column homological relation

$$|A^{p_2q}|_{p_1j}^{-1} |A|_{p_1q} + |A^{p_1q}|_{p_2j}^{-1} |A|_{p_2q} = 0. \tag{A.7}$$

Proposition A.4 (Sylvester's identity). For any $A = (a_{ij}) \in \mathrm{Mat}(N, R)$, define $(N-k) \times (N-k)$ matrix A_0 by

$$A_0 = (a_{ij}) \qquad (k+1 \leqslant i, j \leqslant N)$$

and $k \times k$ matrix C by

$$C = (c_{ij}), \quad c_{ij} = \begin{vmatrix} \boxed{a_{ij}} & a_{i(k+1)} & \cdots & a_{iN} \\ a_{(k+1)j} & & & \\ \vdots & & A_0 & \\ a_{Nj} & & & \end{vmatrix}.$$

Then

$$|A|_{pq} = |C|_{pq}. \tag{A.8}$$

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