博 士 論 文

論文題目

Mathematical and Numerical Analysis for Incompressible Fluid Equations under Friction Boundary Conditions

（摩擦型境界条件下での非圧縮流体の方程式に対する数学解析と数値解析）

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Chapter 1

Finite element analysis for the Stokes equations under the slip boundary condition of friction type

1.1 Introduction

We consider the motion of an incompressible fluid in a bounded two dimensional domain with some nonlinear boundary conditions, specified as the *slip boundary condition of friction type* (SBCF) or the *leak boundary condition of friction type* (LBCF). Intuitively speaking, SBCF (resp. LBCF) represents a boundary condition such that as long as the magnitude of the tangential (resp. normal) stress on the boundary is strictly less than a given threshold, no slip (resp. leak) takes place; however, if it reaches the threshold, then the fluid can slip (resp. leak) at the boundary.

A mathematical formulation of such frictional boundary conditions for fluids was introduced by H. Fujita in [16], and subsequently many studies focused on the properties of the solution, for example, existence, uniqueness, regularity, and continuous dependence on data, for the Stokes and Navier-Stokes equations. Details can be referred to in [16] itself or in [1], [19], [20], [38], [39], [48], [55], [56], [57] and [59], among others. Other related nonlinear boundary conditions are reported in [6], [10] and [35].

The frictional boundary conditions under consideration have been successfully applied to some flow phenomena in environmental problems such as oil flow over or beneath sand layers ([32], [61]). These numerical simulations are implemented by the finite difference method, without rigorous mathematical analysis.

Regarding to numerical methods for SBCF and LBCF in terms of the finite element method, [20] proposed an iterative algorithm of Uzawa type and gave some numerical examples. In [19], the convergence of Uzawa method is proved for continuous problems. However, discretization of continuous problems or error estimate between approximate and exact solutions is not considered in these works.

For theoretical analysis, [40] and [44] proposed a finite element approximation combined with a penalty method for the stationary Stokes and Navier-Stokes equations with SBCF. Low-order finite elements, such as the P1/P1 element with stabilized terms, are applied to those equations in [42], [46], and [47]. Their approach interprets the problem as an elliptic variational inequality of a tuple (u, p), which they regard as just one variable. In these works, they presented error estimates and offered several numerical examples computed by an Uzawa algorithm. The convergence of their Uzawa algorithm is proved for continuous problems in [49]. They also dealt with a semi-discrete approximation for the nonstationary Navier-Stokes equation with a regularized version of SBCF in [41].

Another approach by the P1+/P1 element, based on a saddle-point formulation of the problem, is found in [3]. Some numerical examples are given, and an error estimate is announced without a proof.

The purpose of this work is to present a framework of finite element method for flow problems with SBCF and LBCF, including all of the existence and uniqueness result, error analysis, and numerical implementation. To prove existence and uniqueness, we could exploit the abstract theory of mixed variational inequalities developed by [28] for plasticity problems. In this chapter, however, we give a more direct proof which is analogous to the ones known in continuous problems ([16, 20]). We focus on SBCF here, leaving the topic of LBCF in Chapter 2.

The remainder of this chapter is organized as follows. In Section 1.2, we introduce our notation and symbols, and then review the results of the continuous problems described in [20]. In particular, the weak formulation of the original Stokes problem by a variational inequality is important. We also present the discretized variational inequality, which we are going to analyze, together with our main results. In Section 1.3, we prepare the finite element framework using the P2/P1 element, and state several technical lemmas.

Section 1.4 is devoted to the study of approximate problems for SBCF. We prove that the discretized variational inequality admits a unique solution and that it can be equivalently rewritten as a variational equation. In the error analysis, we first derive a primitive result of the convergence rate $O(h^{\min\{\epsilon,1/4\}})$ under the $H^{1+\epsilon}$-H^{ϵ} regularity assumption with $0 < \epsilon \leq 2$. Second we show that it is improved to $O(h^{\min\{\epsilon,1\}})$ under additional hypotheses of good behavior of the sign of the tangential velocity component on the boundary where SBCF is imposed. A sufficient condition to obtain $O(h^{\epsilon})$, which is of optimal order when $\epsilon = 2$, is also considered. Finally, based on the variational equation formulation mentioned above, we propose an Uzawa-type algorithm to perform numerical computations. What differs from [19], [20], or [49] is that our Uzawa method applies to the discrete problem, not continuous one; thus we prove that the iterative solution converges to the solution of the discretized variational inequality used for the error estimates.

In Section 1.5, several numerical examples are provided to support our theory. We observe that the results of our computation capture the features of SBCF and that the numerically calculated errors decrease at the optimal order $O(h^2)$.

1.2 Settings and results of continuous problems

1.2.1 Basic notation

Let Ω be a *polygonal* domain in $\mathbf{R}^2$. Throughout this chapter, we are concerned with the Stokes equations written in a familiar form

$$-\nu\Delta u + \nabla p = f \quad \text{in} \quad \Omega, \qquad \operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{1.2.1}$$

where $\nu > 0$ is the viscosity constant; u, the velocity field; p, the pressure; and f, the external force. As for the boundary, we assume that $\Gamma := \partial\Omega$ is a union of two non-overlapping parts, that is,

$$\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset,$$

where Γ_0, Γ_1 are relatively nonempty open subsets of Γ. Moreover, $\overline{\Gamma}_1$ is assumed to coincide with *whole one side* of the polygon Ω for the sake of simplicity. Two endpoints of the line segment $\overline{\Gamma}_1$ are denoted by M_1 and M_{m+1}; the meaning of the subscripts is clarified in Section 1.3.1. We note that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \{M_1, M_{m+1}\}$.

We impose the adhesive boundary condition on Γ_0, namely,

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{1.2.2}$$

whereas on Γ_1, we impose the following nonlinear boundary condition:

$$u_n = 0, \qquad |\sigma_\tau| \le g, \qquad \sigma_\tau u_\tau + g|u_\tau| = 0, \qquad \text{on } \Gamma_1, \tag{1.2.3}$$

which is called the *slip boundary condition of friction type* (SBCF). The function g, called the *modulus of friction*, is assumed to be continuous on $\overline{\Gamma}_1$ and strictly positive on Γ_1.

Here, the definitions of the symbols appearing above are as follows:

$$\begin{aligned}
n = {}^t(n_1, n_2) &= \text{ outer unit normal defined a.e. on } \Gamma,\\
\tau = {}^t(n_2, -n_1) &= \text{ unit tangential vector defined a.e. on } \Gamma,\\
u_n = u \cdot n &= \text{ normal component of } u \text{ defined a.e. on } \Gamma,\\
u_\tau = u \cdot \tau &= \text{ tangential component of } u \text{ defined a.e. on } \Gamma,\\
e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) &= \text{ component of rate-of-strain tensor } (1 \le i, j \le 2),\\
T_{ij}(u,p) = -p\delta_{ij} + 2\nu e_{ij}(u) &= \text{ component of Cauchy stress tensor } (1 \le i, j \le 2),\\
\sigma(u,p) = \Big(\sum_{j=1}^{2} T_{ij}(u,p) n_j\Big)_{i=1,2} &= \text{ stress vector defined a.e. on } \Gamma,\\
\sigma_\tau = \sigma_\tau(u) = \sigma(u,p)\cdot\tau &= \text{ tangential component of stress vector.}
\end{aligned}$$

Remark 1.2.1. (i) Since Γ_1 is a segment, n and τ are constant vectors on Γ_1.

(ii) σ_τ does not depend on p, which is verified by a simple calculation.

Remark 1.2.2. Replacing n with τ and vice versa in SBCF, we obtain

$$u_\tau = 0, \qquad |\sigma_n| \le g, \qquad \sigma_n u_n + g|u_n| = 0, \quad \text{on } \Gamma_1, \tag{1.2.4}$$

where $\sigma_n = \sigma_n(u,p) = \sigma(u,p)\cdot n$ denotes the normal component of stress vector. This boundary condition is called the *leak boundary condition of friction type* (LBCF). Since σ_n depends on p unlike σ_τ, it is more complicated to treat LBCF than SBCF, especially in the point that we cannot ignore effects of an additive constant of the pressure. Those situations will be illustrated in Chapter 2.

1.2.2 Function spaces

We use the usual Lebesgue space $L^2(\Omega)$ and Sobolev spaces $H^r(\Omega) = W^{r,2}(\Omega)$ for a nonnegative integer r, together with their standard norms and semi-norms For a space of vector-valued functions, we write $L^2(\Omega)^2$, and so on.

$H^0(\Omega)$ is understood as $L^2(\Omega)$, and $H_0^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. We put

$$Q = L^2(\Omega) \qquad \text{and} \qquad \mathring{Q} = L_0^2(\Omega) = \left\{ \psi \in L^2(\Omega) \,\middle|\, \int_\Omega \psi \, dx = 0 \right\}.$$

$H^s(\Omega)$ is also defined for a non-integer $s > 0$ by the norm

$$\|\psi\|_{H^s(\Omega)} = \left(\|\psi\|_{H^r(\Omega)}^2 + \sum_{|\alpha|=r} \iint_{\Omega\times\Omega} \frac{|\partial^\alpha \psi(x) - \partial^\alpha \psi(y)|^2}{|x-y|^{2+2\theta}} \, dxdy \right)^{1/2},$$

where $\alpha \in \mathbf{N}^2$ is a multi-index and $s = r + \theta$, r and θ being the integer and decimal part of s respectively.

We also use the Lebesgue spaces $L^2(\Gamma)$, $L^\infty(\Gamma)$ and Sobolev spaces $H^s(\Gamma)$ for $s \geq 0$ defined on the boundary Γ. Similarly as before, $H^0(\Gamma)$ means $L^2(\Gamma)$, and $H_0^1(\Gamma_1)$ denotes the closure of $C_0^\infty(\Gamma_1)$ in $H^1(\Gamma_1)$. The norm $\|\eta\|_{H^s(\Gamma)}$ for a non-integer $s > 0$ is defined by

$$\|\eta\|_{H^s(\Gamma)} = \left(\|\eta\|_{H^r(\Gamma)}^2 + \sum_{|\alpha|=r} \iint_{\Gamma\times\Gamma} \frac{|\partial^\alpha \eta(x) - \partial^\alpha \eta(y)|^2}{|x-y|^{1+2\theta}} \, ds(x)ds(y) \right)^{1/2},$$

where ds denotes the surface element of Γ and r is the integer part of s, θ its decimal part. The usual trace operator defined from $H^s(\Omega)$ onto $H^{s-1/2}(\Gamma)$ is denoted by $\psi \mapsto \psi|_\Gamma$ for $s > 1/2$; however, we simply write ψ instead of $\psi|_\Gamma$ when there is no ambiguity. For a vector-valued function ϕ defined on Γ or Γ_1, we write ϕ_n and ϕ_τ to indicate $\phi \cdot n$ and $\phi \cdot \tau$, respectively. Since τ is a unit constant vector on Γ_1, we immediately have

$$\|\phi_\tau\|_{H^s(\Gamma_1)} = \|\phi_\tau \tau\|_{H^s(\Gamma_1)^2} = \|\phi\|_{H^s(\Gamma_1)^2} \qquad (\forall \phi \in H^s(\Gamma_1)^2,\ \phi_n = 0 \text{ on } \Gamma_1). \tag{1.2.5}$$

Furthermore, we need the so-called Lions-Magenes space $H_{00}^{1/2}(\Gamma_1)$ with its norm defined by

$$\|\eta\|_{H_{00}^{1/2}(\Gamma_1)} = \left(\|\eta\|_{H^{1/2}(\Gamma_1)}^2 + \int_{\Gamma_1} \frac{|\eta(x)|^2}{\rho(x)} \, ds \right)^{1/2},$$

where $\rho(x) = \mathrm{dist}(x, \{M_1, M_{m+1}\})$ is the distance from $x \in \Gamma_1$ to the extreme points of Γ_1 along Γ_1. This space is also obtained from the interpolation between $L^2(\Gamma_1)$ and $H_0^1(\Gamma_1)$ (see [50, Section I.11]), and is strictly contained in $H^{1/2}(\Gamma_1)$. It is known ([24, Theorem 1.5.2.3]) that the trace operator maps $\{\psi \in H^1(\Omega) \,|\, \psi = 0 \text{ on } \Gamma_0\}$ onto $H^{1/2}_{00}(\Gamma_1)$. The zero-extension of $\eta \in H^{1/2}_{00}(\Gamma_1)$ to Γ, denoted by $\tilde{\eta}$, belongs to $H^{1/2}(\Gamma)$, and hence it follows from the closed graph theorem that

$$\|\tilde{\eta}\|_{H^{1/2}(\Gamma)} \le C\|\eta\|_{H^{1/2}_{00}(\Gamma_1)} \qquad (\forall \eta \in H^{1/2}_{00}(\Gamma_1)), \tag{1.2.6}$$

with the constant C independent of η.

Now, for a space of velocities which corresponds to SBCF, we introduce

$$V_n = \{v \in H^1(\Omega)^2 \,|\, v = 0 \text{ on } \Gamma_0, \quad v_n = 0 \text{ on } \Gamma_1\}, \tag{1.2.7}$$

with its norm induced from $H^1(\Omega)^2$. The following trace and extension theorem is an easy consequence of [58].

Lemma 1.2.1. (i) *For every $v \in V_n$, it holds that*

$$v_\tau \in H^{1/2}_{00}(\Gamma_1) \qquad \text{and} \qquad \|v_\tau\|_{H^{1/2}_{00}(\Gamma_1)} \le C\|v\|_{H^1(\Omega)^2} \tag{1.2.8}$$

with the constant C independent of v.

(ii) *Every $\eta \in H^{1/2}_{00}(\Gamma_1)$ admits an extension $v \in V_n$ such that*

$$v_\tau = \eta \text{ on } \Gamma_1 \qquad \text{and} \qquad \|v\|_{H^1(\Omega)^2} \le C\|\eta\|_{H^{1/2}_{00}(\Gamma_1)}, \tag{1.2.9}$$

with the constant C independent of η.

Proof. (i) For $v \in V_n$, it follows from [58, Corollary 1.1(i)] that $v|_{\Gamma_1} \in H^{1/2}_{00}(\Gamma_1)^2$ and $\|v|_{\Gamma_1}\|_{H^{1/2}_{00}(\Gamma_1)^2} \le C\|v\|_{H^1(\Omega)^2}$. Since $v = v_\tau\tau$ on Γ_1, in view of (1.2.5) we obtain (1.2.8).

(ii) For $\eta \in H^{1/2}_{00}(\Gamma_1)$, we have $\eta\tau \in H^{1/2}_{00}(\Gamma_1)^2$. By [58, Corollary 1.1(ii)], there exists $v \in H^1(\Omega)^2$, $v|_{\Gamma_0} = 0$ such that $v = \eta\tau$ on Γ_1 and $\|v\|_{H^1(\Omega)^2} \le C\|\eta\tau\|_{H^{1/2}_{00}(\Gamma_1)^2} = C\|\eta\|_{H^{1/2}_{00}(\Gamma_1)}$. Therefore, $v \in V_n$ and we obtain (1.2.9). □

1.2.3 Bilinear forms and barrier terms of friction

Let us introduce

$$a(u, v) = 2\nu \sum_{i,j=1}^{2} \int_\Omega e_{ij}(u)e_{ij}(v)\,dx \qquad (u, v \in H^1(\Omega)^2), \tag{1.2.10}$$

$$b(v, q) = -\int_\Omega \operatorname{div} v\, q\,dx \qquad (v \in H^1(\Omega)^2,\ q \in L^2(\Omega)), \tag{1.2.11}$$

$$j(\eta) = \int_{\Gamma_1} g|\eta|\,ds \qquad (\eta \in L^2(\Gamma_1)). \tag{1.2.12}$$

The bilinear forms a and b are continuous with their operator norms $\|a\|$ and $\|b\|$, respectively, being bounded. As a readily obtainable consequence of Korn's inequality ([33, Lemma 6.2]), there exists a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2_{H^1(\Omega)^2} \qquad (\forall v \in H^1(\Omega)^2,\ v = 0 \text{ on } \Gamma_0). \tag{1.2.13}$$

This implies that a is coercive on V_n. The functional $j(v_\tau|_{\Gamma_1})$, denoted by $j(v_\tau)$ simply, is called the *barrier term of friction.* It is a continuous, positive, and positively homogeneous functional on V_n.

1.2.4 Redefinition of stress vector

For all $(u, p) \in H^{1+\epsilon}(\Omega)^2 \times H^\epsilon(\Omega)$ with $\epsilon > 1/2$ satisfying $\operatorname{div} u = 0$, we obtain the following Green's formula:

$$(-\nu\Delta u + \nabla p, v)_{L^2(\Omega)^2} = a(u, v) + b(v, p) - \int_\Gamma \sigma(u, p) \cdot v \, ds \qquad (\forall v \in H^1(\Omega)^2), \tag{1.2.14}$$

where the stress vector $\sigma(u, p)$ is defined in Section 1.2.1. In fact, the line integral over Γ appearing in the right-hand side is well-defined because $\sigma(u, p) \in H^{\epsilon-1/2}(\Gamma) \subset L^2(\Gamma)$. However, if we have only a lower regularity, say $(u, p) \in H^1(\Omega)^2 \times L^2(\Omega)$, then the definition of $\sigma(u, p)$ in Section 1.2.1 becomes ambiguous. We thus propose a redefinition of $\sigma(u, p)$ as a functional on $H^{1/2}(\Gamma)$, based on (1.2.14), as follows.

Definition 1.2.1. Let $(u, p) \in H^1(\Omega)^2 \times L^2(\Omega)$ with $\operatorname{div} u = 0$. Assume $-\nu\Delta u + \nabla p$ is represented by $f \in L^2(\Omega)^2$ in the distribution sense, that is, in view of (1.2.14),

$$a(u, v) + b(v, p) = (f, v)_{L^2(\Omega)^2} \qquad (\forall v \in H^1_0(\Omega)^2).$$

Then we define $\sigma(u, p) \in (H^{1/2}(\Gamma)^2)'$ by

$$\langle \sigma(u, p), v \rangle_{H^{\frac{1}{2}}(\Gamma)^2} = a(u, v) + b(v, p) - (f, v)_{L^2(\Omega)^2} \qquad (v \in H^1(\Omega)^2). \tag{1.2.15}$$

Here and hereafter, for a Banach space X, we denote the dual space of X by X' and the duality pairing between X and X' by $\langle \cdot, \cdot \rangle_X$.

Remark 1.2.3. The functional $\sigma(u, p)$ is well defined according to the trace theorem and the fact that the right-hand side of (1.2.15) vanishes if $v = 0$ on Γ, i.e. $v \in H^1_0(\Omega)^2$, by virtue of (1.2.14). In addition, this definition of $\sigma(u, p)$ agrees with the previous one if u and p are sufficiently smooth to belong to $H^{1+\epsilon}(\Omega)^2 \times H^\epsilon(\Omega)$ with $\epsilon > 1/2$.

In particular, we see that σ_τ is characterized in $H^{1/2}_{00}(\Gamma_1)'$ by

$$\langle \sigma_\tau, v_\tau \rangle_{H^{1/2}_{00}(\Gamma_1)} = a(u,v) + b(v,p) - (f,v)_{L^2(\Omega)^2} \qquad (v \in V_n),$$

in view of Lemma 1.2.1. This kind of redefinition of σ_τ is also used in [16, Equation (28)] or [20, Lemma 2.1].

1.2.5 Variational formulation to the Stokes problem with SBCF

Let us introduce weak formulations of (1.2.1)–(1.2.3). Herein we assume $f \in L^2(\Omega)^2$ and $g \in C^1(\overline{\Gamma}_1)$ with $g > 0$ on Γ_1. The first formulation, a detailed description of [16, Problem 3] or [20, Problem 1], is as follows:

Problem PDE. Find $(u,p) \in V_n \times \mathring{Q}$ such that $\sigma_\tau = \sigma_\tau(u)$ is well-defined in the sense of Definition 1.2.1 and (1.2.3) is satisfied, that is,

$$\begin{cases} a(u,v) + b(v,p) - (\sigma_\tau, v_\tau)_{L^2(\Gamma_1)} = (f,v)_{L^2(\Omega)^2} & (\forall v \in V_n), \quad (1.2.16) \\ b(u,q) = 0 & (\forall q \in \mathring{Q}), \quad (1.2.17) \\ \sigma_\tau / g \in L^\infty(\Gamma_1) \quad \text{and} \quad |\sigma_\tau| \le g \quad \text{a.e. on } \Gamma_1, & (1.2.18) \\ \sigma_\tau u_\tau + g|u_\tau| = 0 \quad \text{a.e. on } \Gamma_1. & (1.2.19) \end{cases}$$

Note that $\sigma_\tau \in L^2(\Gamma_1)$ follows from (1.2.18), and thus (1.2.16) makes sense.

Another formulation by a variational inequality is proposed in [20, Problem 2]:

Problem VI. Find $(u,p) \in V_n \times \mathring{Q}$ such that, for all $v \in V_n$ and $q \in \mathring{Q}$,

$$\begin{cases} a(u, v-u) + b(v-u, p) + j(v_\tau) - j(u_\tau) \ge (f, v-u)_{L^2(\Omega)^2}, & (1.2.20) \\ b(u,q) = 0. & (1.2.21) \end{cases}$$

The equivalence of Problems PDE and VI, and the existence and uniqueness results are proved in [20, Theorems 2.1 and 2.4]. Their results are collected as follows:

Theorem 1.2.1. (i) *Problems* PDE *and* VI *are equivalent in the sense that* $(u,p) \in V_n \times \mathring{Q}$ *solves Problem* PDE *if and only if it solves Problem* VI.

(ii) *Problem* VI *has a unique solution.*

Remark 1.2.4. In [20], another definition of $\sigma_\tau = \nu \frac{\partial u_\tau}{\partial n}$ is employed and it is supposed that Γ is smooth, with $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$. However, some slight modification, which is not essential, makes the proofs in [20] applicable to our own situation.

1.2.6 Our main results

In order to clarify our objective, we herein state our main theorem. For notations and precise definitions of them, see Sections 1.3 and 1.4. We propose the following variational inequality problem in order to approximate Problem VI:

Problem VI$_h$. Find $(u_h, p_h) \in V_{nh} \times \mathring{Q}_h$ such that, for all $v_h \in V_{nh}$ and $q_h \in \mathring{Q}_h$,

$$\begin{cases} a(u_h, v_h - u_h) + b(v_h - u_h, p_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) \geq (f, v_h - u_h)_{L^2(\Omega)^2}, \\ b(u_h, q_h) = 0. \end{cases}$$

Then we are going to prove:

Theorem 1.2.2. *There exists a unique solution (u_h, p_h) of Problem* VI$_h$. *Under the regularity assumption $(u, p) \in H^{1+\epsilon}(\Omega)^2 \times H^{\epsilon}(\Omega)$ with $\epsilon > 0$, the following error estimate holds:*

$$\|u_h - u\|_{H^1(\Omega)^2} + \|p_h - p\|_{L^2(\Omega)} \leq C h^{\min\{\epsilon, \frac{1}{4}\}}.$$

Here the constant C is independent of h.

The existence and uniqueness will be proved in Theorem 1.4.1, and then the error estimate will be established in Theorem 1.4.2. Under the additional assumptions on u_τ and $u_{h\tau}$ on Γ_1, the above convergence order can be improved; this will be shown in Theorem 1.4.3.

Remark 1.2.5. In our situation where Ω is a polygon and $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 \neq \emptyset$, it is no longer trivial to verify the regularity of (u, p) assumed in the above theorem. In fact, our setting on the boundary is beyond the scope of the known regularity theory of [57]. However, in this thesis we focus on finite element analysis, leaving problems concerning the regularity to a future work.

1.3 Finite element approximation

1.3.1 Triangulation

Let $\{\mathscr{T}_h\}_h$ be a regular family of triangulations of a polygon Ω, where h denotes the greatest length of the sides of the triangles. As usual, we assume that

- For all $T_1, T_2 \in \mathscr{T}_h$ such that $T_1 \neq T_2$, $T_1 \cap T_2$ is a side, a node, or $\emptyset$.

- $\bigcup_{T\in\mathscr{T}_h} T=\overline{\Omega}$, and the boundary vertices belong to Γ.
- Let R_T^1 and R_T^2 be the diameters of the inscribed and circumscribed circles of $T\in\mathscr{T}_h$ respectively. Then $\inf_{T\in\mathscr{T}_h,h>0} R_T^1/R_T^2>0$.
- Each triangle has at least one vertex that is not on Γ.

The one-dimensional meshes of Γ and $\overline{\Gamma}_1$ inherited from the triangulation $\mathscr{T}_h$ are denoted respectively by $\mathscr{E}_h$ and $\mathscr{E}_h|_{\overline{\Gamma}_1}$. For the sets of nodes, we use

$$\begin{aligned}
\Sigma_h' &= \text{set of all vertices of triangles in } \mathscr{T}_h,\\
\Sigma_h'' &= \text{set of all midpoints of sides of triangles in } \mathscr{T}_h,\\
\Sigma_h &= \Sigma_h'\cup\Sigma_h'',\\
\Gamma_{0,h} &= \overline{\Gamma}_0\cap\Sigma_h,\\
\Gamma_{1,h} &= \overline{\Gamma}_1\cap\Sigma_h=\{M_1,M_{3/2},M_2,\cdots,M_m,M_{m+1/2},M_{m+1}\},\\
\mathring{\Gamma}_{1,h} &= \Gamma_1\cap\Sigma_h=\Gamma_{1,h}\setminus\{M_1,M_{m+1}\}.
\end{aligned}$$

Here the subscripts of M_i's are numbered in such a way that

- M_i's, for $i=1,2,\cdots,m+1$, are all vertices of triangles in $\mathscr{T}_h$, which are located in $\overline{\Gamma}_1$ and are arranged in ascending order along $\overline{\Gamma}_1$.
- $M_{i+1/2}$ is the midpoint of M_i and M_{i+1} for $i=1,2,\cdots,m$.

In particular, $\Gamma_{0,h}\cap\Gamma_{1,h}=\overline{\Gamma}_0\cap\overline{\Gamma}_1=\{M_1,M_{m+1}\}$. We denote each side with endpoints M_i,M_{i+1} by $e_i=[M_i,M_{i+1}]$ and its length by $|e_i|=|M_iM_{i+1}|$, for $i=1,2,\cdots,m$.

1.3.2 Approximate function spaces

Hereafter, we denote various constants independent of h by C and those depending on h by $C(h)$, unless otherwise stated.

We employ the P2/P1 element, defining $V_h\subset H^1(\Omega)^2$ and $Q_h\subset Q=L^2(\Omega)$ by

$$\begin{aligned}
V_h &= \left\{v_h\in C^0(\overline{\Omega})^2 \,\middle|\, v_h|_T\in\mathscr{P}_2(T)^2 \quad (\forall T\in\mathscr{T}_h)\right\},\\
Q_h &= \left\{q_h\in C^0(\overline{\Omega}) \,\middle|\, q_h|_T\in\mathscr{P}_1(T) \quad (\forall T\in\mathscr{T}_h)\right\},
\end{aligned}$$

where $\mathscr{P}_k(T)$ denotes the set of all polynomial functions of degree k on T $(k=1,2)$. For $v_h\in V_h$, we let v_{hn} and $v_{h\tau}$ denote $(v_h\cdot n)|_{\Gamma_1}$ and $(v_h\cdot\tau)|_{\Gamma_1}$, respectively. They are piecewise quadratic polynomials on Γ_1 because n and τ are constant vectors on the line segment Γ_1. Now, to approximate V_n and $\mathring{Q}$, we introduce

$$V_{nh} = \left\{ v_h \in V_h \,\middle|\, v_h(M) = 0 \ (\forall M \in \Gamma_{0,h}), \quad v_{hn}(M) = 0 \ (\forall M \in \mathring{\Gamma}_{1,h}) \right\},$$
$$\mathring{Q}_h = Q_h \cap L_0^2(\Omega),$$

together with

$$\mathring{V}_h = V_h \cap H_0^1(\Omega)^2, \quad V_{nh,\sigma} = \left\{ v_h \in V_{nh} \,\middle|\, b(v_h, q_h) = 0 \quad (\forall q_h \in \mathring{Q}_h) \right\}.$$

By a simple observation, we see that $V_{nh} \subset V_n$, $Q_h \subset Q$, $\mathring{Q}_h \subset \mathring{Q}$, and $\mathring{V}_h = V_{nh} \cap H_0^1(\Omega)^2$. We also note that $v_h(M_1) = v_h(M_{m+1}) = 0$ if $v_h \in V_{nh}$, and thus $v_{h\tau} \in H_0^1(\Gamma_1)$.

The quadratic Lagrange interpolation operator $\mathcal{I}_h : C^0(\overline{\Omega})^2 \to V_h$ and L^2-projection operator $\Pi_h : Q \to Q_h$ are defined in the usual sense, that is,

$$\mathcal{I}_h v \in V_h \quad \text{and} \quad (\mathcal{I}_h v)(M) = v(M) \qquad (\forall v \in V, \forall M \in \Sigma_h),$$
$$\Pi_h q \in Q_h \quad \text{and} \quad \int_\Omega (q - \Pi_h q)\, q_h \, dx = 0 \qquad (\forall q \in Q, \forall q_h \in Q_h).$$

It is easy to verify that $\mathcal{I}_h v \in V_{nh}$ if $v \in V_n \cap C^0(\overline{\Omega})^2$ and that $\Pi_h q \in \mathring{Q}_h$ if $q \in \mathring{Q}$. The following results for the interpolation error are standard and will be used without special emphasis in our error analysis :

$$\|v - \mathcal{I}_h v\|_{H^1(\Omega)^2} \le C h^\epsilon \|v\|_{H^{1+\epsilon}(\Omega)^2} \qquad (\forall v \in H^{1+\epsilon}(\Omega)^2), \tag{1.3.1}$$
$$\|q - \Pi_h q\|_{L^2(\Omega)} \le C h^\epsilon \|q\|_{H^\epsilon(\Omega)} \qquad (\forall q \in H^\epsilon(\Omega)), \tag{1.3.2}$$

where $0 < \epsilon \le 2$ and the constant $C > 0$ depends only on Ω. Note that $H^{1+\epsilon}(\Omega)^2 \subset C^0(\overline{\Omega})^2$ by Sobolev's theorem. For (1.3.1) with $1 \le \epsilon \le 2$, see [21, Lemma A.2]; the case $0 < \epsilon < 1$ is proved in [14]. Estimate (1.3.2) is found in [9, Lemma 12.4.3]. The estimate on the boundary, together with the trace theorem, gives

$$\|v_\tau - (\mathcal{I}_h v)_\tau\|_{L^2(\Gamma_1)} \le C h^{1/2+\epsilon} \|v_\tau\|_{H^{1/2+\epsilon}(\Gamma_1)} \le C h^{1/2+\epsilon} \|v\|_{H^{1+\epsilon}(\Omega)^2} \tag{1.3.3}$$

for all $v \in V_n \cap H^{1+\epsilon}(\Omega)^2$.

For approximate functions defined on the boundary Γ_1, we define

$$\Lambda_h = \left\{ \mu_h \in C^0(\overline{\Gamma}_1) \,\middle|\, \mu_h|_e \in \mathscr{P}_2(e) \ (\forall e \in \mathscr{E}_h|_{\overline{\Gamma}_1}), \ \mu_h(M_1) = \mu_h(M_{m+1}) = 0 \right\},$$
$$\tilde{\Lambda}_h = \left\{ \mu_h \in \Lambda_h \,\middle|\, |\mu_h(M)| \le 1 \ (\forall M \in \mathring{\Gamma}_{1,h}) \right\}.$$

By a simple calculation, we find that (see also Lemma 1.3.1(i))

$$\Lambda_h = \{v_{h\tau} \mid v_h \in V_{nh}\} \subset H_0^1(\Gamma_1) \subset H_{00}^{1/2}(\Gamma_1).$$

The space Λ_h becomes a Hilbert space if we define its inner product by

$$(\lambda_h, \mu_h)_{\Lambda_h} = \frac{1}{6}\sum_{i=1}^{m} |e_i| \Big(g_i \lambda_{h,i}\mu_{h,i} + 4 g_{i+\frac{1}{2}} \lambda_{h,i+\frac{1}{2}} \mu_{h,i+\frac{1}{2}} + g_{i+1}\lambda_{h,i+1}\mu_{h,i+1} \Big) \qquad (\lambda_h, \mu_h \in \Lambda_h), \tag{1.3.4}$$

which approximates $\int_{\Gamma_1} g\lambda_h \mu_h \, ds$ by Simpson's formula. Here and in what follows, we occasionally write $g_i, \lambda_{h,i+\frac{1}{2}}, \cdots$ instead of $g(M_i), \lambda_h(M_{i+\frac{1}{2}}), \cdots$, and so on. Since g is assumed to be positive on Γ_1 (particularly, on $\mathring{\Gamma}_{1,h}$), $(\cdot,\cdot)_{\Lambda_h}$ is indeed positive definite. Let us denote the projection operator from the Hilbert space Λ_h onto its closed convex subset $\tilde{\Lambda}_h$ by $\mathrm{Proj}_{\tilde{\Lambda}_h}$. It is explicitly expressed as

$$\mathrm{Proj}_{\tilde{\Lambda}_h}(\mu_h)(M) = \begin{cases} +1 & \text{if } \mu_h(M) > 1 \\ \mu_h(M) & \text{if } |\mu_h(M)| \le 1 \\ -1 & \text{if } \mu_h(M) < -1 \end{cases} \qquad (\forall M \in \Gamma_{1,h}), \tag{1.3.5}$$

for each $\mu_h \in \Lambda_h$.

Finally, to approximate j given in (1.2.12), we introduce j_h as

$$j_h(\eta_h) = \frac{1}{6}\sum_{i=1}^{m} |e_i| \Big(g_i |\eta_{h,i}| + 4 g_{i+\frac{1}{2}} |\eta_{h,i+\frac{1}{2}}| + g_{i+1}|\eta_{h,i+1}| \Big) \qquad (\eta_h \in \Lambda_h), \tag{1.3.6}$$

again with Simpson's formula in mind. Clearly, j_h is a positive, continuous, and positively homogeneous functional defined on Λ_h. This definition of j_h is motivated by [23, Section IV.2.6] and [22, Section II.5.4].

1.3.3 Discrete extension theorems

First let us recall the well-known inf-sup condition ([21, Corollary II.4.1]):

$$\beta \|q_h\|_{L^2(\Omega)} \le \sup_{v_h \in \mathring{V}_h} \frac{b(v_h, q_h)}{\|v_h\|_{H^1(\Omega)^2}} \qquad (\forall q_h \in \mathring{Q}_h), \tag{1.3.7}$$

where $\beta > 0$ is independent of h. Since $\mathring{V}_h \subset V_{nh}$, it is immediate to see

$$\beta \|q_h\|_{L^2(\Omega)} \le \sup_{v_h \in V_{nh}} \frac{b(v_h, q_h)}{\|v_h\|_{H^1(\Omega)^2}} \qquad (\forall q_h \in \mathring{Q}_h). \tag{1.3.8}$$

The following lemma discusses discrete extensions of functions given on the boundary Γ_1 to those defined on the whole domain Ω. The second statement, a solenoidal extension theorem in the discrete sense, will be important in the proofs of Theorems 1.4.1 and 1.4.4.

Lemma 1.3.1. (i) *Every $\eta_h \in \Lambda_h$ admits an extension $u_h \in V_{nh}$ such that*

$$u_{h\tau} = \eta_h \text{ on } \Gamma_1 \quad \text{and} \quad \|u_h\|_{H^1(\Omega)^2} \leq C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}. \tag{1.3.9}$$

(ii) *We can choose u_h in* (i) *in such a way that $u_h \in V_{nh,\sigma}$.*

Proof. (i) For $\eta_h \in \Lambda_h \subset H^{1/2}_{00}(\Gamma_1)$ we have $\eta_h\tau \in H^{1/2}_{00}(\Gamma_1)^2$, whose zero-extension $\widetilde{\eta_h\tau}$ is a piecewise quadratic polynomial defined on Γ. Using a discrete lifting operator (see [8, Theorem 5.1]), we can find $u_h \in V_h$ such that $u_h = \widetilde{\eta_h\tau}$ on Γ and $\|u_h\|_{H^1(\Omega)^2} \leq C\|\widetilde{\eta_h\tau}\|_{H^{1/2}(\Gamma)^2}$. Therefore, $u_h \in V_{nh}$ and $u_{h\tau} = \eta_h$ on Γ_1. In view of (1.2.6), we conclude $\|u_h\|_{H^1(\Omega)^2} \leq C\|\eta_h\tau\|_{H^{1/2}_{00}(\Gamma_1)^2} = C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$.

(ii) For every $\eta_h \in \Lambda_h$, as a result of (i), there exists $\hat{u}_h \in V_{nh}$ such that $\hat{u}_{h\tau} = \eta_h$ on Γ_1 and

$$\|\hat{u}_h\|_{H^1(\Omega)^2} \leq C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}. \tag{1.3.10}$$

For such $\hat{u}_h$, by (1.3.7), we can find $(u_h^*, p_h^*) \in \mathring{V}_h \times \mathring{Q}_h$ satisfying the following discrete Stokes equations (cf. [21, Theorem II.1.1]):

$$\begin{cases} a(u_h^*, v_h) + b(v_h, p_h^*) = 0 & (\forall v_h \in \mathring{V}_h), \quad (1.3.11) \\ b(u_h^*, q_h) = -b(\hat{u}_h, q_h) = (\operatorname{div} \hat{u}_h, q_h)_{L^2(\Omega)} & (\forall q_h \in \mathring{Q}_h). \quad (1.3.12) \end{cases}$$

It also follows that (cf. [21, Remark II.1.3])

$$\|u_h^*\|_{H^1(\Omega)^2} \leq C\|\operatorname{div} \hat{u}_h\|_{L^2(\Omega)} \leq C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}, \tag{1.3.13}$$

where the last inequality results from (1.3.10). Now, choosing $u_h = u_h^* + \hat{u}_h \in V_{nh}$, we deduce that $u_h \in V_{nh,\sigma}$ from (1.3.12), that $u_{h\tau} = \hat{u}_{h\tau} = \eta_h$ because $u_h^* \in \mathring{V}_h$, and that $\|u_h\|_{H^1(\Omega)^2} \leq C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$ from (1.3.10) and (1.3.13). This completes the proof. □

1.3.4 Properties of $(\cdot, \cdot)_{\Lambda_h}$ and j_h

Let us establish several relationships between the inner product of Λ_h and the functional j_h, given by (1.3.4) and (1.3.6), respectively. Although some of them seem to be mentioned in [2, Section XI.4] or [23, Section IV.2], we describe them in detail to make our statements clear. We use the signature function $\mathrm{sgn}(x)$ defined by $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = 0$ if $x = 0$, $\mathrm{sgn}(x) = -1$ if $x < 0$.

Lemma 1.3.2. (i) *If $u_h \in V_{nh}$ and $\lambda_h \in \tilde{\Lambda}_h$, then $|(u_{h\tau}, \lambda_h)_{\Lambda_h}| \le j_h(u_{h\tau})$.*
(ii) *Under the assumptions of* (i)*, the following properties are equivalent:*

(a) $(u_{h\tau}, \lambda_h)_{\Lambda_h} \ge j_h(u_{h\tau})$.

(b) $(u_{h\tau}, \lambda_h)_{\Lambda_h} = j_h(u_{h\tau})$.

(c) $(u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} \le 0$ *for all* $\mu_h \in \tilde{\Lambda}_h$.

(d) *If* $M \in \mathring{\Gamma}_{1,h}$ *and* $u_{h\tau}(M) \neq 0$*, then* $\lambda_h(M) = \mathrm{sgn}(u_{h\tau}(M))$.

(e) $\lambda_h = \mathrm{Proj}_{\tilde{\Lambda}_h}(\lambda_h + \rho u_{h\tau})$ *for all* $\rho \ge 0$.

(iii) *When $\lambda_h \in \Lambda_h$, the following properties are equivalent:*

(a) $\lambda_h \in \tilde{\Lambda}_h$.

(b) $(\eta_h, \lambda_h)_{\Lambda_h} \le j_h(\eta_h)$ *for all* $\eta_h \in \Lambda_h$.

Proof. (i) This is obvious because $|\lambda_h(M)| \le 1$ for all $M \in \Gamma_{1,h}$ if $\lambda_h \in \tilde{\Lambda}_h$.

(ii) (a)⇒(b) Since we have already proved the converse inequality in (i), statement (b) immediately follows from (a).

(b)⇒(c) Let (b) be valid. From (i), it holds that

$$(u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} = (u_{h\tau}, \mu_h)_{\Lambda_h} - j_h(u_{h\tau}) \le 0 \qquad (\forall \mu_h \in \tilde{\Lambda}_h).$$

(c)⇒(d) Assume that (c) is valid and consider an arbitrary $M \in \mathring{\Gamma}_{1,h}$ such that $u_{h\tau}(M) \neq 0$. Let us define $\mu_h \in \tilde{\Lambda}_h$ by

$$\mu_h(N) = \begin{cases} \lambda_h(N) & \text{if} \quad N \in \mathring{\Gamma}_{1,h} \setminus \{M\} \\ \mathrm{sgn}(u_{h\tau}(M)) & \text{if} \quad N = M. \end{cases}$$

When $M \in \Sigma_h'$, we can write $M = M_i$ for some $1 < i < m+1$. Now, by assumption we have

$$(u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} = \frac{g(M)}{6}\Big(|e_{i-1}| + |e_i|\Big)\Big(|u_{h\tau}(M)| - \lambda_h(M) u_{h\tau}(M)\Big) \le 0.$$

This implies that $\lambda_h(M) = \mathrm{sgn}(u_{h\tau}(M))$ because $|\lambda_h(M)| \le 1$ and $u_{h\tau}(M) \ne 0$. Similarly, when $M \in \Sigma_h''$, we can write $M = M_{i+\frac{1}{2}}$ for some $1 \le i \le m$. Then, by assumption we obtain

$$(u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} = \frac{2}{3} g(M)|e_i| \Big(|u_{h\tau}(M)| - \lambda_h(M) u_{h\tau}(M) \Big) \le 0,$$

from which $\lambda_h(M) = \mathrm{sgn}(u_{h\tau}(M))$ follows.

(d)⇒(a) If (d) is true, then we see that

$$(u_{h\tau}, \lambda_h)_{\Lambda_h} = \frac{1}{6} \sum_{i=1}^{m} |e_i| \Big(g_i |u_{h\tau,i}| + 4 g_{i+\frac{1}{2}} |u_{h\tau,i+\frac{1}{2}}| + g_{i+1} |u_{h\tau,i+1}| \Big) = j_h(u_{h\tau}).$$

(c)⇔(e) This is a direct consequence of a general property of projection operators. In fact, we obtain

$$\begin{aligned} & (u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} \le 0 && (\forall \mu_h \in \tilde{\Lambda}_h) \\ \Longleftrightarrow\ & (\lambda_h + \rho u_{h\tau} - \lambda_h, \mu_h - \lambda_h)_{\Lambda_h} \le 0 && (\forall \mu_h \in \tilde{\Lambda}_h, \forall \rho \ge 0) \\ \Longleftrightarrow\ & \lambda_h = \mathrm{Proj}_{\tilde{\Lambda}_h}(\lambda_h + \rho u_{h\tau}) && (\forall \rho \ge 0). \end{aligned}$$

(iii) (a)⇒(b) This is already shown in (i).

(b)⇒(a) Let (b) be valid and consider an arbitrary $M \in \mathring{\Gamma}_{1,h}$. Define $\eta_h \in \Lambda_h$ by

$$\eta_h(N) = \begin{cases} 0 & \text{if } \ N \in \mathring{\Gamma}_{1,h} \setminus \{M\} \\ +1 \text{ or } -1 & \text{if } \ N = M. \end{cases}$$

When $M \in \Sigma_h'$, we can write $M = M_i$ for some $1 < i < m+1$. By assumption, we obtain $(\eta_h, \lambda_h)_{\Lambda_h} \le j_h(\eta_h)$, which leads to

$$\frac{1}{6} \Big(|e_{i-1}| + |e_i| \Big) g(M) \Big(\pm \lambda_h(M) - 1 \Big) \le 0.$$

This implies that $|\lambda_h(M)| \le 1$. We obtain the same result when $M \in \Sigma_h''$ in a similar way. Therefore, we conclude that $\lambda_h \in \tilde{\Lambda}_h$. This completes the proof. □

The following mesh-dependent inf-sup condition is important to deduce the unique existence of a Lagrange multiplier $\lambda_h \in \Lambda_h$, which appears in Section 1.4.

Lemma 1.3.3. *There exists a positive constant β_h depending on h such that*

$$\beta_h \|\eta_h\|_{\Lambda_h} \le \sup_{v_h \in V_{nh}} \frac{(v_{h\tau}, \eta_h)_{\Lambda_h}}{\|v_h\|_{H^1(\Omega)^2}} \qquad (\forall \eta_h \in \Lambda_h).$$

Proof. Because both $\|\cdot\|_{H_{00}^{1/2}(\Gamma_1)}$ and $\|\cdot\|_{\Lambda_h}$ are norms defined on Λ_h, which is of a finite dimension, they are equivalent. Hence there exists a constant $C(h)$ depending on h such that

$$\|\eta_h\|_{H_{00}^{1/2}(\Gamma_1)} \le C(h)\|\eta_h\|_{\Lambda_h} \qquad (\forall \eta_h \in \Lambda_h).$$

Now, we let $\eta_h \in \Lambda_h$ and choose $u_h \in V_{nh}$ satisfying (1.3.9). Then we have

$$\begin{aligned}\sup_{v_h \in V_{nh}} \frac{(v_{h\tau}, \eta_h)_{\Lambda_h}}{\|v_h\|_{H^1(\Omega)^2}} &\ge \frac{(u_{h\tau}, \eta_h)_{\Lambda_h}}{\|u_h\|_{H^1(\Omega)^2}} = \frac{\|\eta_h\|^2_{\Lambda_h}}{\|u_h\|_{H^1(\Omega)^2}} \ge C(h)\frac{\|\eta_h\|_{H_{00}^{1/2}(\Gamma_1)}}{\|u_h\|_{H^1(\Omega)^2}}\|\eta_h\|_{\Lambda_h} \\ &\ge C(h)\|\eta_h\|_{\Lambda_h}.\end{aligned}$$

This completes the proof. □

Remark 1.3.1. This inf-sup condition will be used only to derive the unique existence of a Lagrange multiplier λ_h in the proof of Theorem 1.4.1, where h is always fixed. We will not consider error estimates involving λ_h, and thus there occurs no problem in our theory even if $\beta_h \to 0$ as $h \to 0$.

1.3.5 Error between j and j_h

First we generalize [23, Lemma IV.1.3], where g is assumed to be constant, to the case of non-constant g.

Lemma 1.3.4. (i) *There hold*

$$j_h(\eta_h) \le C\|\eta_h\|_{L^2(\Gamma_1)} \qquad (\forall \eta_h \in \Lambda_h), \tag{1.3.14}$$

$$\|\eta_h\|_{\Lambda_h} \le C\|\eta_h\|_{L^2(\Gamma_1)} \qquad (\forall \eta_h \in \Lambda_h), \tag{1.3.15}$$

with the constant C depending only on g and Γ_1.

(ii) *If $g \in C^1(\overline{\Gamma}_1)$, then for all $0 \le s \le 1$, we have*

$$|j_h(\eta_h) - j(\eta_h)| \le Ch^s\|\eta_h\|_{H^s(\Gamma_1)} \qquad (\forall \eta_h \in \Lambda_h), \tag{1.3.16}$$

with the constant C depending only on g and Γ_1.

Proof. (i) Let $\eta_h \in \Lambda_h$. On each segment $e_i = [M_i, M_{i+1}]$, $i = 1, 2, \cdots, m$, we take two points $M_{i+\frac{1}{6}}$ and $M_{i+\frac{5}{6}}$ such that $|M_iM_{i+\frac{1}{6}}| = \frac{1}{6}|e_i|$ and $|M_iM_{i+\frac{5}{6}}| =$

$\frac{5}{6}|e_i|$, respectively. Let us define a piecewise constant function $r_h(g\eta_h)$ on $\overline{\Gamma}_1$ by

$$r_h(g\eta_h) = \sum_{i=1}^{m} \left\{ g_i \eta_{h,i} \chi_{[M_i, M_{i+\frac{1}{6}}]} + g_{i+\frac{1}{2}} \eta_{h,i+\frac{1}{2}} \chi_{[M_{i+\frac{1}{6}}, M_{i+\frac{5}{6}}]} + g_{i+1} \eta_{h,i+1} \chi_{[M_{i+\frac{5}{6}}, M_{i+1}]} \right\}, \tag{1.3.17}$$

where χ_A denotes the characteristic function of $A \subset \overline{\Gamma}_1$.

Then we have

$$j_h(\eta_h) = \int_{\Gamma_1} |r_h(g\eta_h)| \, ds \le |\Gamma_1|^{1/2} \|r_h(g\eta_h)\|_{L^2(\Gamma_1)}. \tag{1.3.18}$$

By direct computation, it follows that

$$\begin{aligned}
\|r_h(g\eta_h)\|^2_{L^2(\Gamma_1)} &= \sum_{i=1}^{m} \frac{|e_i|}{6} \left(g_i^2 \eta_{h,i}^2 + 4 g_{i+\frac{1}{2}}^2 \eta_{h,i+\frac{1}{2}}^2 + g_{i+1}^2 \eta_{h,i+1}^2 \right) \\
&\le \sup |g|^2 \sum_{i=1}^{m} \frac{|e_i|}{6} \left(\eta_{h,i}^2 + 4\eta_{h,i+\frac{1}{2}}^2 + \eta_{h,i+1}^2 \right) \\
&\le \frac{5}{2} \sup |g|^2 \sum_{i=1}^{m} \frac{|e_i|}{15} \left\{ 2(\eta_{h,i}^2 + \eta_{h,i+1}^2) + 8\eta_{h,i+\frac{1}{2}}^2 - \eta_{h,i}\,\eta_{h,i+1} + 2\eta_{h,i+\frac{1}{2}}(\eta_{h,i} + \eta_{h,i+1}) \right\} \\
&= \frac{5}{2} \sup |g|^2 \, \|\eta_h\|^2_{L^2(\Gamma_1)}.
\end{aligned} \tag{1.3.19}$$

Here we have used the inequality

$$x^2 + 4y^2 + z^2 \le 2(x^2 + z^2) + 8y^2 - xz + 2y(x + z) \tag{1.3.20}$$

to derive the third line. We conclude (1.3.14) from (1.3.18) and (1.3.19).

The estimate (1.3.15) follows similarly if we remark that

$$\|\eta_h\|^2_{\Lambda_h} = \sum_{i=1}^{m} \frac{|e_i|}{6} \left(g_i \eta_{h,i}^2 + 4 g_{i+\frac{1}{2}} \eta_{h,i+\frac{1}{2}}^2 + g_{i+1} \eta_{h,i+1}^2 \right).$$

(ii) Let $\eta_h \in \Lambda_h$. Then,

$$|j_h(\eta_h) - j(\eta_h)| \le \int_{\Gamma_1} |r_h(g\eta_h) - g\eta_h| \, ds \le |\Gamma_1|^{1/2} \|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)}. \tag{1.3.21}$$

It follows from the proof of (i) that

$$\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)} \le \left(\|r_h(g\eta_h)\|_{L^2(\Gamma_1)} + \|g\eta_h\|_{L^2(\Gamma_1)} \right) \le C \|\eta_h\|_{L^2(\Gamma_1)}. \tag{1.3.22}$$

Before giving an estimate of $\|r_h(g\eta_h)-g\eta_h\|_{L^2(\Gamma_1)}$ which involves $\|\eta_h\|_{H^1(\Gamma_1)}$, it should be noted that if $\phi \in \mathscr{P}_2(\mathbf{R})$ we have

$$\phi(x) - \phi(a) = (x-a)\phi'\left(\frac{a+x}{2}\right),$$

so that

$$\begin{aligned}\int_a^b |\phi(x)-\phi(a)|^2\,dx &= \int_a^b |x-a|^2 \left|\phi'\left(\frac{a+x}{2}\right)\right|^2 dx \\ &= 8\int_a^{(a+b)/2} |t-a|^2|\phi'(t)|^2\,dt \\ &\le 2(b-a)^2\int_a^b |\phi'(t)|^2\,dt. \qquad (1.3.23)\end{aligned}$$

In view of the Taylor expansion of g, we apply (1.3.23) to deduce

$$\begin{aligned}I_1(e_i) :=& \int_{M_i}^{M_{i+\frac{1}{6}}} |g_i\eta_{h,i} - g\eta_h|^2\,ds = \int_{M_i}^{M_{i+\frac{1}{6}}} |g_i\eta_{h,i} - g_i\eta_h + g_i\eta_h - g\eta_h|^2\,ds \\ \le& \, 2\int_{M_i}^{M_{i+\frac{1}{6}}} g_i^2(\eta_{h,i}-\eta_h)^2\,ds + 2\int_{M_i}^{M_{i+\frac{1}{6}}} (g_i-g)^2\eta_h^2\,ds \\ \le& \, 2\sup|g|^2\cdot 2\left(\frac{|e_i|}{6}\right)^2 \int_{M_i}^{M_{i+\frac{1}{6}}} |\eta_h'|^2\,ds + 2\left(\frac{|e_i|}{6}\sup|g'|\right)^2 \int_{M_i}^{M_{i+\frac{1}{6}}} |\eta_h|^2\,ds \\ \le& \max\left\{\frac{\sup|g|^2}{9}, \frac{\sup|g'|^2}{18}\right\} |e_i|^2\|\eta_h\|^2_{H^1([M_i,M_{i+\frac{1}{6}}])} \\ \le& \, Ch^2\|\eta_h\|^2_{H^1(e_i)}, \qquad (1.3.24)\end{aligned}$$

for $i = 1, 2, \cdots, m$. By a similar discussion, we have

$$I_2(e_i) := \int_{M_{i+\frac{1}{6}}}^{M_{i+\frac{1}{2}}} |g_{i+\frac{1}{2}}\eta_{h,i+\frac{1}{2}} - g\eta_h|^2\,ds \le Ch^2\|\eta_h\|^2_{H^1(e_i)}, \qquad (1.3.25)$$

$$I_3(e_i) := \int_{M_{i+\frac{1}{2}}}^{M_{i+\frac{5}{6}}} |g_{i+\frac{1}{2}}\eta_{h,i+\frac{1}{2}} - g\eta_h|^2\,ds \le Ch^2\|\eta_h\|^2_{H^1(e_i)}, \qquad (1.3.26)$$

$$I_4(e_i) := \int_{M_{i+\frac{5}{6}}}^{M_{i+1}} |g_{i+1}\eta_{h,i+1} - g\eta_h|^2\,ds \le Ch^2\|\eta_h\|^2_{H^1(e_i)}, \qquad (1.3.27)$$

for each i. Therefore, collecting (1.3.24)–(1.3.27), we obtain

$$\|r_h(g\eta_h) - g\eta_h\|^2_{L^2(\Gamma_1)} = C\sum_{i=1}^{m}\Big(I_1(e_i) + I_2(e_i) + I_3(e_i) + I_4(e_i)\Big)$$
$$\leq Ch^2\sum_{i=1}^{m}\|\eta_h\|^2_{H^1(e_i)} = Ch^2\|\eta_h\|^2_{H^1(\Gamma_1)},$$

so that

$$\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)} \leq Ch\|\eta_h\|_{H^1(\Gamma_1)}. \tag{1.3.28}$$

As a consequence of (1.3.21), (1.3.22), and (1.3.28), the desired inequality (1.3.16) follows from Hilbertian interpolation between $L^2(\Gamma_1)$ and $H^1(\Gamma_1)$. □

As will be shown in Theorem 1.4.2 below, the leading term of the error is that between j_h and j, which is estimated by (1.3.16) with $s = 1/2$. However, under some additional conditions, we can obtain a sharper estimate than (1.3.16).

Definition 1.3.1. An element $\eta_h \in \Lambda_h$ is said to *have a constant sign on every side* if, for any $i = 1, 2, \cdots, m$, either of the following conditions is satisfied:

$$\text{(a) } \eta_h|_{e_i} \geq 0 \qquad \text{or} \qquad \text{(b) } \eta_h|_{e_i} \leq 0.$$

Remark 1.3.2. Let $\eta_h \in \Lambda_h$ have a constant sign on every side. If $\eta_h \geq 0$ on e_{i-1} and $\eta_h \leq 0$ on e_i for some i, then $\eta_h(M_i) = 0$.

Lemma 1.3.5. *Let $g \in C^2(\overline{\Gamma}_1)$. If $\eta_h \in \Lambda_h$ has a constant sign on every side, then*

$$|j_h(\eta_h) - j(\eta_h)| \leq Ch^2\|\eta_h\|_{L^2(\Gamma_1)}.$$

Moreover, if g is a polynomial of degree ≤ 1, then $j_h(\eta_h)$ is exact, that is,

$$j_h(\eta_h) = j(\eta_h). \tag{1.3.29}$$

Proof. Let $\eta_h \in \Lambda_h$ have a constant sign on every side. Because $\eta_h \geq 0$ or $\eta_h \leq 0$ on e_i for each $i = 1, 2, \cdots, m$ and g is positive on Γ_1, we have

$$\int_{e_i} |r_h(g\eta_h)|\, ds = \left|\int_{e_i} r_h(g\eta_h)\, ds\right|,$$
$$\int_{e_i} g|\eta_h|\, ds = \left|\int_{e_i} g\eta_h\, ds\right|,$$

where $r_h(g\eta_h)$ is defined as (1.3.17). Summing up these terms, we obtain

$$j_h(\eta_h) = \int_{\Gamma_1} |r_h(g\eta_h)|\, ds = \sum_{i=1}^{m} \int_{e_i} |r_h(g\eta_h)|\, ds = \sum_{i=1}^{m} \left| \int_{e_i} r_h(g\eta_h)\, ds \right|,$$

$$j(\eta_h) = \int_{\Gamma_1} g|\eta_h|\, ds = \sum_{i=1}^{m} \int_{e_i} g|\eta_h|\, ds = \sum_{i=1}^{m} \left| \int_{e_i} g\eta_h\, ds \right|.$$

Consequently, it follows that

$$|j_h(\eta_h) - j(\eta_h)| \le \sum_{i=1}^{m} \left| \int_{e_i} \big(r_h(g\eta_h) - g\eta_h\big) ds \right|. \tag{1.3.30}$$

Let g_h denote the linear Lagrange interpolation of g using the nodes in $\Sigma'_h \cap \Gamma_{1,h}$. Namely, g_h is continuous on $\overline{\Gamma}_1$ and affine on each side $e_i = [M_i, M_{i+1}]$, satisfying $g_h(M_i) = g(M_i)$ for $i = 1, 2, \cdots, m$. Then the Taylor expansion of g implies

$$|g_h(x) - g(x)| \le \frac{h^2}{8} \sup |g''| \qquad (\forall x \in \overline{\Gamma}_1). \tag{1.3.31}$$

Now, let us estimate each term appearing in the summation on the right-hand side of (1.3.30) by

$$\left| \int_{e_i} \big(r_h(g\eta_h) - g_h\eta_h\big) ds \right| + \int_{e_i} |g_h - g||\eta_h|\, ds. \tag{1.3.32}$$

Since Simpson's formula is exact for cubic polynomials, we can express

$$\begin{aligned}\int_{e_i} g_h\eta_h\, ds &= \frac{|e_i|}{6} \sum_{i=1}^{m} \Big(g_{h,i}\eta_{h,i} + 4g_{h,i+\frac{1}{2}}\eta_{h,i+\frac{1}{2}} + g_{h,i+1}\eta_{h,i+1} \Big) \\ &= \int_{e_i} r_h(g\eta_h)\, ds + \frac{2|e_i|}{3} (g_{h,i+\frac{1}{2}} - g_{i+\frac{1}{2}})\eta_{h,i+\frac{1}{2}}.\end{aligned}$$

Thus, due to (1.3.31), the first term of (1.3.32) is bounded from above by

$$\frac{1}{12} |e_i| h^2 \sup |g''|\, |\eta_{h,i+\frac{1}{2}}|.$$

Note that there holds (cf. (1.3.20))

$$\begin{aligned}&|\eta_{h,i+\frac{1}{2}}|^2 |e_i| \\ \le& \frac{15}{4} \cdot \frac{|e_i|}{15} \Big\{ 2(\eta_{h,i}^2 + \eta_{h,i+1}^2) + 8\eta_{h,i+\frac{1}{2}}^2 - \eta_{h,i}\,\eta_{h,i+1} + 2\eta_{h,i+\frac{1}{2}}(\eta_{h,i} + \eta_{h,i+1}) \Big\} \\ =& \frac{15}{4} \|\eta_h\|_{L^2(e_i)}^2 \le 4\|\eta_h\|_{L^2(e_i)}^2\end{aligned}$$

for $i = 1, 2, \cdots, m$. Then, the sum of the first term of (1.3.32) is estimated as

$$\begin{aligned}
\sum_{i=1}^{m} \left| \int_{e_i} (r_h(g\eta_h) - g_h\eta_h) ds \right| &\le \frac{1}{12} h^2 \sup |g''| \sum_{i=1}^{m} |\eta_{h,i+\frac{1}{2}}| \, |e_i| \\
&\le \frac{1}{12} h^2 \sup |g''| \left(\sum_{i=1}^{m} |\eta_{h,i+\frac{1}{2}}|^2 |e_i| \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} |e_i| \right)^{\frac{1}{2}} \\
&\le \frac{1}{12} h^2 \sup |g''| \cdot 2 \|\eta_h\|_{L^2(\Gamma_1)} |\Gamma_1|^{1/2} \\
&= Ch^2 \|\eta_h\|_{L^2(\Gamma_1)}. \qquad (1.3.33)
\end{aligned}$$

Next, the second term of (1.3.32) is estimated by $\frac{1}{8} h^2 \sup |g''| \int_{e_i} |\eta_h| \, ds$, which gives

$$\begin{aligned}
\sum_{i=1}^{m} \int_{e_i} |g_h - g| \, |\eta_h| \, ds \le \frac{1}{8} h^2 \sup |g''| \int_{\Gamma_1} |\eta_h| \, ds &\le \frac{1}{8} h^2 \sup |g''| \, \|\eta_h\|_{L^2(\Gamma_1)} |\Gamma_1|^{1/2} \\
&= Ch^2 \|\eta_h\|_{L^2(\Gamma_1)}. \qquad (1.3.34)
\end{aligned}$$

Hence we conclude from (1.3.30), (1.3.33), and (1.3.34) that

$$|j_h(\eta_h) - j(\eta_h)| \le Ch^2 \|\eta_h\|_{L^2(\Gamma_1)}.$$

If g is a polynomial of degree ≤ 1, then both terms of (1.3.32) vanish because $g_h = g$, from which (1.3.29) follows. This completes the proof. □

1.4 Discretization of the Stokes problem with SBCF

1.4.1 Existence and uniqueness results

We propose approximate problems for Problem VI as follows (The first one is already mentioned in Section 1.2.6.)

Problem VI$_h$. Find $(u_h, p_h) \in V_{nh} \times \mathring{Q}_h$ such that, for all $v_h \in V_{nh}$ and $q_h \in \mathring{Q}_h$,

$$\begin{cases} a(u_h, v_h - u_h) + b(v_h - u_h, p_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) \ge (f, v_h - u_h)_{L^2(\Omega)^2}, & (1.4.1) \\ b(u_h, q_h) = 0. & (1.4.2) \end{cases}$$

Problem VI$_{h,\sigma}$. Find $u_h \in V_{nh,\sigma}$ such that

$$a(u_h, v_h - u_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) \ge (f, v_h - u_h)_{L^2(\Omega)^2} \qquad (\forall v_h \in V_{nh,\sigma}). \qquad (1.4.3)$$

Problem VE$_h$. Find $(u_h, p_h, \lambda_h) \in V_{nh} \times \mathring{Q}_h \times \tilde{\Lambda}_h$ such that, for all $(v_h, q_h, \mu_h) \in V_{nh} \times \mathring{Q}_h \times \tilde{\Lambda}_h$,

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) + (v_{h\tau}, \lambda_h)_{\Lambda_h} = (f, v_h)_{L^2(\Omega)^2}, & (1.4.4) \\ b(u_h, q_h) = 0, & (1.4.5) \\ (u_{h\tau}, \mu_h - \lambda_h)_{\Lambda_h} \leq 0. & (1.4.6) \end{cases}$$

Recall that we are assuming $f \in L^2(\Omega)^2$ and $g \in C^0(\overline{\Gamma}_1)$. We first establish the existence and uniqueness of these approximate problems.

Theorem 1.4.1. (i) *Problem* $\mathrm{VI}_{h,\sigma}$ *admits a unique solution* $u_h \in V_{nh,\sigma}$. *Furthermore, it satisfies the following equation:*

$$a(u_h, u_h) + j_h(u_{h\tau}) = (f, u_h)_{L^2(\Omega)^2}. \tag{1.4.7}$$

(ii) *Problems* $\mathrm{VI}_{h,\sigma}$, VI_h, *and* VE_h *are equivalent in the following sense.*

(a) *If* $u_h \in V_{nh,\sigma}$ *is a solution of Problem* $\mathrm{VI}_{h,\sigma}$, *then there exists a unique* $p_h \in \mathring{Q}_h$ *such that* (u_h, p_h) *solves Problem* VI_h.

(b) *If* $(u_h, p_h) \in V_{nh} \times \mathring{Q}_h$ *is a solution of Problem* VI_h, *then there exists a unique* $\lambda_h \in \tilde{\Lambda}_h$ *such that* (u_h, p_h, λ_h) *solves Problem* VE_h.

(c) *If* $(u_h, p_h, \lambda_h) \in V_{nh} \times \mathring{Q}_h \times \tilde{\Lambda}_h$ *is a solution of Problem* VE_h, *then* u_h *solves Problem* $\mathrm{VI}_{h,\sigma}$.

Proof. (i) Since the bilinear form a is coercive on V_{nh} and the functional $j_h : V_{nh} \to \mathbf{R}$ is convex, proper, and lower semi-continuous (actually, continuous) with respect to the weak topology, we can apply to Problem $\mathrm{VI}_{h,\sigma}$ a classical existence and uniqueness theorem for second-order elliptic variational inequalities (see [22, Theorem I.4.1]). Thus, there exists a unique $u_h \in V_{nh,\sigma}$ such that (1.4.3) holds. Equation (1.4.7) follows from (1.4.3) with $v_h = 0$ and $2u_h$.

(ii) (a) Let $u_h \in V_{nh,\sigma}$ be a solution of Problem $\mathrm{VI}_{h,\sigma}$. Taking $u_h \pm v_h$ as a test function in (1.4.3), with an arbitrary $v_h \in \mathring{V}_h \cap V_{nh,\sigma}$, we obtain

$$a(u_h, v_h) = (f, v_h)_{L^2(\Omega)^2} \qquad (\forall v_h \in \mathring{V}_h \cap V_{nh,\sigma}).$$

Moreover, we deduce from (1.3.7) the unique existence of $p_h \in \mathring{Q}_h$ such that

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h)_{L^2(\Omega)^2} \qquad (\forall v_h \in \mathring{V}_h) \tag{1.4.8}$$

by a standard argument.

Now we let $v_h \in V_{nh}$ be arbitrary. It follows from Lemma 1.3.1(ii) that there exists some $w_h \in V_{nh,\sigma}$ such that $w_h = v_h$ on Γ, which implies

$$v_h - w_h \in \mathring{V}_h \qquad \text{and} \qquad j_h(v_{h\tau}) = j_h(w_{h\tau}). \tag{1.4.9}$$

Since $u_h, w_h \in V_{nh,\sigma}$, we conclude from (1.4.3), (1.4.8), and (1.4.9) that

$$\begin{aligned} & a(u_h, v_h - u_h) + b(v_h - u_h, p_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) - (f, v_h - u_h)_{L^2(\Omega)^2} \\ =& a(u_h, v_h - w_h) + b(v_h - w_h, p_h) - (f, v_h - w_h)_{L^2(\Omega)^2} \\ & \qquad + a(u_h, w_h - u_h) + j_h(w_{h\tau}) - j_h(u_{h\tau}) - (f, w_h - u_h)_{L^2(\Omega)^2} \\ \geq& 0. \end{aligned}$$

Hence (u_h, p_h) is a solution of VI$_h$.

(b) Let $(u_h, p_h) \in V_{nh} \times \mathring{Q}_h$ be a solution of VI$_h$. Taking $u_h \pm v_h$ as a test function in (1.4.1), with an arbitrary $v_h \in \mathring{V}_h$, we have

$$a(u_h, v_h) + b(v_h, p_h) = (f, v_h)_{L^2(\Omega)^2} \qquad (\forall v_h \in \mathring{V}_h). \tag{1.4.10}$$

Therefore, since $\{v_h \in V_{nh} \,|\, (v_{h\tau}, \eta_h)_{\Lambda_h} = 0 \ (\forall \eta_h \in \Lambda_h)\} = \mathring{V}_h$, the inf-sup condition given in Lemma 1.3.4 asserts the unique existence of $\lambda_h \in \Lambda_h$ (for example, see [21, Lemma I.4.1]) such that

$$a(u_h, v_h) + b(v_h, p_h) + (v_{h\tau}, \lambda_h)_{\Lambda_h} = (f, v_h)_{L^2(\Omega)^2} \qquad (\forall v_h \in V_{nh}). \tag{1.4.11}$$

Combining (1.4.11) with (1.4.1), we obtain

$$(v_{h\tau} - u_{h\tau}, \lambda_h)_{\Lambda_h} \leq j_h(v_{h\tau}) - j_h(u_{h\tau}) \qquad (\forall v_h \in V_{nh}), \tag{1.4.12}$$

which gives, by a triangle inequality, that

$$(v_{h\tau} - u_{h\tau}, \lambda_h)_{\Lambda_h} \leq j_h(v_{h\tau} - u_{h\tau}) \qquad (\forall v_h \in V_{nh}). \tag{1.4.13}$$

From (1.4.13) together with Lemma 1.3.1(i), we deduce

$$(\eta_h, \lambda_h)_{\Lambda_h} \leq j_h(\eta_h) \qquad (\forall \eta_h \in \Lambda_h). \tag{1.4.14}$$

Hence Lemma 1.3.2(iii) implies that $\lambda_h \in \tilde{\Lambda}_h$, and (1.4.4) is established. It remains only to prove (1.4.6). Taking $v_h = 0$ in (1.4.12), we have $j_h(u_{h\tau}) \leq (u_{h\tau}, \lambda_h)_{\Lambda_h}$. This implies (1.4.6) by Lemma 1.3.2(ii). Therefore, (u_h, p_h, λ_h) is a solution of Problem VE$_h$.

(c) Let $(u_h, p_h, \lambda_h) \in V_{nh} \times \mathring{Q}_h \times \tilde{\Lambda}_h$ be a solution of Problem VE$_h$. Then we see that $u_h \in V_{nh,\sigma}$ from (1.4.5), and that

$$(u_{h\tau}, \lambda_h)_{\Lambda_h} = j_h(u_{h\tau}) \tag{1.4.15}$$

from (1.4.6) combined with Lemma 1.3.2(ii). It follows from (1.4.4), (1.4.5), and Lemma 1.3.2(i) that

$$\begin{aligned} &a(u_h, v_h - u_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) - (f, v_h - u_h)_{L^2(\Omega)^2} \\ =& - b(v_h - u_h, p_h) - (v_{h\tau} - u_{h\tau}, \lambda_h)_{\Lambda_h} + j_h(v_{h\tau}) - j_h(u_{h\tau}) \\ =& j_h(v_h) - (v_{h\tau}, \lambda_h)_{\Lambda_h} \geq 0, \end{aligned}$$

for all $v_h \in V_{nh,\sigma}$. Hence u_h is a solution of Problem VI$_{h,\sigma}$. This completes the proof of Theorem 1.4.1. □

1.4.2 Error analysis

Before presenting the rate-of-convergence results, we state the following:

Proposition 1.4.1. *Let (u, p) and (u_h, p_h) be the solutions of Problems* VI *and* VI$_h$ *respectively. Then,*

(i) *it holds that*

$$\|u_h\|_{H^1(\Omega)^2} \leq \|f\|_{L^2(\Omega)^2}/\alpha. \tag{1.4.16}$$

(ii) *for every $v_h \in V_{nh}$ and $q_h \in \mathring{Q}_h$, it holds that*

$$\begin{aligned} \alpha\|u - u_h\|^2_{H^1(\Omega)^2} \leq\ & a(u - u_h, u - v_h) + b(u_h - u, p - q_h) + b(v_h - u, p_h - p) \\ &+ (\sigma_\tau, v_{h\tau} - u_\tau)_{L^2(\Gamma_1)} + j(u_{h\tau}) - j_h(u_{h\tau}) + j_h(v_{h\tau}) - j(u_\tau). \end{aligned} \tag{1.4.17}$$

(iii) *for every $q_h \in \mathring{Q}_h$, it holds that*

$$\|p - p_h\|_{L^2(\Omega)^2} \leq \left(1 + \frac{\|b\|}{\beta}\right) \|p - q_h\|_{L^2(\Omega)^2} + \frac{\|a\|}{\beta} \|u - u_h\|_{H^1(\Omega)^2}. \tag{1.4.18}$$

Proof. (i) Since u_h is the solution of Problem VI$_{h,\sigma}$ by Theorem 1.4.1(ii), it satisfies (1.4.7). Hence Korn's inequality (1.2.13), together with the positiveness of j_h, gives

$$\alpha\|u_h\|^2_{H^1(\Omega)^2} \leq a(u_h, u_h) = (f, u_h)_{L^2(\Omega)^2} - j_h(u_{h\tau}) \leq \|f\|_{L^2(\Omega)^2}\|u_h\|_{L^2(\Omega)^2},$$

which implies (1.4.16).

(ii) Let $v_h \in V_{nh}$ and $q_h \in \mathring{Q}_h$ be arbitrary. We begin with the following equality:

$$a(u-u_h, u-u_h) = a(u-u_h, u-v_h) - a(u, u_h-u) - a(u_h, v_h-u_h) + a(u, v_h-u).$$

Majorize the second term of the right-hand side by (1.2.20) with $v = u_h$, the third one by (1.4.1) with v_h itself, and rewrite the fourth one by (1.2.16) with $v = v_h - u$. Then we have

$$\begin{aligned}
&a(u-u_h, u-u_h)\\
\le\, &a(u-u_h, u-v_h) + b(u_h-u, p) + j(u_{h\tau}) - j(u_\tau) - (f, u_h-u)_{L^2(\Omega)^2}\\
&\qquad + b(v_h-u_h, p_h) + j_h(v_{h\tau}) - j_h(u_{h\tau}) - (f, v_h-u_h)_{L^2(\Omega)^2}\\
&\qquad - b(v_h-u, p) + (\sigma_\tau, v_{h\tau}-u_\tau)_{L^2(\Gamma_1)} + (f, v_h-u)_{L^2(\Omega)^2}\\
=\, &a(u-u_h, u-v_h) + b(u_h-u, p-q_h) + b(v_h-u, p_h-p)\\
&\quad + (\sigma_\tau, v_{h\tau}-u_\tau)_{L^2(\Gamma_1)} + j(u_{h\tau}) - j_h(u_{h\tau}) + j_h(v_{h\tau}) - j(u_\tau).
\end{aligned}$$

Combining this with Korn's inequality (1.2.13), we conclude (1.4.17).

(iii) Taking $u \pm v$ as a test function in (1.2.20), with an arbitrary $v \in H_0^1(\Omega)^2$, gives

$$a(u, v) + b(v, p) = (f, v)_{L^2(\Omega)^2} \qquad (\forall v \in H_0^1(\Omega)^2).$$

On the other hand we know that (1.4.10) holds, and therefore, by subtraction we obtain

$$a(u-u_h, v_h) + b(v_h, p-p_h) = 0 \qquad (\forall v_h \in \mathring{V}_h). \tag{1.4.19}$$

Now let $q_h \in \mathring{Q}_h$. It is clear that

$$\|p-p_h\|_{L^2(\Omega)} \le \|p-q_h\|_{L^2(\Omega)} + \|q_h-p_h\|_{L^2(\Omega)}. \tag{1.4.20}$$

From (1.3.7) and (1.4.19) we have

$$\begin{aligned}
\beta\|q_h-p_h\|_{L^2(\Omega)} &\le \sup_{v_h\in\mathring{V}_h} \frac{b(v_h, q_h-p_h)}{\|v_h\|_{H^1(\Omega)^2}} = \sup_{v_h\in\mathring{V}_h} \frac{b(v_h, q_h-p) + b(v_h, p-p_h)}{\|v_h\|_{H^1(\Omega)^2}}\\
&= \sup_{v_h\in\mathring{V}_h} \frac{b(v_h, q_h-p) - a(u-u_h, v_h)}{\|v_h\|_{H^1(\Omega)^2}}\\
&\le \|b\|\,\|p-q_h\|_{L^2(\Omega)} + \|a\|\,\|u-u_h\|_{H^1(\Omega)^2}.
\end{aligned} \tag{1.4.21}$$

The desired inequality (1.4.18) follows from (1.4.20) and (1.4.21). $\square$

We are now in a position to state the primary result of our error estimates, assuming only the regularity of the exact solution.

Theorem 1.4.2. *Let (u,p) be the solution of Problem* VI *and (u_h,p_h) be that of Problem* VI$_h$ *for $0<h<1$. Suppose $g\in C^1(\overline{\Gamma}_1)$ and $(u,p)\in H^{1+\epsilon}(\Omega)^2\times H^{\epsilon}(\Omega)$ with $0<\epsilon\le 2$. Then we have*

$$\|u-u_h\|_{H^1(\Omega)^2}+\|p-p_h\|_{L^2(\Omega)}\le Ch^{\min\{\epsilon,\frac{1}{4}\}}. \tag{1.4.22}$$

Proof. We recall the interpolation error estimates (1.3.1)–(1.3.3).

Taking $v_h=\mathcal{I}_hu, q_h=\Pi_hp$ in (1.4.17) and (1.4.18), we find that

$$\begin{aligned}\alpha\|u-u_h\|^2_{H^1(\Omega)^2}\le{}& a(u-u_h,u-\mathcal{I}_hu)+b(u_h-u,p-\Pi_hp)+b(\mathcal{I}_hu-u,p_h-p)\\&+(\sigma_\tau,(\mathcal{I}_hu)_\tau-u_\tau)_{L^2(\Gamma_1)}+|j(u_{h\tau})-j_h(u_{h\tau})|\\&+|j_h((\mathcal{I}_hu)_\tau)-j((\mathcal{I}_hu)_\tau)|+|j((\mathcal{I}_hu)_\tau)-j(u_\tau)|,\end{aligned} \tag{1.4.23}$$

and that

$$\begin{aligned}\|p-p_h\|_{L^2(\Omega)}&\le C\left(\|p-\Pi_hp\|_{L^2(\Omega)}+\|u-u_h\|_{H^1(\Omega)^2}\right)\\&\le C(h^\epsilon+\|u-u_h\|_{H^1(\Omega)^2}).\end{aligned} \tag{1.4.24}$$

Each term of the right-hand side in (1.4.23) is estimated as follows:

1.

$$|a(u-u_h,u-\mathcal{I}_hu)|\le\|a\|\,\|u-u_h\|_{H^1(\Omega)^2}\|u-\mathcal{I}_hu\|_{H^1(\Omega)^2}\le Ch^\epsilon\|u-u_h\|_{H^1(\Omega)^2}.$$

2.

$$|b(u_h-u,p-\Pi_hp)|\le\|b\|\,\|u-u_h\|_{H^1(\Omega)^2}\|p-\Pi_hp\|_{L^2(\Omega)}\le Ch^\epsilon\|u-u_h\|_{H^1(\Omega)^2}.$$

3. From (1.4.24),

$$\begin{aligned}|b(\mathcal{I}_hu-u,p_h-p)|&\le\|b\|\,\|\mathcal{I}_hu-u\|_{H^1(\Omega)^2}\|p_h-p\|_{L^2(\Omega)}\\&\le C(h^{2\epsilon}+h^\epsilon\|u-u_h\|_{H^1(\Omega)^2}).\end{aligned}$$

4.

$$\left|(\sigma_\tau,(\mathcal{I}_hu)_\tau-u_\tau)_{L^2(\Gamma_1)}\right|\le\|\sigma_\tau\|_{L^2(\Gamma_1)}\|(\mathcal{I}_hu)_\tau-u_\tau\|_{L^2(\Gamma_1)}\le Ch^{1/2+\epsilon}.$$

5. By Lemma 1.3.4(ii) together with Proposition 1.4.1(i),

$$|j(u_{h\tau})-j_h(u_{h\tau})|\le Ch^{1/2}\|u_{h\tau}\|_{H^{1/2}(\Gamma_1)}\le Ch^{1/2}\|u_h\|_{H^1(\Omega)^2}\le Ch^{1/2}.$$

6. Since $\|\mathcal{I}_h u\|_{H^1(\Omega)^2} \le \|\mathcal{I}_h u - u\|_{H^1(\Omega)^2} + \|u\|_{H^1(\Omega)^2} \le C$, Lemma 1.3.4(ii) implies

$$|j((\mathcal{I}_h u)_\tau) - j_h((\mathcal{I}_h u)_\tau)| \le Ch^{1/2}\|(\mathcal{I}_h u)_\tau\|_{H^{1/2}(\Gamma_1)} \le Ch^{1/2}\|\mathcal{I}_h u\|_{H^1(\Omega)^2} \le Ch^{1/2}.$$

7.

$$|j((\mathcal{I}_h u)_\tau) - j(u_\tau)| \le \|g\|_{L^2(\Gamma_1)}\|(\mathcal{I}_h u)_\tau - u_\tau\|_{L^2(\Gamma_1)} \le Ch^{1/2+\epsilon}.$$

Combining these seven estimates with (1.4.23), we deduce

$$\|u - u_h\|^2_{H^1(\Omega)^2} \le C\left(h^\epsilon\|u - u_h\|_{H^1(\Omega)^2} + h^{2\epsilon} + h^{1/2+\epsilon} + h^{1/2}\right),$$

so that

$$\|u - u_h\|_{H^1(\Omega)^2} \le Ch^{\min\{\epsilon, \frac{1}{4}\}}. \tag{1.4.25}$$

We conclude (1.4.22) from (1.4.24) and (1.4.25), and this completes the proof. □

The previous theorem reveals that the rate of convergence is $O(h^{1/4})$ at best even when the solution is sufficiently smooth. However, it can be improved if additional conditions about the signs of $u_{h\tau}$ and $(\mathcal{I}_h u)_\tau$ on Γ_1 are available. To formulate the result, we make the following assumptions (recall Definition 1.3.1):

(S1) $(\mathcal{I}_h u)_\tau$ has a constant sign on every side.
(S2) $u_{h\tau}$ has a constant sign on every side.
(S3) $\mathrm{sgn}(u_\tau) = \mathrm{sgn}((\mathcal{I}_h u)_\tau)$ on Γ_1.

Theorem 1.4.3. *In addition to the hypotheses in Theorem 1.4.2, we assume $g \in C^2(\overline{\Gamma}_1)$ and that (S1)–(S3) are satisfied. Then we have*

$$\|u - u_h\|_{H^1(\Omega)^2} + \|p - p_h\|_{L^2(\Omega)} \le Ch^{\min\{\epsilon,1\}}. \tag{1.4.26}$$

Moreover, if g is a polynomial function of degree ≤ 1, we have

$$\|u - u_h\|_{H^1(\Omega)^2} + \|p - p_h\|_{L^2(\Omega)} \le Ch^{\epsilon}. \tag{1.4.27}$$

Proof. We first verify that (S3) implies

$$\sigma_\tau(\mathcal{I}_h u)_\tau + g|(\mathcal{I}_h u)_\tau| = 0 \quad \text{a.e. on } \Gamma_1. \tag{1.4.28}$$

In fact, for each side $e \in \mathscr{E}_h|_{\Gamma_1}$, if u_τ vanishes on a subset of e containing more than three points, then the quadratic polynomial $(\mathcal{I}_h u)_\tau$ vanishes on the whole e. Otherwise, we have $|u_\tau| > 0$ a.e. on e; hence we deduce from (1.2.19), namely,

$$\sigma_\tau u_\tau + g|u_\tau| = 0 \quad \text{a.e. on } \Gamma_1, \tag{1.4.29}$$

that $\sigma_\tau = -g\mathrm{sgn}(u_\tau) = -g\mathrm{sgn}((\mathcal{I}_h u)_\tau)$ a.e. on e. In both cases, it follows that $\sigma_\tau(\mathcal{I}_h u)_\tau + g|(\mathcal{I}_h u)_\tau| = 0$ a.e. on e. Thus (1.4.28) is valid.

It follows from (1.4.28) and (1.4.29) that

$$(\sigma_\tau, (\mathcal{I}_h u)_\tau - u_\tau)_{L^2(\Gamma_1)} + j((\mathcal{I}_h u)_\tau) - j(u_\tau) = 0.$$

Therefore, taking $v_h = \mathcal{I}_h u$ and $q_h = \Pi_h p$ in (1.4.17) gives, instead of (1.4.23),

$$\begin{aligned} \alpha\|u - u_h\|^2_{H^1(\Omega)^2} \le\ & a(u - u_h, u - \mathcal{I}_h u) + b(u_h - u, p - \Pi_h p) + b(\mathcal{I}_h u - u, p_h - p) \\ & + |j(u_{h\tau}) - j_h(u_{h\tau})| + |j_h((\mathcal{I}_h u)_\tau) - j((\mathcal{I}_h u)_\tau)|. \end{aligned} \tag{1.4.30}$$

Let us give estimates for each term on the right-hand side. We can evaluate the first three terms by the same way as in the proof of Theorem 1.4.2. By assumptions (S1) and (S2), we can apply Lemma 1.3.5 to estimate the forth and fifth terms as follows:

$$\begin{aligned} |j(u_{h\tau}) - j_h(u_{h\tau})| &\le Ch^2\|u_{h\tau}\|_{L^2(\Gamma_1)} \le Ch^2\|u_h\|_{H^1(\Omega)^2} \le Ch^2, \\ |j_h((\mathcal{I}_h u)_\tau) - j((\mathcal{I}_h u)_\tau)| &\le Ch^2\|(\mathcal{I}_h u)_\tau\|_{L^2(\Gamma_1)} \le Ch^2\|\mathcal{I}_h u\|_{H^1(\Omega)^2} \le Ch^2. \end{aligned}$$

Consequently, we obtain

$$\|u - u_h\|^2_{H^1(\Omega)^2} \le C\left(h^\epsilon\|u - u_h\|_{H^1(\Omega)^2} + h^{2\epsilon} + h^2\right), \tag{1.4.31}$$

which leads to

$$\|u - u_h\|_{H^1(\Omega)^2} \le Ch^{\min\{\epsilon,1\}}.$$

The estimate for $\|p - p_h\|_{L^2(\Omega)}$ is similar to the proof of Theorem 1.4.2, and then, (1.4.26) follows.

Finally, if g is affine then the fourth and fifth terms in (1.4.30) vanish exactly, according to Lemma 1.3.5. Hence we have

$$\|u - u_h\|^2_{H^1(\Omega)^2} \le C\left(h^\epsilon\|u - u_h\|_{H^1(\Omega)^2} + h^{2\epsilon}\right)$$

instead of (1.4.31), from which (1.4.27) follows. □

Remark 1.4.1. Conditions (S1)–(S3) are not so artificial. Assume that u is continuous on $\overline{\Omega}$ and that the isolated zeros of u_τ on Γ_1 are contained in $\Gamma_{1,h}$. If we make h sufficiently small, then we see that (S1) and (S3) are satisfied. Therefore, since Theorem 1.4.2 implies $u_{h\tau} \to u_\tau$ in $H^{1/2}(\Gamma_1)$, we can expect (S2) to also be valid; however, its rigorous proof is not easy.

1.4.3 Numerical realization

We propose the following Uzawa-type method to compute the solution of Problem VE$_h$ (therefore, Problem VI$_h$) numerically.

Algorithm 1.4.1. Choose an arbitrary $\lambda_h^{(1)} \in \tilde{\Lambda}_h$ and $\rho > 0$. Iterate the following two steps for $k = 1, 2, \cdots$:

Step 1: With $\lambda_h^{(k)}$ known, determine $(u_h^{(k)}, p_h^{(k)}) \in V_{nh} \times \mathring{Q}_h$ such that

$$\begin{cases} a(u_h^{(k)}, v_h) + b(v_h, p_h^{(k)}) = (f, v_h)_{L^2(\Omega)^2} - (v_{h\tau}, \lambda_h^{(k)})_{\Lambda_h}, & (1.4.32) \\ b(u_h^{(k)}, q_h) = 0, & (1.4.33) \end{cases}$$

for all $(v_h, q_h) \in V_{nh} \times \mathring{Q}_h$.

Step 2: Renew $\lambda_h^{(k+1)} \in \tilde{\Lambda}_h$ by

$$\lambda_h^{(k+1)} = \mathrm{Proj}_{\tilde{\Lambda}_h}(\lambda_h^{(k)} + \rho u_{h\tau}^{(k)}). \tag{1.4.34}$$

Remark 1.4.2. (i) The unique existence of $(u_h^{(k)}, p_h^{(k)})$ satisfying (1.4.32) and (1.4.33) is guaranteed by the inf-sup condition (1.3.8).

(ii) We can regard (1.4.34) as an approximation of

$$\lambda_h = \mathrm{Proj}_{\tilde{\Lambda}_h}(\lambda_h + \rho u_{h\tau}), \tag{1.4.35}$$

which is equivalent to (1.4.6) by Lemma 1.3.2(ii).

Following standard techniques for Uzawa methods (e.g. [22, p. 66]), we prove the convergence of Algorithm 1.4.1.

Theorem 1.4.4. *Let (u_h, p_h, λ_h) be the solution of Problem* VE$_h$*. Under the same notation as Algorithm 1.4.1, there exists a constant $\rho_0 > 0$ independent of h such that if ρ satisfies $0 < \rho < \rho_0$, then the iterative solution $(u_h^{(k)}, p_h^{(k)}, \lambda_h^{(k)})$ converges to (u_h, p_h, λ_h) in $H^1(\Omega)^2 \times L^2(\Omega) \times \Lambda_h$, as $k \to \infty$.*

Proof. Subtracting (1.4.32) from (1.4.4) with test functions in $V_{nh,\sigma}$, we obtain

$$a(u_h - u_h^{(k)}, v_h) + (v_{h\tau}, \lambda_h - \lambda_h^{(k)})_{\Lambda_h} = 0 \qquad (\forall v_h \in V_{nh,\sigma}). \tag{1.4.36}$$

Take $v_h = u_h^{(k)} - u_h \in V_{nh,\sigma}$ in (1.4.36) and apply Korn's inequality (1.2.13) to obtain

$$(u_{h\tau}^{(k)} - u_{h\tau}, \lambda_h^{(k)} - \lambda_h)_{\Lambda_h} = -a(u_h^{(k)} - u_h, u_h^{(k)} - u_h) \le -\alpha \|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}^2. \tag{1.4.37}$$

Next, we note that $\mathrm{Proj}_{\tilde{\Lambda}_h}$ given in (1.3.5) satisfies

$$\|\mathrm{Proj}_{\tilde{\Lambda}_h}(\mu_h) - \mathrm{Proj}_{\tilde{\Lambda}_h}(\eta_h)\|_{\Lambda_h} \leq \|\mu_h - \eta_h\|_{\Lambda_h} \qquad (\forall \mu_h, \eta_h \in \Lambda_h), \quad (1.4.38)$$

as a result of a general property of a projection operator. It follows from (1.4.38) with $\mu_h = \lambda_h^{(k)} + \rho u_{h\tau}^{(k)}$ and $\eta_h = \lambda_h + \rho u_{h\tau}$, (1.4.34), (1.4.35), and (1.4.37) that

$$\begin{aligned}
\|\lambda_h^{(k+1)} - \lambda_h\|_{\Lambda_h}^2 &\leq \|\lambda_h^{(k)} - \lambda_h + \rho(u_{h\tau}^{(k)} - u_{h\tau})\|_{\Lambda_h}^2 \\
&= \|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}^2 + 2\rho(u_{h\tau}^{(k)} - u_{h\tau}, \lambda_h^{(k)} - \lambda_h)_{\Lambda_h} + \rho^2\|u_{h\tau}^{(k)} - u_{h\tau}\|_{\Lambda_h}^2 \\
&\leq \|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}^2 - 2\alpha\rho\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}^2 + \rho^2\|u_{h\tau}^{(k)} - u_{h\tau}\|_{\Lambda_h}^2.
\end{aligned}$$

Therefore, since $\|u_{h\tau}^{(k)} - u_{h\tau}\|_{\Lambda_h} \leq C\|u_{h\tau}^{(k)} - u_{h\tau}\|_{L^2(\Gamma_1)} \leq C\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}$ in view of Lemmas 1.3.4(i) and 1.2.1(i), we obtain

$$\|\lambda_h^{(k+1)} - \lambda_h\|_{\Lambda_h}^2 \leq \|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}^2 - (2\alpha\rho - C\rho^2)\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}^2, \quad (1.4.39)$$

and thus

$$(2\alpha\rho - C\rho^2)\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}^2 \leq \|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}^2 - \|\lambda_h^{(k+1)} - \lambda_h\|_{\Lambda_h}^2. \quad (1.4.40)$$

On the other hand, by virtue of Lemma 1.3.1(i)(ii), we can choose $w_h \in V_{h,\sigma}$ such that $w_{h\tau} = \lambda_h^{(k)} - \lambda_h$ on Γ_1 and

$$\|w_h\|_{H^1(\Omega)^2} \leq C\|\lambda_h^{(k)} - \lambda_h\|_{H_{00}^{1/2}(\Gamma_1)} \leq C(h)\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}, \qquad (1.4.41)$$

where the constant $C(h)$ concerns the equivalence of the norms on the finite dimensional space Λ_h. Hence, it follows from (1.4.36) with $v_h = w_h$ that

$$\begin{aligned}
\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}^2 = (w_{h\tau}, \lambda_h^{(k)} - \lambda_h)_{\Lambda_h} &= -a(u_h^{(k)} - u_h, w_h) \\
&\leq \|a\| \, \|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}\|w_h\|_{H^1(\Omega)^2} \\
&\leq C(h)\|a\|\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2}\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h},
\end{aligned}$$

so that

$$-C(h)\|a\|\|u_h^{(k)} - u_h\|_{H^1(\Omega)^2} \leq -\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h}. \qquad (1.4.42)$$

Since the constant C in (1.4.39) is independent of ρ (and even of h), if we choose $0 < \rho < \rho_0 := \frac{2\alpha}{C}$ then it follows from (1.4.39) and (1.4.42) that

$$\|\lambda_h^{(k+1)} - \lambda_h\|_{\Lambda_h} \leq \sqrt{1 - \frac{2\alpha\rho - C\rho^2}{C(h)^2\|a\|^2}}\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h},$$

where we may assume $C(h)^2\|a\|^2 \geq 2\alpha\rho - C\rho^2$ (see the remark below). Consequently, we obtain

$$\|\lambda_h^{(k)} - \lambda_h\|_{\Lambda_h} \leq \left(1 - \frac{2\alpha\rho - C\rho^2}{C(h)^2}\right)^{\frac{k-1}{2}} \|\lambda_h^{(1)} - \lambda_h\|_{\Lambda_h} \to 0 \qquad (k \to \infty).$$

Then, from (1.4.40) it also follows that $u_h^{(k)} \to u_h$ in $H^1(\Omega)^2$ as $k \to \infty$.

Finally, subtracting (1.4.42) from (1.4.4) with test functions in $\mathring{V}_h$ gives

$$a(u_h - u_h^{(k)}, v_h) + b(v_h, p_h - p_h^{(k)}) = 0 \qquad (\forall v_h \in \mathring{V}_h).$$

Combining this equation with (1.3.7), we have

$$\beta\|p_h^{(k)} - p_h\|_{L^2(\Omega)} \leq \sup_{v_h \in \mathring{V}_h} \frac{b(v_h, p_h^{(k)} - p_h)}{\|v_h\|_{H^1(\Omega)^2}} = \sup_{v_h \in \mathring{V}_h} \frac{-a(u_h^{(k)} - u_h, v_h)}{\|v_h\|_{H^1(\Omega)^2}} \to 0 \quad (k \to \infty).$$

This completes the proof. □

Remark 1.4.3. (i) If $C(h)^2\|a\|^2 < 2\alpha\rho - C\rho^2$, we replace $C(h)$ by $\sqrt{2\alpha\rho - C\rho^2}/\|a\|$, keeping (1.4.41) valid.

(ii) The convergence speed, which can be evaluated by how much the constant $\sqrt{1 - \frac{2\alpha\rho - C\rho^2}{C(h)^2\|a\|^2}}$ is less than 1, may depend on h.

1.5 Numerical examples

Let Ω be the unit square $(0,1)^2$. The boundary $\Gamma = \partial\Omega$ consists of two portions Γ_0 and Γ_1 given by

$$\begin{aligned}
\Gamma_0 &= \{(0,y) \,|\, 0 < y < 1\} \cup \{(x,0) \,|\, 0 \leq x \leq 1\} \cup \{(1,y) \,|\, 0 < y < 1\}, \\
\Gamma_1 &= \{(x,1) \,|\, 0 < x < 1\}.
\end{aligned}$$

In particular, the set of extreme points is $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \{(0,1), (1,1)\}$.

Let us consider

$$\begin{cases}
u_1(x,y) &= 20x^2(1-x)^2 y(1-y)(1-2y), \\
u_2(x,y) &= -20x(1-x)(1-2x)y^2(1-y)^2, \\
p(x,y) &= 40x(1-x)(1-2x)y(1-y)(1-2y) \\
& \qquad +4(6x^5 - 15x^4 + 10x^3)(2y-1) - 2,
\end{cases} \tag{1.5.1}$$

which turns out to be the solution of the Stokes equations (1.2.1) under the adhesive boundary condition $u|_\Gamma = 0$. Here, we set the viscosity constant $\nu = 1$, and give the external force f by

$$\begin{cases} f_1(x,y) = 0, \\ f_2(x,y) = 120(2x-1)y^2(1-y)^2 \\ \qquad\qquad +80x(1-x)(1-2x)(6y^2-6y+1) + 8(6x^5 - 15x^4 + 10x^3). \end{cases}$$

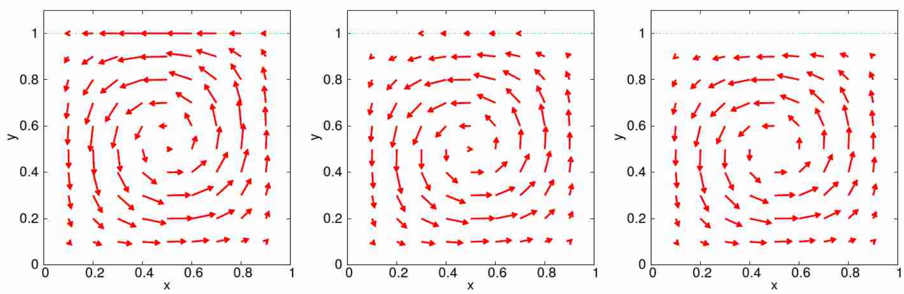
By direct computation, we have

$$\max_{\Gamma_1} |\sigma_\tau| = \max_{0 \le x \le 1} \left|20x^2(1-x)^2\right| = 5/4 = 1.25.$$

Now, instead of the adhesive boundary condition, we impose SBCF on Γ_1, with g being constant. Then it is immediate to see that

$$\begin{cases} g \ge 1.25 & \Longrightarrow \text{(1.5.1) remains a solution.} \\ g < 1.25 & \Longrightarrow \text{(1.5.1) is no longer a solution and a non-trivial slip occurs.} \end{cases}$$

Our numerical solution, shown in Figure 1.5.1, does indicate such a phenomenon. In fact, some slip $u_{h\tau} \neq 0$ takes place on Γ_1 for $g = 0.1$ and $g = 0.8$, whereas no slip is observed for $g = 2.0$. We also find that the bigger (resp. smaller) the threshold g of a tangential stress becomes, the more difficult (resp. easier) it becomes for a non-trivial slip to occur.



(a) $g = 0.1$ (b) $g = 0.8$ (c) $g = 2.0$

Figure 1.5.1: Solution velocity field of the Stokes equations with SBCF

Let us explain details of our numerical experiments. For the triangulation $\mathscr{T}_h$ of $\overline{\Omega}$, we employ a uniform $N \times N$ Friedrichs–Keller type mesh, where N denotes the division number of each side of the square $\overline{\Omega}$. Choosing the parameter $\rho > 0$, fixed for each g, and the starting value $\lambda_h^{(1)}$, we

Table 1.5.1: Values of the Lagrange multiplier λ_h and tangential velocity $u_{h\tau}$ on Γ_1

g	0.1		0.8		2.0		2.0	2.0
ρ	1000.0		50.0		3.0		1.0	3.0
$\lambda_h^{(1)}$	0.0		0.0		0.0		0.0	0.2
x	λ_h	$u_{h\tau}$	λ_h	$u_{h\tau}$	λ_h	$u_{h\tau}$	λ_h	λ_h
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	-1.0	-0.02	-0.26	-8.8E-6	-0.09	-1.6E-5	-0.09	-0.09
0.2	-1.0	-0.05	-0.90	-3.5E-6	-0.25	-9.8E-7	-0.25	-0.25
0.3	-1.0	-0.09	-1.0	-0.01	-0.42	$+9.1$E-6	-0.42	-0.42
0.4	-1.0	-0.12	-1.0	-0.03	-0.55	$+1.7$E-5	-0.55	-0.55
0.5	-1.0	-0.13	-1.0	-0.04	-0.60	$+2.0$E-5	-0.60	-0.59
0.6	-1.0	-0.12	-1.0	-0.03	-0.55	$+1.9$E-5	-0.56	-0.55
0.7	-1.0	-0.09	-1.0	-0.02	-0.43	$+1.4$E-5	-0.43	-0.43
0.8	-1.0	-0.06	-0.94	-1.6E-6	-0.26	$+5.0$E-6	-0.26	-0.25
0.9	-1.0	-0.02	-0.26	-2.0E-6	-0.09	-5.7E-6	-0.09	-0.09
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
k_{itr}	4		18		29		45	52

compute the numerical solution $(u_h^{(k)}, p_h^{(k)}, \lambda_h^{(k)})$ by Algorithm 1.4.1. The stopping criterion for the iteration in Algorithm 1.4.1 is

$$\|u_h^{(k)} - u_h^{(k-1)}\|_{H^1(\Omega)^2} \leq 10^{-5}. \tag{1.5.2}$$

The number of iterations required to attain (1.5.2) is denoted by k_{itr}.

Figure 1.5.1 shows the plots of u_h computed when $N = 10$. The concrete choice for g, ρ, $\lambda_h^{(1)}$ are listed in Table 1.5.1, together with the resulting values of λ_h and $u_{h\tau}$ on $\overline{\Gamma}_1$. The meaning of $\lambda_h^{(1)} = 0.2$ in the last column is that $\lambda_h^{(1)}(M) = 0.2$ for each $M \in \mathring{\Gamma}_{1,h}$. By comparing the three results for $g = 2.0$, we see that the result, except for k_{itr}, does not depend on the choice of ρ or $\lambda_h^{(1)}$.

Next, we consider the behavior of the Lagrange multiplier λ_h. It follows from (1.4.6), combined with Lemma 1.3.2(ii), that for each $M \in \mathring{\Gamma}_{1,h}$

$$\begin{cases} |\lambda_h(M)| \leq 1 & \text{if } u_{h\tau}(M) = 0, \\ \lambda_h(M) = +1 \text{ or } -1 & \text{if } u_{h\tau}(M) \neq 0. \end{cases} \tag{1.5.3}$$

Comparing the values of λ_h with those of $u_{h\tau}$ in Table 1.5.1, we find that our numerical solution indeed reveals behavior like (1.5.3).

Table 1.5.2: Convergence behavior of $\|u_h - u_{\mathrm{ref}}\|_{H^1(\Omega)^2}$ and $\|p_h - p_{\mathrm{ref}}\|_{L^2(\Omega)}$ for $g = 0.8$ and $g = 2.0$

	$g = 0.8$					$g = 2.0$				
N	H^1-error	rate	L^2-error	rate	k_{itr}	H^1-error	rate	L^2-error	rate	k_{itr}
10	1.6E-2	—	1.6E-2	—	18	1.3E-2	—	1.3E-2	—	29
12	1.1E-2	1.9	1.1E-2	2.0	19	9.3E-3	1.9	1.0E-2	2.6	25
15	7.0E-3	2.1	6.3E-3	2.5	22	6.0E-3	1.9	5.9E-3	2.4	19
20	3.9E-3	2.0	3.5E-3	2.1	20	3.4E-3	2.0	3.1E-3	2.3	16
24	2.6E-3	2.1	2.7E-3	1.3	21	2.4E-3	2.0	2.0E-3	2.2	16
30	1.7E-3	2.0	1.5E-3	2.6	18	1.5E-3	2.0	1.3E-3	2.1	16
40	9.0E-4	2.1	8.5E-4	2.0	18	8.7E-4	2.0	7.0E-4	2.1	16

Finally, we evaluate the error between approximate solutions and exact ones as the division number N increases, when $g = 0.8$ and $g = 2.0$. Since we do not know the explicit exact solutions when $g = 0.8$, we employ the approximate solutions with $N = 120$ as the reference solutions $(u_{\mathrm{ref}}, p_{\mathrm{ref}})$, and numerically calculate $\|u_h - u_{\mathrm{ref}}\|_{H^1(\Omega)^2}$ and $\|p_h - p_{\mathrm{ref}}\|_{L^2(\Omega)}$. On the other hand, we know the exact solution (1.5.1) when $g = 2.0$, and thus we take $u_{\mathrm{ref}} = u, p_{\mathrm{ref}} = p$ in this case. Then, as Table 1.5.2 shows, we observe the optimal order convergence $O(h^2)$ for both cases.

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Chapter 2

Finite element analysis for the Stokes equations under the leak boundary condition of friction type

2.1 Introduction

Consider the incompressible Stokes equations

$$-\nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 \qquad \text{in} \quad \Omega, \tag{2.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d\,(d = 2, 3)$ with the Lipschitz boundary $\Gamma = \partial\Omega$. For the boundary condition (abbreviated as b.c. in the sequel), we assume $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, where Γ_0 and Γ_1 are nonempty open subsets of Γ. Γ_0 is subject to the adhesive b.c. (Dirichlet b.c.), that is,

$$u = 0 \qquad \text{on} \quad \Gamma_0, \tag{2.1.2}$$

whereas the following leak b.c. of friction type (LBCF) is imposed on Γ_1:

$$u_\tau = 0, \quad |\sigma_n| \le g, \quad \sigma_n u_n + g|u_n| = 0 \qquad \text{on} \quad \Gamma_1. \tag{2.1.3}$$

Here, n denotes the outer unit normal, and $\sigma = -pn + \nu(\nabla u + (\nabla u)^T)n$ stands for the stress vector. The normal and tangential components of a vector U are indicated as $U_n = U \cdot n$ and $U_\tau = U - U_n n$ respectively. Finally, g is a given positive function on Γ_1 and called the friction parameter.

The physical interpretation of (2.1.3) is as follows. From the second and third conditions, we see that g represents a threshold of the normal stress such that

$$|\sigma| < g \Rightarrow u_n = 0, \quad u_n > 0 \Rightarrow \sigma_n = -g, \quad u_n < 0 \Rightarrow \sigma_n = g. \qquad (2.1.4)$$

If $g \equiv 0$ then (2.1.3) reduces to the usual leak b.c.: $u_\tau = \sigma_n = 0$. Thus LBCF can be regarded as a non-linearized leak b.c. obtained from imposing some friction law against leak phenomena of a fluid.

Although frictional b.c.'s are treated mainly in the context of elasticity (e.g. [33]), they have been considered also in fluid dynamics since the pioneer work [16]. On one hand, the slip b.c. of friction type (SBCF), obtained by replacing n with τ and vice versa in (2.1.3), is a direct transplant of friction problems for solids, and many researchers focus on it from mathematical and numerical points of view (e.g. [3, 39, 41, 43, 55]). On the other hand, LBCF, in which case a material is allowed to penetrate the boundary but not allowed unless the flow is strong enough, is also worth considering in the context of fluid dynamics [51]. Applications of SBCF or LBCF to realistic problems arising in computational fluid dynamics are addressed in [32] and [61]. Nonstationary SBCF and LBCF problems are studied in [18, 30].

Before explaining what our discrete scheme aims at, we review the results shown in the study of continuous problems ([16, 20]). A weak form for (2.1.1) with (2.1.2)–(2.1.3) is the following variational inequality:

Problem VI. Find $(u, p) \in V_n \times Q$ such that, for all $(v, q) \in V_n \times Q$,

$$\begin{cases} a(u, v-u) + b(v-u, p) + j(v_n) - j(u_n) \geq (f, v-u), & (2.1.5a) \\ b(u, q) = 0. & (2.1.5b) \end{cases}$$

Here, employing the standard notation of Lebesgue and Sobolev spaces, we define function spaces and functionals which appear above as follows:

$$\begin{aligned} &V = H^1(\Omega)^d, \\ &V_n = \{v \in V \mid v = 0 \text{ on } \Gamma_0, \quad v_\tau = 0 \text{ on } \Gamma_1\}, \\ &Q = L^2(\Omega), \end{aligned}$$

and

$$a(u, v) = \frac{\nu}{2} \int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) dx, \quad b(u, q) = -\int_\Omega (\operatorname{div} u) q \, dx,$$

$$j(\eta) = \int_{\Gamma_1} g|\eta| \, ds, \quad (\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)^d}, \quad \|\cdot\| = \|\cdot\|_{L^2(\Omega)^d}.$$

If the Lagrange multiplier $\lambda := -\sigma_n/g$ is introduced, Problem VI is proved to be equivalent to the following variational equation problem [20, Theorem 3.4]:

Problem VE. Find $(u, p, \lambda) \in V_n \times Q \times \tilde{\Lambda}$ such that

$$\begin{cases} a(u,v) + b(v,p) + (v_n, \lambda)_\Lambda = (f, v) & (\forall v \in V_n), \quad (2.1.6a) \\ b(u,q) = 0 & (\forall q \in Q), \quad (2.1.6b) \\ (u_n, \mu - \lambda)_\Lambda \le 0 & (\forall \mu \in \tilde{\Lambda}). \quad (2.1.6c) \end{cases}$$

Here, a Hilbert space Λ and its closed convex subset $\tilde{\Lambda}$ are defined by

$$\Lambda = L^2(\Gamma_1), \quad (\cdot,\cdot)_\Lambda = (g\cdot,\cdot)_{L^2(\Gamma_1)}, \quad \tilde{\Lambda} = \{\lambda \in \Lambda \mid \ |\lambda| \le 1 \text{ a.e. on } \Gamma_1\},$$

where $g > 0$ is supposed to belong to $L^\infty(\Gamma_1)$. Further assumptions on g will be specified later in Section 2.3. With this λ, (2.1.4) is expressed as

$$|\lambda| < 1 \Rightarrow u_n = 0, \quad u_n > 0 \Rightarrow \lambda = 1, \quad u_n < 0 \Rightarrow \lambda = -1, \qquad (2.1.7)$$

which we call the *leak/no-leak detecting condition.*

Remark 2.1.1. We refer to (2.1.6a)–(2.1.6c) as a variational *equation* problem since inequality (2.1.6c) can also be interpreted as an equation. In fact, it is equivalent to $\lambda = \tilde{P}(\lambda + \rho u_n)$, $\forall \rho \ge 0$, where $\tilde{P}$ is the projection operator from Λ onto $\tilde{\Lambda}$.

Well-posedness of Problems VI and VE is established in [16, 20]. It should be noted that an additive constant of pressure is uniquely determined if leak occurs, i.e. $u_n \neq 0$ on Γ_1, but that otherwise it may not be unique (see [16, Remark 3.2]).

Summarizing the arguments above, we find that:

(1) Problems VI and VE are equivalent.

(2) λ detects whether leak or no-leak occurs.

(3) Uniqueness of an additive constant of p depends on whether leak occurs or not.

The purpose of this chapter is to propose a discretization of the LBCF problem which preserves (1)–(3) above in a discrete sense. A triangular finite element framework, based on the P1/P1-stabilized, P1b/P1, and P2/P1 elements, is presented. We show that an approximation of $j(\cdot)$ in terms of some numerical integration enables us to accomplish our aim. We notice that such an approximation itself, together with convergence analysis, was already considered in friction problems (e.g. [2, 22, 23, 25, 26, 27]); nevertheless its connection with points (1)–(3) above was not clear.

Preserving properties (1)–(3) is indispensable for our theory, and has more meaning than the fact that it is desirable just numerically. For instance, our error analysis and numerical implementation are respectively based on discrete versions of Problems VI and VE. Thus it is the equivalence between them that guarantees the consistency between our theory and numerical computation.

The plan of this chapter is as follows. In Section 2.2 we introduce notations and lemmas for our FEM. Well-posedness of discrete problems and error analysis are established in Section 2.3. Numerical implementation is discussed in Section 2.4, and we present numerical examples in Section 2.5.

2.2 Finite element framework for the LBCF problem

Henceforth, C denotes a generic constant depending only on Ω unless otherwise stated. When we need to specify its dependency on other quantities, we write $C(f, g), C(h)$ etc.

2.2.1 Finite element spaces

In the rest of this chapter, we assume that Ω is a polygon ($d = 2$) or polyhedron ($d = 3$) and that $\overline{\Gamma}_1$ coincides with whole one side ($d = 2$) or face ($d = 3$) of $\overline{\Omega}$. In particular, n is constant on Γ_1, and $\partial\Gamma_1 = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$ consists of two adjacent vertices of Ω ($d = 2$) or line segments surrounding Γ_1 ($d = 3$).

Remark 2.2.1. When $\overline{\Gamma}_1$ is a finite union of sides or faces, the constraint $u_\tau = 0$ on Γ_1 implies $u = 0$ at corners ($d = 2$) or edges ($d = 3$) contained in Γ_1 (if u is continuous). We can extend our results presented below to such Γ_1, provided that those Dirichlet conditions are incorporated in the approximate spaces.

Let $\mathscr{T}_h$ be a standard triangulation of Ω with $h = \max\{h_T \,|\, T \in \mathscr{T}_h\}$, $h_T = \operatorname{diam} T$. We assume $\{\mathscr{T}_h\}_{h\downarrow 0}$ is regular, i.e. $h_T \le C\rho_T$ for all $T \in \mathscr{T}_h$, where ρ_T denotes the diameter of the inscribed ball of T. To refer to sets of nodes, we put

$$\Sigma_h^1 = \{M \in \overline{\Omega} \mid M \text{ is a vertex of some } T \in \mathscr{T}_h \},$$
$$\Sigma_h^2 = \{M \in \overline{\Omega} \mid M \text{ is the midpoint of an edge of some } T \in \mathscr{T}_h \}.$$

The $d-1$ dimensional triangulation of Γ_1 inherited from $\mathscr{T}_h$ is denoted by $\mathscr{S}_h$, which is also assumed to be regular.

To approximate velocity and pressure, we consider one of the P1/P1, P1b/P1, and P2/P1 elements which we refer to as $l = 1$, $l = 1b$, and $l = 2$ respectively. That is to say, we employ the following approximate spaces:

$$V_h = \begin{cases} \{v_h \in C(\overline{\Omega})^d \mid \quad v_h|_T \in \mathscr{P}_1(T)^d \quad (\forall T \in \mathscr{T}_h)\} & \text{if } \ l = 1, \\ \{v_h \in C(\overline{\Omega})^d \mid \quad v_h|_T \in \mathscr{P}_1(T)^d \oplus \mathscr{B}(T)^d \quad (\forall T \in \mathscr{T}_h)\} & \text{if } \ l = 1b, \\ \{v_h \in C(\overline{\Omega})^d \mid \quad v_h|_T \in \mathscr{P}_2(T)^d \quad (\forall T \in \mathscr{T}_h)\} & \text{if } \ l = 2, \end{cases}$$
$$Q_h = \{q_h \in C(\overline{\Omega}) \mid \quad q_h|_T \in \mathscr{P}_1(T) \quad (\forall T \in \mathscr{T}_h)\},$$

where $\mathscr{P}_k(T)$ and $\mathscr{B}(T)$ respectively denote the space of the polynomials of degree $\leq k$ and that spanned by the bubble function on T. Based on these spaces, we define

$$\mathring{V}_h = V_h \cap H_0^1(\Omega)^d, \quad V_{h,\sigma} = \{v_h \in V_h \mid b(v_h, q_h) = 0 \ (\forall q_h \in V_h)\},$$
$$V_{hn} = V_h \cap V_n, \quad V_{hn,\sigma} = V_{hn} \cap V_{h,\sigma}, \quad \mathring{Q}_h = Q_h \cap L_0^2(\Omega),$$

where $L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0\}$. As is well known, the following inf-sup condition holds with a constant $\beta > 0$ depending only on Ω:

$$\beta \|q_h\|_Q \leq \sup_{v_h \in \mathring{V}_h} \frac{b(v_h, q_h)}{\|v_h\|_V} + \gamma C \|h \nabla q_h\| \qquad (\forall q_h \in \mathring{Q}_h), \tag{2.2.1}$$

where $\gamma = 1$ if $l = 1$, and $\gamma = 0$ if $l = 1b, 2$.

Since n is constant on Γ_1, $\Lambda_h = \{v_{hn}|_{\Gamma_1} \mid v_h \in V_h\}$ coincides with

$$\Lambda_h = \begin{cases} \{\eta_h \in C(\overline{\Gamma}_1) \mid \eta_h = 0 \text{ on } \partial\Gamma_1, \quad \eta_h|_S \in \mathscr{P}_1(S) \quad (\forall S \in \mathscr{S}_h)\} & \text{if } \ l = 1, 1b, \\ \{\eta_h \in C(\overline{\Gamma}_1) \mid \eta_h = 0 \text{ on } \partial\Gamma_1, \quad \eta_h|_S \in \mathscr{P}_2(S) \quad (\forall S \in \mathscr{S}_h)\} & \text{if } \ l = 2. \end{cases}$$

(Note that the bubble function for T vanishes on ∂T in the case $l = 1b$.) The sets of boundary nodes on $\overline{\Gamma}_1$ and $\Gamma_1 = \overline{\Gamma}_1 \setminus \partial\Gamma_1$ are indicated by

$$\Gamma_{1h} = \overline{\Gamma}_1 \cap \Sigma_h^1 \ (\text{if } \ l = 1, 1b), \quad \Gamma_{1h} = \overline{\Gamma}_1 \cap (\Sigma_h^1 \cup \Sigma_h^2) \ (\text{if } \ l = 2), \quad \mathring{\Gamma}_{1h} = \Gamma_{1h} \setminus \partial\Gamma_1.$$

The following discrete trace and lifting theorem is frequently used in the subsequent arguments.

Lemma 2.2.1. (i) *For $v_h \in V_{hn}$, we have $v_{hn} \in \Lambda_h$ and $\|v_{hn}\|_{H^{1/2}_{00}(\Gamma_1)} \le C\|v_h\|_V$.*

(ii) *For $\eta_h \in \Lambda_h$, there exists $v_h \in V_{hn}$ such that $v_{hn} = \eta_h$ on Γ_1 and $\|v_h\|_V \le C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$.*

(iii) *Let $l = 1b$ or 2. In* (ii) *above, if $\int_{\Gamma_1} \eta_h \, ds = 0$, then we can choose v_h in such a way that $v_h \in V_{h,\sigma}$.*

Proof. (i) This follows from a standard trace theorem. For the trace and lifting theorem involving the space $H^{1/2}_{00}(\Gamma_1)$, see [33, Section 5.3].

(ii) Since $\eta_h \in \Lambda_h \subset H^{1/2}_{00}(\Gamma_1)$, the zero-extension of $\eta_h n$ to Γ, denoted by $\tilde{\eta}_h$, belongs to $H^{1/2}(\Gamma)^d$, with its norm bounded by $C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$. Applying a discrete lifting theorem [8, Theorem 5.1] to $\tilde{\eta}_h$, we obtain the conclusion.

(iii) Take an extension $\hat{v}_h$ of η_h given in (ii). By the inf-sup condition (2.2.1), there exists $v_h^* \in \mathring{V}_h$ such that

$$b(v_h^*, q_h) = -b(\hat{v}_h, q_h) \qquad (\forall q_h \in \mathring{Q}_h), \tag{2.2.2}$$

with $\|v_h^*\|_V \le C\|\operatorname{div} \hat{v}_h\|_{L^2(\Omega)} \le C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$. Setting $v_h = \hat{v}_h + v_h^*$, we obtain $v_{hn} = \eta_h$ on Γ_1 and $\|v_h\|_V \le C\|\eta_h\|_{H^{1/2}_{00}(\Gamma_1)}$. Because $\int_{\Gamma_1} \eta_h \, ds = 0$, we have

$$b(v_h, 1) = -\int_{\Gamma_1} v_{hn} \, ds = -\int_{\Gamma_1} \eta_h \, ds = 0,$$

which, combined with (2.2.2), implies $v_h \in V_{h,\sigma}$. This completes the proof. □

Remark 2.2.2. In [8], they present a lifting theorem only in a 2D domain (although extension to $d = 3$ is mentioned at Remark 3 there). Nevertheless, their proof of the theorem is sufficient to prove the 3D case, if combined with an interpolation-operator theory given in [9, Section 4.8] which is valid for all space dimensions.

Next we give a V_{hn}-Q_h type inf-sup condition, which is rather delicate compared with the $\mathring{V}_h$-$\mathring{Q}_h$ type, that is, (2.2.1). Such an inf-sup condition seems to be known (e.g. [15, p. 1687]) but its detailed proof is not found, so that we present it here.

Lemma 2.2.2. *The inf-sup condition of the form* (2.2.1)*, with $\mathring{V}_h$ and $\mathring{Q}_h$ replaced by V_{hn} and Q_h, also holds.*

Proof. First let us prove there exists $\hat{v}_h \in V_{hn}$ such that

$$b(\hat{v}_h, 1)/\|\hat{v}_h\|_V \geq C > 0. \tag{2.2.3}$$

In fact, solving a $d-1$ dimensional finite element problem gives $\eta_h \in \Lambda_h$ such that

$$(\nabla' \eta_h, \nabla' \mu_h)_{L^2(\Gamma_1)^{d-1}} = (-1, \mu_h)_{L^2(\Gamma_1)} \qquad (\forall \mu_h \in \Lambda_h),$$

where ∇' denotes the gradient operator on Γ_1. Then we see that $-\int_{\Gamma_1} \eta_h \, ds = \|\nabla' \eta_h\|^2_{L^2(\Gamma_1)^{d-1}} \geq C$, because η_h converges in $H^1(\Gamma_1)$ to the weak solution of $\Delta' \eta = 1$ in Γ_1, $\eta = 0$ on $\partial\Gamma_1$, which is obviously non-constant (Δ' is the laplacian on Γ_1). Lifting η_h to $\hat{v}_h \in V_{hn}$ by Lemma 2.2.1(ii), we obtain $b(\hat{v}_h, 1) = -\int_{\Gamma_1} \eta_h \, ds \geq C$ and $\|\hat{v}_h\|_V \leq C$, which implies (2.2.3).

Now, we let $q_h \in Q_h$ be arbitrary and split it as $q_h = \mathring{q}_h + \delta_h$, where $\mathring{q}_h \in \mathring{Q}_h$ and $\delta_h = \frac{1}{|\Omega|}(q_h, 1)_Q$. By (2.2.1) there exists $v_h^* \in \mathring{V}_h$ such that

$$\frac{b(v_h^*, \mathring{q}_h)}{\|v_h^*\|_V} \geq \beta \|\mathring{q}_h\|_Q - \gamma C \|h \nabla q_h\|.$$

Setting $v_h = v_h^*/\|v_h^*\|_V + \xi \hat{v}_h/\|\hat{v}_h\|_V$, we deduce that $\|v_h\|_V \leq 1 + \xi$ and that (We may assume $\delta_h \geq 0$ and $\xi \geq 0$; if $\delta_h \leq 0$ then consider $-\xi$ instead of ξ.)

$$\begin{aligned} b(v_h, q_h) &= \frac{b(v_h^*, q_h)}{\|v_h^*\|_V} + \xi \frac{b(\hat{v}_h, q_h)}{\|\hat{v}_h\|_V} = \frac{b(v_h^*, \mathring{q}_h)}{\|v_h^*\|_V} + \xi \frac{b(\hat{v}_h, \mathring{q}_h)}{\|\hat{v}_h\|_V} + \xi \delta_h \frac{b(\hat{v}_h, 1)}{\|\hat{v}_h\|_V} \\ &\geq (\beta - \xi)\|\mathring{q}_h\|_Q + \xi C \delta_h - \gamma C \|h \nabla q_h\|. \end{aligned}$$

Choosing $\xi = \beta/2$, we conclude $b(v_h, q_h) \geq C\|q_h\|_Q - \gamma C \|h \nabla q_h\|$, so that the desired inf-sup condition follows. □

2.2.2 Approximation of $j(\cdot)$ and $(\cdot, \cdot)_\Lambda$ by numerical integration

In what follows we assume $g \in C(\overline{\Gamma}_1)$ and $g > 0$. We label nodes on each $S \in \mathscr{S}_h$ as follows:

- When $d = 2$, the two endpoints of S are denoted by M_S^1, M_S^2; the midpoint of S is indicated by m_S.
- When $d = 3$, the three vertices of S are denoted by M_S^1, M_S^2, M_S^3; the midpoint of the opposite side to M_S^i is indicated by m_S^i.

Then, based on the trapezoidal and Simpson formulas, we define an approximation of $j(\eta)$, for $\eta \in C(\overline{\Gamma}_1)$, by

$$j_h(\eta) = \begin{cases} \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{2} \sum_{i=1}^{2} g|\eta|(M_S^i) & \text{if } d = 2 \text{ and } l = 1, 1b, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{6} \Big(\sum_{i=1}^{2} g|\eta|(M_S^i) + 4g|\eta|(m_S) \Big) & \text{if } d = 2 \text{ and } l = 2, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{3} \sum_{i=1}^{3} g|\eta|(M_S^i) & \text{if } d = 3 \text{ and } l = 1, 1b, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{3} \Big((1-\kappa) \sum_{i=1}^{3} g|\eta|(M_S^i) + \kappa \sum_{i=1}^{3} g|\eta|(m_S^i) \Big) & \text{if } d = 3 \text{ and } l = 2, \end{cases}$$

where $0 < \kappa < 1$ is a constant and $g(M)|\eta(M)|$ is written as $g|\eta|(M)$, etc. Accordingly, we define a scalar product $(\lambda, \mu)_{\Lambda_h}$, for $\lambda, \mu \in C(\overline{\Gamma}_1)$, by

$$(\lambda, \mu)_{\Lambda_h} = \begin{cases} \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{2} \sum_{i=1}^{2} g\lambda\mu(M_S^i) & \text{if } d = 2 \text{ and } l = 1, 1b, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{6} \Big(\sum_{i=1}^{2} g\lambda\mu(M_S^i) + 4g\lambda\mu(m_S) \Big) & \text{if } d = 2 \text{ and } l = 2, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{3} \sum_{i=1}^{3} g\lambda\mu(M_S^i) & \text{if } d = 3 \text{ and } l = 1, 1b, \\ \displaystyle\sum_{S\in\mathscr{S}_h} \frac{|S|}{3} \Big((1-\kappa) \sum_{i=1}^{3} g\lambda\mu(M_S^i) + \kappa \sum_{i=1}^{3} g\lambda\mu(m_S^i) \Big) & \text{if } d = 3 \text{ and } l = 2, \end{cases}$$

where $g\lambda\mu(M)$ means $g(M)\lambda(M)\mu(M)$ etc. We find that $(\cdot, \cdot)_{\Lambda_h}$ is bilinear, symmetric and positive definite, so that Λ_h becomes a Hilbert space equipped with this inner product.

Remark 2.2.3. When $d = 3$ and $l = 2$, the assumption $\kappa \neq 1$ (and $g > 0$) is essential to obtain the positive definiteness of $(\cdot, \cdot)_{\Lambda_h}$. This is why we do not employ the pure Simpson formula which corresponds to $\kappa = 1$. For a concrete value of κ, a formula which is exact for cubic functions [54, p. 402] suggests the use of $\kappa = 17/20$.

Combining Lemma 2.2.1(ii) with the equivalence of $\|\cdot\|_{\Lambda_h}$ and $\|\cdot\|_{H^{1/2}_{00}(\Gamma_1)}$ on the finite dimensional space Λ_h, we have an h-dependent inf-sup condition

$$C(h)\|\eta_h\|_{\Lambda_h} \le \sup_{v_h \in V_{hn}} \frac{(v_{hn}, \eta_h)_{\Lambda_h}}{\|v_h\|_V} \qquad (\forall \eta_h \in \Lambda_h). \tag{2.2.4}$$

This can be proved by the same way as in Lemma 1.3.3 of Chapter 1.

We introduce a closed convex subset of Λ_h by

$$\tilde{\Lambda}_h = \{\lambda_h \in \Lambda_h \mid \quad |\lambda_h| \le 1 \text{ at } \mathring{\Gamma}_{1h}\}.$$

Note that constraints are imposed only at nodes. The projection operator from Λ_h onto $\tilde{\Lambda}_h$, denoted by $\tilde{P}_h$, is characterized by the node-based relation

$$\tilde{P}_h(\eta_h) = \max\{-1, \min\{1, \eta_h\}\} \quad \text{at} \quad \mathring{\Gamma}_{1h}.$$

The rest of this subsection is devoted to the proof of relations between j_h and $(\cdot,\cdot)_{\Lambda_h}$. First, we prove that a discrete version of $j(\eta) = \sup_{|\lambda|\le 1}(\eta, \lambda)_\Lambda$ is valid. Such a relation in a discrete sense would not necessarily hold if we did not introduce numerical-integration approximation of $j(\cdot)$ and $(\cdot,\cdot)_\Lambda$.

Lemma 2.2.3. (i) *Let $\eta_h \in \Lambda_h$ and $\lambda_h \in \tilde{\Lambda}_h$. Then $(\eta_h, \lambda_h)_{\Lambda_h} \le j_h(\eta_h)$.*

(ii) *Under the assumptions of* (i), *following* (a)–(d) *are equivalent to each other:*

(a) $(\eta_h, \lambda_h)_{\Lambda_h} = j_h(\eta_h)$.

(b) $(\eta_h, \mu_h - \lambda_h)_{\Lambda_h} \le 0$ *for all* $\mu_h \in \tilde{\Lambda}_h$.

(c) $\eta_h \lambda_h = |\eta_h|$ *at* $\mathring{\Gamma}_{1h}$.

(d) $\lambda_h = \tilde{P}_h(\lambda_h + \rho\eta_h)$ *for some (actually, all)* $\rho > 0$.

Proof. (i) This is obvious because $\lambda_h \in \tilde{\Lambda}_h$ implies $g\eta_h\lambda_h \le g|\eta_h|$ at $\mathring{\Gamma}_{1h}$.

(ii) (a)⇒(b): Let (a) be valid and $\mu_h \in \tilde{\Lambda}_h$. Since $(\eta_h, \mu_h)_{\Lambda_h} \le j_h(\eta_h)$ by (i), we have $(\eta_h, \mu_h - \lambda_h)_{\Lambda_h} \le 0$.

(b)⇒(c): Let (b) be valid and $M \in \mathring{\Gamma}_{1h}$. Define $\mu_h \in \tilde{\Lambda}_h$ by

$$\mu_h(N) = \begin{cases} \eta_h(M)/|\eta_h(M)| & \text{if } N = M \text{ and } \eta_h(M) \ne 0, \\ 0 & \text{if } N = M \text{ and } \eta_h(M) = 0, \\ \lambda_h(N) & \text{if } N \in \mathring{\Gamma}_{1h} \setminus \{M\}. \end{cases}$$

It follows from (b) that $g|\eta_h|(M) \le g\eta_h\lambda_h(M)$, so that the equality holds since $|\lambda_h(M)| \le 1$. Hence we obtain (c).

(c)⇒(a): It is immediate to see (a) follows from (c).

(b)⇔(d): This is a direct consequence of a general property of the projection onto the convex set $\tilde{\Lambda}_h$. □

Next we state a discrete analogue of the fact that $L^\infty(\Gamma_1)$ is the dual of $L^1(\Gamma_1)$.

Lemma 2.2.4. *Let $\lambda_h \in \Lambda_h$. Then $\lambda_h \in \tilde{\Lambda}_h$ if and only if $(\eta_h, \lambda_h)_{\Lambda_h} \le j_h(\eta_h)$ for all $\eta_h \in \Lambda_h$.*

Proof. We only prove for the case $d = 3$ and $l = 1, 1b$, since a proof for the other cases is similar. From Lemma 2.2.3(i) we already know the "only if" part, and thus we show the other part. For fixed $M \in \mathring{\Gamma}_{1h}$, define $\eta_h \in \tilde{\Lambda}_h$ by

$$\eta_h(M) = \lambda_h(M), \qquad \eta_h(N) = 0 \quad \text{if} \quad N \in \mathring{\Gamma}_{1h} \setminus \{M\}.$$

By assumption we see that

$$g|\lambda_h|^2(M) \sum_{M \in S \in \mathscr{S}_h} |S|/3 \le g|\lambda_h|(M) \sum_{M \in S \in \mathscr{S}_h} |S|/3,$$

so that $|\lambda_h(M)| \le 1$. Hence $\lambda_h \in \tilde{\Lambda}_h$. □

Finally, we establish a discrete counterpart to the fact that the orthogonal complement of $L^2_0(\Gamma_1) := \{\eta_h \in L^2(\Gamma_1) \mid \int_{\Gamma_1} \eta_h \, ds = 0\}$ in $L^2(\Gamma_1)$ is $\mathbb{R}$. When $d = 3$ and $l = 1, 1b$, we assume the following connectivity condition (cf. [34, p. 173]) throughout this chapter; we do not need it if $d = 2$ or $l = 2$. In a practical computation, it is not a restrictive assumption.

(Co) Any two nodes in $\mathring{\Gamma}_{1h}$ can be connected by a polygonal line contained in Γ_1.

Lemma 2.2.5. (i) *Let $l = 1$, $\lambda_h \in \Lambda_h$, and assume* (Co) *if $d = 3$. Then $(\eta_h, \lambda_h)_{\Lambda_h} = 0$ for all $\eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)$ if and only if $g\lambda_h$ is constant at $\mathring{\Gamma}_{1h}$.*

(ii) *Let $d = 2$, $l = 2$ and $\lambda_h \in \Lambda_h$. Then $(\eta_h, \lambda_h)_{\Lambda_h} = 0$ for all $\eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)$ if and only if $g\lambda_h$ is constant at $\mathring{\Gamma}_{1h}$.*

(iii) *Let $d = 3$, $l = 2$ and $\lambda_h \in \Lambda_h$. Then $(\eta_h, \lambda_h)_{\Lambda_h} = 0$ for all $\eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)$ if and only if $\lambda_h = 0$ at $\mathring{\Gamma}_{1h} \cap \Sigma^1_h$ and $g\lambda_h$ is constant at $\mathring{\Gamma}_{1h} \cap \Sigma^2_h$.*

Proof. (i) We prove only for $d = 3$. First, assume $g\lambda_h = \delta \in \mathbb{R}$ at $\mathring{\Gamma}_{1h}$, and let $\eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)$. Then, in view of the trapezoidal formula, we conclude that

$$(\eta_h, \lambda_h)_{\Lambda_h} = \delta \sum_{S \in \mathscr{S}_h} \frac{|S|}{3} \sum_{i=1}^{3} \eta_h(M^i_S) = \delta \int_{\Gamma_1} \eta_h \, ds = 0.$$

Let us prove the converse direction. Let $S \in \mathscr{S}_h$ and $M_1, M_2 \in S \cap \mathring{\Gamma}_{1h}$ be distinct two points. Define $\eta_h \in \Lambda_h$ by

$$\eta_h(M_1) = \xi_2, \qquad \eta_h(M_2) = -\xi_1, \qquad \eta_h(N) = 0 \quad \text{if} \quad N \in \mathring{\Gamma}_{1h} \setminus \{M_1, M_2\},$$

where $\xi_i = \sum_{S \ni M_i} |S|/3$, $i = 1, 2$. Then the trapezoidal formula leads to $\int_{\Gamma_1} \eta_h \, ds = \xi_1 \eta_h(M_1) + \xi_2 \eta_h(M_2) = 0$, i.e. $\eta_h \in L_0^2(\Gamma_1)$. By assumption it follows that

$$0 = (\eta_h, \lambda_h)_{\Lambda_h} = \xi_1 \xi_2 (g\lambda_h(M_1) - g\lambda_h(M_2)),$$

and thus $g\lambda_h(M_1) = g\lambda_h(M_2)$. In view of (Co), we can repeat the same argument to deduce $g\lambda_h = \text{Const.}$ at $\mathring{\Gamma}_{1h}$.

Proofs of (ii)–(iii) are similar, so we omit them. (In (iii), note that $\int_{\Gamma_1} \eta_h \, ds = \sum_{S \in \mathscr{S}_h} |S|/3 \sum_{i=1}^3 \eta_h(m_S^i)$ does not involve values at $\mathring{\Gamma}_{1h} \cap \Sigma_h^1$.) $\square$

2.2.3 Error between j and j_h

For brevity, we prove the results of this subsection only for $d = 3$, but we can treat $d = 2$ in a similar (even easier) way. The $d = 2$ case is studied in [23, Chapter 4] where g is a constant, or in Section 1.3.5 of Chapter 1 where $l = 2$. We begin with the following elementary result.

Lemma 2.2.6. *Let S be a triangle with vertices $O(0,0)$, $A(a_1, a_2)$, $B(b_1, b_2)$, and η be a polynomial of degree ≤ 2 with respect to x_1, x_2. Then it follows that*

$$\int_S \left| \eta(x_1, x_2) - \eta(O) \right|^2 dx_1 dx_2 \le 4(\operatorname{diam} S)^2 \|\nabla \eta\|_{L^2(S)^2}^2.$$

Proof. Any point $P(x_1, x_2)$ in S can be represented as

$$x_1 = t(1-s)a_1 + tsb_1, \quad x_2 = t(1-s)a_2 + tsb_2 \qquad (0 \le t, s \le 1).$$

Writing $\tilde{\eta}(t, s) = \eta(x_1, x_2)$, we have

$$\int_S \left| \eta(x_1, x_2) - \eta(O) \right|^2 dx_1 dx_2 = \int_0^1 \int_0^1 \left| \tilde{\eta}(t,s) - \tilde{\eta}(0,s) \right|^2 J \, dt ds, \qquad (2.2.5)$$

where $J = 2|S|t$ is the Jacobian of the transformation $(t, s) \mapsto (x_1, x_2)$.

Here, since $\eta(x_1, x_2)$ is a quadratic polynomial, $\tilde{\eta}(t, s)$ is a quadratic polynomial of t for fixed s, so that $\partial\tilde{\eta}/\partial t$ is linear in t. By the midpoint formula, we have

$$\tilde{\eta}(t,s) - \tilde{\eta}(0,s) = \int_0^t \frac{\partial \tilde{\eta}}{\partial t}(t', s) \, dt' = t \frac{\partial \tilde{\eta}}{\partial t}\left(\frac{t'}{2}, s\right) = t\overrightarrow{OQ} \cdot \nabla \eta \left(\frac{x_1}{2}, \frac{x_2}{2}\right),$$

where $\overrightarrow{OQ} = (1/t)\overrightarrow{OP}$. Therefore,

$$|\tilde{\eta}(t,s) - \tilde{\eta}(0,s)|^2 \le (\operatorname{diam} S)^2 \left| \nabla \eta \left(\frac{x_1}{2}, \frac{x_2}{2}\right) \right|^2. \qquad (2.2.6)$$

From (2.2.5) and (2.2.6) we deduce

$$\int_S \left|\eta(x_1,x_2)-\eta(O)\right|^2 dx_1dx_2 \le (\operatorname{diam} S)^2 \int_0^1\int_0^1 \left|\nabla\eta\left(\frac{x_1}{2},\frac{x_2}{2}\right)\right|^2 J\,dtds$$
$$= (\operatorname{diam} S)^2 \int_S \left|\nabla\eta\left(\frac{x_1}{2},\frac{x_2}{2}\right)\right|^2 dx_1dx_2 \le 4(\operatorname{diam} S)^2\|\nabla\eta\|^2_{L^2(S)}.$$

This completes the proof. □

Each $S \in \mathscr{S}_h$ is divided into 6 sub-triangles S_i, $i=1,...,6$, as follows:

$$S_1 = \triangle G_S M_S^1 m_S^2, \qquad S_2 = \triangle G_S M_S^1 m_S^3, \qquad S_3 = \triangle G_S M_S^2 m_S^3,$$
$$S_4 = \triangle G_S M_S^2 m_S^1, \qquad S_5 = \triangle G_S M_S^3 m_S^1, \qquad S_6 = \triangle G_S M_S^3 m_S^2,$$

where G_S is the barycenter of S. For $\eta \in C(\overline{\Gamma}_1)$, we introduce the following piecewise constant approximations defined a.e. on Γ_1:

$$r_h^1(\eta) = \begin{cases} \eta(M_S^1) & \text{on } S_1\cup S_2,\\ \eta(M_S^2) & \text{on } S_3\cup S_4,\\ \eta(M_S^3) & \text{on } S_5\cup S_6,\end{cases} \qquad r_h^2(\eta) = \begin{cases} \eta(m_S^1) & \text{on } S_4\cup S_5,\\ \eta(m_S^2) & \text{on } S_6\cup S_1,\\ \eta(m_S^3) & \text{on } S_2\cup S_3.\end{cases}$$

We see that r_h^1 and r_h^2 are kind of lumping operators and that j_h is represented as

$$j_h(\eta) = \begin{cases} \int_{\Gamma_1} |r_h^1(g\eta)|\,ds & \text{if } l=1,1b,\\ (1-\kappa)\int_{\Gamma_1}|r_h^1(g\eta)|\,ds + \kappa\int_{\Gamma_1}|r_h^2(g\eta)|\,ds & \text{if } l=2.\end{cases}$$

Lemma 2.2.7. *Let $\eta_h \in \Lambda_h$. Then*

(i) $\|r_h^1\eta_h\|_{L^2(\Gamma_1)}$ *and* $\|r_h^2\eta_h\|_{L^2(\Gamma_1)}$ *are bounded by* $C\|\eta_h\|_{L^2(\Gamma_1)}$.

(ii) $\|r_h^1\eta_h-\eta_h\|_{L^2(\Gamma_1)}$ *and* $\|r_h^2\eta_h-\eta_h\|_{L^2(\Gamma_1)}$ *are bounded by* $Ch\|\eta_h\|_{H^1(\Gamma_1)}$.

Proof. (i) Consider $l=2$ and estimate $\|r_h^1\eta_h\|_{L^2(\Gamma_1)}$. Combining the concrete expression of $\|\eta_h\|^2_{L^2(\Gamma_1)}$ (see [22, p. 97]) with the following identical inequality

$$\frac{a^2+b^2+c^2}{3} \le 30\cdot\frac{1}{3}\left[\frac{a^2+b^2+c^2}{10} + \frac{8}{15}(d^2+e^2+f^2) - \frac{ab+bc+ca}{30} + \frac{8}{15}(de+ef+fd) - \frac{2}{15}(ae+bf+cd)\right],$$

we obtain $\|r_h^2\eta_h\|^2_{L^2(\Gamma_1)} \le 30\|\eta_h\|^2_{L^2(\Gamma_1)}$. In the same manner we have $\|r_h^1\eta_h\|^2_{L^2(\Gamma_1)} \le 30\|\eta_h\|^2_{L^2(\Gamma_1)}$. For $l=1,1b$, a similar computation yields $\|r_h^1\eta_h\|^2_{L^2(\Gamma_1)} \le 4\|\eta_h\|^2_{L^2(\Gamma_1)}$.

(ii) From Lemma 2.2.6 we obtain

$$\|r_h^1\eta_h - \eta_h\|^2_{L^2(\Gamma_1)} = \sum_{S\in\mathscr{S}_h}\sum_{i=1}^{3}\int_{S_{2i-1}\cup S_{2i}} |\eta_h - \eta_h(M_{iS})|^2\, ds \le 4h^2\|\nabla'\eta_h\|^2_{L^2(\Gamma_1)^2}.$$

$\|r_h^2\eta_h - \eta_h\|_{L^2(\Gamma_1)}$ can be estimated similarly. $\square$

Remark 2.2.4. A similar argument proves $\|\eta_h\|_{\Lambda_h} \le C(g)\|\eta_h\|_{L^2(\Gamma_1)}$.

We are ready to give an estimate for the error $j_h - j$.

Lemma 2.2.8. *Let $\eta_h \in \Lambda_h$ and $g \in W^{1,\infty}(\Gamma_1) \subset C(\overline{\Gamma}_1)$. Then we have*

$$|j_h(\eta_h) - j(\eta_h)| \le C(g)h^s\|\eta_h\|_{H^s(\Gamma_1)} \qquad (0 \le s \le 1).$$

Proof. We prove only for $l = 1, 1b$ since $l = 2$ can be treated similarly. We simply write r_h to indicate r_h^1. First, by a triangle inequality and Schwarz's inequality,

$$|j_h(\eta_h) - j(\eta_h)| \le \int_{\Gamma_1} |r_h(g\eta_h) - g\eta_h|\, ds \le \sqrt{|\Gamma_1|}\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)}. \tag{2.2.7}$$

On one hand, since $r_h(g\eta_h) = r_h g \cdot r_h\eta_h$ and $\|r_h g\|_{L^\infty(\Gamma_1)} \le \|g\|_{L^\infty(\Gamma_1)}$, we see from Lemma 2.2.7(i) that

$$\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)} \le C\|g\|_{L^\infty(\Gamma_1)}\|\eta_h\|_{L^2(\Gamma_1)}. \tag{2.2.8}$$

On the other hand, it holds that

$$\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)} \le \|r_h\eta_h(r_h g - g)\|_{L^2(\Gamma_1)} + \|g(r_h\eta_h - \eta_h)\|_{L^2(\Gamma_1)}.$$

In view of Lemma 2.2.7(i) and the Taylor expansion for g, the first term is bounded by $Ch\|\nabla' g\|_{L^\infty(\Gamma_1)^2}\|\eta_h\|_{L^2(\Gamma_1)}$. Thanks to Lemma 2.2.7(ii), the second term is majorized by $Ch\|g\|_{L^\infty(\Gamma_1)}\|\eta_h\|_{H^1(\Gamma_1)}$. Consequently,

$$\|r_h(g\eta_h) - g\eta_h\|_{L^2(\Gamma_1)} \le Ch\|g\|_{W^{1,\infty}(\Gamma_1)}\|\eta_h\|_{H^1(\Gamma_1)}. \tag{2.2.9}$$

The desired estimate follows from (2.2.7)–(2.2.9) in conjunction with an interpolation inequality between $L^2(\Gamma_1)$ and $H^1(\Gamma_1)$. $\square$

2.3 Analysis of discrete problems for LBCF

2.3.1 Existence and uniqueness

We propose two discrete problems approximating Problems VI and VE as follows:

Problem VI$_h$. Find $(u_h, p_h) \in V_{hn} \times Q_h$ such that, for all $v_h \in V_{hn}$ and $q_h \in Q_h$,

$$\begin{cases} a(u_h, v_h - u_h) + b(v_h - u_h, p_h) + j_h(v_{hn}) - j_h(u_{hn}) \geq (f, v_h - u_h), & (2.3.1a) \\ b(u_h, q_h) = c_h(p_h, q_h). & (2.3.1b) \end{cases}$$

Problem VE$_h$. Find $(u_h, p_h, \lambda_h) \in V_{hn} \times Q_h \times \tilde{\Lambda}_h$ such that

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) + (v_{hn}, \lambda_h)_{\Lambda_h} = (f, v_h) & (\forall v_h \in V_{hn}), \quad (2.3.2a) \\ b(u_h, q_h) = c_h(p_h, q_h) & (\forall q_h \in Q_h), \quad (2.3.2b) \\ (u_{hn}, \mu_h - \lambda_h)_{\Lambda_h} \leq 0 & (\forall \mu_h \in \tilde{\Lambda}_h). \quad (2.3.2c) \end{cases}$$

Here, $c_h(\cdot, \cdot)$ is a pressure-stabilizing term (effective only when $l = 1$) given by

$$c_h(p_h, q_h) = \gamma(h\nabla p_h, h\nabla q_h),$$

where $\gamma = 1$ if $l = 1$, $\gamma = 0$ if $l = 1b, 2$ is the same as in (2.2.1).

Remark 2.3.1. For brevity and in order to focus on essence of LBCF, we work with the simplest form of stabilizing terms. See e.g. [15] for more involved ones.

Theorem 2.3.1. *Problems VI$_h$ and VE$_h$ are equivalent.*

Proof. Let (u_h, p_h) be a solution of Problem VI$_h$. Taking $u_h \pm v_h$ as a test function in (2.3.1a), with $v_h \in \mathring{V}_h$ arbitrary, we have $a(u_h, v_h) + b(v_h, p_h) = (f, v_h)$. Since $\mathring{V}_h = \{v_h \in V_{hn} \mid (v_{hn}, \eta_h)_{\Lambda_h} = 0 \ (\forall \eta_h \in \Lambda_h)\}$, it follows from the inf-sup condition (2.2.4) that there exists a unique $\lambda_h \in \Lambda_h$ such that (2.3.2a) holds. Plugging this into (2.3.1a) gives

$$(v_{hn} - u_{hn}, \lambda_h)_{\Lambda_h} \leq j_h(v_{hn}) - j_h(u_{hn}) \leq j_h(v_{hn} - u_{hn}) \qquad (\forall v_h \in V_{hn}), \quad (2.3.3)$$

so that $(\eta_h, \lambda_h)_{\Lambda_h} \leq j_h(\eta_h)$ for all $\eta_h \in \Lambda_h$. By Lemma 2.2.4 this implies $\lambda_h \in \tilde{\Lambda}_h$. Taking $v_h = 0, 2u_h$ in (2.3.1a) and $v_h = u_h$ in (2.3.2a), we obtain $(u_{hn}, \lambda_h) = j_h(u_{hn})$. Thus (2.3.2c) follows from Lemma 2.2.3(ii). Therefore (u_h, p_h, λ_h) solves Problem VE$_h$.

Conversely, let (u_h, p_h, λ_h) be a solution of Problem VE$_h$. It follows from (2.3.2a) and (2.3.2c), combined with Lemma 2.2.3, that

$$\begin{aligned} & a(u_h, v_h - u_h) + b(v_h - u_h, p_h) + j_h(v_{hn}) - j_h(u_{hn}) - (f, v_h - u_h) \\ &= (u_{hn}, \lambda_h)_{\Lambda_h} - j_h(u_{hn}) - (v_{hn}, \lambda_h)_{\Lambda_h} + j_h(v_{hn}) \geq 0, \end{aligned}$$

which implies (u_h, p_h) solves Problem VI$_h$. $\square$

In the next theorem we show the well-posedness of Problem VE$_h$ and hence that of Problem VI$_h$, together with a discrete analogue of point 3) raised in Introduction. Before stating the results, we remark the coercivity of a, i.e. Korn's inequality:

$$a(v, v) \geq \alpha \|v\|_V^2 \qquad (\forall v \in V,\ v = 0 \text{ on } \Gamma_0), \tag{2.3.4}$$

where $\alpha > 0$ is a constant depending only on Ω (see [33, Lemma 6.2]).

Theorem 2.3.2. (i) *There exists a solution (u_h, p_h, λ_h) of Problem VE$_h$, u_h being uniquely determined.*

(ii) *If $(u_h, p_h^*, \lambda_h^*)$ is another solution, then for some $\delta_h \in \mathbb{R}$ we have:*

- *When $d = 2$ or $l = 1, 1b$, $p_h = p_h^* + \delta_h$ and $\lambda_h = \lambda_h^* + \delta_h/g$ at $\mathring{\Gamma}_{1h}$.*
- *When $d = 3$ and $l = 2$, it holds that $p_h = p_h^* + \kappa\delta_h$ and $\lambda_h = \lambda_h^*$ at $\mathring{\Gamma}_{1h} \cap \Sigma_h^1$, $\lambda_h = \lambda_h^* + \delta_h/g$ at $\mathring{\Gamma}_{1h} \cap \Sigma_h^2$.*

(iii) *In* (ii) *above, if $u_{hn} \not\equiv 0$ on Γ_1, then $\delta_h = 0$. Namely, the uniqueness of Problems VE$_h$ is valid.*

Proof. (i) When $l = 1b, 2$, a standard theory of elliptic variational inequalities (e.g. [22]) leads to the existence and uniqueness of $u_h \in V_{hn,\sigma}$ such that

$$a(u_h, v_h - u_h) + j_h(v_{hn}) - j_h(u_{hn}) \geq (f, v_h - u_h) \qquad (\forall v_h \in V_{hn,\sigma}). \tag{2.3.5}$$

Restricting test functions yields $a(u_h, v_h) = (f, v_h)$, $\forall v_h \in V_{hn,\sigma} \cap \mathring{V}_h$, so that, by (2.2.1), there exists a unique $\mathring{p}_h \in \mathring{Q}_h$ such that

$$a(u_h, v_h) + b(v_h, \mathring{p}_h) = (f, v_h) \qquad (\forall v_h \in \mathring{V}_h).$$

As done in the proof of Theorem 2.3.1, there exists a unique $\mathring{\lambda}_h \in \Lambda_h$ such that

$$a(u_h, v_h) + b(v_h, \mathring{p}_h) + (v_{hn}, \mathring{\lambda}_h)_{\Lambda_h} = (f, v_h) \qquad (\forall v_h \in V_{hn}). \tag{2.3.6}$$

Plugging this into (2.3.5) gives the same as (2.3.3), but this time only for all $v_h \in V_{hn,\sigma}$. Therefore, in view of Lemma 2.2.1(iii), we get $(\eta_h, \mathring{\lambda}_h)_{\Lambda_h} \le j_h(\eta_h)$, $\forall \eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)$. The Hahn-Banach and Riesz representation theorems then assert existence of some $\lambda_h \in \Lambda_h$ satisfying

$$(\eta_h, \lambda_h)_{\Lambda_h} \le j_h(\eta_h) \quad (\forall \eta_h \in \Lambda_h) \quad \text{ and } \quad (\eta_h, \lambda_h - \mathring{\lambda}_h)_{\Lambda_h} = 0 \quad (\forall \eta_h \in \Lambda_h \cap L^2_0(\Gamma_1)).$$

By Lemma 2.2.4, $\lambda_h \in \tilde{\Lambda}_h$. We see from Lemma 2.2.5 that, for some $\mathring{\delta}_h \in \mathbb{R}$,

$$\begin{cases} \lambda_h = \mathring{\lambda}_h + \mathring{\delta}_h/g \quad \text{at } \mathring{\Gamma}_{1h} & \text{if } d = 2 \text{ or } l = 1b, \\ \lambda_h = \mathring{\lambda}_h \quad \text{at } \mathring{\Gamma}_{1h} \cap \Sigma^1_h, \quad \lambda_h = \mathring{\lambda}_h + \mathring{\delta}_h/g \quad \text{at } \mathring{\Gamma}_{1h} \cap \Sigma^2_h & \text{if } d = 3 \text{ and } l = 2. \end{cases}$$

Setting $p_h = \mathring{p}_h + \mathring{\delta}_h$ ($p_h = \mathring{p}_h + \kappa\mathring{\delta}_h$ if $d = 3, l = 2$) and noting that $\int_{\Gamma_1} v_{hn}\, ds$ can be represented by a numerical integration formula corresponding to $(\cdot,\cdot)_{\Lambda_h}$, we deduce $b(v_h, p_h) + (v_{hn}, \lambda_h)_{\Lambda_h} = b(v_h, \mathring{p}_h) + (v_{hn}, \mathring{\lambda}_h)_{\Lambda_h}$, which, combined with (2.3.6), gives (2.3.2a). The derivation of (2.3.2c) is similar to that done in Theorem 2.3.1.

When $l = 1$, there exists a unique $(u_h, \mathring{p}_h) \in \tilde{V}_{hn} \times \mathring{Q}_h$ such that

$$B_h(u_h, p_h; v_h - u_h, p_h - q_h) + j_h(v_{hn}) - j_h(u_{hn}) \ge (f, v_h - u_h) \qquad (\forall (v_h, q_h) \in \tilde{V}_{hn} \times \mathring{Q}_h), \quad (2.3.7)$$

where $\tilde{V}_{hn} = \{v_h \in V_{hn} \,|\, b(v_h, 1) = -\int_{\Gamma_1} v_{hn}\, ds = 0\}$, and

$$B_h(u_h, p_h; v_h, q_h) := a(u_h, v_h) + b(v_h, p_h) - b(u_h, q_h) + c_h(p_h, q_h)$$

is coercive on $V_{hn} \times \mathring{Q}_h$ (but not on $V_{hn} \times Q_h$). Using (2.3.7) instead of (2.3.5), we can proceed as in $l = 1b, 2$ to obtain the conclusion.

(ii) This can be proved by a calculation similar to that in the proof of (i).

(iii) Because (2.3.2c) implies that $u_{hn}\lambda_h = |u_{hn}|$ at $\mathring{\Gamma}_{1h}$ by Lemma 2.2.3(ii), the following discrete leak/no-leak detecting condition is valid at $\mathring{\Gamma}_{1h}$:

$$|\lambda_h| < 1 \Rightarrow u_{hn} = 0, \quad u_{hn} > 0 \Rightarrow \lambda_h = 1, \quad u_{hn} < 0 \Rightarrow \lambda_h = -1. \quad (2.3.8)$$

Therefore, since $u_{hn} \not\equiv 0$ and $\int_{\Gamma_1} u_{hn}\, ds = 0$, λ_h must attain $+1$ and -1 somewhere at $\mathring{\Gamma}_{1h}$. Now let $(u_h, p^*_h, \lambda^*_h)$ be another solution of Problem VE$_h$. Then, since $\lambda^*_h \in \tilde{\Lambda}_h$, $|\lambda^*_h| \le 1$ at $\mathring{\Gamma}_{1h}$. This prevents δ_h in (ii) from being other than 0. □

2.3.2 Convergence proof under minimal regularity

We know the exact solution (u, p) belongs to $H^1(\Omega)^d \times L^2(\Omega)$; however, whether it has higher regularity seems unknown in our setting. (For smooth Ω with non-mixed b.c. case, i.e. $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$, [57] establishes H^2-H^1 regularity, and results for Poisson's equation in a polygon or polyhedron with mixed b.c. are found in [4].) Thereby we prove convergence without assuming higher regularity.

Theorem 2.3.3. *Let $g \in W^{1,\infty}(\Gamma_1)$, and assume (u, p) and (u_h, p_h) are solutions of Problems VI and VI$_h$ respectively. Then $u_h \to u$ strongly in V and $\mathring{p}_h \rightharpoonup \mathring{p}$ weakly in Q, where $\mathring{p} = p - \frac{1}{|\Omega|}(p, 1)_Q \in \mathring{Q}$, etc.*

Proof. Taking $v_h = 0, 2u_h$ in (2.3.1a), we obtain

$$a(u_h, u_h) + j_h(u_{hn}) + \gamma\|h\nabla p_h\|^2 = (f, u_h). \tag{2.3.9}$$

According to (2.3.4) and $j_h(\cdot) \geq 0$, this implies $\|u_h\|_V + \gamma\|h\nabla p_h\| \leq C(f)$. From Lemma 2.2.2 and (2.3.2a) we find that

$$\beta\|p_h\|_Q \leq \sup_{v_h \in V_{hn}} \frac{(f, v_h) - a(u_h, v_h) - (v_{hn}, \lambda_h)_{\Lambda_h}}{\|v_h\|_V} + \gamma C\|h\nabla p_h\| \leq C(f, g), \tag{2.3.10}$$

where we have used $\|v_{hn}\|_{\Lambda_h} \leq C(g)\|v_h\|_V$ (see Remark 2.2.4) and $\lambda_h \in \tilde{\Lambda}_h$.

Therefore, (u_h, p_h) admits a subsequence, denoted by the same symbol, which converges to some (u, p) weakly in $V_n \times Q$. Let us prove (u, p) solves Problem VI. In view of density arguments, it suffices to show (2.1.5a)–(2.1.5b) for smooth v and q. Then there exists $(v_h, q_h) \in V_{hn} \times Q_h$ such that $v_h \to v$ in V, $q_h \to q$ in Q and $h\nabla q_h \to 0$ in $L^2(\Omega)^d$ when $h \to 0$ (see (2.3.13) below). Here, it follows from (2.3.1a)–(2.3.1b) (especially, $b(u_h, p_h) \geq 0$) that

$$a(u_h, v_h) + b(v_h, p_h) + j_h(v_{hn}) - (f, v_h) \geq a(u_h, u_h) + j_h(u_{hn}) - (f, u_h). \tag{2.3.11}$$

Since $\|u_{hn}\|_{H^{1/2}(\Gamma_1)} \leq C(f)$, Lemma 2.2.8 combined with $u_{hn} \to u_n$ in $L^2(\Gamma_1)$ leads to

$$|j_h(u_{hn}) - j(u_n)| \leq |j_h(u_{hn}) - j(u_{hn})| + |j(u_{hn}) - j(u_n)| \to 0.$$

Similarly, $j_h(v_{hn}) \to j(v_n)$. Taking the (lower) limit in (2.3.11) and (2.3.1b) concludes (2.1.5a)–(2.1.5b). The strong convergence with respect to velocity derives from (2.3.9) and its analogue for (2.1.5a) as follows:

$$\begin{aligned}a(u-u_h,u-u_h)+\gamma\|h\nabla p_h\|^2 &= a(u,u)-2a(u,u_h)+a(u_h,u_h)+\gamma\|h\nabla p_h\|^2\\ &= a(u,u)-2a(u,u_h)+(f,u_h)-j_h(u_{hn})\\ &\to -a(u,u)+(f,u)-j(u_n)=0.\end{aligned}$$

By the uniqueness for Problem VI up to an additive constant of p, the whole sequence $\mathring{p}_h$ actually converges weakly. □

Remark 2.3.2. If $u_n \neq 0$ and thus p is also uniquely determined, then we can prove the whole-sequence convergence for p_h instead of $\mathring{p}_h$.

2.3.3 Convergence order estimates

We begin with:

Proposition 2.3.1. *Let $g \in W^{1,\infty}(\Gamma_1)$ and (u,p), (u_h,p_h) be solutions of Problems VI and VI$_h$. Then, for arbitrary $v_h \in V_{hn}$, $q_h \in Q_h$ and $0 \le s \le 1$, we have*

$$\begin{aligned}&\|u-u_h\|_V+\|\mathring{p}-\mathring{p}_h\|_Q+\gamma\|h\nabla p_h\|\\ &\le C(f,g,u,p)\Big(\|u-v_h\|_V+\|p-q_h\|_Q+\gamma h\|\nabla q_h\|+\|u_n-v_{hn}\|_{L^2(\Gamma_1)}^{1/2}\\ &\qquad+h^{1/2}\|v_{hn}\|_{H^1(\Gamma_1)}^{1/2}+h^{s/2}\|u_{hn}\|_{H^s(\Gamma_1)}^{1/2}\Big).\end{aligned}$$

Proof. It is obvious that $a(u-u_h,u-u_h)=a(u-u_h,u-v_h)+a(u,v_h-u)-a(u,u_h-u)-a(u_h,v_h-u_h)$. Using (2.3.4) to the LHS and substituting (2.1.6a), (2.1.5a), and (2.3.1a) into the second, third, and fourth terms of the RHS respectively, we deduce

$$\begin{aligned}\alpha\|u-u_h\|_V^2 &\le a(u-u_h,u-v_h)+b(u_h-v_h,p-p_h)\\ &\qquad-(\lambda,v_{hn}-u_n)_\Lambda+j(u_{hn})-j_h(u_{hn})+j_h(v_{hn})-j(u_n).\end{aligned}$$

Since $b(u_h,q_h)=\gamma h^2(\nabla p_h,\nabla q_h)$ for $q_h\in Q_h$ and $\operatorname{div}u=0$, it follows that

$$\begin{aligned}&\alpha\|u-u_h\|_V^2+\gamma\|h\nabla p_h\|^2\\ &\le a(u-u_h,u-v_h)+b(u-v_h,p-p_h)+b(u_h-u,p-q_h)+\gamma h^2(\nabla p_h,\nabla q_h)\\ &\qquad-(\lambda,v_{hn}-u_n)_\Lambda+j(u_{hn})-j_h(u_{hn})+j_h(v_{hn})-j(u_n).\end{aligned}\tag{2.3.12}$$

Furthermore, denoting by $\|a\|,\|b\|$ the operator norms of a,b and noticing that

$$a(u-u_h,v_h)+b(v_h,\mathring{p}-\mathring{p}_h)=0\qquad(\forall v_h\in\mathring{V}_h),$$

which is obtained thanks to (2.1.6a) and (2.3.2a), we find from (2.2.1) that

$$\begin{aligned}\|\mathring{p} - \mathring{p}_h\|_Q &\le (1 + \frac{\|b\|}{\beta})\|\mathring{p} - \mathring{q}_h\|_Q + \frac{\|a\|}{\beta}\|u - u_h\|_V + \frac{\gamma C}{\beta}\|h\nabla(\mathring{p}_h - \mathring{q}_h)\| \\ &\le C\|p - q_h\|_Q + C\|u - u_h\|_V + \gamma C\|h\nabla p_h\| + \gamma C h\|\nabla q_h\|.\end{aligned}$$

Let us estimate each term appearing in the RHS of (2.3.12) as follows.

1) $|a(u - u_h, u - v_h)| \le \frac{\alpha}{4}\|u - u_h\|_V^2 + C\|u - v_h\|_V^2$.

2) $b(u - v_h, p - p_h) = b(u - v_h, \mathring{p} - \mathring{p}_h) + b(u - v_h, \frac{1}{|\Omega|}(p - p_h, 1)_Q)$, and

- $|b(u - v_h, \mathring{p} - \mathring{p}_h)|$ is bounded by
$$C\|u - v_h\|_V^2 + C\|p - q_h\|_Q^2 + \frac{\alpha}{4}\|u - u_h\|_V^2 + \frac{\gamma}{4}\|h\nabla p_h\|^2 + \gamma C h^2\|\nabla q_h\|^2.$$
- $|b(u - v_h, \frac{1}{|\Omega|}(p - p_h, 1)_Q)|$ is bounded by
$$C(\|p\|_Q + \|p_h\|_Q)\int_{\Gamma_1}|u_n - v_{hn}|\,ds \le C(f, g, p)\|u_n - v_{hn}\|_{L^2(\Gamma_1)}.$$

3) $|b(u_h - u, p - q_h)| \le \frac{\alpha}{4}\|u - u_h\|_V^2 + C\|p - q_h\|_Q^2$.

4) $|\gamma h^2(\nabla p_h, \nabla q_h)| \le \frac{\gamma}{4}\|h\nabla p_h\|^2 + Ch^2\|\nabla q_h\|^2$.

5) $|(\lambda, v_{hn} - u_n)_\Lambda| \le C(g)\|u_n - v_{hn}\|_{L^2(\Gamma_1)}$.

6) By Lemma 2.2.8, $|j(u_{hn}) - j_h(u_{hn})| \le C(g)h^s\|u_{hn}\|_{H^s(\Gamma_1)}$.

7) $j_h(v_{hn}) - j(u_n) = j_h(v_{hn}) - j(v_{hn}) + j(v_{hn}) - j(u_n)$, and

- $|j_h(v_{hn}) - j(v_{hn})| \le C(g)h\|v_{hn}\|_{H^1(\Gamma_1)}$.
- $|j(v_{hn}) - j(u_n)| \le C(g)\|u_n - v_{hn}\|_{L^2(\Gamma_1)}$.

Collecting these estimates, we obtain the conclusion. □

Remark 2.3.3. Estimation for $\|p - p_h\|_Q$ is not trivial, because we have to deal with the error between $(v_{hn}, \lambda_h)_{\Lambda_h}$ and $(v_{hn}, \lambda)_\Lambda$.

Here we presume the regularity $u \in H^2(\Omega)^d$, $p \in H^1(\Omega)$, which is not guaranteed in general as pointed out in Section 2.3.2. However, we can provide a situation where it holds. For instance, it is valid when Ω is convex and $u = 0$ in a neighborhood (w.r.t. Γ) of $\partial\Gamma_1$, which can be achieved if g is sufficiently large there.

Letting $I_h : V_n \cap C(\overline{\Omega})^d \to V_{hn}$ and $R_h : Q \to Q_h$ be the Lagrange interpolation and a local regularization operators respectively, we find that

$$\begin{cases} \|u - I_h u\|_V + h^{-1/2}\|u_n - (I_h u)_n\|_{L^2(\Gamma_1)} \le Ch\|u\|_{H^2(\Omega)^d}, \\ \|p - R_h p\|_Q \le Ch\|p\|_{H^1(\Omega)}, \end{cases} \tag{2.3.13}$$

and that $\|(I_h u)_n\|_{H^1(\Gamma_1)} \le C\|u\|_{H^2(\Omega)^d}$, $\|R_h p\|_{H^1(\Omega)} \le C\|p\|_{H^1(\Omega)}$. Choosing $v_h = I_h u$ and $q_h = R_h p$ in Proposition 2.3.1 with $s = 1/2$ (note that $\|u_{hn}\|_{H^{1/2}(\Gamma_1)} \le C(f)$ by Theorem 2.3.3) leads to the following theorem.

Theorem 2.3.4. *Under the assumptions of Proposition 2.3.1, we assume $u \in H^2(\Omega)^d$ and $p \in H^1(\Omega)$. Then*

$$\|u - u_h\|_V + \|\mathring{p} - \mathring{p}_h\|_Q \le C(f, g, u, p)h^{1/4}.$$

If an inverse inequality between $H^1(\Gamma_1)$ and $H^{1/2}(\Gamma_1)$ is available, one can improve the rate of convergence to $O(h^{1/2})$. In fact, it follows that

$$\begin{aligned} \|u_{hn}\|_{H^1(\Gamma_1)} &\le \|u_{hn} - (I_h u)_n\|_{H^1(\Gamma_1)} + \|(I_h u)_n\|_{H^1(\Gamma_1)} \\ &\le Ch^{-1/2}\|u_{hn} - (I_h u)_n\|_{H^{1/2}(\Gamma_1)} + C\|u\|_{H^2(\Omega)^d} \\ &\le Ch^{-1/2}\|u_h - I_h u\|_{H^1(\Omega)} + C\|u\|_{H^2(\Omega)^d} \\ &\le Ch^{-1/2}\|u_h - u\|_{H^1(\Omega)} + C\|u\|_{H^2(\Omega)^d}. \end{aligned}$$

Therefore,

$$h^{1/2}\|u_{hn}\|_{H^1(\Gamma_1)}^{1/2} \le Ch^{1/4}\|u - u_h\|_{H^1(\Omega)}^{1/2} + C(u)h^{1/2},$$

which combined with Proposition 2.3.1 for $s = 1$ concludes the following result.

Corollary 2.3.1. *Under the assumptions of Theorem 2.3.4, if $\mathscr{S}_h$ is a quasi-uniform mesh of Γ_1, then*

$$\|u - u_h\|_V + \|\mathring{p} - \mathring{p}_h\|_Q \le C(f, g, u, p)h^{1/2}.$$

Remark 2.3.4. In particular, $\|u_{hn}\|_{H^1(\Gamma_1)}$ is bounded uniformly in h.

2.3.4 Sign-conditions to obtain optimal convergence

The error estimate $O(h^{1/2})$ shown in the previous section is suboptimal, but is the best possible one as far as we rely on Lemma 2.2.8 to bound $|j_h(\eta_h) - j(\eta_h)|$. Thereby one might ask if the result of Lemma 2.2.8 can be improved. Unfortunately, this possibility is not promising for a general integrand η_h, in view of [31, Section 4] showing an example of η_h such that $|j_h(\eta_h) - j(\eta_h)| = Ch^s \|\eta_h\|_{H^s(\Gamma_1)}$ for $s = 1/2$.

However, for a special η_h which does not change its sign so rapidly from plus to minus or vice versa, it can be improved. Actually j_h becomes even exact in some cases. For instance, provided that η_h is positive (or negative) definite on Γ_1 and that g is a constant, it is immediate to see $j_h(\eta_h) = j(\eta_h)$ except in the case $d = 3, l = 2$ (cf. Remark 2.2.3).

The aim of this subsection is to present a reasonable sufficient condition to obtain an optimal rate of convergence for $d = 2$, $l = 1$ or $1b$, in terms of the sign of normal component of velocity on Γ_1. Such a strategy is already considered in several unilateral problems (e.g. [2, 5, 7]).

Henceforth we focus on $d = 2$, $l = 1$ or $1b$, and denote by i_h the one-dimensional P1 Lagrange interpolation operator defined on $C(\overline{\Gamma}_1)$. In particular, $i_h(v_n) = (I_h v)_n$ for $v \in V_n \cap C(\overline{\Omega})$, since n is constant on Γ_1.

Lemma 2.3.1. *Let $g \in W^{2,\infty}(\Gamma_1)$. Then, for all $\eta_h \in \Lambda_h$ we have*

$$\|g\eta_h - i_h(g\eta_h)\|_{L^1(\Gamma_1)} \le Ch^2 \|g\|_{W^{2,\infty}(\Gamma_1)} \|\eta_h\|_{H^1(\Gamma_1)}.$$

Proof. Because η_h is piecewise linear, its second derivative vanishes on each $S \in \mathscr{S}_h$. Therefore, a local interpolation result tells us that

$$\begin{aligned}\sum_{S \in \mathscr{S}_h} \|g\eta_h - i_h(g\eta_h)\|_{L^1(S)} &\le \sum_{S \in \mathscr{S}_h} Ch^2 \|g\eta_h\|_{W^{2,1}(S)} \\ &\le Ch^2 \|g\|_{W^{2,\infty}(\Gamma_1)} \|\eta_h\|_{H^1(\Gamma)},\end{aligned}$$

which completes the proof. □

For $\eta \in C(\overline{\Gamma}_1)$, we introduce

$$\begin{aligned}&\mathcal{P}_h(\eta) = \{S \in \mathscr{S}_h \,|\, \eta \ge 0 \text{ on } S\}, \quad \mathcal{N}_h(\eta) = \{S \in \mathscr{S}_h \,|\, \eta \le 0 \text{ on } S\}, \\ &\mathcal{Z}_h(\eta) = \mathscr{S}_h \setminus (\mathcal{P}_h(\eta) \cup \mathcal{N}_h(\eta)).\end{aligned}$$

Because we are considering P1 finite elements on $\overline{\Gamma}_1$, it holds that

$$\mathcal{P}_h(\eta) \subset \mathcal{P}_h(i_h\eta), \quad \mathcal{N}_h(\eta) \subset \mathcal{N}_h(i_h\eta). \tag{2.3.14}$$

We are now ready to state the following result.

Theorem 2.3.5. *In addition to the setting of Corollary 2.3.1, we assume that $d = 2$, $l = 1$ or $1b$, that $g \in W^{2,\infty}(\Gamma_1)$, and that*

(A1) $p - p_h \in \mathring{Q}$.

(A2) $u_n \in W^{1,\infty}(\Gamma_1)$.

(A3) the cardinary numbers of $\mathcal{Z}_h(u_n)$ *and* $\mathcal{Z}_h(u_{hn})$ *are bounded uniformly in* h.

Then we obtain

$$\|u - u_h\|_V + \|p - p_h\|_Q \le C(f, g, u, p)h.$$

Proof. Let us go back to (2.3.12) where $v_h = I_h u$ and $q_h = R_h p$. Assumption (A1) implies $p - p_h = \mathring{p} - \mathring{p}_h$, so that the first four terms in the RHS of (2.3.12) does not cause any sub-optimality if they are treated in the same way as in Proposition 2.3.1. Therefore it suffices to prove that the remaining terms

$$-(\lambda, v_{hn} - u_n)_\Lambda + j(u_{hn}) - j_h(u_{hn}) + j_h(v_{hn}) - j(u_n),$$

can be estimated by $C(f, g, u, p)h^2$. One notices that this expression equals

$$-(\lambda, v_{hn})_\Lambda + j_h(v_{hn}) + j(u_{hn}) - j_h(u_{hn}),$$

since $\sigma_n u_n + g|u_n| = 0$ and $\sigma_n = -g\lambda$.

First we give an estimation for $-(\lambda, v_{hn})_\Lambda + j_h(v_{hn})$. It follows from (2.1.7) and (2.3.14), together with a triangle inequality, that this quantity is bounded by

$$\sum_{S \in \mathcal{P}_h(u_n) \cup \mathcal{N}_h(u_n)} \int_S |-g v_{hn} + i_h(g v_{hn})| \, ds$$
$$+ \sum_{S \in \mathcal{Z}_h(u_n)} \left(\int_S g|v_{hn} - u_n| \, ds + \int_S g|u_n| \, ds + \frac{|S|}{2} \sum_{i=1}^{2} g|u_n|(M_S^i) \right).$$

By Lemma 2.3.1 the first term is estimated as $Ch^2 \|g\|_{W^{2,\infty}(\Gamma_1)} \|u\|_{H^2(\Omega)}$. If $S \in \mathcal{Z}_h(u_n)$, then u_n admits a zero point in S, and hence $|u_n|_{L^\infty(S)} \le h\|u_n\|_{W^{1,\infty}(S)}$. This observation combined with (A3) shows that the second term is bounded by $C(g, u)h^2$. Consequently, $|-(\lambda, v_{hn})_\Lambda + j_h(v_{hn})| \le C(g, u)h^2$.

Next we give an estimation for $j(u_{hn}) - j_h(u_{hn})$. This is bounded by

$$\sum_{S \in \mathcal{P}_h(u_{hn}) \cup \mathcal{N}_h(u_{hn})} \int_S |-g u_{hn} + i_h(g u_{hn})| \, ds + \sum_{S \in \mathcal{Z}_h(u_{hn})} \Bigg\{ \int_S (g - g(m_S))|u_{hn}| \, ds$$
$$+ g(m_S) \underbrace{\Big(\int_S |u_{hn}| \, ds - \frac{|S|}{2} \sum_{i=1}^{2} |u_{hn}(M_S^i)| \Big)}_{\le 0} + \frac{|S|}{2} \sum_{i=1}^{2} (g(m_S) - g(M_S^i))|u_{hn}(M_S^i)| \Bigg\}.$$

By Lemma 2.3.1 and Remark 2.3.4, the first term is estimated as $C(f, g, u, p)h^2$. The second terms is bounded by $Ch\|g\|_{W^{1,\infty}(\Gamma_1)}\|u_{hn}\|_{L^\infty(\Gamma_1)}\sum_{S\in\mathcal{Z}_h(u_{hn})}|S|$ (for the inequality at the underbrace, see [22, Theorem 5.4]). Therefore, assumption (A3) concludes $|j(u_{hn}) - j_h(u_{hn})| \le C(f, g, u, p)h^2$, and this completes the proof of Theorem 2.3.5. □

Remark 2.3.5. (i) If g is a constant, the boundedness of $\|u_{hn}\|_{H^1(\Gamma_1)}$ (and thus the quasi-uniformity of $\mathscr{S}_h$) and that of the cardinary of $\mathcal{Z}_h(u_{hn})$ are not necessary.

(ii) We can drop assumption (A2) at the expense of replacing the estimate $O(h)$ with $O(h|\log h|^{1/2})$, by arguing as in [7, Lemma 2.5].

(iii) Extension of the above strategy to $d = 3$ or $l = 2$ is not trivial. For $d = 2$, $l = 2$, Theorem 1.4.3 of Chapter 1 provides a sufficient condition for the optimal convergence $O(h^2)$ in the slip b.c. problem, which is more restrictive than (A2)–(A3). For $d = 3$, we have to start from finding a good counterpart to (A3), which does not seem to be reported in the existing works.

2.4 Numerical implementation

By virtue of Lemma 2.2.3(ii), Problem VE$_h$ is rewritten as:

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) + (v_{hn}, \lambda_h)_{\Lambda_h} = (f, v_h) & (\forall v_h \in V_{hn}), \quad (2.4.1a) \\ b(u_h, q_h) = c_h(p_h, q_h) & (\forall q_h \in Q_h), \quad (2.4.1b) \\ \lambda_h = \tilde{P}_h(\lambda_h + \rho u_{hn}) & (\rho > 0). \quad (2.4.1c) \end{cases}$$

In what follows, we present two approaches to compute a numerical solution of (2.4.1a)–(2.4.1c). The first method is based on a classical Uzawa algorithm.

Algorithm 2.4.1 (Uzawa method)**.**

Step 1. Choose $\lambda_h^{(1)} \in \tilde{\Lambda}_h$ and $\rho > 0$.

Step 2. With $\lambda_h^{(k)}$ known, determine $(u_h^{(k)}, p_h^{(k)}) \in V_{hn} \times Q_h$ such that

$$\begin{cases} a(u_h^{(k)}, v_h) + b(v_h, p_h^{(k)}) = (f, v_h) - (v_{hn}, \lambda_h^{(k)})_{\Lambda_h} & (\forall v_h \in V_{hn}), (2.4.2a) \\ b(u_h^{(k)}, q_h) = c_h(p_h^{(k)}, q_h) & (\forall q_h \in Q_h). \ (2.4.2b) \end{cases}$$

The well-posedness of (2.4.2a)–(2.4.2b) is guaranteed by Lemma 2.2.2.

Step 3. Set $\lambda_h^{(k+1)} = \tilde{P}_h(\lambda_h^{(k)} + \rho u_{hn}^{(k)})$.

Step 4. Iterate Steps 2–3 until convergence.

The second method is motivated by a primal-dual active set (PDAS) strategy known in optimal control theory (e.g. [60]).

Algorithm 2.4.2 (Active/inactive set method)**.**

Step 1. Choose $\rho > 0$ and set the initial state to the no-leak one, that is, find $u_h^{(1)} \in \mathring{V}_h$, $p_h^{(1)} \in \mathring{Q}_h$, $\lambda_h^{(1)} \in \Lambda_h$ such that (2.4.2a)–(2.4.2b) holds with $k = 1$.

Step 2. With $(u_h^{(k-1)}, p_h^{(k-1)}, \lambda_h^{(k-1)})$ known, define the active set A_k by

$$A_k = \{M \in \mathring{\Gamma}_{1h} \mid |\lambda_h^{(k-1)} + \rho u_h^{(k-1)}| > 1 \ \text{ at } M\}.$$

Determine $\lambda_h^{(k)}$ at A_k by

$$\lambda_h^{(k)} = \tilde{P}_h(\lambda_h^{(k-1)} + \rho u_{hn}^{(k-1)}) \quad \text{at} \ \ A_k.$$

Step 3. Define the inactive set $I_k = \mathring{\Gamma}_{1h} \backslash A_k$. Determine $(u_h^{(k)}, p_h^{(k)}) \in V_{hn} \times Q_h$ such that $u_{hn} = 0$ at I_k and

$$\begin{cases} a(u_h^{(k)}, v_h) + b(v_h, p_h^{(k)}) = (f, v_h) - (v_{hn}, \lambda_h^{(k)})_{\Lambda_h} & (\forall v_h \in V_h,\ v_{hn} = 0 \text{ at } I_k), \\ b(u_h^{(k)}, q_h) = c_h(p_h^{(k)}, q_h) & (\forall q_h \in Q_h). \end{cases}$$

Determine $\lambda_h^{(k)}$ at I_k in such a way that

$$(v_{hn}, \lambda_h^{(k)})_{\Lambda_h} = (f, v_h) - a(u_h^{(k)}, v_h) - b(v_h, p_h^{(k)}) \qquad (\forall v_h \in V_{hn}).$$

Step 4. Iterate Steps 2–3 until convergence.

Let us discuss the convergence of these two algorithms.

Theorem 2.4.1. (i) *In Algorithm 2.4.1, there exists $\rho_0 = \rho_0(g, \Omega)$ such that if $\rho < \rho_0$ then $(u_h^{(k)}, p_h^{(k)}, \lambda_h^{(k)})$ converges to a solution of Problem VE$_h$.*

(ii) *In Algorithm 2.4.2, if A_k is invariant for $k \geq k_0$, then the same conclusion as in* (i) *holds.*

Proof. (i) The convergence proof for $u_h^{(k)}$ is standard, but that for an additive constant requires a delicate argument. Let (u_h, p_h, λ_h) be a solution of Problem VE$_h$. We use the notations $\bar{u}_h^{(k)} = u_h^{(k)} - u_h$ etc., and $\mathring{p}_h = p_h - (p_h, 1)_Q/|\Omega|$ etc. Then, for all $v_h \in V_{hn}$ and $q_h \in Q_h$ one obtains

$$a(\bar{u}_h^{(k)}, v_h) + b(v_h, \bar{p}_h^{(k)}) + (v_{hn}, \bar{\lambda}_h^{(k)})_{\Lambda_h} = 0, \qquad b(\bar{u}_h^{(k)}, q_h) = c_h(\bar{p}_h^{(k)}, q_h). \tag{2.4.3}$$

Since $\tilde{P}_h$ is a contraction, it follows that

$$\|\bar{\lambda}_h^{(k+1)}\|_{\Lambda_h}^2 \le \|\bar{\lambda}_h^{(k)} + \rho \bar{u}_{hn}^{(k)}\|_{\Lambda_h}^2 = \|\bar{\lambda}_h^{(k)}\|_{\Lambda_h}^2 + 2\rho(\bar{\lambda}_h^{(k)}, \bar{u}_{hn}^{(k)})_{\Lambda_h} + \rho^2 \|\bar{u}_{hn}^{(k)}\|_{\Lambda_h}^2,$$

which, combined with the equations above, gives

$$\|\bar{\lambda}_h^{(k+1)}\|_{\Lambda_h}^2 + (2\alpha\rho - C(g)\rho^2)\|\bar{u}_h^{(k)}\|_V^2 + 2\rho\gamma\|h\nabla \bar{p}_h^{(k)}\|^2 \le \|\bar{\lambda}_h^{(k)}\|_{\Lambda_h}^2. \tag{2.4.4}$$

If $0 < \rho < 2\alpha/C(g) =: \rho_0$, we find that $\|\bar{\lambda}_h^{(k)}\|_{\Lambda_h}$ is decreasing in k and that $u_h^{(k)} \to u_h$ when $k \to \infty$. Then the inf-sup condition (2.2.1) yields $\mathring{p}_h^{(k)} \to \mathring{p}_h$.

Let us discuss the convergence of $\delta_h^{(k)} := p_h^{(k)} - \mathring{p}_h^{(k)}$ and $\lambda_h^{(k)}$. The uniform boundedness in k of $\bar{\lambda}_h^{(k)}$ is obvious, and that of $\bar{\delta}_h^{(k)}$ also follows from Lemma 2.2.2. Therefore, there exist subsequences $\bar{\delta}_h^{(k')} \to \bar{\delta}_h$ and $\bar{\lambda}_h^{(k')} \to \bar{\lambda}_h$ for some $\bar{\delta}_h \in \mathbb{R}$ and $\bar{\lambda}_h \in \tilde{\Lambda}_h$. Now we claim that

$$\bar{\lambda}_h = \bar{\delta}_h/g \quad \text{at} \quad \mathring{\Gamma}_{1h}. \tag{2.4.5}$$

When $d = 3$ and $l = 2$, we need to replace (2.4.5) and some relations below by their counterparts (cf. Theorem 2.3.2(ii)), but to save space we omit the detail.

Let us prove (2.4.5) by contradiction. If (2.4.5) is false, then there exists a positive constant $C = C(h, g, \bar{\lambda}_h, \bar{\delta})$, independent of k', such that

$$C(\|\bar{\delta}_h^{(k')}\|_Q + \|\bar{\lambda}_h^{(k')}\|_{\Lambda_h}) \le \sup_{v_h \in V_{hn}} \frac{b(v_h, \bar{\delta}_h^{(k')}) + (v_{hn}, \bar{\lambda}_h^{(k')})_{\Lambda_h}}{\|v_h\|_V}. \tag{2.4.6}$$

In fact, taking $\eta_h \in \Lambda_h$ such that $\eta_h = \bar{\lambda}_h^{(k')} - \bar{\delta}_h^{(k')}/g$ at $\mathring{\Gamma}_{1h}$ and lifting it to $v_h \in V_{hn}$ by Lemma 2.2.1(i), we see that $\|v_h\|_V \le C$ and that

$$\begin{aligned} b(v_h, \bar{\delta}_h^{(k')}) + (v_{hn}, \bar{\lambda}_h^{(k')})_{\Lambda_h} &= -\bar{\delta}_h^{(k')} \int_{\Gamma_1} v_{hn}\, ds + (v_{hn}, \bar{\lambda}_h^{(k')})_{\Lambda_h} \\ = (v_{hn}, \bar{\lambda}_h^{(k')} - \bar{\delta}_h^{(k')}/g)_{\Lambda_h} &= \|\bar{\lambda}_h^{(k')} - \bar{\delta}_h^{(k')}/g\|_{\Lambda_h}^2. \end{aligned}$$

Here, the notation "$1/g$" means $\mu_h \in \Lambda_h$ such that $\mu_h = 1/g$ at $\mathring{\Gamma}_{1h}$ (μ_h is not necessarily equal to $1/g$ everywhere on Γ_1). By assumption, the last line is bounded from below by some $C > 0$. Hence we obtain (2.4.6). Then (2.4.4) and (2.4.6) yield

$$C(\|\bar{\delta}_h^{(k')}\|_Q + \|\bar{\lambda}_h^{(k')}\|_{\Lambda_h}) \leq \sup_{v_h \in V_{hn}} \frac{-a(\bar{u}_h^{(k')}, v_h) - b(v_h, \bar{\mathring{p}}_h^{(k')})}{\|v_h\|_V}$$
$$\leq \|a\| \|u_h^{(k')} - u_h\|_V + \|b\| \|\mathring{p}_h^{(k')} - \mathring{p}_h\|_Q,$$

the RHS converging to 0 when $k' \to \infty$. We have thus $\bar{\delta}_h = \bar{\lambda}_h = 0$, but this contradicts with the assumption $\bar{\lambda}_h \not\equiv \bar{\delta}_h/g$ at $\mathring{\Gamma}_{1h}$. This proves (2.4.5).

Since $\|\bar{\lambda}_h^{(k)}\|_{\Lambda_h}$ is decreasing and hence converges as a whole, we deduce from (2.4.5) that $\bar{\delta}_h$ and $\bar{\lambda}_h$ do not depend on a choice of the subsequences. Therefore, the whole sequences $\bar{\delta}_h^{(k)}$ and $\bar{\lambda}_h^{(k)}$ converge, and consequently,

$$p_h^{(k)} \to p_h + \bar{\delta}_h \qquad \text{and} \qquad \lambda_h^{(k)} \to \lambda_h + \bar{\delta}_h/g \in \tilde{\Lambda}_h.$$

$(u_h, p_h + \bar{\delta}_h, \lambda_h + \bar{\delta}_h/g)$ satisfies (2.4.1a)–(2.4.1c), and this completes the proof of (i).

(ii) The invariant A_k, $k \geq k_0$, is denoted by A, together with $I = \mathring{\Gamma}_{1h} \backslash A$. For $\eta_h \in \Lambda_h$, we denote by $\eta_{h,A}$ the element in Λ_h which equals η_h and 0 at A and I respectively. We use the notation $\bar{u}_h^{(k)} = u_h^{(k)} - u_h^{(k-1)}$ etc. If $A = \emptyset$ and thus $u_h^{(k)} \in \mathring{V}_h$, we can easily get the conclusion; thereby we assume $A \neq \emptyset$ in the following.

A calculation similar to that deriving (2.4.4) shows, for $k \geq k_0 + 1$,

$$\|\bar{\lambda}_{h,A}^{(k+1)}\|_{\Lambda_h}^2 + (2\alpha\rho - C(g)\rho^2)\|\bar{u}_h^{(k)}\|_V^2 + 2\rho\gamma\|h\nabla\bar{p}_h^{(k)}\|^2 \leq \|\bar{\lambda}_{h,A}^{(k)}\|_{\Lambda_h}^2, \quad (2.4.7)$$

where we have used $u_{hn}^{(k)} = u_{hn}^{(k-1)} = 0$ at I. By summation we get $\sum_{k=k_0+1}^{K} \|\bar{u}_h^{(k)}\|_V^2 \leq \|\bar{\lambda}_{h,A}^{(k_0+1)}\|_{\Lambda_h}^2$ for arbitrary K, which implies that $u_h^{(k)}$ is convergent. In view of (2.2.1), $\sum_{k=k_0+1}^{K} \|\bar{\mathring{p}}_h^{(k)}\|_Q^2$ is bounded independently of K, so that $\mathring{p}_h^{(k)}$ is also convergent. In a similar manner, we can continue to proceed as in (i), concluding that $p_h^{(k)}$ and $\lambda_h^{(k)}$ are convergent. The limit satisfies (2.4.1a)–(2.4.1c), and this completes the proof of Theorem 2.4.1. □

2.5 Numerical examples

In the first example, let $\Omega = (0,1)^2$ and $\overline{\Gamma}_1 = \{y = 1\} \cap \Gamma$. We consider the following exact solution satisfying $u = 0$ on Γ:

$$\begin{cases} u_1(x,y) & = 20x^2(1-x)^2 y(1-y)(1-2y), \\ u_2(x,y) & = -20x(1-x)(1-2x)y^2(1-y)^2, \\ p(x,y) & = 40x(1-x)(1-2x)y(1-y)(1-2y) \\ & \qquad\qquad +4(6x^5 - 15x^4 + 10x^3)(2y-1) - 2, \end{cases} \tag{2.5.1}$$

where $\nu = 1$, and the external force f is suitably defined. By a direct computation, $\max_{\overline{\Gamma}_1} |\sigma_n| = 2$. The additive constant "-2" of p in (2.5.1) is chosen in such a way that the minimum $\min_{\delta \in \mathbb{R}} \max_{\overline{\Gamma}_1} |\sigma_n(u, p+\delta)| = 2$ is attained. Now we consider the Stokes problem with f and LBCF on Γ_1, $g > 0$ being a constant. Then one finds that

$$\begin{cases} g \geq 2 \Rightarrow \text{(2.5.1) remains a solution.} \Rightarrow \text{No-leak occurs.} \\ g < 2 \Rightarrow \text{(2.5.1) is no longer a solution.} \Rightarrow \text{Leak occurs.} \end{cases}$$

Such behavior is clearly shown by our numerical solutions in Figure 2.5.1. In addition, from Table 2.5.1 we observe that the leak/no-leak detecting condition:

$$|\lambda_h| < 1 \Rightarrow u_{hn} = 0, \quad u_{hn} > 0 \Rightarrow \lambda_h = 1, \quad u_{hn} < 0 \Rightarrow \lambda_h = -1 \quad \text{at } \mathring{\Gamma}_{1h},$$

which is proved at (2.3.8), is indeed valid numerically. Table 2.5.1 also indicates that:

- If $g = 1.2$ and thus $u_{hn} \neq 0$ on Γ_1, then $\delta_h := p_h(0,0)$ is independent of a choice of $\lambda_h^{(1)}$.
- If $g = 3.0$ and thus $u_{hn} = 0$ on Γ_1, then δ_h varies depending on $\lambda_h^{(1)}$.

This is consistent with the uniqueness results in Theorem 2.3.2.

The detail of our numerical computation is as follows. For the triangulation $\mathscr{T}_h$ of $\overline{\Omega}$, we use a uniform Friedrichs–Keller type mesh with $2N^2$ triangles, where N denotes a division number of each side of the square $\overline{\Omega}$. We employ $l = 2$, i.e. the P2/P1 elements. We compute our numerical solutions based on Algorithm 2.4.1 with the following stopping criterion:

$$\|u_h^{(k)} - u_h^{(k-1)}\|_V \leq 10^{-5}. \tag{2.5.2}$$

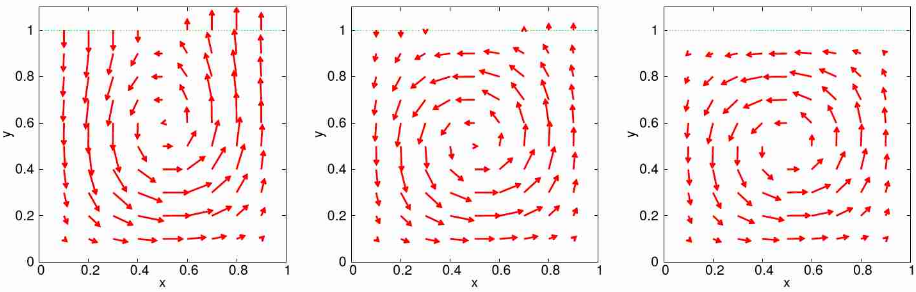

Figure 2.5.1: Solution velocity field in Ω in the first example (left: $g = 0.1$, middle: $g = 1.2$, right: $g = 3.0$).

Table 2.5.1: Values of λ_h, u_{hn}, and $\delta_h = p_h(0,0)$ in the first example.

g	0.1		1.2		1.2	3.0		3.0
ρ	20.0		30.0		30.0	2.0		1.0
$\lambda_h^{(1)}$	0.0		0.0		0.2	0.0		0.2
x	λ_h	u_{hn}	λ_h	u_{hn}	λ_h	λ_h	u_{hn}	λ_h
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1	-1.0	-0.09	-1.0	-0.03	-1.0	-0.63	-4.3E-6	-0.43
0.2	-1.0	-0.11	-1.0	-0.03	-1.0	-0.57	-1.3E-6	-0.37
0.3	-1.0	-0.10	-1.0	-0.01	-1.0	-0.45	-4.3E-7	-0.25
0.4	-1.0	-0.06	-0.83	-1.2E-6	-0.83	-0.25	-6.5E-7	-0.05
0.5	-1.0	-0.002	-0.06	-2.5E-7	-0.06	-0.02	-5.2E-7	$+0.18$
0.6	$+1.0$	$+0.05$	$+0.67$	-7.0E-7	$+0.67$	$+0.22$	-3.0E-7	$+0.42$
0.7	$+1.0$	$+0.10$	$+1.0$	$+0.01$	$+1.0$	$+0.43$	-7.6E-7	$+0.63$
0.8	$+1.0$	$+0.11$	$+1.0$	$+0.03$	$+1.0$	$+0.58$	-5.3E-7	$+0.78$
0.9	$+1.0$	$+0.09$	$+1.0$	$+0.03$	$+1.0$	$+0.66$	-1.2E-6	$+0.86$
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
δ_h	-2.0		-2.0		-2.0	-2.0		-1.4
k_{itr}	21		12		14	30		29

Table 2.5.2: Number of iterations k_{itr} required for Algorithms 2.4.1 and 2.4.2 to converge in the first example.

	$g = 0.1$	$g = 1.2$	$g = 3.0$
Algorithm 2.4.1	21	12	30
Algorithm 2.4.2	7	4	6

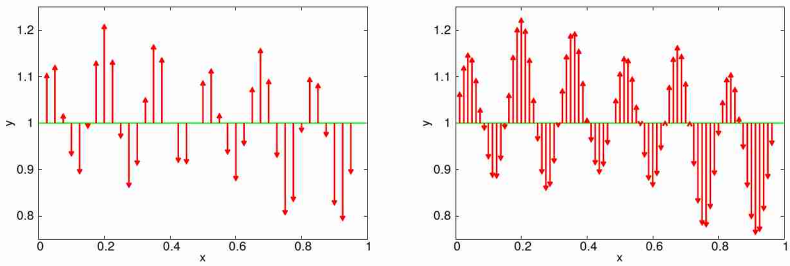

Figure 2.5.2: Solution velocity field on Γ_1 in the second example (left: $N = 40$, right: $N = 80$).

Table 2.5.3: Convergence behavior of $\|u - u_h\|_V$ and $\|p - p_h\|_Q$ in the second example.

N	$\|u_{\mathrm{ref}} - u_h\|_V$	rate	$\|p_{\mathrm{ref}} - p_h\|_Q$	rate	k_{itr}
20	2.63	—	0.437	—	4
30	1.71	1.05	0.364	0.45	4
40	1.32	0.91	0.244	1.39	4
50	1.05	1.01	0.218	0.50	6
60	0.78	1.62	0.168	1.43	6
80	0.64	0.68	0.107	1.59	5
100	0.51	1.08	0.077	1.43	5

The number of iteration required to attain (2.5.2) is denoted by k_{itr}. Figure 2.5.1 presents the plots of $u_h^{(k_{\mathrm{itr}})}$ for three values of g when $N = 10$. The concrete values of $u_{hn}^{(k_{\mathrm{itr}})}$, $\lambda_h^{(k_{\mathrm{itr}})}$ on Γ_1 and $\delta_h^{(k_{\mathrm{itr}})} = p_h^{(k_{\mathrm{itr}})}(0,0)$, together with the parameters ρ and $\lambda_h^{(1)}$, are listed in Table 2.5.1.

Next let us compare the performance of Algorithms 1 and 2 in terms of k_{itr} under the same stopping criterion (2.5.2). The result is reported in Table 2.5.2. We see that Algorithm 2.4.1 is rather slow at convergence and that Algorithm 2.4.2 gives much fewer number of iterations. Here, ρ is chosen to make k_{itr} as small as possible and $\lambda_h^{(1)} = 0$ in Algorithm 2.4.1, whereas $\rho = 1$ in Algorithm 2.4.2.

In the second example, Ω, Γ_1, and the triangulation $\mathscr{T}_h$ are the same as above. We employ $l = 1$ i.e. the P1/P1 finite element method with pressure stabilization (for γ we set $\gamma = 0.1$ instead of 1). The purpose here is to give a numerical evidence for the convergence result presented in Theorem 2.3.5.

For the data, we adopt $\nu = 1$ and

$$f = {}^t(f_1, f_2); \qquad f_1 = 0, \qquad f_2 = \begin{cases} 300\sin(4\pi x) & \text{if } \ y > 5/6, \\ 0 & \text{otherwise}, \end{cases}$$
$$g = 2(1 + \cos(4\pi x)) + 0.1,$$

in order to make u_n change its sign on Γ_1 rather drastically. In fact, the sign changes 11 times according to Figure 2.5.2. Algorithm 2.4.2 (with $\rho = 1$) is used to compute numerical solutions (u_h, p_h). Since the explicit exact solution is unknown, we regard the numerical solution for the mesh $N = 400$ as a reference solution $(u_{\mathrm{ref}}, p_{\mathrm{ref}})$. In Table 2.5.3 we report $\|u_{\mathrm{ref}} - u_h\|_V$ and $\|p_{\mathrm{ref}} - p_h\|_Q$, together with the rate of convergence evaluated by $\frac{\log(E(N_1)/E(N_2))}{\log(N_2/N_1)}$. The result reveals the $O(h)$ convergence and is consistent with Theorem 2.3.5.

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- T. Kashiwabara: Finite element method for Stokes equations under leak boundary condition of friction type, *SIAM J. Numer. Anal.* **51** (2013), pp. 2448-2469.

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Chapter 3

Solvability of the non-stationary Navier-Stokes equations under slip or leak boundary conditions of friction type

3.1 Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^d\,(d = 2, 3)$, and fix $T > 0$. We suppose that the boundary $\Gamma = \partial\Omega$ consists of two nonempty open components Γ_0 and Γ_1, that is, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We are concerned with the non-stationary incompressible Navier-Stokes equations in Ω:

$$\begin{cases} u' + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f & \text{in} \quad \Omega \times (0, T), & (3.1.1) \\ \operatorname{div} u = 0 & \text{in} \quad \Omega \times (0, T), & (3.1.2) \end{cases}$$

with the initial condition

$$u = u_0 \qquad \text{in} \quad \Omega \times \{0\}. \tag{3.1.3}$$

Here, ν, u, p, and f denote a viscosity constant, velocity field, pressure, and external force respectively; u' means the time derivative $\frac{\partial u}{\partial t}$.

As for the boundary condition, we impose the adhesive b.c. on Γ_0:

$$u = 0 \quad \text{on} \quad \Gamma_0. \tag{3.1.4}$$

On the other hand, we consider one of the following nonlinear b.c. on Γ_1:

$$u_n = 0, \qquad |\sigma_\tau| \le g, \qquad \sigma_\tau \cdot u_\tau + g|u_\tau| = 0, \quad \text{on} \quad \Gamma_1, \tag{3.1.5}$$

which is called the *slip boundary condition of friction type* (SBCF), and

$$u_\tau = 0, \qquad |\sigma_n| \le g, \qquad \sigma_n u_n + g|u_n| = 0, \quad \text{on} \quad \Gamma_1, \tag{3.1.6}$$

which is called the *leak boundary condition of friction type* (LBCF). Here, n is the outer unit normal vector defined on Γ, and we write $u_n := u \cdot n$ and $u_\tau := u - u_n n$. The stress tensor $\mathbb{T} = (T_{ij})_{i,j=1,\dots,d}$ is given by $T_{ij} = -p\delta_{ij} + \nu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, δ_{ij} being Kronecker delta. We define the stress vector $\sigma = \sigma(u, p)$ as $\sigma = \mathbb{T}n$, and write $\sigma_n := \sigma \cdot n$ and $\sigma_\tau := \sigma - \sigma_n n$. One can easily see that $\sigma_n = \sigma_n(u, p)$ may depend on p, whereas $\sigma_\tau = \sigma_\tau(u)$ does not.

The function g, given on Γ_1 and assumed to be strictly positive, is called a *modulus of friction*. Its physical meaning is the threshold of the tangential (resp. normal) stress. In fact, if $|\sigma_\tau| < g$ (resp. $|\sigma_n| < g$) then (3.1.5) (resp. (3.1.6)) implies $u_\tau = 0$ (resp. $u_n = 0$), namely, no slip (resp. leak) occurs; otherwise non-trivial slip (resp. leak) can take place. We notice that if we make $g = 0$ formally, (3.1.5) and (3.1.6) reduce to the usual slip and leak b.c. respectively. In summary, SBCF and LBCF are non-linearized slip and leak b.c. obtained from introduction of some friction law on the stress.

It should be also noted that the second and third conditions of (3.1.5) (resp. (3.1.6)) are equivalently rewritten, with the notation of subdifferential, as

$$\sigma_\tau \in -g\partial|u_\tau| \qquad (\text{resp.} \quad \sigma_n \in -g\partial|u_n|).$$

Although we will not pursue this matter further, one can refer to [11, 36] for the Navier-Stokes equations with general subdifferential b.c. See also [12], which considers the motion of a Bingham fluid under b.c. with nonlocal friction against slip.

SBCF and LBCF are first introduced in [16, 20] for the stationary Stokes and Navier-Stokes equations, where existence and uniqueness of weak solutions are established. Generalized SBCF is considered in [38, 39]. The H^2-H^1 regularity for the Stokes equations is proved in [57]. In terms of numerical analysis, [3, 42, 45, 46, 47] deal with finite element methods for SBCF or LBCF. Applications of SBCF and LBCF to realistic problems, together with numerical simulations, are found in [32, 61].

For non-stationary cases, [17, 18] study the time-dependent Stokes equations without external forces under SBCF and LBCF, using a nonlinear semigroup theory. The solvability of nonlinear problems are discussed in [41] for SBCF, and in [1] for a variant of LBCF. They use the Stokes operator associated with the linear slip or leak b.c., and do not take into account a compatibility condition at $t = 0$.

The purpose of this chapter is to prove existence and uniqueness of a strong solution for (3.1.1)–(3.1.4) with (3.1.5) or (3.1.6). We employ the class of solutions of Ladyzhenskaya type (see [37]), searching (u, p) such that

$$\begin{cases} u \in L^\infty(0,T;H^1(\Omega)^d), \quad u' \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^1(\Omega)^d), \\ p \in L^\infty(0,T;L^2(\Omega)). \end{cases}$$

There are several reasons we focus on this strong solution. First, from a viewpoint of numerical analysis, we would like to construct solutions in a class where uniqueness and regularity are assured also for 3D case. Second, we desire an L^∞-estimate with respect to time for p, which may not be obtained for weak solutions of Leray-Hopf type (cf. [62, Proposition III.1.1]). Third, in LBCF, it is not straightforward to deduce a weak solution because of (3.1.7) below. Similar difficulty already comes up in the linear leak b.c. (see [52])

The rest of this chapter is organized as follows. Basic symbols, notation, and function spaces are given in Section 3.2.

In Section 3.3, we investigate the problem with SBCF. The weak formulation is given by a variational inequality, to which we prove uniqueness of solutions. To show existence, we consider a regularized problem, approximate it by Galerkin's method, and derive a priori estimates which allow us to pass on the limit to deduce the desired strong solution. Using the compatibility condition that u_0 must satisfy SBCF, we can adapt u_0 to the regularized problem, which makes an essential point in the estimate.

Section 3.4 is devoted to a study of the problem with LBCF. There are two major differences from SBCF. First, as was pointed out in the stationary case [16, Remark 3.2], we cannot obtain the uniqueness of an additive constant for p if no leak occurs, namely, $u_n = 0$ on Γ_1. Second, under LBCF, the quantity

$$\int_\Omega \Big\{(u \cdot \nabla)v \cdot v\Big\} dx = \frac{1}{2}\int_\Gamma u_n |v|^2 \, ds \qquad (\text{if} \quad \operatorname{div} u = 0) \tag{3.1.7}$$

need not vanish because u_n can be non-zero. This fact affects our a priori estimates badly, and we can extract a solution only when the initial leak $\|u_{0n}\|_{L^2(\Gamma_1)}$ is small enough. Incidentally, if we use the so-called Bernoulli pressure $p + \frac{1}{2}|u|^2$ instead of standard p, the mathematical difficulty arising from (3.1.7) are resolved; nevertheless the leak b.c. involving the Bernoulli pressure is known to cause an unphysical effect in numerical simulations (see [29, p.338]). Thereby we employ the usual formulation.

Finally, in Section 3.5 we conclude this chapter with some remarks on higher regularity.

3.2 Preliminaries

Throughout this chapter, the domain Ω is supposed to be as smooth as required. For the precise regularity of Ω which is sufficient to deduce our main theorems, see Remarks 3.3.5 and 3.4.3. We shall denote by C various generic positive constants depending only on Ω, unless otherwise stated. When we need to specify dependence on a particular parameter, we write as $C = C(f, g, u_0)$, and so on.

We use the Lebesgue space $L^p(\Omega)\,(1 \le p \le \infty)$, and the Sobolev space $H^r(\Omega) = \{\phi \in L^2(\Omega) \mid \|\phi\|^2_{H^r(\Omega)} = \sum_{|\alpha|\le r} \|\partial^\alpha \phi\|^2_{L^2(\Omega)} < \infty\}$ for a nonnegative integer r, where $H^0(\Omega)$ means $L^2(\Omega)$. $H^s(\Omega)$ is also defined for a non-integer $s > 0$ (e.g. [21, Definition 1.2]). We put $L^2_0(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q\,dx = 0\}$. For spaces of vector-valued functions, we write $L^p(\Omega)^d$, and so on.

The Lebesgue and Sobolev spaces on the boundary Γ, Γ_0, or Γ_1, are also used. $H^0(\Gamma_1)$ means $L^2(\Gamma_1)$, and we put $L^2_0(\Gamma_1) = \{\eta \in L^2(\Gamma_1) \mid \int_{\Gamma_1} \eta\,ds = 0\}$, where ds denotes the surface measure. For a positive function g on Γ_1, the weighted Lebesgue spaces $L^1_g(\Gamma_1)$ and $L^\infty_{1/g}(\Gamma_1)$ are defined by the norms

$$\|\eta\|_{L^1_g(\Gamma_1)} = \int_{\Gamma_1} g|\eta|\,ds \qquad \text{and} \qquad \|\eta\|_{L^\infty_{1/g}(\Gamma_1)} = \operatorname{ess.sup}_{\Gamma_1} \frac{|\eta|}{g},$$

respectively. The dual space of $L^1_g(\Gamma_1)$ is $L^\infty_{1/g}(\Gamma_1)$ (see [16, Lemma 2.1]).

The usual trace operator $\phi \mapsto \phi|_\Gamma$ is defined from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$. The restrictions $\phi|_{\Gamma_0}$, $\phi|_{\Gamma_1}$ of $\phi|_\Gamma$, are also considered, and we simply write ϕ to indicate them when there is no fear of confusion. In particular, η_n and η_τ means $(\eta \cdot n)|_\Gamma$ and $(\eta - (\eta \cdot n)n)|_\Gamma$ respectively, for $\eta \in H^{1/2}(\Gamma)^d$. Note that $\|\eta_n\|_{H^{1/2}(\Gamma)} \le C\|\eta\|_{H^{1/2}(\Gamma)^d}$ and $\|\eta_\tau\|_{H^{1/2}(\Gamma)^d} \le C\|\eta\|_{H^{1/2}(\Gamma)^d}$ because n is smooth on Γ.

The inner product of $L^2(\Omega)^d$ is simplified as $(\cdot,\cdot)$, while other inner products and norms are written with clear subscripts, e.g., $(\cdot,\cdot)_{L^2(\Gamma_1)}$ or $\|\cdot\|_{H^1(\Omega)^d}$. For a Banach space X, we denote its dual space by X' and the dual product between X' and X by $\langle\cdot,\cdot\rangle_X$. Moreover, we employ the standard notation of Bochner spaces such as $L^2(0,T;X)$, $H^1(0,T;X)$.

For function spaces corresponding to a velocity and pressure, we introduce closed subspaces of $H^1(\Omega)^d$ or $L^2(\Omega)$ as follows:

$$
\begin{aligned}
V &= \{v \in H^1(\Omega)^d \,|\, v = 0 \text{ on } \Gamma_0\}, & \mathring{V} &= \{v \in H^1(\Omega)^d \,|\, v = 0 \text{ on } \Gamma\},\\
V_n &= \{v \in V \,|\, v_n = 0 \text{ on } \Gamma_1\}, & V_\tau &= \{v \in V \,|\, v_\tau = 0 \text{ on } \Gamma_1\},\\
Q &= L^2(\Omega), & \mathring{Q} &= L^2_0(\Omega).
\end{aligned}
$$

To indicate a divergence-free space, we set $H^1_\sigma(\Omega)^d = \{v \in H^1(\Omega)^d \,|\, \operatorname{div} v = 0\}$. We use the notation $V_\sigma = V \cap H^1_\sigma(\Omega)^d$, $\mathring{V}_\sigma = \mathring{V} \cap H^1_\sigma(\Omega)^d$, $V_{n,\sigma} = V_n \cap H^1_\sigma(\Omega)^d$, and $V_{\tau,\sigma} = V_\tau \cap H^1_\sigma(\Omega)^d$.

Let us define bilinear forms a_0, b, and a trilinear form a_1 by

$$
\begin{aligned}
a_0(u,v) &= \frac{\nu}{2}\sum_{i,j=1}^{d}\int_\Omega \left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx && (u, v \in H^1(\Omega)^d),\\
a_1(u,v,w) &= \int_\Omega \{(u\cdot\nabla)v\}\cdot w\, dx && (u, v, w \in H^1(\Omega)^d),\\
b(v,q) &= -\int_\Omega \operatorname{div} v\, q\, dx && (v \in H^1(\Omega)^d,\ q \in L^2(\Omega)).
\end{aligned}
$$

The bilinear forms a_0, b are continuous, and from Korn's inequality ([33, Lemma 6.2]) there exists a constant $\alpha > 0$ such that

$$
a_0(v,v) \geq \alpha \|v\|^2_{H^1(\Omega)^d} \qquad (\forall v \in V). \tag{3.2.1}
$$

Concerning the trilinear term a_1, we obtain the following two lemmas.

Lemma 3.2.1. (i) *When $d = 2$, for all $u, v, w \in H^1(\Omega)^d$ it holds that*

$$
|a_1(u,v,w)| \leq C\|u\|^{1/2}_{L^2(\Omega)^d}\|u\|^{1/2}_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|^{1/2}_{L^2(\Omega)^d}\|w\|^{1/2}_{H^1(\Omega)^d}. \tag{3.2.2}
$$

(ii) *When $d = 2$ or $d = 3$, for all $u, v, w \in H^1(\Omega)^d$ it holds that*

$$
|a_1(u,v,w)| \leq C\|u\|^{1/4}_{L^2(\Omega)^d}\|u\|^{3/4}_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|^{1/4}_{L^2(\Omega)^d}\|w\|^{3/4}_{H^1(\Omega)^d}. \tag{3.2.3}
$$

Remark 3.2.1. In particular, we see from (3.2.3) that

$$
|a_1(u,v,w)| \leq C\|u\|_{H^1(\Omega)^d}\|v\|_{H^1(\Omega)^d}\|w\|_{H^1(\Omega)^d}. \tag{3.2.4}
$$

Proof. These are well-known classical results; see e.g. [37, 62]. $\square$

Lemma 3.2.2. (i) *For all $u \in V_{n,\sigma}$ and $v \in H^1(\Omega)^d$, $a_1(u, v, v) = 0$.*

(ii) *For all $u \in V_{\tau,\sigma}$ and $v \in H^1(\Omega)^d$, $a_1(u, v, v) = \frac{1}{2}\int_{\Gamma_1} u_n |v|^2\, ds$, and*

$$|a_1(u, v, v)| \leq \gamma_1 \|u_n\|_{L^2(\Gamma_1)} \|v\|^2_{H^1(\Omega)^d}, \tag{3.2.5}$$

where γ_1 is a constant depending only on Ω.

Proof. By integration by parts, we have

$$a_1(u, v, w) + a_1(u, w, v) = -\int_\Omega \operatorname{div} u\, (v \cdot w)\, dx + \int_\Gamma u_n\, (v \cdot w)\, ds,$$

from which the conclusion of (i) and the first assertion of (ii) follow. Combining Hölder's inequality with the continuity of the trace operator $H^1(\Omega) \to L^4(\Gamma_1)$ (see [53, Theorem II.6.2]), we obtain (3.2.5). $\square$

Remark 3.2.2. Whether γ_1 is small or not, especially when compared to α in (3.2.1), is a very crucial point in our a priori estimates for LBCF (see Proposition 3.4.1). This is why we distinguish γ_1 from other constants C and do not combine γ_1 with them. As Lemma 3.2.2(i) shows, this problem does not happen when we consider SBCF.

The following, which are readily obtainable consequences of standard trace and (solenoidal) extension theorems ([21, Theorems I.1.5-6, Lemma I.2.2], see also [33, Section 5.3]), are frequently used in subsequent arguments.

Lemma 3.2.3. (i) For $v \in V_n$, it holds that $\|v_\tau\|_{H^{1/2}(\Gamma_1)^d} \leq C\|v\|_{H^1(\Omega)^d}$.

(ii) For $\eta \in H^{1/2}(\Gamma_1)^d$ satisfying $\eta_n = 0$ on Γ_1, there exists $v \in V_{n,\sigma}$ such that $v_\tau = \eta$ on Γ_1 and $\|v\|_{H^1(\Omega)^d} \leq C\|\eta\|_{H^{1/2}(\Gamma_1)^d}$.

Lemma 3.2.4. (i) For $v \in V_\tau$, it holds that $\|v_n\|_{H^{1/2}(\Gamma_1)} \leq C\|v\|_{H^1(\Omega)^d}$.

(ii) For $\eta \in H^{1/2}(\Gamma_1)$ (resp. $\eta \in H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)$), there exists $v \in V_\tau$ (resp. $v \in V_{\tau,\sigma}$) such that $v_n = \eta$ on Γ_1 and $\|v\|_{H^1(\Omega)^d} \leq C\|\eta\|_{H^{1/2}(\Gamma_1)}$.

The definition of $\sigma(u, p)$ given in Section 3.1 becomes ambiguous when (u, p) has only lower regularity, say $u \in H^1(\Omega)^d$, $p \in L^2(\Omega)$. Thus we propose a redefinition of it, based on the following Green formula:

$$(-\nu\Delta u + \nabla p, v) + \int_\Gamma \sigma(u, p) \cdot v\, ds = a_0(u, v) + b(v, p) \qquad (\text{if} \quad \operatorname{div} u = 0).$$

Definition 3.2.1. Let $u(t) \in V_\sigma$, $p(t) \in Q$, $u'(t) \in L^2(\Omega)^d$, $f(t) \in L^2(\Omega)^d$. If (3.1.1) holds in the distribution sense for a.e. $t \in (0, T)$, that is,

$$(u', v) + a_0(u, v) + a_1(u, u, v) + b(v, p) = (f, v) \qquad (\forall v \in \mathring{V}), \tag{3.2.6}$$

then we define $\sigma = \sigma(u, p) \in (H^{1/2}(\Gamma_1)^d)'$ by

$$\langle \sigma, v \rangle_{H^{1/2}(\Gamma_1)^d} = a_0(u, v) + b(v, p) - \langle F, v \rangle_V \qquad (\forall v \in V), \tag{3.2.7}$$

where $F(t) \in V'$ is given by $\langle F, v \rangle_V = (f, v) - (u', v) - a_1(u, u, v)$.

The above σ is well-defined by virtue of the trace and extension theorem. It coincides with the previous definition when (u, p) is sufficiently smooth. In addition, by Lemmas 3.2.3 and 3.2.4, $\sigma_\tau = \sigma - (\sigma \cdot n)n \in (H^{1/2}(\Gamma_1)^d)'$ and $\sigma_n = \sigma \cdot n \in H^{1/2}(\Gamma_1)'$ are characterized by

$$\begin{cases} \langle \sigma_\tau, \eta n \rangle_{H^{1/2}(\Gamma_1)^d} = 0 & (\forall \eta \in H^{1/2}(\Gamma_1)), \\ \langle \sigma_\tau, v_\tau \rangle_{H^{1/2}(\Gamma_1)^d} = a_0(u, v) + b(v, p) - \langle F, v \rangle_{V_n} & (\forall v \in V_n), \end{cases}$$

and

$$\langle \sigma_n, v_n \rangle_{H^{1/2}(\Gamma_1)} = a_0(u, v) + b(v, p) - \langle F, v \rangle_{V_\tau} \qquad (\forall v \in V_\tau),$$

respectively. By Lemma 3.2.3(ii), σ_τ actually does not depend on p.

3.3 Navier-Stokes Problem with SBCF

3.3.1 Weak formulations

Throughout this section, we assume $f \in L^2(\Omega \times (0, T))^d$, $u_0 \in V_{n,\sigma}$, and $g \in L^2(\Gamma_1 \times (0, T))$ with $g > 0$. Further regularity assumptions on these data will be given before Theorem 3.3.2. In addition, we introduce

$$j_\tau(t; \eta) = \int_{\Gamma_1} g(t)|\eta| \, ds \qquad (\eta \in L^2(\Gamma_1)^d), \tag{3.3.1}$$

which is just written as $j(\eta)$, to simplify notation, until the end of this section. j is obviously nonnegative, positively homogeneous, and Lipschitz continuous for a.e. $t \in (0, T)$. A primal weak formulation of (3.1.1)–(3.1.4) with (3.1.5) is as follows:

Problem PDE-SBCF. For a.e. $t \in (0, T)$, find $(u(t), p(t)) \in V_n \times \mathring{Q}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, σ_τ is well-defined in the sense of Definition 3.2.1, $|\sigma_\tau| \le g$ a.e. on Γ_1, and $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$ a.e. on Γ_1.

Remark 3.3.1. The precise meaning of "$|\sigma_\tau| \le g$" is that $\sigma_\tau \in (H^{1/2}(\Gamma_1)^d)'$ actually belongs to $L^\infty_{1/g}(\Gamma_1)^d$ with $\|\sigma_\tau\|_{L^\infty_{1/g}(\Gamma_1)^d} \le 1$. In particular, one has $\sigma_\tau \in L^2(\Gamma_1)^d$.

Throughout this section, we refer to Problem PDE-SBCF just as Problem PDE. Similar abbreviation will be made for other problems.

One can easily find that a classical solution of (3.1.1)–(3.1.4) with (3.1.5) solves Problem PDE, and that a sufficiently smooth solution of Problem PDE is a classical solution. As the next theorem shows, Problem PDE is equivalent to the following variational inequality problem.

Problem VI$_\sigma$-SBCF. For a.e. $t \in (0, T)$, find $u(t) \in V_{n,\sigma}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, and

$$(u', v-u) + a_0(u, v-u) + a_1(u, u, v-u) + j(v_\tau) - j(u_\tau) \ge (f, v-u) \quad (3.3.2)$$

for all $v \in V_{n,\sigma}$. Here $j = j_\tau(t; \cdot)$ is defined in (3.3.1).

Theorem 3.3.1. *Problems* PDE *and* VI$_\sigma$ *are equivalent.*

Remark 3.3.2. The precise meaning of "equivalent" above is that if (u, p) solves Problem PDE, $u(t)$ solves Problem VI$_\sigma$; if u solves Problem VI$_\sigma$, there exists unique p such that (u, p) solves Problem PDE. Hereafter we will frequently use the terminology "equivalent" in a similar sense.

Proof. Let (u, p) solve Problem PDE. Then, for $v \in V_n$ it follows that

$$(u', v) + a_0(u, v) + a_1(u, u, v) + b(v, p) - (\sigma_\tau, v_\tau)_{L^2(\Gamma_1)^d} = (f, v). \quad (3.3.3)$$

Using this equation together with $|\sigma_\tau| \le g$ and $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$, we have

$$\begin{aligned} &(u', v-u) + a_0(u, v-u) + a_1(u, u, v-u) + j(v_\tau) - j(u_\tau) - (f, v-u) \\ &= -(\sigma_\tau, v_\tau - u_\tau)_{L^2(\Gamma_1)^d} + j(v_\tau) - j(u_\tau) = \int_{\Gamma_1} (g|v_\tau| - \sigma_\tau v_\tau) ds \ge 0, \end{aligned}$$

for all $v \in V_{n,\sigma}$. Hence u is a solution of Problem VI$_\sigma$.

Next, let u be a solution of Problem VI$_\sigma$. Taking $u \pm v$ as a test function in (3.3.2), with arbitrary $v \in \mathring{V}_\sigma$, we find that

$$(u', v) + a_0(u, v) + a_1(u, u, v) = (f, v) \qquad (\forall v \in \mathring{V}_\sigma). \quad (3.3.4)$$

By a standard theory (see [62, Propositions I.1.1 and I.1.2]), there exists unique $p \in \mathring{Q}$ such that (3.2.6) holds. Therefore, $\sigma_\tau \in (H^{1/2}(\Gamma_1)^d)'$ is well-defined, and thus

$$(u', v) + a_0(u, v) + a_1(u, u, v) + b(v, p) - \langle \sigma_\tau, v_\tau \rangle_{H^{1/2}(\Gamma_1)^d} = (f, v) \qquad (\forall v \in V_n).$$

Combining this equation with (3.3.2), we obtain

$$-\langle\sigma_\tau, v_\tau - u_\tau\rangle_{H^{1/2}(\Gamma_1)^d} \le \int_{\Gamma_1} g(|v_\tau| - |u_\tau|)ds \qquad (\forall v \in V_{n,\sigma}), \qquad (3.3.5)$$

and as a result of triangle inequality, $|\langle\sigma_\tau, v_\tau\rangle_{H^{1/2}(\Gamma_1)^d}| \le \int_{\Gamma_1} g|v_\tau|\,ds$ for $v \in V_{n,\sigma}$. In view of Lemma 3.2.3(ii), this implies that for $\eta \in H^{1/2}(\Gamma_1)^d$

$$|\langle\sigma_\tau, \eta\rangle_{H^{1/2}(\Gamma_1)^d}| = |\langle\sigma_\tau, \eta_\tau\rangle_{H^{1/2}(\Gamma_1)^d}| \le \|\eta_\tau\|_{L^1_g(\Gamma_1)^d} \le \|\eta\|_{L^1_g(\Gamma_1)^d}.$$

By a density argument, we can extend σ_τ to an element of $(L^1_g(\Gamma)^d)'$ such that

$$|\langle\sigma_\tau, \eta\rangle_{L^1_g(\Gamma_1)^d}| \le \|\eta\|_{L^1_g(\Gamma_1)^d} \qquad (\forall \eta \in L^1_g(\Gamma_1)^d).$$

Since $(L^1_g(\Gamma_1)^d)' = L^\infty_{1/g}(\Gamma_1)^d$, we conclude $|\sigma_\tau| \le g$. Then $\sigma_\tau \cdot u_\tau + g|u_\tau| = 0$ follows from (3.3.5) with $v = 0$. Hence (u, p) is a solution of Problem PDE. □

3.3.2 Main theorem. Proof of uniqueness.

We are now in a position to state our main theorem. We assume:

(S1) $f \in H^1(0, T; L^2(\Omega)^d)$.

(S2) $g \in H^1(0, T; L^2(\Gamma_1))$ with $g(0) \in H^1(\Gamma_1)$.

(S3) $u_0 \in H^2(\Omega)^d \cap V_{n,\sigma}$, and SBCF is satisfied at $t = 0$, namely,

$$|\sigma_\tau(u_0)| \le g(0) \quad \text{and} \quad \sigma_\tau(u_0) \cdot u_{0\tau} + g(0)|u_{0\tau}| = 0 \quad \text{a.e. on } \Gamma_1.$$

Note that $\sigma_\tau(u_0)$ can be defined in a usual sense because $u_0 \in H^2(\Omega)^d$.

Theorem 3.3.2. *Under* (S1)–(S3), *when* $d = 2$ *there exists a unique solution* u *of Problem* VI_σ *such that*

$$u \in L^\infty(0, T; V_{n,\sigma}), \qquad u' \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; V_{n,\sigma}).$$

When $d = 3$*, the same conclusion holds on some smaller time interval* $(0, T')$.

We call the solution in the above theorem a *strong solution* of Problem VI_σ. First we prove the uniqueness of a strong solution. The existence will be proved in Section 3.3.4 after some additional preparations.

Proposition 3.3.1. *If u_1 and u_2 are strong solutions of Problem* VI_σ*, then* $u_1 = u_2$.

Proof. Taking $v = u_2$ and $v = u_1$ in (3.3.2) for u_1 and that for u_2 respectively, and adding the resulting two inequalities, for a.e. $t \in (0, T)$ we obtain

$$\begin{aligned}
&(u_1' - u_2', u_1 - u_2) + a_0(u_1 - u_2, u_1 - u_2) \\
&\le a_1(u_1, u_1, u_2 - u_1) + a_1(u_2, u_2, u_1 - u_2) \\
&= -a_1(u_1 - u_2, u_2, u_1 - u_2) - a_1(u_2, u_1 - u_2, u_1 - u_2).
\end{aligned} \tag{3.3.6}$$

We deduce from (3.2.3), together with Young's inequality, that

$$\begin{aligned}
|a_1(u_1 - u_2, u_2, u_1 - u_2)| &\le C\|u_1 - u_2\|_{L^2(\Omega)^d}^{1/2} \|u_1 - u_2\|_{H^1(\Omega)^d}^{3/2} \|u_2\|_{H^1(\Omega)^d} \\
&\le \frac{\alpha}{2}\|u_1 - u_2\|_{H^1(\Omega)^d}^2 + C\|u_2\|_{H^1(\Omega)^d}^2 \|u_1 - u_2\|_{L^2(\Omega)^d}^2,
\end{aligned}$$

$$\begin{aligned}
|a_1(u_2, u_1 - u_2, u_1 - u_2)| &\le C\|u_2\|_{H^1(\Omega)^d} \|u_1 - u_2\|_{H^1(\Omega)^d}^{7/4} \|u_1 - u_2\|_{L^2(\Omega)^d}^{1/4} \\
&\le \frac{\alpha}{2}\|u_1 - u_2\|_{H^1(\Omega)^d}^2 + C\|u_2\|_{H^1(\Omega)^d}^8 \|u_1 - u_2\|_{L^2(\Omega)^d}^2.
\end{aligned}$$

Combining (3.2.1) and these estimates with (3.3.6), we have

$$\frac{d}{dt}\|u_1 - u_2\|_{L^2(\Omega)^d}^2 \le C(\|u_2\|_{H^1(\Omega)^d}^2 + \|u_2\|_{H^1(\Omega)^d}^8)\|u_1 - u_2\|_{L^2(\Omega)^d}^2.$$

By Gronwall's inequality, we conclude

$$\|u_1(t) - u_2(t)\|_{L^2(\Omega)^d}^2 \le e^{\int_0^t C(\|u_2\|_{H^1(\Omega)^d}^2 + \|u_2\|_{H^1(\Omega)^d}^8)\,dt} \|u_1(0) - u_2(0)\|_{L^2(\Omega)^d}^2 = 0,$$

since $u_1(0) = u_2(0) = u_0$. (Note that $\int_0^t (\|u_2\|_{H^1(\Omega)^d}^2 + \|u_2\|_{H^1(\Omega)^d}^8)\,dt$ remains finite because $u \in L^\infty(0, T; H^1(\Omega)^d)$.) Thus $u_1(t) = u_2(t)$. □

Remark 3.3.3. In the case of SBCF here, the last term of (3.3.6) vanishes, according to Lemma 3.2.2(i). We did not use that fact because we would like to make our proof of uniqueness remain unchanged when we deal with LBCF.

Concerning the associated pressure, we find:

Proposition 3.3.2. *Under the assumptions of Theorem* 3.3.2*, let u be the strong solution of Problem* VI_σ*, and p be the associated pressure obtained in the proof of Theorem* 3.3.1*. Then $p \in L^\infty(0, T; \mathring{Q})$.*

Proof. For a.e. $t \in (0, T)$, the well-known inf-sup condition (see [21, I.(5.14)]), together with (3.3.3), (3.2.4), and $|\sigma_\tau| \le g$ a.e. on Γ_1, yields

$$\begin{aligned}\|p\|_{L^2(\Omega)} &\le \sup_{v \in \mathring{V}} \frac{b(v,p)}{\|v\|_{H^1(\Omega)^d}} \\ &\le \|u'\|_{L^2(\Omega)^d} + C\|u\|_{H^1(\Omega)^d} + C\|u\|^2_{H^1(\Omega)^d} + C\|g\|_{L^2(\Gamma_1)} + \|f\|_{L^2(\Omega)^d}.\end{aligned}$$

Since RHS is bounded uniformly in t, p is in $L^\infty(0, T; \mathring{Q})$. □

3.3.3 Regularized problem

To prove the solvability of Problem VI$_\sigma$, we consider a regularized variational inequality, which is shown to be equivalent to a variational equation.

Before stating those problems in detail, for fixed $\epsilon > 0$ we introduce

$$j_\epsilon(\eta) = \int_{\Gamma_1} g\rho_\epsilon(\eta)\, ds \qquad (\eta \in L^2(\Gamma_1)^d),$$

where ρ_ϵ is a regularization of $|\cdot|$ having the following properties:

(a) $\rho_\epsilon \in C^2(\mathbb{R}^d)$ is a nonnegative convex function.

(b) It holds that

$$\big|\rho_\epsilon(z) - |z|\big| \le \epsilon \qquad (\forall z \in \mathbb{R}^d). \tag{3.3.7}$$

(c) If α_ϵ denotes $\nabla\rho_\epsilon$, then

$$|\alpha_\epsilon(z)| \le 1 \quad \text{and} \quad \alpha_\epsilon(z) \cdot z \ge 0 \qquad (\forall z \in \mathbb{R}^d). \tag{3.3.8}$$

In particular, as a result of the convexity, the Hessian of ρ_ϵ, denoted by β_ϵ, is semi-positive definite, that is,

$${}^t y \beta_\epsilon(z) y \ge 0 \qquad (\forall y, z \in \mathbb{R}^d), \tag{3.3.9}$$

where ${}^t y$ means the transpose of y. Such ρ_ϵ does exist; for example, $\rho_\epsilon(z) = \sqrt{|z|^2 + \epsilon^2}$ enjoys all of (a)–(c) above.

Remark 3.3.4. One could use the Moreau-Yoshida approximation of $|\cdot|$ as ρ_ϵ, which is considered in [57], but it is only in $C^1(\mathbb{R}^d)$, not in $C^2(\mathbb{R}^d)$.

Since ρ_ϵ is differentiable, the functional j_ϵ is Gâteaux differentiable, with its derivative $Dj_\epsilon(\eta) \in (H^{1/2}(\Gamma_1)^d)'$ computed by

$$\langle Dj_\epsilon(\eta), \xi \rangle_{H^{1/2}(\Gamma_1)^d} = \int_{\Gamma_1} g\alpha_\epsilon(\eta) \cdot \xi \, ds \qquad (\eta, \xi \in H^{1/2}(\Gamma_1)^d).$$

We are ready to state the regularized problems mentioned above.

Problem VI$_\sigma^\epsilon$-SBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{n,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$\begin{aligned}&(u'_\epsilon, v - u_\epsilon) + a_0(u_\epsilon, v - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, v - u_\epsilon) + j_\epsilon(v_\tau) - j_\epsilon(u_{\epsilon\tau}) \\ \geq&(f, v - u_\epsilon) \qquad\qquad (\forall v \in V_{n,\sigma}).\end{aligned} \tag{3.3.10}$$

Problem VE$_\sigma^\epsilon$-SBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{n,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$(u'_\epsilon, v) + a_0(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \int_{\Gamma_1} g\alpha_\epsilon(u_{\epsilon\tau}) \cdot v_\tau ds = (f, v) \qquad (\forall v \in V_{n,\sigma}). \tag{3.3.11}$$

Here, u_0^ϵ is a perturbation of the original initial velocity u_0. The way one obtains u_0^ϵ from u_0 is described later. By an elementary observation (e.g. [13, Section 3.3] or [57, Lemma 3.3]), we see that:

Proposition 3.3.3. *Problems* VI$_\sigma^\epsilon$ *and* VE$_\sigma^\epsilon$ *are equivalent.*

Now we focus on the construction of a perturbed initial velocity u_0^ϵ. Since $u_0 \in H^2(\Omega)^d$ satisfies SBCF by (S3), it follows from the Green formula $a_0(u_0, v) = (-\nu\Delta u_0, v) + \int_{\Gamma_1} \sigma_\tau(u_0) \cdot v_\tau \, ds$, for $v \in V_{n,\sigma}$, that

$$a_0(u_0, v - u_0) + \int_{\Gamma_1} g(0)|v_\tau| \, ds - \int_{\Gamma_1} g(0)|u_{0\tau}| \, ds \geq (-\nu\Delta u_0, v - u_0). \tag{3.3.12}$$

Here we consider the regularized problem: find $u_0^\epsilon \in V_{n,\sigma}$ such that

$$\begin{aligned}a_0(u_0^\epsilon, v - u_0^\epsilon) + \int_{\Gamma_1} g(0)\rho_\epsilon(v_\tau) \, ds - \int_{\Gamma_1} g(0)\rho_\epsilon(u_{0\tau}^\epsilon) \, ds \geq (-\nu\Delta u_0, v - u_0^\epsilon) \\ (\forall v \in V_{n,\sigma}),\end{aligned} \tag{3.3.13}$$

which is equivalent to (cf. Proposition 3.3.3)

$$a_0(u_0^\epsilon, v) + \int_{\Gamma_1} g(0)\alpha_\epsilon(u_{0\tau}^\epsilon) \cdot v_\tau \, ds = (-\nu\Delta u_0, v) \qquad (\forall v \in V_{n,\sigma}). \tag{3.3.14}$$

By a standard theory of elliptic variational inequalities [22], (3.3.13) admits a unique solution u_0^ϵ, which is the perturbation of u_0 in question. With this setting, we find:

Lemma 3.3.1. (i) *When* $\epsilon \to 0$, $u_0^\epsilon \to u_0$ *strongly in* $H^1(\Omega)^d$.

(ii) $u_0^\epsilon \in H^2(\Omega)^d$ *and*

$$\|u_0^\epsilon\|_{H^2(\Omega)^d} \le C(\|\nu\Delta u_0\| + \|g(0)\|_{H^1(\Gamma_1)}). \tag{3.3.15}$$

Proof. (i) Taking $v = u_0$ in (3.3.13) and $v = u_0^\epsilon$ in (3.3.12), adding the resulting two inequalities, applying Korn's inequality, and using (3.3.7), we conclude

$$\begin{aligned}\alpha\|u_0^\epsilon - u_0\|_{H^1(\Omega)^d}^2 &\le \int_{\Gamma_1} g(0)\big(|u_0^\epsilon| - \rho_\epsilon(u_0^\epsilon)\big)\,ds + \int_{\Gamma_1} g(0)\big(\rho_\epsilon(u_0) - |u_0|\big)\,ds \\ &\le 2\epsilon \int_{\Gamma_1} g(0)\,ds \to 0 \qquad (\epsilon \to 0).\end{aligned}$$

(ii) Since $g(0) \in H^1(\Gamma_1)$ by (S2), we can directly apply the regularity result [57, Lemma 5.2] to the elliptic variational inequality (3.3.13), and obtain (3.3.15). Though our ρ_ϵ and α_ϵ are different from those of [57], it makes no difference in the proof of that lemma. □

Remark 3.3.5. (i) As a result of (i) above, for sufficiently small $\epsilon > 0$ we have

$$\|u_0^\epsilon\|_{L^2(\Omega)^d} \le 2\|u_0\|_{L^2(\Omega)^d} \qquad \text{and} \qquad \|u_0^\epsilon\|_{H^1(\Omega)^d} \le 2\|u_0\|_{H^1(\Omega)^d}. \tag{3.3.16}$$

(ii) Concerning the regularity of the domain, [57] assumes that Γ_0 and Γ_1 are class of C^2 and C^4 respectively, which is sufficient for our theory as well.

Remark 3.3.6. In [57], dealing with the stationary problem, the author stated that $g \in H^{1/2}(\Gamma_1)$ was enough to derive $u \in H^2(\Omega)^d$ and $p \in H^1(\Omega)$. However, it turned out that his proof presented there worked only for $g \in H^1(\Gamma_1)$; see the errata by the same author. This is why we have assumed $g(0) \in H^1(\Gamma_1)$ in (S2), not $g(0) \in H^{1/2}(\Gamma_1)$.

3.3.4 Proof of existence

Due to Proposition 3.3.3, we concentrate on solving Problem $\mathrm{VE}_\sigma^\epsilon$. In doing so, we construct approximate solutions by Galerkin's method. Since $V_{n,\sigma} \subset H^1(\Omega)^d$ is separable, there exist members $w_1, w_2, \ldots \in V_{n,\sigma}$, linear independent to each other, such that $\bigcup_{m=1}^{\infty} \operatorname{span}\{w_k\}_{k=1}^m \subset V_{n,\sigma}$ dense in $H^1(\Omega)^d$. Here ϵ is fixed, and thus we may assume $w_1 = u_0^\epsilon$.

Problem VE$_\sigma^{\epsilon,m}$-SBCF. Find $c_k \in C^2([0,T])\,(k = 1,...,m)$ such that $u_m \in V_{n,\sigma}$ defined by $u_m = \sum_{k=1}^m c_k(t)w_k$ satisfies $u_m(0) = u_0^\epsilon$ and

$$(u_m', w_k) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau} ds = (f, w_k) \qquad (k = 1, ..., m). \tag{3.3.17}$$

Since $\alpha_\epsilon \in C^1(\mathbb{R}^d)^d$, the system of ordinal differential equations (3.3.17) admits unique solutions $c_k \in C^2([0,\tilde{T}])\,(k = 1,...,m)$ for some $\tilde{T} \le T$. The a priori estimate below shows $\tilde{T}$ can be taken as T, so that we write T instead of $\tilde{T}$ from the beginning.

Proposition 3.3.4. *Let* (S1)–(S3) *be valid and ϵ be small enough so that* (3.3.16) *holds.*

(i) *When $d = 2$, $u_m \in L^\infty(0,T;V_{n,\sigma})$ and $u_m' \in L^\infty(0,T;L^2(\Omega)^d) \cap L^2(0,T;V_{n,\sigma})$ are bounded independently of m and ϵ.*

(ii) *When $d = 3$, the same conclusion holds for some smaller interval $(0,T')$, which can be taken independently of m and ϵ.*

Proof. Due to space limitations, we simply write $\|u\|_{L^2}$, $\|g\|_{L^2}$, $\|f\|_{L^2}$, etc. instead of $\|u\|_{L^2(\Omega)^d}$, $\|g\|_{L^2(\Gamma_1)}$, $\|f\|_{L^2(\Omega)^d}$, etc.

(i) Multiplying (3.3.17) by $c_k(t)$, and adding the resulting equations for $k = 1,\dots,m$, we obtain

$$(u_m', u_m) + a_0(u_m, u_m) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot u_{m\tau} ds = (f, u_m),$$

where we have used Lemma 3.2.2(i). It follows from (3.2.1) and (3.3.8) that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \le (f, u_m) \le \|f\|_{L^2}\|u_m\|_{H^1} \le \frac{\alpha}{2}\|u_m\|_{H^1}^2 + \frac{1}{2\alpha}\|f\|_{L^2}^2,$$

which gives

$$\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \le C\|f\|_{L^2}^2. \tag{3.3.18}$$

Consequently, for $0 \le t \le T$,

$$\|u_m(t)\|_{L^2}^2 + \alpha\int_0^T \|u_m\|_{H^1}^2 dt \le \|u_0^\epsilon\|_{L^2}^2 + C\int_0^T \|f\|_{L^2}^2 dt. \tag{3.3.19}$$

From (3.3.16), we find that $\|u_m\|_{L^\infty(0,T;L^2)}$ and $\|u_m\|_{L^2(0,T;V_{n,\sigma})}$ are bounded by $C(f,u_0)$ independently of m and ϵ.

Next, we differentiate (3.3.17) with respect to t, which is possible because $c_k(t)$'s are in $C^2([0,T])$, to deduce

$$(u_m'', w_k) + a_0(u_m', w_k) + a_1(u_m', u_m, w_k) + a_1(u_m, u_m', w_k)$$
$$+ \int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau} ds + \int_{\Gamma_1} g\, {}^t u_{m\tau}' \beta_\epsilon w_{k\tau}\, ds = (f', w_k) \quad (k = 1, ..., m).$$

Multiplying this by $c_k'(t)$, and adding the resulting equations, we obtain

$$(u_m'', u_m') + a_0(u_m', u_m') + a_1(u_m', u_m, u_m') + \int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau}) \cdot u_{m\tau}' ds$$
$$+ \int_{\Gamma_1} g\, {}^t u_{m\tau}' \beta_\epsilon(u_{m\tau}) u_{m\tau}'\, ds = (f', u_m'), \quad (3.3.20)$$

where we have again used Lemma 3.2.2(i). Here,

$$\begin{aligned} a_1(u_m', u_m, u_m') &\le C\|u_m'\|_{L^2}\|u_m\|_{H^1}\|u_m'\|_{H^1} \qquad \text{(by (3.2.2))} \\ &\le \frac{\alpha}{6}\|u_m'\|_{H^1}^2 + C\|u_m\|_{H^1}^2\|u_m'\|_{L^2}^2, \qquad (3.3.21) \end{aligned}$$

$$\begin{aligned} \left| \int_{\Gamma_1} g'\alpha_\epsilon(u_{m\tau}) \cdot u_{m\tau}' ds \right| &\le \|g'\|_{L^2}\|u_{m\tau}'\|_{L^2(\Gamma_1)^d} \qquad \text{(by (3.3.8))} \\ &\le C\|g'\|_{L^2}\|u_m'\|_{H^1} \qquad \text{(by Lemma 3.2.3(i))} \\ &\le \frac{\alpha}{6}\|u_m'\|_{H^1}^2 + C\|g'\|_{L^2}^2, \end{aligned}$$

$$\int_{\Gamma_1} g\, {}^t u_{m\tau}' \beta_\epsilon(u_{m\tau}) u_{m\tau}'\, ds \ge 0, \qquad \text{(by } g > 0 \text{ and (3.3.9))}$$

$$|(f', u_m')| \le \|f'\|_{L^2}\|u_m'\|_{H^1} \le \frac{\alpha}{6}\|u_m'\|_{H^1}^2 + C\|f'\|_{L^2}^2.$$

Collecting these estimates, it follows from (3.3.20) that for $0 \le t \le T$

$$\frac{d}{dt}\|u_m'\|_{L^2}^2 + \alpha\|u_m'\|_{H^1}^2 \le C(\|f'\|_{L^2}^2 + \|g'\|_{L^2}^2) + C\|u_m\|_{H^1}^2\|u_m'\|_{L^2}^2. \quad (3.3.22)$$

If the second term of LHS is neglected, Gronwall's inequality leads to

$$\|u_m'(t)\|_{L^2}^2 \le \left(\|u_m'(0)\|_{L^2}^2 + C\int_0^T (\|f'\|_{L^2}^2 + \|g'\|_{L^2}^2) dt \right) e^{C\int_0^T \|u_m\|_{H^1}^2 dt}. \quad (3.3.23)$$

Provided that $\|u_m'(0)\|_{L^2}^2$ is bounded independently of m and ϵ, estimate (3.3.23) gives the boundedness of $\|u_m'\|_{L^\infty(0,T;L^2)}$ because we already know that of $\|u_m\|_{L^2(0,T;V_{n,\sigma})}$ due to (3.3.19). Then, by (3.3.18) and (3.3.19) we have

$$\alpha\|u_m(t)\|_{H^1}^2 \le C\|f\|_{L^2}^2 + \|u_m'\|_{L^2}\|u_m\|_{L^2} \le C(f, g, u_0),$$

which implies $\|u_m\|_{L^\infty(0,T;V_{n,\sigma})}$ is bounded. Finally, integrating (3.3.22), we see that $\|u_m'\|_{L^2(0,T;V_{n,\sigma})}$ is also bounded.

To show the boundedness of $\|u_m'(0)\|_{L^2}^2$, we multiply (3.3.17) by $c_k'(t)$, add the resulting equations, and make $t = 0$, arriving at

$$\begin{aligned}&\|u_m'(0)\|_{L^2}^2 + a_0(u_0^\epsilon, u_m'(0)) + a_1(u_0^\epsilon, u_0^\epsilon, u_m'(0)) + \int_{\Gamma_1} g(0)\alpha_\epsilon(u_{0\tau}^\epsilon)\cdot u_{m\tau}'(0)ds \\ &= (f(0), u_m'(0)).\end{aligned} \tag{3.3.24}$$

From the construction of u_0^ϵ, especially (3.3.14), we have

$$\begin{aligned}\left|a_0(u_0^\epsilon, u_m'(0)) + \int_{\Gamma_1} g(0)\alpha_\epsilon(u_{0\tau}^\epsilon)\cdot u_{m\tau}'(0)ds\right| &= |(-\nu\Delta u_0, u_m'(0))| \\ &\le C\|u_0\|_{H^2}\|u_m'(0)\|_{L^2}.\end{aligned} \tag{3.3.25}$$

Furthermore, by Schwarz's inequality, Sobolev's inequality and (3.3.15),

$$\begin{aligned}|a_1(u_0^\epsilon, u_0^\epsilon, u_m'(0))| &\le C\|u_0^\epsilon\|_{L^\infty}\|u_0^\epsilon\|_{H^1}\|u_m'(0)\|_{L^2} \le C\|u_0^\epsilon\|_{H^2}^2\|u_m'(0)\|_{L^2} \\ &\le C(\|u_0\|_{H^2} + \|g(0)\|_{H^1})^2\|u_m'(0)\|_{L^2}.\end{aligned}$$

Combining these estimates with (3.3.24), we obtain

$$\|u_m'(0)\|_{L^2} \le \|f(0)\|_{L^2} + C\|u_0\|_{H^2} + C(\|u_0\|_{H^2} + \|g(0)\|_{H^1})^2,$$

which proves the boundedness of $\|u_m'(0)\|_{L^2}^2$. This completes the proof of (i).

(ii) The discussion before (3.3.21) and the observation for $\|u_m'(0)\|_{L^2}$ are the same as (i). What changes from the case $d = 2$ is that when $d = 3$, instead of (3.3.21), we only have (by (3.2.3) and Young's inequality)

$$\begin{aligned}|a_1(u_m', u_m, u_m')| &\le C\|u_m'\|_{L^2}^{1/2}\|u_m\|_{H^1}\|u_m'\|_{H^1}^{3/2} \\ &\le \gamma\|u_m\|_{H^1}\|u_m'\|_{H^1}^2 + C\|u_m\|_{H^1}\|u_m'\|_{L^2}^2,\end{aligned}$$

for a constant $\gamma > 0$ which can be arbitrarily small. We choose γ satisfying $\gamma\|u_0\|_{H^1} \le \frac{\alpha}{24}$, and from (3.3.16) we obtain $\gamma\|u_0^\epsilon\|_{H^1} \le \frac{\alpha}{12}$. Let $T' > 0$, which may depend on m, ϵ at this stage, be the maximum value of t such that $\gamma\|u_m(t)\|_{H^1} \le \frac{\alpha}{6}$. If $\gamma\|u_m(t)\|_{H^1} < \frac{\alpha}{6}$ for all $0 \le t \le T$, we set $T' = T$. Since $\gamma\|u_m(0)\|_{H^1} < \frac{\alpha}{6}$ and $u_m(t)$ is continuous with respect to t, such T' does exist, and furthermore if $T' < T$ then $\gamma\|u_m(t)\|_{H^1} = \frac{\alpha}{6}$.

Therefore, in place of (3.3.22) we obtain

$$\frac{d}{dt}\|u_m'\|_{L^2}^2 + \alpha\|u_m'\|_{H^1}^2 \le C(\|f'\|_{L^2}^2 + \|g'\|_{L^2}^2) + C\|u_m\|_{H^1}\|u_m'\|_{L^2}^2 \quad (0 \le t \le T'),$$

which leads to the boundedness of $\|u_m'\|_{L^2(0,T';V_{n,\sigma})}$ and $\|u_m'\|_{L^\infty(0,T';L^2)}$, together with $\|u_m\|_{L^\infty(0,T';V_{n,\sigma})}$.

Finally, let us prove that T' is bounded from below independently of m and ϵ. In fact, if $T' < T$ then we see that

$$\begin{aligned}\frac{\alpha}{12\gamma} &\le \|u_m(T')\|_{H^1} - \|u_m(0)\|_{H^1} \le \|u_m(T') - u_m(0)\|_{H^1} = \left\| \int_0^{T'} u_m'(t)\, dt \right\|_{H^1} \\ &\le \int_0^{T'} \|u_m'(t)\|_{H^1} dt \le \sqrt{T'}\|u_m'\|_{L^2(0,T';V_{n,\sigma})}.\end{aligned}$$

Since we already know $\|u_m'\|_{L^2(0,T';V_{n,\sigma})}$ is bounded, we obtain the lower bound for T'. This completes the proof of Proposition 3.3.4. □

Remark 3.3.7. (i) A naive computation gives, by (3.3.8),

$$\left| \int_{\Gamma_1} g(0)\alpha_\epsilon(u_{0\tau}^\epsilon) \cdot u_{m\tau}'(0)\, ds \right| \le \|g(0)\|_{L^2(\Gamma_1)}\|u_{m\tau}'(0)\|_{L^2(\Gamma_1)^d},$$

but $\|u_{m\tau}'(0)\|_{L^2(\Gamma_1)^d}$ cannot be bounded by $\|u_m'(0)\|_{L^2(\Omega)^d}$ in general. Therefore, the perturbation of u_0, which is based on the compatibility condition in (S3), is essential in deriving (3.3.25).

(ii) If $d = 3$ and f, g, u_0 are sufficiently small, we can prove $\gamma\|u_m(t)\|_{H^1(\Omega)^d} \le \frac{\alpha}{6}$ for all $0 \le t \le T$, and consequently the existence of a global solution.

As a final step for our proof of the existence, we discuss passing to the limits $m \to \infty$ and $\epsilon \to 0$. The proof below is valid for both $d = 2, 3$, except that when $d = 3$ we have to replace T with T' given in Proposition 3.3.4.

Proposition 3.3.5. (i) *Under the assumptions of Proposition* 3.3.4, *there exists a solution u_ϵ of Problem* VI$_\sigma^\epsilon$ *such that all of $\|u_\epsilon\|_{L^\infty(0,T;V_{n,\sigma})}$, $\|u_\epsilon'\|_{L^2(0,T;V_{n,\sigma})}$, and $\|u_\epsilon'\|_{L^\infty(0,T;L^2(\Omega)^d)}$ are bounded independently of ϵ.*

(ii) *There exists a strong solution of Problem* VI$_\sigma$.

Proof. (i) As a consequence of Proposition 3.3.4, we can extract a subsequence of $\{u_m\}_{m=1}^{\infty}$, denoted by the same symbol, such that

$$\begin{aligned} u_m &\rightharpoonup u_\epsilon \quad \text{weakly-}* \text{ in } L^\infty(0,T;V_{n,\sigma}), \\ u'_m &\rightharpoonup u'_\epsilon \quad \text{weakly in } L^2(0,T;V_{n,\sigma}) \text{ and weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \end{aligned}$$

for some $u_\epsilon \in L^\infty(0,T;V_{n,\sigma})$, $u'_\epsilon \in L^2(0,T;V_{n,\sigma}) \cap L^\infty(0,T;L^2(\Omega)^d)$. The norms of u_ϵ and u'_ϵ in those spaces are uniformly bounded in ϵ.

Let us prove u_ϵ solves Problem VI$^\epsilon_\sigma$. By Proposition 3.3.3, it suffices to show u_ϵ solves Problem VE$^\epsilon_\sigma$. For $\phi \in C_0^\infty(0,T)$, it follows from (3.3.17) that

$$\int_0^T \phi(t)\Big\{(u'_m, w_k) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{m\tau}) \cdot w_{k\tau} ds \\ - (f, w_k)\Big\} dt = 0 \qquad (k = 1, ..., m). \tag{3.3.26}$$

By standard compactness results (see [62, Theorem III.2.1], [53, Theorem II.6.2]), $u_m \to u_\epsilon$ strongly in $L^2(0,T;L^4(\Omega)^d)$ and $u_{m\tau} \to u_{\epsilon\tau}$ strongly in $L^2(\Gamma_1 \times (0,T))^d$. In particular, $u_{m\tau} \to u_{\epsilon\tau}$ a.e. on $\Gamma_1 \times (0,T)$, and thus the continuity of $\alpha_\epsilon(z)$ yields $\alpha_\epsilon(u_{m\tau}) \to \alpha_\epsilon(u_{\epsilon\tau})$ a.e. From Lebesgue's convergence theorem combined with a density argument, we see that (3.3.26) holds, with u_m and w_k replaced by u_ϵ and arbitrary $v \in V_{n,\sigma}$ respectively. Hence (3.3.11) holds for a.e. t, which implies that u_ϵ solves Problem VE$^\epsilon_\sigma$.

(ii) As a result of (i), we can extract a subsequence of $\{u_\epsilon\}_{\epsilon\downarrow 0}$, denoted by the same symbol, such that

$$\begin{aligned} u_\epsilon &\rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0,T;V_{n,\sigma}), \\ u'_\epsilon &\rightharpoonup u' \quad \text{weakly in } L^2(0,T;V_{n,\sigma}) \text{ and weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)^d), \end{aligned}$$

for some $u \in L^\infty(0,T;V_{n,\sigma})$, $u' \in L^2(0,T;V_{n,\sigma}) \cap L^\infty(0,T;L^2(\Omega)^d)$. As before, one sees that $u_\epsilon \to u$ strongly in $L^2(0,T;L^4(\Omega)^d)$ and $u_{\epsilon\tau} \to u_\tau$ strongly in $L^2(\Gamma_1 \times (0,T))$. In addition, $u_\epsilon \rightharpoonup u$ weakly in $L^2(0,T;V_{n,\sigma})$, and thus it follows that $\int_0^T a_0(u,u)\,dt \le \underline{\lim}_{\epsilon\to 0} \int_0^T a_0(u_\epsilon, u_\epsilon)\,dt$.

Let $\tilde{v} \in L^2(0,T;V_{n,\sigma})$ be arbitrary. We take $v = \tilde{v}(t)$ in (3.3.10) and integrate the resulting equation over $(0,T)$ to deduce

$$\int_0^T \Big\{(u'_\epsilon, \tilde{v} - u_\epsilon) + a_0(u_\epsilon, \tilde{v} - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, \tilde{v} - u_\epsilon) \\ + j_\epsilon(\tilde{v}_\tau) - j_\epsilon(u_{\epsilon\tau}) - (f, \tilde{v} - u_\epsilon)\Big\} dt \ge 0. \tag{3.3.27}$$

In view of (3.3.7), together with triangle inequality and Lipschitz continuity of j, we have $\int_0^T j_\epsilon(\tilde{v}_\tau)\,dt \to \int_0^T j(\tilde{v}_\tau)\,dt$ and $\int_0^T j_\epsilon(u_{\epsilon\tau})\,dt \to \int_0^T j(u_\tau)\,dt$ when $\epsilon \to 0$. Therefore, taking the lower limit, we see that (3.3.27), with u_ϵ replaced by u, holds. Then, a technique using the Lebesgue differentiation theorem (see [13, p.57]) enables us to conclude that u satisfies (3.3.2) at a.e. t.

For the initial condition, Lemma 3.3.1(i) leads to $u(0) = \lim_{\epsilon\to 0} u_\epsilon(0) = \lim_{\epsilon\to 0} u_0^\epsilon = u_0$. Hence u is a strong solution of Problem VI$_\sigma$. $\square$

Propositions 3.3.1 and 3.3.5(ii) complete the proof of Theorem 3.3.2.

3.4 Navier-Stokes Problem with LBCF

3.4.1 Weak formulations

Throughout this section, we assume $f \in L^2(\Omega \times (0,T))^d$, $u_0 \in V_{\tau,\sigma}$, and $g \in L^2(\Gamma_1 \times (0,T))$ with $g > 0$. Further regularity assumptions on these data will be given before Theorem 3.4.2. As in SBCF, we introduce

$$j_n(t;\eta) = \int_{\Gamma_1} g(t)|\eta|\,ds \qquad (\eta \in L^2(\Gamma_1)), \tag{3.4.1}$$

which is simply written as $j(\eta)$ until the end of this section (note that η is scalar). A primal weak formulation of (3.1.1)–(3.1.4) with (3.1.6) is as follows:

Problem PDE-LBCF. For a.e. $t \in (0,T)$, find $(u(t), p(t)) \in V_\tau \times Q$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$, σ_n is well-defined in the sense of Definition 3.2.1, $|\sigma_n| \le g$ a.e. on Γ_1, and $\sigma_n u_n + g|u_n| = 0$ a.e. on Γ_1.

Throughout this section, we refer to Problem PDE-LBCF just as Problem PDE. Similar abbreviation will be made for other problems. Next, as in SBCF, we propose a variational inequality problem:

Problem VI$_\sigma$-LBCF. For a.e. $t \in (0,T)$, find $u(t) \in V_{\tau,\sigma}$ such that $u'(t) \in L^2(\Omega)^d$, $u(0) = u_0$ and

$$(u', v-u) + a_0(u, v-u) + a_1(u,u,v-u) + j(v_n) - j(u_n) \ge (f, v-u) \tag{3.4.2}$$

for all $v \in V_{\tau,\sigma}$. Here $j = j_n(t;\cdot)$ is defined in (3.4.1).

Unlike the case of SBCF, Problem VI$_\sigma$ is not exactly equivalent to Problem PDE, as is shown in the following theorem.

Theorem 3.4.1. (i) *If* (u, p) *solves Problem* PDE, *then* u *solves Problem* VI_σ.

(ii) *If* u *solves Problem* VI_σ, *then there exists at least one* p *such that* (u, p) *solves Problem* PDE. *If another* p^* *satisfies the same condition, then for a.e.* $t \in (0, T)$ *there exists a unique* $\delta(t) \in \mathbb{R}$ *such that*

$$p(t) = p^*(t) + \delta(t) \quad \text{and} \quad \sigma_n(u(t), p(t)) = \sigma_n(u(t), p^*(t)) - \delta(t). \tag{3.4.3}$$

(iii) *In* (ii), *if we assume furthermore* $u_n(t) \neq 0$, *then* $\delta(t) = 0$. *Namely, the associated pressure is uniquely determined.*

Proof. (i) This can be proved by the same way as Theorem 3.3.1.

(ii) For a.e. $t \in (0, T)$ and $v \in \mathring{V}_\sigma$, it follows from (3.4.2) that $(u', v) + a_0(u, v) + a_1(u, u, v) = (f, v)$, and thus there exists unique $\mathring{p} \in \mathring{Q}$ such that

$$(u', v) + a_0(u, v) + a_1(u, u, v) + b(v, \mathring{p}) = (f, v) \qquad (\forall v \in \mathring{V}).$$

According to Definition 3.2.1, $\mathring{\sigma}_n = \sigma_n(u, \mathring{p})$ is well-defined, so that

$$(u', v) + a_0(u, v) + b(v, \mathring{p}) + a_1(u, u, v) - \langle \mathring{\sigma}_n, v_n \rangle_{H^{1/2}(\Gamma_1)} = (f, v) \qquad (\forall v \in V_\tau).$$

Substituting this equation into (3.4.2), we obtain $-\langle \mathring{\sigma}_n, v_n - u_n \rangle_{H^{1/2}(\Gamma_1)} \leq j(v_n) - j(u_n)$ for all $v \in V_{\tau,\sigma}$. It follows from Lemma 3.2.4(ii) that

$$|\langle \mathring{\sigma}_n, \eta \rangle_{H^{1/2}(\Gamma_1)}| \leq \int_{\Gamma_1} g|\eta| \, ds \qquad (\forall \eta \in H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)).$$

The Hahn-Banach theorem allows us to extend $\mathring{\sigma}_n$ to a linear functional $\sigma_n : L^1_g(\Gamma_1) \to \mathbb{R}$ satisfying the same inequality as above for all $\eta \in L^1_g(\Gamma_1)$. Therefore, $\sigma_n \in L^\infty_{1/g}(\Gamma_1)$ and $|\sigma_n| \leq g$. In addition, $\sigma_n u_n + g|u_n| = 0$ follows.

Since $\mathring{\sigma}_n - \sigma_n$ vanishes on $H^{1/2}(\Gamma_1) \cap L^2_0(\Gamma_1)$, there exists a constant $\delta(t)$ such that $\mathring{\sigma}_n - \sigma_n = \delta(t)$. Now, by setting $p(t) = \mathring{p}(t) + \delta(t)$, it follows that σ_n given above actually equals $\sigma_n(u(t), p(t))$ and that $(u(t), p(t))$ solves Problem PDE. Relation (3.4.3) can be verified by a similar argument.

(iii) Since $\int_{\Gamma_1} u_n \, ds = \int_\Omega \operatorname{div} u \, dx = 0$, the assumption $u_n(t) \neq 0$ implies that there exist subsets A_+, A_- of Γ_1 with positive $d - 1$ dimensional Lebesgue measure satisfying $u_n(t) > 0$ on A_+ and $u_n(t) < 0$ on A_-. Because $|\sigma_n| \leq g$ and $\sigma_n u_n + g|u_n| = 0$ on Γ_1, $\sigma_n = -g(t)$ on A_+ and $\sigma_n = g(t)$ on A_-. Hence $\delta(t)$ in (3.4.3) cannot be other than zero. □

Remark 3.4.1. Since $|\sigma_n| \leq g$, $\delta(t)$ is no more than $2g(t)$ nor less than $-2g(t)$.

3.4.2 Main theorem

Let us state our main theorems for the case of LBCF. As in SBCF, some compatibility condition is necessary; it is rather complicated because normal stress at $t = 0$ involves a pressure at $t = 0$, which is not given as a data. The precise description is as follows: we say that LBCF is satisfied at $t = 0$ if $u_0 \in H^2(\Omega)^d \cap V_{\tau,\sigma}$ and there exists $p_0 \in H^1(\Omega)$ such that

$$|\sigma_n(u_0, p_0)| \le g(0) \quad \text{and} \quad \sigma_n(u_0, p_0)u_{0n} + g(0)|u_{0n}| = 0 \quad \text{a.e. on } \Gamma_1 \tag{3.4.4}$$

We remark that a similar compatibility condition appears in nonlinear semi-group approaches (see [17, 18]).

Furthermore, in order to overcome a difficulty arising from (3.1.7), we need no-leak condition at $t = 0$, that is, $u_{0n} = 0$ on Γ_1. In view of (3.4.4), this is automatically satisfied if $|\sigma_n(u_0, p_0)| < g(0)$ on Γ_1. Examining our proof of the a priori estimates carefully, one finds that this assumption can be weaken to the condition that $\|u_{0n}\|_{L^2(\Gamma_1)}$ is sufficiently small.

Including what we have discussed above, we assume the following:

(L1) $f \in H^1(0, T; L^2(\Omega)^d)$.

(L2) $g \in H^1(0, T; L^2(\Gamma_1))$ with $g(0) \in H^1(\Gamma_1)$.

(L3) $u_0 \in H^2(\Omega)^d \cap V_{\tau,\sigma}$, and LBCF is satisfied at $t = 0$.

(L4) $u_{0n} = 0$ a.e. on Γ_1.

Theorem 3.4.2. *Under* (L1)–(L4) *above, there exists a unique solution* u *of Problem* VI$_\sigma$ *on some interval* $(0, T')$*, with* $T' \le T$*, such that*

$$u \in L^\infty(0, T'; V_{\tau,\sigma}), \qquad u' \in L^\infty(0, T'; L^2(\Omega)^d) \cap L^2(0, T'; V_{\tau,\sigma}).$$

The uniqueness can be proved by the same way as Proposition 3.3.1. We can also obtain $p \in L^\infty(0, T'; L^2(\Omega))$ by a similar manner to Proposition 3.3.2, using the rather infamous inf-sup condition (see [57, Lemma 2.2])

$$C\|p\|_{L^2(\Omega)} \le \sup_{v \in V_\tau} \frac{b(v, p)}{\|v\|_{H^1(\Omega)^d}} \qquad (\forall p \in L^2(\Omega)).$$

The rest of this section is devoted to the proof of the existence. To state regularized problems, for fixed $\epsilon > 0$ we introduce

$$j_\epsilon(\eta) = \int_{\Gamma_1} g\rho_\epsilon(\eta)\, ds \qquad (\eta \in L^2(\Gamma_1)),$$

where ρ_ϵ is a function satisfying properties (a)–(c) for the case $d = 1$, considered at the beginning of Section 3.3.3. We use the notation introduced there such as $\alpha_\epsilon = d\rho/dz$ and $\beta_\epsilon = d^2\rho/dz^2$.

Now let us state the regularized problems.

Problem VI$^\epsilon_\sigma$-LBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{\tau,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$\begin{aligned}&(u'_\epsilon, v - u_\epsilon) + a_0(u_\epsilon, v - u_\epsilon) + a_1(u_\epsilon, u_\epsilon, v - u_\epsilon) + j_\epsilon(v_n) - j_\epsilon(u_{\epsilon n})\\ \geq&(f, v - u_\epsilon) \qquad\qquad (\forall v \in V_{\tau,\sigma}).\end{aligned}$$

Problem VE$^\epsilon_\sigma$-LBCF. For a.e. $t \in (0, T)$, find $u_\epsilon(t) \in V_{\tau,\sigma}$ such that $u'_\epsilon(t) \in L^2(\Omega)^d$, $u_\epsilon(0) = u_0^\epsilon$ and

$$(u'_\epsilon, v) + a_0(u_\epsilon, v) + a_1(u_\epsilon, u_\epsilon, v) + \int_{\Gamma_1} g\alpha_\epsilon(u_{\epsilon n})v_n ds = (f, v) \qquad (\forall v \in V_{\tau,\sigma}).$$

As in Proposition 3.3.3, Problems VI$^\epsilon_\sigma$ and VE$^\epsilon_\sigma$ are equivalent. The construction of the perturbed initial velocity u_0^ϵ is similar to that of SBCF. In fact, since LBCF holds at $t = 0$ by (L3), the Green formula leads to

$$a_0(u_0, v - u_0) + \int_{\Gamma_1} g(0)|v_n|\, ds - \int_{\Gamma_1} g(0)|u_{0n}|\, ds \geq (-\nu\Delta u_0 + \nabla p_0, v - u_0),$$

for $v \in V_{\tau,\sigma}$. Consider the regularized problem: find $u_0^\epsilon \in V_{\tau,\sigma}$ such that

$$\begin{aligned}&a_0(u_0^\epsilon, v - u_0^\epsilon) + \int_{\Gamma_1} g(0)\rho_\epsilon(v_n)\, ds - \int_{\Gamma_1} g(0)\rho_\epsilon(u_{0n}^\epsilon)\, ds\\ \geq&(-\nu\Delta u_0 + \nabla p_0, v - u_0^\epsilon) \qquad\qquad (\forall v \in V_{\tau,\sigma}),\end{aligned} \tag{3.4.5}$$

which is equivalent to (cf. Proposition 3.3.3)

$$a_0(u_0^\epsilon, v) + \int_{\Gamma_1} g(0)\alpha_\epsilon(u_{0n}^\epsilon)v_n\, ds = (-\nu\Delta u_0 + \nabla p_0, v) \qquad (\forall v \in V_{\tau,\sigma}). \tag{3.4.6}$$

The elliptic variational inequality (3.4.5) admits a unique solution u_0^ϵ, which is the perturbation of u_0 in question. With this setting, we find:

Lemma 3.4.1. (i) *When $\epsilon \to 0$, $u_0^\epsilon \to u_0$ strongly in $H^1(\Omega)^d$. In particular, it follows that $u_0^\epsilon \to 0$ in $L^2(\Gamma_1)$.*

(ii) *$u_0^\epsilon \in H^2(\Omega)^d$ and*

$$\|u_0^\epsilon\|_{H^2(\Omega)^d} \leq C(\|\nu\Delta u_0 + \nabla p_0\|_{L^2(\Omega)^d} + \|g(0)\|_{H^1(\Gamma_1)}). \tag{3.4.7}$$

Proof. (i) is proved by the same way as Lemma 3.3.1(i). Since $g(0) \in H^1(\Gamma_1)$ by (L3), (ii) is a direct consequence of [57, Lemma 4.1]. $\square$

Remark 3.4.2. By (i) and (L4), for sufficiently small $\epsilon > 0$ we have

$$\|u_0^\epsilon\|_{L^2(\Omega)^d} \leq 2\|u_0\|_{L^2(\Omega)^d}, \quad \|u_0^\epsilon\|_{H^1(\Omega)^d} \leq 2\|u_0\|_{H^1(\Omega)^d}, \quad \|u_{0n}^\epsilon\|_{L^2(\Gamma_1)} \leq \frac{\alpha}{8\gamma_1}, \tag{3.4.8}$$

where α and γ_1 are the constants in (3.2.1) and (3.2.5) respectively.

Remark 3.4.3. As in SBCF, if Γ_0 is C^2 and Γ_1 is C^4, then we can apply Lemma 4.1 of [57]. On the other hand, $g(0) \in H^{1/2}(\Gamma_1)$, stated in [57], is actually insufficient to deduce the H^2-H^1 regularity (see the errata of [57]).

To solve Problem $\mathrm{VE}_\sigma^\epsilon$, we construct approximate solutions by Galerkin's method. Since $V_{\tau,\sigma} \subset H^1(\Omega)^d$ is separable, there exist $w_1, w_2, ... \in V_{\tau,\sigma}$, linear independent to each other, such that $\bigcup_{m=1}^\infty \mathrm{span}\{w_k\}_{k=1}^m \subset V_{\tau,\sigma}$ dense in $H^1(\Omega)^d$. Here we may assume $w_1 = u_0^\epsilon$.

Problem $\mathrm{VE}_\sigma^{\epsilon,m}$-LBCF. Find $c_k \in C^2([0,T])$ $(k = 1, ..., m)$ such that $u_m \in V_{\tau,\sigma}$ defined by $u_m = \sum_{k=1}^m c_k(t) w_k$ satisfies $u_m(0) = u_0^\epsilon$ and

$$(u_m', w_k) + a_0(u_m, w_k) + a_1(u_m, u_m, w_k) + \int_{\Gamma_1} g\alpha_\epsilon(u_{mn}) w_{kn} ds = (f, w_k) \qquad (k = 1, ..., m). \tag{3.4.9}$$

Since $\alpha_\epsilon \in C^1(\mathbb{R})$, there exist unique solutions $c_k \in C^2([0,\tilde{T}])$ $(k = 1, ..., m)$ for some $\tilde{T}$, which may depend on m and ϵ at this stage.

Proposition 3.4.1. *Assume* (L1)–(L4), *and let $\epsilon > 0$ be sufficiently small so that* (3.4.8) *holds. Then there exists some interval* $(0, T')$ *such that $u_m \in L^\infty(0, T'; V_{\tau,\sigma})$ and $u_m' \in L^\infty(0, T'; L^2(\Omega)^d) \cap L^2(0, T'; V_{\tau,\sigma})$ are uniformly bounded with respect to m and ϵ. Here, T' is independent of m and ϵ.*

Proof. Due to space limitations, we sometimes simply write $\|u\|_{L^2}, \|g\|_{L^2}, ...,$ instead of $\|u\|_{L^2(\Omega)^2}, \|g\|_{L^2(\Gamma_1)}, ...,$ when there is no fear of confusion.

First we consider the case $d = 2$. Multiplying (3.4.9) by $c_k(t)$ for $k = 1, ..., m$, adding them, and using (3.2.1), (3.2.5) and (3.3.8), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|_{L^2}^2 + (\alpha - \gamma_1\|u_{mn}\|_{L^2(\Gamma_1)})\|u_m\|_{H^1}^2 \leq (f, u_m). \tag{3.4.10}$$

Since $\|u_{mn}(t)\|_{L^2(\Gamma_1)}$ is continuous with respect to t and (3.4.8) holds, there exists a maximum value $T_1 \in (0, \tilde{T}]$ of t such that $\gamma_1\|u_{mn}(t)\|_{L^2(\Gamma_1)} \leq \frac{\alpha}{4}$. If this inequality holds for all $0 \le t \le \tilde{T}$, we take $T_1 = \tilde{T}$. Noting $|(f, u_m)| \le \frac{\alpha}{4}\|u_m\|_{H^1}^2 + \frac{1}{\alpha}\|f\|_{L^2}^2$, we find from (3.4.10) that

$$\frac{d}{dt}\|u_m\|_{L^2}^2 + \alpha\|u_m\|_{H^1}^2 \le C\|f\|_{L^2}^2 \qquad (0 \le t \le T_1).$$

Hence $u_m \in L^\infty(0, T_1; L^2) \cap L^2(0, T_1; V_{\tau,\sigma})$ is uniformly bounded in m and ϵ.

Next, differentiating (3.4.9), multiplying the resulting equation by $c_k'(t)$, and adding them, we obtain

$$\begin{aligned}(u_m'', u_m') &+ a_0(u_m', u_m') + a_1(u_m', u_m, u_m') + a_1(u_m, u_m', u_m') \\ &+ \int_{\Gamma_1} g'\alpha_\epsilon(u_{mn})u_{mn}'\,ds + \int_{\Gamma_1} g\beta_\epsilon(u_{mn})|u_{mn}'|^2\,ds = (f', u_m'). \quad (3.4.11)\end{aligned}$$

Here, we estimate each term in (3.4.11) as follows:

$$\begin{aligned}|a_1(u_m', u_m, u_m')| &\le C\|u_m'\|_{L^2}\|u_m\|_{H^1}\|u_m'\|_{L^2} \\ &\le \frac{\alpha}{12}\|u_m'\|_{H^1}^2 + C\|u_m\|_{H^1}^2\|u_m'\|_{L^2}, \qquad (3.4.12) \\ |a_1(u_m, u_m', u_m')| &\le \gamma_1\|u_{mn}\|_{L^2(\Gamma_1)}\|u_m'\|_{H^1}^2 \le \frac{\alpha}{4}\|u_m'\|_{H^1}^2, \\ \left|\int_{\Gamma_1} g'\alpha_\epsilon(u_{mn})u_{mn}'\,ds\right| &\le C\|g'\|_{L^2}\|u_m'\|_{H^1} \le \frac{\alpha}{12}\|u_m'\|_{H^1}^2 + C\|g'\|_{L^2}^2, \\ \int_{\Gamma_1} g\beta_\epsilon(u_{mn})|u_{mn}'|^2\,ds &\ge 0, \\ |(f', u_m')| &\le \frac{\alpha}{12}\|u_m'\|_{H^1}^2 + C\|f'\|_{L^2}^2.\end{aligned}$$

Collecting these estimates, we derive from (3.4.11) that for $0 \le t \le T_1$

$$\frac{d}{dt}\|u_m'\|_{L^2} + \alpha\|u_m'\|_{H^1}^2 \le C(\|f'\|_{L^2}^2 + \|g'\|_{L^2}^2) + C\|u_m\|_{H^1}^2\|u_m'\|_{L^2}^2. \quad (3.4.13)$$

Combining the technique used in Proposition 3.3.4 with (3.4.6) and (3.4.7), we observe that $\|u_m'\|_{L^\infty(0,T_1;L^2)}$, $\|u_m'\|_{L^2(0,T_1;V_{\tau,\sigma})}$, and $\|u_m\|_{L^\infty(0,T_1;V_{\tau,\sigma})}$ are bounded by $C(f, g, u_0, p_0)$.

It remains to show that T_1 is bounded from below independently of m, ϵ. If $\gamma_1\|u_{mn}(T_1)\|_{L^2(\Gamma_1)} < \alpha/4$ and thus $T_1 = \tilde{T}$, we can extend $u_m(t)$ beyond $t = \tilde{T}$ and repeat the above discussion until we reach either

$$\max_{0\le t\le T} \gamma_1\|u_{mn}(t)\|_{L^2(\Gamma_1)} \le \alpha/4 \quad \text{or} \quad \gamma_1\|u_{mn}(T_1)\|_{L^2(\Gamma_1)} = \alpha/4.$$

In the former case $T_1 = T$. In the latter case, we have

$$\frac{\alpha}{8\gamma_1} \leq \|u_{mn}(T_1)\|_{L^2(\Gamma_1)} - \|u_{mn}(0)\|_{L^2(\Gamma_1)} \leq \|u_{mn}(T_1) - u_{mn}(0)\|_{L^2(\Gamma_1)}$$
$$\leq \int_0^{T_1} \|u'_{mn}(t)\|_{L^2(\Gamma_1)} dt \leq C \int_0^{T_1} \|u'_m\|_{H^1(\Omega)^d} dt \leq C\sqrt{T_1}\|u'_m\|_{L^2(0,T_1;V_{\tau,\sigma})}.$$

Hence T_1 is bounded from below, and we complete the proof for $d = 2$.

Second let us consider the case $d = 3$. What changes from $d = 2$ is that (3.4.12) is replaced with

$$\begin{aligned}|a_1(u'_m, u_m, u'_m)| &\leq C\|u'_m\|_{L^2}^{1/2}\|u_m\|_{H^1}\|u'_m\|_{H^1}^{3/2} \\ &\leq \gamma_2\|u_m\|_{H^1}\|u'_m\|_{H^1}^2 + C\|u_m\|_{H^1}\|u'_m\|_{L^2}^2,\end{aligned}$$

where γ_2 can be arbitrarily small. We choose γ_2 satisfying $\gamma_2\|u_0\|_{H^1} \leq \frac{\alpha}{48}$, so that $\gamma_2\|u_0^\epsilon\|_{H^1} \leq \frac{\alpha}{24}$ by virtue of (3.4.8). Let T_2 be the maximum value of $t \in (0, \tilde{T}]$ such that $\gamma_2\|u_m(t)\|_{H^1} \leq \frac{\alpha}{12}$. If this inequality holds for all $t \in (0, \tilde{T}]$, we set $T_2 = \tilde{T}$. Such T_2 does exist, and if $T_2 < \tilde{T}$ then $\gamma_2\|u_m(T_2)\|_{H^1} = \frac{\alpha}{12}$.

Therefore, setting $T' = \min(T_1, T_2)$, instead of (3.4.13) we get

$$\frac{d}{dt}\|u'_m\|_{L^2} + \alpha\|u'_m\|_{H^1}^2 \leq C(\|f'\|_{L^2}^2 + \|g'\|_{L^2}^2) + C\|u_m\|_{H^1}\|u'_m\|_{L^2}^2 \quad (0 \leq t \leq T').$$

As a consequence, we observe that $\|u'_m\|_{L^2(0,T';V_{\tau,\sigma})}$, $\|u'_m\|_{L^\infty(0,T';L^2)}$, and $\|u_m\|_{L^\infty(0,T';V_{\tau,\sigma})}$ are bounded by $C(f, g, u_0, p_0)$.

Now, if $T_1 < \tilde{T}$ or $T_2 < \tilde{T}$ then T' are bounded from below as follows:

$$\frac{\alpha}{12\gamma_1} \leq \|u_{mn}(T')\|_{L^2(\Gamma_1)} - \|u_{mn}(0)\|_{L^2(\Gamma_1)} \leq \int_0^{T'} \|u'_{mn}\|_{L^2(\Gamma_1)} dt$$
$$\leq C\int_0^{T'} \|u'_m\|_{H^1} dt \leq C\sqrt{T_1}\|u'_m\|_{L^2(0,T';V_{\tau,\sigma})},$$
$$\frac{\alpha}{24\gamma_2} \leq \|u_m(T')\|_{H^1} - \|u_m(0)\|_{H^1} \leq \int_0^{T'} \|u'_m\|_{H^1} dt \leq \sqrt{T'}\|u'_m\|_{L^2(0,T';V_{\tau,\sigma})}.$$

When $T_1 = \tilde{T}$ and $T_2 = \tilde{T}$, we can extend $u_m(t)$ beyond $t = \tilde{T}$ and repeat the above discussion. This completes the proof of Proposition 3.4.1. □

The last step of the proof, i.e. passing to the limits $m \to \infty$ and $\epsilon \to 0$, proceeds in the same way as Proposition 3.3.5, with n replaced by τ and vice versa. This proves that a solution of Problem VI$_\sigma$ exists, which, combined with the uniqueness result, completes the proof of Theorem 3.4.2.

Remark 3.4.4. At first glance one may think Theorem 3.4.2, where we get only a time-local solution in spite of a smallness assumption on u_0 even if $d = 2$, is too poor. However, in view of the fact that we obtain only time-local solutions in 2D case under the linear leak b.c. (see [29, Theorem 6] or [52]), such limitations cannot be avoided to some extent.

Remark 3.4.5. Under additional smallness assumptions on the data f, g, u_0, p_0, we can derive global existence results for both $d = 2$ and $d = 3$.

3.5 Concluding Remarks

By the discussion presented above, we have established the existence and uniqueness, while we did not get in touch with higher regularity, such as $u \in L^\infty(0,T; H^2(\Omega)^d)$, $p \in L^\infty(0,T; H^1(\Omega))$. This is because some regularity results for the elliptic cases are not available. For instance, Problem VI$_\sigma$-SBCF is rewritten as

$$\begin{aligned} a_0(u, v-u) + j_\tau(t; v_\tau) - j_\tau(t; u_\tau) &\geq (f, v-u) - (u', v-u) - a_1(u, u, v-u) \\ &=: \langle F(t), v-u \rangle_{V_{n,\sigma}} \qquad (\forall v \in V_{n,\sigma}), \end{aligned}$$

with $F(t) \in L^p(\Omega)^d$ for some $p < 2$. If we prove this elliptic variational inequality has a unique solution in $W^{2,p}(\Omega)^d$ when $p < 2$, then a technique similar to [62, Theorems III.3.6 and III.3.8] allows us to deduce $u(t) \in H^2(\Omega)^d$. Thereby, we need to extend the regularity theory of [57] to cases $p \neq 2$.

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