A New Element in Discrete Analysis of Plane Strain Problems

平間板問題の離散化解析に対する新しい要素

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Summary

Recently one of the author has proposed a new physical model for bending analysis of beam and plate structures. The same idea is now extended to finite element analysis of the plane strain problems. This element consists of rigid plates connected with various types of springs and size of the stiffness matrix of this element is only $3 \times 3$ and therefore considerable reduction of computational time can be expected in nonlinear analysis of plane strain problems.

1. A New Physical Model in Plane Strain Problem

Consider two rigid plates which are connected by three different types of springs $k_d, k_l$, and $k_r$ as shown in Fig. 1.

\[ u = u_1 + (y - y_1) \theta_1 \]
\[ v = v_1 - (x - x_1) \theta_1 \]

where the rotational displacement $\theta_1$ is assumed to be very small. After some loading, two plates are displaced to positions as shown in Fig. 2.

Fig. 2 Two plates positions after deformation

The relative displacement of the edge $\overline{AC}$ in each plate is given by $\delta_r, \delta_h$, and $\varphi$. Coordinates of the mid point $M$ of the edge $\overline{AC}$ is given by

\[ (\frac{1}{2}(x_3 + x_5), \frac{1}{2}(y_3 + y_5)) \]

This point will be displaced to $M'$ in $\overline{ABC}$ and $M'$ in $\overline{ACD}$ respectively after loading.

Coordinates of these points are given as follows:

\[ x_{u'} = \frac{1}{2} \left[ (x_3 + x_5) + 2u_1 + (y_3 + y_5) \theta_1 \right] \]
\[ y_{u'} = \frac{1}{2} \left[ (y_3 + y_5) + 2v_1 - (x_3 + x_5) \theta_1 \right] \]
\[ x_{u'} = \frac{1}{2} \left[ (x_3 + x_5) + 2u_2 + (y_3 + y_5) \theta_2 \right] \]
\[ y_{u'} = \frac{1}{2} \left[ (y_3 + y_5) + 2v_2 - (x_3 + x_5) \theta_2 \right] \]

where $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$

Therefore a vector $M'M''$ is given by the following equations:

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\[
(M' M^\prime)^\prime_y = \frac{1}{2} \left\{ \begin{array}{l}
2 u_{21} + (y_{32} + y_{52}) \theta_2 \\
- (y_{31} + y_{51}) \theta_1 
\end{array} \right. \\
(M' M^\prime)^\prime_x = \frac{1}{2} \left\{ \begin{array}{l}
2 u_{21} - (x_{32} + x_{52}) \theta_2 \\
+ (x_{31} + x_{51}) \theta_1 
\end{array} \right. 
\]

\tag{3}

Again, the following notations are employed.

\[ u_i = u_i - u_1, \quad v_i = v_i - v_1 \]

Denoting an unit vector along the edge \( \overrightarrow{AC} \) by \( t \) before deformation as shown in Fig. 1, the displacement component \( \delta_N \) of the vector \( M' M^\prime \) along the edge \( \overrightarrow{AC} \) can be given as follows:

\[
\delta_N = (M' M^\prime, t) = \frac{1}{2} I_{55} \left\{ \begin{array}{l}
x_{53} (2 u_{21} + (y_{32} + y_{52}) \theta_2 - (y_{31} + y_{51}) \theta_1) \\
+ y_{53} (2 v_{21} - (x_{32} + x_{52}) \theta_2 + (x_{31} + x_{51}) \theta_1) 
\end{array} \right. 
\]

\tag{4}

Similarly, the displacement component \( \delta_V \) of the vector \( M' M^\prime \) perpendicular to the edge \( \overrightarrow{AC} \) can be given by the following equation:

\[
\delta_V^2 = |t \times M' M^\prime|^2 = \frac{1}{4} I_{55} \left\{ \begin{array}{l}
x_{53} (2 v_{21} - (x_{32} + x_{52}) \theta_2 + (x_{31} + x_{51}) \theta_1) \\
- y_{53} (2 u_{21} + (y_{32} + y_{52}) \theta_2 - (y_{31} + y_{51}) \theta_1) 
\end{array} \right. 
\]

\tag{5}

Relative angle change \( \phi \) of the edges \( \overrightarrow{AC} \) and \( \overrightarrow{A'C'} \) is also obtained from following equation:

\[
\cos \phi = \frac{(t, t')} = 1 - \frac{\phi^2}{2} 
\]

where \( t' \) is the unit vector along the edge \( \overrightarrow{A'C'} \) after deformation.

\[
\frac{\phi^2}{2} = \frac{1}{k_{55}} \left[ (u_{53} + u_{53}') (x_{53} + (v_{53} + v_{53}')) y_{53} + u_{53} u_{53}' + v_{53} v_{53}' \right] \]

\tag{6}

Now strain energy \( V \) to be stored in the spring \( k_d, k_2 \) and \( k_r \) after deformation will be given as follows:

\[
V = \frac{1}{2} k_d \delta_N^2 + \frac{1}{2} k_2 \delta_V^2 + \frac{1}{2} k_r \phi^2 
\]

\tag{7}

In view of (5),(6),(7) and (8) it is clearly seen that the strain energy \( V \) is a quadratic function of \( u_1, v_1, \theta_1 \) and \( u_2, v_2, \theta_2 \) and therefore applying Castigliano's theorem, the following reaction force vector \( R \) can be derived.

\[
R = \frac{\partial V}{\partial u} = Ku 
\]

\tag{8}

where \( K \) is the stiffness matrix to be obtained and \( u \) is the displacement vector given as follows:

\[
u^T = [u_1, v_1, \theta_1, u_2, v_2, \theta_2] 
\]

\tag{9}

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( y_1 )</th>
<th>( \theta_1 )</th>
<th>( u_2 )</th>
<th>( v_2 )</th>
<th>( \theta_2 )</th>
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<td>( k_d y_1^2 + k_r x_1 x_1 )</td>
<td>( k_d x_1^2 + k_r y_1 y_1 )</td>
<td>( 2 d_{11} = x_{11} (x_{11} + x_{12}) + y_{11} (y_{11} + y_{12}) )</td>
<td>( 2 d_{12} = x_{12} (y_{11} + y_{12}) - y_{12} (x_{11} + x_{12}) )</td>
<td>( 2 d_{13} = -x_{11} (y_{11} + y_{12}) + y_{11} (x_{11} + x_{12}) )</td>
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<td>( k_d y_2^2 + k_r x_2 x_2 )</td>
<td>( k_d x_2^2 + k_r y_2 y_2 )</td>
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<td>( k_d y_3^2 + k_r x_3 x_3 )</td>
<td>( k_d x_3^2 + k_r y_3 y_3 )</td>
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The final form of the stiffness matrix is given in the Table 1.

Spring constant \( k_d, k_2 \) and \( k_r \) are determined in the following way. Considering two plates shown in Fig. 3 the normal strains \( \varepsilon_d \) and shearing strain \( \gamma \) in these springs may be given by the following formulae.
This problem is a standard plane strain problem in plasticity and it was studied by many investigators for various cases of $h/b$ ratios using the so-called "slip line theory".

In present analysis a given material is ideal plastic and the maximum shearing stress theory is used as the yield criterion. Since the material is assumed incompressible after yielding, the plastic strain increment is purely shearing deformation. Analysis was made for three different cases of $h/b$ ratios as follows:

1. $h/b = 1$
2. $h/b = 2$
3. $h/b = 1$

The load deformation curves obtained, assumed mesh division and slip lines as well as computing time (w. r. t. *HITAC 8700−8800 approximately comparable to IBM 360−195) are shown in Fig. 5, 6, and 7.

It should be mentioned here that the rotational component was neglected in this analysis so that the size of stiffness matrices used were only $2 \times 2$ and yet the ultimate loads and slip lines obtained

![Diagram](image)

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2. Some numerical examples on the elasto-plastic plane strain problems

To show validity of this new element, elasto-plastic analysis of the punch problem and a slit notch specimen under tensile load are made.

1. Punch problem

Punch problem of an elasto-plastic slab as shown in Fig. 4 is considered.
were in good agreement with the results obtained by previous authors.

(2) Elasto-plastic analysis of a slit notch tensile specimen

Analysis was made for the specimen as shown in Fig. 8. Elasto-plastic incremental analysis was made under the assumption of no rotational displacement and the maximum shearing stress theory. The result obtained was extremely in good agreement with the result of previous investigators.

3. Conclusion

A new element suitable for nonlinear analysis of plane strain problems is proposed in this paper. Results of numerical analysis on some simple problems duly justified use of this element for elasto-plastic analysis of complex plane strain problems. The authors would like to express their thanks to Messrs. K. Kondou and M. Watanabe for their valuable discussions.

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Reference