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ANALYSIS OF ONE-DIMENSIONAL STRESS WAVE BY THE FINITE ELEMENT METHOD

動的応答の一次元有限要素法解析

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Summary

This paper presents the solution of one-dimensional elastic and visco-elastic wave propagation by the finite element method as a basis for pursuing the general two or three dimensional stress wave problems. The numerical analysis is executed by the following procedures: (i) The elapse of natural time t is divided into small finite segments of equal interval Δt. The acceleration of the element mass is assumed to vary linearly with time within each interval, i.e. the time derivative of acceleration or third derivative of displacement being assumed constant. (ii) Whole bar is divided into segments with equal length Δl that have either lumped or consistent masses at the points of division or nodes. Mechanical behavior of the bar element may be elastic or visco-elastic.

Various combinations of time-to-length segment ratios $\Delta t/\Delta l$ and mass matrices have been examined, and it is found that the consistent mass matrix associated with the characteristic time-to-length ratio $\Delta t/\Delta l=1/c$ (c: velocity of wave propagation) gives numerical solution which correlates satisfactorily with the analytical solution. It is due to the hyperbolic nature of basic equation in the stress wave problems.

1. Introduction

One-dimensional elastic or visco-elastic wave propagation has been analized conventionally by the Laplace-transformation or the finite difference method. The former procedure is not practicable when the number of constituent elements which represent the mechanical behavior of the material is large. Moreover, it is well known that the correspondence principle based on the Laplace transformation fails to cover the boundary conditions that changes with time. The latter finite difference method is rather clumsy and may not be suitable for the wave propagation problems in two or three dimensions. Fatal deficiency of the method is that it is practically impossible to formulate a unified or general purpose solution procedure applicable to a variety of problem areas.

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We shall examine, in this paper, the appropriateness of the finite element approach to wave propagation analysis. As the first step, one-dimensional wave shall be studied to elucidate the accuracy of numerical solutions as influenced by mass matrices adopted as well as time-to-space segments ratio in the computation.

2. Method of Analysis

(1) One-dimentional elastic wave

We adopt notations $\lfloor A \rfloor$ and $\{B\}$ for row vector and column vector respectively. By the virtual work principle, the nodal force $\{F\}_t^e$ of each element at time t is expressed in terms of the stress σ_t in the element and apparent nodal forces due to distributing load p_t[1]:

$$\{F\}_t^e=\int\{B\}\sigma_t d(\mathrm{vol})-\int\{N\}p_t d(\mathrm{vol}) \qquad (1)$$

Suffix t indicates quantities at time t and $\{B\}$ and $\{N\}$ are vectors which combine respectively the strain ε and displacement f in the element with nodal displacement $\{\delta\}^e$. Note σ and ε are scalar quantities in one-dimensional problem. We employ the Zienkiewicz's notation and assume that the displacement in the element is linear function of condinates along the axis of the bar, so that

$$\varepsilon = \lfloor B \rfloor \{\delta\}^e, \quad f = \lfloor N \rfloor \{\delta\}^e \qquad (2)$$

Similarly at time $t+\Delta t$.

$$\{F\}^e_{t+\Delta t} = \int \{B\} \sigma_{t+\Delta t} d(\mathrm{vol}) - \int \{N\} p_{t+\Delta t} d(\mathrm{vol}) \qquad (3)$$

Subtracting equation (1) from (3), we obtain the following equation which relates the increments of basic variables within time interval Δt

$$\{\Delta F\}^e = \int \{B\} \Delta\sigma d(\mathrm{vol}) - \int \{N\} \Delta p d(\mathrm{vol}) \qquad (4)$$

where $\{\Delta F\}^e = \{F\}^e_{t+\Delta t} - \{F\}^e_t$ etc.

Increments of stress and distributing load (inertia force) in dynamic problems are given from (2), by noting $\lfloor B \rfloor$ as well as $\lfloor N \rfloor$ in the present formulation are not dependent on time and space coordinates

$$\Delta\sigma = E\Delta\varepsilon = E\lfloor B \rfloor \{\Delta\delta\},$$
$$\Delta p = -\rho \Delta \ddot{f} = -\rho \lfloor N \rfloor \{\Delta\ddot{\delta}\} \qquad (5)$$

where $\Delta \ddot{f} = \left(\frac{d^2 f}{dt^2}\right)_{t+\Delta t} - \left(\frac{d^2 f}{dt^2}\right)_t$ etc.

E and ρ are Young's modulus and density of the bar element. Substituting $\Delta\sigma$ and Δp from equation (5) into (4), we obtain

$$\{\Delta F\}^e = [k]^e \{\Delta\delta\}^e + [m]^e \{\Delta\ddot{\delta}\}^e \qquad (6)$$

where $[k]^e = E\int \{B\} \lfloor B \rfloor d(\mathrm{vol}) = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$[m]^e = \rho \int \{N\} \lfloor N \rfloor d(\mathrm{vol}) = \rho A l \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

The matrix $[m]^e$ in equation (6) is the one that is called consistent mass matrix. Alternatively the lumped mass matrix has been used in many applications where the element mass is allotted equally to each nodal point as follows:

$$[m^*]^e = \rho A l \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

By synthesizing or assembling equation (6) for each element, we obtain the equation of motion of whole system, i.e.

$$[M]\{\Delta\ddot{\delta}\} + [K]\{\Delta\delta\} = \{\Delta F\} \qquad (7)$$

where $[M]$ and $[K]$ denote mass and stiffness matrices of the system respectively. Vector $\{\Delta F\}$ corresponds to the external force increment, if any, at the nodes including both ends of the bar. Material damping may be taken into account in the dynamic equilibrium by adding a term expressed conventionally by $[c]\{\Delta\dot{\delta}\}$. In our formulation, however, material damping is incorporated in the material constitutive equation, giving rise to a term corresponding to the apparent force applied at nodes of each element (see Section 2.2). The solution of equation (7) has been obtained by a kind of step-by-step procedure.

By the assumption of linear variation of acceleration with time, we can replace equation (7), which is basically simultaneous differential equations of second order, by linear algebraic equations. We first note, due to the assumption of linear variation of acceleration, that the incremeuts $\Delta\dot{\delta}$ and $\Delta\ddot{\delta}$ respectively of velocity and acceleration during time interval Δt can be expressed in terms of displacement increment $\Delta\delta$ together with time derivatives $\dot{\delta}$ and $\ddot{\delta}$ at time t[2,3]:

$$\Delta\dot{\delta} = (3/\Delta t)\Delta\delta - 3\dot{\delta} - (\Delta t/2)\ddot{\delta} \qquad (8)$$

and

$$\Delta\ddot{\delta} = (6/\Delta t^2)\Delta\delta - (6/\Delta t)\dot{\delta} - 3\ddot{\delta} \qquad (9)$$

Substitution of equation (9) into equation of motion (7) pertaining to *each* bar element yields a set of linear algebraic equations for unknown displacement vector $\{\Delta\delta\}$ at nodal points as follows:

$$\left([K] + \frac{6}{\Delta t^2}[M]\right)\{\Delta\delta\}$$
$$= \{\Delta F\} + 3[M]\left\{\frac{2}{\Delta t}\dot{\delta} + \ddot{\delta}\right\} \qquad (10)$$

The solution of simultaneous equations (10) gives the increment of displacement $\{\Delta\delta\}$, and the increments of velocity and acceleration $\{\Delta\dot{\delta}\}$, $\{\Delta\ddot{\delta}\}$ can be evaluated by equations (8) and (9). The displacements $\{\delta\}$, velocities $\{\dot{\delta}\}$ and accelerations $\{\ddot{\delta}\}$ at time $t+\Delta t$ are calculated by adding these increments to corresponding quantities at time t. Then we can proceed the next time step Δt of computation.

(2) Visco-elastic wave

We consider here specifically the visco-elastic material expressed by three-element model of Fig. 1. (Note, however, that the material is a special case of generalized Maxwell model of Fig. 2.)

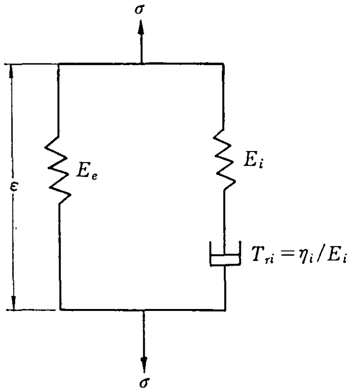

Fig. 1 Three-element model.

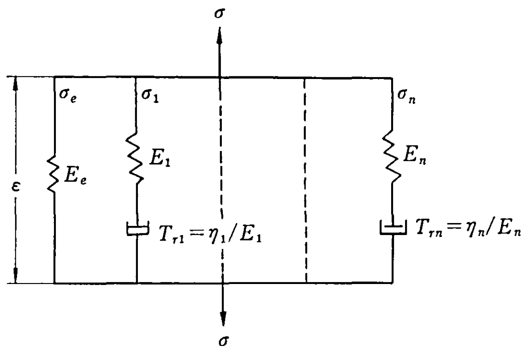

Fig. 2 Generalized Maxwell model.

Stress-strain relations or constitutive equation of the three element model is given by the following differential equation at time t.

$$\sigma+T_{ri}\dot{\sigma}=E_e\varepsilon+(E_i+E_e)T_{ri}\dot{\varepsilon} \tag{11}$$

Incremental form of equation (11) is

$$\Delta\sigma+T_{ri}\Delta\dot{\sigma}=E_e\Delta\varepsilon+(E_i+E_e)T_{ri}\Delta\dot{\varepsilon} \tag{12}$$

In the following, we assume that the stress and strain velocities $\dot{\sigma}$ and $\dot{\varepsilon}$ vary linearly with time in each inteoval Δt, i.e. the second time derivatives of stress and strain being constant such that

$$\Delta\dot{\sigma}=\frac{2}{\Delta t}\Delta\sigma-2\dot{\sigma},\quad \Delta\dot{\varepsilon}=\frac{2}{\Delta t}\Delta\varepsilon-2\dot{\varepsilon} \tag{13}$$

Assumption of linear variation of $\dot{\varepsilon}$ is consistent with that made for $\ddot{\delta}$ in deriving equations (7) and (8) of the preceding section. Substituting $\Delta\dot{\sigma}$ and $\Delta\dot{\varepsilon}$ from (13) into (12) and using the relation given by equation (11), we obtain

$$\Delta\sigma=\left(E_e+\frac{1}{1+\frac{\Delta t}{2T_{ri}}}E_i\right)\Delta\varepsilon-\frac{\Delta t/T_{ri}}{1+\frac{\Delta t}{2T_{ri}}}\sigma_i \tag{14}$$

where $\sigma_i=\sigma_t-E_e\varepsilon$

Equation (14) is extended, for generalised Maxwell model of Fig. 2, as

$$\Delta\sigma=\left(E_e+\sum_i\frac{1}{1+\frac{\Delta t}{2T_{ri}}}E_i\right)\Delta\varepsilon-\sum_i\frac{\Delta t/T_{ri}}{1+\frac{\Delta t}{2T_{ri}}}\sigma_i \tag{15}$$

In visco-elastic wave analysis, equation (14) substitutes the relation $\Delta\sigma=E\Delta\varepsilon$ for elastic case. Thus modifing equations (5) through (10) appropriately, we can compute visco-elastic stress wave by the computer program developed for elastic wave.

3. Results of Computation

Our first example concerns with the longitudinal wave propagation in an elastic bar of finite length whose one end ($x=0$) is rigidly fixed (Figs. 3 through 6); the external loading being applied at the other end ($x=l$). The type of loading considered was the displacement type with a constant tensile speed $v(l, t)$ or displacement $u(l, t)=v(l, t)t$. Parameters or data in the computation are:

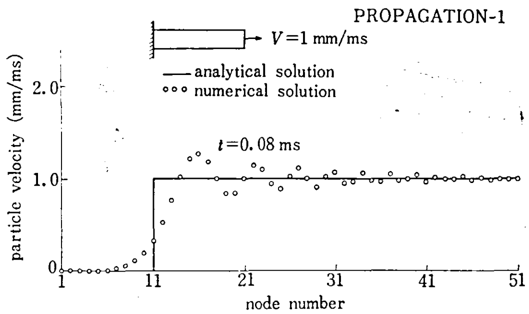

Fig. 3 LUMPED MASS with $\Delta t=\Delta l/c$

total number of nodes 51

total number of elements 50

mass matrices (1) lumped (2) consistent

time increment (1) $\Delta t=\Delta l/c$,

(2) $\Delta t=\frac{1}{2}\Delta l/c$

Young's modulus $E=20000\ \text{kg/mm}^2$

density of the bar material

$\rho=0.0008\ \text{kg/msec}^2/\text{mm}^4$

velocity of elastic wave propagation

$c=5000\ \text{mm/msec}$

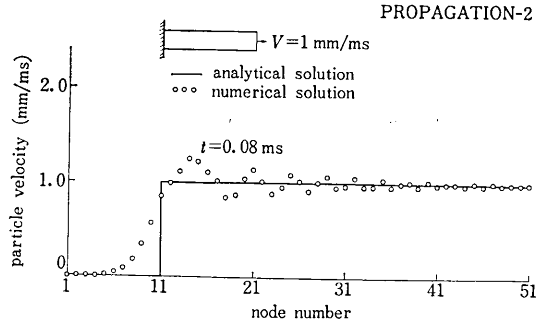

Fig. 4 LUMPED MASS with $\Delta t=0.5\Delta l/c$

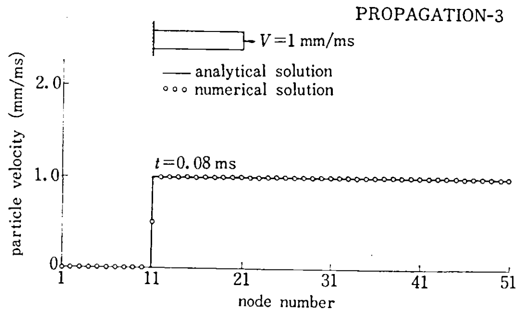

Fig. 5 CONSISTENT MASS with $\Delta t=\Delta l/c$

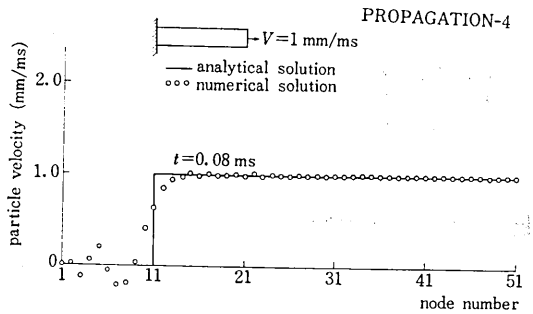

Fig. 6 CONSISTENT MASS with $\Delta t=0.5\Delta l/c$

length of each bar element

$$\Delta l=10\text{ mm}$$

tensile speed at bar end　$v(l,\ t)=1$ mm/msec

Results of computation are portrayed in Figs. 3 through 6, giving particle velocities pertaining each element in the bar along longitudinal axis x.

The accuracy of calculation has been found unsatisfactory when we adopt lumped mas matrix, irrespective of the size of time division Δt (Figs. 3 and 4). Instability occured when consistent mass matrix is adopted with $\Delta t>\Delta l/c$, similarly as in the paper of Chiu[4] where the computation was carried out by the finite difference procedure. However, consistent mass matrix with $\Delta t<\Delta l/c$ predicts correct wave except near the wave front of propagation. Particularly, it can be seen from Fig. 5 that characteristic time interval $\Delta t=\Delta l/c$ gives exact solutions for whole region of the bar in consistent with the hyperbolic nature of governing equation of stress wave.

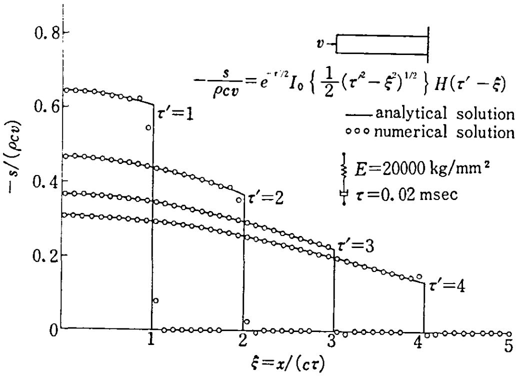

Fig. 7 VISCO-ELASTIC WAVE PROPAGATION

Fig. 7 exemplifies the visco-elastic wave. Bar material is of simple Maxwell type that is the special case of three-element model of Fig. 1 with $E_e=0$. Inset of Fig. 7 gives the characteristic values of elastic spring constant E and relaxation time τ for the Maxwell model in the computation, together with exact solution[5] of stress induced in the bar.　　(Manuscript received Feb. 18, 1971)

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