Inductance Computation of Microscopic Superconducting Loop

T. Nakazato and Y. Okabe
Research Center for Advanced Science and Technology, The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153, Japan

Abstract—**Aimed for the design of superconducting digital circuits, a direct method is proposed to estimate the inductance of three-dimensional microscopic superconducting loop. This method directly computes current-density distribution by using the Maxwell equations and the expression of the momentum, which are both discretized; without free-energy minimization technique, we just solve a set of linear equations considering a spatially-discrete model. Computer simulation was carried out for various shapes of superconductors, and the simulated results agreed well with the Chang's formula in a model which can be regarded as two-dimensional. The magnetic field distribution also agreed well with the theoretical value.**

I. Introduction

Recently, Single Flux Quantum Logic (SFQL) in which the logic states are decided whether one flux quantum exists in a one-junction interferometer or not has been studied actively [1]. The accurate estimation of the loop inductance is needed for designing the SFQL because the value of the inductance decides the logic operation of the circuit. Some calculation methods for two-dimensional models have been reported so far (e.g., the Chang's formula [2]). However, for designing the actual complicated circuits, three-dimensional analyses are required. Moreover, in the case where the size of devices becomes as small as several hundred nanometers, the penetration of the magnetic flux must be considered.

In this paper, we propose a new method to estimate the inductance of three-dimensional microscopic superconducting loop. The effects of the penetration of magnetic flux are accurately reflected in this method. A spatially-discrete model is set up, and the loop inductance including the kinetic inductance is calculated with the distribution of the current caused by the trapped flux. The characteristic of this method is that the current-density distribution is directly given as the solution of simultaneous equations. They are derived from the Maxwell equations and the expression of the momentum, which both are linear in respect of the vector potential and the current-density. Not only the inductance but also the magnetic flux distribution is given by a very simple expression.

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T. Nakazato, +81-3-3481-4460, fax +81-3-3481-4461, nakazato@okabe.rcast.u-tokyo.ac.jp, http://www.okabe.racst.u-tokyo.ac.jp/~nakazato/HomePage.html

II. Calculation Method

A. Theoretical background

When the fluxoid trapped in a superconducting loop is $n\Phi_0$, the loop inductance L is given by

$$L = \frac{n\Phi_0}{\iint_S \boldsymbol{J}(\boldsymbol{r})dS} \tag{1}$$

where $\boldsymbol{J}(\boldsymbol{r})$ is the current-density, $\Phi_0(= h/2e)$ is a flux quantum, and S is a cross section of the loop. The kinetic inductance is also included in L (see the Appendix). As (1) shows, $\boldsymbol{J}(\boldsymbol{r})$ must be known before we calculate the loop inductance.

Current-density distribution in a superconducting loop is obtained as follows. When the density of the Cooper pairs n_s is distributed uniformly, the expression of the momentum is written as

$$\hbar\nabla\theta = m^*\boldsymbol{v}_s + e^*\boldsymbol{A} \tag{2}$$

where θ is the phase, m^* is the effective mass of the Cooper pairs, $\boldsymbol{v}_s$ is the velocity of the Cooper pairs, $e^* = -2e$ is the effective charge of the Cooper pairs, and $\boldsymbol{A}$ is the vector potential.

By substituting the relation between $\boldsymbol{v}_s$ and the current-density

$$\boldsymbol{v}_s = \frac{\boldsymbol{J}}{n_s e^*} \tag{3}$$

for (2), the following expression is obtained:

$$\boldsymbol{A} = -\Lambda\boldsymbol{J} + \frac{\hbar\nabla\theta}{e^*} \tag{4}$$

where $\Lambda = m^*/n_s e^{*2} = \mu_0\lambda^2$, and λ is the London penetration depth.

The total vector potential $\boldsymbol{A}$ is the sum of $\boldsymbol{A}_c$, the vector potential made by the current, and $\boldsymbol{A}_f$, the vector potential made by the external magnetic field;

$$\boldsymbol{A} = \boldsymbol{A}_c + \boldsymbol{A}_f \tag{5}$$

$\boldsymbol{A}_c$ is expressed with the Maxwell equations as

$$\boldsymbol{A}_c(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int_{\mathbf{SC}} \frac{\boldsymbol{J}(\boldsymbol{r}')dv'}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{6}$$

where the integration is over the entire superconductor.

$\boldsymbol{J}(\boldsymbol{r})$ is obtained by solving (4)~(6) which are linear in respect of the vector potential and the current-density, but it is impossible in most cases to solve these equations analytically. In the next section, a spatially-discrete model is set up.

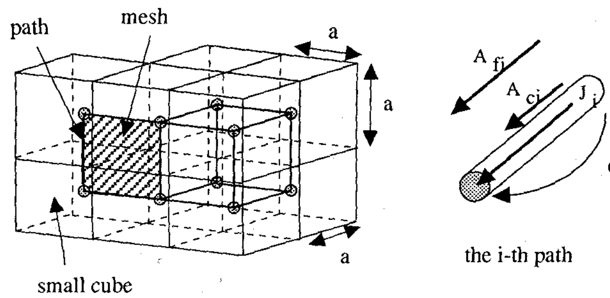
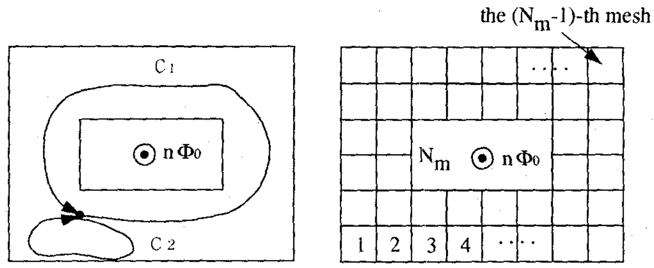


Fig. 1. Spatially-discrete model



(a) Superconducting loop (b) Current network

Fig. 2. Two-dimensional current network

B. Spatially-discrete model

A superconductor is divided into small cubes whose edge is a[m] long, and the center of adjacent cubes are connected by "path"s. The structure made of all these paths is called a "current network", and the smallest loop made of paths is called a "mesh" (Fig. 1). Assuming that the current flows only in these paths, the current distribution is represented by the current flowing this network [3]. We also assume that the number of all paths is N_p, the number of all meshes is N_m, and the i-th path has following parameters; J_i as the current-density, A_{ci} as the vector potential made by the current, A_{fi} as the vector potential made by the external magnetic field, and ϕ_i as the phase difference.

Under these assumptions, (4)~(6) are discretized respectively;

$$|A\rangle = -\Lambda|J\rangle - \frac{\Phi_0}{2\pi a}|\phi\rangle \qquad (7)$$

$$|A\rangle = |A_\text{c}\rangle + |A_\text{f}\rangle \qquad (8)$$

$$|A_\text{c}\rangle = M\,|J\rangle \qquad (9)$$

where $|X\rangle = {}^t(X_1, X_2, ..., X_{N_\text{p}})$ is a column vector, and M is an $N_\text{p} \times N_\text{p}$ matrix which depends on only the shape of the superconductor. We used $\nabla\theta = \Delta\theta/a = \phi/a$ to derive (7). From (7)~(9), the following expression is derived;

$$(\Lambda E + M)\,|J\rangle = -|A_\text{f}\rangle - \frac{\Phi_0}{2\pi a}|\phi\rangle \qquad (10)$$

where E is identity matrix.

The unknowns in (10) are $|J\rangle$ and $|\phi\rangle$, so the total number of them is $2N_\text{p}$ though the number of equations is only N_p. We must add the next two conditions to solve; (i) the continuity of current and (ii) the quantization of fluxoid.

The first condition means that $|J\rangle$ can be expressed with less parameters because the current must satisfy the Kirchhoff's Current Law. Actually, the independent variables are the density of the loop current which flows around a mesh, so $|J\rangle$ is expressed as

$$|J\rangle = R\,|m\rangle \qquad (11)$$

where $|m\rangle = {}^t(m_1, m_2, ..., m_{N_\text{m}})$ is a column vector, m_i is the loop-current-density of the i-th mesh, and R is $N_\text{p} \times N_\text{m}$ matrix whose elements are 0,-1, or 1. This condition reduces the total number of the unknowns from $2N_\text{p}$ to $N_\text{p} + N_\text{m}$.

The second condition puts restrictions on $|\phi\rangle$. For instance, let us consider that n flux quanta are trapped in a loop whose current network is two-dimensional (Fig. 2). The phase change of the Cooper pair along an arbitrary anticlockwise route is $-2n\pi$ if the route contains the loop inside (e.g. C_1), 0 otherwise (e.g. C_2). In the discretized model, this fact is expressed as

$$\sum_{\text{the } i\text{-th mesh}} \phi = \begin{cases} 0 & (1 \leq i \leq N_\text{m} - 1) \\ -2n\pi & (i = N_\text{m}) \end{cases} \qquad (12)$$

where the N_m-th mesh is the loop in which the fluxoid exist. ${}^tR\,|\phi\rangle$ is the column vector whose i-th element is the sum of the phase difference on the paths which compose the i-th mesh. Therefore, ${}^tR\,|\phi\rangle$ is written as

$${}^tR\,|\phi\rangle = {}^t(0, 0, \ldots, 0, -2n\pi) \qquad (13)$$

As this expression is a set of N_m equations, the total number of the equations becomes $N_\text{p} + N_\text{m}$, which equals to the number of the unknowns. We now can solve the set of linear equations.

By substituting (11) for (10) and multiplying (10) by ${}^tR$ from left, a set of N_m equations is derived;

$$\begin{aligned} &{}^tR\,(\Lambda E + M)\,R\,|m\rangle \\ &= -{}^tR\,|A_\text{f}\rangle - \frac{\Phi_0}{2\pi a}{}^t(0, 0, \ldots, 0, -2n\pi) \end{aligned} \qquad (14)$$

where all parameters are known except $|m\rangle$. $|m\rangle$ is obtained by solving (14).

I_loop, the total current which flows around the loop, is given by $a^2 m_{N_\text{m}}$ because the other mesh-currents flow locally and don't effect the value of the loop current. Therefore, the loop inductance is given by

$$L = \frac{n\Phi_0}{a^2 m_{N_\text{m}}} \qquad (15)$$

Φ_i, the magnetic flux which penetrates the i-th mesh, is equal to the i-th element of $a\,{}^tR\,|A\rangle$, because Φ_i is the contour integral of the vector potential $\boldsymbol{A}$ along the i-th mesh. By multiplying ${}^tR$ by (7) from left and using (11) and (13), $a\,{}^tR\,|A\rangle$ is given by

$$a\,{}^tR\,|A\rangle = -a\Lambda\,{}^tRR\,|m\rangle + {}^t(0, 0, \ldots, 0, n\Phi_0) \qquad (16)$$

Thus the loop inductance and the magnetic flux distribution are calculated with these simple expressions.

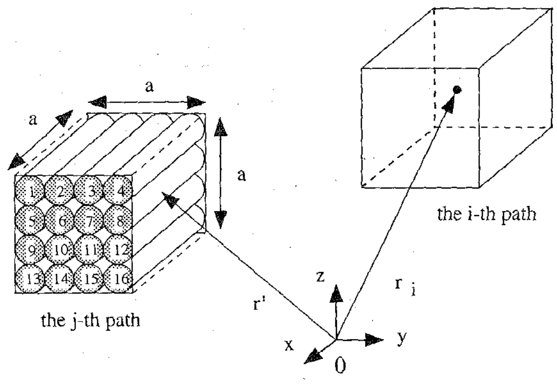


Fig. 3. Positions of the i-th and j-th path

The matrix M is calculated as follows. $M_{ij}J_j$ is the vector potential on the i-th path made by the current which flows on the j-th path. As the current and the vector potential made by the current are parallel, M_{ij} is 0 when the i-th path and the j-th path are orthogonal. Though the shape of the path is described as one cylinder in Fig. 1, the actual shape is a cube whose edge is a[m]. We regard that it is made of the 16 cylinders as an approximation to make the calculation easy. When the i-th and j-th path are along the x direction and located like Fig. 3, M_{ij} is calculated as

$$M_{ij} = \frac{\mu_0}{4\pi} \int_{V_j} \frac{d\boldsymbol{v}'}{|\boldsymbol{r}_i - \boldsymbol{r}'|} \tag{17}$$

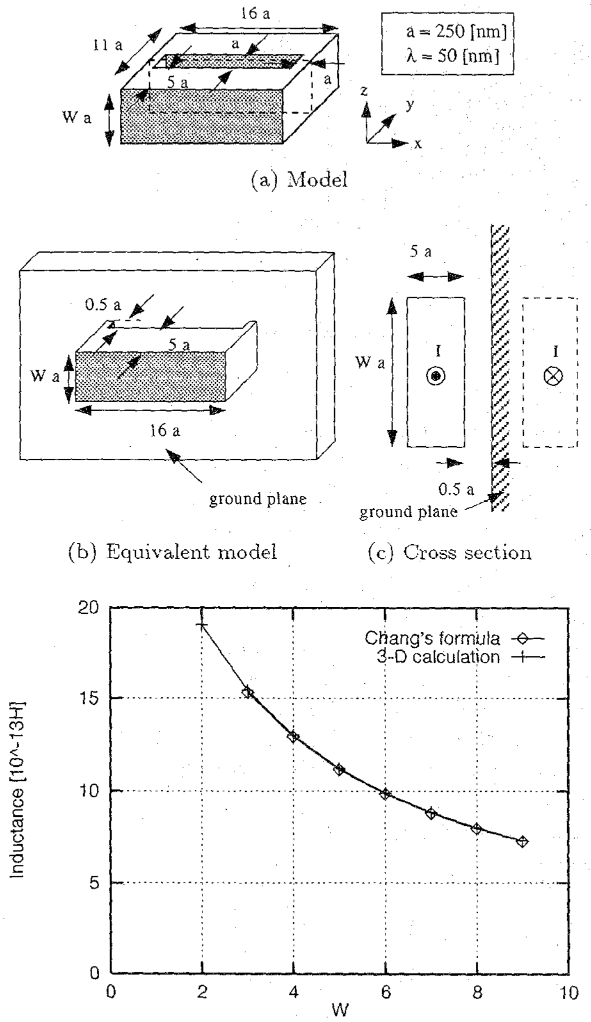
$$\doteqdot \sum_{k=1}^{16} \int_{x_j - \frac{a}{2}}^{x_j + \frac{a}{2}} \frac{\frac{\mu_0}{\pi} \times dx' \times \left(\frac{a}{8}\right)^2}{\sqrt{(x_i - x')^2 + (y_i - y_{jk})^2 + (z_i - z_{jk})^2}} \tag{18}$$

where (x_i, y_i, z_i) is the coordinates of the center of the i-th path, and (y_{jk}, z_{jk}) is the coordinates of the center axis of the k-th cylinder in the j-th path.

III. Results of Calculation

A. Inductance

With our method and the Chang's formula [2], the loop inductance of the model shown in Fig. 4(a) is calculated as a function of the thickness of the superconductor W. Though this formula is only for two-dimensional models, we apply the formula to this three-dimensional model with an ingenious way, which is as follows. This model has a plane of symmetry which is perpendicular to the y axis. It is shown with broken lines. If the flux is trapped in the loop, it makes the current distribution. When the current-density vector at a point is expressed as (J_x, J_y, J_z), that of the symmetrical point with respect to the plane becomes $(-J_x, J_y, J_z)$. Therefore, assuming that the superconductor is divided by this plane, one side plays a role of the ideal ground plane to the other. The equivalent model of Fig. 4(a) becomes Fig. 4(b) where the other side is replaced by the ground plane. As its cross section is like Fig. 4(c), the self inductance per the unit length is calculated with the Chang's formula. Considering the



(a) Model

(b) Equivalent model

(c) Cross section

(d) Calculated results

Fig. 4. Inductance

current network of this model, the length of the arm is regarded as $15a$. The total inductance of the two arm is calculated as

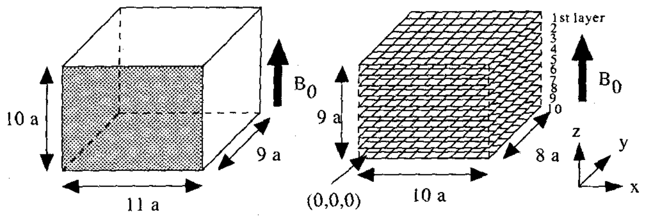
$$L = 15a \times 2 \times \frac{\mu_0}{K_t W a} \left[0.5a + \lambda \coth\left(\frac{4a}{\lambda}\right) + \lambda \coth\left(\frac{\infty}{\lambda}\right) + \beta\lambda\,\mathrm{cosech}\left(\frac{4a}{\lambda}\right) \right] \tag{19}$$

where K_t is the fringe field factor, β is the parameter which depends on the shape of the strip line, and the thickness of the ground plane is regarded as infinity. This value is expected to be equal to the loop inductance.

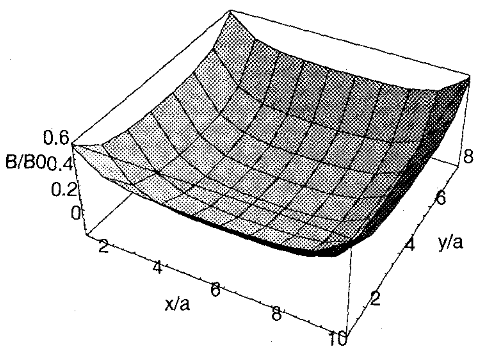
As this formula is applicable in the case where the superconductor is much larger than the London penetration depth, the parameters are chosen as $a = 250$[nm], $\lambda = 50$[nm]. The results of the computation and the Chang's formula are shown in Fig. 4(d). Agreement between the two is quite good.

B. Magnetic field distribution

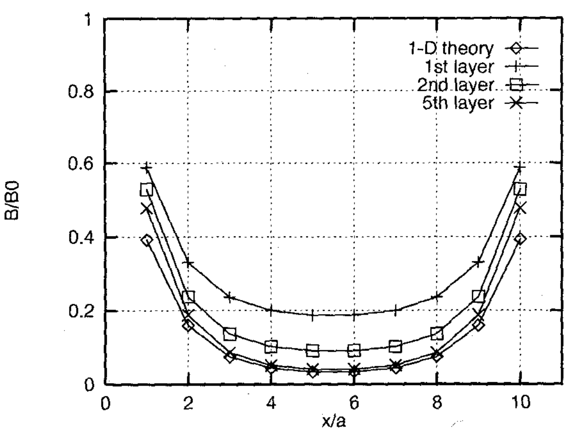
The magnetic field distribution in a superconducting block is calculated. The model and its current networ



(a) Superconductor (b) Current network



(c) Magnetic field distribution at the 5th layer



(d) Calculated results and theoretical value at $y = 4a$

Fig. 5. Magnetic field distribution

are shown in Fig. 5(a) and (b). The parameters are set as $a = \lambda = 50$[nm], and the block is in the external uniform magnetic field B_0. In Fig. 5(c), the calculated magnetic field distribution at the 5th layer ($z = 5a$) is shown. We can see how the magnetic flux is excluded from the block. The numerical analysis of each layer at $y = 4a$ is shown in Fig. 5(d), where "1-D theory" is calculated assuming that flux decays exponentially from the surface. As Fig. 5(d) shows, according as the magnetic flux goes the center of the block along the z direction, the flux is excluded more and the profile becomes like that of the one-dimensional theoretical value.

IV. Conclusion

We have proposed a method to calculate the inductance of a microscopic superconducting loop. In this method, the superconductor is divided into small cubes, and the current-density distribution is directly given as the solution of linear simultaneous equations which are derived from the Maxwell equations and the expression of momentum. The results agreed well with the Chang's formula in a model which can be regarded as two-dimensional. The magnetic field distribution also agreed well with the theoretical value.

Appendix

It is explained as follows that L given by $n\Phi_0/I_{\text{loop}}$ includes the kinetic inductance. The total energy of this system W is the sum of the magnetic energy $W_m = \frac{1}{2}\int_{sc} \boldsymbol{A}\cdot\boldsymbol{J}dv$ and the kinetic energy $W_k = \frac{1}{2}\int_{sc}\Lambda\boldsymbol{J}\cdot\boldsymbol{J}dv$. In the discrete model, W is written as

$$W = \frac{1}{2}\langle J|A\rangle a^3 + \frac{1}{2}\Lambda\langle J|J\rangle a^3 \tag{20}$$

$$= \frac{a^3}{2}\{\langle J|(\Lambda E + M)|J\rangle + \langle J|A_{\text{f}}\rangle\} \tag{21}$$

$$= \frac{a^3}{2}\frac{-\Phi_0}{2\pi a}\langle J|\phi\rangle \tag{22}$$

$$= -\frac{a^2\Phi_0}{4\pi}\langle m|^t R|\phi\rangle \tag{23}$$

$$= -\frac{a^2\Phi_0}{4\pi}\langle m|^t(0,0,\ldots,0,-2n\pi) \tag{24}$$

$$= \frac{1}{2}n\Phi_0\, a^2 m_{N_m} \tag{25}$$

$$= \frac{1}{2}n\Phi_0\, I_{\text{loop}} \tag{26}$$

where $\langle X|$ is a row vector, and we used (7)~(14) for the transformations. On the other hand, W is expressed as

$$W = \frac{1}{2}L_m I_{\text{loop}}^2 + \frac{1}{2}L_k I_{\text{loop}}^2 \tag{27}$$

where L_m is the magnetic inductance and L_k is the kinetic inductance. From (26) and (27), $n\Phi_0/I_{\text{loop}} = L_m + L_k$ is derived.

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