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17. *Synthesis of the Non-dipole Components of the Earth's Magnetic Field from Spherical Harmonic Coefficients.*

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Summary

The non-dipole components of the earth's magnetic field are synthesized from particular sets of spherical harmonic coefficients for 1885 and 1945, and compared with those of charted data. It has been confirmed that the synthesized field can well approximate the observed field. The standard deviations of the differences between the synthesized and charted vertical components are obtained as 1360γ for 1885 and 778γ for 1945, indicating that the total approximation errors are about 19 percent of the root-mean-square intensity of the non-dipole field itself for 1885 and 11 percent for 1945.

Gauss-Schmidt coefficients of different analyses are examined to estimate between-analyses errors. It follows that for most of the analyses in the 19th century the Gauss-Schmidt coefficients can be synthesized up to $n=m=4$ with an uncertainty of about 20%, whilst for those in the 20th century, they can safely be used at least up to $n=m=6$ with an uncertainty from several to ten percent.

1. Introduction

The earth's magnetic field is approximated predominatingly by a dipole type field with its axis deviating from the axis of rotation by about 11.5°. Its secular variation, on the other hand, is characterized by irregular distributions of several foci where the rates of change take a maximum or a minimum, and it seems closely related to the non-dipole field obtained after removing the dipole component from the observed field. For the purpose of investigating the secular variation, therefore, it is convenient to treat the non-dipole fields separately from that of the dipole.

The non-dipole fields have so far been obtained by subtracting the field due to the centred inclined dipole represented by the lowest terms of a spherical harmonic analysis from the field values read at grid points properly spaced on magnetic charts. Employing Schmidt's analysis for 1885, Bauer[1] computed the dipole fields at 10 degree intervals of longitude and 20 degree intervals of latitude, and then subtracted them from the values read on Schmidt and Petersen's map. Thus he obtained the non-dipole fields (the residual fields) for the north (X) east (Y) and vertical (Z) component for 1885. Based on Vestine et al.'s extensive work on the geomagnetic field[2)3)], Bullard et al.[4] repeated the procedure to compute the non-dipole fields for 1907.5 and 1945.

The above procedure based on charted values is not a convenient way to obtain the non-dipole fields for the past epochs, because in most of the cases the original data of the earth's magnetic field or the magnetic charts of the past on which the component values should be read are very difficult to acquire, whereas we have a sufficient number of spherical harmonic analyses readily available. If the non-dipole field can be relevantly synthesized, it will provide useful results to investigate the major features of the geomagnetic secular variation extending back to possibly the 17th century.

Two syntheses of the non-dipole field have been carried out very recently. One is by Vestine et al.[5] for 1955 and the other by McDonald[6] for 1835 and 1965, indicating the very similar features between the epochs. Until then, no one seems to have attempted the synthesis of the non-dipole field, supposedly due to the uncertainties involved in the harmonic analyses of the field.

Although Vestine et al. and McDonald computed the non-dipole field without detailed examination of errors involved in the analyses, it is highly important to pay careful consideration to them in order to investi-

1) L. A. BAUER, "The Physical Decomposition of the Earth's Permanent Magnetic Field—No. 1. The Assumed Normal Magnetization and the Characteristics of the Resulting Residual Field," *Terr. Mag.*, **4** (1899), 32-52.

2) E. H. VESTINE, C. COOPER, I. LANGE and W. C. HENDRIX, "Description of the Earth's Main Magnetic Field and its Secular Change, 1905-1945," *Carnegie Inst. Wash. Publ.*, **578** (1947), 1-532.

3) E. H. VESTINE, L. LAPORTE, I. LANGE and W. E. SCOTT, "The Geomagnetic Field, Its Description and Analysis," *Carnegie Inst. Wash. Publ.*, **580** (1947), 1-390.

4) E. C. BULLARD, C. FREEDMAN, H. GELLMAN and J. NIXON, "The Westward Drift of the Earth's Magnetic Field," *Phil. Trans. Roy. Soc. London*, **A**, **243** (1950), 67-92.

5) E. H. VESTINE, W. L. SIBLEY, J. W. KERN and J. L. CARLSTEDT, "Integral and Spherical-harmonic Analysis of the Geomagnetic Field for 1955.0 Part 1, "*Jour. Geomag. Geoelect.*, **15** (1963), 47-72.

6) K. L. MCDONALD and R. H. GUNST, "An Analysis of the Earth's Magnetic Field from 1835 to 1965," *ESSA Technical Report IER* 46-*IES* 1, (1967), 1-87.

gate the secular variation on the basis of the synthesized field. In this paper, it is attempted to estimate how closely the observed field can be fitted by the synthesized field and how far the apparent differences between the fields calculated from different analyses can be attributed to the incompleteness of data and to the different methods employed.

2. Comparison of the synthesized fields with the observed ones

In order to estimate the uncertainties involved in the synthesis of the field, it is desirable to compare the synthesized field with the original measurements at many places. However, it is not an easy matter to collect the original data and reduce them to the field values at a specific epoch in a form relevant to the comparison for the present purpose. In place of individual measurements, field values read from magnetic charts are assumed as standards of the comparison, and Bauer's or Bullard et al.'s non-dipole fields are referred to as the observed ones for the respective epochs.

In this section, non-dipole fields were computed for 1885 and 1945, and compared with Bauer's and Bullard's fields respectively. For the synthesis of 1885 fields, Schmidt's analysis[7] in which the earth's field was expanded in a series up to $n=m=6$ was adopted. For 1945 fields, Vestine et al.'s analysis[8] and that of Fanselau and Kautzleben[9] were used. Fanselau-Kautzleben's analysis is based on the same data as Vestine et al.'s. The difference lies in that Fanselau and Kautzleben approximated the field in a series up to $n=m=15$, while Vestine et al. terminated the series at $n=m=6$.

In order to see quickly how far the synthesized field can approximate the observed one, both the synthesized and the observed non-dipole fields are shown in Fig. 1 for the vertical component of 1945. The upper diagram is the field synthesized up to $n=m=6$ from Vestine et al.'s analysis, and the lower one is the observed field obtained by Bullard et al. On comparison of the two, we can see that the general features are well preserved in the synthesized field.

7) Cited from P. MAUERSBERGER, "Mathematische Beschreibung und Statistische Untersuchung des Hauptfeldes und der Säkularvariation," in: G. FANSELAU (Editor), *Geomagnetismus und Aeronomie.* VEB Deutscher Verlag der Wiss., Berlin, **3** (1959), 95-213.

8) Cited from A. B. KAHLE, J. W. KERN and E. H. VESTINE, "Spherical Harmonic Analyses for the Spheroidal Earth (II)," *Jour. Geomag. Geoelect.*, **18** (1966), 349-354.

9) G. FANSELAU and H. KAUTZLEBEN, "Die Analytische Darstellung des Geomagnetischen Feldes," *Geofis. Pura Applicata, Milano,* **41** (1958), 33-72.

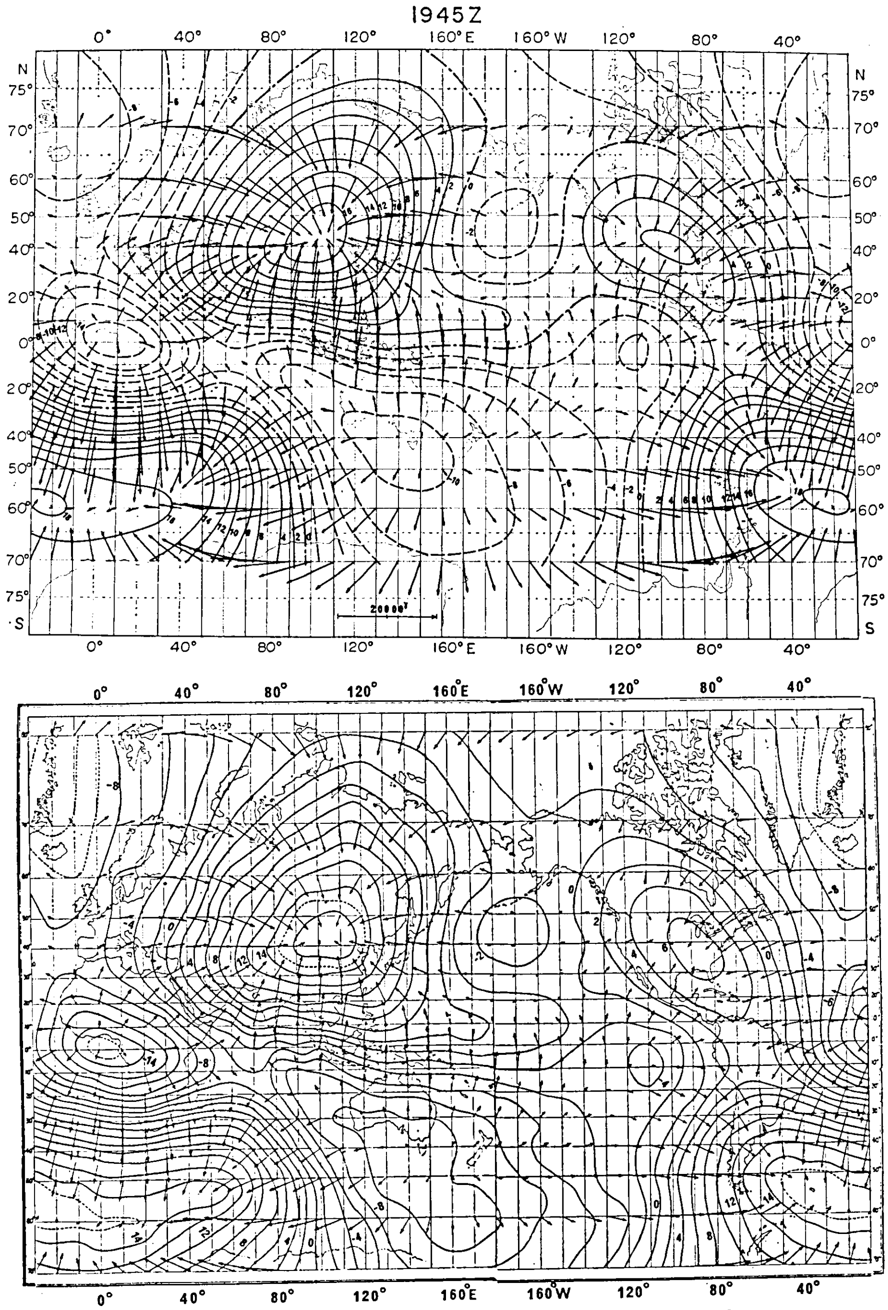

Fig. 1. The non-dipole vertical components for 1945. The upper diagram is the field synthesized up to $n=m=6$ from Vestine et al.'s analysis, and the lower one is the observed field obtained by Bullard et al. In units of 1000γ.

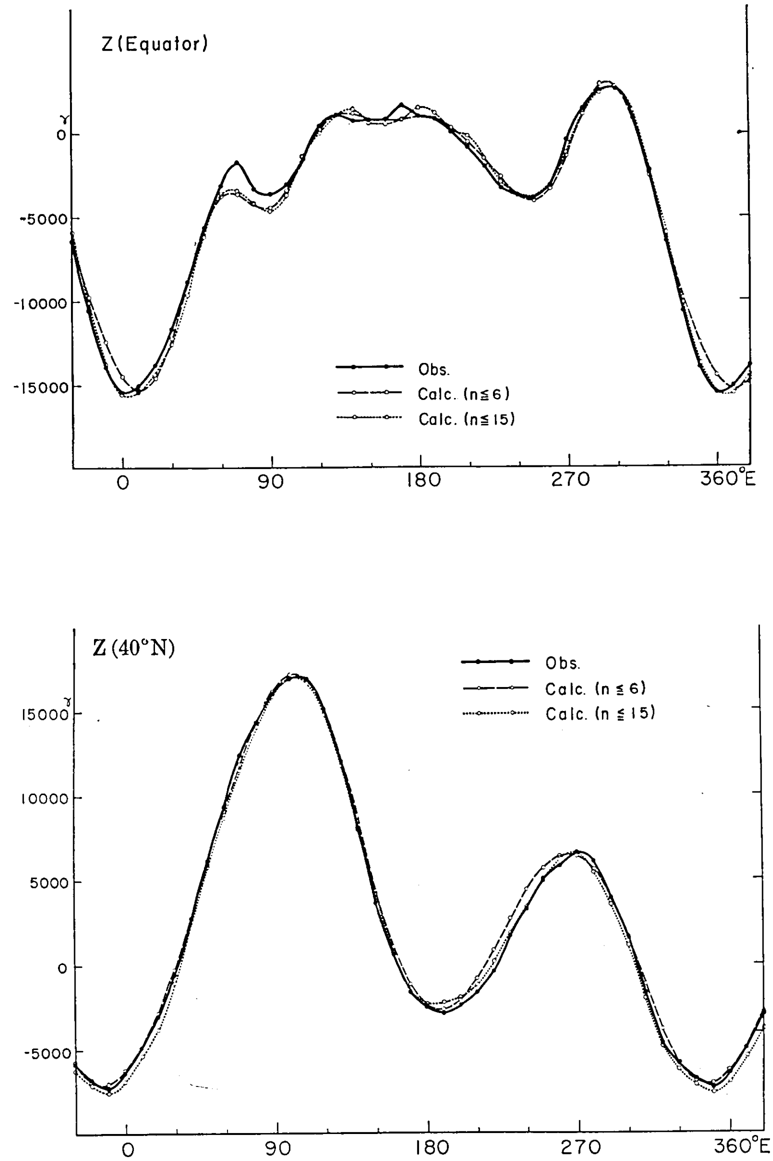

Fig. 2. The profiles of the non-dipole vertical components for 1945 along the parallels of latitude. The solid lines represent the observed fields by Bullard et al.'s, the broken lines denote those synthesized up to $n=m=6$ fromVestine et al.'s analysis and the dotted lines those synthesized up to $n=m=15$ from Fanselau-Kautzleben's analysis.

Upper diagram: A profile along the equator.

Lower diagram: A profile along the parallel of 40°N latitude.

In Fig. 2, the profiles of the non-dipole vertical components along the circles of the equator and the 40°N parallel are illustrated for 1945. The fields synthesized from Fanselau-Kautzleben's analysis including the terms up to $n=m=15$ are shown together with those of Vestine et al. terminating the series with $n=m=6$. The synthesized field can well represent the observed ones except for a small swelling around 60°E on the equator. Even the higher term approximation including the terms up to $n=m=15$ failed to approximate the swell. An improvement by the higher term approximation is seen at the minimum around 0°E on the equator. The longitude of the minimum synthesized from the higher terms agrees with that of the observed within an accuracy of 5°, while that from the lower approximation differs approximately by 10°. However, for such a smooth variation as along the circle of 40° N, no appreciable difference can be observed between the two approximations.

For examining the over all approximation by the synthesized fields, differences between the observed and the computed were taken for the three components at 10° intervals of longitude and latitude and then their standard deviations were calculated, giving equal weight to the difference at each grid point. Histograms of the differences for the vertical component all over the world are shown for 1945 in Fig. 3 and Fig. 4. The differences distribute in a Gaussian normal form and range from -30000γ to 30000γ. The means of the differences are given as 75.0γ for Vestine et al.'s analysis and 92.2γ for Fanselau and Kautzleben's and the standard deviations are 778γ and 1005γ respectively.

The average differences and their standard deviations for the three components are listed in Table 1 for the respective latitudes. The table indicates that the fields are not sufficiently approximated at the high latitudes in the southern hemisphere. When world averages are taken, the Y-component has the best fit and the Z-component the worst. This is probably due to the errors involved in the original material, suggesting that the vertical components are most seriously contaminated. The differences and the standard deviations for the vertical components for Fanselau-Kautzleben's analysis are shown in the last column of Table 1. The histogram of the differences for all the data over the world is shown in Fig. 4. Over all fit is somewhat worse than Vestine et al.'s case ($n=m=6$), when equal weight is given to each latitude. However, a close examination indicates that Fanselau-Kautzleben's analysis shows better fit in the lower latitudes. This seems to suggest that different weights might have been attached to the original data, though

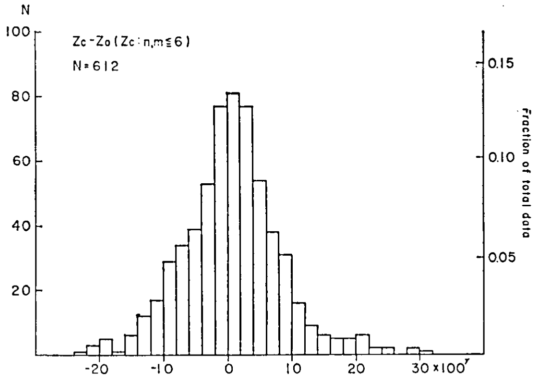

Fig. 3. A histogram of differences between the vertical components observed and those synthesized at the grid points spaced at 10° intervals in latitude and longitude for the epoch 1945. Vestine et al.'s analysis terminating with $n=m=6$ is used for the synthesis. The mean of the differences is 75γ and the standard deviation is 778γ.

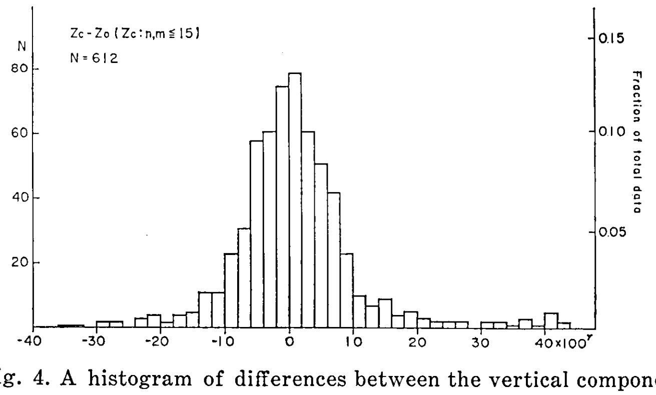

Fig. 4. A histogram of differences between the vertical components observed and those synthesized at the grid points spaced at 10° intervals in longitude and latitude for the epoch 1945. Fanselau-Kautzleben's analysis terminating with $n=m=15$ is used for the synthesis. The mean of the differences is 92γ and the standard deviation is 1005γ.

the data used were the same. Accordingly the present comparison suggests that, even if more terms are employed beyond a certain order and degree of a harmonic series, the approximation cannot appreciably be improved.

A similar tendency was also shown by Vestine et al.[10] They conducted the spherical harmonic analysis for American and Russian world charts, terminating the series in degrees and orders 4, 6, 8, 10 and 12, and examined the root-mean-square errors in the fit between the synthesized and the charted data. The averages of the three components obtained for the American chart are reproduced in Table 2. As is seen in the table, no appreciable improvement can be expected beyond degree 8.

Comparison of the observed and the computed results was also made for the 1885 field, based on Bauer's field and Schmidt's analysis. The differences and the standard deviations are tabulated in Table 3. The standard deviation of the differences is the largest for the vertical com-

Table 1. Differences between the synthesized and the observed fields for 1945 and the standard deviations.

	Vestine et al.'s analysis ($n=m\leqq 6$)						Fanselau-Kautzleben's analysis ($n=m\leqq 15$)	
	$\overline{\Delta X}$	σ_x	$\overline{\Delta Y}$	σ_y	$\overline{\Delta Z}$	σ_z	$\overline{\Delta Z}$	σ_z
80°N	543γ	406γ	−114γ	525γ	552γ	455γ	559γ	694γ
70	427	282	83	398	−3	655	327	534
60	−110	257	72	332	−398	496	−65	536
50	−260	315	39	351	11	400	−101	445
40	−86	220	−22	318	251	419	−124	395
30	171	278	−28	307	120	454	−24	407
20	190	239	−33	376	−71	668	64	478
10	−40	232	−42	406	−210	613	14	515
0	−154	265	0	419	−97	612	−159	554
−10	−70	229	0	305	110	564	−74	606
−20	127	357	11	351	130	469	15	532
−30	214	294	−64	364	−232	672	−153	565
−40	57	298	−47	315	−525	609	−182	715
−50	−339	459	−67	368	−524	807	−445	842
−60	−270	693	36	450	325	774	71	1241
−70	265	1017	522	1246	850	1232	466	2482
−80	1744	909	814	1341	986	841	1380	1552
World mean	142	667	68	610	75	778	92	1005

$\overline{\Delta X}$, $\overline{\Delta Y}$, $\overline{\Delta Z}$: The means of the differences (computed minus observed) for the respective components.

σ_x, σ_y, σ_z: Standard deviations of the differences ΔX, ΔY, ΔZ.

10) *loc. cit.*, 5)

Table 2. The means of the root-mean-square errors for the three elements in fit between the synthesized and the charted American data for 1955 (after Vestine et al.).

Maximum degree	The mean of the root-mean-square errors
4	256 γ
6	110
8	87
10	82
12	78

Table 3. Differences between the synthesized and the observed fields for 1885 and the standard deviations.

	$\overline{\Delta X}$	σ_x	$\overline{\Delta Y}$	σ_y	$\overline{\Delta Z}$	σ_z
60°N	$+3\gamma$	568γ	-32γ	464γ	-454γ	2078γ
40	−197	408	−186	705	564	908
20	−177	684	−10	566	127	814
0	−261	604	110	509	−52	809
−20	−297	455	87	392	−47	1027
−40	−219	543	284	554	−204	1042
−60	−105	1321	−142	628	−212	2006
World mean	−179	712	−16	566	−40	1360

$\overline{\Delta X}$, $\overline{\Delta Y}$, $\overline{\Delta Z}$: The means of the differences (computed minus observed) for the respective components.

σ_x, σ_y, σ_z: Standard deviations of the differences ΔX, ΔY, ΔZ.

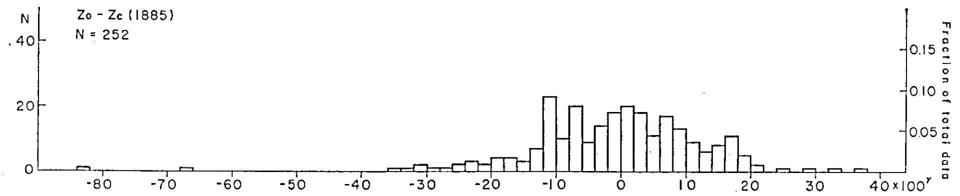

Fig. 5. A histogram of differences between the vertical components observed and those synthesized for the epoch 1885. The field values are synthesized up to $n=m=6$ from Schmidt's analysis and compared at the grid points spaced at 10° intervals of longitude and at 20° intervals of latitude. The 80° parallels were excluded from the comparison. The means of the differences is -40γ, the standard deviation being 1360γ.

ponent as is the case for the 1945 field. The approximation becomes poorer especially for the vertical component at high latitudes. It is noted that the overall world approximation for the three components is definitely worse than that for 1945. This can also be seen in Fig. 5, where the histogram of the differences of the vertical component is shown. The poor approximation for the 1885 field seems to arise partly from the incompleteness of the original data, and partly from the dipole component used by Bauer which is different from that of Schmidt's analysis employed for synthesizing the non-dipole field. Bauer computed the dipole component from $g_1^0=-31657\gamma$, $g_1^1=-2345\gamma$, $h_1^1=5955\gamma$, whilst Schmidt's analysis gives $g_1^0=-31919\gamma$, $g_1^1=-2117\gamma$, $h_1^1=5981\gamma$. The differences may easily yield root-mean-square errors of several hundred gammas. When identical values had been assumed for the dipole component, the approximation must have been much better.

It should be noted, at the end of this section, that the root-mean-square value of the non-dipole vertical forces itself is about 7300γ when Bullard et al.'s fields are averaged all over the world. The scatters of the differences between the observed and the computed are, therefore, approximately 19% of the non-dipole itself for the vertical component of 1885 and about 11% for that of 1945.

3. Comparison of the different analyses

For the estimate of the uncertainty involved in the synthesized field, it is important to evaluate what the errors are due to the incompleteness of the data and what are those due to the methods employed for the analysis of the data. Although there are analyses independently made for the same epoch, it is usually uncertain whether they are based on identical data or not. Therefore it is difficult to separate the errors originating from the above two sources merely by comparing the different analyses for the same epoch. In most of the cases, the *between-analyses errors* thus estimated include both kind of errors.

However, it is possible to calculate the Gauss-Schmidt coefficients independently from different components of the field. Averages of the coefficients from the north and the east components are most widely used for representing the geomagnetic field. It is probably because some of the data for the vertical components involve errors difficult to assess. In this case, the discrepancies between the coefficients derived from the different components are purely dependent on the quality of the data. Consequently the relevant test of the internal self-consistency of the analysis will give some indication of the incompleteness of the data.

In the first place, we shall estimate the root-mean-square differences between the fields synthesized from different analyses. Let V_1 and V_2 be two different magnetic potentials terminated with the same degrees and orders. Then the difference in the vertical component of the non-dipole field $\varDelta Z$ can be written as

$$\varDelta Z=\frac{\partial V_1}{\partial r}-\frac{\partial V_2}{\partial r}=\frac{\partial}{\partial r}\varDelta V \tag{1}$$

where

$$\varDelta V=V_1-V_2=a\sum_{n=2}^{N}\sum_{m=0}^{n}(\varDelta g_n^m \cos m\lambda+\varDelta h_n^m \sin m\lambda)P_n^m(\theta)$$

and $\varDelta g_n^m$ and $\varDelta h_n^m$ are the differences between the Gauss-Schmidt coefficients of V_1 and V_2, a is the mean radius of the earth, and θ, λ denote the colatitude and the longitude respectively. On calculating the root-mean-square values of $\varDelta Z$ over the earth's surface, we have

$$\begin{aligned}\langle \varDelta Z^2\rangle &=\left[\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(\varDelta Z)^2 \sin\theta d\theta d\varphi\right]^{\frac{1}{2}}\\ &=\left[\sum_{n=2}^{N}\sum_{m=0}^{n}\frac{(n+1)^2}{2n+1}\{(\varDelta g_n^m)^2+(\varDelta h_n^m)^2\}\right]^{\frac{1}{2}}\end{aligned} \tag{2}$$

For the 1885 field, analyses by Fritsche[11] and Schmidt[12] were compared, and the root-mean-square differences of 298 γ for $\langle \varDelta Z^2\rangle$ were obtained. For 1945, Vestine et al.'s analysis was compared with that of Fanselau and Kautzleben. Although Fanselau and Kautzleben's analysis includes higher terms up to $n=m=15$, only the terms up to $n=m=6$ were compared. Then the root-mean-square difference $\langle \varDelta Z^2\rangle$ became 168 γ. Since the root-mean-square value of the non-dipole vertical component itself, as has previouly been mentioned, is approximately 7300 γ, inconsistency between the analyses may be regarded to be less than 4% of the non-dipole field.

Relative errors of different analyses were also computed for an estimate of the accuracy of the spherical harmonic analysis. We define C_n^m by

11) H. FRITSCHE, "Die Elemente des Erdmagnetismus für die Epochen 1600, 1650, 1700, 1780, 1842 und 1885, und Ihre Saecularen Aenderungen," 1899, St. Petersburg.

12) *loc. cit.*, 7)

$$C_n^m = [(g_n^m)^2 + (h_n^m)^2]^{\frac{1}{2}}$$

Then $|\Delta g_n^m|/C_n^m$ and $|\Delta h_n^m|/C_n^m$ give a measure of the inconsistency of the coefficients. If they are close to or exceed unity, either coefficient of the two analyses are erroneous and meaningless.

The examination was made for the data of 1885, 1945 and 1960. For 1885, the analyses by Schmidt, Fritsche and Neumayer-Petersen[13] were compared, for 1945, Vestine et al.'s and Fanselau-Kautzleben's analyses and for 1960, the analyses by Cain et al.[14] and Hendricks-Cain[15] were examined. A few of the results are shown in Figs. 6 (a) to (c). In the figures, full circles denote $|\Delta g_n^m|/C_n^m$ and open circles $|\Delta h_n^m|/C_n^m$. In Fig. 6(a), showing the comparison between Schmidt's and Fritsche's analyses for 1885, the ratios remain less than 0.1 up to $n=m=3$, except

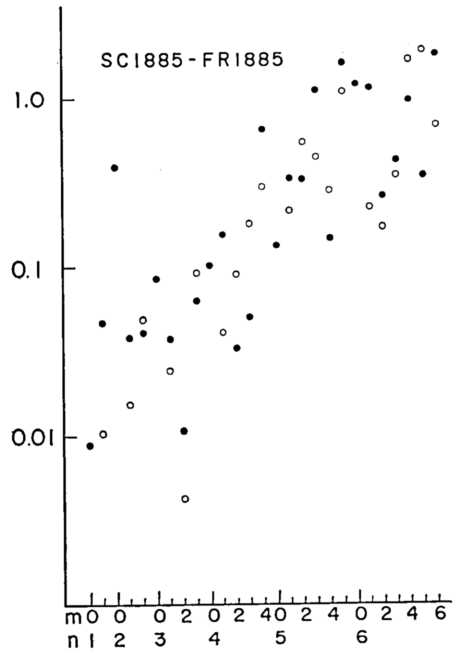

Fig. 6a. Comparison between Schmidt's and Fritsche's analyses for 1885.

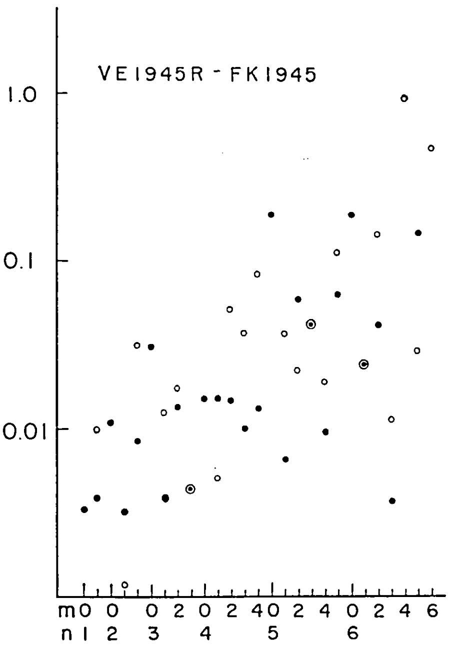

Fig. 6b. Comparison between Vestine et al.'s and Fanselau-Kautzleben's analyses for 1945.

13) *loc. cit.*, 7)

14) J. C. CAIN, W. E. DANIELS and S. J. HENDRICKS, "An Evaluation of the Main Geomagnetic Field, 1940-1962," *Jour. Geophys. Res.*, **70** (1965), 3647-3674.

15) S. J. HENDRICKS and J. C. CAIN, "Magnetic Field Data for Trapped-Particle Evaluations," *Jour. Geophys. Res.*, **71** (1966), 346-347.

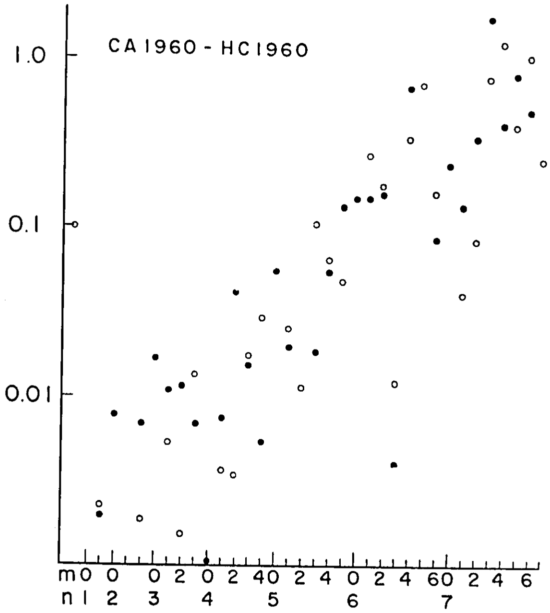

Fig. 6c. Comparison between Cain et al.'s and Hendricks-Cain's analyses for 1960.

Fig. 6. Comparison of the Gauss-Schmidt coefficients between different analyses. The ordinates are the differences between Gauss-Schmidt coefficients (Δg_n^m, Δh_n^m) normalized by C_n^m. Full circles denote $|\Delta g_n^m|/C_n^m$ and open ones $|\Delta h_n^m|/C_n^m$.

for the term $n=2$, $m=0$. For the coefficients from $n=4, m=0$ to $n=5$, $m=2$, most of the ratios distribute between 0.1 and 0.7 and for the terms higher than $n=5, m=2$ some points exceed unity. A similar tendency can be pointed out for the other combinations of 1885 data. Consequently Gauss-Schmidt coefficients of the 1885 analyses can be regarded as bearing some physical significance at least up to $n=m=4$. The 1945 data, on the other hand, indicate that all the ratios up to $n=m=6$ are less than unity. For the 1960 analyses, terms up to $n=7, m=2$ are regarded to be significant.

The relative error estimate can be applied to the coefficients obtained from different components to examine an internal self-consistency of the analysis. The internal self-consistency tests were made for Vestine et al.'s analysis (1945) and Leaton et al.'s[16] (1965). Results are shown in

16) B. R. LEATON, S. R. C. MALIN and M. J. EVANS, "An Analytical Representation of the Estimated Geomagnetic Field and Its Secular Change for the Epoch 1965.0," *Jour. Geomag. Geoelect.*, **17** (1965), 187-194.

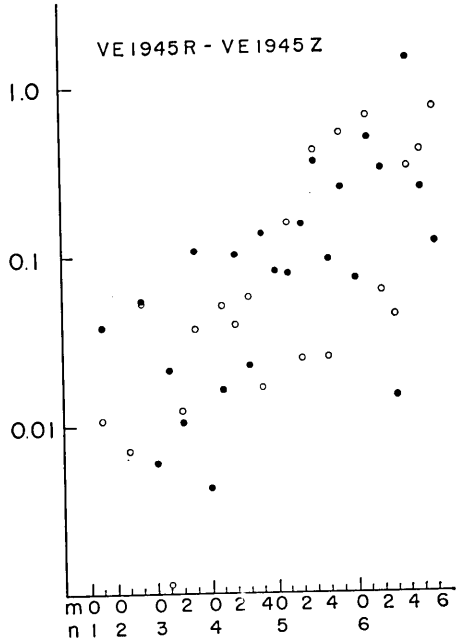

Fig. 7a. Comparison between the Gauss-Schmidt coefficients of the X and Y components and of the Z component for Vestine et al.'s analysis for 1945.

Figs. 7 (a) and (b). General features of Figs. 7 (a) and (b) are very similar to those of Fig. 6 (b) and (c) respectively, indicating that between-analyses errors do not greatly exceed the internal self-consistency errors. Therefore it may be concluded that the discrepancies between the different analyses for the same epoch are mostly due to the quality of the data used rather than the technique employed for the analysis.

Without detailed examination, it is difficult to decide the highest degree term up to which the Gauss-Schmidt coefficients of a specific analysis can be used for the synthesis, but the above investigation roughly suggests that for very recent analyses, such as for 1960 and 1965, coefficients up to $n=m=7$ may be significantly synthesized, for the earlier data of this century, such as 1945, 1922, up to $n=m=6$ may be employed, whilst for the older analyses in the 19th century only terms up to $n=m=4$ have significance.

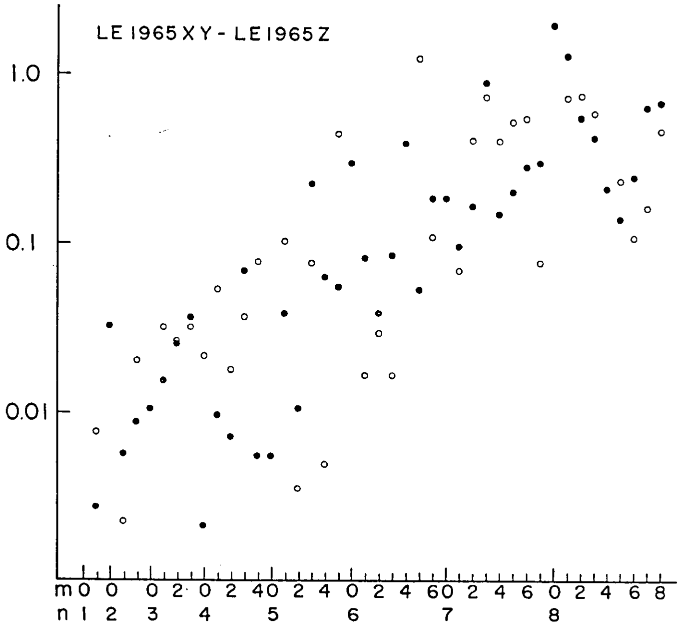

Fig. 7b. Comparison between the Gauss-Schmidt coefficients of the X and Y components and of the Z component for Leaton et al.'s analysis for 1965.

Fig. 7. Internal self-consistency test. Gauss-Schmidt coefficients obtained from different components based on the same data are compared. Full circles denote $|\Delta g_n^m|/C_n^m$ and open ones $|\Delta h_n^m|/C_n^m$.

4. Structure of the geomagnetic field

As is well known, the earth's magnetic field is roughly that of a dipole, the quadrupole and other multipole terms being about 10% of the dipole term. When the geomagnetic field is expressed in a spherical harmonic series, and if the series converges rapidly, the magnetic field can be well approximated with a few lower terms.

Figs. 8 and 9 show a type of spatial power and amplitude spectra of the geomagnetic field. $R_n^2\left(=\sum_{m=0}^{n}(g_n^{m^2}+h_n^{m^2})\right)$ were calculated for three different analyses (Vestine et al.'s for 1955 ($n, m=6$); Vestine et al.'s for 1955 ($n, m=12$); Fanselau-Kautzleben's for 1945 ($n, m=15$)) and showed a rapid decrease with increasing n. On comparison with equation (2), Fig. 8 gives a rough estimate of the order of the magnitude of truncation errors of a series, provided that the truncation does not seriously influence the existing terms. When a spherical harmonic series is terminated at a certain degree, $n=6$ for example, then the root-mean-square of truncated errors for the vertical component will be given by

$$\langle \Delta Z^2 \rangle = \left[\sum_{n=7}^{15} \sum_{m=0}^{n} \frac{(n+1)^2}{2n+1} \{(g_n^m)^2 + (h_n^m)^2\} \right]^{\frac{1}{2}} .$$

Visual check of Fig. 8 gives this being on the order of 100γ.

Fig. 9 shows the dependence of the amplitude C_n^m on degree (n) and order (m) for three different analyses terminating in different degrees and orders. The amplitude decreases rapidly with the increase in the degree and the order. The agreement of the three different analyses is remarkable as far as $n=m=4$, but beyond $n=7$, $m=0$, the discrepancies become considerable.

From the considerations in the previous section, it has been suggested that the Gauss-Schmidt coefficient has some significance up to $n=m=7$ for very recent analyses, whereas the old analyses of the 19th century may be reliable only as far as $n=m=4$. When the non-dipole fields are constructed by synthesizing the Gauss-Schmidt coefficients up to $n=m=7$, truncation errors are of the order of several tens of gammas from Fig. 9. Since the largest amplitude

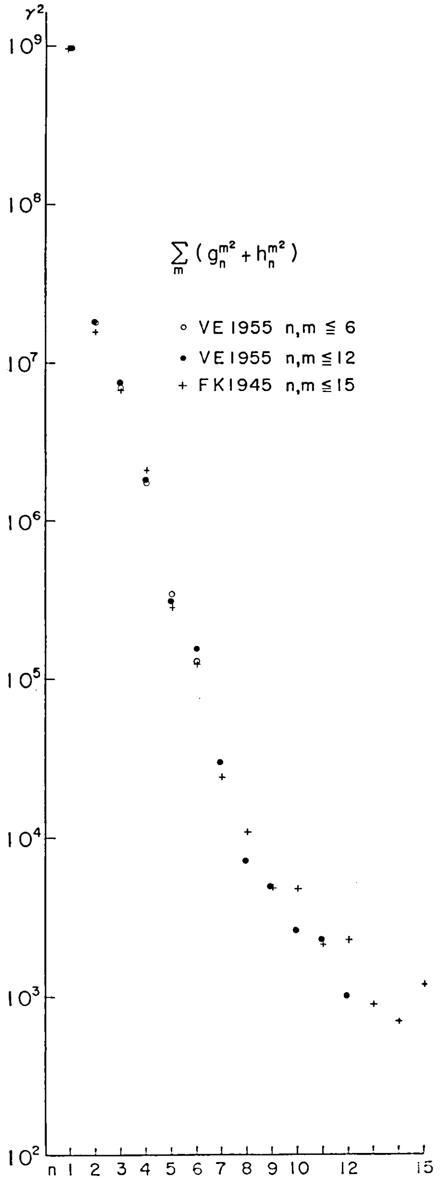

Fig. 8. $\sum_{m=0}^{n}(g_n^{m^2}+h_n^{m^2})$ as a function of degree n. Open circles are obtained from Vestine et al.'s analysis for 1955 terminating the series in $n=m=6$, full circles are from Vestine et al.'s for 1955 terminating in $n=m=12$ and plus signs from Fanselau-Kautzleben's for 1945 terminating in $n=m=15$.

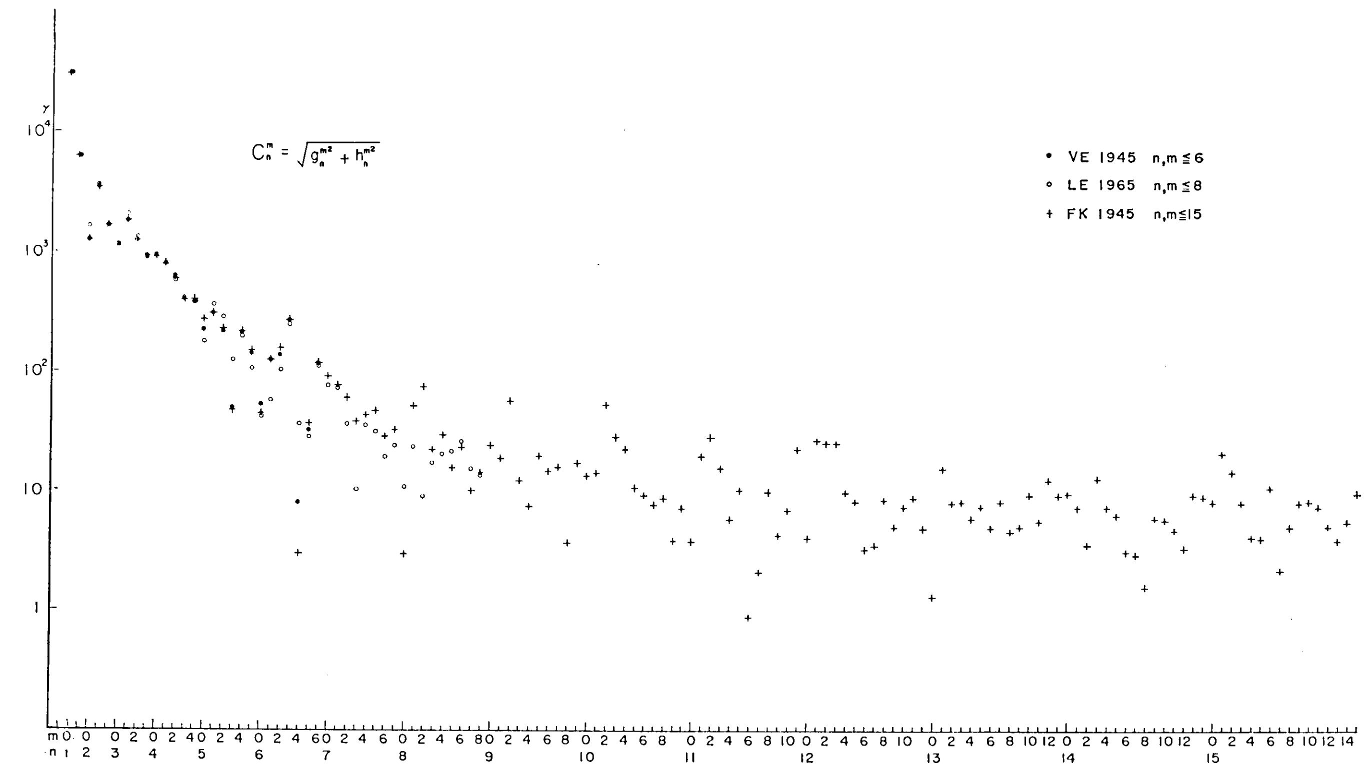

Fig. 9. The dependence of the amplitude of the Gauss-Schmidt coefficients (C_n^m) upon the degree and order. Full circles are from Vestine et al.'s analysis for 1945, open circles are from Leaton et al.'s for 1965 and plus signs from Fanselau-Kautzleben's for 1945.

of the non-dipole component is approximately 2500 γ, truncation errors would be a few percent of the total non-dipole field. Similarly, when a series terminates with $n=m=6$, truncation errors would be several percent, and when coefficients merely up to $n=m=4$ are taken, errors would be as much as 10% of the actual field.

There is another type of truncation error when the field is approximated by a finite number of harmonic series terms. Spherical harmonic coefficients are dependent on the number of terms adopted, even if the data used are the same. For the 1955 field, Vestine et al.[17] demonstrated that, when only the terms up to $n=m=4$ were employed, the spherical harmonic coefficients differed notably from those when the series was terminated with the degree and order 8. The maximum difference amounted to 800 γ for g_1^1, and the largest of the non-dipole components was 300 γ for g_3^1. However, as far as the non-dipole field is concerned, this does not mean a new restriction to the approximation of the synthesis. Even when a series terminated with $n=m=4$ is synthesized, the truncation error does not seem to exceed the limit 1360 γ estimated by the comparison of the synthesized and charted results of section 2.

5. Summary of the results

It is very difficult to estimate the synthesis errors definitely, because the reliability of original measurements are not always the same. Actually the quality of the data, on which the analysis is based, has been strikingly improved with the passing of time.

In this paper, the non-dipole fields synthesized from particular analyses were compared with charted fields for the epochs 1885 and 1945. Over all standard deviations of the differences were given as 1360 γ for the vertical component of 1885, and 778 γ for 1945. Between-analyses errors were then examined for the vertical component. The root-mean-square differences were obtained as 298 γ for 1885 and 168 γ for 1945 analyses. The relative error evaluation was also made for different analyses for the same epoch, with the result that the spherical harmonic series are permissibly synthesized up to $n=m=4$ for the epoch 1885, up to $n=m=6$ for 1945 and up to $n=7$, $m=2$ for 1960 field. When the structure of the present geomagnetic field is taken into consideration, the errors caused by terminating the series with the above degrees and orders are about 10% of the non-dipole field for 1885, several percent

17) *loc. cit.*, 5)

Table 4. Uncertainties involved in the synthesis of the field.

Epochs	1885	1945	1960	1965
Root-mean-square of the differences between the charted and the synthesized	1360γ (19 %)	778γ (11 %)		
Between-analyses errors				
Root-mean-squares $\langle \Delta Z^2 \rangle$	298γ (4%)	168γ (2%)		
Relative errors $\Delta g_n^m/C_n^m$, $\Delta h_n^m/C_n^m$	10%	several percent	a few percent	
(The highest term allowable for the synthesis)	(up to $n=m=4$)	(up to $n=m=6$)	(up to $n=7, m=2$)	
Internal self-consistency (The highest term allowable for the synthesis)		several percent (up to $n=m=6$)		less than a few percent (up to $n=m=7$ possibly up to $n=m=8$)

Percentages in parentheses are the ratios of the errors to the amplitude of the non-dipole vertical components.

for 1945 and a few percent of the original non-dipole field for 1960. The test of internal self-consistencies made by means of the relative error technique has indicated that the between-analyses errors are mostly due to the incompleteness of the data. The various types of errors estimated in this paper are summarized in Table 4, showing together, in percentages, what part of the non-dipole field these errors correspond to.

I would like to express my thanks to Miss H. Tachinaka for her assistance in preparing the manuscripts and figures.

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17. ガウス・シュミット係数による地球磁場非双極子部分の合成

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球函数解析結果を逆に合成して，地球磁場を十分な精度で近似できれば，かなり古い時代にさかのぼつて地球磁場の永年変化を詳細に調べることができる．1885年と1945年のいくつかの解析をとりあげ，ガウス・シュミット係数から非双極子磁場を合成して磁気図より読み取つた値と比較した．その結果合成磁場でかなりよく近似できることがわかつた．合成磁場と観測磁場の差をとり，その標準偏差を求めると，1885年に対して1360γ，1945年に対して778γで，非双極子磁場自体の自乗平均のそれぞれ19%および11%となる．

異なる解析間の誤差を見積るために，同時代の解析で相対応するガウス・シュミット係数を比較検討した．19世紀の解析については，$n=m=4$まで合成することが可能で約20%の誤差で磁場を近似できる．これに対して，20世紀の解析では少くとも$n=m=6$迄とることができて，数%から10%の誤差におさまることが明かになつた．