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32. *Rayleigh Wave Propagation along an Undulatory Surface.—Comparison with Waves through a Heterogeneous Medium with a Periodic Structure.*

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1. Introduction

The study of the surface wave propagated along an uneven surface is important to geophysicists, but seems to be intractable because of mathematical difficulties. In a previous paper (Onda, 1967 *b*), the effect of an undulatory surface on surface wave propagation was investigated by using an appropriate conformal mapping. In that paper, an approximate equation for a wave passing along this surface is equivalent to the equation of waves propagated through a periodic structure. It is found by careful investigations that the latter waves have the following characteristics (Yoshiyama, 1960; Onda, 1966, 1967 *a*):

1) It is very difficult to calculate the amplitude of a progressive wave in a periodic structure;

2) The wave with a wave length twice the structural wave length is unstable, and the unstable wave results from the resonance in this structures; and

3) The apparent and characteristic attenuation of progressive waves appears near the frequency of the unstable wave.

Therefore, it follows that the main feature of surface waves propagated along the undulatory surface is given by these characteristics. When it is uncertain whether the solution is satisfactory or not, it will be necessary that the solution obtained is examined from other standpoints.

The procedure treated in this paper is the perturbation method, according to the classification of the previous paper (Onda, 1967 *b*). This title in the classification is ascribed to expanding the boundary condition given upon an uneven surface about the mean level. Let us assume first that an original wave as the first approximation fulfills the free stress on the mean level, and the stress on the given boundary does not vanish on account of the uneven surface. Then a secondary wave is calculated so as to give free stress upon the uneven surface. This method has been studied by Mr. Homma (1941), Dr. Sato (1957) and many other authors.

The wave treated in the previous paper was of the SH type. In this procedure, however, since we replace the unevenness of a topography by a corresponding stress distribution applied on the even surface, we cannot calculate the secondary SH wave propagated along the surface. Therefore, the wave treated in this paper is not of SH type but of Rayleigh type.

In section 2, a brief sketch of the perturbation method is given. In section 3, when the original wave is of the Rayleigh type, the secondary Rayleigh wave, which will constitute the main part of the secondary wave, is calculated. The sum of the original wave and the secondary wave travelling in the same direction as the original one corresponds to the transmitted wave, and the secondary wave travelling in the opposite direction becomes the reflected wave. Therefore, when the surface is sinusoidal, the equivalent transmission and reflection coefficients are easily calculated. As already stated, the nature of the surface wave propagated along an undulatory surface is similar to waves propagated through a periodic structure. In section 4, the coefficients obtained here are compared with these for the periodic structure.

2. The perturbation method

We assume that the medium is isotropic and homogeneous and that the motion is in the (x, z) plane. The wave in the medium is calculated by two potentials which are solutions of the equations

$$(\nabla^2+h^2)\phi=0\,, \qquad (\nabla^2+k^2)\psi=0\,, \tag{1}$$

where

$$h=\omega/V_p\,, \qquad k=\omega/V_s\,, \tag{2}$$

and V_p, V_s are velocities of the longitudinal and transverse waves and ω is the circular frequency. Let V_R and κ be the velocity and wave number of the Rayleigh wave. The displacement components of the x- and the z- axes are

$$u=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z}\,, \qquad w=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z}\,. \tag{3}$$

In addition we assume that the surface is uneven in bounded extent in the x direction, that the elevation of the surface is small compared with the wave length and that the surface gradient is much smaller than unity. The topography of the surface is given by

$$z=f(x)\,, \tag{4}$$

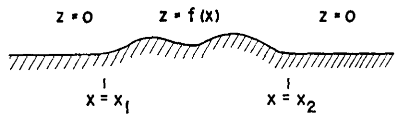

Fig. 1. Geometry of the uneven topography.

where $f(x)$ is some function for $x_1<x<x_2$ only and is zero for other values of x (Fig. 1). Since $\kappa>k>h$, these assumptions are expressed as

$$|f'|\ll 1 \quad \text{and} \quad |\kappa f|\ll 1\,. \tag{5}$$

If the maximum of the gradient, $|(f')_m|$, is smaller than the product of the maximum of the elevation, $|f_m|$, and a constant with the dimension of the wave number of the unevenness, γ, the first of expressions (5) is rewritten as

$$|(f')_m|\leq|\gamma\cdot f_m|=\gamma/\kappa\cdot|\kappa f_m|\ll 1;$$

hence it follows from both parts of (5) that κ is of the order γ.

If the tangential and normal stresses upon the surface $f(x)$ are $\mathfrak{T}(x)$ and $\mathfrak{N}(x)$ respectively, these are calculated (Love, 1952; p. 80) as

$$\left.\begin{aligned}\mathfrak{T}(x)&=[\widehat{xz}+(\widehat{zz}-\widehat{xx})\cdot f'-\widehat{xz}\cdot f'^2]/(1+f'^2)\,,\\ \mathfrak{N}(x)&=[\widehat{zz}-2\widehat{xz}\cdot f'+\widehat{xx}\cdot f'^2]/(1+f'^2)\,.\end{aligned}\right\} \tag{6}$$

The boundary condition in our problem is given by

$$\mathfrak{T}(x)=0 \quad \text{and} \quad \mathfrak{N}(x)=0 \quad \text{at} \quad z=f(x)\,. \tag{7}$$

As the first step of solving the problem, we consider, as the first approximation, an original wave which satisfies free stresses upon the plane surface $z=0$. Let the wave be satisfied by the potentials (ϕ^0, ψ^0). However, the condition (7) is not fulfilled by the wave (ϕ^0, ψ^0) alone, by reason of the uneven surface. The secondary wave determined by the potentials (ϕ^s, ψ^s) is calculated so that the sum of them $(\phi^0+\phi^s, \psi^0+\psi^s)$ satisfies the condition (7). Here, it is assumed that the disturbances in the secondary wave are small compared with the motions in the original wave.

Under the assumption (5), substitution of the Taylor expansion of the right-hand side of (6) into (7) yields, neglecting higher order terms,

$$\left.\begin{aligned}[\widehat{xz^s}]_{z=0} &= -\left[\frac{\partial \widehat{xz^0}}{\partial z}\right]_{z=0}\cdot f + [\widehat{xx^0}]_{z=0}\cdot f' ,\\ [\widehat{zz^s}]_{z=0} &= -\left[\frac{\partial \widehat{zz^0}}{\partial z}\right]_{z=0}\cdot f ,\end{aligned}\right\} \quad (8)$$

where the stresses of the left-hand and the right-hand sides are calculated by means of (ϕ^s, ψ^s) and (ϕ^0, ψ^0), respectively. We have essentially replaced the free-stress condition on the perturbed surface by an equivalent stress distribution on the surface $z=0$. All quantities in equation (8) are calculated at $z=0$. We put

$$\left.\begin{aligned}[\widehat{xz^s}]_{z=0} &= T(x)e^{i\omega t-ik'x} ,\\ [\widehat{zz^s}]_{z=0} &= N(x)e^{i\omega t-ik'x} ,\end{aligned}\right\} \quad (9)$$

where k' is the wave number component of the x-direction of the incident wave (ϕ^0, ψ^0). If the Fourier transform with respect to x is applied to these stresses and the potentials (ϕ^s, ψ^s), then together with equation (9), after interchanging the order of integration, we have the final expressions for (ϕ^s, ψ^s):

$$\left.\begin{aligned}\phi^s &= \frac{1}{2\pi\mu}\int_{-\infty}^{\infty} T(\eta)\phi_1^s(\eta)e^{i\omega t-ik'\eta}\,d\eta + \frac{1}{2\pi\mu}\int_{-\infty}^{\infty} N(\eta)\phi_2^s(\eta)e^{i\omega t-ik'\eta}\,d\eta ,\\ \psi^s &= \frac{1}{2\pi\mu}\int_{-\infty}^{\infty} T(\eta)\psi_1^s(\eta)e^{i\omega t-ik'\eta}\,d\eta + \frac{1}{2\pi\mu}\int_{-\infty}^{\infty} N(\eta)\psi_2^s(\eta)e^{i\omega t-ik'\eta}\,d\eta ,\end{aligned}\right\} \quad (10)$$

where

$$\left.\begin{aligned}\phi_1^s(\eta) &= \int_{-\infty}^{\infty} \frac{-2i\xi\lambda_\beta}{F(\xi)} e^{i\xi(x-\eta)-\lambda_\alpha z}\,d\xi ,\\ \phi_2^s(\eta) &= \int_{-\infty}^{\infty} \frac{(2\xi^2-k^2)}{F(\xi)} e^{i\xi(x-\eta)-\lambda_\alpha z}\,d\xi ,\\ \psi_1^s(\eta) &= \int_{-\infty}^{\infty} \frac{(2\xi^2-k^2)}{F(\xi)} e^{i\xi(x-\eta)-\lambda_\beta z}\,d\xi ,\\ \psi_2^s(\eta) &= \int_{-\infty}^{\infty} \frac{-2i\xi\lambda_\alpha}{F(\xi)} e^{i\xi(x-\eta)-\lambda_\beta z}\,d\xi ,\end{aligned}\right\} \quad (11)$$

and

$$\left.\begin{aligned}&F(\xi) = (2\xi^2-k^2)^2 - 4\xi^2\lambda_\alpha\lambda_\beta ,\\ &\lambda_\alpha = \sqrt{\xi^2-h^2} , \quad \lambda_\beta = \sqrt{\xi^2-k^2} .\end{aligned}\right\} \quad (12)$$

In these expressions, (ϕ_1^s, ψ_1^s) are the potentials observed at $x=x$ with application of a tangential stress with unit magnitude at $x=\eta$, whereas (ϕ_2^s, ψ_2^s) are observed at $x=x$ with application of a normal stress at $x=\eta$. The displacement is obtained by substituting equations (10) into (3).

3. Reflection and transmission of Rayleigh waves passing along an undulatory surface

Let the original wave be a Rayleigh wave travelling in the $+x$ direction, then k' is κ and

$$\left.\begin{aligned} \phi^0 &= \phi_0 e^{i\omega t - i\kappa x - \nu z}, \\ \psi^0 &= \psi_0 e^{i\omega t - i\kappa x - \nu' z}, \end{aligned}\right\} \tag{13}$$

where

$$\psi_0 = \frac{2\kappa^2 - k^2}{2i\kappa\nu}\phi_0, \quad \nu = \sqrt{\kappa^2 - h^2}, \quad \nu' = \sqrt{\kappa^2 - k^2}, \quad F(\pm\kappa) = 0. \tag{14}$$

Whence, from equations (8) and (9),

$$\left.\begin{aligned} T(x) &= i\mu\phi_0(\nu - \nu')\{2\kappa\nu f(x) + i2(\nu + \nu')f'(x)\}, \\ N(x) &= \mu\phi_0(\nu - \nu')(2\kappa^2 - k^2)f'(x). \end{aligned}\right\} \tag{15}$$

Substituting these expressions into equations (10), we can calculate the secondary wave generated by an uneven surface $z=f(x)$. In this paper the Rayleigh wave in the secondary wave is studied, because it predominates at a sufficiently great distance from an uneven area. We shall call it the secondary Rayleigh wave.

The path of integration in equations (10) is transformed into Sommerfeld's contour developed by Lapwood (1949). A Riemann sheet is defined as the real parts of λ_α and λ_β being positive, for a complex variable ζ whose real part is ξ, and the branch cuts are taken by letting the imaginary parts of λ_α and λ_β be zero. As is well known, the Rayleigh wave is estimated by the residue on this sheet. On this sheet both $\exp(-\lambda_\alpha z)$ and $\exp(-\lambda_\beta z)$ become zero when $|\zeta| \to \infty$, so that the path of integration is taken separately, accordingly as $x-\eta>0$ or $x-\eta<0$. If $x-\eta>0$, the path is taken in the upper half plane, and the pole is at $\zeta=-\kappa$; while if $x-\eta<0$, the path is in the lower half plane, and the pole is at $\zeta=\kappa$. If the Rayleigh potential resulting from $x-\eta>0$ and $x-\eta<0$ are noted by $(\phi_{jR}^{+s}, \psi_{jR}^{+s})$ and $(\phi_{jR}^{-s}, \psi_{jR}^{-s})$ $(j=1, 2)$ respectively, we have, from the calculation of residues,

$$\left.\begin{aligned}\phi_{1R}^{+s} &= \frac{-4\pi\kappa\nu'}{F'(-\kappa)}e^{-i\kappa(x-\eta)-\nu z}, & \phi_{2R}^{+s} &= \frac{2\pi i(2\kappa^2-k^2)}{F'(-\kappa)}e^{-i\kappa(x-\eta)-\nu z},\\ \psi_{1R}^{+s} &= \frac{2\pi i(2\kappa^2-k^2)}{F'(-\kappa)}e^{-i\kappa(x-\eta)-\nu' z}, & \psi_{2R}^{+s} &= \frac{4\pi\kappa\nu}{F'(-\kappa)}e^{-i\kappa(x-\eta)-\nu' z};\end{aligned}\right\} \quad (16)$$

and

$$\left.\begin{aligned}\phi_{1R}^{-s} &= \frac{-4\pi\kappa\nu'}{F'(\kappa)}e^{i\kappa(x-\eta)-\nu z}, & \phi_{2R}^{-s} &= \frac{-2\pi i(2\kappa^2-k^2)}{F'(\kappa)}e^{i\kappa(x-\eta)-\nu z},\\ \psi_{1R}^{-s} &= \frac{-2\pi i(2\kappa^2-k^2)}{F'(\kappa)}e^{i\kappa(x-\eta)-\nu' z}, & \psi_{2R}^{-s} &= \frac{4\pi\kappa\nu}{F'(\kappa)}e^{i\kappa(x-\eta)-\nu' z};\end{aligned}\right\} \quad (17)$$

where

$$\left.\begin{aligned}F'(\kappa) &= \frac{2(4\kappa^4-k^4)}{\kappa} - 4\kappa^3\left(\frac{\nu}{\nu'}+\frac{\nu'}{\nu}\right),\\ F'(-\kappa) &= -F'(\kappa).\end{aligned}\right\} \quad (18)$$

Substitution of expressions (15), (16), (17) into equations (10) yields

$$\left.\begin{aligned}\phi_R^{+s} &= \frac{-\kappa\nu'(k^2-h^2)}{F'(-\kappa)}\phi_0\, e^{i\omega t-i\kappa x-\nu z}\int_{-\infty}^{\infty} f'(\eta)d\eta,\\ \psi_R^{+s} &= \frac{2i(\kappa^2-k^2)(k^2-h^2)}{F'(-\kappa)}\phi_0\, e^{i\omega t-i\kappa x-\nu' z}\int_{-\infty}^{\infty} f'(\eta)d\eta;\end{aligned}\right\} \quad (19)$$

and, by means of the formula

$$\int_{-\infty}^{\infty} f'(\eta)\cdot g(x-\eta)d\eta = -\int_{-\infty}^{\infty} f(\eta)g'(x-\eta)d\eta$$

for functions $f(\pm\infty)=0$,

$$\left.\begin{aligned}\phi_R^{-s} &= i\frac{8\kappa^2\nu'^2(\nu-\nu')}{F'(\kappa)}\phi_0 e^{i\omega t+i\kappa x-\nu z}\int_{-\infty}^{\infty} f(\eta)e^{-2i\kappa\eta}\,d\eta,\\ \psi_R^{-s} &= \frac{4\kappa\nu'(\nu-\nu')(2\kappa^2-k^2)}{F'(\kappa)}\phi_0 e^{i\omega t+i\kappa x-\nu' z}\int_{-\infty}^{\infty} f(\eta)e^{-2i\kappa\eta}\,d\eta\\ &= i\frac{8\kappa^2\nu'^2(\nu-\nu')}{F'(\kappa)}\psi_0 e^{i\omega t+i\kappa x-\nu' z}\int_{-\infty}^{\infty} f(\eta)e^{-2i\kappa\eta}\,d\eta.\end{aligned}\right\} \quad (20)$$

Since we have assumed that the gradient of the topography is small everywhere, we have

$$\int_{-\infty}^{\infty} f'(\eta)d\eta \approx 0; \tag{21}$$

hence the secondary Rayleigh wave travelling in the $+x$ direction is neglected under the accuracy treated in this paper. That is, the Rayleigh wave propagated past an uneven surface is unaffected. Accordingly the modulus of the transmission coefficient past the uneven surface is unity:

$$T_R = 1 \,. \qquad (22)^{1)}$$

The Rayleigh wave evaluated from the potentials $(\phi_R^{-s}, \psi_R^{-s})$ is one travelling in the $-x$ direction; that is, it is the reflected Rayleigh wave from the uneven surface. The factor in expression (20), excluding the phase variation for time and space, noted as R_R, is regarded as the complex reflection coefficient for the uneven surface, and then the reflected potentials are expressed by

$$\begin{pmatrix} \phi_R^{-s} \\ \psi_R^{-s} \end{pmatrix} = R_R \begin{pmatrix} \phi_0 e^{-\nu z} \\ \psi_0 e^{-\nu' z} \end{pmatrix} e^{i\omega t + i\kappa x} \,, \tag{23}$$

where

$$R_R = \frac{8\kappa^2 \nu'^2 (\nu - \nu')}{F'(\kappa)} \int_{-\infty}^{\infty} f(\eta) e^{-2i\kappa\eta} d\eta \,.$$

If an uneven surface is specified by (*cf.* Fig. 2)

$$f(x) = \Gamma(1 - \cos\gamma x) \quad \text{for} \quad 0 < x < x_0 = 2n\pi/\gamma \,, \tag{24}$$

we get

1) This form may be understood more easily in Mr. Homma's work (1941) than in this paper. In that paper, the secondary Rayleigh wave for the original Rayleigh wave generated by the surface $\zeta \cos p(x-\eta)$ is expressed, in the notation of Eq. 38 of Mr. Homma's paper, by

$$\phi_{\pm} = i\zeta \frac{G(p)}{\Delta(l \pm p)} e^{iat + i(l \pm p)x} \,,$$

where $G(p)$ is an analytic function with the property $G(0)=0$, Δ corresponds to $F(\xi)$ in this paper, and the original Rayleigh wave has the factor $\exp(iat + ilx)$. Then, when $p=0$, the surface is perfectly flat, the phase variation of the secondary wave in Eq. (38) is equal to that of the original wave and the wave is travelling in the same direction as the original one. However, along the flat surface, there cannot be the secondary Rayleigh wave of this kind. As a result, we obtain the potential of the secondary wave

$$\phi_{\pm}(p=0) = 0 \,,$$

and, similarly,

$$\psi_{\pm}(p=0) = 0 \,.$$

$$T_R=1$$

$$R_R=i\frac{8\Gamma\kappa^2\nu'^2(\nu-\nu')}{F'(\kappa)}\left[e^{-i\kappa x_0}\frac{\sin\kappa x_0}{\kappa}+e^{i(2\kappa-\gamma)x_0/2}\frac{\sin(2\kappa-\gamma)x_0/2}{2\kappa-\gamma}\right.$$
$$\left.+e^{-i(2\kappa+\gamma)x_0/2}\frac{\sin(2\kappa+\gamma)x_0/2}{2\kappa+\gamma}\right]. \tag{25}$$

Since the wave number κ is always positive, the reflection coefficient becomes a maximum when $2\kappa\rightarrow\gamma$. In particular, if $\lambda=\mu$, those coefficients when $\kappa=\gamma/2$ are expressed by

$$T_{Rm}=1\,,\qquad R_{Rm}=-i\cdot 0.1\kappa^2\Gamma x_0\,. \tag{26}$$

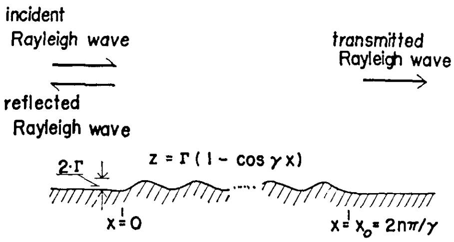

Fig. 2. Geometry of the undulatory surface.

If x_0 is large, the coefficient R_{Rm} is in some cases larger than unity, and it contradicts the assumption on the secondary wave, so that the undulatory surface should be confined to a relatively bounded extent.

Consequently, we find that though the transmitted Rayleigh wave is the same for all wave frequencies, the amplitude of the reflected Rayleigh wave is proportional to that of the incident one, the surface undulation Γ and the linear extent x_0 of the undulatory surface, near the resonant wave frequency.

4. Comparison with waves propagated through a periodic structure

In a previous paper (Onda, 1967 *b*), it was found that the effect of the undulatory surface on the surface wave propagation corresponds to that of the horizontal heterogeneity of the medium on the wave propagation. In this section, the result obtained in the preceding section are compared with those in a periodic structure, where the velocity varies periodically in a limited extent as (*cf.* Fig. 3)

$$\left.\begin{array}{lll} c(x)=c_0(1+\varepsilon) & \text{for} & x<0\,, \\ \quad=c_0(1+\varepsilon\cos\gamma x) & \quad ” & 0<x<x_0=2n\pi/\gamma\,, \\ \quad=c_0(1+\varepsilon) & \quad ” & x_0<x\,. \end{array}\right\} \tag{27}$$

In the transmission coefficient of waves through this structure, a characteristic attenuation appears near the specified frequency $\omega=\gamma c_0/2$, and waves with other frequencies are transmitted without being affected by the structure (Onda, 1966). If we consider the reflected wave, it follows that the reflection coefficient is the greatest near the specified frequency, and the reflected wave with another frequency is apparently not observed. The relation $\omega=\gamma c_0/2$ implies that $k=\gamma/2$, so that the greatest reflection at $k=\gamma/2$ agrees with that obtained in the preceding section.

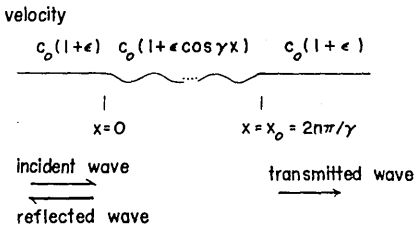

Fig. 3. Schematic illustration of the periodic structure.

Now, the transmission and reflection coefficients for this characteristic wave are expressed as follows (Yoshiyama, 1960),

$$\left.\begin{aligned} T_p &= \frac{\exp(-ikx_0)}{\cosh(\eta\xi_0)} \\ R_p &= -i\tanh(\eta\xi_0) \end{aligned}\right\} \quad \text{for} \quad k=\gamma/2\,, \tag{28}$$

$$\xi_0=\gamma x_0/2=kx_0\,, \quad k=\omega/c_0\,, \quad \eta=\varepsilon/2\,.$$

Providing that $\eta\xi_0$ is small, we obtain the approximate expressions

$$\left.\begin{aligned} T_p &= \exp(-ikx_0)\,, \\ R_p &= -i\frac{\varepsilon}{2}kx_0\,, \end{aligned}\right\} \quad \text{for} \quad k=\gamma/2\,. \tag{29}$$

Since the phase of these coefficients is measured from $x=0$, the transmission coefficient T_p should be regarded as unity, in order to be compared with the coefficient T_{Rm}. It appears that expressions (29) agree with expressions (26), if $\varepsilon/2$ in the former corresponds to $0.1\,\kappa\Gamma$ in the latter.

In the calculation on Rayleigh waves, since the surface undulation was replaced by an equivalent stress distribution on the plane surface, the ratio of the stress amplitude to the displacement amplitude of the incident wave is calculated, for the purpose of comparing the velocity undulation in a periodic structure with the surface undulation. The displacement in the periodic structure has been obtained by Prof. Yoshiyama (1960). If u_1 is the displacement of the incident wave and

S_2 is the stress in the periodic structure induced by the incident wave, we have, when $k=\gamma/2$,

$$S_2=-i\frac{\gamma c_0\sqrt{\rho c_2}\,A_1}{2\cosh(\gamma\xi_0)}\cosh\{\gamma(\xi_0-\xi)\}\exp(-i\xi)\,. \tag{30}$$

where

$$u_1=\frac{A_1}{\sqrt{\rho c_1}}\exp(-ikx)\,, \tag{31}$$

and, if $\gamma\xi_0$ ($>\gamma\xi$) is small, we have

$$S_2\doteqdot -i\omega\sqrt{\rho c_2}\,A_1 e^{-i\xi}\,.$$

Here, as $c_2=c_0(1+\varepsilon\cos\gamma x)$, if ΔS_2 is the amplitude of the stress variation in the periodic structure, the absolute value of the ratio is

$$|\Delta S_2/u_1|=\omega\rho c_0\cdot\varepsilon/2\,. \tag{32}$$

On the other hand, in Rayleigh waves passing along the undulatory surface, we have from equations (9), when Poisson's ratio is 1/4 and $\kappa=\gamma/2$,

$$\left.\begin{aligned} T(x)&=\mu\phi_0\Gamma\kappa^3(i\cdot 0.8\cos\gamma x-2.3\sin\gamma x)\,,\\ N(x)&=\mu\phi_0\Gamma\kappa^3\cdot 0.5\cos\gamma x\,. \end{aligned}\right\} \tag{33}$$

Hence the stress amplitude in this case may be estimated as about 2.5 $\cdot\mu\phi_0\Gamma\kappa^3$. The displacement at the surface of the original Rayleigh wave, $|u_0|$, is given by

$$|u_0|=\sqrt{|u^0|^2+|w^0|^2}=0.75\phi_0\kappa\,, \tag{34}$$

whence

$$\begin{aligned}\left|\frac{\Delta S}{u_0}\right|&=\frac{2.5}{0.75}\mu\Gamma\kappa^3\doteqdot 3.3\rho\omega V_R\kappa\Gamma\left(\frac{V_s}{V_R}\right)^2\\ &=3.5\,\omega\rho V_R\kappa\Gamma\,. \end{aligned} \tag{35}$$

Therefore the results in the calculation on the Rayleigh wave must be multiplied by about 3.5, so as to be compared with ones from the periodic structure. In the result, the direct correspondence of the product $\kappa\Gamma$ to the velocity undulation ε, submitted in the previous paper (Onda, 1967 *b*), has about a 30 per cent error. It seems that this difference is caused by the converted body waves radiated from the uneven surface into the medium.

5. Concluding remarks

As a continuation of the study of waves propagated through a periodic structure, a surface wave propagating along an undulatory surface was investigated in an earlier paper (Onda, 1967 *b*); it was found that the main features of the surface wave may be analysed from the solution of the equation of waves propagated through a periodic structure. The problem of surface waves passing along an uneven surface, which is a boundary value problem, has not been satisfactorily solved. In this paper, the solution obtained in the previous paper is justified by using another procedure, that is, by means of an approximate expansion by perturbation of the boundary condition given upon the uneven surface. In this calculation, we assume that the elevation is small compared with the wave length of a propagated wave, its gradient is much smaller than unity, the wave length of the surface undulation is of the same order as that of the propagated wave and the undulatory surface covers a limited extent. In the result, although there is some uncertainty in the precision, we reach a conclusion similar to the one of the characteristics of the wave propagation obtained in the previous paper. Then it follows that the effect of the undulatory surface on the surface wave is as follows: The wave with the wave length twice the wave length of the surface undulation is unstable, the instability resulting from the resonance along the undulatory surface, and the apparent and characteristic attenuation appears near this specified wave frequency.

The perturbation method has several advantages, *e. g.*, to elucidate the effect of the topography and the interaction between the incident wave and the other waves generated at the uneven surface and to estimate the degree of approximation. However, it should be noted that the solution derived from this method is not valid when we treat the problems of the uneven surface with a wide range and a steep slope.

On the other hand, notwithstanding that the conformal mapping method has not been sufficiently studied up to date, is appears that this method gives useful information for the study of the surface wave propagation.

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References

HOMMA, S., 1941, "On the Effect of Topography on the Surface Oscillation," *Quart. J. Seism.*, **11**, 349-364, (in Japanese).

LAPWOOD, E. R., 1949, "The Disturbance due to a Line Source in a Semi-infinite Elastic Medium," *Phil. Trans. Roy. Soc. London A*, **242**, 63-100.

LOVE, A. E. H., 1952, *A Treatise on the Mathematical Theory of Elasticity*, (4th ed.), Cambridge U. P., 643pp.

ONDA, I., 1966, "Plane Pulse Propagation through a Heterogeneous Medium with a Periodic Structure," *Bull. Earthq. Res. Inst.*, **44**, 1-12.

ONDA, I., 1967*a*, "Notes on Propagation of Elastic Waves through a Heterogeneous Medium with Periodic Structures," *Bull. Earthq. Res. Inst.*, **45**, 559-566.

ONDA, I., 1967*b*, "Propagation of Elastic Waves through a Heterogeneous Medium with Periodic Structures.—Application to Certain Seismological Problems," *Bull. Earthq. Res. Inst.*, **45**, 567-588.

SATO, R., 1967, "On Rayleigh Waves Generated at Rough Surface (1)," *Zisin*, [ii], **8**, 121-137.

YOSHIYAMA, R., 1960, "Stability of Waves through a Heterogeneous Medium and Apparent Internal Friction," *Bull. Earthq. Res. Inst.*, **38**, 467-478.

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32. 規則的に起伏している表面を伝わるレイリー波

——周期構造を伝わる波との比較——

地震研究所 音 田 功

表題の問題は地震学において，非常に重要であるが，まだ満足のゆく解は得られていない．前の論文 (Onda, 1967*b*) で，この種の波は近似的に周期構造に伝わる波動と同じ性質をもつていることを導いた．そこで，得られた解の性質を，それとは別の方法，すなわち所謂“摂動法”によつて調べてみた．

平面で境された半無限媒質に伝わるレイリー波を第一近似の波と考えると，それだけでは平らでない表面の上で，応力は消えない．そこで，次の仮定を措く；1) 表面の起伏は伝わる波の波長に比べて小さい，2) 地表面の傾斜もまた小さい，3) 起伏している表面の波長は，伝わる波の波長と同じ大きさである．これらの仮定のもとで，自由表面の境界条件を満足するように，二次的に発生する波を求めた．その波のうち，始めの波と同じ方向に進むレイリー波は，ここで取扱つた近似のもとでは現われない．いいかえると，起伏している表面を越えて伝わるレイリー波は，起伏とその勾配が共に小さいならば，形を変えずに伝わる．それに対して，始めの波と逆方向に進む二次波の振幅は，始めの波の波長が表面起伏の波長の倍に相当する長さをもつときに，最大となる．この性質は周期構造の厚さが厚くないと仮定して，その反射係数を展開したときの第 1 項から得られるものと同じである．従つて，周期構造を伝わる波の性質から，その特定の周波数の波が一種の共鳴を起すことが期待される．そして，この論文で取扱った方法では計算できないけれども，さらに高次の項まで考慮すると，起伏のある表面を越えて伝わる波には，その周波数を中心にして，見掛けの減衰が現われると推定される．