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41. *Note on Earthquake Energy.*

By Ryoichi YOSHIYAMA,

Earthquake Research Institute.

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1. Introduction

Strictly speaking, in understanding earthquake energy there are some theoretical ambiguities. Answers to the problem would be found out in the results of studies of "Elastic waves animated by potential energy".[1] On the other hand, repeated comments by Prof. K. Sezawa[2] on the results conclude by saying, "It is, thus, impossible to evolve wave energy from the strained part of an elastic body even in sudden release of the force at the origin. Half the value of the energy at the origin, however, is consumed in resolving the statical strain, the remaining half alone being transmitted in the form of pure elastic waves to infinity." "If the stress were applied or released extremely slowly, no wave energy would be transmitted to infinity, whereas, if done with infinite rapidity, exactly half the energy given at the origin will be transmitted to infinity." According to his theory, ambiguities concerned in this paper do not exist. However, it seems that the comments are not based on proper mathematics, mathematical procedure being diverted into the solution of particular problems. Consequently, soon afterwards, a brief note to confirm our results against the above comments was submitted by the present writer[3] and, later, Mr. C. Y. Fu[4] pointed out a misuse of the "law of superposition" in Prof. Sezawa's paper. Nevertheless, it appears that Prof. Sezawa's opinion still affects some seismologists as regards energy estimation of an earthquake, while there would be scarcely any seismologist at present who doubts the possibility of the evolution of strain energy into earthquake energy.

The problem is studied over again to clear away the confusion and

1) H. KAWASUMI and R. YOSHIYAMA, *Bull. Earthq. Res. Inst.*, **13** (1935), 496.
2) K. SEZAWA, *Bull. Earthq. Res. Inst.*, **13** (1935), 740.
K. SEZAWA and K. KANAI, *Bull. Earthq. Res. Inst.*, **14** (1936), 10.
K. SEZAWA, *Bull. Earthq. Res. Inst.*, **14** (1936), 149.
3) R. YOSHIYAMA, "Note on the wave equation." (in Japanese). *Zisin*, **8** (1936), 443.
4) C. Y. FU, *Bull. Seism. Soc. Amer.*, **35** (1945), 37.

to understand better the theoretical process of estimation of earthquake energy.

2. General description of diverging wave

Suppose a purely longitudinal wave diverging uniformly in all directions from a spherical origin. Putting displacement $u=\partial\varphi/\partial r$, equation of motion is

$$\frac{\partial^2}{\partial t^2}(r\varphi)=c^2\frac{\partial^2}{\partial r^2}(r\varphi)\,, \tag{1}$$

where $c^2=(\lambda+2\mu)/\rho$. General solution of (1) is composed of two progressive waves, the one diverging wave and the other converging wave. In order to study a disturbance spreading in an infinite medium out of a source, the latter is discarded. Let a be the radius of the source,

then
$$\varphi=\frac{a}{r}f\left(t-\frac{r-a}{c}\right), \tag{2}$$

or,

$$\frac{\partial}{\partial t}(r\varphi)=-c\frac{\partial}{\partial r}(r\varphi)\,. \tag{3}$$

Simple as it is, (2) or (3) is a fundamental formula of a diverging wave from a source. Mathematically speaking, a may be any constant regardless of its physical meaning: it is preferable to say that the radius of the source is defined equal to a by (2).

The space within $r\leqq a$ is termed "the source" in this paper; the physical meaning of the space and suitability of the term can be discerned from the results of the following studies. $f(t)$, that is φ at $r=a$ and that gives the wave form, should be determined by the wave generating mechanism: boundary condition assigned on the surface of the source, whether it be an energy source or merely a wave source, determines $f(t)$ and the wave form is given in consequence. It is remarkable that since $\partial\varphi/\partial r$ is displacement, the first derivative of φ should be continuous and the second derivative should be bounded in time and in space, provided $r\geqq a$.

In virtue of (2) or (3), it is easily proved that displacement and component of stress are related to φ by the following formulas,

$$u = \frac{\partial \varphi}{\partial r}$$

$$= -\frac{1}{c}\frac{\partial \varphi}{\partial t} - \frac{\varphi}{r}, \tag{4}$$

$$\widehat{rr} = \lambda \Delta + 2\mu e_{rr}$$

$$= \rho \frac{\partial^2 \varphi}{\partial t^2} - \frac{4\mu}{r} u \tag{5}$$

or,

$$= \rho \frac{\partial^2 \varphi}{\partial t^2} + \frac{4\mu}{rc}\frac{\partial \varphi}{\partial t} + \frac{4\mu}{r^2}\varphi . \tag{6}$$

If the boundary condition is such that $u=G(t)$ at $r=a$, f in (2) is given by a particular solution of the following differential equation, obtained by putting $r=a$ in the right hand side of (4),

$$\frac{1}{c}\frac{df}{dt} + \frac{1}{a} f = -G(t) . \tag{7}$$

If the boundary condition is such that $\widehat{rr}=G(t)$ at $r=a$, the differential equation, in place of (7), is obtained from (6) as follows,

$$\rho \frac{d^2 f}{dt^2} + \frac{4\mu}{ac}\frac{df}{dt} + \frac{4\mu}{a^2} f = G(t) . \tag{8}$$

It is remarkable that (8) is similar to the equation of motion of a pendulum with a damping force; this is a theoretical proof of a conclusion deduced by Kawasumi from the solution of the special example in the previous paper.

3. Energy flow and energy from the source

Outward flow of energy across a spherical surface with a radius r and concentric with the surface of the source is given by

$$E(r) = -4\pi r^2 \int_{\tau_1}^{\tau_2} \widehat{rr} \frac{\partial u}{\partial t} dt , \tag{9}$$

where $\tau = t-(r-a)/c$; τ_1 and τ_2 are respectively the time for the initial state and for the final state; $\widehat{rr}$ is a normal stress to the surface, tangential stresses are zero; time integration at a distance r is performed with a time lag of $(r-a)/c$.

$E(a)$, putting $r=a$ in (9), is no doubt energy supply outward from inside of the source; the debatable point is the relation between that energy and earthquake energy. It would appear reasonable to use $E(a)$ for "earthquake energy", unless it is proved in the previous paper that, when stress on the surface of the source is suddenly removed to a free state, $E(a)$ is zero, while wave is generated, and differs from that when a stress is suddenly applied, contrary to Sezawa's opinion. To merely understand the difference, a slight knowledge of physics will be sufficient. For further studies, $E(r)$ is calculated directly by (2), (4) and (5) or (6), saving elaborate calculation in the previous paper.

$$-\widehat{rr}\frac{\partial u}{\partial t}=-\left(\rho\frac{\partial^2\varphi}{\partial t^2}-\frac{4\mu}{r}u\right)\frac{\partial u}{\partial t}$$
$$=\rho\frac{\partial^2\varphi}{\partial t^2}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\varphi}{\partial t}+\frac{\varphi}{r}\right)+\frac{4\mu}{r}u\frac{\partial u}{\partial t}$$
$$=\frac{\rho}{c}\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2+\frac{\rho}{r}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\varphi}{\partial t}+\frac{4\mu}{r}u\frac{\partial u}{\partial t}\ . \qquad (10)$$

Since u and $\partial\varphi/\partial t$ are continuous and $\partial^2\varphi/\partial t^2$ is finite,

$$\int_{\tau_1}^{\tau_2}\frac{\partial^2\varphi}{\partial t^2}\frac{\partial\varphi}{\partial t}dt=\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2\right]_{\tau_1}^{\tau_2}\ ,\qquad \int_{\tau_1}^{\tau_2}u\frac{\partial u}{\partial t}dt=\frac{1}{2}[u^2]_{\tau_1}^{\tau_2}\ . \qquad (11)$$

So that, substituting (10) into (9) and using (2), we get

$$E(r)=E_0+E_1(r)+E_2(r)\ , \qquad (12)$$

where

$$\left.\begin{aligned}E_0&=\frac{4\pi\rho}{c}r^2\int_{\tau_1}^{\tau_2}\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2dt=\frac{4\pi a^2\rho}{c}\int_{\tau_1}^{\tau_2}\left(\frac{\partial^2 f}{\partial t^2}\right)^2dt\ ,\\ E_1(r)&=2\pi\rho r\left[\left(\frac{\partial\varphi}{\partial t}\right)^2\right]_{\tau_1}^{\tau_2}=\frac{2\pi a^2\rho}{r}\left[\left(\frac{\partial f}{\partial t}\right)^2\right]_{\tau_1}^{\tau_2}\ ,\\ E_2(r)&=8\pi\mu r[u^2]_{\tau_1}^{\tau_2}\ .\end{aligned}\right\} \qquad (13)$$

We must remember that, to derive (13), some assumptions are assigned to φ and its derivatives, from which (11) results and, at the same time, that those assumptions are natural so far as stress is finite in (8) and are, in general, supported in physics.

From (13), important characteristics of E_0, E_1 and E_2 are deduced: E_1 and E_2 depend on r, but E_0 does not; E_0 depends on the process of the change of the state, but E_1 and E_2 do not, depending utterly upon the initial and final states; E_0 is always positive, but E_1 and E_2 not always; E_1 and E_2 are not defined by mere relative difference of the two states, because of quadratic form. While it seems there is no doubtful point in the reasoning above stated, discrepancy between our conclusion and Sezawa's originates in short from whether these characteristics, especially of E_2, are accepted or not.

E_1 and E_2 tend to zero as $r \to \infty$, so that

$$E(\infty)=E_0 \ . \tag{14}$$

It is remarkable that E_0 is not only independent of r, but is also equal to $E(r)$ at $r=\infty$.

4. From statical state to statical state

If the initial and final states are both statical ones, clearly $E_1(r)=0$, so that $E(r)=E_0+E_2(r)$ and, using (14),

$$E(a)=E(\infty)+E_2(a) \ , \tag{15}$$

while $E_2(r)$ is positive or negative according as absolute magnitude of displacement in the final state is larger or smaller than that in the initial state. On the other hand, the displacement in either state is derived from φ such that $\partial^2(r\varphi)/\partial r^2=0$, because of continuity of displacement in time. So that $u(\tau_1)=a^2u_0(\tau_1)/r^2$ and $u(\tau_2)=a^2u_0(\tau_2)/r^2$, where $u_0(\tau_1)$ and $u_0(\tau_2)$ are respectively the displacement on the surface of the source in the initial and in the final states. $u_0(\tau_1)$ for a statical pressure P_1 is equal to $aP_1/4\mu$. Summing up,

$$\begin{aligned} E_2(r) &= 8\pi\mu a^4\{u_0(\tau_2)^2-u_0(\tau_1)^2\}/r^3 \ , \\ \text{or,} \quad &= \pi a^6\{P_2^2-P_1^2\}/2\mu r^3 \ , \end{aligned} \tag{16}$$

where P_1 and P_2 represent pressure within the source respectively in the initial and in the final state. Whether P_1 or P_2 is a pressure or a tension $E_2(r)$ is unchanged, depending simply on its magnitude, while the displacement naturally changes its sign. $E_2(r)$ and, therefore, $E_2(a)$ also depend on the absolute displacement from an unstressed state and are indeterminable by mere observation of relative displacement or by mere assumption of relative change of pressure within the source in the two states. According as $|P_2|>|P_1|$ or $|P_2|<|P_1|$, regardless whether these are plus or minus, $E_2>0$ or $E_2<0$. So that, when the process is from a free or less stressed state to a more stressed state, since $E_2>0$, $E(a)>E(\infty)$: a part of energy equal to $E(a)-E(\infty)=E_2(a)$ from inside of the source must be stored up in the medium as an elastic strain energy. In this case, the space inside of $r=a$ clearly stands for the energy source of the wave. On the other hand, when the process is from a stressed state to a free or a less stressed state, since $E_2<0$, $E(a)<E(\infty)$: it means that energy stored up in the medium where $\infty>r\geqq a$ is released into an energy flow by the amount of $E(\infty)-E(a)=-E_2(a)$. The above stated process in either case is not necessarily monotonous; there may be a maximum as well as a minimum stressed state during the process. Not only when the final state is the same as the initial state, but also when $P_1=-P_2$, $E_2=0$.

It seems that some part of the comment by Prof. Sezawa results from overlooking the significance of $E_2(a)$ in *earthquake energy* and a lack of careful reasoning of mathematical results. Otherwise, he would not have referred to wave propagation in a stressed medium in a discussion of the problem or have been puzzled by an apparent paradox. In the problem of wave propagation, $P_1=P_2$, so that $E_2(a)$ has nothing to do with the flow of energy: $E_2(a)$ has clearly a special importance in a problem of wave generation, especially as regards earthquake or underground explosion in which a semi-permanent set would be left behind at the wave source.

5. Wave energy E_0

E_0 is the most important term among the three. No doubt $E_2(a)$ has an important significance in the definition of earthquake energy and its magnitude may be very large in some cases. However, its significance and magnitude are undeterminable at this present stage of seismological research. On the other hand, the contribution of E_0 to the earthquake energy is determinate in character and magnitude. That is; $E_0=E(\infty)$ as already mentioned: E_0 depends entirely upon the process, or, to speak plainly, the speed of change of the state, and is always positive except when the process is quasi-stationary and $E_0=0$: it is clear looking at the integrand of its formula (13) that any kind of statical component has no contribution to E_0. The difference between the two states which E_2 completely depends upon affects E_0 but slightly; zero difference does not necessarily mean $E_0=0$, nor large difference does large E_0. Because of those properties, E_0 is classified as " wave energy ", and, in practice, it is estimated as follows. Originally, from (13),

$$E_0=\frac{4\pi r^2}{c}\rho\int\left(\frac{\partial^2\varphi}{\partial t^2}\right)^2 dt \quad . \tag{17}$$

Since E_0 is independent of r, it is estimated at $r\to\infty$, where, in virtue of (3),

$$\frac{\partial^2\varphi}{\partial t^2}=-c\frac{\partial u}{\partial t}-\frac{ac}{r^2}\frac{\partial f}{\partial t}\doteqdot -c\frac{\partial u}{\partial t}\ .$$

So that,

$$E_0=4\pi r^2\rho c\int\left(\frac{\partial u}{\partial t}\right)^2 dt\ , \tag{18}$$

provided r is infinitely large compared with a and wave length. $\partial u/\partial t$ is the velocity of wave motion, and thus E_0 is estimated from the seismogram. It is remarkable that E_0 is, as expected, twice the flow of kinetic energy, *but at a large distance from the source.*

6. Energy source and wave source

Going back to (15) or to its original formula, $E(a)=E_0+E_2(a)$, suppose a process from a stressed state to a less stressed state is performed very slowly, whether monotonous or not. Then E_0 is small, though positive, and $E_2(a)<0$, so that $E(a)<0$. At a certain distance $r=r_0$, $E(r_0)=0$. At any place within that range where $r<r_0$, $E(r)<0$; that means energy flow is directed inward there. In that case, too, E_0, calculated by (17) or approximately by (18), is positive and is constant universally for any position, provided $r\geqq a$, that is even on the surface of the source. $E(a)<0$ means that energy equal to $|E(a)|$ is consumed within or on the surface of the source for the work necessary in the process of the change of the state there. It means also that the energy is supplied from the strain energy outside the source. Energy $-E_2(a)$, because $E_2(a)<0$, is equal to the difference of strain energy at the two states, the stressed initial state and the less stressed final state; it is partly, by the amount of $-E(a)$, consumed for the work in the process and partly, by the amount of E_0, for the wave; the energy relation above stated is represented by the following equation,

$$-E_2(a)=E_0+\{-E(a)\}\ , \tag{19}$$

which is obtained from the general equation (15).

If $E(a)>0$, space within $r=a$ may well be interpreted as an energy source as already mentioned. However, if $E(a)<0$ or $E(a)=0$, the space can not be an energy source. The boundary condition assigned to the surface $r=a$ is a starting action of the wave. And, when energy is necessary for that action, for instance, for breaking of material or for a growth of a fault plain, naturally it follws that $E(a)<0$. In spite of $E(a)\leqq 0$, a wave starts on the surface $r=a$ with outward flow of wave energy E_0, which remains unchanged and positive everywhere and is equal to the total outward flow at $r=\infty$: $E_0=E(\infty)$. If $\partial^2\varphi/\partial t^2=0$ on the surface of the source all the time, $E_0=0$ on the surface, therefore $E_0=0$ everywhere, because of its independency of r, that indicates $\partial^2\varphi/\partial t^2=0$ everywhere. So that the wave, if any, is generated on the surface $r=a$ and not on any other surface on the way such that $r>a$: the space within $r\leqq a$ is a wave source, if not an energy source. And, as far as E_0 is concerned, it appears it comes from the inside of the source across the surface $r=a$. That the energy comes in fact from the strain energy in the medium outside, and not from inside, of the source is indicated by the fact that $E(a)\leqq 0$. However, by mere observation of the wave, we can not determine whether $E_2(a)>0$ or <0, and, therefore, whether $E(a)=0$, >0 or <0. If relative stress change P_2-P_1 were estimated by a minute study of wave motion, estimation of $E_2(a)$ is impossible, unless P_1 or P_2 is assumed : determination of $E_2(a)$ is not a problem of accuracy of observation alone, but is that of a theory to stand on. However, in practice, it is generally, but implicitly, assumed that $u_1=0$, so that $P_1=0$, in an analysis of seismograms.

Anyway, $E(\infty)=E_0$ and $E_0>0$. So that, when the wave is observed at a large distance from the source, it appears, regardless of the sign of $E_2(a)$, as if its energy comes from the source, the energy being determined by the process of how the state changes at the source. The results just mentioned would be quite concordant with the general idea of an earthquake, and, together with the fact that a stable wave function f is always determined from (8) for any reasonable assumption of $G(t)$, would prove the advisability of the discarding of the converging wave to define the primary attributes, at least, as regards an earthquake. If, on the contrary, we assume a solution of converging wave instead of a diverging wave, f is determined by

$$\rho\frac{d^2 f}{dt^2}-\frac{4\mu}{ac}\frac{df}{dt}+\frac{4\mu}{a^2}f=G(t) \tag{20}$$

in place of (8). The wave form f thus determined is unstable, and is inadequate for the solution of the problem. Moreover, we obtain $E_0<0$, which indicates an inward flow of energy at a large distance from the source, contrary to the generally accepted idea of an earthquake.

7. Earthquake wave energy and earthquake energy

The way to define earthquake energy is examined from the results above obtained, though there still remains some theoretical question in the applicability of them, because of an unequal azimuthal distribution of wave energy in an earthquake.

If "earthquake energy" is understood as the energy of earthquake waves, it is determinate and represented by a sum of E_0 for each kind of earthquake wave. On the other hand, if it is understood as the energy necessary for the earthquake occurrence, it is undeterminable, because of $E_2(a)$ or something equivalent to it. It is not only because the magnitude of $E_2(a)$ is undeterminable, but also because its sense is undeterminable, as already explained.

As regards the mechanism of earthquake occurrence, at least two kinds of process (A) and (B) are conceivable, which correspond respectively to "stress increasing process" and "stress decreasing process."

A: Sudden increase of pressure within the source causes an earthquake and accumulation of strain energy.→Discharge of the strain energy by a non-elastic process or by a sequence of after-shocks.

B: Accumulation of strain energy due to a slow quasi-stationary increase of pressure within the source.→Elastic breakdown of the material causes a discharge of the strain energy as an earthquake, and, in some cases, with a sequence of after-shocks.

Process (A) needs an energy decidedly larger than (B) for the main earthquake; in the process (A), an extra energy $E_2(a)$ is necessary, while, in the process (B), energy is necessary to break the material.

There is another stress decreasing process which is accepted by some seismologists. In that process, an external force at $r=\infty$ and a weak point at $r=0$ are assumed. Mathematically, the process may be treated in a way similar to that of (B), but the physical meaning of the process would be quite different. Anyway, it seems the stress decreasing process is accepted more widely than (A) at present.

Process (A) is suggested by Prof. Ishimoto in his "Magma intrusion theory of earthquake occurrence." In his theory, cone type distribution of P-wave motion and sequence of after-shocks are explained reasonably well. The most important point of the difference between (B) and (A) lies in whether fracture is necessary or not for the occurrence of an earthquake. However, even when a fault is apparent after an earthquake, it is difficult to reject (A) decidedly for (B) in our knowledge of earthquake fault at present.

There would be a process, too, in which (A) or (B) plays a function of trigger for each other.

Considering such circumstances, it is a debatable point how to treat, in relation to earthquake energy, the apparent strain energy, which is calculated by some seismologists as based on surface change left behind an earhquake, setting aside the question whether it is duely estimated or not. So far as it is considered with respect to earthquake occurrence, the energy corresponds in nature to $E_2(a)$. If we assume process (A), the change of surface would indicate a strained state and the energy concerned would be a part of that necessary for the earthquake; addition of the energy to earthquake wave energy may be one step towards a correct estimation of earthquake energy. Whereas, if we assume process (B), the energy is not a real one, but the surface change merely indicates that a strain energy was already released for earthquake wave and partly for the work of fracture: addition of the energy to wave energy is quite meaningless. On the other hand, it is impossible to discriminate between (A) and (B) by mere observation of earth movement in an earthquake, and, moreover, it is generally assumed $u_1=0$ in an analysis of the seismograms, unless a special consideration is given. So that, if energy flow is calculated by (9) from the seismograms, a result is obtained automatically always as if a process similar to (A) that starts from a zero stressed state is assumed. And, if the estimation of energy flow by (9) is extended up to the source $r=a$, $E(a)$ of the process (A) appears always, independent of the fact, as the energy of the earthquake, apparently supplied from inside of the source, while the process (B) is widely accepted. Such a conclusion may appear to be concordant with the theory of Prof. Sezawa, but, however, it would be readily understood that there is an essential difference between the two.

No doubt it is one of the important problems of seismology to determine which is the most probable process of earthquake occurrence among (A), (B) or any other process, but such would only be solved through a careful study in future of earth movement before and after an earthquake.

In conclusion, under these circumstances, it would be preferable, for the present at least, to take earthquake wave energy E_0 for earthquake energy and to give attention independent of it to earthquake strain energy or earthquake potential energy in general, if any. If we adopt such a definition of earthquake energy, in spite of its formal appearance, (9) loses its theoretical authenticity, even against an approximate formula (18), as a formula of energy estimation, to say nothing of its remote relation to the observation.

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41. 地震のエネルギーについて

地震研究所　吉山良一

地震のエネルギーといえば一般には地震波動のエネルギーと解される．そうきまつていればよいが，時には地震を起すに必要なエネルギーと解釈される場合もある．地震学上重要なものはむしろ後者で，地震波動のエネルギーを求める公式に苦労するのも結局は地震を起すに必要なエネルギーを求めるためであろう．この二種のエネルギーの違いには静的エネルギーが動的エネルギーに変る際の効率が実際上は最大の影響を与えるであろうが，そのほかに本質的に違う面もある．ここにはそれについて所見を述べ将来この方面における理論の整理に資したいと思う．