35. *Stoneley Wave—Its Velocity, Orbit and the Distribution of Amplitude.*

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Introduction

About thirty years ago, Stoneley proved the possible existence of boundary waves of the Rayleigh-type being transmitted along the plane surface of separation of two solid media, and also proved that it is impossible for waves of the Love-type to be so transmitted[1].

Later, the nature of the wave, now called the Stoneley wave, was investigated by K. Sezawa and K. Kanai[2], and the range of possible existence was elucidated. But even its velocity, to say nothing of the orbit and the distribution of amplitude, has not yet been studied, although more complicated problems concerning the Rayleigh and Sezawa waves propagated in a stratified medium have been approached by a number of authors[3].

Hence, in this paper, we will give the results of our calculations concerning the said properties, and will discuss the nature and the meaning of the Stoneley wave.

Velocity

Let the axes of x and z be drawn in coincidence with and perpendicular to the surface of discontinuity, the density and elastic constants in the media on the positive and negative side of z being $\rho, \lambda, \mu; \rho', \lambda', \mu'$

1) R. STONELEY, "Elastic Waves at the Surface of Separation of Two Solid," *Proc. Roy. Soc. London*, **106** (1924), 416.

2) K. SEZAWA and K. KANAI, "The Range of possible Existence of Stoneley-waves and Some Related Problem," *Bull. Earthq. Res. Inst.*, **17** (1939), 1. With respect to the generation of waves in a medium consisting of two semi-infinite elastic body, K. SEZAWA and K. KANAI published another paper. "The Formation of Boundary Waves at the Surface of a Discontinuity within the Earth Crust I," *ibid.*, **16** (1938), 504.

3) e.g. K. KANAI, "On the M_2-waves (Sezawa-waves)," *Bull. Earthq. Res. Inst.*, **29** (1951), 39.

respectively. In this case the displacement potentials of the dilatational and distortional waves assume the forms

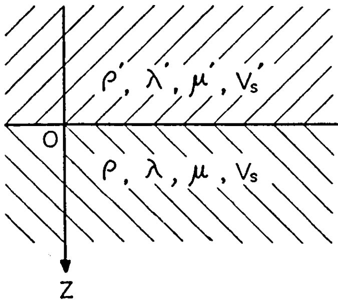

Fig. 1. We always assume the relation $\rho'<\rho$, which does not impair the generality. In the numerical calculations we also assumed the relation $\lambda=\mu$, $\lambda'=\mu'$.

$$\begin{aligned} \phi &= A \exp(-rz+ifx-ipt), \\ \psi &= B \exp(-sz+ifx-ipt), \\ \phi' &= A' \exp(r'z+ifx-ipt), \\ \psi' &= B' \exp(s'z+ifx-ipt), \end{aligned} \tag{1}$$

where

$$\begin{aligned} & r^2=f^2-h^2, \quad s^2=f^2-k^2, \\ & r'^2=f^2-h'^2, \quad s'^2=f^2-k'^2, \\ & h^2=\rho p^2/(\lambda+2\mu), \quad k^2=\rho p^2/\mu, \\ & h'^2=\rho' p^2/(\lambda'+2\mu'), \quad k'^2=\rho' p^2/\mu'. \end{aligned} \tag{2}$$

The displacements in both media are then expressed by

$$\begin{aligned} & u=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z}, \qquad w=\frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial x}, \\ & u'=\frac{\partial\phi'}{\partial x}+\frac{\partial\psi'}{\partial z}, \qquad w'=\frac{\partial\phi'}{\partial z}-\frac{\partial\psi'}{\partial x}. \end{aligned} \tag{3}$$

Since the two media are continuous, the boundary conditions which must be satisfied at the surface of separation are the continuity of the displacements and the stress, viz.,

$$\begin{aligned} & u=u', \\ & w=w', \\ & \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)=\mu'\left(\frac{\partial w'}{\partial x}+\frac{\partial u'}{\partial z}\right), \\ & \lambda\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}=\lambda'\left(\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}\right)+2\mu'\frac{\partial w'}{\partial z}. \end{aligned} \tag{4}$$

Putting (1) and (3) into (4), we get

$$\begin{aligned} & ifA-sB-ifA'-s'B'=0, \\ & rA+ifB+r'A'-ifB'=0, \\ & i2\mu rfA-\mu(f^2+s^2)B+i2\mu'r'fA'+\mu'(f^2+s'^2)B'=0, \\ & (-\lambda h^2+2\mu r^2)A+i2\mu sfB+(\lambda'h'^2-2\mu'r'^2)A'+i2\mu's'fB'=0. \end{aligned} \tag{5}$$

Thus the velocity equation of the wave transmitted along the boundary surface under consideration is

$$\begin{vmatrix} if & -s & -if & -s' \\ r & if & r' & -if \\ i2\mu rf & -\mu(f^2+s^2) & i2\mu' r' f & \mu'(f^2+s'^2) \\ -\lambda h^2+2\mu r^2 & i2\mu sf & \lambda' h'^2-2\mu' r'^2 & i2\mu' s' f \end{vmatrix} = 0 \quad (6)$$

Assuming various values of ρ'/ρ and putting them into the above equation, we calculated the relation between $(V/V_s)^2$ and μ'/μ. Here we can assume, without losing generality, that the condition $\rho' < \rho$ holds. Further, we employed the assumption that λ and λ' are always equal to μ and μ' respectively.

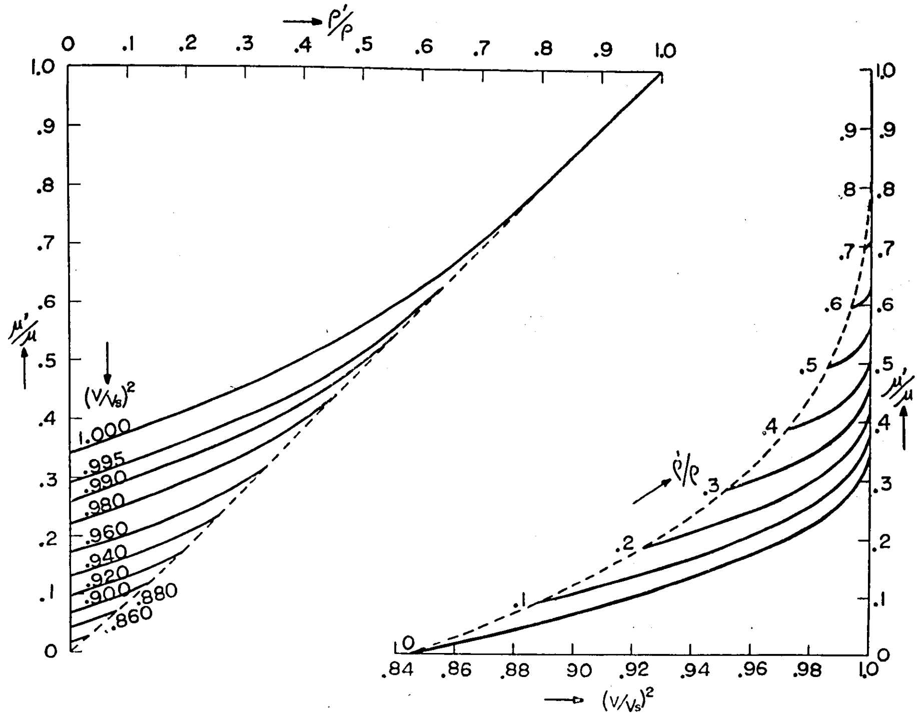

Fig. 2. The relation between ρ'/ρ, μ'/μ and $(V/V_s)^2$. V being the velocity of Stoneley wave. Above two figures have the same content, only the parameter and the abscissa are interchanged. On the broken lines V is equal to V_s'. In the right figure the parameter is ρ'/ρ, the value of which is given on the left and of the full lines. $\rho'/\rho=0$, $\mu'/\mu=0$ gives the Rayleigh wave.

The results of the calculation are given in Table I, and illustrated in Fig. 2.

Table I. The values of μ'/μ and w_0/u_0 when the conditions ρ'/ρ and $(V/V_s)^2$ are given. (w_0 and u_0 are the vertical and horizontal displacements at the plane of separation.)

ρ'/ρ → / $(V/V_s)^2$ ↓	0		0.1		0.2		0.3	
	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0
1.0000	.3435	1.608	.3772	1.685	.4153	1.789	.4579	1.932
.9900	—	—	—	—	—	—	.3766	1.977
.9800	.2221	1.592	.2559	1.678	.2955	1.800	.3425	1.999
.9600	.1710	1.575	.2043	1.659	.2449	1.799	.2965	2.066
.9519	—	—	—	—	—	—	.2856	2.164
.9400	.1312	1.555	.1646	1.643	.2070	1.803		
.9240	—	—	—	—	.1848	1.864		
.9200	.0976	1.535	.1311	1.627				
.9000	.0680	1.517	.1024	1.619				
.8882	—	—	.0888	1.638				
.8800	.0414	1.499						
.8600	.0169	1.480						
.8453	0	1.468						

ρ'/ρ → / $(V/V_s)^2$ ↓	0.4		0.5		0.6		0.7	
	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0
1.0000	.5065	2.138	.5625	2.458	.6290	3.011	.7088	4.104
.9990	—	—	—	—	—	—	.7005	4.805
.9980	—	—	—	—	—	—	.6987	5.220
.9978	—	—	—	—	—	—	.6984	5.389
.9975	—	—	—	—	.6046	3.432		
.9950	—	—	.5160	2.690	.5978	3.758		
.9935	—	—	—	—	.5961	3.986		
.9900	.4307	2.271	.4997	2.876				
.9854	—	—	.4927	3.139				
.9800	.4015	2.379						
.9720	.3888	2.570						

ρ'/ρ → / $(V/V_s)^2$ ↓	0.8		0.9		1.0	
	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0	μ'/μ	w_0/u_0
1.00000	.8013	6.648	.90005	14.73	1.00000	∞
.99997	—	—	.89997	16.27		
.9998	.8000	7.485				
.9995	.7996	8.141				

Orbit

When the characteristic equation (6) holds, only three of the four equations in the expression (5) are independent, so we selected the upper three.

$$\begin{aligned} &ifA-sB-ifA'-s'B'=0\,, \\ &rA+ifB+r'A'-ifB'=0\,, \\ &i2\mu rfA-\mu(f^2+s^2)B+i2\mu'r'fA'+\mu'(f^2+s'^2)B'=0\,. \end{aligned} \tag{7}$$

The amplitude constants A, B and A', B' can be then written as follows:

$$\begin{aligned} A&=K\Delta_A\,, \quad B=K\Delta_B\,, \\ A'&=K\Delta_{A'}\,, \quad B'=K\Delta_{B'}\,. \end{aligned} \tag{8}$$

In these expressions Δ_A, Δ_B and $\Delta_{A'}$, $\Delta_{B'}$ are the minor determinants made by the coefficients of A, B and A', B' in the equation (7). They may be written as follows:

$$\begin{aligned} \Delta_A&=\mu(f^2-r's')(f^2+s^2)-\mu'(f^2+r's)(f^2+s'^2)+2\mu'r'f^2(s+s')\,, \\ \Delta_B&=-if\{\mu'(r+r')(f^2+s'^2)-2\mu r(f^2-r's')-2\mu'r'(f^2+rs')\}\,, \\ \Delta_{A'}&=\mu(f^2+rs')(f^2+s^2)-\mu'(f^2-rs)(f^2+s'^2)-2\mu rf^2(s+s')\,, \\ \Delta_{B'}&=-if\{\mu(r+r')(f^2+s^2)-2\mu'r'(f^2-rs)-2\mu r(f^2+r's)\}\,. \end{aligned} \tag{9}$$

Using these expressions, we can describe the displacements in both media as follows:

$$\begin{aligned} u&=K(\Delta_A\, if\, e^{-rz}-\Delta_B\, s\, e^{-sz})\cdot\exp(ifx-ipt)\,, \\ w&=K(-\Delta_A\, r\, e^{-rz}-\Delta_B\, if\, e^{-sz})\cdot\exp(ifx-ipt)\,, \\ u'&=K(\Delta_{A'}\, if\, e^{r'z}+\Delta_{B'}\, s'\, e^{s'z})\cdot\exp(ifx-ipt)\,, \\ w'&=K(\Delta_{A'}\, r'\, e^{r'z}-\Delta_{B'}\, if\, e^{s'z})\cdot\exp(ifx-ipt)\,. \end{aligned} \tag{10}$$

Taking the real part,

$$\begin{aligned} u&=K(\Delta_A f e^{-rz}+\Delta_B\, is\, e^{-sz})\sin(pt-fx)\,, \\ w&=K(-\Delta_A\, r\, e^{-rz}-\Delta_B\, if\, e^{-sz})\cos(pt-fx)\,, \\ u'&=K(\Delta_{A'} f e^{r'z}-\Delta_{B'}\, is'\, e^{s'z})\sin(pt-fx)\,, \\ w'&=K(\Delta_{A'}\, r'\, e^{r'z}-\Delta_{B'}\, if\, e^{s'z})\cos(pt-fx). \end{aligned} \tag{11}$$

The ratio of the displacements u and w at $z=0$ is

$$\frac{w_0}{u_0}=\frac{r_1(1-r_1's_1')(V/V_s)^2+r_1'(1-r_1s_1)(\rho'/\rho)(V/V_s)^2}{2(1-\mu'/\mu)(1-r_1s_1)(1-r_1's_1')-(1-r_1's_1')(V/V_s)^2+(1-r_1s_1)(\rho'/\rho)(V/V_s)^2}, \tag{12}$$

where the quantities with suffix 1 have the following meaning:

$$r_1=r/f=\sqrt{1-V^2/\gamma^2V_s^2}, \qquad s_1=s/f=\sqrt{1-(V/V_s)^2},$$
$$r_1'=r'/f=\sqrt{1-(V^2/\gamma'^2V_s^2)(\rho'/\rho)(\mu/\mu')}, \quad s_1'=s'/f=\sqrt{1-(V/V_s)^2(\rho'/\rho)(\mu/\mu')}, \tag{13}$$

(when $\lambda=\mu$ and $\lambda'=\mu'$, $\gamma^2=\gamma'^2=3$)

We put various values of ρ'/ρ, μ'/μ and $(V/V_s)^2$ which satisfy the condition (6) into the above expressions and performed the calculation. The results are shown in Fig. 3. In this figure, the ordinate is the ratio w_0/u_0, the abscissa is the square of the velocity of the Stoneley wave measured by the unit V_s, while the parameter is the ratio of the densities of the two media. The numerical values are tabulated in Table I. From the sign of the ratio we can infer the direction of the rotation of orbit which is shown in Fig. 4.

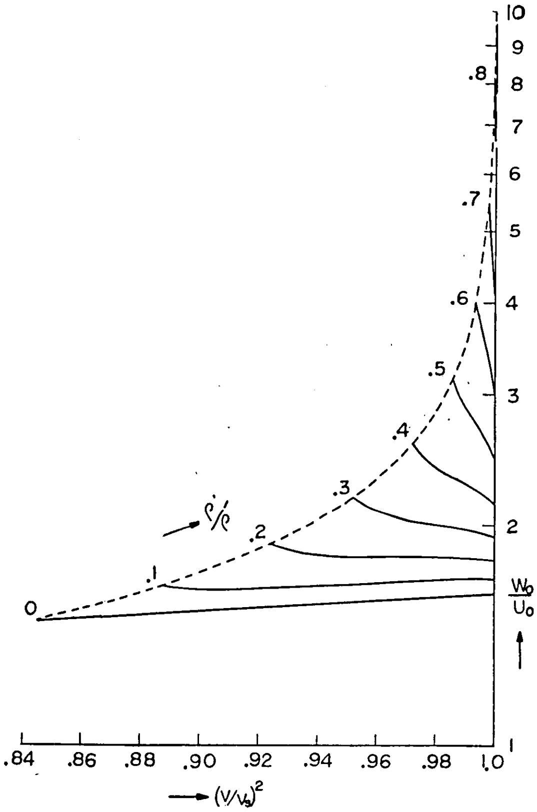

Fig. 3. Ratio of the amplitude of vertical and horizontal displacements at the boundary surface. Numbers beside the broken line are the parameter ρ'/ρ. On the broken line V is equal to V_s'.

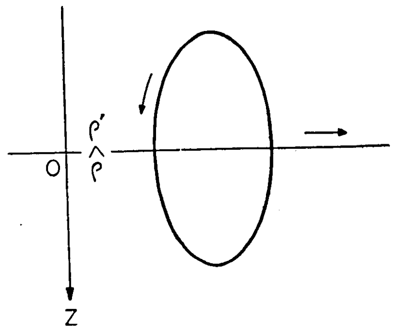

Fig. 4. The direction of the rotation of particle orbit at the surface of separation of two media.

Distribution of Amplitude

We also studied the distribution of amplitude at different depths. Using the equation (11), the horizontal and vertical displacements in both media are calculated corresponding to the given values of z/L. (L implies wave length.) The results are shown in Figs. 5–13. (In these figures the vertical displacement at $z=0$ is taken as unit.)

Conclusion

1. In the previous sections we have tried to elucidate the relation between ρ'/ρ, μ'/μ and $(V/V_s)^2$. If two of the above three quantities are given, we can obtain the remaining one using Fig. 2. This study also makes clear that the Rayleigh wave (the velocity of which is equal to $\sqrt{0.8453\cdots\cdots}\cdot V_s$) is the limiting case of the Stoneley wave in which the values ρ'/ρ and μ'/μ both tend towards zero. (See Fig. 2.)

2. We next considered the ratio of the amplitude of vertical and horizontal displacements at $z=0$. When $(V/V_s)^2=0.8453\cdots$ the ratio w_0/u_0 gives the minimum value and is equal to that of the Rayleigh wave ($1.468\cdots$). As the parameter ρ'/ρ increases, so does the ratio w_0/u_0. These properties can be seen from Fig. 3. When the parameter ρ'/ρ approaches to unity the change of the value of w_0/u_0 becomes very remarkable, because the horizontal displacement becomes small and finally vanishes when ρ' becomes equal to ρ.

3. The sence of the rotation of particle orbit at the surface of separation of two media is determined easily by means of the expression (11). With respect to the denser medium (the lower one), the sense of rotation is the same with that of the Rayleigh wave, consequently it is reverse with regard to the upper medium. (See Fig. 4.)

In the aspect of the distribution of the amplitude, we also find a feature similar to that of the Rayleigh wave in the lower denser medium, excluding the case when V/V_s is nearly equal to 1. Of course the Rayleigh wave is the limiting case of $\rho'/\rho\to 0$ and $\mu'/\mu\to 0$ (V tends towards V_s'). (See Figs. 2 and 11.) The general tendency can be seen in Figs. 5–13.

4. Lastly we will consider the significance of the Stoneley wave a little. The Stoneley wave is a wave propagated along the surface of separation of two semi-infinite medium and has the type $\left[\frac{EE}{EE}\right]$[4].

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4) Y. Satô, "Study on Surface Waves XI. Definition and Classification of Surface Waves," *Bull. Earthq. Res. Inst.*, **32** (1954), 161.

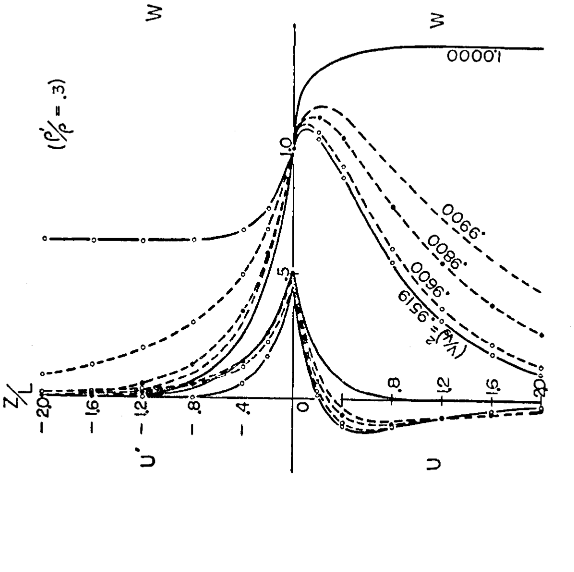

Fig. 5.

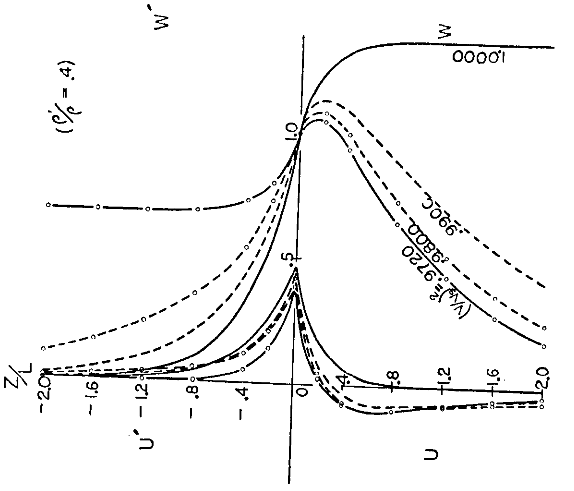

Fig. 6.

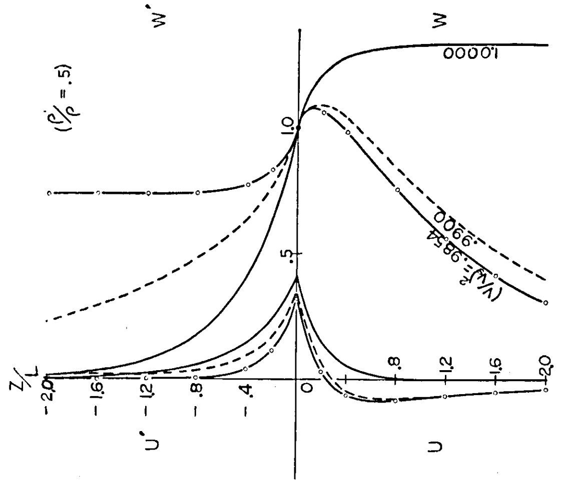

Fig. 7.

Distribution of the amplitude U, U' (horizontal component) and W, W' (vertical component) when $\rho'/\rho=0.3$, 0.4 and 0.5. In these figures the vertical displeasement at $z=0$ is taken as unit.

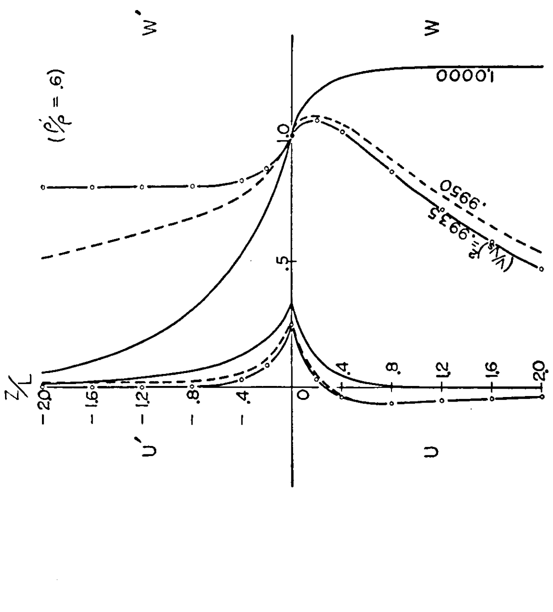

Fig. 8.

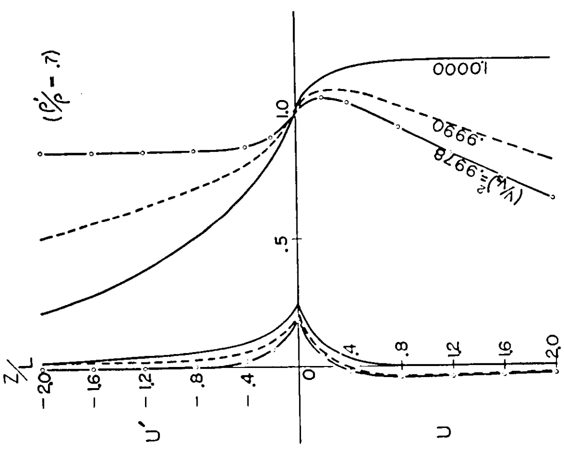

Fig. 9.

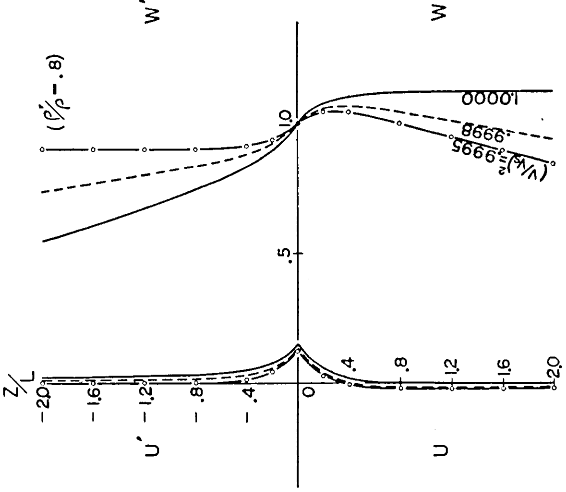

Fig. 10

Distribution of amplitude U, U' (horizontal component) and W, W' (vertical component) when $\rho'/\rho=0.6$, 0.7 and 0.8. In these figures the vertical displacement at $z=0$ is taken as unit.

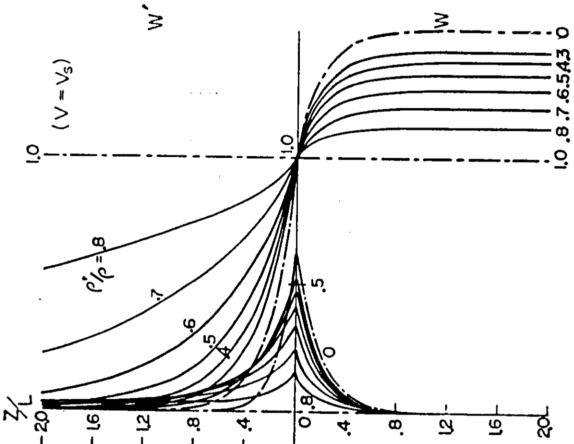

Fig. 13.

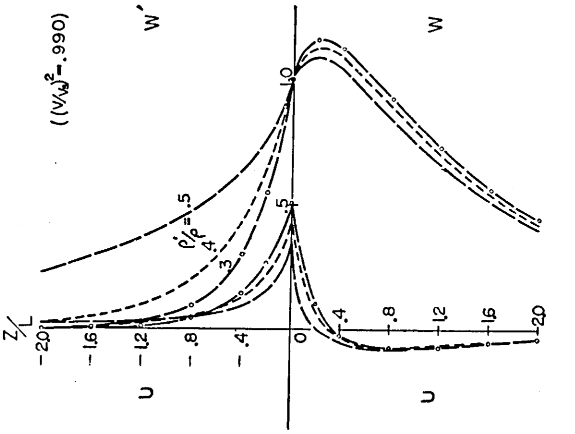

Fig. 12.

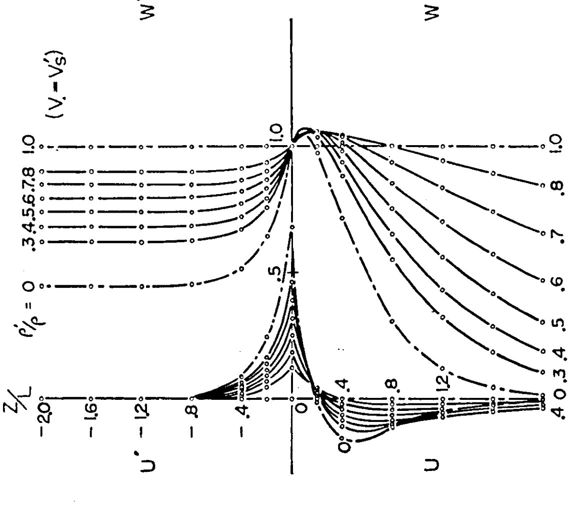

Fig. 11.

Distribution of amplitude U, U' (horizontal component) and W, W' (vertical component) when $V=V_s{}'$, $(V/V_s)^2=0.990$ and $V=V_s$.
In these figures the vertical displacement at $z=0$ is taken as unit.

Consequently it cannot practically exist in the strict sense. Therefore, when the term "Stoneley wave" is used with reference to a stratified medium, the authors are of opinion that it should stand for the phenomenon that approximately shows the properties of the Stoneley wave, because the thickness of the layer is sufficiently large compared with the wave length. Such a wave, however, can hardly be observed unless we install the measuring instruments deeply in the ground.

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35. Stoneley 波——その速度，軌道及び振巾の分布

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1. 今日 Stoneley 波と名づけられている表面波の可能性について，初めて論文が書かれてから既に 30 年以上を経過する．

その後，妹沢・金井らによつて存在限界についての議論はなされたが，速度・軌道・振幅の分布等については未だに手が着けられていない．本研究はこの欠を補う為になされたものである．

2. 速度．速度方程式は既に得られているが，この式によつて決定される ρ'/ρ, μ'/μ と $(V/V_S)^2$ の関係を求めた．(ρ, ρ'; μ, μ'; V_S は慣用の意味を持つ． V は Stoneley 波の速さ．第 1 図参照.) 即ち，存在限界の範囲内において，ρ'/ρ と μ'/μ を与えれば V/V_S が，ρ'/ρ, μ'/μ の一方と V/V_S とを与えれば残りの一つの量が求められる．(第 2 図.)

3. 軌道．速度方程式が成立つ場合，境界面における変位は容易に計算する事ができる．上下，水平両成分の振幅を計算して，その比 (第 3 図) 及び軌道の回転方向 (第 4 図) を求めた．

上下動と水平動の比は $\rho'/\rho\to 0$, $\mu'/\mu\to 0$ の時 (即ち Rayleigh 波) が最も小さくて 1.468･･･,他の場合にはこれより大きくなる． ρ'/ρ が大きくなるとこの値は急激に増し，又 V/V_S の変化に伴う変動も急になる．軌道の回転方向は密度の大きい方の媒質に関して常にレーリー波と同じ，従つて軽い方の媒質に関しては逆になる．

4. 振幅の分布．既に求めた計算式によつて，境界面から離れるに伴つて振幅が変る状態をも求める事ができる．(第 5～13 図) 図に見られるように，V/V_S が 1 に極めて近い場合をのぞき重い方の媒質内の分布は，Rayleigh 波と似た点がすくなくない．当然の事ながら，$\rho'/\rho\to 0$, $\mu'/\mu\to 0$ とすると Rayleigh 波の振幅分布に近づく．(第 11 図)

5. Stoneley 波の意義．Stoneley 波は二つの半無限体の境界にそつて伝わる波であり，$\left[\frac{EE}{EE}\right]$ なる型を持つ．従つて厳密な意味で現実には成立ちがたい．時に表面層を持つ構造内に於いても "Stoneley 波" なる言葉が使用されるが，これは層が十分あつくて，自由表面の影響が無視しうる時に，近似的に成立つ現象に附せらるべき名称と解すべきであろう．但し，観測は地下深い境界面の附近に限られ，一般にはその機会に乏しいと考えられる．