36. **On the Plastic Properties of the Earth’s Core.**

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1. **Introduction.**

It seems that the Earth's core transmits only longitudinal waves, not transverse waves. It is however uncertain whether this results from the extremely small torsion modulus of the material within the core or from particularly high viscous resistance of the same material against distortional movement. Since, as a matter of fact, the Earth’s core is subjected to enormously high pressure, the material in that core is probably in a certain plastic state. The special features of a general plastic body are its increased viscous resistance against distortional force (for any quickness) and its almost incompressible condition in any (statical) deformation, whence it follows from either one of these features of the plastic body that, were the Earth's core a plastic body, transmission of transverse waves (relatively long waves) through the same core would be improbable.

Now it is well known that the velocity of the core waves is about \(1/\sqrt{3}\) of that of the longitudinal waves transmitted through the rocky shell immediately next to the core. If the material within the core were in a plastic state, and that in the rocky shell is in an elastic state, both being of the same density, the relation cited above would then be a most likely one, regardless of the denser character of the Earth’s core. Whether or not the material forming the core is as dense as iron or nickel is another problem calling for different treatment.

As already mentioned, we shall assume that, although the Earth's central core is in a plastic condition, the outer shell is in a purely elastic state, the density of the material being assumed uniform throughout the Earth. Gravity is taken as a body force, but the effect of the Earth’s rotation is neglected. Although there are a number of theories defining the plastic state of materials, yet in accordance with the present vogue among investigators of plastic bodies, we shall take the maximum shear stress theory, which was originated by Saint Venant.  \[1\]

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1) St. Venant, *C. R.*, 70 (1870), 73 (1871).
and developed by certain mathematicians eminent in applied mechanics\(^2\). Although the criteria of the plastic state differ with authors, as the quantitative difference between them is not very marked, we shall use the maximum shear stress theory throughout this paper.

2. *Equilibrium of the elastic shell.*

Assuming that the material displacement is invariably symmetrical with respect to the Earth’s centre, that is, that displacement \(u\) is perfectly radial, the strain and stress components are then expressed by

\[
\begin{align*}
\varepsilon_r &= \frac{\partial u}{\partial r}, \quad \varepsilon_\theta = \varepsilon_r = \frac{u}{r}, \quad \Delta = \frac{\partial^2 u}{\partial r^2} + \frac{2u}{r}, \\
\sigma_r &= \lambda \Delta + 2\mu \frac{\partial u}{\partial r} = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{\partial^2 u}{\partial r^2}, \\
\sigma_\theta &= \frac{\partial u}{\partial r} = \lambda \Delta + 2\mu \frac{u}{r} = \lambda \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) + 2\mu \frac{u}{r},
\end{align*}
\]

where \(\lambda, \mu\) are Lamé’s elastic constants. The gravitational potential \(V\) and the resulting radial pressure \(p\) are connected by the formulae

\[
\begin{align*}
V &= \frac{2}{3} \pi \tau \rho (3a^2 - r^2), \quad -\frac{1}{p} \frac{\partial p}{\partial r} + \frac{\partial V}{\partial r} = 0, \\
\frac{\partial p}{\partial r} &= -\frac{4}{3} \pi \tau \rho \varrho r, \quad p = \frac{2}{3} \pi \tau \rho \varrho (a^2 - r^2),
\end{align*}
\]

where \(a\) is the Earth’s radius, \(\rho\) the Earth’s mean density (assumed almost uniform), and \(\tau\) the gravitational constant. The equation of equilibrium of the elastic shell is

\[
\rho \frac{\partial V}{\partial r} + \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \left( 2\sigma_r - \sigma_\theta - \sigma_\phi \right) = 0,
\]

the effect of pressure on the change of density being neglected. Substituting from (1), (2) in (3), we obtain

\[
-\frac{4}{3} \pi \tau \rho \varrho r + (\lambda + 2\mu) \left( \frac{\partial^2 u}{\partial r^2} + \frac{2u}{r} \frac{\partial u}{\partial r} - \frac{2u}{r^2} \right) = 0.
\]

The solution for the present statical problem is then

\[
u = \frac{2}{15} \frac{\pi \tau \rho ^2}{\lambda + 2\mu} r^3 + Br + \frac{C}{r^2},
\]

\(^2\) A. NADAI, *Der bildsame Zustand der Werkstoffe* (Berlin, 1927), *Handbuch d. Physik*, 6, Plastizität u. Erdruck. Nadai’s problem and various plastic theories dealt by other people are involved in these books.
where \( B, C \) are determined by boundary conditions. The stress components are

\[
\begin{align*}
\tilde{r} & = \frac{\pi \mu \rho^2 (10 \lambda + 12 \mu)}{15 (\lambda + 2 \mu)} r^2 + \left( 3 \lambda + 2 \mu \right) B - \frac{4 \mu C}{r^3}, \\
\partial \theta = \phi & = \frac{\pi \mu \rho^2 (10 \lambda + 4 \mu)}{15 (\lambda + 2 \mu)} r^2 + \left( 3 \lambda + 2 \mu \right) B + \frac{2 \mu C}{r^3}.
\end{align*}
\]

(6)

The boundary conditions are such that the normal stress is zero at the free surface, while the maximum shear stress at the boundary between the shell and the core assumes a definite value \( k \). Mathematically these conditions are expressed by

\[
\begin{align*}
r = a; & \quad \tilde{r} = 0, \\
r = b; & \quad \tilde{r} - \frac{\partial \theta}{\partial r} = \pm 2k, \quad \tilde{r} - \phi = \pm 2k,
\end{align*}
\]

(7)

where \( b \) is the radius of the Earth's core. An additional condition is

\[
r = c (a > c > b); \quad |\tilde{r} - \frac{\partial \theta}{\partial r}| \geq 2k, \quad |\tilde{r} - \phi| \geq 2k.
\]

(8)

After reading this paper, these conditions were found to be rather needless.

Substituting (6) in (7) we obtain

\[
\begin{align*}
\frac{6 \mu C}{b^3} = & \frac{8}{15} \pi \mu \rho^2 \frac{\mu}{\lambda + 2 \mu} b^2 \pm 2k, \\
(3 \lambda + 2 \mu) B = & \frac{-\pi \mu \rho^2 a^2}{15 (\lambda + 2 \mu)} \left\{ \frac{10 \lambda + 12 \mu}{3 a^5} - \frac{16 \mu b^5}{3 a^5} \right\} \pm \frac{4}{3} \frac{k b^3}{a^3}.
\end{align*}
\]

(9)

The stress components are therefore

\[
\begin{align*}
\tilde{r} & = \frac{-\pi \mu \rho^2 (10 \lambda + 12 \mu)}{15 (\lambda + 2 \mu)} (a^2 - r^2) + \frac{2}{3} b^3 \left( \frac{1}{a^3} - \frac{1}{r^3} \right) \left\{ \frac{8 \pi \mu \rho^2 \mu}{15 (\lambda + 2 \mu)} b^2 \pm 2k \right\}, \\
\partial \theta = \phi & = \frac{-\pi \mu \rho^2 (10 \lambda + 12 \mu)}{15 (\lambda + 2 \mu)} a^2 + \frac{\pi \mu \rho^2 (10 \lambda + 4 \mu)}{15 (\lambda + 2 \mu)} r^2 \\
& \quad + \left( \frac{2}{3} b^3 + \frac{b^3}{3 r^3} \right) \left\{ \frac{8 \pi \mu \rho^2 \mu}{15 (\lambda + 2 \mu)} b^2 \pm 2k \right\}.
\end{align*}
\]

(10)

It is to be borne in mind that the stresses in (10) should satisfy the condition (8).

In the special case \( \mu/\lambda \rightarrow 0 \), equations in (10) reduce to

\[
\begin{align*}
\tilde{r} & = \frac{-2 \pi \mu \rho^2}{3} (a^2 - r^2) \pm \frac{4}{3} \frac{k b^3}{a^3} \left( \frac{1}{a^3} - \frac{1}{r^3} \right), \\
\partial \theta = \phi & = \frac{-2 \pi \mu \rho^2}{3} (a^2 - r^2) \pm \frac{4}{3} \frac{k b^3}{a^3} \left( \frac{1}{a^3} + \frac{1}{2 r^3} \right).
\end{align*}
\]

(11)
3. The equilibrium of the plastic core.

The equation of the equilibrium of the plastic core is the same as (3), namely,
\[- \frac{4}{3} \pi \rho^2 r + \frac{\delta \hat{r}}{\delta r} + \frac{1}{r} \left( 2 \hat{r} - \hat{\theta} - \hat{\phi} \right) = 0,\]
(3')
where \( \hat{\theta} = \hat{\phi} \), and the equation of the plastic condition is
\[\hat{r}r - \hat{\theta} = \mp 2k\]
(12)
for any \( r (r < b) \). From (3'), (12) we obtain
\[- \frac{4}{3} \pi \rho^2 r + \frac{\delta \hat{r}}{\delta r} + \frac{4k}{r} = 0.\]
(13)
Integrating this we get
\[\hat{r}r + 4k \log_e r - \frac{2}{3} \pi \rho^2 r^2 + I = 0,\]
(14)
where \( I \) is the integration constant. From (10) the boundary condition is
\[r = b; \quad \hat{r} = -\frac{10 \lambda + 12 \mu}{15(\lambda + 2 \mu)} (a^2 - b^2)\]
\[+ \frac{2}{3} \left( \frac{b^3}{a^3} + 1 \right) \left( \frac{8}{15} \pi \rho^2 \frac{\mu}{\lambda + 2 \mu} b^2 + 2k \right).\]
(15)
Determining \( I \) in (14) by means of (15), we obtain
\[\hat{r} = \pm 4k \log_e \frac{r}{b} - \frac{2}{3} \pi \rho^2 (b^2 - r^2) - \frac{10 \lambda + 12 \mu}{15(\lambda + 2 \mu)} (a^2 - b^2)\]
\[+ \frac{2}{3} \left( \frac{b^3}{a^3} + 1 \right) \left( \frac{8}{15} \pi \rho^2 \frac{\mu}{\lambda + 2 \mu} b^2 + 2k \right).\]
(16)
\[\hat{\theta} = -\hat{\phi} = \pm 2k + \hat{r}.\]

When \( \mu / \lambda \to 0 \), these equations reduce to
\[\hat{r} = \pm 4k \log_e \frac{r}{b} - \frac{2}{3} \pi \rho^2 (a^2 - r^2) + \frac{4}{3} k \left( \frac{b^3}{a^3} + 1 \right),\]
\[\hat{\theta} = -\hat{\phi} = \pm 2k + \hat{r}.\]
(17)
In the case of the Earth's core the upper signs in these solutions should always be used.

By comparing (11) and (17), we find that when \( \mu / \lambda \to 0 \), the additional stresses in the core are
\[
\pm 4k \log_e \frac{r}{b} \pm \frac{4}{3} k \left(1 + \frac{b^3}{r^3}\right), \quad \pm 4k \log_e \frac{r}{b} \pm \frac{4}{3} k \left(1 - \frac{b^3}{2r^3}\right)
\]

for \( \theta \) and \( \phi \) respectively.

4. Possible slip planes in the plastic core.

The angle \( \beta \) which the slip surfaces in the plastic core make with any radius of the Earth is obtained by the formula

\[\tan 2\beta = \frac{\theta - \varphi}{2r \theta} \quad (19)\]

where \( \theta \) is the shear stress between \( r \) and \( \theta \), and \( \varphi \) that between \( r \) and \( \phi \).

Since \( \theta = 0 (\varphi = 0) \) in the present case, we get \( 2\beta = \pi / 2 \), so that

\[\beta = \pi / 4\]

namely \( \tan \beta = 1 \). The equation of the slip surfaces is such that

\[\frac{r \theta}{dr} \left(= \frac{r \phi}{dr}\right) = \tan \beta\]

from which we have

\[\theta = A \log_e r, \quad \phi = B \log_e r\]

The surfaces are therefore of the type of logarithmic spiral. It is immaterial whether the sections of the surfaces are directed in the sense of \( \theta \) or \( \phi \) or in that of a certain azimuth between \( \theta \) and \( \phi \).

5. The numerical solution and its interpretation.

The approximate values of the radii of the Earth and its core are given by \( a = 6370 \) km, \( a - b = 2900 \) km, so that \( b/a = 0.545 \). It is possible to assume \( \gamma_r a^2 = 8.10^{11} \) in C.G.S. units. The value of the plastic constant \( k \) of the Earth's core is not known. According to Griggs's tests\(^3\) of solenhofen limestone and marble under high hydrostatic pressure, the yield points due to a longitudinal force do not differ markedly, though increasing somewhat, in the hydrostatic pressure applied.

(i) If we assume that the materials in the core and in the shell immediately next to it have plastic constants of the same value as their respective plastic constants under atmospheric pressure (the actual temperature and melting point being both raised), we may then approximately put

\[\text{3) D. T. Griggs, Journ. Geol., 44 (1936), 541~577.}\]
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2\(k=5.10^9\) (C. G. S.). \hspace{1cm} (23)

Thus, from (9)

\[
\frac{6\mu C}{b^3} = 5.10^9 + 4.10^{11} \frac{\mu}{\lambda},
\]

\[
(3\lambda + 2\mu)B = 0.436.10^{11} \frac{\mu}{\lambda} - 16.8.10^{11},
\]

from which \(\mu C\) and \((3\lambda + 2\mu)B\) are determined as functions of \(\mu/\lambda\). In both equations of (24), \(\mu/\lambda\) is assumed to be relatively small. Comparing these results with the condition (8), we get

\[
\frac{\mu}{\lambda} < 1.36.10^{-2}.
\]

(25)

The result in (24) is therefore

\[
\frac{6\mu C}{b^3} < 10.4.10^9, \hspace{1cm} (C > 0)
\]

\[
(3\lambda + 2\mu)B > -16.8.10^{11}. \hspace{1cm} (B < 0)
\]

(26)

The condition for the Poisson's ratio is

\[
\frac{1}{2} > \sigma > 0.493.
\]

(27)

The last condition has resulted from the assumption (23). Although in the present calculation the value of \(2k\) has been given, no assumption regarding the absolute values of \(\lambda\) and \(\mu\) has been made. Although the result well conforms with the properties of a general plastic body and also with the nature of seismic core waves, since the radius of the Earth and that of its core are of comparable order and not so much different as in the case of Mars\(^{40}\), there still remains some question as to whether the Earth's state shall change from elastic to plastic just on its core's boundary.

(ii) The analysis of seismic waves shows that \(\sigma = 1/4\) even at such a deep part of the Earth's shell as immediately next to the core. Thus, if we assume that Poisson's ratio \(\sigma\) is \(1/4\), namely \(\lambda = \mu\), in the rocky shell as well as in the core, we then have

\[
\frac{6\mu C}{b^3} = 1.33.10^{11} + 2k,
\]

\[
(3\lambda + 2\mu)\beta = -12.3.10^{11} + 0.213 \hspace{1cm} k,
\]

(28)

from which \(\mu C\) and \((3\lambda + 2\mu)B\) are determined as functions of \(k\). Re-

ferring to the condition (8), we get

\[ 2k > 1.22 \times 10^{11}. \]

The result in (28) is therefore

\[
\begin{aligned}
\frac{6\mu C}{b^3} &> 2.55 \times 10^{11}, \\
(3\lambda + 2\mu)B &> -12.1 \times 10^{11}.
\end{aligned}
\]  \( (30) \)

From the first hypothesis it follows that the material in the rocky shell, owing to some transformation of that material, changes its state from compressible to incompressible in the immediate vicinity of the boundary of the core, that is, \( \sigma \) changes from \( 1/4 \) to \( 1/2 \), the other properties of the same material changing from elastic to plastic also in the same vicinity. An alternative explanation resulting from the second hypothesis, though impossible to confirm it even with the data of high pressure experiments that have been made, would be such that while the plastic constant \( k \) in a fairly deep part in the Earth is as large as \( 2k = 10^{11} \) (C. G. S.), the Poisson's ratio remains nearly \( \sigma = 1/4 \) even within the same core. It is now possible to assume that the actual condition of the Earth's core (and the shell next to it) is probably intermediating between that in (i) and that in (ii).

Added July 7, 1937.—From our continued study (the paper read July 6, 1937), it was found that the condition (8) is rather needless. The plastic state of the core is naturally accompanied by the plastic state of the Earth's surface stratum of thickness of a few hundred kilometers, the layer intermediating between the surface stratum in question and the core being possibly elastic. The full explanation will be shown in next Bulletin.
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地球の核には弾性のみが通って横波が通らねというある。しかし核の物質の剛性が極めて小さい何かであるか、或いは振動を受けて核性が特に小さく変わるかは不明である。地球の核が高圧を受けてる鉱にプラスチックの状態にある事は明らかである。一般のプラスチック側の特性の一つは振るの力に対して核性が強く働く事であり、他の一つは非圧縮性である。従って地球の核がプラスチックであるとすれば横波が通る難しい事は当然なように思われる。核のみがプラスチックであるのを通って非圧縮性であるとすると、核の内部とその外部との波の速度の差も説明できるようである。

今、地球の核のみがプラスチックであるとし、仮に波を考慮にあてて理論的計算を試みて見ると、プラスチックの理論は最もよく用いられて来る要因の深大見通力を的を用いて計算する。他の理論でも数値の程度はそれぞれには遠いよう。核の中ではすべての状況が詳細毎仮の性質を備える事は容易に説明される。

理論的計算をあって見ると、核のある実験結果に従い、プラスチックの常数が地球の核の近辺でし別に弾性の性質を同じであると仮定して見ると（温度の低い方、傾斜が斜めに増加するから想定して置いた）、核の性質は非圧縮性でなければならない事がわかるのである。之は一般のプラスチック側と共通の性質であり、且つ地球の核を通る地震波の性質とも一致しており、一寸面白しという方が、これは実際の理論からいふと、弾性からプラスチック側に変する事が、必ずしも実際の核の面でなくてよい事の自由さが残されをるものである。之は地球の半径とその核の半径が同じ程度の値を持って居り、火星のそれ等のやや近達がないからである。それで別に考へて Poisson 比が核の中まで 1/4 であることを問題を取扱ふ事もやって見たのである。

そのようなにして考察して見ると、地球の核の面が柄度弾性からプラスチックに変わる面であるとすると、プラスチックの常数が前に仮定した地震波面での値よりも 100 倍も大きくなければならぬ事が出るのである。実際の上高らいない圧力で実験でも常数が増加する傾向にはあるが、そのような高圧の下での実験結果は未だがないから、ある大きなプラスチック性はおそらくは断言できない。しかし如何に大き過ぎるようなに見える事は事実である。

質疑問題としては、地球の核及びその附近では Poisson 比が 1/4 は 1/2 の間にあり、プラスチックの常数は地震波面で測定したそれの何倍か又は何十倍かのものであるから考へ方の方がよいのであらうか。

1937 年 7 月 7 日述記—この研究を進めた結果（7月 6 日発表）、核のプラスチック性を前記のやうに大きく取る、普通の値にすると、地球の表面に数百鉄の厚さのプラスチック性の層があり、この層と地球の核との間には弾性であればよい事がわかった。この方向は反面種類の事柄をよく説明するやうである。要する事は次の報告で述べるべき。