On a conjecture of L. Solomon

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(Communicated by N. Iwahori)

§ 1. Introduction and statement of result

Let K be a field, let E be an n-dimensional vector space over K and let G be a finite group of linear transformations of E. Let g_r be the number of elements of G with an $(n-r)$-dimensional fixed-point set. Now we will consider the polynomial $P(t)=\sum_r g_r t^r$. It is well known that $P(t)$ is the product of linear factors of the form $1+mt$ (m is a natural number) when K is the set of real numbers and when G is a group generated by reflections. Solomon [1] proved that $P(t)$ is not in general the product of such linear factors when K is a finite field F_q (or F_{q^2}) and when G is a symplectic group, or an orthogonal group (or a unitary group), and also Solomon [1] conjectured that $P(t)$ has linear factors of the form $1+q^i t$ (i is a natural number).

In this paper we prove the following three theorems, and they show that the conjecture of Solomon [1] is true.

THEOREM 1. *Let E be an n-dimensional vector space over F_{q^2} with a nonsingular sesquilinear form which is hermitian with respect to the automorphism $\alpha\to\alpha^q$ of F_{q^2}. Let $G(n)=U(n,q^2)$ be the unitary group and let $g_r(n)$ be the number of elements of $G(n)$ with an $(n-r)$-dimensional fixed-point set. Then the polynomial $P_n(t)=\sum_r g_r(n)t^r$ is given by the following recurrence formula*

$$P_0(t)=1,$$

$$(1)\qquad P_{2m+1}(t)=q^{2m}(q^{2m+1}+1)tP_{2m}(t)+\prod_{j=0}^{m}(1-q^{2j}t)\prod_{j=1}^{m}(1+q^{2j-1}t),$$

$$(2)\qquad P_{2m}(t)=q^{2m-1}(q^{2m}-1)tP_{2m-1}(t)+\prod_{j=0}^{m-1}(1-q^{2j}t)\prod_{j=1}^{m}(1+q^{2j-1}t).$$

Furthermore if v is the index of the sesquilinear bilinear form, $P_n(t)$ has $n-v$ linear factors of the form $1+q^i t$ $(i=1,3,\dots,2(n-v)-1)$.

THEOREM 2. *Let E be an n-dimensional vector space (n is even) over F_q with a nonsingular alternating bilinear form. Let $G(n)=Sp(n,q)$ be the symplectic group* *and let $g_r(n)$ be the number of elements of $G(n)$ with an $(n-r)$-dimensional fixed-point set. Then the polynomial $P_n(t)=\sum_r g_r(n)t^r$ is given by the following recurrence formula.*

$$P_0(t)=1,$$

$$P_{2m+2}(t)=q^{2m+1}(q^{2m+2}-1)t^2P_{2m}(t)+(1+q^{2m+2}t)\prod_{j=0}^{m}(1-q^jt)\prod_{j=1}^{m}(1+q^jt).$$

Furthermore if v is the index of the alternating bilinear form, $P_n(t)$ has $n-v$ linear factors of the form $1+q^it$ $(i=1,2,\dots,n-v)$.

THEOREM 3. *Let E be an n-dimensional vector space over F_q (char $F_q\neq 2$) with a nonsingular symmetric bilinear form Φ, and let $G(n,\Phi)=O(n,q,\Phi)$ be the orthogonal group which leaves Φ invariant. Nonsingular symmetric bilinear forms on E are classified into four types according to the following scheme:*

Type	n	*Discriminant*	*Index*
1	*odd*	$(-1)^{(n-1)/2}$	$(n-1)/2$
2	*odd*	$(-1)^{(n-1)/2}w$	$(n-1)/2$
3	*even*	$(-1)^{n/2}$	$n/2$
4	*even*	$(-1)^{n/2}$	$n/2-1$

where w is a nonsquare in F_q.

Let $g_r(n,i)$ $(i=1,2,3,4)$ be the number of elements of $G(n,\Phi)$, where Φ is of type i, with an $(n-r)$-dimensional fixed-point set. Then the polynomial $P_n^i(t)=\sum_r g_r(n,i)t^r$ is given by the following recurrence formula.

(1) $$\begin{cases} P_1^1(t)=1+t, \\ P_{2m+1}^1(t)=q^{2m-1}(q^{2m}-1)t^2P_{2m-1}^1(t)+(1+q^{2m}t)\prod_{j=0}^{m-1}(1-q^jt)\prod_{j=0}^{m-1}(1+q^jt), \end{cases}$$

(2) $$P_n^2(t)=P_n^1(t),$$

(3) $$\begin{cases} P_0^3(t)=1, \\ P_{2m+2}^3(t)=q^{2m}(q^{m+1}-1)(q^m+1)t^2P_{2m}^3(t)+(1+q^{2m+1}t)\prod_{j=0}^{m}(1-q^jt)\prod_{j=0}^{m-1}(1+q^jt), \end{cases}$$

(4) $$\begin{cases} P_0^4(t)=0, \\ P_{2m+2}^4(t)=q^{2m}(q^{m+1}+1)(q^m-1)t^2P_{2m}^4(t)+(1+q^{2m+1}t)\prod_{j=0}^{m-1}(1-q^jt)\prod_{j=0}^{m}(1+q^jt). \end{cases}$$

Furthermore if v is the index of the symmetric bilinear form, $P_n(t)$ has $n-v$ linear factors of the form $1+q^it$ $(i=0,1,\dots,n-v-1)$.

§2. The calculation of the polynomial

If S is a finite set, we let $|S|$ denote the number of elements of S. Let F_q^n denote an n-dimensional vector space over F_q, and let G be a subgroup of $GL(F_q^n)$. Now we introduce the following notations.

If B is a subspace of F_q^n,

$$H(G, B)=\{g\in G;\ \text{the restriction of } g \text{ to } B \text{ is identity}\},$$

$$v(n, r)=|\{B;\ B \text{ is an } r\text{-dimensional subspace of } F_q^n\}|,$$

$$X(n, r)=q^{r(r-1)/2}v(n, r)=q^{r(r-1)/2}\frac{\prod_{k=1}^{n}(q^k-1)}{\prod_{k=1}^{r}(q^k-1)\prod_{k=1}^{n-r}(q^k-1)},$$

$$s(G, r)=\sum_{\dim B=r}|H(G, B)|.$$

$g_r(G)$ is the number of elements of G with an $(n-r)$-dimensional fixed-point set.

$$P_G(t)=\sum_{r=0}^{n} g_r(G)t^r.$$

Then we obtain the next proposition.

PROPOSITION 1.

$$P_G(t)=\sum_{r=0}^{n} s(G, r)t^{n-r}\prod_{j=0}^{r-1}(1-q^jt).$$

PROOF. We can easily see that

$$s(G, r)=\sum_{k=r}^{n} g_{n-k}(G)v(k, r). \tag{2.1}$$

On $X(n, r)$ and $v(n, r)$ the next lemma holds.

LEMMA 1.

$$\sum_{r=0}^{n} X(n, r)t^r=\prod_{k=0}^{n-1}(1+q^kt), \tag{1}$$

$$\sum_{r=0}^{n}(-1)^{n+r}X(n, n-r)\prod_{k=0}^{r-1}(1+q^kt)=q^{n(n-1)/2}t^n. \tag{2}$$

COROLLARY OF LEMMA 1. *If the real numbers* $y_0, y_1, \dots, y_n, z_0, z_1, \dots, z_n$ *satisfy the relations* $y_r=\sum_{k=r}^{n} z_k v(k, r)$, $r=0, 1, \dots, n$, *then* $z_k=\sum_{r=k}^{n}(-1)^{r+k}y_r X(r, r-k)$.

PROOF OF LEMMA 1. From $X(n, 0)=1$, $X(n, n)=q^{n(n-1)/2}$ and $X(n+1, r)=X(n, r)+q^nX(n, r-1)$, $r=1, 2, \dots, n$, it can be easily seen by an induction method.

PROOF OF COROLLARY. From (1) of Lemma 1,

$$\sum_{r=0}^{n} y_r q^{r(r-1)/2} t^r = \sum_{k=0}^{n} z_k \prod_{j=0}^{k-1} (1+q^j t).$$

From (2) of Lemma 1,

$$\sum_{r=0}^{n} y_r q^{r(r-1)/2} t^r = \sum_{k=0}^{n} \prod_{j=0}^{k-1} (1+q^j t) \sum_{r=k}^{n} (-1)^{r+k} y_r X(r, r-k).$$

So it is proved.

From (2.1) and Corollary, $g_{n-k}(G) = \sum_{r=k}^{n} (-1)^{r+k} s(G, r) X(r, r-k)$. So

$$\begin{aligned}\sum_{k=0}^{n} g_{n-k}(G) t^{n-k} &= \sum_{r=0}^{n} \sum_{k=0}^{r} (-1)^{r+k} s(G, r) X(r, r-k) t^{n-k} \\ &= \sum_{r=0}^{n} s(G, r) t^{n-r} \sum_{k=0}^{r} (-t)^{r-k} X(r, r-k) \\ &= \sum_{r=0}^{n} s(G, r) t^{n-r} \prod_{j=0}^{r-1} (1-q^j t)\end{aligned}$$

which proves the proposition.

Now we modify the Proposition 1 in order to obtain the recurrence formula. Let $\{n_m\}_{m=1,2,\dots}$ be a strictly increasing sequence of natural numbers, let G_m be a subgroup of $GL(F_q^{n_m})$, and let $D(m, r)$ denote

$$|G_{m+1}|^{-1} s(G_{m+1}, r) - |G_m|^{-1} s(G_m, r).$$

From Proposition 1 we see that

$$\frac{P_{G_{m+1}}(t)}{|G_{m+1}|} - t^{n_{m+1}-n_m} \frac{P_{G_m}(t)}{|G_m|} = \sum_r D(m, r) t^{n_{m+1}-r} \prod_{j=0}^{r-1} (1-q^j t).$$

Now we obtain the next proposition.

PROPOSITION 2.

$$P_{G_{m+1}}(t) = \frac{|G_{m+1}|}{|G_m|} t^{n_{m+1}-n_m} P_{G_m}(t) + D_m(t)$$

where

$$D_m(t) = |G_{m+1}| \sum_r D(m, r) t^{n_{m+1}-r} \prod_{j=0}^{r-1} (1-q^j t).$$

§ 3. The counting argument of $s(G, r)$ in the case when G is a classical group

Let E be an n-dimensional vector space and let Φ be a sesquilinear, symmetric or alternative nonsingular bilinear form on E. Now we will consider a group

$$G=\{g\in GL(E)\,;\,\Phi(gu, gv)=\Phi(u, v) \text{ for all } u, v\in E\}.$$

Here we introduce the following notations.

For a subspace B of E,

$$G(B)=\{f\in GL(B)\,;\,\Phi(gu, gv)=\Phi(u, v) \text{ for all } u, v\in B\},$$
$$G(E, B)=\{g\in G;\ gB\subseteq B\},$$
$$V_r=\{B;\ B \text{ is an } r\text{-dimensional subspace of } E\}.$$

We write $A\perp B$ for the Witt sum of two subspaces A, B of E.

The element of G operates on V_r as a permutation. Let Z_r be a set of representatives for those orbits of V_r under G. Then it can be obviously seen that the number of elements of $H(G, B)$ which is defined in the previous section depends upon the orbit of V_r to which B belongs, and that the number of elements in the orbit of B under G is $|G|\cdot|G(E, B)|^{-1}$. So

$$s(G, r)=|G|\sum_{B\in Z_r}|G(E, B)|^{-1}|H(G, B)|.$$

But there is a homomorphism $G(E, B)\rightarrow G(B)$ defined by restriction of $G(E, B)$ to B. By Witt's theorem this is an epimorphism and the kernel is $H(G, B)$. Thus

$$(3.1)\qquad |G|^{-1}s(G, r)=\sum_{B\in Z_r}|G(B)|^{-1}.$$

The next lemma helps us to compute $|G(B)|$.

LEMMA 2. *Let $B\perp X$ be a subspace of E, where B is nonisotropic and where X is totally isotropic. Then*

$$|G(B\perp X)|=|GL(X)|\cdot|\mathrm{Hom}(B, X)|\cdot|G(B)|.$$

PROOF. To $\varsigma\in G(B)$, $\gamma\in GL(X)$ and $\varepsilon\in\mathrm{Hom}(B, X)$, we associate $\beta\in G(B\perp X)$ which is defined as follows: $\beta e=\varsigma e+\varepsilon e$ if $e\in B$, $\beta e=\gamma e$ if $e\in X$.

§ 4. The unitary group; proof of Theorem 1.

Let E_n be an n-dimensional vector space over F_{q^2} with a nonsingular sesquilinear bilinear form Φ_n, and let $G_n=U(n, q^2)$ be the unitary group which leaves Φ_n invariant. Let $\beta(n)$ denote $|G_n|$ and let $\gamma_2(n)$ denote $|GL(n, q^2)|$. It is well known that $\beta(n)=q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-(-1)^i)$ and $\gamma_2(n)=q^{n(n-1)}\prod_{i=1}^{n}(q^{2i}-1)$.

Let A and B be r-dimensional subspaces of E_n. A and B belong to the same orbit under G_n, if and only if the dimensions of the maximal nonisotropic subspaces of A and B are the same, and the necessary and sufficient condition that there is an r-dimensional subspace of E_n with a c-dimensional maximal nonisotropic subspace is $0\le c\le r$ and $0\le c+2(r-c)\le n$. Let y denote $r-c$. From (3.1) and Lemma 2

$$|G_n|^{-1}s(n, r)=\sum_{y=0}^{\min(n-r,r)}[\gamma_2(y)q^{2y(r-y)}\beta(r-y)]^{-1}.$$

So $D(n, r)=|G_{n+1}|^{-1}s(n+1, r)-|G_n|^{-1}s(n, r)$ can be handled similarly for even and odd n except a little difference.

The case of $n=2m$ (m is a natural number).

$$D(n, r)=\begin{cases}0 & \text{if } 0\le r\le m\\ [\gamma_2(2m+1-r)q^{2(2m+1-r)(2r-2m-1)}\beta(2r-2m-1)]^{-1} & \text{if } m+1\le r\le 2m+1.\end{cases}$$

By easy calculation we can see that

$$\begin{aligned}D_n(t)&=|G_{n+1}|\sum_{r=0}^{n+1}D(n, r)t^{n+1-r}\prod_{i=0}^{r-1}(1-q^{2i}t)\\&=|G_{n+1}|\sum_{j=0}^{m}[\gamma_2(m-j)q^{2(m-j)(2j+1)}\beta(2j+1)]^{-1}t^{m-j}\prod_{i=0}^{m+j}(1-q^{2i}t)\\&=q^{m^2}\sum_{j=0}^{m}X_2(m, j)q^{-j(2m-1)}\prod_{i=j+2}^{m+1}(q^{2i-1}+1)t^{m-j}\prod_{i=0}^{m+j}(1-q^{2i}t)\end{aligned}$$

where

$$X_2(m, j)=q^{j(j-1)}\frac{\prod_{k=1}^{m}(q^{2k}-1)}{\prod_{k=1}^{j}(q^{2k}-1)\prod_{k=1}^{m-j}(q^{2k}-1)}.$$

$X_2(m, j)$ is $X(m, j)$ in which q is replaced by q^2.

Obviously $1-q^{2i}t$ $(i=0, 1, \dots, m)$ are factors of $D_n(t)$. We will prove that $1+q^{2i+1}t$ $(i=1, 2, \dots, m)$ are factors of $D_n(t)$.

Let t_r be $-q^{-(2r-1)}$ $(r=0, 1, \dots, m-1)$. Then,

$$\begin{aligned}D_n(t_r)&=q^{m^2}t_r^m\prod_{i=0}^{m}(1-q^it_r)\left\{\sum_{j=0}^{m}X_2(m, j)(-q^{2(m-r-1)})^{-j}\prod_{i=j+2}^{m+1}(q^{2i-1}+1)\prod_{i=m+1-r}^{m+j-r}(q^{2i}-1)\right\}\\&=q^{m^2}t_r^m\prod_{i=0}^{m}(1-q^it_r)\prod_{i=m+1-r}^{m+1}(q^{2i-1}+1)\left\{\sum_{j=0}^{m}X_2(m, j)(-q^{2(m-r-1)})^{-j}\prod_{i=j+2}^{m+j-r}(q^{2i-1}+1)\right\}\\&=q^{m^2}t_r^m\prod_{i=0}^{m}(1-q^it_r)\prod_{i=m+1-r}^{m+1}(q^{2i-1}+1)\\&\quad\times\left\{\sum_{j=0}^{m}X_2(m, j)(-q^{2(m-r-1)})^{-j}\sum_{k=0}^{m-r-1}X_2(m-r-1, k)(q^{2j+3})^k\right\}\quad\text{(from Lemma 1 (1))}\\&=q^{m^2}t_r^m\prod_{i=0}^{m}(1-q^it_r)\prod_{i=m+1-r}^{m+1}(q^{2i-1}+1)\\&\quad\times\left\{\sum_{k=0}^{m-r-1}X_2(m-r-1, k)q^{3k}\sum_{j=0}^{m}X_2(m, j)(-q^{2(m-r-1-k)})^{-j}\right\}.\end{aligned}$$

But

$$\sum_{j=0}^{m} X_2(m,j)(-q^{2(m-r-1-k)})^{-j}=\prod_{j=0}^{m-1}[1-q^{2j}(q^{2(m-r-1-k)})^{-1}]=0 \qquad (k=0,1,\ldots,m-r-1).$$

So $D_n(t_r)=0$, which proves our assertion.

The degree of the polynomial $D_n(t)$ is $n+1$ $(=2m+1)$, and $D_n(0)=1$. Thus $D_n(t)=\prod_{i=0}^{m}(1-q^{2i}t)\prod_{i=1}^{m}(1+q^{2i-1}t)$. This proves (1) of Theorem 1.

We can similarly prove (2) of Theorem 1 in the case where n is odd. Now we will prove the last part of Theorem 1. It suffices to show that $1+q^{2m+1}t$ is a factor of $P_{2m+1}(t)$. From (1) and (2) it follows that

$$P_{2m+1}(t)=q^{4m-1}(q^{2m+1}+1)(q^{2m}-1)t^2P_{2m-1}(t)+(1+q^{4m+1}t)\prod_{i=0}^{m-1}(1-q^{2i}t)\prod_{i=0}^{m-1}(1+q^{2i+1}t)$$

so that

$$P_{2m+1}(t)=\sum_{j=0}^{m}\prod_{i=j+1}^{m}[q^{4i-1}(q^{2i+1}+1)(q^{2i}-1)t^2](1+q^{4j+1}t)\prod_{i=0}^{j-1}(1-q^{2i}t)\prod_{i=0}^{j-1}(1+q^{2i+1}t).$$

Let t_0 be $-q^{-(2m+1)}$, then

$$\begin{aligned}P_{2m+1}(t_0)&=\sum_{j=0}^{m}\prod_{i=j+1}^{m}[q^{-(4(m-i)+3)}(q^{2i+1}+1)(q^{2i}-1)](1+q^{4j+1}t_0)\\&\quad\times\prod_{i=0}^{j-1}(1+q^{-(2(m-i)+1)})\prod_{i=0}^{j-1}(1-q^{-2(m-i)})\\&=\sum_{j=0}^{m}\left[\prod_{i=1}^{m}q^{-(4(m-i)+3)}\right]q^{-2j}(1+q^{4j+1}t_0)\prod_{i=j+1}^{m}[(q^{2i+1}+1)(q^{2i}-1)]\\&\quad\times\prod_{i=m-j+1}^{m}[(q^{2i+1}+1)(q^{2i}-1)]\\&=\left[\prod_{i=0}^{m-1}q^{-(4i+3)}\right]\sum_{j=0}^{m}(q^{-2j}-q^{-2(m-j)})\varsigma(j)\end{aligned}$$

where

$$\varsigma(j)=\prod_{i=j+1}^{m}[(q^{2i+1}+1)(q^{2i}-1)]\prod_{i=m-j+1}^{m}[(q^{2i+1}+1)(q^{2i}-1)].$$

But $\varsigma(j)=\varsigma(m-j)$, so that $P_{2m+1}(t_0)=0$. This completes the proof of Theorem 1.

§ 5. The symplectic group; proof of Theorem 2.

We sketch those parts of the argument which differ from the unitary case. Let E_{2m} be a $2m$-dimensional vector space over F_q with a nonsingular alternative bilinear form Φ_{2m} and let $G_m=Sp(2m,q)$ be the symplectic group which leaves Φ_{2m} invariant. Let A and B be r-dimensional subspaces of E_{2m}. Then A and B belong to the same orbit under G_m, if and only if the dimension of the maximal nonisotropic subspace of A and that of B are the same. And the necessary and sufficient condition that there is an r-dimensional subspace of E_{2m} with a c-dimensional maximal nonisotropic subspace is $0\leq c\leq r$, $0\leq c+2(r-c)\leq n$ and that c is even. Let $\varepsilon(k)$ denote $|Sp(2k,q)|=q^{k^2}\prod_{j=1}^{k}(q^{2j}-1)$ and let $\gamma(k)$ denote $|GL(k,q)|=q^{k(k-1)/2}\prod_{j=1}^{k}(q^j-1)$. From (3.1) and Lemma 2 we have

$$|G_m|^{-1}s(m,r)=\sum_{k=\max(0,r-m)}^{[r/2]}(\gamma(r-2k)q^{2k(r-k)}\varepsilon(k))^{-1}$$

where [ ] is Gauss' notation. Thus

$$D(m,r)=\begin{cases}0 & \text{if } 0\leq r\leq m,\\ (\gamma(2m+2-r)q^{2(r-m-1)(2m+2-r)}\varepsilon(r-m-1))^{-1} & \text{if } m+1\leq r\leq 2m+2.\end{cases}$$

So

$$\begin{aligned}D_m(t)&=|G_{m+1}|\sum_{r=0}^{2m+2}D(m,r)t^{2m+2-r}\prod_{k=0}^{r-1}(1-q^kt)\\ &=q^{(m+1)(m+2)/2}\sum_{j=0}^{m+1}X(m+1,j)q^{-(m+1)j}\prod_{i=j+1}^{m+1}(q^i+1)\cdot t^{m+1-j}\cdot\prod_{k=0}^{m+j}(1-q^kt).\end{aligned}$$

And by the same argument as the proof of Theorem 1 we can easily see $D_m(-q^{-h})=0$ $(h=1,2,\ldots,m)$. And by comparing coefficients we have

$$D_m(t)=(1+q^{2m+2}t)\prod_{i=0}^{m}(1-q^it)\prod_{i=1}^{m}(1+q^it).$$

The proof of the last part of Theorem 2 is similar to that of Theorem 1.

§6. The orthogonal group; proof of Theorem 3.

We sketch those parts of the argument which differ from the unitary case. Let E_n be an n-dimensional vector space over F_q (char $F_q\neq 2$) with a nonsingular symmetric bilinear form Φ and let $O(n,q,\Phi)$ be the orthogonal group which leaves Φ invariant. The nonsingular symmetric bilinear forms on E_n are classified into four types. (See the statement of Theorem 3.)

Let G_m^i $(i=1,2,3,4)$ denote $O(n,q,\Phi)$ where Φ is of type i and where $n=2m-1$ if $i=1$ or 2 and $n=2m$ if $i=3$ or 4. (m is a natural number.) Then we adapt Proposition 2 to each $\{G_m^i\}_{m=1,2,\ldots}$. Let A and B be r-dimensional subspaces of E_n. A and B belong to the same orbit under $O(n,q,\Phi)$, if and only if the dimensions of the maximal nonisotropic subspaces of A and B are the same and the types of the nonsingular symmetric bilinear forms which are the restriction of Φ to those maximal nonisotropic subspaces are the same, and the necessary and sufficient condition that there exists an r-dimensional subspace U of E_n which has a c-dimensional maximal nonisotropic subspace and the restriction of Φ to U is of type i is as follows:

$0\leq c\leq r$ and $0\leq c+2(r-c)\leq n$, or $0\leq c\leq r$, $c+2(r-c)=n$ and Φ is of type i. Let $g(n,i)$ denote $|O(n,q,\Phi)|$ where Φ is of type i and let $\gamma(n)$ denote $GL(n,q)$. It is well known that

$$g(n,i)=2q^{(n-1)^2/4}\prod_{i=1}^{(n-1)/2}(q^{2i}-1) \qquad \text{if } i=1 \text{ or } 2,$$

$$g(n,i)=2q^{n(n-2)/4}(q^{n/2}-\varepsilon)\prod_{i=1}^{(n-2)/2}(q^{2i}-1) \qquad \text{if } i=3 \text{ or } 4,$$

where $\varepsilon=+1$ if $i=3$ and $\varepsilon=-1$ if $i=4$.

From (3.1) and Lemma 2 we have

$$\begin{aligned}|G_m^i|^{-1}s(G_m^i,r)=&\sum_{k=\max(0,[r+1-n/2])}^{[r/2]}(\gamma(r-2k)q^{2k(r-2k)})^{-1}(g(2k,3)^{-1}+g(2k,4)^{-1})\\&+\sum_{k=\max(0,[r-(n-1)/2])}^{[(r-1)/2]}(\gamma(r-2k-1)q^{(2k+1)(r-2k-1)})^{-1}\\&\times(g(2k+1,1)^{-1}+g(2k+1,2)^{-1})+\alpha_r(\gamma(n-r)q^{(n-r)(2r-n)}g(2r-n,i))^{-1}\end{aligned}$$

where $n=2m-1$ if $i=1,2$ and $n=2m$ if $i=3,4$, where $\alpha_r=0$ if $0\leq 2r\leq n-1$ and $\alpha_r=1$ if $n\leq 2r$, and where we set $g(0,3)^{-1}=1$ and $g(0,4)^{-1}=0$ formally.

We write $D_m^i(m,r)$ for $|G_{m+1}^i|^{-1}s(G_{m+1}^i,r)-|G_m^i|^{-1}s(G_m^i,r)$ and we will consider $D_m^i(t)=|G_{m+1}^i|\sum_{r=0}^{n+2}D^i(m,r)t^{n+2-r}\prod_{j=0}^{r-1}(1-q^jt)$. If $i=1$ or 2,

$$\begin{aligned}&|G_{m+1}^i|^{-1}D_m^i(t)\\&=\sum_{r=m}^{2m}(\gamma(2m-r)q^{2(r-m)(2m-r)})^{-1}(g(2(r-m),3)^{-1}+g(2(r-m),4)^{-1})t^{2m+1-r}\prod_{j=0}^{r-1}(1-q^jt)\\&\quad+\sum_{r=m}^{2m-1}(\gamma(2m-r-1)q^{(2r-2m+1)(2m-r+1)})^{-1}\\&\qquad\times(g(2r-2m+1,1)^{-1}+g(2r-2m+1,2)^{-1})t^{2m+1-r}\prod_{j=0}^{r-1}(1-q^jt)\\&\quad+\sum_{r=m+1}^{2m+1}(\gamma(2m+1-r)q^{(2m+1-r)(2r-2m-1)})^{-1}g(2r-2m-1,i)^{-1}t^{2m+1-r}\prod_{j=0}^{r-1}(1-q^jt)\\&\quad-\sum_{r=m}^{2m}(\gamma(2m-1-r)q^{(2m-1-r)(2r-2m+1)})^{-1}g(2r-2m-1,i)^{-1}t^{2m+1-r}\prod_{j=0}^{r-1}(1-q^jt).\end{aligned}$$

And if $i=3$ or 4,

$$\begin{aligned}&|G_{m+1}^i|^{-1}D_m^i(t)\\&=\sum_{r=m}^{2m}(\gamma(2m-r)q^{2(r-m)(2m-r)})^{-1}(g(2(r-m),3)^{-1}+g(2(r-m),4)^{-1})t^{2m+2-r}\prod_{j=0}^{r-1}(1-q^jt)\end{aligned}$$

$$+\sum_{r=m+1}^{2m+1}(\gamma(2m-r+1)q^{(2r-2m-1)(2m-r+1)})^{-1}$$
$$\times(g(2r-2m-1,1)^{-1}+g(2r-2m-1,2)^{-1})t^{2m+2-r}\prod_{j=0}^{r-1}(1-q^jt)$$
$$+\sum_{r=m+1}^{2m+2}(\gamma(2m+2-r)q^{(2m+2-r)(2r-2m-2)})^{-1}g(2r-2m-2,i)^{-1}t^{2m+2-r}\prod_{j=0}^{r-1}(1-q^jt)$$
$$-\sum_{r=m}^{2m}(\gamma(2m-r)q^{(2m-r)(2r-2m)})^{-1}g(2r-2m)^{-1}t^{2m+2-r}\prod_{j=0}^{r-1}(1-q^jt).$$

It is obvious that

$$g(2c+1,1)^{-1}=g(2c+1,2)^{-1}=\frac{1}{2}\left(q^{c^2}\prod_{j=1}^{c}(q^{2j}-1)\right)^{-1},$$
$$g(2c,3)^{-1}=\frac{1}{2}\left(q^{c^2}\prod_{j=1}^{c}(q^{2j}-1)\right)^{-1}(q^{2c}+q^c),$$
$$g(2c,4)^{-1}=\frac{1}{2}\left(q^{c^2}\prod_{j=1}^{c}(q^{2j}-1)\right)^{-1}(q^{2c}-q^c).$$

And by the same argument as the proof of Theorem 1 we can easily see the next proposition.

PROPOSITION 3. *Let* $h(r,k)=\left(\gamma(r-2k)q^{2k(r-2k)}q^{k^2}\prod_{j=1}^{k}(q^{2j}-1)\right)^{-1}$. *Then*

$$\sum_{j=0}^{m}h(m+j,j)q^jt^{m-j}\prod_{k=0}^{m+j-1}(1-q^kt)=h(2m,m)q^m\prod_{k=0}^{m-1}(1-q^kt)\prod_{k=0}^{m-1}(1+q^kt)$$
$$\sum_{j=0}^{m+1}h(m+1+j,j)q^{2j}t^{m+1-j}\prod_{k=0}^{m+k}(1-q^kt)-\sum_{j=0}^{m}h(m+j,j)q^{2j}t^{m+2-j}\prod_{k=0}^{m+j-1}(1-q^kt)$$
$$=h(2m+2,m+1)(-q^{4m+2}t^2-q^{2m+1}(q^{2m+2}-1)t+q^{2m+2})\prod_{k=0}^{m-1}(1-q^kt)\prod_{k=0}^{m-1}(1+q^kt).$$

From Proposition 3 we have

$$D_m^1(t)=D_m^2(t)=(1+q^{2m}t)\prod_{j=0}^{m-1}(1-q^jt)\prod_{j=0}^{m-1}(1+q^jt),$$
$$D_m^3(t)=(1+q^{2m+1}t)\prod_{j=0}^{m}(1-q^jt)\prod_{j=0}^{m-1}(1+q^jt),$$
$$D_m^4(t)=(1+q^{2m+1}t)\prod_{j=0}^{m-1}(1-q^jt)\prod_{j=0}^{m}(1+q^jt).$$

This proves the recurrence formula of Theorem 3.

The proof of the last part of Theorem 3 is similar to that of Theorem 1. So the proof of Theorem 3 is complete.

Reference

[1] Solomon, L., A fixed-point formula for the classical groups over a finite field, Trans. Amer. Math. Soc. **117** (1965), 423–440.

(Received September 28, 1976)

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