On coverings and hyperalgebras of affine algebraic groups, II

By Mitsuhiro TAKEUCHI

(Communicated by N. Iwahori)

Introduction. In the previous paper [3] we considered a relationship between the *hyperalgebra* and the *universal group covering* of a connected affine algebraic group scheme over a field.

Let k be a field of characteristic $p \geq 0$ and $\mathfrak{G} = \mathrm{Spec}\,(A)$ a connected affine algebraic k-group scheme corresponding to the commutative k-Hopf algebra A. Let A° be the *dual Hopf algebra* to A [3, p. 254] and $\mathrm{hy}(\mathfrak{G})$ the hyperalgebra of $\mathfrak{G}$ [3, p. 259]. A canonical map of Hopf algebras: $A \rightarrow \mathrm{hy}(\mathfrak{G})^{\circ}$ corresponds to the inclusion: $\mathrm{hy}(\mathfrak{G}) \hookrightarrow A^{\circ}$. Let

$$\gamma : \mathfrak{G}^* = \mathrm{Spec}\,(\mathrm{hy}(\mathfrak{G})^{\circ}) \rightarrow \mathfrak{G} = \mathrm{Spec}\,(A)$$

be the associated morphism of affine k-group schemes. Since $\mathfrak{G}$ is connected, γ is an epimorphism (or equivalently $A \rightarrow \mathrm{hy}(\mathfrak{G})^{\circ}$ is injective) and each algebraic quotient of $\mathfrak{G}^*$ is connected by [3, 0. 3. 1 (g)]. Hence the affine k-group scheme $\mathfrak{G}^*$ is connected [1, III, § 3, n° 7].

In this paper we first show that $(\mathfrak{G}^*, \gamma)$ is a *central extension* of $\mathfrak{G}$, i.e., the kernel $\mathfrak{Ker}\,(\gamma)$ is contained in the center of $\mathfrak{G}^*$.

An affine k-group scheme is *proetale* if each algebraic quotient is etale [1, ibid.].

When $p > 0$, we prove that $(\mathfrak{G}^*, \gamma)$ is a *proetale extension* of $\mathfrak{G}$, (i.e., the kernel $\mathfrak{Ker}\,(\gamma)$ is proetale) if and only if the quotient group scheme $\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is *finite*, where $[\mathfrak{G}, \mathfrak{G}]$ denotes the *derived* group of $\mathfrak{G}$ [3, p. 257].

If this is the case, the pair $(\mathfrak{G}^*, \gamma)$ clearly satisfies the following universal mapping property: Let $\eta : \mathfrak{H} \rightarrow \mathfrak{G}$ be an epimorphism, where $\mathfrak{H}$ is a connected affine k-group scheme and $\mathfrak{Ker}\,(\eta)$ is proetale. There is a *unique* morphism of k-group schemes $\eta^* : \mathfrak{G}^* \rightarrow \mathfrak{H}$ such that $\eta \circ \eta^* = \gamma$.

Hence, in this case, $(\mathfrak{G}^*, \gamma)$ is a *universal proetale extension* of $\mathfrak{G}$. In particular, if $\mathfrak{G}^*$ is algebraic (or equivalently, if the commutative Hopf algebra $\mathrm{hy}(\mathfrak{G})^{\circ}$ is finitely generated), then (G^*, γ) is a *universal group covering* of $\mathfrak{G}$ in the sense of [3].

Therefore, combined with [3, Th. 1.9], we have the following:

COROLLARY. *Let k be perfect with $p>0$. For each connected affine algebraic k-group scheme $\mathfrak{G}$, the following are equivalent:*

i) *$\mathfrak{G}$ has a universal group covering.*

ii) *$(\mathfrak{G}^*, \gamma)$ is the universal group covering of $\mathfrak{G}$.*

iii) *$(\mathfrak{G}^*, \gamma)$ is an etale group covering* [3] *of $\mathfrak{G}$.*

iv) *$\mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$ is finite and* $\mathrm{hy}(\mathfrak{G})^\circ$ *is finitely generated.*

For example, if $\mathfrak{G}$ is a *semisimple* k-group scheme, then these equivalent conditions are satisfied [3, Th. 3.1].

We fix a ground field k of characteristic $p \geq 0$. We shall freely use the notations and the terminology of [3].

We prove that $(\mathfrak{G}^*, \gamma)$ is a central extension of $\mathfrak{G}$ in two different ways. One depends on the hyperalgebra theory for algebraic groups [4]. The other on the Hopf algebra techniques [2].

1. The extension $(\mathfrak{G}^*, \gamma)$ is central; Proof based upon the hyperalgebra theory

We summarized in [3, § 0.3] the hyperalgebra theory for algebraic groups. We must recall in addition *the underlying coalgebra* of a *k-functor*.

A covariant functor from $\boldsymbol{M}_k$ the category of commutative k-algebras to $\boldsymbol{E}$ the category of sets is called a k-functor [1].

Let $\boldsymbol{W}_k$ and $\boldsymbol{M}_k^f$ denote the categories of cocommutative k-coalgebras and finite dimensional commutative k-algebras respectively. If $R \in \boldsymbol{M}_k^f$, then $R^* \in \boldsymbol{W}_k$ [3, p. 254].

Let $\mathfrak{X}$ be a k-functor. The underlying coalgebra $T(\mathfrak{X})$ [4, 2.1.1] is a uniquely determined object of $\boldsymbol{W}_k$ by the natural isomorphisms

$$\mathfrak{X}(R) \simeq \boldsymbol{W}_k(R^*, T(\mathfrak{X})), \ {}^\forall R \in \boldsymbol{M}_k^f.$$

The coalgebra $T(\mathfrak{X})$ exists if and only if the restricted functor $\mathfrak{X} | \boldsymbol{M}_k^f$ preserves all pullback diagrams and the final object [5, 5.1.2.3]. For example if $\mathfrak{X}$ is a *k-scheme*, $T(\mathfrak{X})$ exists [4, 2.1.6].

Let V be a k-vector space and V_a the k-functor: $R \mapsto R \otimes V$, $R \in \boldsymbol{M}_k$ [1, II, § 1, 2.1]. The coalgebra $T(V_a)$ exists and $T(V_a) = C_a(V)$ [4, 3.2.7], where the cocommutative coalgebra $C_a(V)$ satisfies the following universal mapping property:

$$\boldsymbol{W}_k(C, C_a(V)) \simeq \mathrm{Hom}_k(C, V), \ {}^\forall C \in \boldsymbol{W}_k.$$

Let $\pi: C_a(V) \to V$ be the k-linear map associated with the identity $I: C_a(V) \to C_a(V)$.

Let W be another k-vector space and $\mathfrak{Mod}(V, W)$ the k-functor: $R \mapsto \mathrm{Hom}_R(R\otimes V, R\otimes W)$, $R\in \boldsymbol{M}_k$ [1, II, §1, 2.4]. We have

$$T(\mathfrak{Mod}(V, W)) \simeq C_a(\mathrm{Hom}_k(V, W)),$$

since $\mathfrak{Mod}(V, W)(R) \simeq \mathrm{Hom}_k(V, R\otimes W) \simeq R\otimes \mathrm{Hom}_k(V, W)$ for $R\in \boldsymbol{M}_k^f$.

Let $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Y}$ be a morphism of k-functors, where the coalgebras $T(\mathfrak{X})$ and $T(\mathfrak{Y})$ both exist. There is a unique W_k-map $T(\mathfrak{f}): T(\mathfrak{X})\to T(\mathfrak{Y})$ [4, 2.1.1] which makes commute the diagrams

$$\begin{array}{ccc} \mathfrak{X}(R) & \xrightarrow{\mathfrak{f}(R)} & \mathfrak{Y}(R) \\ \wr\| & & \wr\| \\ W_k(R^*, T(\mathfrak{X})) & \xrightarrow[T(\mathfrak{f})]{} & W_k(R^*, T(\mathfrak{Y})), \end{array} \qquad {}^\forall R\in \boldsymbol{M}_k^f.$$

A morphism of k-functors

$$\mathfrak{u}: \mathfrak{X}\times V_a \to W_a$$

is called *linear* if for each $x\in\mathfrak{X}(R)$ with $R\in \boldsymbol{M}_k$, the induced map $\mathfrak{u}(x, ?): R\otimes V\to R\otimes W$ is R-linear. Such linear morphisms correspond bijectively with morphisms: $\mathfrak{X}\to\mathfrak{Mod}(V, W)$. Suppose the k-functor $\mathfrak{X}$ has the coalgebra $T(\mathfrak{X})$. We define the k-linear map

$$\bar{\mathfrak{u}}: T(\mathfrak{X})\otimes V\to W$$

as follows: For each $v\in V$, the linear map $\bar{\mathfrak{u}}(?\otimes v): T(\mathfrak{X})\to W$ is the composite

$$T(\mathfrak{X}) \xrightarrow{T(\mathfrak{u}(?, V))} T(W_a) = C_a(W) \xrightarrow{\pi} W$$

where π denotes the canonical projection. Since $v\mapsto \bar{\mathfrak{u}}(?\otimes v)$ is k-linear, the linear map $\bar{\mathfrak{u}}$ is well-defined and called *associated with* $\mathfrak{u}$.

If we identify the linear morphism $\mathfrak{u}$ with a morphism $\mathfrak{f}: \mathfrak{X}\to\mathfrak{Mod}(V, W)$, then the associated map $\bar{\mathfrak{u}}$ is identified with the composite

$$\bar{\mathfrak{f}}: T(\mathfrak{X}) \xrightarrow{T(\mathfrak{f})} T(\mathfrak{Mod}(V, W)) = C_a(\mathrm{Hom}_k(V, W)) \xrightarrow{\pi} \mathrm{Hom}_k(V, W).$$

Let W' be a subspace of W and

$$\mathfrak{X}' = \mathfrak{Transp}_{\mathfrak{u}}(V_a, W'_a) \quad [1, \text{I}, §2, 7.4],$$

i.e., $\mathfrak{X}'(R) = \{x\in\mathfrak{X}(R) \mid \mathfrak{u}(x, R\otimes V)\subset R\otimes W'\}$, $R\in \boldsymbol{M}_k$. It is easy to see that $\mathfrak{X}'$ is a *closed* subfunctor of $\mathfrak{X}$ (cf. [1, I, §2, 7.5]).

1.1 LEMMA. *Let* $\mathfrak{u}:\mathfrak{X}\times V_a\rightarrow W_a$ *be a linear morphism of k-functors, where* $T(\mathfrak{X})$ *exists and V and W are k-vector spaces. Let* $\bar{\mathfrak{u}}:T(\mathfrak{X})\otimes V\rightarrow W$ *be the associated k-linear map. Let W′ be a subspace of W and* $\mathfrak{X}'=\mathfrak{Transp}_{\mathfrak{u}}(V_a, W'_a)$. *Then* $T(\mathfrak{X}')$ *exists and equals the largest subcoalgebra of* $T(\mathfrak{X})$ *contained in* $\{a\in T(\mathfrak{X})\,|\,\bar{\mathfrak{u}}(a\otimes V)\subset W'\}$.

PROOF. Let $x\in\mathfrak{X}(R)$ with $R\in M_k^f$. We view x as a coalgebra map $\bar{x}:R^*\rightarrow T(\mathfrak{X})$. The k-linear map $\mathfrak{u}(x, ?):V\rightarrow R\otimes W$ is identified with the composite

$$R^*\otimes V\xrightarrow{\bar{x}\otimes I} T(\mathfrak{X})\otimes V\xrightarrow{\bar{\mathfrak{u}}} W.$$

Hence $x\in\mathfrak{X}'(R)$ if and only if $\bar{\mathfrak{u}}(\bar{x}(R^*)\otimes V)\subset W'$. Q.E.D.

Let $\mathfrak{X}$ be a *locally algebraic k-scheme* and $\mathfrak{X}'$ a *subscheme* of $\mathfrak{X}$. Then $\mathfrak{X}=\mathfrak{X}'$ if and only if $T(\mathfrak{X})=T(\mathfrak{X}')$ [4, 2.3.3]. Therefore:

1.2 COROLLARY. *With the same notations as (1.1), let* $\mathfrak{X}$ *be a locally algebraic k-scheme. Then* $\mathfrak{X}=\mathfrak{X}'$ *if and only if* $\bar{\mathfrak{u}}(T(\mathfrak{X})\otimes V)\subset W'$.

A covariant functor from $\boldsymbol{M}_k$ to $\boldsymbol{Gr}$ the category of groups is called a *k-group functor* [1].

Let $\mathfrak{G}$ be a k-group functor. Suppose the coalgebra $T(\mathfrak{G})$ exists. The product $\mathfrak{p}:\mathfrak{G}\times\mathfrak{G}\rightarrow\mathfrak{G}$, the unit $\mathfrak{u}:\mathrm{Spec}\,(k)\rightarrow\mathfrak{G}$ and the inverse $\mathfrak{i}:\mathfrak{G}\rightarrow\mathfrak{G}$ induce the coalgebra maps respectively

$$T(\mathfrak{p}):T(\mathfrak{G})\otimes T(\mathfrak{G})\simeq T(\mathfrak{G}\times\mathfrak{G})\rightarrow T(\mathfrak{G})$$

$$T(\mathfrak{u}):k\simeq T(\mathrm{Spec}\,(k))\rightarrow T(\mathfrak{G})$$

$$T(\mathfrak{i}):T(\mathfrak{G})\rightarrow T(\mathfrak{G}).$$

The triple $(T(\mathfrak{G}), T(\mathfrak{p}), T(\mathfrak{u}))$ is a *cocommutative k-Hopf algebra* with *antipode* $S=T(\mathfrak{i})$ [4, 3.1.1].

Let V be a k-vector space. A linear morphism

$$\mathfrak{u}:\mathfrak{G}\times V_a\rightarrow V_a$$

is a *linear action* if for each $R\in\boldsymbol{M}_k$, the group $\mathfrak{G}(R)$ operates on the left on $R\otimes V$, R-linearly via $\mathfrak{u}(R)$. This is equivalent to saying that $\mathfrak{u}$ determines a morphism of k-group functors $\rho:\mathfrak{G}\rightarrow\mathfrak{GL}(V)$, where $\mathfrak{GL}(V)(R)=GL_R(R\otimes V)$, $R\in\boldsymbol{M}_k$ [1, II, § 2. 1.1].

Suppose the Hopf algebra $T(\mathfrak{G})$ exists. If $\mathfrak{u}:\mathfrak{G}\times V_a\rightarrow V_a$ is a linear action, the associated linear map

$$\bar{\mathfrak{u}}: T(\mathfrak{G})\otimes V \to V$$

clearly makes V into a left $T(\mathfrak{G})$-module (cf. [4, 3.2.5]).

Let H be a cocommutative Hopf algebra with antipode S. We put

$$\mathrm{ad}(x)(y) = \sum_{(x)} x_{(1)} y S(x_{(2)})$$
$$[x, y] = \sum_{(x,y)} x_{(1)} y_{(1)} S(x_{(2)}) S(y_{(2)}), \quad x, y \in H$$

where the 'sigma' notation [2] is used. A sub-Hopf algebra $K \subset H$ is *normal* if $\mathrm{ad}(H)(K) \subset K$ or equivalently if $[H, K] \subset K$. K is *central* if $[H, K] \subset k$. The irreducible component [3, p. 254] H^1 of H containing 1 is a normal sub-Hopf algebra [5, 5.5.1.2].

Let $\mathfrak{G}$ be a *locally algebraic k-group scheme*. The hyperalgebra $\mathrm{hy}(\mathfrak{G})$ is the irreducible component of $T(\mathfrak{G})$ containing 1 [4, 3.1.4]. The *adjoint representation* [3, p. 260]

$$\mathfrak{Ad}: \mathfrak{G} \times \mathrm{hy}(\mathfrak{G})_a \to \mathrm{hy}(\mathfrak{G})_a$$

is a *unique* linear action with which is associated the adjoint action

$$\overline{\mathfrak{Ad}} = \mathrm{ad}: T(\mathfrak{G}) \otimes \mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G}).$$

The uniqueness follows from (1.2).

Based upon the above preliminaries we give a first proof for $(\mathfrak{G}^*, \gamma)$ being a central extension of $\mathfrak{G}$.

Let $\mathfrak{G} = \mathrm{Spec}(A)$ be a connected affine algebraic k-group scheme, where $A = \mathcal{O}(\mathfrak{G})$ is the corresponding finitely generated commutative k-Hopf algebra. Since the hyperalgebra $\mathrm{hy}(\mathfrak{G})$ is the irreducible component of A° containing 1 [4, 3.2.2], a canonical injective homomorphism of commutative Hopf algebras $A \to \mathrm{hy}(\mathfrak{G})^\circ$ is associated with the inclusion $\mathrm{hy}(\mathfrak{G}) \hookrightarrow A^\circ$. The injectivity follows from [3, 0.3.1(g)].

We view A as a sub-Hopf algebra of $\mathrm{hy}(\mathfrak{G})^\circ$. Let B be a finitely generated sub-Hopf algebra of $\mathrm{hy}(\mathfrak{G})^\circ$ containing A and $\mathfrak{G}' = \mathrm{Spec}(B)$ the corresponding affine algebraic k-group scheme. The inclusion $A \hookrightarrow B$ determines an epimorphism of k-group schemes $\mathfrak{f}: \mathfrak{G}' \to \mathfrak{G}$.

Let $j: \mathrm{hy}(\mathfrak{G}) \to B^\circ$ be the Hopf algebra map corresponding to the inclusion $B \hookrightarrow \mathrm{hy}(\mathfrak{G})^\circ$. Clearly $\mathrm{Im}(j) \subset \mathrm{hy}(\mathfrak{G}')$ and the composite

$$\mathrm{hy}(\mathfrak{G}) \xrightarrow{j} \mathrm{hy}(\mathfrak{G}') \xrightarrow{\mathrm{hy}(\mathfrak{f})} \mathrm{hy}(\mathfrak{G})$$

where $\mathrm{hy}(\mathfrak{f})$ denotes the induced map of hyperalgebras [3, p. 259], is the identity by definition. This means in particular that $\mathfrak{G}'$ is connected by [3, 0.3.1(g)]. Let $\mathfrak{G}^*=\mathrm{Spec}(\mathrm{hy}(\mathfrak{G})^\circ)$ and $\gamma:\mathfrak{G}^*\to\mathfrak{G}$ be the epimorphism of affine k-group schemes determined by the inclusion $A\hookrightarrow\mathrm{hy}(\mathfrak{G})^\circ$. Since $\mathfrak{G}^*$ is the projective limit of $\mathfrak{G}'$, where B runs through all the finitely generated sub-Hopf algebras of $\mathrm{hy}(\mathfrak{G})^\circ$ containing A, the affine k-group scheme $\mathfrak{G}^*$ is connected [1, III, § 3, n°7].

In order to prove that $(\mathfrak{G}^*,\gamma)$ is a central extension, we have only to show that so is $(\mathfrak{G}',\mathfrak{f})$ for each B.

A sub-hyperalgebra $J\subset\mathrm{hy}(\mathfrak{G}')$ is *dense* if $\mathrm{hy}(\mathfrak{G}')=A(J)$ the *algebraic hull* of J [3, p. 261] or equivalently if the corresponding Hopf algebra map $B\to J^\circ$ is injective.

Since $B\hookrightarrow\mathrm{hy}(\mathfrak{G})^\circ$, $\mathrm{Im}(j)$ is a dense sub-hyperalgebra of $\mathrm{hy}(\mathfrak{G}')$ and $\mathrm{hy}(\mathfrak{f})$ is bijective on $\mathrm{Im}(j)$.

Therefore the extension $\mathfrak{f}:\mathfrak{G}'\to\mathfrak{G}$ satisfies the hypothesis of the following:

1.3 PROPOSITION. *Let $\mathfrak{f}:\mathfrak{G}'\to\mathfrak{G}$ be a morphism of connected algebraic k-group schemes and* $\mathrm{hy}(\mathfrak{f}):\mathrm{hy}(\mathfrak{G}')\to\mathrm{hy}(\mathfrak{G})$ *the induced map of hyperalgebras. If there is a dense sub-hyperalgebra* $J\subset\mathrm{hy}(\mathfrak{G}')$ *on which* $\mathrm{hy}(\mathfrak{f})$ *is injective, then the kernel* $\mathfrak{Ker}(\mathfrak{f})$ *is contained in the center of* $\mathfrak{G}'$.

PROOF. Let $\mathfrak{Ad}:\mathfrak{G}'\times\mathrm{hy}(\mathfrak{G}')_a\to\mathrm{hy}(\mathfrak{G}')_a$ denote the adjoint representation for $\mathfrak{G}'$. The normalizer $\mathfrak{N}_{\mathfrak{G}'}(J)$ in $\mathfrak{G}'$ of J_a with respect to $\mathfrak{Ad}$ is a closed subgroup scheme of $\mathfrak{G}'$ [1, II, § 2, 1.4]. By (1.1), $T(\mathfrak{N}_{\mathfrak{G}'}(J))$ is the largest sub-Hopf algebra of $T(\mathfrak{G}')$ which normalizes J. In particular $J\subset\mathrm{hy}(\mathfrak{N}_{\mathfrak{G}'}(J))$. Since $A(J)=\mathrm{hy}(\mathfrak{G}')$, $\mathrm{hy}(\mathfrak{N}_{\mathfrak{G}'}(J))=\mathrm{hy}(\mathfrak{G}')$. Since $\mathfrak{G}'$ is connected, $\mathfrak{G}'=\mathfrak{N}_{\mathfrak{G}'}(J)$ [3, 0.3.1 (f)]. Therefore $T(\mathfrak{G}')=T(\mathfrak{N}_{\mathfrak{G}'}(J))$ or equivalently J is a *normal* sub-Hopf algebra of $T(\mathfrak{G}')$.

The Hopf algebra $T(\mathfrak{Ker}(f))$ is the *Hopf kernel* [3, p. 255] of $T(\mathfrak{f})$: $T(\mathfrak{G}')\to T(\mathfrak{G})$ [4, 3.1.5] and a normal sub-Hopf algebra of $T(\mathfrak{G}')$. Since $T(\mathfrak{Ker}(\mathfrak{f}))\cap J=k$ by hypothesis, we have

$$[J,T(\mathfrak{Ker}(\mathfrak{f}))]\subset J\cap T(\mathfrak{Ker}(\mathfrak{f}))\subset k.$$

Hence the sub-Hopf algebras J and $T(\mathfrak{Ker}(\mathfrak{f}))$ centralize each other. Therefore $\mathfrak{Ker}(\mathfrak{f})$ operates trivially on J_a via $\mathfrak{Ad}$ by (1.2).

Let $\mathfrak{C}=\mathfrak{C}_{\mathfrak{G}'}(\mathfrak{Ker}(\mathfrak{f}))$ be the centralizer in $\mathfrak{G}'$ of $\mathfrak{Ker}(\mathfrak{f})$, which is a closed subgroup scheme [1, II, § 1, 3.7]. The hyperalgebra $\mathrm{hy}(\mathfrak{C})$ is the largest subcoalgebra D of $\mathrm{hy}(\mathfrak{G}')$ on which $\mathfrak{Ker}(\mathfrak{f})$ operates trivially via $\mathfrak{Ad}$ [3, 0.3.3 (a)]. Hence $J\subset\mathrm{hy}(\mathfrak{C})$. Since J is dense in $\mathrm{hy}(\mathfrak{G}')$, it follows similarly that $\mathfrak{G}'=\mathfrak{C}$. This means that $\mathfrak{Ker}(\mathfrak{f})$ is contained in the center of $\mathfrak{G}'$. Q.E.D.

2. The extension $(\mathfrak{G}^*, \gamma)$ is central; Proof based upon the Hopf algebra theory

Here we give a second proof for $(\mathfrak{G}^*, \gamma)$ being a central extension of $\mathfrak{G}$ based on the Hopf algebra techniques [2].

Let C be a k-coalgebra and V a right C-comodule [2, § 2.0] with structure map $\lambda\colon V \to V \otimes C$. V is a left C^*-module via $X \cdot v = (I \otimes X)\lambda(v)$, $X \in C^*$, $v \in V$ [2, § 2.1]. Hence the dual space V^* is a right C^*-module by transpose.

Let W be a left C-comodule with structure $\rho\colon W \to C \otimes W$. Similarly W is a right C^*-module.

A subalgebra $B \subset C^*$ is *dense* if the corresponding linear map $C \to B^*$ is injective.

2.1 LEMMA. *Suppose there are a dense subalgebra $B \subset C^*$, an injective right B-linear map $\iota\colon W \hookrightarrow V^*$ and a subcoalgebra $D \subset C$ such that $\lambda(V) \subset V \otimes D$. Then $\rho(W) \subset D \otimes W$.*

PROOF. We view W as a right B-submodule of V^* via ι. For each $b \in B$, the transpose of the composite

$$V \xrightarrow{\lambda} V \otimes C \xrightarrow{I \otimes b} V$$

induces the composite

$$W \xrightarrow{\rho} C \otimes W \xrightarrow{b \otimes I} W,$$

i.e., $\langle v, (b \otimes I)\rho(w) \rangle = \langle (I \otimes b)\lambda(v), w \rangle$, $\forall v \in V$, $w \in W$, $b \in B$. Since B is dense in C^* (hence $C \hookrightarrow B^*$), it follows that

$$\langle I \otimes v, \rho(w) \rangle = \langle \lambda(v), w \otimes I \rangle \in C, \ \forall v \in V, \ w \in W.$$

If $\lambda(V) \subset V \otimes D$ for some subcoalgebra $D \subset C$, we have

$$\langle I \otimes V, \rho(w) \rangle \subset D.$$

Since $D \otimes W = (C \otimes W) \cap \mathrm{Hom}_k(V, D)$ in $\mathrm{Hom}_k(V, C)$, where we view $C \otimes W \subset C \otimes V^* \subset \mathrm{Hom}_k(V, C)$, it follows that $\rho(W) \subset D \otimes W$. Q.E.D.

Let A be a commutative k-Hopf algebra and $\mathfrak{G} = \mathrm{Spec}(A)$ the corresponding affine k-group scheme. The algebra map

$$\rho\colon A \to A \otimes A, \quad \rho(a) = \sum_{(a)} a_{(1)} S(a_{(3)}) \otimes a_{(2)}$$

where S denotes the antipode of A, represents the inner action

$$\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G},\ (g,h)\mapsto ghg^{-1}$$

and hence makes A a *left* A-comodule.

If A is finitely generated, the adjoint representation

$$\mathfrak{Ad}:\ \mathfrak{G}\times \mathrm{hy}(\mathfrak{G})_a\to \mathrm{hy}(\mathfrak{G})_a$$

corresponds to a *right* A-comodule structure map

$$\lambda:\ \mathrm{hy}(\mathfrak{G})\to \mathrm{hy}(\mathfrak{G})\otimes A.$$

This is described as follows: Let $M=\{a\in A\mid \varepsilon(a)=0\}$ where ε is the unit of $\mathfrak{G}(k)$. M is an ideal of A and $[A/M^n:k]<\infty$ for all $n\geq 0$, since A is finitely generated. Since $\rho(M)\subset A\otimes M$, $\rho(M^n)\subset A\otimes M^n$. Hence A/M^n are finite dimensional left A-comodules. In general if V is a finite dimensional left C-comodule, V^* has a natural right C-comodule structure [2, 5.1.4] such that the corresponding left C^*-module V^* is the transpose of the right C^*-module V. Hence $(A/M^n)^*$ form an inductive system of *right* A-comodules. Since

$$\mathrm{hy}(\mathfrak{G})=\varinjlim_n (A/M^n)^* \quad [4,\ 2.1.11],$$

the space $\mathrm{hy}(\mathfrak{G})$ has a right A-comodule structure. Let this be λ.

Since A is finitely generated, A° is dense in A^* [2, § 6.1]. $\mathrm{hy}(\mathfrak{G})$ is a left A°-module by λ and A a right A°-module by ρ. Hence A^* is a left A°-module by transpose. The inclusion $\mathrm{hy}(\mathfrak{G})\hookrightarrow A^*$ is left A°-linear by definition. This means that the adjoint action

$$\mathrm{ad}:\ A^\circ\otimes \mathrm{hy}(\mathfrak{G})\to \mathrm{hy}(\mathfrak{G}),\ x\otimes y\mapsto \sum_{(x)} x_{(1)}yS(x_{(2)})$$

corresponds to λ. Hence λ represents the adjoint representation $\mathfrak{Ad}$.

Suppose further $\mathfrak{G}$ is connected algebraic, or equivalently $\mathrm{hy}(\mathfrak{G})$ is dense in A°. Let J be a dense subhyperalgebra of $\mathrm{hy}(\mathfrak{G})$ and ι: $A\hookrightarrow J^\circ$ the Hopf algebra injection associated with the inclusion $J\hookrightarrow A^\circ$. Let

$$\rho':\ J^\circ\to J^\circ\otimes J^\circ,\ \rho'(x)=\sum_{(x)} x_{(1)}S(x_{(3)})\otimes x_{(2)}$$

be the algebra map representing the inner action of $\mathrm{Spec}(J^\circ)$. Then

$$\rho'\circ\iota=(\iota\otimes\iota)\circ\rho$$

clearly. Let

$$\lambda': \mathrm{hy}(\mathfrak{G}) \xrightarrow{\lambda} \mathrm{hy}(\mathfrak{G})\otimes A \xrightarrow{I\otimes\iota} \mathrm{hy}(\mathfrak{G})\otimes J^{\circ}$$

be the composite.

The associated left J-module structure on $\mathrm{hy}(\mathfrak{G})$ is obtained by restricting the adjoint action ad: $\mathrm{hy}(\mathfrak{G})\otimes \mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G})$ to $J\otimes\mathrm{hy}(\mathfrak{G})$. Hence J is left J-stable, or equivalently we have

$$\lambda'(J)\subset J\otimes J^{\circ}.$$

Since ι is injective, it follows that $\lambda(J)\subset J\otimes A$.

Consider the following data:

$$\begin{array}{ll} \rho': J^{\circ}\to J^{\circ}\otimes J^{\circ} & \text{(left } J^{\circ}\text{-comodule structure)} \\ \lambda': J\to J\otimes J^{\circ} & \text{(right } J^{\circ}\text{-comodule structure)} \\ J\subset (J^{\circ})^{*} & \text{(a dense subalgebra)} \\ \iota(A)\subset J^{\circ} & \text{(a subcoalgebra).} \end{array}$$

Hence J° is a right J-module by ρ', J a left J-module by λ' and J^{*} a right J-module by transpose.

2.2 LEMMA. *The inclusion $J^{\circ}\hookrightarrow J^{*}$ is right J-linear.*

PROOF. Let $x\in J^{\circ}$ and $a\in J$. We want to show

$$\langle I\otimes a, \rho'(x)\rangle = \langle \lambda'(a), x\otimes I\rangle \in J^{\circ}.$$

But for $b\in J$

$$\begin{aligned} &\langle b, \langle I\otimes a, \rho'(x)\rangle\rangle = \langle b\otimes a, \rho'(x)\rangle \\ &= \sum_{(x)} \langle b, x_{(1)}S(x_{(3)})\rangle\langle a, x_{(2)}\rangle = \sum_{(x)} \langle b_{(1)}, x_{(1)}\rangle\langle a, x_{(2)}\rangle\langle S(b_{(2)}), x_{(3)}\rangle \\ &= \sum_{(b)} \langle b_{(1)}aS(b_{(2)}), x\rangle = \langle\langle I\otimes b, \lambda'(a)\rangle, x\rangle \\ &= \langle b, \langle \lambda'(a), x\otimes I\rangle\rangle. \end{aligned}$$

Q.E.D.

Since $\lambda'(J)\subset J\otimes\iota(A)$, it follows from (2.1) that

$$\rho'(J^{\circ})\subset \iota(A)\otimes J^{\circ}.$$

This implies that the inner action $\mathrm{Spec}(J^{\circ})\times\mathrm{Spec}(J^{\circ})\to\mathrm{Spec}(J^{\circ})$ induces a left action $\mathfrak{G}\times\mathrm{Spec}(J^{\circ})\to\mathrm{Spec}(J^{\circ})$ through the projection $\mathrm{Spec}(\iota):\mathrm{Spec}(J^{\circ})\to\mathfrak{G}$. This proves that the kernel of Spec (ι) is contained in the center of Spec (J°). Hence

2.3 THEOREM. *Let $\mathfrak{G}=\mathrm{Spec}\,(A)$ be a connected affine algebraic k-group scheme corresponding to the commutative Hopf algebra A. Let J be a dense subhyperalgebra of* hy $(\mathfrak{G})$ *and ι: $A\hookrightarrow J^{\circ}$ the Hopf algebra injection corresponding to the* *inclusion* $J \hookrightarrow A^\circ$. *Then* (Spec (J°), Spec (ι)) *is a central extension of* $\mathfrak{G}$.

3. (PE) affine algebraic group schemes

3.1 PROPOSITION. *A connected affine algebraic k-group scheme* $\mathfrak{G}$ *is* (*PE*) *if equivalently*:

(i) $(\mathfrak{G}^*, \gamma)$ *is a proetale extension,*

(ii) *Let* $\mathfrak{G}'$ *be an affine algebraic k-group scheme and* $\mathfrak{f}$: $\mathfrak{G}' \to \mathfrak{G}$ *a morphism of k-group schemes. If there is a map of hyperalgebras* s: $\mathrm{hy}(\mathfrak{G}) \to \mathrm{hy}(\mathfrak{G}')$ *such that* $\mathrm{hy}(\mathfrak{f}) \circ s = I$ *then* $\mathrm{Im}(s)$ *is a closed* [3, p. 261] *subhyperalgebra of* $\mathrm{hy}(\mathfrak{G}')$.

PROOF. (i) $\Rightarrow$ (ii). Let $\mathfrak{G}''$ be the connected closed subgroup scheme of $\mathfrak{G}'$ such that $\mathrm{hy}(\mathfrak{G}'') = A(\mathrm{Im}(s))$ the algebraic hull of $\mathrm{Im}(s)$ [3, p. 261]. Let $\mathfrak{G}'' = \mathrm{Spec}(B)$ and ν: $B \to \mathrm{hy}(\mathfrak{G}^\circ)$ correspond to the Hopf algebra map s: $\mathrm{hy}(\mathfrak{G}) \to B^\circ$. The Hopf algebra map ν is injective, since $\mathrm{Im}(s)$ is dense in $\mathrm{hy}(\mathfrak{G}'')$. Hence $\mathfrak{G}''$ is an algebraic quotient of $\mathfrak{G}^* = \mathrm{Spec}\,(\mathrm{hy}(\mathfrak{G})^\circ)$ via $\mathrm{Spec}(\nu)$. Since the composite

$$\mathfrak{G}^* \xrightarrow{\mathrm{Spec}\,(\nu)} \mathfrak{G}'' \xrightarrow{\mathfrak{f}} \mathfrak{G}$$

equals γ by definition, it follows that $\mathfrak{f}$: $\mathfrak{G}'' \to \mathfrak{G}$ is an etale morphism. Hence $\mathrm{hy}\,(\mathfrak{f})$: $\mathrm{hy}\,(\mathfrak{G}'') \to \mathrm{hy}\,(\mathfrak{G})$ is bijective by [3, 1.1]. This implies $\mathrm{Im}\,(s) = \mathrm{hy}\,(\mathfrak{G}'')$.

(ii) $\Rightarrow$ (i). Let $\mathfrak{G} = \mathrm{Spec}(A)$. View A as a sub-Hopf algebra of $\mathrm{hy}(\mathfrak{G})^\circ$. Let B be a finitely generated sub-Hopf algebra of $\mathrm{hy}(\mathfrak{G})^\circ$ containing A. Let $\mathfrak{G}' = \mathrm{Spec}\,(B)$ and $\mathfrak{f}$: $\mathfrak{G}' \to \mathfrak{G}$ correspond to the inclusion $A \hookrightarrow B$. A hyperalgebra map s: $\mathrm{hy}(\mathfrak{G}) \to B^\circ$ corresponds to the inclusion $B \hookrightarrow \mathrm{hy}(\mathfrak{G})^\circ$. Since $\mathrm{Im}(s) \subset \mathrm{hy}(\mathfrak{G}')$ and $\mathrm{hy}(\mathfrak{f}) \circ s = I$ by definition, it follows that $\mathrm{Im}(s)$ is a closed sub-hyperalgebra of $\mathrm{hy}(\mathfrak{G}')$. But since $\mathrm{Im}(s)$ is dense in $\mathrm{hy}(\mathfrak{G}')$, we have $\mathrm{Im}(s) = \mathrm{hy}(\mathfrak{G}')$. Hence $\mathfrak{f}$: $\mathfrak{G}' \to \mathfrak{G}$ is an etale covering by [3, 1.1]. Since $\mathfrak{G}^*$ is the projective limit of $\mathfrak{G}'$, $(\mathfrak{G}^*, \gamma)$ is a proetale extension.
Q.E.D.

If $\mathfrak{G}$ is (PE), $(\mathfrak{G}^*, \gamma)$ is a *universal proetale extension* of $\mathfrak{G}$ (cf. [1, V, §3, 4.1]) in the sense:

3.2 PROPOSITION. *Let* $\mathfrak{G}$ *be a connected* (*PE*) *affine algebraic k-group scheme. Let* η: $\mathfrak{H} \to \mathfrak{G}$ *be an epimorphism, where* $\mathfrak{H}$ *is a connected affine k-group scheme and* $\mathfrak{Ker}\,(\eta)$ *is proetale. There is a unique morphism of k-group schemes* η^*: $\mathfrak{G}^* \to \mathfrak{H}$ *such that* $\eta \circ \eta^* = \gamma$.

PROOF. Let $\mathfrak{H} \to \mathfrak{H}'$ be an algebraic quotient of $\mathfrak{H}$ through which η factors. $\mathfrak{H}$ is the projective limit of such quotients $\mathfrak{H}'$. Let η': $\mathfrak{H}' \to \mathfrak{G}$ be the induced epimorphism. Then $\mathfrak{H}'$ is connected affine algebraic and $\mathfrak{Ker}(\eta')$ is etale, since it is an algebraic quotient of $\mathfrak{Ker}(\eta)$. Hence $\mathrm{hy}(\eta')\colon \mathrm{hy}(\mathfrak{H}')\to\mathrm{hy}(\mathfrak{G})$ is bijective [3, 1.1]. The Hopf algebra map $\mathcal{O}(\mathfrak{H}')\to\mathrm{hy}(\mathfrak{G})^{\circ}$ corresponding to $\mathrm{hy}(\mathfrak{G})\xrightarrow{\sim}\mathrm{hy}(\mathfrak{H}')\hookrightarrow\mathcal{O}(\mathfrak{H}')^{\circ}$ determines a unique morphism $\eta'^{*}\colon \mathfrak{G}^{*}\to\mathfrak{H}'$ such that $\eta'\circ\eta'^{*}=\gamma$. They determine a unique morphism $\eta^{*}\colon \mathfrak{G}^{*}\to\mathfrak{H}$ with $\eta\circ\eta^{*}=\gamma$ going to $\varprojlim$. Q.E.D.

The purpose of the rest of this paper is to show that when $p>0$ the connected affine algebraic k-group scheme $\mathfrak{G}$ is (PE) if and only if the quotient group scheme $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$ is finite.

3.3 Proposition. *Let $\mathfrak{G}$ be a connected affine algebraic k-group scheme. If $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$ is finite, then $\mathfrak{G}$ is (PE).*

Proof. Let $\mathfrak{H}'$ be a locally algebraic k-group scheme and $\mathfrak{f}\colon \mathfrak{G}'\to\mathfrak{G}$ a morphism of k-group schemes. Suppose $s\colon \mathrm{hy}(\mathfrak{G})\to\mathrm{hy}(\mathfrak{G}')$ is a hyperalgebra map such that $\mathrm{hy}(\mathfrak{f})\circ s=I$. Let $J=\mathrm{Im}(s)$. Then $[J,J]$ is a closed subhyperalgebra of $\mathrm{hy}(\mathfrak{G}')$ by [3, 0.3.4 (f)]. Since $J/\!/[J,J]$ is finite dimensional, J is a closed subhyperalgebra of $\mathrm{hy}(\mathfrak{G}')$ by [3, 0.3.4 (b)]. Hence $\mathfrak{G}$ is (PE). Q.E.D.

3.4 Proposition. *Each quotient of a connected (PE) affine algebraic k-group scheme is (PE).*

Proof. Let $\mathfrak{H}$ be a quotient group scheme of a connected (PE) affine algebraic k-group scheme $\mathfrak{G}$. Then $\mathfrak{H}$ is connected affine algebraic. Let $\mathfrak{g}\colon \mathfrak{H}'\to\mathfrak{H}$ be a morphism of k-group schemes, where $\mathfrak{H}'$ is affine algebraic. Suppose there is a hyperalgebra map $t\colon \mathrm{hy}(\mathfrak{H})\to\mathrm{hy}(\mathfrak{H}')$ with $\mathrm{hy}(\mathfrak{g})\circ t=I$. Construct the pullback diagram

$$\begin{array}{ccc} \mathfrak{G}' & \xrightarrow{\ \mathfrak{f}\ } & \mathfrak{G} \\ {\scriptstyle \mathfrak{p}'}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}} \\ \mathfrak{H}' & \xrightarrow{\ \mathfrak{g}\ } & \mathfrak{H} \end{array}$$

where $\mathfrak{G}'$ is also an affine algebraic k-group scheme. The induced diagram

$$\begin{array}{ccc} T(\mathfrak{G}') & \xrightarrow{\ T(\mathfrak{f})\ } & T(\mathfrak{G}) \\ {\scriptstyle T(\mathfrak{p}')}\downarrow & & \downarrow{\scriptstyle T(\mathfrak{p})} \\ T(\mathfrak{H}') & \xrightarrow{\ T(\mathfrak{g})\ } & T(\mathfrak{H}) \end{array}$$

is a pullback diagram in the category W_k by [4, 2.1.1]. Let $s\colon\mathrm{hy}(\mathfrak{G})\to T(\mathfrak{G}')$ be a unique coalgebra map such that $T(\mathfrak{p}')\circ s=t\circ\mathrm{hy}(\mathfrak{p})$ and $T(\mathfrak{f})\circ s=I$. From the uniqueness follows that s is a Hopf algebra map. Hence $\mathrm{Im}(s)\subset\mathrm{hy}(\mathfrak{G}')$. Since $\mathfrak{G}$ is (PE), Im (s) is a closed subhyperalgebra of hy $(\mathfrak{G}')$. Hence by [3, 0.3.2 (b)], Im $(t)=$ hy $(\mathfrak{p}')$(Im (s)) is a closed subhyperalgebra of hy $(\mathfrak{H}')$. Therefore $\mathfrak{H}$ is (PE). Q.E.D.

3.5 PROPOSITION. *Let l/k be a finite field extension. A connected affine algebraic k-group scheme $\mathfrak{G}$ is (PE) if and only if so is the l-group scheme $\mathfrak{G}\otimes l$.*

PROOF. The l-group scheme $\mathfrak{G}\otimes l$ is connected affine algebraic and $\mathrm{hy}_l(\mathfrak{G}\otimes l)=\mathrm{hy}(\mathfrak{G})\otimes l$ [3, 0.3.1 (b)]. Since l/k is finite, the dual l-Hopf algebra $\mathrm{hy}_l(\mathfrak{G}\otimes l)^\circ$ equals $\mathrm{hy}(\mathfrak{G})^\circ\otimes l$ [3, p. 269]. Hence the l-group scheme $\mathfrak{G}\otimes l$ is (PE) if and only if the extension $\gamma\otimes l$: $\mathfrak{G}^*\otimes l\rightarrow\mathfrak{G}\otimes l$ is proetale. Since an affine k-group scheme $\mathfrak{H}$ is proetale if and only if so is the l-group scheme $\mathfrak{H}\otimes l$ [1, III, § 3, 7.7], $\gamma\otimes l$ is proetale if and only if $\mathfrak{G}$ is (PE). Q.E.D.

Let $p>0$. We show that $\mathfrak{G}_a$ and $\mathfrak{G}_m$ are *not* (PE) in the next section.

3.6 THEOREM. *Let $\mathfrak{G}$ be a connected affine algebraic k-group scheme, where $p>0$. $\mathfrak{G}$ is (PE) if and only if the quotient group scheme $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$ is finite.*

PROOF. 'If' part follows from (3.3). Let $\mathfrak{G}$ be (PE). Then so is $\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$ by (3.4). Suppose $\mathfrak{G}$ is commutative (PE). There is a finite normal closed subgroup scheme $\mathfrak{N}\lhd\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{N}$ is smooth [1, III, § 3, 6.10]. Since $\mathfrak{G}/\mathfrak{N}$ is (PE) by (3.4), we have only to prove that a connected commutative smooth (PE) affine algebraic k-group scheme should be trivial. Let $\mathfrak{G}$ be such a group scheme with the multiplicative part $\mathfrak{G}^m$ [1, IV, § 3, 1.1].

If $\mathfrak{G}/\mathfrak{G}^m\neq 0$, there is a nontrivial morphism of k-group schemes $\mathfrak{f}$: $\mathfrak{G}/\mathfrak{G}^m\rightarrow\mathfrak{G}_a$ by [1, IV, § 2, 2.1]. Since the image $\mathfrak{f}(\mathfrak{G}/\mathfrak{G}^m)$ is a smooth subgroup scheme of $\mathfrak{G}_a$, $\mathfrak{f}$ is an epimorphism by [1, IV, § 2, 1.1]. This contradicts the fact for $\mathfrak{G}_a$ being not (PE). Hence $\mathfrak{G}=\mathfrak{G}^m$. Since $\mathfrak{G}$ is connected smooth, $\mathfrak{G}$ is a k-torus [1, IV, § 1, 3.9]. Hence there are a finite extension of fields l/k and an integer $n\geq 0$ such that $\mathfrak{G}\otimes l\simeq(\mathfrak{G}_m)^n\otimes l$ as l-group schemes [1, IV, § 1, 3.8]. The l-group scheme $\mathfrak{G}\otimes l$ is (PE) by (3.5). Since $\mathfrak{G}_m$ is not (PE), n must be 0. Hence $\mathfrak{G}$ is trivial. Q.E.D.

3.7 COROLLARY. *Let k be perfect with $p>0$. For each connected affine algebraic k-group scheme $\mathfrak{G}$, the following are equivalent:*

i) *$\mathfrak{G}$ has a universal group covering,*

ii) *$(\mathfrak{G}^*,\gamma)$ is the universal group covering of $\mathfrak{G}$,*

iii) *$(\mathfrak{G}^*,\gamma)$ is an etale group covering of $\mathfrak{G}$,*

iv) *$\mathfrak{G}/[\mathfrak{G},\mathfrak{G}]$ is finite and $\mathrm{hy}(\mathfrak{G})^\circ$ is finitely generated.*

PROOF. This follows immediately from [3, Th. 1.9] and (3.6).

4. $\mathfrak{G}_a$ and $\mathfrak{G}_m$ are not (PE)

Let $\mathfrak{G}$ be a locally algebraic k-group scheme and $\mathcal{O}_e$ its local ring at unit e with the maximal ideal m_e. Since $\mathrm{hy}(\mathfrak{G})=(\mathcal{O}_e)^\circ=\varinjlim_n (\mathcal{O}_e/m_n^e)^*$ [3, p. 259] [4, 2.1.11], we have

$$\mathrm{hy}(\mathfrak{G})^*=\hat{\mathcal{O}}_e=\text{the } m_e\text{-adic completion of } \mathcal{O}_e.$$

Let $\mathfrak{G}=\mathrm{Spec}\,(A)$ be an affine algebraic k-group scheme and $M=\mathfrak{Ker}\,(\varepsilon)$. Since $\mathcal{O}_e=A_M=$ the M-adic localization of A, we have

$$\mathrm{hy}(\mathfrak{G})^*=\hat{A}=\text{the } M\text{-adic completion of } A.$$

Similarly we have

$$\begin{aligned}(\mathrm{hy}(\mathfrak{G})\otimes\mathrm{hy}(\mathfrak{G}))^*&=\mathrm{hy}(\mathfrak{G}\times\mathfrak{G})^*\\ &=\widehat{A\otimes A}=\text{the } (A\otimes M+M\otimes A)\text{-adic completion of } A\otimes A.\end{aligned}$$

The coproduct $\varDelta$: $A\to A\otimes A$ induces $\hat{\varDelta}$: $\hat{A}\to\widehat{A\otimes A}$. This is the transpose of the product: $\mathrm{hy}(\mathfrak{G})\otimes\mathrm{hy}(\mathfrak{G})\to\mathrm{hy}(\mathfrak{G})$.

By [2, 6.0.3] we have

$$\mathrm{hy}(\mathfrak{G})^\circ=\hat{\varDelta}^{-1}(\mathrm{hy}(\mathfrak{G})^*\otimes\mathrm{hy}(\mathfrak{G})^*).$$

The map $\hat{\varDelta}$ restricted to $\mathrm{hy}(\mathfrak{G})^\circ$ is the coproduct

$$\varDelta:\ \mathrm{hy}(\mathfrak{G})^\circ\to\mathrm{hy}(\mathfrak{G})^\circ\otimes\mathrm{hy}(\mathfrak{G})^\circ.$$

In particular if $x\in\hat{A}$ satisfies $\hat{\varDelta}(x)=x\otimes 1+1\otimes x\in\widehat{A\otimes A}$, then $x\in\mathrm{hy}(\mathfrak{G})^\circ$ and $\varDelta(x)=x\otimes 1+1\otimes x\in\mathrm{hy}(\mathfrak{G})^\circ\otimes\mathrm{hy}(\mathfrak{G})^\circ$. Similarly if $y\in\hat{A}$ satisfies $\hat{\varDelta}(y)=y\otimes y\in\widehat{A\otimes A}$, then $y\in\mathrm{hy}(\mathfrak{G})^\circ$ and $\varDelta(y)=y\otimes y\in\mathrm{hy}(\mathfrak{G})^\circ\otimes\mathrm{hy}(\mathfrak{G})^\circ$, so y is invertible in $\hat{A}$ unless $y=0$ [2, 9.2.5].

4.1 PROPOSITION. *If $p>0$, the additive group scheme $\mathfrak{G}_a$ is not (PE).*

PROOF. Recall $\mathfrak{G}_a=\mathrm{Spec}\,(k[T])$ [3, p. 256] where $\varDelta(T)=T\otimes 1+1\otimes T$, $\varepsilon(T)=0$ and $S(T)=-T$. Hence

$$\mathrm{hy}(\mathfrak{G}_a)^*=k[[T]]=\text{the } (T)\text{-adic completion of } k[T]$$

and

$$\begin{aligned}(\mathrm{hy}(\mathfrak{G}_a)\otimes\mathrm{hy}(\mathfrak{G}_a))^*&=k[[T\otimes 1, 1\otimes T]]\\ &=\text{the } (T\otimes 1, 1\otimes T)\text{-adic completion of } k[T]\otimes k[T].\end{aligned}$$

The diagonal map $\hat{\varDelta}$: $k[[T]]\to k[[T\otimes 1, 1\otimes T]]$ is determined by $\hat{\varDelta}(T)=T\otimes 1+1\otimes T$.

If $f(T)=\lambda_0 T+\lambda_1 T^p+\lambda_2 T^{p^2}+\cdots+\lambda_n T^{p^n}+\cdots$ is a p-power power series with $\lambda_i \in k$, then $\hat{\Delta}(f(T))=f(T)\otimes 1+1\otimes f(T)$. Hence $f(T)\in \mathrm{hy}(\mathfrak{G}_a)^\circ$. There is a p-power power series $f(T)$ such that T and $f(T)$ are algebraically independent over k [6, § 5]. Hence there is an injective Hopf algebra map β: $k[T]\otimes k[T]\hookrightarrow \mathrm{hy}(\mathfrak{G}_a)^\circ$ such that $\beta(T\otimes 1)=T$, $\beta(1\otimes T)=f(T)$, where the Hopf structure on $k[T]\otimes k[T]$ corresponds to the direct product $\mathfrak{G}_a\times\mathfrak{G}_a=\mathrm{Spec}\,(k[T]\otimes k[T])$. The composite

$$\mathfrak{G}_a^*=\mathrm{Spec}\,(\mathrm{hy}\,(\mathfrak{G}_a)^\circ)\xrightarrow{\mathrm{spec}\,(\beta)}\mathfrak{G}_a\times\mathfrak{G}_a\xrightarrow{\mathfrak{pr}_1}\mathfrak{G}_a$$

where $\mathfrak{pr}_1$ denotes the projection onto the first term, equals γ. Since $\mathfrak{pr}_1$ is not an etale group covering, γ is not proetale. Hence $\mathfrak{G}_a$ is not (PE). Q.E.D.

4.2 PROPOSITION. *If $p>0$, the multiplicative group scheme $\mathfrak{G}_m$ is not (PE).*

PROOF. Recall that $\mathfrak{G}_m=\mathrm{Spec}\,(k[X,X^{-1}])$ [3, p. 256] where $\Delta(X)=X\otimes X$, $\varepsilon(X)=1$ and $S(X)=X^{-1}$. Put $T=X-1$. Then $k[X,X^{-1}]$ is the localization of $k[T]$ with respect to one element $1+T$. Hence the (T)-adic localizations of $k[T]$ and $k[X,X^{-1}]$ are the same. Similarly the $(T\otimes 1, 1\otimes T)$-adic localizations of $k[T]\otimes k[T]$ and $k[X,X^{-1}]\otimes k[X,X^{-1}]$ are the same. Therefore

$$\mathrm{hy}(\mathfrak{G}_m)^*=k[[T]] \quad \text{and}$$
$$(\mathrm{hy}(\mathfrak{G}_m)\otimes\mathrm{hy}(\mathfrak{G}_m))^*=k[[T\otimes 1, 1\otimes T]].$$

The diagonal map $\hat{\Delta}$: $k[[T]]\to k[[T\otimes 1,1\otimes T]]$ is determined by $\hat{\Delta}(T)=T\otimes T+T\otimes 1+1\otimes T$. The inclusion $k[X,X^{-1}]\hookrightarrow k[[T]]$ by $X\mapsto 1+T$.

Let

$$G(k[[T]])=\{x\in k[[T]]\mid \hat{\Delta}(x)=x\otimes x,\ x\neq 0\}.$$

This is a subgroup of units $k[[T]]^\times$ and equal to

$$G(\mathrm{hy}(\mathfrak{G}_m)^\circ)=\{x\in \mathrm{hy}(\mathfrak{G}_m)^\circ\mid \Delta(x)=x\otimes x,\quad x\neq 0\}.$$

Since $1+T\in G(k[[T]])$, it follows that $1+T^{p^n}=(1+T)^{p^n}\in G(k[[T]])$ for all $n\geq 0$.

For each family of integers $a_n\geq 0$, the infinite product

$$\textstyle\prod_n (1+T^{p^n})^{a_n}$$

is a well-defined element of $G(k[[T]])$.

Let $\hat{\boldsymbol{Z}}_{(p)}=\varprojlim_n \boldsymbol{Z}/(p^n)$ be the (p)-adic completion of $\boldsymbol{Z}$. Each element of $\hat{\boldsymbol{Z}}_{(p)}$ can be uniquely written as $\sum_n a_n p^n$, where $0\leq a_n<p$. The map

$$\chi:\ \hat{\boldsymbol{Z}}_{(p)}\to G(k[[T]]),\ \chi(\sum_n a_n p^n)=\textstyle\prod_n (1+T^{p^n})^{a_n}$$

is well-defined. We claim that χ is an injective group homomorphism.

Indeed the multiplicative order of $1+T$ in $(k[T]/T^{p^n})^{\times}$ is p^n. Hence there is an injective group homomorphism

$$Z/(p^n) \hookrightarrow (k[T]/T^{p^n})^{\times}, \ 1 \mapsto 1+T.$$

Taking $\varprojlim$ we obtain an injective group homomorphism

$$\hat{Z}_{(p)} \hookrightarrow k[[T]]^{\times}$$

which is χ.

The quotient group $\hat{Z}_{(p)}/Z$ contains at least one torsion-free element $x \bmod Z$ with $x \in \hat{Z}_{(p)}$ by (4.3). Hence we have an injective group homomorphism

$$\alpha: \ Z \times Z \hookrightarrow \hat{Z}_{(p)}, \ \alpha(1,0)=1, \ \alpha(0,1)=x.$$

This induces an injective Hopf algebra map

$$\bar{\alpha}: \ k[X, X^{-1}] \otimes k[X, X^{-1}] \hookrightarrow \mathrm{hy}\,(\mathfrak{G}_m)^{\circ}$$

where $\bar{\alpha}(X \otimes 1)=X$, $\bar{\alpha}(1 \otimes X)=\chi(x)$. The composite

$$\mathfrak{G}_m^* = \mathrm{Spec}\,(\mathrm{hy}\,(\mathfrak{G}_m)^{\circ}) \xrightarrow{\mathrm{Spec}\,(\bar{\alpha})} \mathfrak{G}_m \times \mathfrak{G}_m \xrightarrow{\mathfrak{pr}_1} \mathfrak{G}_m$$

where $\mathfrak{pr}_1$ denotes the projection onto the first term, is γ. Since $\mathfrak{pr}_1$ is not etale, $\mathfrak{G}_m$ is not (PE). Q.E.D.

4.3 Lemma. *The group $\hat{Z}_{(p)}/Z$ has a torsion-free element.*

Proof. This is perhaps a known fact. We give an elementary proof. Let $e>1$ be an integer and

$$x = p + p^e + p^{e^2} + \cdots + p^{e^n} + \cdots \in \hat{Z}_{(p)}.$$

We claim that $x \bmod Z$ is torsion-free in $\hat{Z}_{(p)}/Z$.

Suppose there is an integer $m>0$ such that $mx \in Z$. Write $m=a_0+a_1p+\cdots+a_np^n$ where $0 \le a_i < p$. Take an integer $N>1$ so that $e^{N+1}-e^N-n>1$. Let

$$\sum_{\substack{i \le n \\ j \le N}} a_i p^{i+e^j} = b_0 + b_1 p + \cdots + b_M p^M$$

where $0 \le b_r < p$. Take an integer $l \ge N$ so that $n+e^l \ge M$. Thus $b_0+b_1p+\cdots+b_Mp^M < p^{n+e^l+1}$. On the other hand

$$\sum_{\substack{i \le n \\ N<j \le l}} a_i p^{i+e^j} < p^{n+e^l+1}$$

since $i_1+e^{j_1}=i_2+e^{j_2}$ with $i_1, i_2 \le n$ and $j_1, j_2 > N$ implies $i_1=i_2$ and $j_1=j_2$. Hence

$$\sum_{\substack{i\leq n\\ j\leq l}} a_i p^{i+e^j} < 2p^{n+e^l+1} \leq p^{n+e^l+2} \leq p^{e^{l+1}}.$$

Therefore

$$\sum_{\substack{i\leq n\\ j\leq l}} a_i p^{i+e^j} = c_0 + c_1 p + \cdots + c_L p^L,$$

where $0\leq c_r<p$ and $L<e^{l+1}$. We have

$$mx = c_0 + c_1 p + \cdots + c_L p^L + \sum_{\substack{i\leq n\\ j>l}} a_i p^{i+e^j}.$$

Here $i_1+e^{j_1}=i_2+e^{j_2}$ with $j_1, j_2>l$ implies $i_1=i_2$ and $j_1=j_2$ and $L<e^{l+1}\leq i+e^j$ for all $j>l$. Hence $mx\notin \boldsymbol{Z}$ a contradiction. Q.E.D.

References

[1] Demazure, M. and P. Gabriel, Groupes algébriques, Tome I, North-Holland, Amsterdam, 1970.
[2] Sweedler, M. E., Hopf algebras, Benjamin, New York, 1969.
[3] Takeuchi, M., On coverings and hyperalgebras of affine algebraic groups, Trans. Amer. Math. Soc. **211** (1975), 249-275.
[4] Takeuchi, M., Tangent coalgebras and hyperalgebras, I, Japan. J. Math. **42** (1974), 1-143.
[5] Takeuchi, M., Tangent coalgebras and hyperalgebras, II, to appear.
[6] Takeuchi, M., A characterization of the Galois subbialgebras $H_k(K/F)$, J. Algebra (to appear).

(Received January 30, 1976)

University of Tsukuba
Ibaraki
300-31 Japan