The structure of the Hecke algebra $H_C(G_\sigma, U_\sigma)$ for a finite Steinberg group G_σ

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§1. Introduction.

Let G be a finite universal Chevalley group of type A_l, D_l $(l \geqq 4)$, or E_6. By virtue of the suitable automorphism σ of G of order 2, we can define a finite Steinberg group $G_\sigma = \{g \in G | \sigma(g) = g\}$ as usual (see [9]).

Let B be a Borel subgroup of G. It is well-known that $B = UH$ where U is a maximal connected unipotent subgroup of G and H is a maximal torus of G. Of course there are many possibilities for B, U and H, but we adopt the same subgroups B, U and H as in [9].

Let $B_\sigma = B \cap G_\sigma$, $U_\sigma = U \cap G_\sigma$ and $H_\sigma = H \cap G_\sigma$. Let λ be a linear character of H_σ into the complex field $\boldsymbol{C}$. Since $B_\sigma / U_\sigma \cong H_\sigma$, we can consider λ a linear character of B_σ into $\boldsymbol{C}$ canonically.

In this paper, we discuss the structure of certain Hecke algebras, and we obtain the following theorems. In the first place, we give in Theorem 1 a presentation of the Hecke algebra $H_C(G_\sigma, U_\sigma)$, i.e., the centralizer ring of $1_{U_\sigma}^{G_\sigma}$ (see [3]), with certain generators and relations. We also show in Proposition 1 that the Hecke algebra $H_C(G_\sigma, B_\sigma)_\lambda$, i.e., the centralizer ring of $\lambda_{B_\sigma}^{G_\sigma}$, can be considered as a subalgebra of $H_C(G_\sigma, U_\sigma)$ with unity element

$$\varepsilon_\lambda = \frac{1}{|B_\sigma|} \sum_{b \in B_\sigma} \lambda(b^{-1}) b .$$

In the second place, it is proved in Theorem 2 that there exists a generic algebra A of $H_C(G_\sigma, U_\sigma)$. By virtue of the existence of the generic algebra A, this theorem also shows that $H_C(G_\sigma, U_\sigma) \cong \boldsymbol{C}N_\sigma$ where N is the same subgroup of G as in [9] and $\boldsymbol{C}N_\sigma$ is the group algebra of N_σ over $\boldsymbol{C}$.

In the last theorem, Theorem 3, we construct a generic algebra A_λ of $H_C(G_\sigma, B_\sigma)_\lambda$ from A. From the existence of A_λ we can also show that $H_C(G_\sigma, B_\sigma)_\lambda$ is isomorphic to the centralizer ring of $\lambda_{H_\sigma}^{N_\sigma}$.

It seems to be very important, in the complex representation theory of the simple groups of Lie type, to consider the irreducible characters of G_σ which appear as irreducible components of $\lambda_{B_\sigma}^{G_\sigma}$ (see [2] and [8]). The above theorems suggest a general method of treating these characters and a way of generalizing the theory of the irreducible characters of G_σ which are irreducible components of $1_{B_\sigma}^{G_\sigma}$.

In §2, we introduce the notations and conventions of this paper, and in §3, we show the precise statements of the results. We prove Theorem 1 in §4, Theorem 2 in §5 and Theorem 3 in §6.

About the structure of Hecke algebras of a finite Chevalley group, the similar results have been obtained (see [10] and [11]).

I wish to thank Prof. T. Yokonuma for his kind advice, which was very useful for computations.

§2. Notations and conventions.

Let $\mathfrak{g}$ be a simple Lie algebra over $\boldsymbol{C}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let Φ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and Π a basis of Φ, then there exists a Chevalley basis $\{X_\alpha, H_\beta \mid \alpha \in \Phi \text{ and } \beta \in \Pi\}$ of $\mathfrak{g}$.

With respect to a faithful representation ψ of $\mathfrak{g}$ there is a lattice M in the representation space of ψ which is invariant under $U_{\boldsymbol{Z}}$ and $\mathfrak{g}_{\boldsymbol{Z}}=\sum_{\alpha\in\Phi} \boldsymbol{Z}X_\alpha+\mathfrak{h}_{\boldsymbol{Z}}$ where $\boldsymbol{Z}$ is the ring of integers and $U_{\boldsymbol{Z}}$ is the $\boldsymbol{Z}$-subalgebra of a universal enveloping algebra of $\mathfrak{g}$ generated by all $X_\alpha^n/n!$ ($\alpha\in\Phi$, $\boldsymbol{Z}\ni n\geqq 0$) and $\mathfrak{h}_{\boldsymbol{Z}}=\{H\in\mathfrak{h} \mid \mu(H)\in\boldsymbol{Z}$ for all weights μ of $\psi\}$.

Then we can define a Chevalley group over an arbitrary field K as a subgroup of $\mathrm{Aut}\,(M\otimes_{\boldsymbol{Z}}K)$ generated by all

$$x_\alpha(t)=\exp tX_\alpha=\sum_{n=0}^{\infty} t^n X_\alpha^n/n!$$

where $\alpha\in\Phi$ and $t\in K$.

We call G universal if the additive group generated by all weights of ψ equals the additive group generated by all fundamental weights of $\mathfrak{g}$.

For the more precise definition of Chevalley groups see [9].

In this paper we use the following notations.

(1) k: finite field with an automorphism θ of order 2

$k_\theta=\{t\in k \mid t^\theta=t\}$,

$q=|k_\theta|$,

Φ: root system of type A_l, D_l, or E_6 with a basis Π,

G: universal Chevalley group of type Φ over k

$\{x_\alpha(t)\,|\,\alpha\in\Phi,\ t\in k\}$: set of generators of G as before (see [9])

$w_\alpha(t)=x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$

$h_\alpha(t)\ =w_\alpha(t)w_\alpha(1)^{-1}$

$\omega(\alpha)\ =w_\alpha(1)$

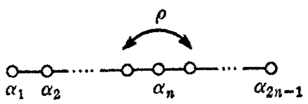
$U=\langle x_\alpha(t)\,|\,\alpha>0,\ t\in k\rangle$: subgroup of G generated by all $x_\alpha(t)$'s where α is a positive root of Φ and t is an element of k

$H=\langle h_\alpha(t)\,|\,\alpha\in\Phi,\ t\in k^*\rangle$ where $k^*=k-\{0\}$

$B=UH$

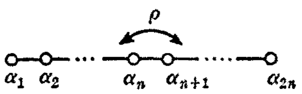
$N=\langle w_\alpha(t)\,|\,\alpha\in\Phi,\ t\in k^*\rangle$.

(2) Let ρ be an angle-preserving permutation of Π such that $\rho\neq 1$ and $\rho^2=1$. In this paper, the possibilities for Π and ρ are:

A_{2n-1} 

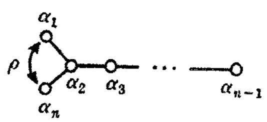
i.e. $\rho\alpha_i=\alpha_{2n-i}$ $(i=1,\cdots,n)$ and

$\Pi=\{\alpha_i\,|\,i=1,\cdots,2n-1\}$,

A_{2n} 

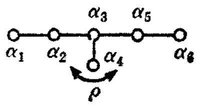
i.e. $\rho\alpha_i=\alpha_{2n+1-i}$ $(i=1,\cdots,n)$ and

$\Pi=\{\alpha_i\,|\,i=1,\cdots,2n\}$,

$D_n\ (n\geq 4)$ 

i.e. $\rho\alpha_1=\alpha_n$, $\rho\alpha_i=\alpha_i$ $(i=2,\cdots,n-1)$ and

$\Pi=\{\alpha_i\,|\,i=1,\cdots,n\}$,

E_6 

i.e. $\rho\alpha_i=\alpha_{7-i}$ $(i=1,2)$, $\rho\alpha_i=\alpha_i$ $(i=3,4)$ and

$\Pi=\{\alpha_i\,|\,i=1,\cdots,6\}$.

Let $\pi=\{\alpha_i,\rho\alpha_i\}$ be a ρ-orbit of Π. When we write $\pi=\pi_i$, we assume $\rho\alpha_i=\alpha_j$ and $i\leqq j$. We can classify these ρ-orbits into three types. When $\pi=\{\alpha_i\}$, we call π of type 1. When

$$\pi=\left\{\begin{matrix}\circ & \circ\\ \alpha_i & \rho\alpha_i\end{matrix}\right\}$$

we call π of type 2. If

$$\pi=\left\{\begin{matrix}\circ\!\!-\!\!\!-\!\!\circ\\ \alpha_i\quad\rho\alpha_i\end{matrix}\right\}$$

we call π of type 3.

σ: automorphism of G such that $\sigma(x_\alpha(t))=x_{\rho\alpha}(t^\theta)$ for all $\alpha\in\Pi$ and $t\in k$ (see [9])

$G_\sigma=\{g\in G|\sigma(g)=g\}$: finite Steinberg group.

Let D be an arbitrary subset of G, then we write $D_\sigma=D\cap G_\sigma$.

Since U is normal in B, $B=UH$ and $U\cap H=1$, U_σ is normal in B_σ, $B_\sigma=U_\sigma H_\sigma$ and $U_\sigma\cap H_\sigma=1$.

(3) $\boldsymbol{C}$: complex field

λ: $H_\sigma\to\boldsymbol{C}^*$ linear character.

Since the sequence $1\to U_\sigma\to B_\sigma\to H_\sigma\to 1$ is exact, we consider λ a linear character of B_σ.

$$\varepsilon_\lambda=\frac{1}{|B_\sigma|}\sum_{b\in B_\sigma}\lambda(b^{-1})b$$

$$e=\frac{1}{|U_\sigma|}\sum_{u\in U_\sigma}\lambda(u^{-1})u=\frac{1}{|U_\sigma|}\sum_{u\in U_\sigma}u$$

$$\varepsilon_\lambda(H_\sigma)=\frac{1}{|H_\sigma|}\sum_{h\in H_\sigma}\lambda(h^{-1})h$$

ζ: $N_\sigma\to W_\sigma=N_\sigma/H_\sigma$ canonical homomorphism. Let n be an element of N_σ and w be an element of W_σ. If $\zeta(n)=w$, then we write $n=n_w$.

$w\lambda(h)=\lambda(n_w h n_w^{-1})$ where $w\in W_\sigma$ and $h\in H_\sigma$

$W_{\sigma,\lambda}=\{w\in W_\sigma|w\lambda=\lambda\}$

$H_{\boldsymbol{C}}(G_\sigma, U_\sigma)=e\boldsymbol{C}G_\sigma e=\sum_{n\in N_\sigma}\boldsymbol{C}ene$ where $\boldsymbol{C}G_\sigma$ is the group algebra of G_σ over $\boldsymbol{C}$. A Hecke algebra $H_{\boldsymbol{C}}(G_\sigma, U_\sigma)$ is also called the centralizer ring (see [3]) or the commutor algebra of $1_{U_\sigma}^{G_\sigma}$ (see [6]).

$H_{\boldsymbol{C}}(G_\sigma, B_\sigma)_\lambda=\varepsilon_\lambda\boldsymbol{C}G_\sigma\varepsilon_\lambda=\sum_{w\in W_{\sigma,\lambda}}\boldsymbol{C}\varepsilon_\lambda n_w\varepsilon_\lambda$. It is also called the centralizer ring or the commutor algebra of $\lambda_{B_\sigma}^{G_\sigma}$.

$\mathrm{ind}\,(n_w)=|B_\sigma\backslash B_\sigma n_w B_\sigma|=|U_\sigma\backslash U_\sigma n_w U_\sigma|$ where $w\in W_\sigma$

$\alpha(n_w)=\mathrm{ind}\,(n_w)en_we$ where $w\in W_\sigma$

$\beta(n_w)=\mathrm{ind}\,(n_w)\varepsilon_\lambda n_w\varepsilon_\lambda$ where $w\in W_{\sigma,\lambda}$.

(4) With respect to the ρ-orbits we define the elements ω_i's of N_σ, the constants $c(i)$'s and the elements $h_i(t)$'s of H_σ as follows.

If π_i is of type 1, $\omega_i=\omega(\alpha_i)$, $c(i)=1$ and $h_i(t)=h_\alpha(t)$ where $\alpha\in\pi_i$ and $t\in k_\theta^*$.

If π_i is of type 2, $\omega_i=\omega(\alpha_i)\omega(\rho\alpha_i)$, $c(i)=2$ and $h_i(t)=h_\alpha(t)h_{\rho\alpha}(t^\theta)$ where $\alpha=\alpha_i\in\pi_i$ and $t\in k^*$.

If π_i is of type 3, $\omega_i=\omega(\alpha_i)\omega(\rho\alpha_i)\omega(\alpha_i)$, $c(i)=3$ and $h_i(t)=h_\alpha(t)h_{\rho\alpha}(t^\theta)$ where $\alpha=$

$\alpha_i \in \pi_i$ and $t \in k^*$.

Let u be an indeterminate over $\boldsymbol{C}$ and $\boldsymbol{C}[u]$ be the polynomial ring over $\boldsymbol{C}$. Then with respect to a ρ-orbit π_i, we define a polynomial $p_i(u)$ by

$$p_i(u)=(u^{c(i)}-1)/(q^{c(i)}-1).$$

For a fixed Steinberg group G_σ, we define $r_i=\omega_i H_\sigma$ ($r_i \in W_\sigma$), $R=\{r_i\}$ and $R'=\{\omega_i\}$, then the pair (W_σ, R) is a Coxeter system (see [1] and [9]). $r(i)$ and $\omega(i)$ mean r_i and w_i respectively.

§ 3. Statements of results.

PROPOSITION 1.

(1) $G_\sigma=\bigcup_{n \in N_\sigma} U_\sigma n U_\sigma$ *(disjoint union).*

(2) *The elements* $\{\alpha(n)=\mathrm{ind}\,(n)ene \mid n \in N_\sigma\}$ *form a basis for* $Hc(G_\sigma, U_\sigma)$.

(3) *Let* $N_{\sigma,\lambda}$ *be a system of representatives for* $W_{\sigma,\lambda}$ *in* N_σ *such that* $n_1=1$ *i.e.* $N_{\sigma,\lambda}=\{n_w \in N_\sigma \mid w \in W_{\sigma,\lambda},\ n_w H_\sigma=n_{w'}H_\sigma \Longrightarrow n_w=n_{w'}$ *and* $n_1=1\}$, *then the elements* $\{\beta(n_w) \mid n_w \in N_{\sigma,\lambda}\}$ *form a basis for* $H_C(G_\sigma, B_\sigma)_\lambda$.

(4) ε_λ *is an elements of* $H_C(G_\sigma, U_\sigma)$, *and we can regard* $H_C(G_\sigma, B_\sigma)_\lambda$ *as a subalgebra* $\varepsilon_\lambda H_C(G_\sigma, U_\sigma)\varepsilon_\lambda$ *of* $H_C(G_\sigma, U_\sigma)$ *with unity element* ε_λ.

THEOREM 1. *The algebra* $H_C(G_\sigma, U_\sigma)$ *has a presentation with generators* $\{\alpha(h),\ \alpha(\omega_i) \mid h \in H_\sigma,\ \omega_i \in R'\}$ *and relations*

(I) $\alpha(h)\alpha(h')=\alpha(hh')\quad(\forall h,\ \forall h' \in H_\sigma)$,

(II) $\alpha(\omega_i)\alpha(h)=\alpha(\omega_i h\omega_i^{-1})\alpha(\omega_i)\quad(\forall h \in H_\sigma,\ \forall\omega_i \in R')$,

(III) $\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{t \in k_\theta^*}\alpha(h_i(t))\alpha(\omega_i)$ *if* π_i *is of type* 1,

$\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{t \in k^*}\alpha(h_i(t))\alpha(\omega_i)$ *if* π_i *is of type* 2,

$\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{\substack{t \in k^* \\ t+t^\theta=0}}\alpha(h_i(t))\alpha(\omega_i)+(q+1)\sum_{\substack{t \in k^* \\ t+t^\theta\neq 0}}\alpha(h_i(t))\alpha(\omega_i)$

if π_i *is of type* 3, $(\forall\omega_i \in R')$ *and*

(IV) $\underbrace{\alpha(\omega_i)\alpha(w_j)\alpha(\omega_i)\cdots}_{m_{ij}}=\underbrace{\alpha(\omega_j)\alpha(\omega_i)\alpha(\omega_j)\cdots}_{m_{ij}}\quad(\forall\omega_i,\ \forall\omega_j \in R'$ *such that* $\omega_i\neq\omega_j)$,

where m_{ij} *denotes the order of* $r_i r_j$ *in* W_σ.

THEOREM 2.

(1) *With respect to a given finite Steinberg group* G_σ, *there exists an associative* $\boldsymbol{C}[u]$-*algebra* A *with unity element, which has a free basis* $\{a(n) \mid n \in N_\sigma\}$ *over* $\boldsymbol{C}[u]$, *and in which the multiplication is determined by the following formulas, for* $h \in H_\sigma$, $n \in N_\sigma$ *and* $\omega_i \in R'$:

$a(h)a(n)=a(hn)$,

$a(\omega_i)a(n)=a(\omega_i n)$ *if* $l(r_i\zeta(n))>l(\zeta(n))$ *where* $l(\zeta(n))$ *denotes the minimal length* l *of all expressions* $\zeta(n)=r(j_1)\cdots r(j_l)$ $(r(j_1),\cdots,r(j_l)\in R)$ *(see* [1]*),*

$a(\omega_i)a(n)=u^{c(i)}a(\omega_i n)+p_i(u)\sum_{t\in k_\theta^*}a(h_i(t)n)$ *if* $l(r_i\zeta(n))<l(\zeta(n))$ *and* π_i *is of type* 1,

$a(\omega_i)a(n)=u^{c(i)}a(\omega_i n)+p_i(u)\sum_{t\in k^*}a(h_i(t)n)$ *if* $l(r_i\zeta(n))<l(\zeta(n))$ *and* π_i *is of type* 2,

$a(\omega_i)a(n)=u^{c(i)}a(\omega_i n)+p_i(u)(\sum_{\substack{t\in k^*\\ t+t^\theta=0}}a(h_i(t)n)+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}a(h_i(t)n))$

if $l(r_i\zeta(n))<l(\zeta(n))$ *and* π_i *is of type* 3.

(2) *As in the case of the Hecke algebras, A has a presentation with generators* $\{a(h),\ a(\omega_i)\,|\,h\in H_\sigma,\ \omega_i\in R'\}$ *and relations*

(I) $a(h)a(h')=a(hh')$ $(\forall h,\ \forall h'\in H_\sigma)$,

(II) $a(\omega_i)a(h)=a(\omega_i h\omega_i^{-1})a(\omega_i)$ $(\forall h\in H_\sigma,\ \forall\omega_i\in R')$,

(III) $a(\omega_i)^2=u^{c(i)}a(\omega_i^2)+p_i(u)\sum_{t\in k_\theta^*}a(h_i(t))a(\omega_i)$ *if* π_i *is of type* 1,

$a(\omega_i)^2=u^{c(i)}a(\omega_i^2)+p_i(u)\sum_{t\in k^*}a(h_i(t))a(\omega_i)$ *if* π_i *is of type* 2,

$a(\omega_i)^2=u^{c(i)}a(\omega_i^2)+p_i(u)\sum_{\substack{t\in k^*\\ t+t^\theta=0}}a(h_i(t))a(\omega_i)+(q+1)p_i(u)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}a(h_i(t))a(\omega_i)$

if π_i *is of type* 3, $(\forall\omega_i\in R')$ *and*

(IV) $\underbrace{a(\omega_i)a(\omega_j)a(\omega_i)\cdots}_{m_{ij}}=\underbrace{a(\omega_j)a(\omega_i)a(\omega_j)\cdots}_{m_{ij}}$ $(\forall\omega_i,\ \forall\omega_j\in R'$ *such that* $\omega_i\neq\omega_j)$.

COROLLARY TO THEOREM 2. *Let* $f_1, f_q: \boldsymbol{C}[u]\to\boldsymbol{C}$ *be defined by* $f_1(u)=1$ *and* $f_q(u)=q$ *respectively, then* $\boldsymbol{C}[u]$ *becomes a* $(\boldsymbol{C},\boldsymbol{C}[u])$*-bimodule via* $(c, c', t(u))\mapsto cc't(f(u))$, $c, c'\in\boldsymbol{C}$, $t(u)\in\boldsymbol{C}[u]$ *where* $f=f_1$ *or* f_q.

Let $A_{f_1,\boldsymbol{C}}=\boldsymbol{C}\otimes_{\boldsymbol{C}[u]}A$ *and* $A_{f_q,\boldsymbol{C}}=\boldsymbol{C}\otimes_{\boldsymbol{C}[u]}A$, *then*

$$A_{f_1,\boldsymbol{C}}\cong\boldsymbol{C}N_\sigma,\ A_{f_q,\boldsymbol{C}}\cong H_{\boldsymbol{C}}(G_\sigma, U_\sigma)\ \text{and}\ H_{\boldsymbol{C}}(G_\sigma, U_\sigma)\cong\boldsymbol{C}N_\sigma.$$

DEFINITION 1. *We call A the generic algebra of* $H_{\boldsymbol{C}}(G_\sigma, U_\sigma)$.

THEOREM 3. *Let*

$$\tilde{\varepsilon}_\lambda(H_\sigma)=\frac{1}{|H_\sigma|}\sum_{h\in H_\sigma}\lambda(h^{-1})a(h)\quad\text{and}\quad A_\lambda=\tilde{\varepsilon}_\lambda(H_\sigma)A\tilde{\varepsilon}_\lambda(H_\sigma)\ .$$

Then

(1) A_λ *is an associative algebra over* $\boldsymbol{C}[u]$ *with unity element* $\tilde{\varepsilon}_\lambda(H_\sigma)$.

(2) *Let* $\{n_w\}$ *be as in Proposition* 1, *then the elements* $\{\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)\,|\,n_w\in N_{\sigma,\lambda}\}$ *form a basis for* A_λ.

(3) $(A_\lambda)_{f_1,\boldsymbol{C}}\cong\varepsilon_\lambda(H_\sigma)\boldsymbol{C}N_\sigma\varepsilon_\lambda(H_\sigma)$ *and* $(A_\lambda)_{f_q,\boldsymbol{C}}\cong H_{\boldsymbol{C}}(G_\sigma, B_\sigma)_\lambda$.

(4) $H_{\boldsymbol{C}}(G_\sigma, B_\sigma)_\lambda\cong\varepsilon_\lambda(H_\sigma)\boldsymbol{C}N_\sigma\varepsilon_\lambda(H_\sigma)$.

DEFINITION 2. *We call A_λ the generic algebra of $H_C(G_\sigma, B_\sigma)_\lambda$.*

§ 4. Presentation of $H_C(G_\sigma, U_\sigma)$.

LEMMA 1. *Let*

$$\varepsilon_\lambda = \frac{1}{|B_\sigma|} \sum_{b \in B_\sigma} \lambda(b^{-1})b, \quad e = \frac{1}{|U_\sigma|} \sum_{u \in U_\sigma} u \quad and \quad \varepsilon_\lambda(H_\sigma) = \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(h^{-1})h .$$

Then

$$\begin{cases} \varepsilon_\lambda = e\varepsilon_\lambda(H_\sigma) = \varepsilon_\lambda(H_\sigma)e, \\ ue = eu = e \ \text{for } \forall u \in U_\sigma, \\ h\varepsilon_\lambda(H_\sigma) = \varepsilon_\lambda(H_\sigma)h = \lambda(h)\varepsilon_\lambda(H_\sigma) \ \text{for } \forall h \in H_\sigma, \\ e^2 = e, \ \varepsilon_\lambda(H_\sigma)^2 = \varepsilon_\lambda(H_\sigma). \end{cases}$$

PROOF. It is clear from the definitions of idempotents, $B_\sigma = U_\sigma H_\sigma$ and $\lambda | U_\sigma = 1$ where $\lambda | U_\sigma$ is the restriction of λ to U_σ. Q.E.D.

PROOF OF PROPOSITION 1. From [9, Theorem 33] we can prove (1) straightforwardly, and (2) and (3) are clear from [3]. (4) Since $H_C(G_\sigma, U_\sigma) = e\boldsymbol{C}G_\sigma e$ and $\boldsymbol{C}G_\sigma \ni \varepsilon_\lambda(H_\sigma)$, it can be proved from Lemma 1. Q.E.D.

LEMMA 2.

(1) $n_w\varepsilon_\lambda(H_\sigma) = \varepsilon_\lambda(H_\sigma)n_w$ *if* $w \in W_{\sigma,\lambda}$,
$\varepsilon_\lambda(H_\sigma)n_w\varepsilon_\lambda(H_\sigma) = 0$ *if* $w \notin W_{\sigma,\lambda}$.

(2) $\beta(n_w) = \varepsilon_\lambda(H_\sigma)\alpha(n_w)\varepsilon_\lambda(H_\sigma) = \alpha(n_w)\varepsilon_\lambda(H_\sigma) = \varepsilon_\lambda(H_\sigma)\alpha(n_w)$ *if* $w \in W_{\sigma,\lambda}$,
$\varepsilon_\lambda(H_\sigma)\alpha(n_w)\varepsilon_\lambda(H_\sigma) = 0$ *if* $w \notin W_{\sigma,\lambda}$.

PROOF. For any $w \in W_\sigma$, we have

$$n_w^{-1}\varepsilon_\lambda(H_\sigma)n_w = \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} w\lambda(h^{-1})h = \varepsilon_{w\lambda}(H_\sigma) .$$

Hence $n_w^{-1}\varepsilon_\lambda(H_\sigma)n_w = \varepsilon_\lambda(H_\sigma)$ if $w \in W_{\sigma,\lambda}$ and $n_w^{-1}\varepsilon_\lambda(H_\sigma)n_w\varepsilon_\lambda(H_\sigma) = 0$ if $w \notin W_{\sigma,\lambda}$.

The rest of this lemma can be proved from definitions and Lemma 1.

Q.E.D.

LEMMA 3. *Let $\nu: \boldsymbol{C}G_\sigma \to \boldsymbol{C}$ be defined by $\nu(\sum_{g \in G_\sigma} \alpha_g g) = \sum_{g \in G_\sigma} \alpha_g$ where α_g's are elements of $\boldsymbol{C}$, then ν becomes a $\boldsymbol{C}$-algebra homomorphism, and $\nu(e) = 1$ and $\nu(\alpha(n)) =$* ind(n) *for all $n \in N_\sigma$.*

PROOF. It is clear that ν is a $\boldsymbol{C}$-algebra homomorphism. By definition, $\nu(e) = 1$ and $\nu(\alpha(n)) = \text{ind}\,(n)$. Q.E.D.

LEMMA 4.

$$\alpha(h)\alpha(n) = \alpha(hn) \quad for \ \forall h \in H_\sigma \ and \ \forall n \in N_\sigma ,$$

$$\alpha(n)\alpha(h)=\alpha(nh) \quad \textit{for } \forall h \in H_\sigma \textit{ and } \forall n \in N_\sigma .$$

PROOF.

$$U_\sigma h U_\sigma U_\sigma n U_\sigma = U_\sigma h U_\sigma n U_\sigma = U_\sigma h U_\sigma h^{-1} h n U_\sigma = U_\sigma h n U_\sigma .$$

Similarly it is shown that $U_\sigma n U_\sigma U_\sigma h U_\sigma = U_\sigma n h U_\sigma$. Hence $\alpha(h)\alpha(n)=c\alpha(hn)$ and $\alpha(n)\alpha(h)=c'\alpha(nh)$ for some algebraic integers c and c' (see [3] and [6]).

Now, since

$$\nu(\alpha(h)\alpha(n))=\nu(\alpha(h))\nu(\alpha(n))=\mathrm{ind}(n)$$

and $\nu(c\alpha(hn))=c\,\mathrm{ind}(n)$, we have $c=1$. Similarly we have $c'=1$. Q.E.D.

LEMMA 5. *Let n be an arbitrary element of N_σ and r_i be an element of R such that $l(r_i\zeta(n))>l(\zeta(n))$, then $\alpha(\omega_i)\alpha(n)=\alpha(\omega_i n)$.*

PROOF. Let $\zeta(n)=w$ and $r_i=r$. We have

$$U_\sigma \omega_i U_\sigma U_\sigma n U_\sigma \subset B_\sigma \omega_i B_\sigma B_\sigma n B_\sigma = B_\sigma \omega_i n B_\sigma .$$

Hence

$$U_\sigma \omega_i U_\sigma U_\sigma n U_\sigma \subset \bigcup_{h \in H_\sigma} U_\sigma \omega_i n h U .$$

From [6] we have

$$\alpha(\omega_i)\alpha(n)=\sum_{h \in H_\sigma} c_h \alpha(\omega_i n h)$$

where

$$c_h=\frac{1}{|U_\sigma|}\,|U_\sigma \omega_i^{-1} U_\sigma \omega_i n h \cap U_\sigma n U_\sigma| .$$

Let $\lambda=1_{B_\sigma}$, then from [6] $\beta(n_r)\beta(n_w)=\beta(n_{rw})$ in $Hc(G_\sigma, B_\sigma)_\lambda$. Since

$$\alpha(\omega_i)\alpha(n)\varepsilon_\lambda(H_\sigma)=\beta(n_r)\beta(n_w)=(\sum_{h \in H_\sigma} c_h)\beta(n_{rw})=\beta(n_{rw})$$

from Lemma 2, we have $\sum_{h \in H_\sigma} c_h=1$. Since c_h's are non-negative integers and $c_1>0$, we have $c_1=1$ and $c_h=0$ where $h\neq 1$. Q.E.D.

LEMMA 6. *Let $\omega_i \in R'$.*

(1) *If π_i is of type 1, then*

$$\omega_i U_\sigma \omega_i \subset U_\sigma \omega_i^2 U_\sigma \cup (\bigcup_{t \in k_0^*} U_\sigma h_i(t) \omega_i U_\sigma) .$$

(2) *If π_i is of type 2 or 3, then*

$$\omega_i U_\sigma \omega_i \subset U_\sigma \omega_i^2 U_\sigma \cup (\bigcup_{t \in k^*} U_\sigma h_i(t) \omega_i U_\sigma) .$$

PROOF. Let P be the set of positive roots of Φ with respect to Π. Let $T=P\cap r_i^{-1}P$ and $Q=P\cap r_i^{-1}(-P)$, then we have $U=U_T U_Q$ and $U_\sigma=U_{T,\sigma}U_{Q,\sigma}$ (see [9, Theorem 4′]). We write $U_{i,\sigma}$ instead of $U_{Q,\sigma}$.

If π_i is of type 1, then

$$U_{i,\sigma}=\{x_\alpha(t)\,|\,t\in k_\theta \text{ and } \alpha=\alpha_i\}\ .$$

If π_i is of type 2, then

$$U_{i,\sigma}=\{x_\alpha(t)\sigma x_\alpha(t)\,|\,t\in k \text{ and } \alpha=\alpha_i\}\ .$$

If π_i is of type 3, then

$$U_{i,\sigma}=\{x_\alpha(t)x_{\rho\alpha}(t^\theta)x_{\alpha+\rho\alpha}(u)\,|\,N_{\alpha,\rho\alpha}tt^\theta+u+u^\theta=0 \text{ and } \alpha=\alpha_i\}$$

(see [9, Lemma 63]).

Now,

$$\omega_i U_\sigma \omega_i=\omega_i U_{T,\sigma}U_{i,\sigma}\omega_i=\omega_i U_{T,\sigma}\omega_i^{-1}\omega_i U_{i,\sigma}\omega_i\ .$$

Hence $\omega_i U_\sigma \omega_i=\omega_i U_{T,\sigma}\omega_i^{-1}\omega_i U_{i,\sigma}\omega_i^{-1}h_0$ where $h_0=\omega_i^2$.

Let $\omega_i X\omega_i^{-1}\omega_i Y\omega_i\in\omega_i U_\sigma\omega_i$ where $X\in U_{T,\sigma}$ and $Y\in U_{i,\sigma}$. If $Y=1$, we have

$$\omega_i X\omega_i^{-1}\omega_i^2\in U_\sigma\omega_i^2\subset U_\sigma\omega_i^2 U_\sigma\ .$$

Case 1. π_i is of type 1. In this case $h_0=h_i(-1)$ and

$$\omega_i U_{i,\sigma}\omega_i^{-1}=\{x_{-\alpha}(t)\,|\,t\in k_\theta \text{ and } \alpha=\alpha_i\}\ .$$

$$t\neq 0 \Longrightarrow x_{-\alpha}(-t^{-1})=x_\alpha(-t)h_\alpha(t)\omega_\alpha x_\alpha(-t)\in U_\sigma h_i(t)\omega_i U_\sigma\ .$$

Thus we have (1) in this case.

Case 2. π_i is of type 2. In this case $h_0=h_i(-1)$ and

$$\omega_i U_{i,\sigma}\omega_i^{-1}=\{x_{-\alpha}(t)x_{-\rho\alpha}(t^\theta)\,|\,t\in k \text{ and } \alpha=\alpha_i\}\ .$$

$$t\neq 0 \Longrightarrow x_{-\alpha}(-t^{-1})x_{-\rho\alpha}(-t^{-\theta})=x_\alpha(-t)x_{\rho\alpha}(-t^\theta)h_i(t)\omega_\alpha\omega_{\rho\alpha}x_\alpha(-t)x_{\rho\alpha}(-t^\theta)\in U_\sigma h_i(t)\omega_i U_\sigma$$

where $-t^{-\theta}=(-t^{-1})^\theta$.

Case 3. π_i is of type 3. This case occurs only when Φ is of type A_{2n} and $i=n$. In this case $h_0=1$ and

$$\omega_i U_{i,\sigma}\omega_i^{-1}=\{x_{-\rho\alpha}(t)x_{-\alpha}(t^\theta)x_{-\alpha-\rho\alpha}(u)\,|\,tt^\theta+u+u^\theta=0,\ \alpha=\alpha_n\}$$
$$(N_{-\rho\alpha,-\alpha}=-N_{\rho\alpha,\alpha}=N_{\alpha,\rho\alpha}=1)\ .$$

From the fact that $G\cong SL(2n+1, k)$ and

$G'=\langle x_\alpha(t_1), x_{\rho\alpha}(t_2), x_{\alpha+\rho\alpha}(t_3), x_{-\alpha}(t_1'), x_{-\rho\alpha}(t_2'), x_{-\alpha-\rho\alpha}(t_3')\,|\,t_1, t_2, t_3, t_1', t_2', t_3'\in k\rangle\cong SL(3, k)$

(see [9]), we have

$$\begin{aligned}&x_{-\rho\alpha}(t)x_{-\alpha}(t^\theta)x_{-\alpha-\rho\alpha}(u)\\&\quad=x_\alpha(-u^{-1}t)^\rho x_{\rho\alpha}(-u^{-\theta}t^\theta)x_{\alpha+\rho\alpha}(u^{-1})h_\alpha(-u^{-1})h_{\rho\alpha}(-u^{-\theta})\omega_\alpha\omega_{\rho\alpha}\omega_\alpha\\&\qquad\times x_\alpha(-u^{-\theta}t)x_{\rho\alpha}(-u^{-1}t^\theta)x_{\alpha+\rho\alpha}(u^{-1})\in U_\sigma h_i(-u^{-1})\omega_i U_\sigma\end{aligned}$$

where $tt^{\theta}+u+u^{\theta}=0$, $u\neq 0$ and $-u^{-\theta}$ means $(-u^{-1})^{\theta}$. Q.E.D.

From the above proof we have the following lemma.

LEMMA 7.

(1) *If G is of type A_{2n} and $tt^{\theta}+u+u^{\theta}=0$ where $t\in k$ and $u\in k^*$, then*

$$x_{-\rho\alpha}(t)x_{-\alpha}(t^{\theta})x_{-\alpha-\rho\alpha}(u)\in U_{\alpha}h_n(-u^{-1})\omega_n U_{\sigma}$$

where $\pi_i=\pi_n$ and $\alpha=\alpha_n$.

(2) $\mathrm{ind}(\omega_i)=q^{c(i)}$ *where $\omega_i\in R'$* (see [9, Lemma 65]).

LEMMA 8. *Let $\omega_i\in R'$.*

(1) *If π_i is of type 1, then*

$$\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{t\in k_{\theta}^*}\alpha(h_i(t))\alpha(\omega_i)\ .$$

(2) *If π_i is of type 2, then*

$$\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{t\in k^*}\alpha(h_i(t))\alpha(\omega_i)\ .$$

(3) *If π_i is of type 3, then*

$$\alpha(\omega_i)^2=q^{c(i)}\alpha(\omega_i^2)+\sum_{\substack{t\in k^*\\ t+t^{\theta}=0}}\alpha(h_i(t))\alpha(\omega_i)+(q+1)\sum_{\substack{t\in k^*\\ t+t^{\theta}\neq 0}}\alpha(h_i(t))\alpha(\omega_i)\ .$$

PROOF.

(1) We assume that π_i is of type 1. From (1) of Lemma 6 and [6] we have

$$\alpha(\omega_i)^2=c(\omega_i^2)\alpha(\omega_i^2)+\sum_{t\in k_{\theta}^*}c(h_i(t)\omega_i)\alpha(h_i(t)\omega_i)$$

where

$$c(\omega_i^2)=\frac{1}{|U_{\sigma}|}\,|U_{\sigma}\omega_i^{-1}U_{\sigma}\omega_i^2\cap U_{\sigma}\omega_i U_{\sigma}|$$

and

$$c(h_i(t)\omega_i)=\frac{1}{|U_{\sigma}|}\,|U_{\sigma}\omega_i^{-1}U_{\sigma}h_i(t)\omega_i\cap U_{\sigma}\omega_i U_{\sigma}|\ .$$

It is clear that $c(\omega_i^2)=\mathrm{ind}\,(\omega_i)=q^{c(i)}$ from $\omega_i^2\in H$.

Now,

$$\nu(\alpha(\omega_i)^2)=(q^{c(i)})^2=q^{c(i)}\nu(\alpha(\omega_i^2))+\sum_{t\in k_{\theta}^*}c(h_i(t)\omega_i)\nu(\alpha(h_i(t)\omega_i))\ .$$

Hence

$$q^{2c(i)}=q^{c(i)}+q^{c(i)}\sum_{t\in k_{\theta}^*}c(h_i(t)\omega_i)\ .$$

Therefore we have

$$\sum_{t\in k_{\theta}^*}c(h_i(t)\omega_i)=q^{c(i)}-1\ .$$

Since

$$c(h_i(t)\omega_i)=\frac{1}{|U_\sigma|}\,|U_\sigma h_i(-t^{-1})\omega_i U_\sigma \cap U_\sigma\omega_i U_\sigma\omega_i^{-1}|$$

and

$$U_\sigma h_i(-t^{-1})\omega_i U_\sigma \cap U_\sigma\omega_i U_\sigma\omega_i^{-1} \ni x_{-\alpha}(t)$$

where $\alpha=\alpha_i$ (see the proof of Lemma 6), $c(h_i(t)\omega_i)$ is a positive integer for each $t\in k_\theta^*$. Hence we have $c(h_i(t)\omega_i)=1$ for all $t\in k_\theta^*$.

(2) We assume that π_i is of type 2. From (2) of Lemma 6 and [6] we have

$$\alpha(\omega_i)^2=c(\omega_i^2)\alpha(\omega_i^2)+\sum_{t\in k^*}c(h_i(t)\omega_i)\alpha(h_i(t)\omega_i)$$

where

$$c(\omega_i^2)=\frac{1}{|U_\sigma|}\,|U_\sigma\omega_i^{-1}U_\sigma\omega_i^2\cap U_\sigma\omega_i U_\sigma|=q^{c(i)}$$

and

$$c(h_i(t)\omega_i)=\frac{1}{|U_\sigma|}\,|U_\sigma\omega_i^{-1}U_\sigma h_i(t)\omega_i\cap U_\sigma\omega_i U_\sigma|\ .$$

Now,

$$\nu(\alpha(\omega_i)^2)=(q^{c(i)})^2=q^{c(i)}\nu(\alpha(\omega_i^2))+\sum_{t\in k^*}c(h_i(t)\omega_i)\nu(\alpha(h_i(t)\omega_i))\ .$$

Hence

$$\sum_{t\in k^*}c(h_i(t)\omega_i)=q^{c(i)}-1\ .$$

Since

$$c(h_i(t)\omega_i)=\frac{1}{|U_\sigma|}\,|U_\sigma h_i(-t^{-1})\omega_i U_\sigma\cap U_\sigma\omega_i U_\sigma\omega_i^{-1}|$$

and

$$U_\sigma h_i(-t^{-1})\omega_i U_\sigma\cap U_\sigma\omega_i U_\sigma\omega_i^{-1}\ni x_{-\alpha}(t)x_{-\rho\alpha}(t^\theta)$$

where $\alpha=\alpha_i$ (see the proof of Lemma 6), $c(h_i(t)\omega_i)$ is a positive integer for each $t\in k^*$. Hence $c(h_i(t)\omega_i)=1$ for all $t\in k^*$.

(3) We assume that π_i is of type 3. From (2) of Lemma 6 and [6] we have

$$\alpha(\omega_i)^2=c(\omega_i^2)\alpha(\omega_i^2)+\sum_{t\in k^*}c(h_i(t)\omega_i)\alpha(h_i(t)\omega_i)$$

where $c(\omega_i^2)=q^{c(i)}$ and

$$c(h_i(t)\omega_i)=\frac{1}{|U_\sigma|}\,|U_\sigma\omega_i^{-1}U_\sigma h_i(t)\omega_i\cap U_\sigma\omega_i U_\sigma|\ .$$

Now we have

$$c(h_i(t)\omega_i)=\frac{1}{|U_\sigma|}\,|U_\sigma h_i(t^{-\theta})\omega_i U_\sigma\cap U_\sigma\omega_i U_\sigma\omega_i^{-1}|$$

where $t^{-\theta}=(t^{-1})^{\theta}$, and the all elements of

$$\omega_i U_{i,\sigma}\omega_i^{-1}=\{x_{-\rho\alpha}(t)x_{-\alpha}(t^{\theta})x_{-\alpha-\rho\alpha}(u)\mid tt^{\theta}+u+u^{\theta}=0,\ \alpha=\alpha_n\}$$

(see the proof of Lemma 6) form the set of all right coset representatives of U_σ in $U_\sigma\omega_i U_\sigma\omega_i^{-1}$, i.e., $U_\sigma\omega_i U_\sigma\omega_i^{-1}=\bigcup_{x\in\omega_i U_{i,\sigma}\omega_i^{-1}} U_\sigma x$ (disjoint union).

Since

$$x_{-\rho\alpha}(t)x_{-\alpha}(t^{\theta})x_{-\alpha-\rho\alpha}(u)\in U_\sigma h_i(-u^{-1})\omega_i U_\sigma$$

where $tt^{\theta}+u+u^{\theta}=0$ and $u\neq 0$ (see (1) of Lemma 7), we have

$$c(h_i(-u^{\theta})\omega_i)=|\{t\in k\mid tt^{\theta}+u+u^{\theta}=0\}|$$

from (1) of Proposition 1.

By the way, for any $u\in k^*$ we have

$$|\{t\in k\mid tt^{\theta}+u+u^{\theta}=0\}|=1 \text{ if } u+u^{\theta}=0 \qquad \text{and}$$

$$|\{t\in k\mid tt^{\theta}+u+u^{\theta}=0\}|=q+1 \text{ if } u+u^{\theta}\neq 0\,.$$

Hence $c(h_i(-u^{\theta})\omega_i)=1$ if $u+u^{\theta}=0$ and $c(h_i(-u^{\theta})\omega_i)=q+1$ if $u+u^{\theta}\neq 0$.

Therefore, for any $t\in k^*$, $c(h_i(t)\omega_i)=1$ if $t+t^{\theta}=0$ and $c(h_i(t)\omega_i)=q+1$ if $t+t^{\theta}\neq 0$.

Q.E.D.

PROPOSITION 2. *Let $h\in H_\sigma$, $n\in N_\sigma$ and $\zeta(n)=w\in W_\sigma$, then*

(1) $\alpha(h)\alpha(n)=\alpha(hn)$;

(2) $\alpha(\omega_i)\alpha(n)=\alpha(\omega_i n)$ *if* $l(r_i w)>l(w)$;

(3) $\alpha(\omega_i)\alpha(n)=q^{c(i)}\alpha(\omega_i n)+\sum_{t\in k_0^*}\alpha(h_i(t))\alpha(n)$ *if* $l(r_i w)<l(w)$ *and* π_i *is of type* 1,

$\alpha(\omega_i)\alpha(n)=q^{c(i)}\alpha(\omega_i n)+\sum_{t\in k^*}\alpha(h_i(t))\alpha(n)$ *if* $l(r_i w)<l(w)$ *and* π_i *is of type* 2,

$$\alpha(\omega_i)\alpha(n)=q^{c(i)}\alpha(\omega_i n)+\sum_{\substack{t\in k^*\\ t+t^{\theta}=0}}\alpha(h_i(t))\alpha(n)+(q+1)\sum_{\substack{t\in k^*\\ t+t^{\theta}\neq 0}}\alpha(h_i(t))\alpha(n)$$

if $l(r_i w)<l(w)$ *and* π_i *is of type* 3.

PROOF. From Lemmas 4 and 5, (1) and (2) are obtained.

Assume $l(r_i w)<l(\mathrm{w})$. Let $r_i w=r(j_1)\cdots r(j_k)$ be a reduced expression of $r_i w$ where $r(j_1),\cdots,r(j_k)\in R$, then $w=r_i r(j_1)\cdots r(j_k)$ becomes a reduced expression of w (see [1]).

Hence we can write $n=\omega_i\omega(j_1)\cdots\omega(j_k)h$ for some $h\in H_\sigma$. From (2) we have

$$\alpha(n)=\alpha(\omega_i)\alpha(\omega(j_1))\cdots\alpha(\omega(j_k))\alpha(h).$$

Therefore

$$\alpha(\omega_i)\alpha(n)=(\alpha(\omega_i)\alpha(\omega_i))\alpha(\omega(j_1))\cdots\alpha(\omega(j_k))\alpha(h)\,.$$

The rest of the proposition can be easily verified from Lemma 8. Q.E.D.

LEMMA 9.

(1) *Let m_{ij} be the order of $r_i r_j$ in W_σ where $i \neq j$, then*

$$\underbrace{\omega_i\omega_j\omega_i\cdots}_{m_{ij}}=\underbrace{\omega_j\omega_i\omega_j\cdots}_{m_{ij}}.$$

(2) *Let $w \in W_\sigma$ and $r(j_1)\cdots r(j_k)$ be a reduced expression of w, then the mapping Ω of W_σ into N_σ defined by $\Omega(w)=\omega(j_1)\cdots\omega(j_k)$ is well defined.*

PROOF. Using [9, Lemma 56], one can prove (1) by direct calculation. One can also prove (2) from [1, Proposition 5 of Ch. IV] and (1). Q.E.D.

PROOF OF THEOREM 1. With respect to $H_C(G_\sigma, U_\sigma)$ we define a C-algebra $\mathfrak{A}$ such that it has a presentation with generators $\{\Delta(h), \Delta(\omega_i) \mid h \in H_\sigma, \omega_i \in R'\}$ and relations

(I) $\Delta(h)\Delta(h')=\Delta(hh')$ $(\forall h, \forall h' \in H_\sigma)$,

(II) $\Delta(\omega_i)\Delta(h)=\Delta(\omega_i h\omega_i^{-1})\Delta(\omega_i)$ $(\forall h \in H_\sigma, \forall\omega_i \in R')$,

(III) $\Delta(\omega_i)^2=q^{c(i)}\Delta(\omega_i^2)+\sum_{t\in k_0^*}\Delta(h_i(t))\Delta(\omega_i)$ if π_i is of type 1,

$\Delta(\omega_i)^2=q^{c(i)}\Delta(\omega_i^2)+\sum_{t\in k^*}\Delta(h_i(t))\Delta(\omega_i)$ if π_i is of type 2,

$\Delta(\omega_i)^2=q^{c(i)}\Delta(\omega_i^2)+\sum_{\substack{t\in k^*\\ t+t^\theta=0}}\Delta(h_i(t))\Delta(\omega_i)+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}\Delta(h_i(t))\Delta(\omega_i)$ if π_i is of type 3, $(\forall\omega_i \in R')$ and

(IV) $\underbrace{\Delta(\omega_i)\Delta(\omega_j)\Delta(\omega_i)\cdots}_{m_{ij}}=\underbrace{\Delta(\omega_j)\Delta(\omega_i)\Delta(\omega_j)\cdots}_{m_{ij}}$ $(\forall\omega_i, \forall\omega_j \in R'$ such that $\omega_i \neq \omega_j)$.

Let $n \in N_\sigma$, $\zeta(n)=w$ and $r(j_1)\cdots r(j_k)$ be a reduced expression of w where $r(j_1), \cdots, r(j_k) \in R$, then we have $n=\omega(j_1)\cdots\omega(j_k)h$ for some $h \in H_\sigma$. From Lemma 9 h is uniquely determined by n, and from [1, Proposition 5 of Ch. IV] we can define $\Delta(n)=\Delta(\omega(j_1)\cdots\Delta(\omega(j_k))\Delta(h)$.

From the above relations it is shown that $\mathfrak{A}$ is generated by $\{\Delta(n) \mid n \in N_\sigma\}$ as a C-module.

Now from the definition of $\mathfrak{A}$ there exists a C-algebra homomorphism φ of $\mathfrak{A}$ onto $H_C(G_\sigma, U_\sigma)$ such that $\varphi(\Delta(h))=\alpha(h)$ and $\varphi(\Delta(\omega_i))=\alpha(\omega_i)$ for all $h \in H_\sigma$ and $\omega_i \in R'$. Since $\varphi(\Delta(n))=\alpha(n)$ and $\{\alpha(n) \mid n \in N_\sigma\}$ is a C-basis for $H_C(G_\sigma, U_\sigma)$, φ becomes a bijective homomorphism. Hence Theorem 1 is proved. Q.E.D.

§5. Construction of the generic algebra of $H_C(G_\sigma, U_\sigma)$.

Let A be a free $C[u]$-module with a free basis $\{a(n) \mid n \in N_\sigma\}$. We will define the algebra structure on A, by virtue of the existence of the bijection ρ^* of $\mathfrak{P}$ onto A where $\mathfrak{P}$ is a subalgebra of $\mathrm{End}_{C[u]}(A)$ generated by $\{P_h, P_i | h \in H, 1 \leqq i \leqq |R'|\}$ and $\rho^*(\tilde{P})=(\tilde{P})(a(1))$ (see Lemma 13 and the proof of (1) of Theorem 2), as required.

LEMMA 10 (See [7, Appendix]). *Let w be an element of W_σ with a reduced expression $r(j_1) \cdots r(j_k)$, and r_i, r_j be elements of R. If $l(r_i w r_j)=l(w)$ and $l(r_i w)=l(w r_j)$, then $r_i w = w r_j$ and $\omega_i \omega = \omega \omega_j$ where $\omega=\omega(j_1) \cdots \omega(j_k)$.*

PROOF. (For the convenience of the readers we prove it.)

Case 1. $l(r_i w)=l(w r_j)=k+1$. Since $r(j_1) \cdots r(j_k) r_j$ is a reduced expression and $r_i r(j_1) \cdots r(j_k) r_j$ is not a reduced one, there exists s such that $1 \leqq s \leqq k+1$ and

$$r_i r(j_1) \cdots r(j_{s-1})=r(j_1) \cdots r(j_s)$$

where $j_{k+1}=j$ (see [1, Ch. IV]). Hence

$$r_i r(j_1) \cdots r(j_k) r_j = r(j_1) \cdots r(\check{j_s}) \cdots r(j_k) r_j \, .$$

If $1 \leqq s \leqq k$, then

$$r_i r(j_1) \cdots r(j_k)=r(j_1) \cdots r(\check{j_s}) \cdots r(j_k)$$

and it contradicts $l(r_i w)=k+1$. Hence $s=k+1$ and

$$r_i r(j_1) \cdots r(j_k) r_j = r(j_1) \cdots r(j_k) = w \, .$$

Since

$$r_i r(j_1) \cdots r(j_k) = r(j_1) \cdots r(j_k) r_j$$

are reduced expressions of $r_i w$, we have $\omega_i \omega = \omega \omega_j$ from Lemma 9.

Case 2. $l(r_i w)=l(w r_j)=k-1$. Let $w'=r_i w$, then

$$l(r_i w' r_j)=l(w r_j)=k-1=l(w') \quad \text{and} \quad l(r_i w')=l(w)=l(w' r_j)=k \, .$$

Hence we have $r_i w' = w' r_j$ from Case 1. Therefore $w=r_i w r_j$. Similarly we have $\omega_i \omega = \omega \omega_j$. Q.E.D.

LEMMA 11. *Let $n \in N_\sigma$ and r_i, $r_j \in R$ such that $r_i w = w r_j$ where $w=\zeta(n)$.*

(1) *If π_j is of type 1, then π_i is also of type 1 and $w\alpha_j = \pm \alpha_i$.*

(2) *If π_j is of type 2, then π_i is also of type 2 and $w\alpha_j = \pm \alpha_i$ or $w\alpha_j = \pm \rho\alpha_i$.*

(3) *If π_j is of type 3, then π_i is also of type 3 and $w\alpha_j = \pm \alpha_i$ or $w\alpha_j = \pm \rho\alpha_i$.*

PROOF. Let (,) be the same inner product as in [9] of the vector space V over the rational field $\boldsymbol{Q}$ which is generated by Φ (see [9, §1]). Let $\alpha \in \Phi$ and r_α be the reflection of V given by $r_\alpha(v)=v-2(v, \alpha)/(\alpha, \alpha)\alpha$ for all $v \in V$, then $W=\langle r_\alpha | \alpha \in \Phi \rangle$ from the definition.

It is clear that $(wv, wv)=(v, v)$ and $wr_\alpha w^{-1}=r_{w\alpha}$ for all $w\in W$ and $v\in V$. From the diagrams of (2) of §2 we can conclude that π_i and π_j are of the same type if $r_i=wr_jw^{-1}$ (see [1, Proposition 3 of Ch. IV]).

(1) Since $r_i=r_\alpha$ and $r_j=r_\beta$ where $\alpha=\alpha_i$ and $\beta=\alpha_j$, we have $wr_jw^{-1}=r_{w\beta}=r_\alpha$. Now $\{v\in V|r_\alpha(v)=-v\}=\boldsymbol{Q}\alpha$ and $r_\alpha(w\beta)=-w\beta$. Hence $w\beta=\pm\alpha$.

(2) Since $r_i=r_\alpha r_{\rho\alpha}$ and $r_j=r_\beta r_{\rho\beta}$ where $\alpha=\alpha_i$ and $\beta=\alpha_j$, we have

$$wr_jw^{-1}=r_{w\beta}r_{w\rho\beta}=r_\alpha r_{\rho\alpha}\ .$$

Now $r_{w\beta}r_{\rho w\beta}(w\beta)=-w\beta$ and

$$r_\alpha r_{\rho\alpha}(w\beta)=w\beta-2\frac{(w\beta,\ \alpha)}{(\alpha,\ \alpha)}\alpha-2\frac{(w\beta,\ \rho\alpha)}{(\rho\alpha,\ \rho\alpha)}\rho\alpha\ .$$

Hence $w\beta=a\alpha+b\rho\alpha$ where $2a,\ 2b\in\boldsymbol{Z}$. Therefore $w\beta=\pm\alpha$ or $\pm\rho\alpha$.

(3) Since $r_i=r_\alpha r_{\rho\alpha}r_\alpha$ and $r_i=r_j$ where $\alpha=\alpha_i=\alpha_j$ we have

$$wr_jw^{-1}=r_{w(\alpha+\rho\alpha)}=r_{\alpha+\rho\alpha}\ .$$

Now

$$\{v\in V|r_{\alpha+\rho\alpha}(v)=-v\}=\boldsymbol{Q}(\alpha+\rho\alpha)\ .$$

Hence

$$w(\alpha+\rho\alpha)=\pm(\alpha+\rho\alpha)\ .$$

Let

$$w\alpha_j=\sum_{s=1}^{2n}\tau_s\alpha_s\ ,\quad w\rho\alpha_j=\sum_{s=1}^{2n}\tau_s\rho\alpha_s\ ,$$

then

$$w\alpha_j+w\rho\alpha_j=\sum_{s=1}^{2n}\tau_s(\alpha_s+\rho\alpha_s)=\sum_{s=1}^{n}(\tau_s+\tau_{2n+1-s})(\alpha_s+\rho\alpha_s)\ .$$

Since $\tau_j+\tau_{2n+1-j}=\pm1$ and $\tau_l+\tau_{2n+1-l}=0$ where $1\leq l\leq n$ and $l\neq j$, we have $w\alpha_j=\pm\alpha_i$ or $\pm\rho\alpha_i$. Q.E.D.

LEMMA 12. *Let* $h\in H_\sigma$.

(1) *If* π_i *is of type* 1, *then* $\omega_ih\omega_i^{-1}=hh_i(t_0)$ *for some* $t_0\in k_0^*$.

(2) *If* π_i *is of type* 2, *then* $\omega_ih\omega_i^{-1}=hh_i(t_0)$ *for some* $t_0\in k^*$.

(3) *If* π_i *is of type* 3, *then* $\omega_ih\omega_i^{-1}=hh_i(t_0)$ *for some* $t_0\in k_0^*$.

PROOF. Since H_σ is generated by all $h_j(t)$'s, it is sufficient to prove the formulas when $h=h_j(t)$.

(1) If π_j is of type 1,

$$\omega_ih_j(t)\omega_i^{-1}=h_j(t)h_i(t^{-\langle\alpha_i,\alpha_j\rangle})$$

where $t\in k_0^*$ and $\langle\alpha_i, \alpha_j\rangle=2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$ (see [9, Lemma 20]).

If π_j is of type 2,

$$\omega_i h_j(t)\omega_i^{-1}=h_j(t)h_i(t^{-\langle\alpha_i,\alpha_j\rangle}(t^{-\langle\alpha_i,\alpha_j\rangle})^\theta)$$

where $t\in k^*$.

(2) If π_j is of type 1,

$$\omega_i h_j(t)\omega_i^{-1}=h_j(t)h_i(t^{-\langle\alpha_i,\alpha_j\rangle})$$

where $t\in k_0^*$.

If π_j is of type 2 or 3,

$$\omega_i h_j(t)\omega_i^{-1}=h_j(t)h_i(t^{-\langle\alpha_i,\alpha_j\rangle}(t^{-\langle\alpha_i,\rho\alpha_j\rangle})^\theta)$$

where $t\in k^*$.

(3) If π_j is of type 2 or 3,

$$\omega_i h_j(t)\omega_i^{-1}=h_j(t)h_i(t^{-\langle\alpha_i+\rho\alpha_i,\alpha_j\rangle}(t^{-\langle\alpha_i+\rho\alpha_i,\alpha_j\rangle})^\theta)$$

where $t\in k^*$. Q.E.D.

LEMMA 13. *Let $n\in N_\sigma$ and $\zeta(n)=w$. We define the $\mathbf{C}[u]$-module homomorphisms*

$$\{P_h, P_i, Q_{h'}, Q_j \mid h, h'\in H_\sigma, 1\leqq i, j\leqq |R'|\}$$

of A as follows:

$P_h(a(n))=a(hn)$,

$P_i(a(n))=a(\omega_i n)$ *if* $l(r_i w)>l(w)$,

$P_i(a(n))=u^{c(i)}a(\omega_i n)+p_i(u)\sum_{t\in k_0^*}a(h_i(t)n)$ *if* $l(r_i w)<l(w)$ *and* π_i *is of type* 1,

$P_i(a(n))=u^{c(i)}a(\omega_i n)+p_i(u)\sum_{t\in k^*}a(h_i(t)n)$ *if* $l(r_i w)<l(w)$ *and* π_i *is of type* 2,

$P_i(a(n))=u^{c(i)}a(w_i n)+p_i(u)\Big(\sum_{\substack{t\in k^*\\ t+t^\theta=0}}a(h_i(t)n)+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}a(h_i(t)n)\Big)$ *if* $l(r_i w)<l(w)$ *and* π_i *is of type* 3;

$Q_{h'}(a(n))=a(nh')$

$Q_j(a(n))=a(n\omega_j)$ *if* $l(wr_j)>l(w)$,

$Q_j(a(n))=u^{c(j)}a(n\omega_j)+p_j(u)\sum_{t\in k_0^*}a(nh_j(t))$ *if* $l(wr_j)<l(w)$ *and* π_j *is of type* 1,

$Q_j(a(n))=u^{c(j)}a(n\omega_j)+p_j(u)\sum_{t\in k^*}a(nh_j(t))$ *if* $l(wr_j)<l(w)$ *and* π_j *is of type* 2,

$Q_j(a(n))=u^{c(j)}a(n\omega_j)+p_j(u)\Big(\sum_{\substack{t\in k^*\\ t+t^\theta=0}}a(nh_j(t))+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}a(nh_j(t))\Big)$ *if* $l(wr_j)<l(w)$ *and* π_j *is of type* 3.

Then for all $h, h'\in H_\sigma$ and $1\leqq i, j\leqq |R'|$ we have

$$P_hQ_{h'}=Q_{h'}P_h,\ P_hQ_j=Q_jP_h,\ P_iQ_{h'}=Q_{h'}P_i \text{ and } P_iQ_j=Q_jP_i .$$

PROOF. We can prove $P_hQ_{h'}=Q_{h'}P_h$, $P_hQ_j=Q_jP_h$ and $P_iQ_{h'}=Q_{h'}P_i$ easily.

To prove $P_iQ_j=Q_jP_i$, it is sufficient to consider the next seven cases: Case 1. π_i and π_j are of type 1; Case 2. π_i is of type 1 and π_j is of type 2; Case 3. π_i is of type 2 and π_j is of type 1; Case 4. π_i and π_j are of type 2; Case 5. π_i is of type 2 and π_j is of type 3; Case 6. π_i is of type 3 and π_j is of type 2; Case 7. π_i and π_j are of type 3.

Using Lemmas 10, 11 and 12, we prove here only the most complicated case, Case 7. The others can be proved similarly.

Case 7. π_i and π_j are of type 3. In this case $i=j$.

From the definition $P_iQ_j(a(n))$ and $Q_jP_i(a(n))$ are as follows:

$$P_iQ_j(a(n))=a(\omega_i n\omega_j)=(\mathrm{A}) \text{ if } l(wr_j)>l(w) \text{ and } l(r_iwr_j)>l(wr_j),$$

$$P_iQ_j(a(n))=u^{c(i)}a(\omega_i n\omega_j)+p_i(u)\Big(\sum_{\substack{t'\in k^*\\ t'+t'^\theta=0}} a(h_i(t')n\omega_j)+(\mathrm{q}+1)\sum_{\substack{t'\in k^*\\ t'+t'^\theta\neq 0}} a(h_i(t')n\omega_j)\Big)$$

$$=(\mathrm{B}) \quad \text{if } l(wr_j)>l(w) \text{ and } l(r_iwr_j)<l(wr_j),$$

$$P_iQ_j(a(n))=u^{c(j)}a(\omega_i n\omega_j)+p_j(u)\sum_{\substack{t\in k^*\\ t+t^\theta=0}} P_i(a(nh_j(t)))$$

$$+p_j(u)(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}} P_i(a(nh_j(t)))$$

$$=(\mathrm{C}) \quad \text{if } l(wr_j)<l(w) \text{ and } l(r_iwr_j)>l(wr_j),$$

$$P_iQ_j(a(n))=u^{c(j)}\Big(u^{c(i)}a(\omega_i n\omega_j)+p_i(u)\Big(\sum_{\substack{t'\in k^*\\ t'+t'^\theta=0}} a(h_i(t')n\omega_j)+(q+1)\sum_{\substack{t'\in k^*\\ t'+t'^\theta\neq 0}} a(h_i(t')n\omega_j)\Big)\Big)$$

$$+p_j(u)\Big(\sum_{\substack{t\in k^*\\ t+t^\theta=0}} P_i(a(nh_j(t)))+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}} P_i(a(nh_j(t)))\Big)$$

$$=(\mathrm{D}) \quad \text{if } l(wr_j)<l(w) \text{ and } l(r_iwr_j)<l(wr_j);$$

$$Q_jP_i(a(n))=a(\omega_i n\omega_j)=(\mathrm{E}) \text{ if } l(r_iw)>l(w) \text{ and } l(r_iwr_j)>l(r_iw),$$

$$Q_jP_i(a(n))=u^{c(j)}a(\omega_i n\omega_j)+p_j(u)\Big(\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(\omega_i nh_j(t))+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}} a(\omega_i nh_j(t))\Big)$$

$$=(\mathrm{F}) \quad \text{if } l(r_iw)>l(w) \text{ and } l(r_iwr_j)<l(r_iw),$$

$$Q_jP_i(a(n))=u^{c(i)}a(\omega_i n\omega_j)+p_i(u)\sum_{\substack{t'\in k^*\\ t'+t'^\theta=0}} Q_j(a(h_i(t')n))$$

$$+p_i(u)(q+1)\sum_{\substack{t'\in k^*\\ t'+t'^\theta\neq 0}} Q_j(a(h_i(t')n))$$

$$=(\mathrm{G}) \quad \text{if } l(r_iw)<l(w) \text{ and } l(r_iwr_j)>l(r_iw),$$

$$Q_jP_i(a(n))=u^{c(i)}\Big(u^{c(j)}a(\omega_i n\omega_j)+p_j(u)\Big(\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(\omega_i nh_j(t))+(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(\omega_i nh_j(t))\Big)\Big),$$

$$+p_i(u)\Big(\sum_{\substack{t'\in k^*\\ t'+t'^\theta=0}} Q_j(a(h_i(t')n))+(q+1)\sum_{\substack{t'\in k^*\\ t'+t'^\theta\neq 0}} Q_j(a(h_i(t')n))\Big)$$

$=(\mathrm{H})$ if $l(r_i w)<l(w)$ and $l(r_i w r_j)<l(r_i w)$.

(1) If $P_iQ_j(a(n))=(\mathrm{A})$, then $Q_jP_i(a(n))=(\mathrm{E})$ and $(\mathrm{A})=(\mathrm{E})$.

(2) If $P_iQ_j(a(n))=(\mathrm{B})$, then $Q_jP_i(a(n))=(\mathrm{F})$ or (G).

Assume $Q_jP_i(a(n))=(\mathrm{F})$, i.e., $l(wr_j)>l(w)$, $l(r_iwr_j)<l(wr_j)$, $l(r_iw)>l(w)$ and $l(r_iwr_j)<l(r_iw)$. Then, from Lemma 10 $r_iw=wr_j$. Hence $c(i)=c(j)$. Let $r(j_1)\cdots r(j_k)$ be a reduced expression of w and $\omega=\omega(j_1)\cdots\omega(j_k)$, then $n=\omega h$ for some $h\in H_\sigma$. From Lemma 10 we have $\omega_i n=\omega_i\omega h=\omega\omega_j h$, and from Lemma 12

$$\omega_i n=\omega\omega_j h\omega_j^{-1}\omega_j=\omega h h_j(t_0)\omega_j=nh_j(t_0)\omega_j$$

for some $t_0\in k_\theta^*$.

Hence
$$\begin{aligned}\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(\omega_i n h_j(t)) &= \sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(nh_j(t_0)\omega_j h_j(t))\\ &=\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(nh_j((t_0t^{-1})^\theta)\omega_j)=\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(nh_j(t)\omega_j)\\ &=\sum_{\substack{t\in k^*\\ t+t^\theta=0}} a(h_i(t)n\omega_j)\ \text{(see Lemma 11)}.\end{aligned}$$

Therefore $(\mathrm{B})=(\mathrm{F})$.

If $Q_jP_i(a(n))=(\mathrm{G})$, it is clear that $(\mathrm{B})=(\mathrm{G})$ from $l(wr_j)>l(w)$.

(3) If $P_iQ_j(a(n))=(\mathrm{C})$, then $Q_jP_i(a(n))=(\mathrm{F})$ or (G).

If $Q_jP_i(a(n))=(\mathrm{F})$, we have $(\mathrm{C})=(\mathrm{F})$ from $l(r_iw)>l(w)$.

If $Q_jP_i(a(n))=(\mathrm{G})$, then $l(r_iw)<l(w)$ and $l(wr_j)<l(w)$. By virtue of this fact we can show that $(\mathrm{C})=(\mathrm{G})$.

(4) If $P_iQ_j(a(n))=(\mathrm{D})$, then $Q_jP_i(a(n))=(\mathrm{H})$. In this case $l(wr_j)<l(w)$ and $l(r_iw)<l(w)$, and from this fact, it follows that $(\mathrm{D})=(\mathrm{H})$. Q.E.D.

PROOF OF THEOREM 2.

(1) We use the essentially same method of Appendix of [7]. Let

$$\mathfrak{P}=\langle P_h,\ P_i\in \mathrm{End}_{C[u]}(A)\,|\,h\in H_\sigma,\ 1\leqq i\leqq |R'|\rangle$$

be a subalgebra of $\mathrm{End}_{C[u]}(A)$ generated by

$$\{P_h,\ P_i\,|\,h\in H_\sigma,\ 1\leqq i\leqq |R'|\}$$

and

$$\mathfrak{Q}=\langle Q_h,\ Q_j\in \mathrm{End}_{C[u]}(A)\,|\,h\in H_\sigma,\ 1\leqq j\leqq |R'|\rangle\ .$$

Let ρ^* and λ^* be two $C[u]$-module homomorphisms such that

$$\begin{array}{ccc} \rho^*:\mathfrak{P}\longrightarrow A \\ \quad\ \ \Psi \qquad\ \Psi \\ \quad\ \tilde{P}\mapsto \tilde{P}(a(1)) \end{array} \quad\text{and}\quad \begin{array}{ccc} \lambda^*:\mathfrak{Q}\longrightarrow A \\ \quad\ \ \Psi \qquad\ \Psi \\ \quad\ \tilde{Q}\mapsto \tilde{Q}(a(1)) \end{array}\ .$$

Let $n\in N_\sigma$, $\zeta(n)=w$ and $r(j_1)\cdots r(j_k)$ be a reduced expression of w, then $n=h_0\omega(j_1)\cdots\omega(j_k)$ for some $h_0\in H_\sigma$.

Since $\rho^*(P_{h_0}P_{j_1}\cdots P_{j_k})=a(n)$ and $\lambda^*(Q_{j_k}\cdots Q_{j_1}Q_{h_0})=a(n)$, ρ^* and λ^* are surjective homomorphisms.

Next we show that ρ^* is bijective. Let $\tilde{P}\in\mathfrak{P}$ such that $\rho^*(\tilde{P})=0$, then $\tilde{P}(a(1))=0$. Hence $\tilde{Q}\tilde{P}(a(1))=0$ for all $\tilde{Q}\in\mathfrak{Q}$. From Lemma 13 we have $\tilde{Q}\tilde{P}(a(1))=\tilde{P}\tilde{Q}(a(1))$ for all $\tilde{Q}\in\mathfrak{Q}$. Since λ^* is surjective, $\tilde{P}(A)=0$. Hence $\tilde{P}=0$ and ρ^* becomes bijective.

Therefore we can define an algebra structure on A as follows: Let $a, b\in A$, then the product ab is defined by the formula,

$$ab=\rho^*((\rho^*)^{-1}(a)(\rho^*)^{-1}(b)) .$$

One can easily check that this structure satisfies the conditions of (1) of Theorem 2.

(2) With respect to the above A we define a $C[u]$-algebra $\tilde{A}$ which has a presentation with generators $\{A(h), A(\omega)\,|\,h\in H_\sigma, \omega_i\in R'\}$ and relations

(I) $A(h)A(h')=A(hh')$ $\quad({}^\forall h, {}^\forall h'\in H_\sigma)$,

(II) $A(\omega_i)A(h)=A(\omega_i h\omega_i^{-1})A(\omega_i)$ $\quad({}^\forall h\in H_\sigma, {}^\forall\omega_i\in R')$,

(III) $A(\omega_i)^2=u^{c(i)}A(\omega_i^2)+p_i(u)\sum_{t\in k_\theta^*}A(h_i(t))A(\omega_i)$ if π_i is of type 1,

$$A(\omega_i)^2=u^{c(i)}A(\omega_i^2)+p_i(u)\sum_{t\in k'}A(h_i(t))A(\omega_i)\quad\text{if } \pi_i \text{ is of type 2,}$$

$$A(\omega_i)^2=u^{c(i)}A(\omega_i^2)+p_i(u)\sum_{\substack{t\in k^*\\ t+t^\theta=0}}A^*(h_i(t))A(\omega_i)$$

$$+p_i(u)(q+1)\sum_{\substack{t\in k^*\\ t+t^\theta\neq 0}}A(h_i(t))A(\omega_i)\ \text{if } \pi_i \text{ is of type 3, } ({}^\forall\omega_i\in R')$$

and

(IV) $\underbrace{A(\omega_i)A(\omega_j)A(\omega_i)\cdots}_{m_{ij}}=\underbrace{A(\omega_j)A(\omega_i)A(\omega_j)\cdots}_{m_{ij}}$ $\quad({}^\forall\omega_i, {}^\forall\omega_j\in R'$ such that $\omega_i\neq\omega_j)$.

Then we can prove (2) as Theorem 1. Q.E.D.

PROOF OF COROLLARY TO THEOREM 2. It is clear from the above discussion that $A_{f_1,C}\cong CN_\sigma$ and $A_{f_q,C}\cong H_C(G_\sigma, U_\sigma)$. The rest of the corollary is the straight consequence of the next theorem.

DEFINITION 3 (see [4]). We define, for a separable algebra S over a field K, the numerical invariants of S to be the integers $\{n_i\}$ such that

$$S^{\bar{K}}\cong\sum_i M_{n_i}(\bar{K})\quad\text{(direct sum)},$$

where $\bar{K}$ is an algebraic closure of K and $M_{n_i}(\bar{K})$ are the total matrix algebras over $\bar{K}$.

THEOREM 4 (see [4, Theorem (I, II)] or [9, Lemma 85]). *Let A be an associative algebra over an integral domain I with quotient field K, such that A has a finite basis over I. Let L be a field, and $f: I \to L$ a homomorphism, then L becomes an (L, I)-bimodule via $(\lambda, \lambda', x) \mapsto \lambda\lambda' f(x)$, $\lambda, \lambda' \in L$, $x \in I$.*

If $A_{f,L} = L \otimes_I A$ is a separable algebra over L, then A^K is separable over K, and A^K and $A_{f,L}$ have the same numerical invariants.

§6. Construction of the generic algebra of $H_C(G_\sigma, B_\sigma)_\lambda$.

Let

$$\bar{\varepsilon}_\lambda(H_\sigma) = \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(h^{-1}) a(h) .$$

In this section we show that $\bar{\varepsilon}_\lambda(H_\sigma)$ is an idempotent of A and $A_\lambda = \bar{\varepsilon}_\lambda(H_\sigma) A \bar{\varepsilon}_\lambda(H_\sigma)$ becomes the generic algebra of $H_C(G_\sigma, B_\sigma)_\lambda$.

LEMMA 14.

(1) *$a(h_0)\bar{\varepsilon}_\lambda(H_\sigma) = \lambda(h_0)\bar{\varepsilon}_\lambda(H_\sigma) = \bar{\varepsilon}_\lambda(H_\sigma) a(h_0)$ for all $h_0 \in H_\sigma$. $\bar{\varepsilon}_\lambda(H_\sigma)^2 = \bar{\varepsilon}_\lambda(H_\sigma)$.*

(2) *Let $n \in N_\sigma$ and $\zeta(n) = w$. Then*
$\bar{\varepsilon}_\lambda(H_\sigma) a(n) \bar{\varepsilon}_\lambda(H_\sigma) = \bar{\varepsilon}_\lambda(H_\sigma) a(n) = a(n) \bar{\varepsilon}_\lambda(H_\sigma)$ if $w \in W_{\sigma,\lambda}$, and
$\bar{\varepsilon}_\lambda(H_\sigma) a(n) \bar{\varepsilon}_\lambda(H_\sigma) = 0$ if $w \notin W_{\sigma,\lambda}$.

PROOF. Let δ be the bijective C-linear mapping of CH_σ onto $E = \sum_{h \in H_\sigma} Ca(h)$ defined by $\delta(h) = a(h)$, then from Theorem 2 δ becomes a C-algebra isomorphism.

(1) Since $CH_\sigma \cong E$ and $\delta(\varepsilon_\lambda(H_\sigma)) = \bar{\varepsilon}_\lambda(H_\sigma)$, (1) is clear from Lemma 1.

(2) Generally

$$\bar{\varepsilon}_\lambda(H_\sigma) a(n) = \left(\frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(h^{-1}) a(h) \right) a(n) = \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(h^{-1}) a(hn)$$
$$= a(n) \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(h^{-1}) a(n^{-1}hn) = a(n) \frac{1}{|H_\sigma|} \sum_{h \in H_\sigma} \lambda(nh^{-1}n^{-1}) a(h) = a(n) \bar{\varepsilon}_{w\lambda}(H_\sigma)$$

for any $n \in N_\sigma$.

Hence

$$\bar{\varepsilon}_\lambda(H_\sigma) a(n) \bar{\varepsilon}_\lambda(H_\sigma) = \bar{\varepsilon}_\lambda(H_\sigma) a(n) = a(n) \bar{\varepsilon}_\lambda(H_\sigma)$$

if $\zeta(n) = w \in W_{\sigma,\lambda}$, and

$$\bar{\varepsilon}_\lambda(H_\sigma) a(n) \bar{\varepsilon}_\lambda(H_\sigma) = a(n) \bar{\varepsilon}_{w\lambda}(H_\sigma) \bar{\varepsilon}_\lambda(H_\sigma) = a(n) \delta(\varepsilon_{w\lambda}(H_\sigma) \varepsilon_\lambda(H_\sigma)) = 0$$

if $\zeta(n) = w \notin W_{\sigma,\lambda}$ (see the proof of Lemma 2). Q.E.D.

PROOF OF THEOREM 3.

(1) is clear from (1) of Lemma 14.

(2) $A_\lambda=\tilde{\varepsilon}_\lambda(H_\sigma)A\tilde{\varepsilon}_\lambda(H_\sigma)=\sum_{n\in N_\sigma}\boldsymbol{C}[u]\tilde{\varepsilon}_\lambda(H_\sigma)a(n)\tilde{\varepsilon}_\lambda(H_\sigma)=\sum_{\substack{n\in N_\sigma\\ \zeta(n)\in W_{\sigma,\lambda}}}\boldsymbol{C}[u]\tilde{\varepsilon}_\lambda(H_\sigma)a(n)\tilde{\varepsilon}_\lambda(H_\sigma)$

(see Lemma 14).

Suppose $n\in N_\sigma$ and $\zeta(n)=w\in W_{\sigma,\lambda}$, then we have $n=n_wh$ for some $h\in H_\sigma$ and

$$\tilde{\varepsilon}_\lambda(H_\sigma)a(n)\tilde{\varepsilon}_\lambda(H_\sigma)=\tilde{\varepsilon}_\lambda(H_\sigma)a(n_wh)\tilde{\varepsilon}_\lambda(H_\sigma)=\lambda(h)\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)$$

from (1) of Lemma 14. Hence

$$A_\lambda=\sum_{w\in W_{\sigma,\lambda}}\boldsymbol{C}[u]\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)\ .$$

Since

$$\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)=\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)=\frac{1}{|H_\sigma|}\sum_{h\in H_\sigma}\lambda(h^{-1})a(hn_w)$$

from (2) of Lemma 14 and $\{hn_w|h\in H_\sigma\}\cap\{hn_{w'}|h\in H_\sigma\}=\emptyset$ if $w\neq w'$, the elements $\{\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)|w\in W_{\sigma,\lambda}\}$ form a $\boldsymbol{C}[u]$-basis for A_λ.

(3) $(A_\lambda)_{f_q,C}\supset\{1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)|w\in W_{\sigma,\lambda},\ n_w\text{: as in (2)}\}$. Since the elements $\{1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)\}$ form a basis for $(A_\lambda)_{f_q,C}$, we can define a bijective $\boldsymbol{C}$-linear mapping γ such that

$$\begin{array}{rcl}\gamma:\boldsymbol{C}\otimes A_\lambda & \longrightarrow & H_C(G_\sigma,B_\sigma)_\lambda\\ \Downarrow & & \Downarrow\\ 1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma) & \mapsto & \varepsilon_\lambda(H_\sigma)\alpha(n_w)\varepsilon_\lambda(H_\sigma)=\beta(n_w)\ .\end{array}$$

Now

$$1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})\tilde{\varepsilon}_\lambda(H_\sigma)1\otimes\tilde{\varepsilon}_\lambda(H)a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma)=1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma)$$

in $(A_\lambda)_{f_q\cdot C}$ where $w',\ w''\in W_{\sigma,\lambda}$. Assume that we have

$$a(n_{w'})a(n_{w''})=\sum_{\substack{n\in N_\sigma\\ f_n(u)\in C[u]}}f_n(u)a(n)\quad\text{in } A\ .$$

Then

$$\begin{aligned}\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma)&=\sum_{n\in N_\sigma}f_n(u)\tilde{\varepsilon}_\lambda(H_\sigma)a(n)\tilde{\varepsilon}_\lambda(H_\sigma)\\ &=\sum_{\substack{n\in N_\sigma\\ \zeta(n)\in W_{\sigma,\lambda}}}f_n(u)\tilde{\varepsilon}_\lambda(H_\sigma)a(n)\tilde{\varepsilon}_\lambda(H_\sigma)=\sum_{w\in W_{\sigma,\lambda}}\Big(\sum_{h\in H_\sigma}f_{hn(w)}(u)\tilde{\varepsilon}_\lambda(H_\sigma)a(hn_w)\tilde{\varepsilon}_\lambda(H_\sigma)\Big)\\ &=\sum_{w\in W_{\sigma,\lambda}}\Big(\sum_{h\in H_\sigma}\lambda(h)f_{hn(w)}(u)\Big)\tilde{\varepsilon}_\lambda(H_\sigma)a(n_w)\tilde{\varepsilon}_\lambda(H_\sigma)\end{aligned}$$

where $n(w)$ means n_w. Hence

$$\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma))=\sum_{w\in W_{\sigma,\lambda}}(\sum_{h\in H_\sigma}\lambda(h)f_{hn(w)}(q))\beta(n_w)\ .$$

On the other hand

$$\begin{aligned}\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})\tilde{\varepsilon}_\lambda(H))\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma))&=\beta(n_{w'})\beta(n_{w''})\\&=\varepsilon_\lambda(H_\sigma)\alpha(n_{w'})\alpha(n_{w''})\varepsilon_\lambda(H_\sigma)\ .\end{aligned}$$

Since $A_{f_q,C}\cong H_C(G_\sigma,\ U_\sigma)$, we have $\alpha(n_{w'})\alpha(n_{w''})=\sum_{n\in N_\sigma}f_n(q)\alpha(n)$. Therefore

$$\begin{aligned}\varepsilon_\lambda(H_\sigma)\alpha(n_{w'})\alpha(n_{w''})\varepsilon_\lambda(H_\sigma)&=\sum_{n\in N_\sigma}f_n(q)\varepsilon_\lambda(H_\sigma)\alpha(n)\varepsilon_\lambda(H_\sigma)\\&=\sum_{\substack{n\in N_\sigma\\ \zeta(n)\in W_{\sigma,\lambda}}}f_n(q)\varepsilon_\lambda(H_\sigma)\alpha(n)\varepsilon_\lambda(H_\sigma)\\&=\sum_{w\in W_{\sigma,\lambda}}(\sum_{h\in H_\sigma}f_{hn(w)}(q)\varepsilon_\lambda(H_\sigma)\alpha(hn_w)\varepsilon_\lambda(H_\sigma))\ .\end{aligned}$$

Since $\alpha(hn_w)=\alpha(h)\alpha(n_w)$ and

$$\varepsilon_\lambda(H_\sigma)\alpha(h)=\varepsilon_\lambda(H_\sigma)\ \mathrm{ind}(h)ehe=e\varepsilon_\lambda(H_\sigma)he=\lambda(h)\varepsilon_\lambda(H_\sigma)e\ ,$$

we have

$$\varepsilon_\lambda(H_\sigma)\alpha(hn_w)\varepsilon_\lambda(H_\sigma)=\lambda(h)\varepsilon_\lambda(H_\sigma)\alpha(n_w)\varepsilon_\lambda(H_\sigma)\ .$$

Hence

$$\varepsilon_\lambda(H_\sigma)\alpha(n_{w'})\alpha(n_{w''})\varepsilon_\lambda(H_\sigma)=\sum_{w\in W_{\sigma,\lambda}}(\sum_{h\in H_\sigma}\lambda(h)f_{hn(w)}(q))\varepsilon_\lambda(H_\sigma)\alpha(n_w)\varepsilon_\lambda(H_\sigma))$$

and

$$\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma))=\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w'})\tilde{\varepsilon}_\lambda(H_\sigma))\gamma(1\otimes\tilde{\varepsilon}_\lambda(H_\sigma)a(n_{w''})\tilde{\varepsilon}_\lambda(H_\sigma))\ .$$

Therefore γ becomes a $\boldsymbol{C}$-algebra isomorphism of $(A_\lambda)_{f_q,C}$ onto $H_C(G_\sigma,\ B_\sigma)_\lambda$.

Similarly we can show $(A_\lambda)_{f_q,C}\cong\varepsilon_\lambda(H_\sigma)\boldsymbol{C}N_\sigma\varepsilon_\lambda(H_\sigma)$, because the elements

$$\{\varepsilon_\lambda(H_\sigma)n_w\varepsilon_\lambda(H_\sigma)|w\in W_{\sigma,\lambda}\}$$

form a basis for $\varepsilon_\lambda(H_\sigma)\boldsymbol{C}N_\sigma\varepsilon_\lambda(H_\sigma)$.

(4) is clear from Theorem 4 (see [5, Theorem (71.2)]). Q.E.D.

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