On unitary representations of exponential groups

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(Communicated by N. Iwahori)

§0. In this paper, one theorem about unitary representations of exponential groups (see the definition in §1) will be proved. It gives an affirmative answer to an open problem in Quint [7] concerning irreducibility and equivalence of holomorphically induced representations (see below), under certain conditions in the case of exponential groups.

Auslander-Kostant [1] extended the Kirillov theory for nilpotent Lie groups to solvable Lie groups. The following is one of their main results.

"Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ be a nilpotent ideal in $\mathfrak{g}$ which contains $[\mathfrak{g}, \mathfrak{g}]$. Given an integral form $f \in \mathfrak{g}^*$, the corresponding character is denoted by $\eta_f: G_f \to \boldsymbol{T}$. Then, if $\mathfrak{h}$ is a strongly $\mathfrak{n}$-admissible positive polarization of G at f (hence $\mathfrak{h}$ satisfies the strong Pukanszky condition), the holomorphically induced representation $\rho(f, \eta_f, \mathfrak{h}, G)$ is irreducible and is independent (up to unitary equivalence) of $\mathfrak{h}$ and $\mathfrak{n}$."

Duflo [6] generalized this theorem; if $\mathfrak{n}$ is a nilpotent ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{n}$ is nilpotent, the same conclusion holds.

But these results are not so complete as Kirillov's result for nilpotent Lie groups. If we consider a polarization $\mathfrak{h}$ which is not necessarily strongly $\mathfrak{n}$-admissible for any $\mathfrak{n}$ as above, many problems come to arise. For example, let $\mathfrak{h}$ be a positive polarization of G at $f \in \mathfrak{g}^*$ which is not necessarily strongly admissible for any $\mathfrak{n}$ of above type. 1) When is the space of $\rho(f, \eta_f, \mathfrak{h}, G)$ not zero? 2) When is $\rho(f, \eta_f, \mathfrak{h}, G) \neq 0$ irreducible? 3) Is $\rho(f, \eta_f, \mathfrak{h}, G)$ (supposed to be irreducible) independent of $\mathfrak{h}$?

We give an affirmative answer to the last two problems under certain conditions; namely, if G is exponential and if $\mathfrak{h}$ satisfies the strong Pukanszky condition (see the definition below), then $\rho(f, \mathfrak{h}, G) = \rho(f, \eta_f, \mathfrak{h}, G)$ (supposed to be non zero) is irreducible and independent of $\mathfrak{h}$, and actually is equivalent to the Kirillov-Bernat representation associated to f.

In preparing this paper, I have received many advices from Professor M. Saito. I would like to express my cordial thanks to him. I must be also grateful to

Professor M. Vergne and the referee to whom I owe many improvements of the first version of the manuscript.

Notations. In this paper, we have the following conventions.

1. The letter $\boldsymbol{R}$ (resp. $\boldsymbol{C}$) designates the field of real numbers (resp. the field of complex numbers).

2. Lie groups (resp. Lie algebras) are always of finite dimension over $\boldsymbol{R}$ (resp. $\boldsymbol{R}$ or $\boldsymbol{C}$).

3. Let E, F be sets, $\phi: E \to F$ a mapping and let $A \subset E$, then $\phi | A$ stands for the restriction of ϕ to A.

4. If $\mathfrak{g}$ is a Lie algebra, we denote its dual space by $\mathfrak{g}^*$ and for $f \in \mathfrak{g}^*$, we define an alternating bilinear form B_f on $\mathfrak{g}$ by $B_f(X, Y) = \langle f, [X, Y] \rangle = f([X, Y])$ for $X, Y \in \mathfrak{g}$. If $\mathfrak{a}$ is a vector subspace of $\mathfrak{g}$, we define $\mathfrak{a}^{\perp, \mathfrak{g}^*}$ and $\mathfrak{a}_f$ respectively by $\mathfrak{a}^{\perp, \mathfrak{g}^*} = \{f \in \mathfrak{g}^*;\ f | \mathfrak{a} = 0\}$ and $\mathfrak{a}_f = \{X \in \mathfrak{g};\ B_f(X, Y) = 0$ for all $Y \in \mathfrak{a}\}$. When there is no danger of confusion, we write $\mathfrak{a}^{\perp}$ instead of $\mathfrak{a}^{\perp, \mathfrak{g}^*}$. A subspace $\mathfrak{a}$ is called isotropic (with respect to B_f) when $\mathfrak{a} \subset \mathfrak{a}_f$. The set of subalgebras of $\mathfrak{g}$ which are isotropic subspaces will be denoted by $S(f, \mathfrak{g})$ and the subset of $S(f, \mathfrak{g})$ which consists of all maximal isotropic subspaces will be denoted by $M(f, \mathfrak{g})$. A subalgebra $\mathfrak{h}$ in $S(f, \mathfrak{g})$ belongs to $M(f, \mathfrak{g})$ if and only if $\dim \mathfrak{h} = \frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}_f)$.

5. If V is a vector space over $\boldsymbol{R}$, V_C is its complexification: $V_C = V + iV$. For $X \in V_C$, $X \to \bar{X}$ denotes the conjugation with respect to V. If W is a subspace of V, $\bar{W} = \{\bar{X};\ X \in W\}$.

6. Let G be a Lie group with Lie algebra $\mathfrak{g}$, then G acts on $\mathfrak{g}^*$ by the coadjoint representation and its action will be denoted by $a.f (a \in G, f \in \mathfrak{g}^*)$.

7. The unitary equivalence of representations will be denoted by $\simeq$.

§1. We define at first some concepts following Auslander-Kostant [1]. In this section, unless otherwise stated, G will be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_C = \mathfrak{g} + i\mathfrak{g}$ and consider $f \in \mathfrak{g}^*$ as a complex-valued linear functional on $\mathfrak{g}_C$, then B_f is considered as an alternating bilinear form on $\mathfrak{g}_C$.

DEFINITION. A complex subalgebra $\mathfrak{h} \subset \mathfrak{g}_C$ is called a *positive polarization* of G at $f \in \mathfrak{g}^*$ if $\mathfrak{h}$ has the following properties:

1) $\mathfrak{h}$ is a maximal isotropic subspace of $\mathfrak{g}_C$ with respect to B_f.

2) $\mathfrak{h} + \bar{\mathfrak{h}}$ is a subalgebra of $\mathfrak{g}_C$.

3) $\mathfrak{h}$ is stable under $\mathrm{Ad}_G G_f$, where G_f is the isotropic subgroup of G at f.

4) If $X \in \mathfrak{h}$, then $if([X, \bar{X}]) \geqq 0$.

Let $\mathfrak{g}$ be a Lie algebra over $\boldsymbol{R}$ and let $f \in \mathfrak{g}^*$. A positive polarization of $\mathfrak{g}$ at

f will mean a positive polarization at f of the simply connected Lie group with Lie algebra $\mathfrak{g}$. We denote by $P^+(f,G)$ the set of positive polarizations of G at f.

DEFINITION. For $\mathfrak{h}\in P^+(f,G)$, we define two subalgebras of $\mathfrak{g}$ by $\mathfrak{d}=\mathfrak{h}\cap\mathfrak{g}$ and $\mathfrak{e}=(\mathfrak{h}+\bar{\mathfrak{h}})\cap\mathfrak{g}$.

Let $\mathfrak{h}\in P^+(f,G)$ and let $\mathfrak{d}$ and $\mathfrak{e}$ be defined as in the above definition. Let D_0 (resp. E_0) be the connected Lie subgroup of G with Lie algebra $\mathfrak{d}$ (resp. $\mathfrak{e}$). Since $\mathfrak{h}$ is stable under $\mathrm{Ad}_G G_f$, it follows that D_0 and E_0 are normalized by G_f, so that $D=G_f D_0$ and $E=G_f E_0$ are subgroups of G.

DEFINITION. We shall say that $\mathfrak{h}\in P^+(f,G)$ satisfies the *strong Pukanszky condition* if $E.f$ is closed in $\mathfrak{g}^*$, and that $\mathfrak{h}$ satisfies the *weak Pukanszky condition* if $D.f$ is closed in $\mathfrak{g}^*$, or equivalently, if $f+\mathfrak{e}^{\perp}\subset O(f)$ (See [3]).

The strong Pukanszky condition is the Pukanszky condition in the sense of Auslander-Kostant [1], and the weak Pukanszky condition is the Pukanszky condition in the sense of Bernat and others [3]. The former is effectively stronger than the latter. Next we define the concept of an exponential group.

DEFINITION [5]. Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. G is called an *exponential group* if the exponential mapping $\exp: \mathfrak{g}\to G$ is surjective.

As to other equivalent definitions of an exponential group, see [5] and [8]. Henceforth in this paper, G is always an exponential group with Lie algebra $\mathfrak{g}$.

Let $f\in\mathfrak{g}^*$ and let $\mathfrak{k}\in S(f,\mathfrak{g})$, then $\chi(f,\mathfrak{k})(\exp x)=e^{if(x)}(x\in\mathfrak{k})$ gives a character (1-dimensional unitary representation) of the connected Lie subgroup $K=\exp\mathfrak{k}$ of G corresponding to $\mathfrak{k}$. We denote by $\hat{\rho}(f,\mathfrak{k},G)$ the unitary representation $\mathrm{ind}_{K\uparrow G}\,\chi(f,\mathfrak{k})$ of G induced from $\chi(f,\mathfrak{k})$, by $\hat{\mathscr{H}}(f,\mathfrak{k},G)$ the representation space of $\hat{\rho}(f,\mathfrak{k},G)$ and by $I(f,\mathfrak{g})$ the set of $\mathfrak{k}\in S(f,\mathfrak{g})$ such that $\hat{\rho}(f,\mathfrak{k},G)$ is irreducible. We have $I(f,\mathfrak{g})\subset M(f,\mathfrak{g})$ (see [2]). Let $O(f)$ be the orbit through f with respect to the coadjoint representation of G and let $\hat{\rho}(O(f))$ be the equivalence class of irreducible unitary representations of G corresponding to $O(f)$ in the sense of Kirillov-Bernat [2].

REMARK. If $\mathfrak{k}\in M(f,\mathfrak{g})$, then $\mathfrak{k}_C\in P^+(f,G)$. The following conditions are equivalent: 1) $\mathfrak{k}\in I(f,\mathfrak{g})$; 2) $\mathfrak{k}_C$ satisfies the weak Pukanszky condition; 3) $\mathfrak{k}_C$ satisfies the strong Pukanszky condition.

§2. Let G be an exponential group with Lie algebra $\mathfrak{g}$ and let $f\in\mathfrak{g}^*$. If $\mathfrak{h}\in P^+(f,G)$ satisfies the strong Pukanszky condition, the holomorphically induced representation $\rho(f,\mathfrak{h},G)$ can be constructed from $\mathfrak{h}$ just as in [1], since every $f\in\mathfrak{g}^*$

is integral and the corresponding character η_f of G_f is uniquely determined by the simply-connectedness of G_f. The representation space of $\rho(f, \mathfrak{h}, G)$ will be denoted by $\mathscr{H}(f, \mathfrak{h}, G)$. Now we prove the following theorem.

THEOREM. *Let G be an exponential group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h}$ be a positive polarization of G at f satisfying the strong Pukanszky condition. If $\mathscr{H}(f, \mathfrak{h}, G) \neq \{0\}$, then $\rho(f, \mathfrak{h}, G)$ is irreducible and $\rho(f, \mathfrak{h}, G) \in \hat{\rho}(O(f))$. In particular, $\rho(f, \mathfrak{h}, G)$ is independent of $\mathfrak{h}$.*

PROOF. The theorem is trivial when $\dim G=1$, so we prove it by induction on $\dim G$, and assume $\dim G=n$.

Case 1. There is an ideal $\mathfrak{a} \neq \{0\}$ in $\mathfrak{g}$ such that $f(\mathfrak{a})=0$.

Let $A=\exp \mathfrak{a}$, $\tilde{G}=G/A$ and let $\pi: G \rightarrow \tilde{G}$ be the canonical projection. Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{G}$ and let $d\pi$: $\mathfrak{g}_C \rightarrow (\tilde{\mathfrak{g}})_C$ be the differential of π. Now we consider the exact sequence of exponential groups $1 \rightarrow A \rightarrow G \xrightarrow{\pi} \tilde{G} \rightarrow 1$.

Let $\tilde{f} \in (\tilde{\mathfrak{g}})^*$ be such that $\tilde{f} \circ d\pi=f$ and let $\tilde{\mathfrak{h}}=d\pi(\mathfrak{h})$. Since $\mathfrak{h} \supset \mathfrak{g}_f \supset \mathfrak{a}$, it is clear that $\tilde{\mathfrak{h}} \in P^+(\tilde{f}, \tilde{G})$ and $(\tilde{\mathfrak{g}})^*$ is naturally isomorphic to $\mathfrak{a}^{\perp, \mathfrak{g}^*}$, so that $\tilde{\mathfrak{h}}$ satisfies the strong Pukanszky condition. So by Proposition I. 5.13 in [1],

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \circ \pi \simeq \rho(f, \mathfrak{h}, G). \tag{1}$$

Hence by our assumption, $\mathscr{H}(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \neq \{0\}$. Since $\dim \tilde{G} < \dim G$, the induction hypothesis implies that $\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \in \hat{\rho}(O(\tilde{f}))$. That is, there is an $\tilde{\mathfrak{h}}_0 \in I(\tilde{f}, \tilde{\mathfrak{g}})$ such that

$$\rho(\tilde{f}, \tilde{\mathfrak{h}}, \tilde{G}) \simeq \hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}). \tag{2}$$

Let $\mathfrak{h}_0=d\pi^{-1}(\tilde{\mathfrak{h}}_0)$. Then it is obvious that

$$\hat{\rho}(\tilde{f}, \tilde{\mathfrak{h}}_0, \tilde{G}) \circ \pi \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \text{ and } \mathfrak{h}_0 \in I(f, \mathfrak{g}). \tag{3}$$

From (1), (2) and (3), $\rho(f, \mathfrak{h}, G) \simeq \hat{\rho}(f, \mathfrak{h}_0, G) \in \hat{\rho}(O(f))$.

Case 2. There is no ideal $\mathfrak{a} \neq \{0\}$ in $\mathfrak{g}$ such that $f(\mathfrak{a})=0$. This case is divided into two subcases.

i) $\mathfrak{e} \subsetneqq \mathfrak{g}$. We choose and fix one complementary linear subspace $\mathfrak{m}$ of $\mathfrak{e}$ in $\mathfrak{g}$, and let $j: \mathfrak{e}^* \rightarrow \mathfrak{g}^*$ be an injection such that $j(h)(x)=h(y)$ for $h \in \mathfrak{e}^*$ and $x=y+z$ with $y \in \mathfrak{e}$, $z \in \mathfrak{m}$. From now on, we identify $\mathfrak{e}^*$ with its image $j(\mathfrak{e}^*)$, so that $\mathfrak{e}^* \subset \mathfrak{g}^*$. Let $\pi: \mathfrak{g}^* \rightarrow \mathfrak{e}^*$ be the restriction mapping such that $\pi(l)=l'=l|\mathfrak{e}$ for $l \in \mathfrak{g}^*$. Then $\pi|\mathfrak{e}^*=I$ (the identity mapping of $\mathfrak{e}^*$) under the above identification. Now $\mathfrak{h}$ is clearly a positive polarization of $\mathfrak{e}$ at $\pi(f)=f'=f|\mathfrak{e}$. We shall show that $\mathfrak{h}$ satisfies the strong Pukanszky condition as a polarization of $\mathfrak{e}$ at f'.

At first, since $\mathfrak{e}$ is a subalgebra of $\mathfrak{g}$,

$$\pi(E.f)=(E.f)'=O(f)', \tag{4}$$

where $O(f')$ is the orbit through f' with respect to the co-adjoint representation of E. Next, we prove that $\pi^{-1}(O(f'))=E.f$. If $a\in E$ and $l\in\mathfrak{e}^{\perp}$, one has $a.l\in\mathfrak{e}^{\perp}$. For $a\in E$, we write $a.f=(a.f)'+\hat{f}$, where $\hat{f}\in\mathfrak{e}^{\perp}$. Then we have, for any $h\in\mathfrak{e}^{\perp}$, $a^{-1}.(\hat{f}-h)\in\mathfrak{e}^{\perp}$ and $a.(f-a^{-1}.(\hat{f}-h))=(a.f)'+h$. Since $\mathfrak{h}$ satisfies the strong Pukanszky condition as a polarization of G at f, Proposition I.5.6 in [1] shows that one has $f-a^{-1}.(\hat{f}-h)=b.f$ for some $b\in D\subset E$. Hence

$$(ab).f=(a.f)'+h. \tag{5}$$

Since $a\in E$ and $h\in\mathfrak{e}^{\perp}$ are arbitrary, (4) and (5) imply that

$$\pi^{-1}(O(f'))=E.f. \tag{6}$$

Hence $O(f')=E.f\cap\mathfrak{e}^*$. Therefore $O(f')$ is closed in $\mathfrak{e}^*$ and $\mathfrak{h}$ satisfies the strong Pukanszky condition as a polarization of $\mathfrak{e}$ at f'.

Since $\rho(f,\mathfrak{h},G)=\operatorname{ind}_{E\uparrow G}\rho(f',\mathfrak{h},E)$, $\mathscr{H}(f',\mathfrak{h},E)\neq\{0\}$ provided $\mathscr{H}(f,\mathfrak{h},G)\neq\{0\}$. Since $\dim E<\dim G$, we can apply the induction hypothesis to have $\rho(f',\mathfrak{h},E)\in\hat{\rho}(O(f'))$. That is, there is an $\mathfrak{h}_0\in I(f',\mathfrak{e})$ such that

$$\rho(f',\mathfrak{h},E)\simeq\hat{\rho}(f',\mathfrak{h}_0,E). \tag{7}$$

Lemma 2.2.3 in [2] implies that $\mathfrak{h}_0\in M(f',\mathfrak{e})$. And since $\mathfrak{h}\in P^+(f',E)$, $\dim_R\mathfrak{h}_0=\dim_C\mathfrak{h}=\frac{1}{2}(\dim_R\mathfrak{g}+\dim_R\mathfrak{g}_f)$. Thus $\mathfrak{h}_0\in M(f,\mathfrak{g})$.

We show next that $(\mathfrak{h}_0)_C$ satisfies the weak Pukanszky condition as a polarization of $\mathfrak{g}$. We first notice that

$$f+\mathfrak{h}_0^{\perp,\mathfrak{g}^*}=f'+\mathfrak{h}_0^{\perp,\mathfrak{e}^*}+\mathfrak{e}^{\perp}. \tag{8}$$

But since $\mathfrak{h}_0\in I(f',\mathfrak{e})$, Proposition 3.2 in Chap. VI of [3] asserts that $(\mathfrak{h}_0)_C$ satisfies the weak Pukanszky condition as an element of $P^+(f',E)$ so that

$$f'+\mathfrak{h}_0^{\perp,\mathfrak{e}^*}\subset O(f'). \tag{9}$$

One knows from (6), (8) and (9) that $f+\mathfrak{h}_0^{\perp,\mathfrak{g}^*}\subset O(f)$. Thus $(\mathfrak{h}_0)_C$ satisfies the weak Pukanszky condition as a polarization of $\mathfrak{g}$.

Hence Proposition 3.2 in Chap. VI of [3] implies that $\mathfrak{h}_0\in I(f,\mathfrak{g})$. That is, $\operatorname{ind}_{E\uparrow G}\hat{\rho}(f',\mathfrak{h}_0,E)=\hat{\rho}(f_0,\mathfrak{h}_0,G)$ is irreducible. So by (7), $\rho(f,\mathfrak{h},G)=\operatorname{ind}_{E\uparrow G}\rho(f',\mathfrak{h},E)\simeq\operatorname{ind}_{E\uparrow G}\hat{\rho}(f',\mathfrak{h}_0,E)\in\hat{\rho}(O(f))$.

ii) $\mathfrak{e}=\mathfrak{g}$ (i.e., $\mathfrak{h}$ is totally complex in the sense of Blattner [4]). In this case, Blattneer [4] shows that $\rho(f, \mathfrak{h}, G)$ is irreducible. So we can assume that $\rho(f, \mathfrak{h}, G) \in \hat{\rho}(O(f_0))$ for some $f_0 \in \mathfrak{g}^*$, and it suffices to see that $f_0 \in O(f)$.

The first part of the proof of Theorem I. 4.10 in [1] for nilpotent Lie groups is also valid for exponential groups and we have the following lemma.

LEMMA 1. *When G is an exponential group, $\mathfrak{d}$ is an ideal in $\mathfrak{e}$.*

PROOF. For $x \in \mathfrak{d}$, let $\pi(x) \in \mathrm{End}\, \mathfrak{e}/\mathfrak{d}$ be the operator on $\mathfrak{e}/\mathfrak{d}$ induced by $\mathrm{ad}\, x$. Then $\pi(x)$ is a skew-symmetric operator with respect to a positive non-degenerate bilinear form on $\mathfrak{e}/\mathfrak{d}$ (see the proof of Theorem I. 4.10 in [1]). Thus its eigenvalues are 0 or purely imaginary. On the other hand, $\mathfrak{e}/\mathfrak{d}$, considered as a $\mathfrak{d}$-module with respect to π, is of exponential type (see Chap. I in [3]). Hence $\pi(x)=0$. So $[\mathfrak{d}, \mathfrak{e}] \subset \mathfrak{d}$. q.e.d.

LEMMA 2. *$\mathfrak{d}$ is an ideal in $\mathfrak{g}$ and $\dim \mathfrak{d} \leqq 1$. Further if we denote by $\mathfrak{z}$ the center of $\mathfrak{g}$, then $\mathfrak{d}=\mathfrak{g}_f=\mathfrak{z}$.*

PROOF. Since $\mathfrak{e}=\mathfrak{g}$, we first notice that $\mathfrak{d}$ is an ideal in $\mathfrak{g}$ from Lemma 1. We put $\mathfrak{b}=\mathfrak{d} \cap \ker f$, then $[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{b}$, because $[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{d}$ and $f([\mathfrak{g}, \mathfrak{b}])=f([\mathfrak{e}, \mathfrak{b}])=0$. Thus $\mathfrak{b}$ is an ideal in $\mathfrak{g}$ and $f(\mathfrak{b})=0$. So, from our assumption, it follows that $\mathfrak{b}=\{0\}$. Hence $\dim \mathfrak{d} \leqq 1$ and $\mathfrak{d} \subset \mathfrak{z}$. On the other hand, it is clear that $\mathfrak{d} \supset \mathfrak{g}_f \supset \mathfrak{z}$. Hence $\mathfrak{d}=\mathfrak{g}_f=\mathfrak{z}$. q.e.d.

Now we continue the proof of our theorem. If we assume $\dim \mathfrak{d}=0$, then $\mathfrak{g}_f=\{0\}$ and $\dim O(f)=\dim \mathfrak{g}=n$. Therefore the differential $X \mapsto X.f$ of the mapping $g \mapsto g.f$ is bijective, so that $O(f)$ is open in $\mathfrak{g}^*$. On the other hand, since $\mathfrak{h}$ satisfies the strong Pukanszky condition, $O(f)=G.f=E.f$ is closed in $\mathfrak{g}^*$. It follows from the connectedness of $\mathfrak{g}^*$ that $O(f)=\mathfrak{g}^*$, which is a contradiction. Thus $\dim \mathfrak{d}=1$.

We can put $\mathfrak{d}=\mathfrak{g}_f=\mathfrak{z}=\{\boldsymbol{R}z\}$ with $f(z) \neq 0$ and then $\dim O(f)=\dim \mathfrak{g}-\dim \mathfrak{g}_f=n-1$. The set $V=\{h \in \mathfrak{g}^*;\ h(z)=f(z)\}$ is an $(n-1)$-dimensional hyperplane in $\mathfrak{g}^*$. Since $z \in \mathfrak{z}$, $(a.f)(z)=f(z)$ for any $a \in G$; i.e., $O(f) \subset V$. Here we can repeat the above argument to conclude that

$$O(f)=V. \tag{10}$$

Let $\mathfrak{h}_0 \in I(f_0, \mathfrak{g})$. Then $\mathfrak{h}_0 \supset \mathfrak{z}$ (see Chap. II in [2]). We denote an intertwining operator between $\hat{\rho}(f_0, \mathfrak{h}_0, G) \in \hat{\rho}(O(f_0))$ and $\rho(f, \mathfrak{h}, G)$ by $R: \hat{\mathscr{H}}(f_0, \mathfrak{h}_0, G) \rightarrow \mathscr{H}(f, \mathfrak{h}, G)$. For brevity, we write $\hat{\rho}(f_0, \mathfrak{h}_0, G)=\hat{L}$ and $\rho(f, \mathfrak{h}, G)=L$. Thus, if $\phi \in \hat{\mathscr{H}}(f_0, \mathfrak{h}_0, G)$ and $a \in G$, then $(R \circ \hat{L}_a)\phi=(L_a \circ R)\phi$. Let $t_0 \in \boldsymbol{R}$ be fixed and put $a_0=\exp t_0 z$. Since a_0 belongs to the center of G,

$$(\hat{L}_{a_0}\phi)(a)=\phi(\exp(-t_0z)a)=\phi(a\exp(-t_0z))=e^{it_0\langle f_0,z\rangle}\phi(a)$$

for $a\in G$. Hence $(R\circ\hat{L}_{a_0})\phi=e^{it_0\langle f_0,z\rangle}R\phi$. On the other hand, $((L_{a_0}\circ R)\phi)(a)=(R\phi)(\exp(-t_0z)a)=(R\phi)(a\exp(-t_0z))=e^{it_0\langle f,z\rangle}(R\phi)(a)$. Hence $(L_{a_0}\circ R)\phi=e^{it\langle f_0,z\rangle}R\phi$. It follows that $e^{it_0\langle f_0,z\rangle}\psi(a)=e^{it_0\langle f,z\rangle}\psi(a)$ for some complex-valued C^∞-function $\psi\not\equiv 0$ on G, and for all $a\in G$. Thus $e^{it_0\langle f_0,z\rangle}=e^{it_0\langle f,z\rangle}$. Since $t_0\in \boldsymbol{R}$ is arbitrary, it follows that

$$\langle f_0,z\rangle=\langle f,z\rangle. \tag{11}$$

We can conclude from (10) and (11) that $f_0\in O(f)$. q.e.d.

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(Received June 7, 1974)

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