A transformation group whose orbits are homeomorphic to a circle or a point

By Hideki Omori

In his suggestive paper [2], D. Montgomery proved that a homeomorphism $T$ of a connected manifold $M$ is finitely periodic, if there is an integer $k = k(x)$ such that $T^k(x) = x$ for every point. This result can not be extended to the case of one parameter transformation groups, that is, a one parameter transformation group acting effectively on $M$ is not necessarily a circle group, even if every orbit of the group is homeomorphic to a circle $S^1$. A simple example of this fact can be made easily on a two-dimensional torus.

The topology, however, of the one parameter group is affected by the condition that every orbit is homeomorphic to $S^1$. The following theorem, which will be proved in this paper, shows a thing of this kind.

For convenience, by $M$ we mean a connected manifold with the second countability axiom and by $H(M)$ the group of all the homeomorphisms from $M$ onto $M$ with compact open topology. These notations are fixed throughout this paper.

**Theorem A.** Let $(L, T_0)$ be a vector group of finite dimension, where $L$ is the underlying additive group and $T_0$ is the topology for $L$. Let $\varphi$ be a non-trivial continuous homomorphism from $(L, T_0)$ into $H(M)$. If every orbit of $\varphi(L)$ is homeomorphic to $S^1$ or a point, then $\varphi(L)$ is closed in $H(M)$.

More precisely, $\varphi(L) \cong (L', T') \times S^1$ or $(L', T')$ for some vector group $(L', T')$.

Since $H(M)$ is a set of second category [1], the above theorem means that $\varphi$ is an open mapping from $(L, T_0)$ onto $\varphi(L)$. Thus, $\varphi(L)$ is a Lie group under compact open topology.

Now, we consider the case where the above homomorphism is a monomorphism.

Let $\varphi$ be a continuous monomorphism from $(L, T_0)$ into $H(M)$. If $\varphi(L)$ is not closed in $H(M)$, then the relative topology for $\varphi(L)$ in $H(M)$ introduces a new topology $T$ for $L$ such that (i) $(L, T)$ satisfies the first countability axiom,

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(ii) \((L, \mathcal{T})\) satisfies Hausdorff's separation axiom, (iii) \(\mathcal{T}\) is weaker than \(\mathcal{T}_0\), (iv) \((L, \mathcal{T})\) is a topological additive group and (v) \((L, \mathcal{T}) \neq (L, \mathcal{T}_0)\).

For a fixed underlying group \(L\), we denote by \(T(L, \mathcal{T}_0)\) the collection of all the pairs of the fixed abstract group \(L\) and a topology \(\mathcal{T}\) for \(L\) satisfying (i)\(\sim\)(iv) above.

For a subgroup \(L'\) of \(L\), \((L', \mathcal{T})\) means the subgroup \(L'\) with the relative topology in \((L, \mathcal{T})\).

Under these notations, an element \((L, \mathcal{T}) \in T(L, \mathcal{T}_0)\) is said to be irreducible, if for any proper vector subgroup \(L'\), \((L', \mathcal{T}) = (L', \mathcal{T}_0)\) but \((L, \mathcal{T}) \neq (L, \mathcal{T}_0)\).

Since \(\dim L < \infty\), we see easily that if \((L, \mathcal{T}) \in T(L, \mathcal{T}_0)\) and \((L, \mathcal{T}) \neq (L, \mathcal{T}_0)\), there is a vector subgroup \((L', \mathcal{T})\) which is irreducible. We know in [4] that there is an example of topology \(\mathcal{T}\) for two-dimensional vector group \(L\) such that \((L, \mathcal{T}) \in T(L, \mathcal{T}_0)\) and \((L, \mathcal{T})\) is irreducible.

Now, in the case of monomorphic \(\varphi\), Theorem A is obtained as an immediate consequence of the following Theorem B.

**Theorem B.** Let \((L, \mathcal{T}) \in T(L, \mathcal{T}_0)\) be irreducible. Assume furthermore that there is a non-trivial continuous homomorphism \(\varphi\) from \((L, \mathcal{T})\) into \(H(M)\) such that every orbit \(\varphi(L)(x)\) is homeomorphic to \(S^1\) or a point. Then \(\varphi(L)\) is isomorphic to \(S^1\).

**Corollary.** Notations and assumptions being as in Theorem A, if \(\varphi\) is monomorphic, then \(\varphi(L)\) is closed in \(H(M)\).

The proof of Theorem B, which will be given later, is similar to that of the following well-known theorem.

**Theorem C.** Let \(\varphi\) be a non-trivial homomorphism from a toroidal group \(T\) into \(H(M)\) such that every orbit \(\varphi(T)(x)\) is homeomorphic to a circle or a point. Then \(\varphi(T) \cong S^1\).

The proof of this theorem consists of the following three steps, which correspond to those of the proof of our Theorem B.

a) It is well-known that the Pontryagin dual group \(\text{Hom}(T, S^1)\) is a discrete group.

b) Let \(T_x^0(\subset T)\) be the connected component containing \(0\) of the full-inverse of the isotropy subgroup of \(\varphi(T)\) at \(x \in M\) and let \(M'\) be the set of the points such that \(\varphi(T)(x)\) is homeomorphic to a circle. Then \(T/T_x^0 \cong S^1\) for \(x \in M'\).

Therefore, there is a homomorphism \(\varphi_x\) from \(T\) onto \(S^1\) depending continuously on \(x \in M'\). Thus, from a) we have that \(\varphi_x\) is constant on every connected component of \(M'\). In other words, \(T\) operates as a circle group on each connected component of \(M'\).
c) $M'$ is connected and dense in $M$, for the fixed point set of a compact Lie transformation group acting effectively on $M$ has no interior point of $M$.

Each step of the proof of Theorem B corresponds to each of a), b) and c), that is, §1, §2 and §3 correspond a), b) and c) respectively. In our case, however, since the transformation group in question is a vector group, we have not to take a connected component of the isotropy group at $x$ but the isotropy group itself works well in our purpose.

Theorem A follows quite naturally from Theorems B and C. This will be seen in §4.

1. As for an element $(L, \mathcal{F}) \in T(L, \mathcal{F}_0)$, we have the following lemmas whose proofs are seen in [3].

**Lemma 1.** If $(L, \mathcal{F}) \in T(L, \mathcal{F}_0)$ and $(L, \mathcal{F}) \neq (L, \mathcal{F}_0)$, then for any neighborhood $U$ of 0 in $(L, \mathcal{F})$ and for any positive number $r$,

$$U \cap \{ (x_1, \cdots, x_k) \in L; \sum x_i^2 > r \} \neq \emptyset .$$

**Lemma 2.** Assumptions being as above, for any $\varepsilon > 0$, there is a neighborhood $V$ of 0 in $(L, \mathcal{F})$ such that the diameter of any connected component of $V$ is smaller than $\varepsilon$, where the metric on $L$ is the natural euclidean metric.

Using these lemmas, we have the following lemma on $(L, \mathcal{F})$ which is irreducible.

**Lemma 3.** Let $i$ be an integer such that $1 \leq i \leq k = \dim L$. If $(L, \mathcal{F}) \in \mathcal{T}(L, \mathcal{F}_0)$ is irreducible, then for any $K > 0$ and for any neighborhood $U$ of 0 in $(L, \mathcal{F})$, there is $y = (y_1, \cdots, y_k) \in U$ such that $|y_i| > K$.

**Proof.** Assume there are a neighborhood $U$ of 0 in $(L, \mathcal{F})$, a positive $K$ and an integer $j$, $1 \leq j \leq k = \dim L$ such that $|y_j| \leq K$ for any point $(y_1, \cdots, y_k) \in U$. Without loss of generality, we assume that $j = 1$.

Let $\rho$ be the metric on $L$ defined by

$$\rho(x, y)^2 = \sum (x_i - y_i)^2 .$$

From the condition that $(L, \mathcal{F})$ satisfies the first countability axiom and from Lemma 2, there is a basis $\{ V_i \}$ of the neighborhoods of the identity 0 in $(L, \mathcal{F})$ satisfying a) $U \supset V_i$, b) $V_i \supset 2V_{i+1}$, c) $- V_i = V_i$, d) the diameter of each connected component of $V_i$ is less than $K$.

Let $E_q = \prod_{i=1}^k [-qK, qK]$ be a cube in $L$ containing 0 in the center of $E_q$ and let $F_q = L - E_q$. By Lemma 1, we see that $V_i \cap F_q \neq \emptyset$ for any $i$ and $q$, because $(L, \mathcal{F}) \neq (L, \mathcal{F}_0)$.

Let $V_i^{(q)}$ be the union of the connected components of $V_i$ which intersect $F_q$. From the condition d) of $\{ V_i \}$, we have $V_i^{(q)} \cap F_{q-i} = \emptyset$ for every $i$ and $q$. 
Considering the projection \( \text{Pr} \) from \( L \) onto \( R \) (real number field) defined by
\[
\text{Pr}(y_1, \cdots, y_n) = y_i,
\]
we see that \( \text{Pr}(V_i^{(q)}) \subset [-K, K] \) for all \( i \) and \( q \). Thus, there is \( \hat{y} \in [-K, K] \) such that \( \hat{y} \in \bigcap_{q} \text{Cl}(\text{Pr}(V_i^{(q)})) \), where \( \text{Cl}(A) \) is the closure of \( A \) in \([ -K, K ] \). This implies that for any \( \varepsilon, q \) and \( i \), there is \( y \in V_i^{(q)} \) satisfying
\[
|\text{Pr}(y) - \hat{y}| < \varepsilon.
\]
It follows that \( (y + V(\varepsilon)) \cap H(\hat{y}) \neq \emptyset \), where \( V(\varepsilon) \) is an \( \varepsilon \)-neighborhood of 0 under the metric \( \rho \) and \( H(\hat{y}) \) is the hyperplane defined by \( y_i = \hat{y} \). On the other hand, since the identity mapping from \((L, \mathcal{I}_0)\) onto \((L, \mathcal{I})\) is continuous and \( V(\varepsilon) \) is connected, we can choose sufficiently small \( \varepsilon_i \) such that \( V(\varepsilon_i) \) is contained in the connected component of \( V \) containing \( 0 \). Thus, we have
\[
y + V(\varepsilon_i) \subset V_i^{(q_i)} + V(\varepsilon_i) \subset V_i^{(q_i-1)}.
\]
It follows that \( V_i^{(q_i)} \cap H(\hat{y}) \neq \emptyset \) for all \( i \) and \( q \), because \( (y + V(\varepsilon_i)) \cap H(\hat{y}) \neq \emptyset \). Therefore, there are \( y, y' \in V_i \cap H(\hat{y}) \) such that \( \rho(y, y') \geq N \) for any positive number \( N \).

Let \( L' \) be the vector subspace defined by \( y_i = 0 \). As for \( y, y' \) above, we see that \( y - y' \in L' \), \( y - y' \in 2V_i \subset V_{i-1} \) and \( \rho(y - y', 0) \geq N \). This implies that
\[
L' \cap F_q \cap V \neq \emptyset
\]
for all \( i \) and \( q \). It follows that \( (L', \mathcal{I}) \neq (L', \mathcal{I}_0) \), contradicting the assumption that \( (L, \mathcal{I}) \) is irreducible.

Let \( \langle x, y \rangle \) be the ordinary inner product in \( L \) i.e. \( \langle x, y \rangle = \sum x_i y_i \). Then, as a consequence of Lemma 3, we have the following:

**Corollary.** Let \((L, \mathcal{I}) \subset T(L, \mathcal{I}_0)\) be irreducible. If \( x \in L \) satisfies
\[
|\langle x, y \rangle| < \hat{u} \quad \text{(bounded for any \( y \) of some neighborhood \( U \) of 0 in \( L, \mathcal{I} \))},\]
then \( x = 0 \).

Let \( S^1 = \{ e^{it} \} \) be the unit circle with the natural topology and let Hom \((\mathcal{L}, \mathcal{I}_0), S^1\) be the set of the continuous homomorphisms from \((L, \mathcal{I}_0)\) into \( S^1 \) with compact open topology. For any \((L, \mathcal{I}) \subset T(L, \mathcal{I}_0)\), a homomorphism \( \varphi \) from \((L, \mathcal{I})\) into \( S^1 \) can be considered as a homomorphism from \((L, \mathcal{I}_0)\) into \( S^1 \). By Hom \((\mathcal{L}, \mathcal{I}), S^1\) we mean the set of the continuous homomorphisms from \((L, \mathcal{I})\) into \( S^1 \) with relative topology in Hom \((\mathcal{L}, \mathcal{I}_0), S^1\).

It is well-known that \((L, \mathcal{I}_0)\) is isomorphic to Hom \((\mathcal{L}, \mathcal{I}_0), S^1\). The isomorphism \( \eta \) is given by \( \eta(x)(y) = e^{i(x,y)} \).

For a neighborhood \( U \) of 0 in \((L, \mathcal{I}), 0 < \varepsilon < \frac{\pi}{2}\) and \( \varepsilon \)-neighborhood \( V(\varepsilon) \) of
0 in $S^1$, we denote

$$\mathcal{S}(U, \varepsilon) = \{ \psi \in \text{Hom}((L, \mathcal{T}), S^1); \psi(U) \subset V(\varepsilon) \}.$$ 

**Proposition 1.** Notations being as above, if $(L, \mathcal{T})$ is irreducible, then $\mathcal{S}(U, \varepsilon)$ is totally disconnected.

**Proof.** Let $W_\psi$ be the connected component of $\mathcal{S}(U, \varepsilon)$ containing $\psi$. For every $\psi' \in W_\psi$, $\psi'(x) = e^{i\alpha \cdot \psi(x)}$. Thus, if $x \in U$, then $e^{i\alpha \cdot \psi(x)} \in V(\varepsilon)$. Since $\varepsilon < \frac{\pi}{2}$, there is an integer $m_\alpha(\psi')$ such that

$$|\langle x, \psi^{-1}\psi' \rangle - 2\pi m_\alpha(\psi')| < \varepsilon.$$

It follows that $m_\alpha(\psi')$ is constant on $W_\psi$ for every $x \in U$. Therefore

$$|\langle x, \psi^{-1}\psi' - \psi^{-1}\psi \rangle| < 2\varepsilon$$

for every $x \in U$. Since $(L, \mathcal{T})$ is irreducible, by Corollary to Lemma 3 we have $\psi = \psi'$.

As an application of the Proposition 1 to transformation groups, we consider $(L, \mathcal{T})$ operating continuously on a metric space $X$. The operation is denoted by $f$. Assume furthermore that there is continuous operation $\tilde{f}$ of $S^1$ on $X$ such that if $\tilde{f}(s, x) = x$ for a point $x \in X$, then $s = 0$, and that there is a mapping $\mathcal{V} : (L, \mathcal{T}) \times X \to S^1 \times X$ satisfying (1) $\tilde{f}_s \mathcal{V} = f_s \mathcal{V}$ (2) $\mathcal{V}(l, x) = (\mathcal{V}_s(l), x)$ and $\mathcal{V}_s$ is a homomorphism from $(L, \mathcal{T})$ onto $S^1$.

Since $f, \tilde{f}$ are continuous, so is $\mathcal{V}$. In fact, if $\lim (l_n, x_n) = (l_0, x_0)$, then by compactness of $S^1$, there is a subsequence $(l_{n'}, x_{n'})$ such that

$$\lim (\mathcal{V}_{x_{n'}}(l_{n'}), x_{n'}) = (s_0, x_0).$$

On the other hand,

$$\tilde{f}(s_0, x_0) = \lim \tilde{f}(\mathcal{V}_{x_{n'}}(l_{n'}), x_{n'}) = \lim f(l_{n'}, x_{n'}) = f(l_0, x_0).$$

Therefore $\mathcal{V}_{x_0}(l_0) = s_0$.

From the continuity of $\mathcal{V}$, we see that for an $\varepsilon$-neighborhood $V(\varepsilon)$ of 0 in $S^1$ there are a neighborhood $U$ of 0 in $(L, \mathcal{T})$ and an open set $Y$ of $X$ such that $\mathcal{V}(U, Y) \subset (V(\varepsilon), Y)$. This means that the mapping $x \to \mathcal{V}_x$ is continuous from $Y$ into $\mathcal{S}(U, \varepsilon)$. Thus, we have

**Corollary.** If $(L, \mathcal{T})$ is irreducible, then the mapping $x \to \mathcal{V}_x$ is constant on every connected component of $Y$. Moreover, if $X$ is locally connected and connected, then $x \to \mathcal{V}_x$ is constant on $X$. In other words, $(L, \mathcal{T})$ operates on $X$ as a circle group.

2. Now we consider $(L, \mathcal{T}) \in T(L, \mathcal{T}_0)$ acting effectively and continuously on a manifold $M$ as a transformation group. Assume that every orbit is homeo-
morphic to a circle or a point. Clearly, the subset of $M$ consisting of all the points $x$ such that the orbit of $x$ is homeomorphic to $S^1$ is $L$-invariant and is acted on by $(L, \mathcal{F})$ as a transformation group. Since the identity mapping from $(L, \mathcal{F}_0)$ onto $(L, \mathcal{F})$ is continuous, $(L, \mathcal{F}_0)$ acts naturally on $M$. Thus, we assume from the beginning, to simplify the argument below, that $(L, \mathcal{F}_0)$ acts effectively and continuously on $M$ itself as a transformation group and that every orbit is homeomorphic to $S^1$.

Denote by $L_x^0$ the connected component containing the identity of the isotropy subgroup $L_x$ of $L$ at $x$.

Clearly $L_x^0$ is continuous, that is, if $x_n \to x_0$, then

$$\lim L_x^0 = \{ \lim k_n; k_n \in L_x^0 \} = L_{x_0}^0.$$

Since $M$ is locally simply connected, there is a connected open set $M'$ on which the unit vector $n(x)$ orthogonal to $L_x^0$ can be chosen in such a way that it is continuous with respect to the variable $x$. Since $L_x^0$ is constant on every orbit, we may assume that $M'$ is an $L$-invariant open connected subset of $M$.

**Lemma 4.** Let $\lambda(x) = \min \{ \lambda > 0; \lambda n(x) \in L_x \}$. Then $\lambda(x)$ is lower semi-continuous, the points of continuity are open and dense in $M'$, and $\lambda(x)$ is $L$-invariant.

**Proof.** It is easy to see that $\lambda(x)$ is $L$-invariant and lower semi-continuous.

Let $x$ be a point of continuity. Then there is an open neighborhood $V$, such that $|\lambda(x) - \lambda(y)| < \varepsilon$ for any $y \in V$. If there is a sequence $\{y_n\}$ in $V$ converging to $y$ in $V$, and $\lim \lambda(y_n) \neq \lambda(y)$, then we have

$$\lim \lambda(y_n) \geq 2\lambda(y).$$

Thus, for sufficiently large $n$, we have

$$\lambda(y_n) - \lambda(y) \geq \varepsilon \geq \lambda(x) - 2\varepsilon.$$

On the other hand, $|\lambda(y_n) - \lambda(y)| < 2\varepsilon$. It follows that if $\varepsilon < \frac{1}{4} \lambda(x)$, then $\lim \lambda(y_n) = \lambda(y)$. Thus, the point of continuity of $\lambda(x)$ are open.

This argument shows that if $\lambda(x)$ is bounded on an open set $U$ and $\lambda_0 = \sup \{ \lambda(x); x \in U \}$, then for a sufficiently small $\varepsilon > 0$, a point $x \in U$ satisfying $\lambda(x) \geq \lambda_0 - \varepsilon$ is a point of continuity. Since every open set in $M'$ is a set of second category, a category argument gives that the points of continuity is dense in $M'$.

Let $M''$ be an open $L$-invariant subset of $M'$ on which $\lambda(x)$ is continuous. Let $\psi$ be a mapping from $(L, \mathcal{F}_0) \times M''$ into $S^1 \times M''$ defined by

$$\psi(l, x) = (e^{i\lambda_l(x)} - i\lambda(x), l, x),$$

and $f$ be a mapping from $S^1 \times M''$ into $M''$ defined by
where \( f \) is the continuous operation of \((L, \mathcal{F})\) on \( M \). It is easy to see that \( \hat{f} \) is a continuous operation of \( S' \) on \( M'' \) with \( f - \hat{f} \) and if \( \hat{f}(s, x) = x \) for some \( x \in M'' \), then \( s = 0 \).

Now, let \((L, \mathcal{F})\) be irreducible and act on \( M \) as a transformation group. Assume that every orbit is homeomorphic to \( S' \). By the same argument as above, there is an \( L \)-invariant open subset \( M'' \) on which \( \lambda(x) \) is continuous and the continuous mappings \( \hat{f} \) and \( \mathcal{F} \) above are defined. Thus, we have from Corollary to Proposition 1 the following

**Lemma 5.** \((L, \mathcal{F})\) operates as a circle group on every connected component of \( M' \). More precisely, let \( A \) be a connected \( L \)-invariant subset of \( M' \) on which \( \lambda(x) \) is continuous. Then \((L, \mathcal{F})\) operates on \( A \) as a circle group.

3. In this section, it will be proved that \( M'' = M' \). The fundamental fact used in proving this is that if a compact Lie group \( G \) acts effectively and continuously on a connected manifold and if the fixed point set of \( G \) contains an interior point, then \( G = \{e\} \).

Let \( K \) be the collection of points \( x \in M' \) such that on every neighborhood of \( x \), \( \lambda(x) \) has an infinite least upper bound. Then \( K \) is a closed and \( L \)-invariant subset of \( M' \) and is nowhere dense.

**Lemma 6.** On every connected component \( R \) of \( M' - K \), the function \( \lambda(x)n(x) \) is constant, that is, \((L, \mathcal{F})\) operates on \( R \) as a circle group.

**Proof.** Clearly \( R \) is an \( L \)-invariant open subset. Since the points of continuity is dense and open, there is a connected open set \( H \) in \( R \) such that \( \lambda(x)n(x) \) is constant on \( H \). If \( H \) is not all of \( R \), let \( b \) be a point of \( R \) on the boundary of \( H \). There are an open neighborhood \( U \) of \( b \) in \( R \) and a positive number \( m \) such that \( \sup(\lambda(x); x \in U) = m \), because \( \lambda(x) \) has a finite least upper bound at \( b \). There is a point \( y \in U \) such that \( \lambda(y) \geq m - \varepsilon \) and we see easily that for sufficiently small \( \varepsilon \), such \( y \) is a point of continuity. It follows that there exists an open connected subset \( V \) of \( H \cup U \) on which \( \lambda(x)n(x) \) is constant. Since \( \lambda(x)n(x) \) is \( L \)-invariant, the set \( V \) can be assumed to be \( L \)-invariant.

Assume furthermore that \( V \) is a maximal open connected subset on which \( \lambda(x)n(x) \) is constant and equal to \( \lambda(b')n(b') \) for a point \( b' \in V \). From Lemma 4, we have that \( \lambda(x) < \lambda(b') \) on the boundary point of \( V \) in \( H \cup U \). It follows that \( \lambda(x) = \frac{1}{k} \lambda(b') \) for some integer \( k = k(x) \geq 2 \). Since the boundary \( B \) of \( V \) is closed in \( H \cup U \) and then a set of second category, we see by a category argument
that there is an $L$-invariant open subset $W$ in $H \cup U$ such that $\lambda(x)$ is constant and equal to $\frac{1}{k} \lambda(b')$ on $W \cap B$ for some integer $k \geq 2$. Then, an operation of $Z_l := \{e^{2\pi i k} \}$ on $V \cup W$ is defined as follows:

$$g(e^{2\pi i k}, x) = \begin{cases} f(\frac{1}{k} \lambda(b') n(b'), x) & \text{if } x \in V \\ x & \text{if } x \in V \cup W \setminus V, \end{cases}$$

where $f$ is the operation of $(L, \mathcal{T})$ on $M$ and $l$ is an integer. It is easy to see that $g : Z_l \times (V \cup W) \to V \cup W$ is continuous.

Since $V \cup W$ is a connected manifold and $V \cup W \setminus V$ has an interior point, $Z_l$ operates trivially on $V \cup W$. This contradicts the definition of $\lambda(x)$. Thus, we have $V \setminus R = H$.

**Proposition 2.** Let $(L, \mathcal{T}) \in T(L_{\mathcal{T}^0})$ be irreducible and act on a connected manifold $M$. If every orbit is homeomorphic to $S^1$, then $(L, \mathcal{T})$ acts on $M$ as a circle group.

**Proof.** Notations being as above, it is easy to see that for every point $x \in M$ there is an open, connected and $L$-invariant subset $M'$ containing $x$ on which $n(y)$ is continuous. We have only to show that $\lambda(x) n(x)$ is constant on $M'$. If $\lambda(x)$ is bounded on $M'$, then by Lemma 6 we have $\lambda(x) n(x)$ is constant. The proposition will now be proved by the method of contradiction. Assume that $\lambda(x)$ is unbounded on $M'$. On the basis of this assumption the lemma above shows that $M' \setminus K$ is not connected and therefore that the closed set $K$ is not vacuous.

Let $\lambda(x|K)$ denote $\lambda(x)$ restricted to $K$; $\lambda(x|K)$ is lower semi-continuous on $K$. Since $K$ is a set of second category, we have by the same argument as in Lemma 4 that the set of continuity of $\lambda(x|K)$ is open and dense. Thus, there is an $L$-invariant connected open subset $U$ in $M'$ such that $\lambda(x|K)$ is continuous on $U \cap K \setminus \emptyset$. Let $R$ be any connected component of $M' \setminus K$ and $\lambda(x) = \lambda_0$ on $R$. The boundary $B_R$ of $R$ is contained in $K$ and $\lambda(x|K) = \frac{1}{k} \lambda_0$ on $B_R$ for some integer $k = k(x)$. Since $\lambda(x|K)$ is continuous on $B_R \cap U$, we see that $\lambda(x|K)$ is constant and equal to $\lambda_0$ on $B_R \cap U$ by the same reason as in Lemma 6 because the set of the points $x$ where $\lambda(x|K) = \frac{1}{k} \lambda_0$ is open in $B_R \cap U$ for every fixed $k$.

It will be shown below that $\lambda(x)$ is continuous on $U$. Let $(x_n)$ be a sequence converging to a point $x_0$ in $U$. If $x_0 \notin K$, then $\lim \lambda(x_n) = \lambda(x_0)$ because any connected component of $M' \setminus K$ is an open subset. Assume $x_0 \in K \cap U$. There is an arc $C : [0, 1] \to U$ such that $C(t_0) = x_0$, $C(1) = x_0$ and $\lim_{t \to 1} C(t) = x_0$. For every $t_n$ there is $t_n'$ such that $t_n \leq t_n'$, $C(t_n') \in K \cap U$ and $C([t_n, t_n'])$ is contained in the
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closure of a connected component of \( M - K \). Since \( \tilde{\lambda}(x|K) \) is continuous on \( U \cap K \), we see \( \text{lim} \lambda(C(t_u)) = \lambda(x_0) \). From this and the fact that \( \lambda(x_0) = \lambda(C(t_u)) \), we have \( \text{lim} \lambda(x_u) = \lambda(x_0) \). Thus, \( \lambda(x) \) is continuous on \( U \) and then constant on \( U \). This contradicts the definition of the set \( K \). It follows that \( K \) is vacuous. Then \( \lambda(x)\mathbb{I}(x) \) is constant on \( M' \) and then on \( M \). This means that \( (L, \mathcal{F}) \) acts as a circle group on \( M \).

**Proof of Theorem B.**

Let \( K \) be the set of points such that \( \varphi(L)(x) \) is a point. Clearly \( K \) is a closed subset. By Proposition 2, we see that \( \lambda(x) \) is constant on a connected component \( M' \) of \( M - K \). That is, \( (L, \mathcal{F}) \) acts as a circle group on \( M' \). Define an operation \( f' \) of \( (L, \mathcal{F}) \) as follows:

\[
f'(l, x) = \begin{cases} f(l, x) & \text{if } x \in M' \\ x & \text{if } x \in M - M', \end{cases}
\]

where \( f \) is the operation of \( (L, \mathcal{F}) \) on \( M \). Clearly \( f' \) is an operation of \( (L, \mathcal{F}) \) on \( M \) as a circle group. Therefore \( M - M' \) contains no interior point. This means \( f = f' \), completing the proof.

4. **Proof of Theorem A.**

Let \( \varphi \) be a homomorphism from \( (L, \mathcal{T}_0) \) into \( H(M) \) and \( K \) be the kernel of \( \varphi \). The factor group \( (L, \mathcal{T}_0)/K \) is isomorphic to \( (L', \mathcal{T}_0) \times T \) where \( T \) is a toroidal group. Naturally, there is a monomorphism \( \tilde{\varphi} \) from \( (L', \mathcal{T}_0) \times T \) into \( H(M) \) such that \( \tilde{\varphi} \circ \pi = \varphi \) where \( \pi \) is the natural projection from \( (L, \mathcal{T}_0) \) onto \( (L', \mathcal{T}_0) \times T \). Assume furthermore that every orbit \( \varphi(L)(x) \) is homeomorphic to a circle or a point. Then, we see that every orbit \( \tilde{\varphi}(T)(x) \) is homeomorphic to a circle or a point. Thus, from Theorem C we have \( T = S^1 \).

We have only to show that \( \tilde{\varphi}(L') \) is closed in \( H(M) \). Assume that \( \tilde{\varphi}(L') \) is not closed in \( H(M) \). Then the relative topology for \( \tilde{\varphi}(L') \) introduces a topology \( \mathcal{F} \) for \( L' \) such that \( (L', \mathcal{F}) \in T(L', \mathcal{T}_0) \) and \( (L', \mathcal{F}) \cong (L', \mathcal{T}_0) \). It follows that there is a vector subspace \( L'' \) of \( L' \) such that \( (L'', \mathcal{F}) \) is irreducible. From the irreducibility, we see that every orbit \( \tilde{\varphi}(L'') \) is homeomorphic to a circle or a point. It follows by Theorem B that \( \tilde{\varphi}(L'') = S^1 \), contradicting the fact that \( \tilde{\varphi} \) is isomorphic. Thus, we see that \( \varphi(L) \) is closed and isomorphic to \( (L', \mathcal{T}_0) \times S^1 \).

Tokyo Metropolitan University
References


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