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Uniform Estimates for Distributions of the Sum of i.i.d. Random Variables with Fat Tail in the Threshold Case

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Abstract. We show uniform estimates for distributions of the sum of i.i.d. random variables in the threshold case. Rozovskii showed several uniform estimates but the speed of convergence was not known. Our main uniform estimate implies a speed of convergence. We also compare our estimates with Nagaev's estimate which is valid in the non-threshold case and, moreover, give a necessary and sufficient condition for Nagaev's estimate to hold in the threshold case.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X_n, n = 1, 2, \ldots,$ be independent identically distributed random variables whose probability laws are μ. Let $F : \mathbb{R} \to [0,1]$ and $\bar{F} : \mathbb{R} \to [0,1]$ be given by $F(x) = \mu((-\infty, x])$ and $\bar{F}(x) = \mu((x,\infty))$, $x \in \mathbb{R}$. We assume the following.
(A1) $\bar{F}(x)$ is a regularly varying function of index $-\alpha$ for some $\alpha \geq 2$, as $x \to \infty$,i.e., if we let

$$L(x) = x^{\alpha}\bar{F}(x)\ ,\ x \geq 1,$$

then $L(x) > 0$ for any $x \geq 1$, and for any $a > 0$

$$\frac{L(ax)}{L(x)} \to 1\ ,\ x \to \infty.$$

(A2) $\int_{-\infty}^{0} |x|^{\alpha+\delta_0}\mu(dx) < \infty$ for some $\delta_0 \in (0,1)$, $\int_{\mathbb{R}} x^2\mu(dx) = 1$ and $\int_{\mathbb{R}} x\mu(dx) = 0$.

S.V. Nagaev [5] proved the following theorem.

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THEOREM 1 (Nagaev). *Assume* (A1) *for* $\alpha > 2$ *and* (A2). *Then we have*

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^{n} X_k > n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1| \to 0, \qquad n \to \infty. \tag{1}$$

Here $\Phi_0 : \mathbb{R} \to \mathbb{R}$ is given by

$$\Phi_0(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-\frac{y^2}{2})dy, \qquad x \in \mathbb{R}.$$

In this paper, we assume (A1) for $\alpha = 2$ (threshold case), (A2) and the following.
(A3) The probability law μ is absolutely continuous and has a density function $\rho : \mathbb{R} \to [0, \infty)$ which is right continuous and has a finite total variation.

We show two uniform estimates. Our main estimate gives the speed of convergence. The other one is similar to (1).

Let us define $\Phi_k : \mathbb{R} \to \mathbb{R}, k = 1, 2, 3$ by

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}) = -\frac{d}{dx}\Phi_0(x),$$

and

$$\Phi_k(x) = (-1)^{k-1} \frac{d^{k-1}}{dx^{k-1}}\Phi_1(x), \qquad k = 2, 3.$$

Let $v_n = \int_{-\infty}^{n^{1/2}} x^2\mu(dx)$ for $n \geq 1$.

Our main result is the following.

THEOREM 2. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a constant* $C > 0$ *such that*

$$\sup_{s\in[1,\infty)} |\frac{P(\sum_{k=1}^{n} X_k > n^{1/2}s)}{H(n, v_n^{-1/2}s)} - 1| \leq CL(n^{1/2})^{1-\delta}. \tag{2}$$

Here

$$\begin{aligned} H(n, s) \;=\; & \Phi_0(s) + n \int_{-\infty}^{s} \bar{F}((s - x)v_n^{1/2}n^{1/2})\Phi_1(x)dx \\ & - \left(v_n^{-1/2}n^{1/2}\Phi_1(s) \int_0^\infty x\mu(dx) + v_n^{-1}\frac{\Phi_2(s)}{2} \int_0^{n^{1/2}} x^2\mu(dx) \right). \end{aligned}$$

We also show a similar uniform estimate to (1) and a necessary and sufficient condition for (1) to hold under the three assumptions.

THEOREM 3. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then we have*

$$(3) \qquad \sup_{s\in[1,\infty)} \left|\frac{P(\sum_{k=1}^n X_k > n^{1/2}s)}{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty.$$

Rozovskii [6] showed different types of uniform estimate. (see Theorems 1, 2 and 3b in [6].) The estimates in Theorems 1 and 2 in [6] were proved under more general assumptions but they were complex and the speed of convergence was not proved. The estimate in Theorem 3b in [6] is strongly related to (3) but does not necessarily imply our result. The proof of uniform estimates in [6] is different from ours.

We also prove the following.

THEOREM 4. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *If* $\limsup_{n\to\infty}(1 - v_n)\log\dfrac{1}{L(n^{1/2})} = 0$, *then we have*

$$(4) \qquad \sup_{s\in[1,\infty)} \left|\frac{\Phi_0(v_n^{-1/2}s) + n\bar{F}(n^{1/2}s)}{\Phi_0(s) + n\bar{F}(n^{1/2}s)} - 1\right| \to 0, \qquad n \to \infty.$$

If $\limsup_{n\to\infty}(1 - v_n)\log\dfrac{1}{L(n^{1/2})} > 0$, *then* (4) *does not hold.*

Combining Theorems 2 and 3 gives a necessary and sufficient condition for (1) to hold, i.e. if we assume (A1) for $\alpha = 2$, (A2), (A3) and $\limsup_{n\to\infty}(1 - v_n)\log\dfrac{1}{L(n^{1/2})} = 0$, then (3) holds, namely

$$P(\sum_{k=1}^n X_k > s) \sim \Phi_0(n^{-1/2}s) + n\bar{F}(s), \qquad for\ s > n^{1/2}.$$

The condition $\limsup_{n\to\infty}(1 - v_n)\log\dfrac{1}{L(n^{1/2})} = 0$ corresponds to (56) in [6]. Hence the estimate with $B_n = n^{1/2}$ in Theorem 3b in [6] is not valid under our assumptions.

We also prove the following to obtain Theorem 2.

THEOREM 5. *Assume* (A1) *for* $\alpha = 2$, (A2) *and* (A3). *Then for any* $\delta \in (0, 1)$, *there is a constant* $C > 0$ *such that*

$$|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2} s)| \leq CL(n^{1/2})^{2-\delta}, \qquad s \geq 1.$$

Throughout this paper we assume (A1) for $\alpha = 2$, (A2) and (A3). Then we see that $L(t) \to 0$, $t \to \infty$ and $\dfrac{1 - v_n}{L(n^{1/2})} \to \infty$, $n \to \infty$ (see (5) and (6)).

2. Preliminary Facts

We summarize several known facts (c.f. Fushiya-Kusuoka[2]).

PROPOSITION 1. *We have*

$$\sup_{1/2 \leq a \leq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty,$$

and

$$\inf_{1/2 \leq a \leq 2} \frac{L(ax)}{L(x)} \to 1, \qquad x \to \infty.$$

PROPOSITION 2. *For any* $\varepsilon \in (0, 1)$, *there is an* $M(\varepsilon) \geq 1$ *such that*

$$M(\varepsilon)^{-1} y^{-\varepsilon} \leq \frac{L(yx)}{L(x)} \leq M(\varepsilon) y^{\varepsilon} \qquad x, y \geq 1.$$

PROPOSITION 3. (1) *For any* $\beta < -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_t^{\infty} x^{\beta} L(x) dx \to -\frac{1}{\beta + 1}, \qquad t \to \infty.$$

(2) *For any* $\beta > -1$,

$$\frac{1}{t^{\beta+1} L(t)} \int_1^{t} x^{\beta} L(x) dx \to \frac{1}{\beta + 1}, \qquad t \to \infty.$$

(3) *Let* $f : [1,\infty) \to (0,\infty)$ *be given by*

$$f(t) = \int_1^t x^{-1} L(x)dx \qquad t \geq 1.$$

Then f *is slowly varying. Moreover if* $\lim_{t\to\infty} f(t) < \infty$, *we have*

$$\frac{1}{L(t)} \int_t^\infty x^{-1} L(x)dx \to \infty, \qquad t \to \infty.$$

Proposition 4. *There is a constant* $C_0 > 0$ *such that*

$$|\Phi_k(x)| \leq C_0(1+x)^{k-1}\Phi_1(x), \qquad x \geq 0, k = 1, 2$$

and

$$C_0^{-1}\Phi_1(x) \leq x\Phi_0(x) \leq C_0\Phi_1(x), \qquad x \geq 1/2.$$

Proposition 5. (1) *For any* $m \geq 1$, *let* $r_{e,m} : \mathbb{R} \to \mathbb{C}$ *be given by*

$$r_{e,m}(t) = \exp(it) - (1 + \sum_{k=1}^m \frac{(it)^k}{k!}), \qquad t \in \mathbb{R}.$$

Then we have

$$|r_{e,m}(t)| \leq \frac{\min(|t|^{m+1}, 2(m+1)|t|^m)}{(m+1)!}, \qquad t \in \mathbb{R}.$$

(2) *For any* $m \geq 1$, *let* $r_{l,m} : \{z \in \mathbb{C};\ |z| \leq 1/2\} \to \mathbb{C}$ *be given by*

$$r_{l,m}(z) = \log(1+z) - \sum_{k=1}^m \frac{(-1)^{k-1}}{k} z^k, \qquad z \in \mathbb{C}, |z| \leq 1/2.$$

Then we have

$$|r_{l,m}(z)| \leq 2|z|^{m+1}, \qquad z \in \mathbb{C}, |z| \leq 1/2.$$

Let $\mu(t), \nu(t)$, $t > 0$, be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\begin{aligned} \mu(t)(A) &= (1 - \bar{F}(t))^{-1}\mu(A \cap (-\infty, t]), \\ \nu(t)(A) &= \bar{F}(t)^{-1}\mu(A \cap (t, \infty]), \end{aligned}$$

for any $A \in \mathcal{B}(\mathbb{R})$. Let $\varphi(\cdot; \mu(t))$ (resp. $\varphi(\cdot; \nu(t))$), $t > 0$, be the characteristic function of the probability measure $\mu(t)$ (resp. $\nu(t)$),i.e.,

$$\varphi(\xi; \mu(t)) = \int_{\mathbb{R}} \exp(ix\xi)\mu(t)(dx), \qquad \xi \in \mathbb{R}.$$

Proposition 6. *There is a constant $c_0 > 0$ such that for any $t \geq 2$, $\xi \in \mathbb{R}$ and positive integers n, m with $n \geq m$,*

$$|\varphi(n^{-1/2}\xi; \mu(t))|^n \leq (1 + \frac{c_0}{m}|\xi|^2)^{-m/4}.$$

Proposition 7. *Let ν be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\int_{\mathbb{R}} x^2\nu(dx) < \infty$. Also, assume that there is a constant $C > 0$ such that the characteristic function $\varphi(\cdot; \nu) : \mathbb{R} \to \mathbb{C}$ satisfies*

$$|\varphi(\xi; \nu)| \leq C(1 + |\xi|)^{-2}, \qquad \xi \in \mathbb{R}.$$

Then for any $x \in \mathbb{R}$ and $v > 0$

$$\nu((x, \infty)) = \Phi_0(v^{-1/2}x) + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-ix\xi}}{i\xi}(\varphi(\xi, \nu) - \exp(-\frac{v\xi^2}{2}))d\xi.$$

3. Estimate for Moments and Characteristic Functions

Let

$$\eta_k(t) = \int_{-\infty}^{t} x^k\mu(dx), \qquad t > 0, \ k = 1, 2,$$

and

$$\eta_3(t) = \int_{1}^{t} x^3\mu(dx), \qquad t > 1.$$

Then we see that

$$-\eta_1(t) = \int_t^\infty x\mu(dx) = \int_t^\infty \bar{F}(x)dx + t\bar{F}(t), \qquad t > 0,$$

$$1 - \eta_2(t) = \int_t^\infty x^2\mu(dx) = 2\int_t^\infty x\bar{F}(x)dx + t^2\bar{F}(t), \qquad t > 0,$$

and

$$\eta_3(t) = \bar{F}(1) - t^3\bar{F}(t) + 3\int_1^t x^2\bar{F}(x)dx \qquad t > 1.$$

In particular, we see that

$$L(t) \le 1 - \eta_2(t) \to 0, \quad t \to \infty, \tag{5}$$

$$\frac{1 - \eta_2(t)}{L(t)} \to \infty, \quad t \to \infty. \tag{6}$$

For any $\delta > 0$, let $t_n = n^{1/2}L(n^{1/2})^\delta$. Note that $n^{-1/2}t_n \to 0, \quad n \to \infty$.

PROPOSITION 8. *For any $\varepsilon > 0$, there is a constant $C > 0$ such that*

$$\frac{L(t_n)}{L(n^{1/2})} \le CL(n^{1/2})^{-\varepsilon\delta} \tag{7}$$

$$n\bar{F}(t_n) \le CL(n^{1/2})^{1-2\delta-\varepsilon\delta} \tag{8}$$

$$\eta_2(n^{1/2}) - \eta_2(t_n) \le CL(n^{1/2})^{1-2\varepsilon\delta} \tag{9}$$

$$-n^{1/2}\eta_1(t_n) \le CL(n^{1/2})^{1-2\delta} \tag{10}$$

$$n^{-1/2}\eta_3(t_n) \le CL(n^{1/2}) \tag{11}$$

for any $n \ge 1$.

PROOF. From Proposition 2, there is an $M(\varepsilon) > 0$ such that

$$\frac{L(t_n)}{L(n^{1/2})} = \frac{L(t_n)}{L(t_nL(n^{1/2})^{-\delta})} \le M(\varepsilon)L(n^{1/2})^{-\varepsilon\delta}.$$

Hence we have (7). Similarly, we see that

$$n\bar{F}(t_n) = L(n^{1/2})^{-2\delta}L(t_n) = L(n^{1/2})^{1-2\delta}\frac{L(t_n)}{L(n^{1/2})}$$

and

$$\begin{aligned}\eta_2(n^{1/2}) - \eta_2(t_n) &= L(t_n) - L(n^{1/2}) + 2\int_{t_n}^{n^{1/2}} \frac{L(z)}{z}dz \\ &= L(t_n) - L(n^{1/2}) + 2L(t_n)\int_1^{L(t_n)^{-\delta}} \frac{L(t_n y)}{L(t_n)y}dy \\ &\le L(t_n) - L(n^{1/2}) + 2L(t_n)M(\varepsilon)\int_1^{L(t_n)^{-\delta}} y^{-1+\varepsilon}dy \\ &\le L(t_n) - L(n^{1/2}) + 2\frac{M(\varepsilon)}{\varepsilon}L(t_n)(L(n^{1/2})^{-\varepsilon\delta} - 1).\end{aligned}$$

Therefore by (7), we have (8) and (9).

Let

$$\varepsilon_1(t) = \frac{1}{t^{-1}L(t)}\int_t^{\infty} x^{-2}L(x)dx - 1$$

and

$$\varepsilon_3(t) = \frac{1}{tL(t)}\int_1^t L(x)dx - 1.$$

Then from Proposition 3 (1) and (2) we have $\varepsilon_1(t) \to 0$ and $\varepsilon_3(t) \to 0$ as $t \to \infty$.

Hence we see that

$$\begin{aligned}-n^{1/2}\eta_1(t_n) &= n^{1/2}\left(t_n\bar{F}(t_n) + \int_{t_n}^{\infty} \bar{F}(x)dx\right) \\ &= L(n^{1/2})^{-\delta}L(t_n)(2 + \varepsilon_1(t_n)) \\ &= (2 + \varepsilon_1(t_n))L(n^{1/2})^{1-\delta}\frac{L(t_n)}{L(n^{1/2})}\end{aligned}$$

and

$$\begin{aligned}n^{-1/2}\eta_3(t_n) &= n^{-1/2}\bar{F}(1) + (2 + \varepsilon_3(t_n))L(n^{1/2})^{\delta}L(t_n) \\ &= n^{-1/2}\bar{F}(1) + (2 + \varepsilon_3(t_n))L(n^{1/2})^{1+\delta}\frac{L(t_n)}{L(n^{1/2})}.\end{aligned}$$

From (7), we have (10) and (11). □

4. Asymptotic Expansion of Characteristic Functions

Remind that $v_n = \int_{-\infty}^{n^{1/2}} x^2\mu(dx)$ and $t_n = n^{1/2}L(n^{1/2})^{\delta}$. In this section, we prove the following lemma.

LEMMA 1. *Let*

$$\begin{aligned} R_{n,0}(\xi) &= \exp(\frac{v_n}{2}\xi^2)(1-\bar{F}(t_n))^n\varphi(n^{-1/2}\xi;\mu(t_n))^n \\ &\quad -(1+n((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1)+\frac{v_n}{2}\xi^2), \\ R_{n,1}(\xi) &= \exp(\frac{v_n}{2}\xi^2)(1-\bar{F}(t_n))^n\varphi(n^{-1/2}\xi;\mu(t_n))^n-1, \\ R_{n,2}(\xi) &= \exp(\frac{v_n}{2}\xi^2)(1-\bar{F}(t_n))^{n-1}\varphi(n^{-1/2}\xi;\mu(t_n))^{n-1}-1. \end{aligned}$$

Then there is a constant $C>0$ such that

$$|R_{n,0}(\xi)| \le CL(n^{1/2})^{2-5\delta}|\xi| \tag{12}$$

and

$$|R_{n,1}(\xi)|+|R_{n,2}(\xi)| \le CL(n^{1/2})^{1-2\delta}|\xi| \tag{13}$$

for any $n\ge 8$ and $\xi\in\mathbb{R}$ with $|\xi|\le L(n^{1/2})^{-\delta}$.

As a corollary to Lemma 1, we have the following.

COROLLARY 1. *Let*

$$\begin{aligned} \tilde{R}_0(n,s) &= (1-\bar{F}(t_n))^n\mu(t_n)^{*n}((sn^{1/2},\infty))-\Phi_0(v_n^{-1/2}s) \\ &\quad -\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{-is\xi}}{i\xi}\left(n((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1)+\frac{v_n\xi^2}{2}\right) \\ &\quad \times e^{-v_n\xi^2/2}d\xi \end{aligned}$$

and

$$\begin{aligned} \tilde{R}_{1,k}(n,s) &= (1-\bar{F}(t_n))^{n-k}\mu(t_n)^{*(n-k)}((sn^{1/2},\infty)) \\ &\quad -\Phi_0(v_n^{-1/2}s), \qquad k=0,1. \end{aligned}$$

Then there is a constant $C > 0$ such that for any $n \geq 1$, we have for $s \in \mathbb{R}$

$$|\tilde{R}_0(n,s)| \leq CL(n^{1/2})^{2-6\delta} \tag{14}$$

and

$$|\tilde{R}_{1,0}(n,s)| + |\tilde{R}_{1,1}(n,s)| \leq CL(n^{1/2})^{1-4\delta}. \tag{15}$$

PROOF. From Proposition 8, we see that

$$\begin{aligned}
&\tilde{R}_0(n,s) \\
= \ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\Big((1-\bar{F}(t_n))^n \varphi(n^{-1/2}\xi;\mu(t_n))^n - e^{-\frac{v_n\xi^2}{2}} \\
&-\big(n\big((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n)) - 1\big) + \frac{v_n\xi^2}{2}\big)e^{-\frac{v_n\xi^2}{2}}\Big)d\xi \\
= \ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} R_{n,0}(\xi) e^{-v_n\xi^2/2} d\xi.
\end{aligned}$$

By Lemma 1, there is a constant $C_0 > 0$ such that

$$\int_{|\xi| \leq L(n^{1/2})^{-\delta}} \frac{|R_{n,0}(\xi)|}{|\xi|} d\xi \leq C_0 L(n^{1/2})^{2-6\delta}.$$

It is easy to see from Proposition 5 (1) that

$$n|(1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n)) - 1| \leq n^{1/2}|\eta_1(t_n)||\xi| + \frac{|\xi|^2}{2}, \qquad \xi \in \mathbb{R}.$$

From the above inequality and Proposition 7, we see that for any $m \geq 2/\delta$, there is a constant $C_1 > 0$ such that for any $n \geq 2m$ and $\xi \in \mathbb{R}$ with $|\xi| \geq L(n^{1/2})^{-\delta}$,

$$\begin{aligned}
&|\varphi(n^{-1/2}\xi;\mu(t_n))|^n \\
&+ \left|n((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n)) - 1) + 1 + \frac{v_n\xi^2}{2}\right| e^{-\frac{v_n\xi^2}{2}} \leq C_1|\xi|^{-m}.
\end{aligned}$$

Hence we have

$$\int_{|\xi| > L(n^{1/2})^{-\delta}} |\xi|^{-1}\Big|(1-\bar{F}(t_n))^n\varphi(n^{-1/2}\xi;\mu(t_n))^n - e^{-\frac{v_n\xi^2}{2}}$$

$$- \left(n((1 - \bar{F}(t_n))\varphi(n^{-1/2}\xi; \mu(t_n)) - 1) + \frac{v_n \xi^2}{2} \right)$$
$$\times e^{-\frac{v_n \xi^2}{2}} \Big| d\xi$$
$$\leq \; 2C_1 \int_{L(n^{1/2})^{-\delta}}^{\infty} |\xi|^{-m-1} d\xi = \frac{2C_1}{m} L(n^{1/2})^{m\delta} \leq \frac{2C_1}{m} L(n^{1/2})^2.$$

Therefore we have (14). We see also that

$$\begin{aligned}
\tilde{R}_{1,k}(n, s) &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} \\
&\quad \times \left((1 - \bar{F}(t_n))^{n-k} \varphi(n^{-1/2}\xi; \mu(t_n))^{n-k} - e^{-\frac{v_n \xi^2}{2}} \right) d\xi \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi} R_{n,1+k}(\xi) e^{-v_n \xi^2/2} d\xi.
\end{aligned}$$

Similarly to the first equation, we have (15). □

We make some preparations to prove Lemma 1. Let

$$R_0(n, \xi) = (1 - \bar{F}(t_n))\varphi(n^{-1/2}\xi, \mu(t_n)) - (1 - v_n \frac{\xi^2}{2n}).$$

First we prove the following.

Proposition 9. *There is a constant $C > 0$ such that for any $n \geq 8$, and $\xi \in \mathbb{R}$ with $|\xi| \leq L(n^{1/2})^{-\delta}$,*

$$|nR_0(n, \xi)| \leq CL(n^{1/2})^{1-2\delta}|\xi|$$

and

$$n|(1 - \bar{F}(t_n))\varphi(n^{-1/2}\xi; \mu(t_n)) - 1| \;\; \leq \;\; CL(n^{1/2})^{-\delta}|\xi|.$$

In particular,

$$\sup\{|nR_0(n, \xi)|; |\xi| \leq L(n^{1/2})^{-\delta}\} \to 0, \qquad n \to \infty. \tag{16}$$

Proof. We can easily see that

$$\begin{aligned}&\varphi(\xi;\mu(t)) = \int_{\mathbb{R}} \exp(ix\xi)\mu(t)(dx)\\ =\ & 1+\eta_1(t)(i\xi)+\eta_2(t)\frac{(i\xi)^2}{2}+\int_{-\infty}^{1} r_{e,2}(\xi x)\mu(dx)\\ &+\int_1^t r_{e,2}(\xi x)\mu(dx)+\frac{\bar{F}(t)}{1-\bar{F}(t)}\int_{-\infty}^{t} r_{e,0}(\xi x)\mu(dx).\end{aligned}$$

Hence we have that

$$\begin{aligned}(1-\bar{F}(t_n))^{-1}R_0(n,\xi) \;=\;& n^{-1/2}\eta_1(t_n)(i\xi)+(\eta_2(t_n)-\eta_2(n^{1/2}))\frac{(i\xi)^2}{2n}\\ &+\int_{-\infty}^{1} r_{e,2}(n^{-1/2}\xi x)\mu(dx)\\ &+\int_1^{t_n} r_{e,2}(n^{-1/2}\xi x)\mu(dx)\\ &+\frac{\bar{F}(t_n)}{1-\bar{F}(t_n)}\int_{-\infty}^{t} r_{e,0}(n^{-1/2}\xi x)\mu(dx).\end{aligned}$$

Then we see that

$$\begin{aligned}n|R_0(n,\xi)| \;\le\;& n^{1/2}|\eta_1(t_n)||\xi|+(\eta_2(n^{1/2})-\eta_2(t_n))\frac{|\xi|^2}{2}\\ &+n^{-\delta_1/2}\int_{-\infty}^{1}|x|^{2+\delta_0}\mu(dx)|\xi|^{2+\delta_0}\\ &+\frac{1}{6}n^{-1/2}\eta_3(t_n)|\xi|^3\\ &+n^{1/2}\bar{F}(t_n)\int_{\mathbb{R}}|x|\mu(dx)|\xi|, \qquad \xi\in\mathbb{R}, t\ge 2,\end{aligned}$$

where δ_0 is in (A2). Hence from Proposition 5, we see that there is a constant $C>0$ such that

$$\begin{aligned}&|nR_0(n,\xi)|\\ \le\;& C\left(L(n^{1/2})^{1-2\delta}|\xi|+L(n^{1/2})^{1-\delta}|\xi|^2+n^{-\delta_0/2}|\xi|^{2+\delta_0}+L(n^{1/2})|\xi|^3\right).\end{aligned}$$

Therefore we have the first inequality.

Since $n\big((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1\big) = nR_0(n,\xi) - \eta_2(n^{1/2})\xi^2/2$, we have the second inequality. □

For $k = 0, 1$, let

$$\begin{aligned} R_{1,k}(n,\xi) &= (n-k)\log\big((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))\big) \\ &\quad - n((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1). \end{aligned}$$

PROPOSITION 10. *There is a constant $C > 0$ such that for any $\xi \in \mathbb{R}$ with $|\xi| \le L(n^{1/2})^{-\delta}$ and $k = 0, 1$,*

$$|R_{1,k}(n,\xi)| \le Cn^{-1}L(n^{1/2})^{-3\delta}|\xi|.$$

In particular, for $k = 0, 1$ we have

$$\sup\{|R_{1,k}(n,\xi)|; |\xi| \le L(n^{1/2})^{-\delta}\} \to 0, \qquad n \to \infty. \tag{17}$$

PROOF. First, for any $\xi \in \mathbb{R}$ with $|\xi| \le L(n^{1/2})^{-\delta}$, we have

$$\begin{aligned} \log\big((1-\bar{F}(t_n))\varphi(\xi,\mu(t))\big) &= (1-\bar{F}(t_n))\varphi(\xi,\mu(t)) \\ &\quad -1 + r_{l,1}((1-\bar{F}(t_n))\varphi(\xi,\mu(t))-1). \end{aligned}$$

Hence we have

$$\begin{aligned} R_{1,k}(n,\xi) &= -k\log\big((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))\big) \\ &\quad + nr_{l,1}((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1). \end{aligned}$$

From Proposition 9, we see that there is a constant $C >$ such that

$$\begin{aligned} |R_{1,k}(n,\xi)| &\le |(1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1| \\ &\quad + 2n|(1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n))-1|^2 \\ &\le Cn^{-1}L(n^{1/2})^{-3\delta}|\xi|, \qquad |\xi| \le L(n^{1/2})^{-\delta}. \ \square \end{aligned}$$

Let us prove Lemma 1. Note that for $k = 0, 1$ we have

$$\log(e^{v_n\xi^2/2}(1-\bar{F}(t_n))^{n-k}\varphi(n^{-1/2}\xi;\mu(t_n))^{n-k}) = nR_0(n,\xi) + R_{1,k}(n,\xi).$$

We see that

$$e^{v_n\xi^2/2}(1-\bar{F}(t_n))^{n-k}\varphi(n^{-1/2}\xi;\mu(t_n))^{n-k} = \exp(nR_0(n,\xi)+R_{1,k}(n,\xi)).$$

Hence we see that

$$\begin{aligned} R_{n,0}(\xi) &= e^{v_n\xi^2/2}(1-\bar{F}(t_n))^n\varphi(n^{-1/2}\xi;\mu(t_n))^n - (1+nR_0(n,\xi)) \\ &= \exp(nR_0(n,\xi)) - (1+nR_0(n,\xi)) \\ &\quad + \exp(nR_0(n,\xi))(\exp(R_{1,0}(n,\xi))-1). \end{aligned}$$

By (16), we see that there is a constant $C>0$ such that

$$|R_{n,0}(\xi)| \le C\left(|nR_0(n,\xi)|^2 + |R_{1,0}(n,\xi)|\right).$$

Therefore we have (14) from Propositions 9 and 10. The proof of (15) is similar to (14).

5. Proof of Theorem 5

Note that

$$P(\sum_{l=1}^n X_l > sn^{1/2}) = \sum_{k=0}^n I_k(n,s),$$

where

$$I_k(n,s) = P(\sum_{l=1}^n X_l > sn^{1/2}, \sum_{l=1}^n 1_{\{X_l>t_n\}} = k), \qquad k=0,1,\dots,n.$$

Then we have

$$\begin{aligned} I_k(n,s) &= \binom{n}{k} P(\sum_{l=1}^n X_l > sn^{1/2}, X_i>t_n, i=1,\dots,k, \\ &\qquad X_j \le t_n, j=k+1,\dots,n) \\ &= \binom{n}{k}\bar{F}(t_n)^k(1-\bar{F}(t_n))^{n-k}\mu(t_n)^{*(n-k)} * \nu(t_n)^{*k}((n^{1/2}s,\infty)), \end{aligned}$$

for $k=0,1,\dots,n$. We estimate $I_1(n,s)$, $I_2(n,s)$ and $\sum_{k=2}^n I_k(n,s)$ one by one. This approach was originally used in A.V. Nagaev's papers ([3], [4]).

Let $\bar{F}_{n,0}(x) = P(X_1 > n^{1/2}x, X_1 \le t_n) = (1-\bar{F}(t_n))\mu(t_n)((n^{1/2}x,\infty))$ and $\bar{F}_{n,1}(x) = P(X_1 > n^{1/2}x, X_1 > t_n)$. Note that $\bar{F}_{n,0}(x)+\bar{F}_{n,1}(x) = \bar{F}(n^{1/2}x)$.

PROPOSITION 11. *There is a constant $C>0$ such that*

$$\begin{aligned} & |I_0(n,s) - (1-n)\Phi_0(v_n^{-1/2}s) - \frac{1}{2}\Phi_2(v_n^{-1/2}s) \\ & \quad - n\int_{\mathbb{R}} \bar{F}_{n,0}(s - v_n^{1/2}x)\Phi_1(x)dx| \\ \le\ & CL(n^{1/2})^{2-5\delta}, \qquad n\ge 1, s\ge 1. \end{aligned}$$

PROOF. First, we see that

$$\begin{aligned} & \int_{\mathbb{R}} \bar{F}_{n,0}(s - v_n^{1/2}x)\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s) \\ =\ & \int_s^{\infty} \frac{1}{2\pi}\left(\int_{\mathbb{R}} e^{-ix\xi}((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n)) - 1)e^{-\frac{v_n}{2}\xi^2}d\xi\right)dx \\ =\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}((1-\bar{F}(t_n)\varphi(n^{-1/2}\xi;\mu(t_n)) - 1)e^{-\frac{v_n}{2}\xi^2}d\xi. \end{aligned}$$

Hence we have

$$\begin{aligned} & n\Big(\int_{\mathbb{R}} \bar{F}_{n,0}(s - v_n^{1/2}x)\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s)\Big) + \frac{1}{2}\Phi_2(v_n^{-1/2}s) \\ =\ & \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{-is\xi}}{i\xi}\Big(n((1-\bar{F}(t_n))\varphi(n^{-1/2}\xi;\mu(t_n)) - 1) + \frac{v_n\xi^2}{2}\Big)e^{-v_n\xi^2/2}d\xi. \end{aligned}$$

By Corollary 1, we have our assertion. □

PROPOSITION 12. *There is a constant $C>0$ such that*

$$|I_1(n,s) - n\int_{\mathbb{R}} \bar{F}_{n,1}(s - v_n^{1/2}x)\Phi_1(x)dx| \le CL(n^{1/2})^{2-6\delta}, \qquad n\ge 1, s\ge 1.$$

PROOF. We see that

$$\begin{aligned} I_1(n,s) &= n\bar{F}(t_n)(1-\bar{F}(t_n))^{n-1}\nu(t_n) * \mu(t_n)^{*(n-1)}((n^{1/2}s,\infty)) \\ &= n\bar{F}(t_n)\int_{\mathbb{R}}(1-\bar{F}(t_n))^{n-1}\mu(t_n)^{*(n-1)} \\ &\qquad \times ((n^{1/2}(s - n^{-1/2}x),\infty))\nu(t_n)(dx) \end{aligned}$$

and

$$\begin{aligned} & n\int_{\mathbb{R}} \bar{F}_{n,1}(s - v_n^{1/2}x)\Phi_1(x)dx \\ = \ & n\bar{F}(t_n)\int_{\mathbb{R}} \nu(t_n)((n^{1/2}(s - v_n^{1/2}x), \infty))\Phi_1(x)dx \\ = \ & n\bar{F}(t_n)\int_{\mathbb{R}} \nu(t_n)((n^{1/2}s - x, \infty))\Phi_1(n^{-1/2}v_n^{-1/2}x)n^{-1/2}v_n^{-1/2}dx \\ = \ & n\bar{F}(t_n)\int_{\mathbb{R}} \Phi_0(v_n^{-1/2}s - n^{-1/2}v_n^{-1/2}x)\nu(t_n)(dx). \end{aligned}$$

Hence we have

$$\begin{aligned} & I_1(n,s) - n\int_{\mathbb{R}} \bar{F}_{n,1}(s - v_n^{1/2}x)\Phi_1(x)dx \\ = \ & n\bar{F}(t_n)\int_{\mathbb{R}} \Big((1-\bar{F}(t_n))^{n-1}\mu(t_n)^{*(n-1)}((n^{1/2}s - x), \infty)) \\ & \qquad - \Phi_0(v_n^{-1/2}(s - n^{-1/2}x))\Big)\,\nu(t_n)(dx). \end{aligned}$$

Therefore, by Corollary 1, we have our assertion. □

Let us prove Theorem 5. From Propositions 11 and 12, we see that there is a constant $C > 0$ such that

$$\begin{aligned} & |I_0(n,s) + I_1(n,s) - (1-n)\Phi_0(v_n^{-1/2}s) - \frac{1}{2}\Phi_2(v_n^{-1/2}s) \\ & \quad - n\int_{\mathbb{R}} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \\ \leq \ & CL(n^{1/2})^{2-6\delta}. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\mathbb{R}} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx - \Phi_0(v_n^{-1/2}s) \\ = \ & \int_{-\infty}^{v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx \\ & + \int_{v_n^{-1/2}s}^{\infty} (\bar{F}(n^{1/2}(s - v_n^{1/2}x)) - 1_{\{v_n^{1/2}x > s\}})\Phi_1(x)dx \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{v_n^{-1/2}s} \bar{F}(n^{1/2}v_n^{1/2}(v_n^{-1/2}s - x))\Phi_1(x)dx \\
&\quad - \int_{v_n^{-1/2}s}^{\infty} F(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx
\end{aligned}
$$

and

$$
\begin{aligned}
& n\int_{v_n^{-1/2}s}^{\infty} F((n^{1/2}(s - v_n x))\Phi_1(x)dx \\
= \;& n^{1/2}\int_{-\infty}^{0} F(y)\Phi_1(v_n^{-1/2}s - n^{-1/2}v_n^{-1/2}y)v_n^{-1/2}dy.
\end{aligned}
$$

Let $R(z, y) = \Phi_1(z - y) - \Phi_1(z) - \Phi_2(z)y$, for $z > 0, y \leq 0$, then we see that there is a constant $C_1 > 0$ such that

$$|R(s, y)| \leq C_1|y|^{1+\delta_0}.$$

Hence we have

$$
\begin{aligned}
& n|\int_{v_n^{-1/2}s}^{\infty} F(n^{1/2}(s - v_n x))\Phi_1(x)dx \\
& \quad - \sum_{k=1}^{2} v_n^{-k/2}n^{-k/2}\Phi_k(v_n^{-1/2}s)\int_{-\infty}^{0} y^{k-1}F(y)dy| \\
= \;& |n^{1/2}\int_{-\infty}^{0} R(v_n^{-1/2}s, n^{-1/2}v_n^{-1/2}y)F(y)dy| \\
\leq \;& C_1 n^{-\delta_1/2}v_n^{-(1+\delta_1)/2}\int_{-\infty}^{0} |y|^{1+\delta_0}F(y)dy \\
\leq \;& Cn^{-\delta_1/2},
\end{aligned}
$$

where $C = C_1 v_1^{-(1+\delta_1)/2}\int_{-\infty}^{0} y^{1+\delta_1}F(y)dy < \infty$. Since

$$\int_{-\infty}^{0} F(y)dy = \int_{-\infty}^{0} y\mu(dy) = -\int_{0}^{\infty} y\mu(dy)$$

and

$$-\int_{-\infty}^{0} yF(y)dy = \frac{1}{2}\int_{-\infty}^{0} y^2\mu(dy) = \frac{v_n}{2} - \frac{1}{2}\int_{0}^{n^{1/2}} y^2\mu(dy),$$

we see that

$$\begin{aligned}&\frac{1}{2}\Phi_2(v_n^{-1/2}s)+v_n^{-1}\Phi_2(v_n^{-1/2}s)\int_{-\infty}^0 y^2F(y)dy\\=\ & v_n^{-1}\frac{\Phi_2(v_n^{-1/2}s)}{2}\int_0^{n^{1/2}} y^2\mu(dy).\end{aligned}$$

Therefore we have

$$\begin{aligned}&|(1-n)\Phi_0(v_n^{-1/2}s)+\frac{1}{2}\Phi_2(v_n^{-1/2}s)\\&\quad +n\int_{\mathbb{R}}\bar{F}(n^{1/2}(s-v_n^{1/2}x))\Phi_1(x)dx-H(n,v_n^{-1/2}s)|\\\leq\ & Cn^{-\delta_0/2}.\end{aligned}$$

We also see that

$$\begin{aligned}\sum_{k=2}^n I_k(n,s) &\leq \sum_{k=2}^n\frac{n(n-1)}{k(k-1)}\binom{n-2}{k-2}\bar{F}(t_n)^k(1-\bar{F}(t_n))^{n-k}\\&\leq \frac{n(n-1)}{2}\bar{F}(t_n)^2\leq L(n^{1/2})^{2-5\delta}.\end{aligned}$$

This completes the proof of Theorem 5. □

6. Some Estimations

Let

$$\begin{aligned}\hat{F}_n(s) &= \int_{-\infty}^s\bar{F}((s-x)v_n^{1/2}n^{1/2})\Phi_1(x)dx,\\ A(n,s) &= n\hat{F}_n(s)-v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty x\mu(dx)\\&\quad -\frac{v_n^{-1}}{2}\Phi_2(s)\int_0^{n^{1/2}}x^2\mu(dx),\\&= n\hat{F}_n(s)-v_n^{-1/2}n^{1/2}\Phi_1(s)\int_0^\infty\bar{F}(x)dx\\&\quad -v_n^{-1}\Phi_2(s)\left(\int_0^{n^{1/2}}x\bar{F}(x)dx-\frac{L(n^{1/2})}{2}\right).\end{aligned}$$

Then we have

$$H(n,s) \quad = \quad \Phi_0(s) + A(n,s).$$

Let

$$H_0(n,s) = \Phi_0(s) + n\bar{F}(v_n^{1/2}n^{1/2}s).$$

In this section we prove the following lemma.

LEMMA 2.

$$\sup_{s\in[1,\infty)}\left|\frac{H(n,s)}{H_0(n,s)} - 1\right| \to 0, \qquad n \to \infty.$$

Let $u_n = v_n^{1/2}n^{1/2}$, $\alpha_n = L(u_n)^{1/3}$ and $\beta_n = L(u_n)^{-1/12}$.

PROPOSITION 13. *For any $\varepsilon > 0$, there is a constant $C > 0$ such that*

$$\frac{1}{nF(u_n s)} \leq CL(u_n)^{-1}s^{2+\varepsilon}, \qquad s \in [1,\infty).$$

In particular, for $s > \beta_n$ we have

$$\frac{1}{nF(u_n s)} \leq Cs^{14+\varepsilon}.$$

PROOF. From Proposition 8 we see that for any $\varepsilon > 0$ there is a constant $C > 0$ such that

$$\begin{aligned}\frac{1}{nF(u_n s)} &= v_n s^2 \frac{1}{L(u_n)}\frac{L(u_n)}{L(u_n s)} \\ &\leq CL(u_n)^{-1}s^{2+\varepsilon}.\end{aligned}$$

Since $L(u_n)^{-1} = \beta_n^{12} \leq s^{12}$ for $s > \beta_n$, we have the second inequality. □

Let $n\hat{F}_n(s) = \sum_{k=1}^4 I_k(n,s)$, where

$$\begin{aligned}I_1(n,s) &= n\int_{s-\alpha_n}^{s} \bar{F}((s-x)u_n)\Phi_1(x)dx, \\ I_2(n,s) &= n\int_{\sqrt{7/8}s}^{s-\alpha_n} \bar{F}((s-x)u_n)\Phi_1(x)dx, \\ I_3(n,s) &= n\int_{-s}^{\sqrt{7/8}s} \bar{F}((s-x)u_n)\Phi_1(x)dx\end{aligned}$$

and

$$I_4(n,s) \quad = \quad n\int_{-\infty}^{-s} \bar{F}((s-x)u_n)\Phi_1(x)dx.$$

Let

$$\begin{aligned} R(n,s,y) \quad = \quad & \Phi_1(s-u_n^{-1}y) \\ & -(\Phi_1(s)+u_n^{-1}y\Phi_2(s)), \qquad \textit{for } n \geq 1,\ s,y \in [1,\infty). \end{aligned}$$

PROPOSITION 14.

$$\begin{aligned} \sup_{s\in[1,\infty)} H_0(n,s)^{-1}|I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s) \\ \times \int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \to 0, \qquad n\to\infty. \end{aligned}$$

PROOF. We see that

$$\begin{aligned} & I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s)\int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy \\ = \quad & n u_n^{-1}\int_0^{\alpha_n u_n} \bar{F}(y)\left(\Phi_1(s-u_n^{-1}y)-\Phi_1(s)-u_n^{-1}y\Phi_2(s)\right)dy \\ = \quad & n u_n^{-1}\int_0^{\alpha_n u_n} \bar{F}(y)R(n,s,y)dy. \end{aligned}$$

Note that for any $y\in[0,\alpha_n u_n]$,

$$\begin{aligned} |R(n,s,y)| \quad & \leq \quad u_n^{-2}y^2 \sup_{z\in[s-\alpha_n,s]} |\Phi_3(z)| \\ & \leq \quad C_0 n^{-1}y^2(1+s)^2\Phi_1(s-\alpha_n) \\ & \leq \quad C_0^2 n^{-1}y^2(1+s)^3\Phi_0(s)\exp(\alpha_n s). \end{aligned}$$

Hence for all $s\in[1,\infty)$

$$\begin{aligned} & |I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s)\int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \\ \leq \quad & 8C_0\sup\{z^2\bar{F}(z);\ z\geq 0\}\alpha_n s^3\Phi_0(s)\exp(\alpha_n s). \end{aligned}$$

Since $\alpha_n\beta_n^3 = L(u_n)^{1/12} \to 0$, $n \to \infty$, we have

$$\sup_{s\le\beta_n} \Phi_0(s)^{-1}|I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s) \times \int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \to 0,\ n \to \infty.$$

From Proposition 13, we see that for any $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that

$$(n\bar{F}(u_n s))^{-1} \le C(\varepsilon)s^{14+\varepsilon}.$$

Hence we see that for $s > \beta_n$,

$$\begin{aligned} & (n\bar{F}(u_n s))^{-1}|I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s) \int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \\ \le\ & 8C(\varepsilon)C_0^2 \sup\{z^2\bar{F}(z);\ z \ge 0\}\alpha_n s^{17+\varepsilon}\Phi_0(s)\exp(\alpha_n s). \end{aligned}$$

Since $\sup_{n\ge1}\sup_{s>\beta_n} s^{17+\varepsilon}\Phi_0(s)\exp(\alpha_n s) < \infty$, we have

$$\sup_{s>\beta_n} (n\bar{F}(u_n s))^{-1}|I_1(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{-(k-2)/2}\Phi_k(s) \times \int_0^{\alpha_n u_n} y^{k-1}\bar{F}(y)dy| \to 0,\ n \to \infty.$$

Therefore we have our assertion. □

PROPOSITION 15.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2}\Phi_k(s) \times \int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} y^{k-1}\bar{F}(y)dy| \to 0, \qquad n \to \infty.$$

PROOF. Similarly to Proposition 14, we see that

$$\begin{aligned}
&|I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} y^{k-1} \bar{F}(y) dy| \\
\le\ & n u_n^{-1} \int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} \bar{F}(y) |R(n,s,y)| dy \\
\le\ & n u_n^{-3} \bar{F}(u_n \alpha_n) C_0 (1+s)^2 (\sup_{z \in [\sqrt{7/8}s, s]} |\Phi_1(z)|) \int_{u_n \alpha_n}^{(1-\sqrt{7/8})u_n s} y^2 dy \\
\le\ & 4 C_0 n \bar{F}(u_n \alpha_n) s^5 \Phi_1(\sqrt{7/8}s) \\
\le\ & 4 C_0^{1+7/8} n \bar{F}(u_n \alpha_n) s^6 \Phi_0(s)^{7/8}.
\end{aligned}$$

Since $H_0(n,s)^{-1} \le \Phi_0(s)^{-6/7} (n\bar{F}(u_n s))^{-1/7}$, it is easy to see that for any $\varepsilon \in (0, 4/7)$, there is a constant $C_1 > 0$ such that

$$\begin{aligned}
&H_0(n,s)^{-1} |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} y^{k-1} \bar{F}(y) dy| \\
\le\ & C_1 s^{6+2/7+\varepsilon} \Phi_0(s)^{7/8-6/7} L(u_n)^{(1-\varepsilon)/3-1/7}.
\end{aligned}$$

Since $\sup_{s\ge 1}\{s^{6+2/7+\varepsilon}\Phi_0(s)^{7/8-6/7}\} < \infty$ and $(1-\varepsilon)/3 - 1/7 > 0$, we have

$$\begin{aligned}
&\sup_{s\ge 1} H_0(n,s)^{-1} |I_2(n,s) - \sum_{k=1}^{2} v_n^{-k/2} n^{(2-k)/2} \Phi_k(s) \\
&\qquad \times \int_{\alpha_n u_n}^{(1-\sqrt{7/8})u_n s} y^{k-1} \bar{F}(y) dy| \to 0, \ n \to \infty. \ \square
\end{aligned}$$

PROPOSITION 16.

$$\sup_{s \in [1,\infty)} H_0(n,s)^{-1} |I_3(n,s) - n\bar{F}(u_n s)| \to 0, \qquad n \to \infty.$$

PROOF.

$$\begin{aligned}
I_3(n,s) &= n\bar{F}(u_n s) \int_{-s}^{\sqrt{7/8}s} \frac{\bar{F}(u_n(s-x))}{\bar{F}(u_n s)} \Phi_1(x) dx \\
&= n\bar{F}(u_n s) \int_{-s}^{\sqrt{7/8}s} (1 - \frac{x}{s})^{-2} \frac{L(u_n(s-x))}{L(u_n s)} \Phi_1(x) dx.
\end{aligned}$$

It is easy to see that there is a constant $C_1 > 0$ such that

$$\Phi_0(s)^{-1} \le C_1 L(u_n)^{-2/3}, \qquad n \ge 1, s \in [1, (-\log L(u_n))^{1/2}],$$
$$|\int_{-s}^{\sqrt{7/8}s} \frac{\bar{F}(u_n(s-x))}{\bar{F}(u_n s)} \Phi_1(x)dx| \le C_1, \qquad n \ge 1, s \in [1, \infty).$$

Then we have

$$\begin{aligned} &\sup_{s \le (-\log L(u_n))^{1/2}} H_0(n,s)^{-1} |I_3(n,s) - n\bar{F}(u_n s)| \\ \le\ & C_1(C_1+1) L(u_n)^{-2/3} n\bar{F}(u_n) \\ \le\ & C_1(C_1+1) v_n^{-1} L(u_n)^{1/3} \to 0, \qquad n \to \infty. \end{aligned}$$

We take $M > 1$ arbitrarily, then $(-\log L(u_n))^{1/4} > M$ for sufficiently large n. Hence we see that for $s > (-\log L(u_n))^{1/2}$

$$\begin{aligned} & \left| \int_{-s}^{\sqrt{7/8}s} (1 - \frac{x}{s})^{-2} \frac{L(u_n(s-x))}{L(u_n s)} \Phi_1(x)dx - 1 \right| \\ \le\ & \left| \int_{-s}^{\sqrt{7/8}s} \{(1 - \frac{x}{s})^{-2} - 1\} \frac{L(u_n(s-x))}{L(u_n s)} \Phi_1(x)dx \right| \\ & + \left| \int_{-s}^{\sqrt{7/8}s} (\frac{L(u_n(s-x))}{L(u_n s)} - 1) \Phi_1(x)dx \right| \\ & + \int_{[-s, \sqrt{7/8}s]^C} \Phi_1(x)dx \\ \le\ & 2\left(\int_{-M}^{M} |(1 - \frac{x}{s})^{-2} - 1| \Phi_1(x)dx + 8\Phi_0(M) \right) \\ & + \sup_{t > (-\log L(u_n))^{1/2}} \sup_{1-\sqrt{7/8} \le a \le 1} |\frac{L(at)}{L(t)} - 1| + 2\Phi_0(\sqrt{7/8}s). \end{aligned}$$

Hence we have

$$\sup_{s > (-\log L(u_n))^{1/2}} |n\bar{F}(u_n s)|^{-1} |I_3(n,s) - n\bar{F}(u_n s)| \to 0, \qquad n \to \infty.$$

So we have our assertion. □

PROPOSITION 17.

$$\sup_{s\in[1,\infty)} \frac{I_4(n,s)}{H_0(n,s)} \to 0, \qquad n\to\infty.$$

PROOF. $|I_4(n,s)| \le n\bar{F}(2u_n s)\Phi_0(s)$. Hence we have

$$\Phi_0(s)^{-1}|I_4(n,s)| \le n\bar{F}(2u_n s) \le n\bar{F}(u_n) \to 0, \qquad n\to\infty. \square$$

PROPOSITION 18.

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}|v_n^{-1/2}n^{1/2}\Phi_1(s)\int_{\sqrt{7/8}u_n s}^{\infty}\bar{F}(y)dy| \to 0, \qquad n\to\infty$$

and

$$\sup_{s\in[1,\infty)} H_0(n,s)^{-1}$$
$$\times\, |v_n^{-1}\Phi_2(s)\left(\int_{\sqrt{7/8}u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n\to\infty.$$

PROOF. From Proposition 3 (2), we see that there is a constant $C_1>0$ such that

$$n^{1/2}\int_{(1-\sqrt{7/8})u_n s}^{\infty}\bar{F}(y)dy \le C_1 s^{-1}L((1-\sqrt{7/8})u_n s).$$

We can easily see that

$$\sup_{s\in[1,\beta_n)} \Phi_0(s)^{-1}n^{1/2}\Phi_1(s)\int_{(1-\sqrt{7/8})n^{1/2}s}^{\infty}\bar{F}(y)dy \to 0, \qquad n\to\infty$$

and

$$\sup_{s\in[\beta_n,\infty)} (n\bar{F}(n^{1/2}s))^{-1}n^{1/2}\Phi_1(s)\int_{(1-\sqrt{7/8})n^{1/2}s}^{\infty}\bar{F}(y)dy \to 0, \qquad n\to\infty.$$

Also we see that for any $\varepsilon \in (0,1)$, there is a constant $C_2 > 0$ such that

$$\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy = \int_{(1-\sqrt{7/8})v_n^{1/2} s}^{1} \frac{L(n^{1/2}x)}{x}dx \le C_2 L(n^{1/2})s^{\varepsilon}.$$

Hence we can easily see that

$$\sup_{s\in[1,\beta_n)} \Phi_0(s)^{-1} \times |\Phi_2(s)\left(\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n\to\infty$$

and

$$\sup_{s\in[\beta_n,\infty)} (n\bar{F}(n^{1/2}s))^{-1} \times |\Phi_2(s)\left(\int_{(1-\sqrt{7/8})u_n s}^{n^{1/2}} y\bar{F}(y)dy + L(n^{1/2})\right)| \to 0, \qquad n\to\infty.$$

Therefore we have our assertion. □

Now let us prove Lemma 2. Note that $H(n,s) - H_0(n,s) = A(n,s) - n\bar{F}(sn^{1/2})$. So Propositions 14, 15, 16, 17 and 18 imply Lemma 2.

7. Proof of Theorem 2 and 4

First we prove the following lemma.

LEMMA 3. *For any $\beta > 0$ and $\delta \in (0,1)$, there is a constant $C > 0$ such that*

$$\sup_{s>L(n^{1/2})^{-\beta}} \left|\frac{P(\sum_{k=1}^n X_k > sn^{1/2})}{H(n, v_n^{-1/2}s)} - 1\right| \le CL(n^{1/2})^{1-\delta}.$$

We make some preparation to prove Lemma 3. Similarly to Proposition 26 in Fushiya-Kusuoka [2], we can prove the following.

Proposition 19. (1) *For any* $t, s > 0$, *and* $n \geq 2$,

$$P(\sum_{k=2}^{n} X_k 1_{\{X_k \leq tn^{1/2}\}} > sn^{1/2}) \leq \exp(\frac{6}{t^2} - \frac{s}{t}).$$

(2) *For any* $s, t > 0$, $\varepsilon \in (0,1)$ *with* $t < (1-\varepsilon)s$,

$$\begin{aligned}
&|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - nP(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \leq tn^{1/2}\}} > sn^{1/2}, \\
&\quad \sum_{k=2}^{n} X_k 1_{\{X_k \leq tn^{1/2}\}} \leq \varepsilon sn^{1/2})| \\
&\leq 2n(n-1)\bar{F}(tn^{1/2})^2 + \exp(\frac{6}{t^2} - \frac{s}{t}) + n\bar{F}(tn^{1/2}) \exp(\frac{6}{t^2} - \frac{\varepsilon s}{2t}).
\end{aligned}$$

Also we prove the following for the proof of Lemma 3.

Proposition 20. *For any* γ, δ, $\varepsilon \in (0,1)$ *and* $\beta > 0$, *there is a constant* $C > 0$ *such that*

$$\begin{aligned}
&|P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \leq s^{\gamma} n^{1/2}\}} > sn^{1/2}, \sum_{k=2}^{n} X_k 1_{\{X_k \leq s^{\gamma} n^{1/2}\}} \leq \varepsilon sn^{1/2}) \\
&- \int_{-\infty}^{\varepsilon v_n^{-1/2} s} \bar{F}(n^{1/2}(s - v_n^{1/2} x)) \Phi_1(x) dx| \\
&\leq C\bar{F}((1-\varepsilon)n^{1/2}s) L(n^{1/2})^{1-3\delta}, \qquad \textit{for } s > L(n^{1/2})^{-\beta}.
\end{aligned}$$

Proof. It is easy to see that there is a constant $C_1 > 0$ such that

$$\begin{aligned}
&|P(X_1 + \sum_{k=2}^{n} X_k 1_{\{X_k \leq s^{\gamma} n^{1/2}\}} > sn^{1/2}, \sum_{k=2}^{n} X_k 1_{\{X_k \leq s^{\gamma} n^{1/2}\}} \leq \varepsilon sn^{1/2}) \\
&- P(X_1 + \sum_{k=2}^{n} X_k > sn^{1/2}, \sum_{k=2}^{n} X_k \leq \varepsilon sn^{1/2}, \\
&X_2 \leq L(n^{1/2})^{\delta} n^{1/2}, \ldots, X_n \leq L(n^{1/2})^{\delta} n^{1/2})| \\
&\leq C_1 \bar{F}((1-\varepsilon)n^{1/2}s) L(n^{1/2})^{1-3\delta}, \qquad \textit{for } s > L(n^{1/2})^{-\beta}.
\end{aligned}$$

We see that

$$P(X_1 + \sum_{k=2}^n X_k > sn^{1/2}, \sum_{k=2}^n X_k \le \varepsilon sn^{1/2}, X_2 \le t_n, \ldots, X_n \le t_n)$$
$$= (1 - \bar{F}(t_n))^{n-1} \int_{-\infty}^{\varepsilon s} \bar{F}(n^{1/2}(s-x))\mu(t_n)^{*(n-1)}(dx),$$

here $t_n = L(n^{1/2})^\delta n^{1/2}$. Similarly to the proof of Proposition 12, we have our assertion. □

Now let us prove Lemma 3. Since

$$\begin{aligned}
& H(n, v_n^{-1/2}s) - n \int_{-\infty}^{\varepsilon v_n^{-1/2} s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx \\
= \; & \Phi_0(v_n^{-1/2}s) + n \int_{\varepsilon v_n^{-1/2} s}^{v_n^{-1/2} s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx \\
& - v_n^{-1/2} n^{1/2} \Phi_1(v_n^{-1/2}s) \int_0^\infty x\mu(dx) - v_n^{-1} \frac{\Phi_2(v_n^{-1/2}s)}{2} \int_0^{n^{1/2}} x^2 \mu(dx) \\
= \; & \Phi_0(v_n^{-1/2}s) - v_n^{-1} \frac{\Phi_2(v_n^{-1/2}s)}{2} \int_0^{n^{1/2}} x^2 \mu(dx) \\
& + v_n^{-1/2} n^{1/2} \eta_1((1-\varepsilon)n^{1/2}s)\Phi_1(v_n^{-1/2}s) \\
& + v_n^{-1/2} n^{1/2} \\
& \times \left(\int_0^{(1-\varepsilon)n^{1/2}s} \bar{F}(z)(\Phi_1(v_n^{-1/2}s - n^{-1/2}v_n^{-1/2}z) - \Phi_1(v_n^{-1/2}s))dz \right),
\end{aligned}$$

it is easy to see that there is a constant $C_1 > 0$ such that

$$\begin{aligned}
& |H(n, v_n^{-1/2}s) - n \int_{-\infty}^{\varepsilon v_n^{-1/2} s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \\
\le \; & C_1 s \Phi_1(\varepsilon v_n^{-1/2}s), \; for \; s \ge 1.
\end{aligned}$$

Combining Proposition 19 (2) and 20, we see that there is a constant $C_2 > 0$

such that

$$\begin{aligned}
&|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - n\int_{-\infty}^{\varepsilon v_n^{-1/2}s} \bar{F}(n^{1/2}(s - v_n^{1/2}x))\Phi_1(x)dx| \\
\le\ & 2n(n-1)\bar{F}(s^{\gamma}n^{1/2})^2 + \exp(\frac{6}{s^{2\gamma}} - \frac{s}{s^{\gamma}}) + n\bar{F}(s^{\gamma}n^{1/2})\exp(\frac{6}{s^{2\gamma}} - \frac{\varepsilon s}{2s^{\gamma}}) \\
& + C_2\bar{F}((1-\varepsilon)n^{1/2}s)L(n^{1/2})^{1-\delta}.
\end{aligned}$$

Hence we see that there is a constant $C > 0$ such that

$$\begin{aligned}
&\sup_{s > L(n^{1/2})^{-\beta}} (n\bar{F}(n^{1/2}s))^{-1}|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2}s)| \\
&\le CL(n^{1/2})^{1-\delta}.
\end{aligned}$$

Therefore by Lemma 2, we have our assertion.

Now let us prove Theorem 4. By Theorem 2, we see that there is a constant $C_1 > 0$ such that

$$|P(\sum_{k=1}^{n} X_k > sn^{1/2}) - H(n, v_n^{-1/2}s)| \le C_1 L(n^{1/2})^{2-\delta/2}, \qquad s \ge 1.$$

Note that for any $\varepsilon > 0$, there is a constant $C_2 > 0$ such that $n\bar{F}(n^{1/2}s) \ge C_2^{-1}s^{-3}L(n^{1/2}) \ge C_2^{-1}L(n^{1/2})^{1+\delta/2}$ for $s \le L(n^{1/2})^{-\delta/6}$. Hence by Lemma 2, we see that there is a constant $C_3 > 0$ such that

$$\begin{aligned}
H(n, v_n^{-1/2}s)^{-1} &\le C_3(n\bar{F}(n^{1/2}s))^{-1} \\
&\le C_2C_3L(n^{1/2})^{1+\delta/2}, \qquad s \le L(n^{1/2})^{-\delta/6}.
\end{aligned}$$

So we have

$$\sup_{s \le L(n^{1/2})^{-\delta/6}} |\frac{P(\sum_{k=1}^{n} X_k > sn^{1/2})}{H(n, v_n^{-1/2}s)} - 1| \le C_1C_2C_3L(n^{1/2})^{1-\delta}.$$

From this and Lemma 3, we have Theorem 4. Theorem 2 is an easy consequence of Theorem 4 and Lemma 2.

8. Proof of Theorem 3

First let us assume $\limsup_{n\to\infty}(1-v_n)\log\dfrac{1}{L(n^{1/2})}=0$. Then we see that

$$\begin{aligned}\Phi_0(s)-\Phi_0(v_n^{-1/2}s)=\int_s^{v_n^{-1/2}s}\Phi_1(z)dz &= \int_{s^2}^{v_n^{-1}s^2}\frac{1}{\sqrt{2\pi}}e^{-y/2}\frac{dy}{2\sqrt{y}}\\ &\le \frac{s}{2v_1}(1-v_n)\Phi_1(s)\\ &\le C_0\frac{s^2}{2v_1}(1-v_n)\Phi_0(s).\end{aligned}$$

Let $z_n=\dfrac{1}{L(n^{1/2})}$, then we have $\limsup_{n\to\infty}(1-v_n)\log z_n=0$. Hence we have

$$\begin{aligned}&\sup_{s\in[1,\sqrt{3\log z_n})}\left|\frac{\Phi_0(v_n^{-1/2}s)+n\bar F(n^{1/2}s)}{\Phi_0(s)+n\bar F(n^{1/2}s)}-1\right|\\ &\le\frac{3C_0}{2v_1}(1-v_n)\log z_n\to 0,\qquad n\to\infty.\end{aligned}$$

We also see that for $s>\sqrt{3\log z_n}$,

$$\begin{aligned}&\left|\frac{\Phi_0(v_n^{-1/2}s)+n\bar F(n^{1/2}s)}{\Phi_0(s)+n\bar F(n^{1/2}s)}-1\right|\le\frac{C_0}{2v_1}\frac{(1-v_n)s^2\Phi_0(s)}{\Phi_0(s)+n\bar F(n^{1/2}s)}\\ \le\ &\frac{C_0}{2v_1}\frac{(1-v_n)s^4\Phi_0(s)}{L(n^{1/2}s)}\le\frac{C_0^2}{2\sqrt{2\pi}v_1}s^5\exp(-s^2/2)\frac{L(n^{1/2})}{L(n^{1/2}s)}z_n\\ \le\ &\frac{C_0^2}{2\sqrt{2\pi}v_1}s^6\exp(-s^2/2)z_n\\ \le\ &\frac{C_0^2}{2\sqrt{2\pi}v_1}\sup_{s\ge\sqrt{3\log z_n}}s^6\exp(-s^2/6)\to 0,\qquad n\to\infty.\end{aligned}$$

Hence we have $\sup_{s\in[1,\infty)}\left|\dfrac{\Phi_0(v_n^{-1/2}s)+n\bar F(n^{1/2}s)}{\Phi_0(s)+n\bar F(n^{1/2}s)}-1\right|\to 0$, $n\to 0$.

Next, we assume $\limsup_{n\to\infty}(1-v_n)\log\dfrac{1}{L(n^{1/2})}>0$. Let $y_n=(1-$

$v_n)\log z_n$ and $s_n = \sqrt{\log z_n}$. Then $\limsup_{n\to\infty} y_n > 0$. Hence we see that

$$\begin{aligned}\liminf_{n\to\infty} \Phi_0(s_n)^{-1}\Phi_0(v_n^{-1/2}s_n) &= \liminf_{n\to\infty} v_n^{1/2}\Phi_1(s_n)^{-1}\Phi_1(v_n^{-1/2}s_n)\\ &\le \liminf_{n\to\infty} \exp(-v_n^{-1}(1-v_n)s_n^2) = \exp(-\limsup_{n\to\infty} y_n) < 1\end{aligned}$$

and

$$\begin{aligned}\Phi_0(s_n)^{-1}n\bar{F}(n^{1/2}s_n) &\le C_0 s_n \Phi_1(s_n)^{-1}s_n^{-2}L(n^{1/2}s_n)\\ &\le \sqrt{2\pi}C_0 M(1)L(n^{1/2})^{1/2} \to 0, \qquad n\to\infty.\end{aligned}$$

So we have

$$\liminf_{n\to\infty} \frac{\Phi_0(v_n^{-1/2}s_n) + n\bar{F}(n^{1/2}s_n)}{\Phi_0(s_n) + n\bar{F}(n^{1/2}s_n)} < 1.$$

Therefore we have Theorem 3.

We give an example in the rest of this section. Let $x_0 \ge 1$ and $L : [x_0,\infty) \to (0,\infty)$ be a C^2 slowly varying function satisfying

$$\begin{aligned}&\int_{x_0}^{\infty} \frac{L(x)}{x}dx < \infty, \qquad L(x) \to 0, x\to\infty,\\ &\sup_{x\ge x_0} (|L'(x)| + |L''(x)|) < \infty.\end{aligned}$$

Then we can find $F : \mathbb{R} \to [0,1]$ non-decreasing C^2 function with $F(-\infty) = 0$, $F(\infty) = 1$, $\int_{\mathbb{R}} |F''(x)|dx < \infty$ and $F(x) = x^{-2}L(x)$ for sufficient large $x > 0$. Let μ be a probability measure whose distribution function is F. Then we see that μ satisfies (A3). Let $L(x) = (\log x)^{-1}(\log\log x)^{-1-b}$, $b > 0$ for sufficiently large $x > 0$. We can easily see that $L(x)$ satisfies the above conditions. For sufficiently large $n \ge 1$, we see that

$$\begin{aligned}1 - v_n &= \int_{n^{1/2}}^{\infty} x^2\mu(dx) = L(n^{1/2}) + 2\int_{n^{1/2}}^{\infty} \frac{L(x)}{x}dx\\ &= L(n^{1/2}) + \frac{2}{b}(\log\log n - \log 2)^{-b}\\ &\sim \frac{2}{b}(\log\log n)^{-b}.\end{aligned}$$

Hence we have the following.

Proposition 21. *Let* $L(x) = (\log x)^{-1}(\log\log x)^{-1-b}$, $b > 0$ *for sufficiently large* $x > 0$. *Then we have*

$$\limsup_{n\to\infty}(1 - v_n)\log\frac{1}{L(n^{1/2})} > 0, \qquad for\ b \in (0, 1]$$

and

$$\lim_{n\to\infty}(1 - v_n)\log\frac{1}{L(n^{1/2})} = 0, \qquad for\ b \in (1, \infty).$$

Therefore (1) does not hold for $b \in (0, 1]$.

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