SURFACE LINKS WHICH ARE COVERINGS OF A TRIVIAL TORUS KNOT

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Abstract

We consider surface links in the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we call torus-covering-links. Torus-covering-links contain spun \( T^2 \)-knots, turned spun \( T^2 \)-knots, symmetry-spun tori and torus \( T^2 \)-knots. In this paper we study the braid indices, the link groups, the unknotting numbers etc. of torus-covering-links.

0 Introduction

Locally flatly embedded closed 2-manifolds in the 4-space \( \mathbb{R}^4 \) are called surface links. It is known that any oriented surface link can be deformed to the closure of a simple surface braid, that is, a simple branched covering of the 2-sphere ([25]).

As surface knots of genus one which can be made from classical knots, there are spun \( T^2 \)-knots, turned spun \( T^2 \)-knots, symmetry-spun tori and torus \( T^2 \)-knots. Consider \( \mathbb{R}^4 \) as obtained by rotating \( \mathbb{R}^3_+ \) around the boundary \( \mathbb{R}^2 \). Then a spun \( T^2 \)-knot is obtained by rotating a classical knot ([4]), a turned spun \( T^2 \)-knot by turning it once while rotating ([4]), a symmetry-spun torus by turning a classical knot with periodicity rationally while rotating ([36]), and a torus \( T^2 \)-knot is a surface knot on the boundary of a neighborhood of a solid torus in \( \mathbb{R}^4 \) ([20]). Symmetry-spun tori include spun \( T^2 \)-knots, turned spun \( T^2 \)-knots and torus \( T^2 \)-knots. We call the link version of a symmetry-spun torus, a spun \( T^2 \)-knot, and a turned spun \( T^2 \)-knot a symmetry-spun \( T^2 \)-link, a spun \( T^2 \)-link and a turned spun \( T^2 \)-link respectively. We enumerate several properties of symmetry-spun \( T^2 \)-links.

(0.1) If a symmetry-spun \( T^2 \)-link is ribbon, then it is a spun \( T^2 \)-link (cf. [36]).
(0.2) The turned spun $T^2$-link of a non-trivial classical link is not ribbon (cf. [4], [35]).

(0.3) A symmetry-spin $T^2$-link is pseudo-ribbon and has the triple point number zero.

(0.4) A symmetry-spin $T^2$-link has a classical link group (cf. [36]).

Now we consider surface links in the 4-sphere or the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we will define as torus-covering-links (see Definition 2.1). Until Proposition 2.7, we consider torus-covering-links in $S^4$, and from Theorem 2.8 throughout this paper we consider torus-covering-links in $\mathbb{R}^4$. By definition, a torus-covering-link is described by a torus-covering-chart, which is a chart on the trivial torus knot. Torus-covering-links include symmetry $T^2$-links and spun $T^2$-links, turned spun $T^2$-links, and torus $T^2$-links. A torus-covering-link has no 2-knot component. Each component of a torus-covering-link is of genus at least one.

There are several natural questions.
Is there a surface knot of genus one which is not a torus-covering-knot?
What are the properties of torus-covering-links and knots?
What difference from symmetry-spin $T^2$-links do torus-covering-links have?

For the first question we show that “There are surface knots of genus one which are not torus-covering-knots” (Corollary 4.11).

In Section 2 we define torus-covering-links (Definition 2.1), the turned torus-covering-links (Definition 2.4) and the normal forms of ribbon torus-covering-knots of genus one (Theorem 2.8). Moreover we show that “There is a torus-covering-knot with positive triple point number” (Theorem 2.11 (cf. (0.3))), which means there is a torus-covering-knot which is not pseudo-ribbon.

In Section 3 we deform the torus-covering-link associated with a torus-covering-chart of degree $m$ to the closure of a simple surface braid, from which we obtain its (surface link) chart description, which is of degree $2m$, and give an upper estimate of its braid index. In particular, we see that the turned spun $T^2$-knot of the torus $(2,p)$-knot has the braid index four.

In Section 4 we consider torus-covering-links which are $T^2$-links, that is, torus-covering-links whose each component is of genus one. Such torus-covering-links are associated with torus-covering-charts without black vertices. We study link groups of torus-covering-links associated with torus-covering-charts without black vertices and show that “There are infinitely many torus-covering-links with two components such that each component
is of genus one and the link groups are not classical link groups” (Proposition 4.5 and Theorem 4.6 (cf. (0.4) and (0.1))), which means that they are not symmetry-spun \( T^2 \)-links. Moreover, they are ribbon, which means that there are infinitely many torus-covering-links which are ribbon but not spun \( T^2 \)-links. We show its knot version as well: “There are infinitely many torus-covering-knots of genus one and whose knot groups are not classical link groups” (Theorem 4.9), which means that they are not symmetry-spun \( T^2 \)-knots. Moreover, they are ribbon, which means that there are infinitely many torus-covering-knots which are ribbon but not spun \( T^2 \)-knots.

In Section 5 we study the unknotting numbers of torus-covering-links. In particular, we give an alternative proof of the fact that the spun (or the turned spun) \( T^2 \)-knot of a classical \((p, q)\)-torus knot has the unknotting number one (Proposition 5.7 (cf. [26])).

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1 Definitions and Preliminaries

**Definition 1.1.** A locally flatly embedded closed 2-manifold in \( S^4 \) or \( \mathbb{R}^4 \) is called a surface link. A surface link with one component is called a surface knot. A surface link whose each component is of genus zero (resp. one) is called a 2-link (resp. \( T^2 \)-link). In particular a surface knot of genus zero (resp. one) is called a 2-knot (resp. \( T^2 \)-knot).

An orientable surface link \( F \) is trivial (or unknotted) if there is an embedded 3-manifold \( M \) with \( \partial M = F \) such that each component of \( M \) is a handlebody.

An oriented surface link \( F \) is called pseudo-ribbon if there is a surface link diagram of \( F \) whose singularity set consists of double points and ribbon if \( F \) is obtained from a trivial 2-link \( F_0 \) by 1-handle surgeries along a finite number of mutually disjoint 1-handles attaching to \( F_0 \). By definition, a ribbon surface link is pseudo-ribbon.

Two surface links are equivalent if there is an ambient isotopy or an orientation-preserving diffeomorphism of \( S^4 \) or \( \mathbb{R}^4 \) which deforms one to the other.

**Definition 1.2.** A compact and oriented 2-manifold \( S \) embedded properly and locally flatly in \( D^2_1 \times D^2_2 \) is called a braided surface of degree \( m \) if \( S \) satisfies the following conditions:

(i) \( \text{pr}_2|_S : S \to D^2_2 \) is a branched covering map of degree \( m \),

\( \text{pr}_2 \) is the projection from \( D^2_1 \times D^2_2 \) to the second coordinate.
(ii) $\partial S$ is a closed $m$-braid in $D_1^2 \times \partial D_2^2$, where $D_1^2, D_2^2$ are 2-disks, and $\text{pr}_2 : D_1^2 \times D_2^2 \to D_2^2$ is the projection to the second factor.

A braided surface $S$ is called a surface braid if $\partial S$ is the trivial closed braid. Moreover, $S$ is called simple if every singular index is two.

Two braided surfaces are equivalent if there is a fiber-preserving ambient isotopy of $D_1^2 \times D_2^2 \rel D_1^2 \times \partial D_2^2$ which carries one to the other.

There is a theorem which corresponds to Alexander’s theorem for classical oriented links.

**Theorem 1.3 (Kamada [25]).** Any oriented surface link can be deformed by an ambient isotopy of $\mathbb{R}^4$ to the closure of a simple surface braid.

There is a chart which represents a simple surface braid.

**Definition 1.4.** Let $m$ be a positive integer, and $\Gamma$ be a graph on a 2-disk $D_2^2$. Then $\Gamma$ is called a surface link chart of degree $m$ if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2^2 = \emptyset$.

(ii) Every edge is oriented and labeled, and the label is in $\{1, \ldots, m - 1\}$.

(iii) Every vertex has degree 1, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled $i$ and $i + 1$ alternately for some $i$.

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels $i$ and $j$ of the diagonals satisfy $|i - j| > 1$.

\[
\begin{align*}
&\text{black vertex} & &\text{white vertex} \\
&\text{black vertex} & &\text{white vertex}
\end{align*}
\]

| $i - j$ | 1 |
| ~~~~~~~~~~~~~~~~~~ |

Fig. 1.1
Vertices of degree 1 and 6 are called a black vertex and a white vertex. A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple surface braid associated with the chart.

An edge without end points is called a loop. An edge whose end points are black vertices is called a free edge, and a configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an oval nest.

An unknotted chart is a chart presented by a configuration consisting of free edges. A trivial oriented surface link is presented by an unknotted chart ([25]).

A ribbon chart is a chart presented by a configuration consisting of oval nests. A ribbon surface link is presented by a ribbon chart ([25]).

A chart with a boundary represents a simple braided surface.

Definition 1.5. Two charts in $D_2^2$ of the same degree are C-move equivalent if we can deform one to the other by a finite sequence of ambient isotopies of $D_2^2$ and chart moves (or C-moves).

Let $\Gamma$ and $\Gamma'$ be two charts in $D_2^2$ of the same degree. Then $\Gamma'$ is said to be obtained from $\Gamma$ (or $\Gamma$ is said to be obtained from $\Gamma'$) by a chart move of type I, II or III, or by a CI-move, CII-move or CIII-move if there exists a 2-disk $E$ in $D_2^2$ such that the loop $\partial E$ is in general position with respect to $\Gamma$ and $\Gamma'$ and $\Gamma \cap (D_2^2 - E) = \Gamma' \cap (D_2^2 - E)$ and the following condition holds:

(CI) There are no black vertices in $\Gamma \cap E$ nor $\Gamma' \cap E$. Any CI-move can be presented by a finite sequence of CI-moves of types (1), (2),... , and (7) (cf. [37]). Fig. 1.2 gives CI-moves of types (1), (2),... , and (7).
Fig. 1.2. CI-moves of types (1), (2), and (3)

(CII) \( \Gamma \cap E \) and \( \Gamma' \cap E \) are as in Fig. 1.3, where \( |i - j| > 1 \).

Fig. 1.3. CII-moves

(CIII) \( \Gamma \cap E \) and \( \Gamma' \cap E \) are as in Fig. 1.4, where \( |i - j| = 1 \).

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Theorem 1.6 ([25]). Two charts of the same degree are C-move equivalent if and only if their associated simple braided surfaces are equivalent.

Throughout this paper, let us denote the oval nest with a free edge with the label \( i \) and its surrounding loops with the labels \( i_1, \ldots, i_n \) and the orientation \( \epsilon_1, \ldots, \epsilon_n \) from the free edge outward by \( O(i; i_1^* i_2^* \cdots i_n^*) \), where \( \epsilon_j = \pm 1 \) and \( i_j^* = i_j \) (resp. \( \overline{i_j} \)) if \( \epsilon_j = +1 \) (resp. \(-1\)). In particular, let us denote the free edge \( O(i; \emptyset) \) by \( F_i \). For \( 0 < i < j \), let us denote \( i(i+1) \cdots j \) (resp. \( \overline{i(i+1) \cdots j} \)) by \( i \nearrow j \) (resp. \( \overline{i \nearrow j} \)), and for \( 0 < j < i \), let us denote \( i(i-1) \cdots j \) (resp. \( \overline{i(i-1) \cdots j} \)) by \( i \searrow j \) (resp. \( \overline{i \searrow j} \)). Moreover, let us denote a disjoint union of charts by \( \cup \).

Let us define the braid group relations between two sequences of integers as follows:
1. $\emptyset = i \cdot i = i$, for a positive integer $i$,

2. $i \cdot j = j \cdot i$, for positive integers $i, j$ with $|i - j| > 1$,

3. $i \cdot j \cdot i = j \cdot i \cdot j$, for positive integers $i, j$ with $|i - j| = 1$.

Then for positive integers $i, j$ and two sequences of integers $b$ and $b'$ which are braid group equivalent, the following holds:

(1.7.1) $O(i; b) = O(i; b')$,

(1.7.2) $O(i, i) = O(i; \emptyset) = F_i$ (Fig. 1.6),

(1.7.3) $O(i; j) = O(i; \emptyset) = F_i$, where $|i - j| > 1$ (Fig. 1.7),

(1.7.4) $O(i; j) = O(j; i)$, where $|i - j| = 1$ (Fig. 1.8),

(1.7.5) $O(i; b \cdot j) \cup F_j = O(i; b) \cup F_j$,

(1.7.6) $O(i; j \cdot b) \cup O(j; b) = O(i; b) \cup O(j; b)$.

![Fig. 1.6](image1.png)

![Fig. 1.7](image2.png)
2 Torus-covering-links

In this section, we greatly rely on [36]. Until Proposition 2.7, we consider torus-covering-links in $S^4$, and from Theorem 2.8 throughout this paper we consider torus-covering-links in $\mathbb{R}^4$.

**Definition 2.1. (Torus-covering-links)** First, embed $D^2 \times S^1 \times S^1$ into $S^4$ or $\mathbb{R}^4$ naturally, or more precisely, consider as follows (cf. [36], [4] and [29]). Let $S^1 \times S^1$ be a standardly embedded torus in $S^4$ and let $D^2 \times S^1 \times S^1$ be a tubular neighborhood of $S^1 \times S^1$ in $S^4$. We can assume that its framing is canonical, that is, the homomorphism induced by the inclusion map $H_1 (\{0\} \times S^1 \times S^1; \mathbb{Z}) \to H_1 (\{p\} \times S^1 \times S^1; \mathbb{Z}) \to H_1 (S^4 - S^1 \times S^1; \mathbb{Z})$ where $p \in \partial D^2$, is zero. Let $\tilde{l} = \partial D^2 \times 0 \times 0$, $\tilde{s} = 0 \times S^1 \times 0$, and $\tilde{r} = 0 \times 0 \times S^1$ be curves on $\partial E^4$.

Let $l, s$, and $r$ be canonical curves on $\partial E^4$, which are identified with $\tilde{l}$, $\tilde{s}$ and $\tilde{r}$ under the natural identification map $i = \partial D^2 \times S^1 \times S^1 \to \partial E^4$. Then $l$, $r$, and $s$ represent a basis of $H_1 (\partial E^4; \mathbb{Z})$.

Let $f : \partial E^4 \to E^4$ be a diffeomorphism with $f_# ( l \ s \ r ) = ( l \ s \ r ) A^f$, where $A^f \in GL(3, \mathbb{Z}) \cong \pi_0 \text{Diff}(\partial E^4)$. Then $f$ can be extended to a diffeomorphism $\tilde{f} : E^4 \to E^4$ if and only if $A^f \in H$, where

$$H = \left\{ \left( \begin{array}{ccc} \pm 1 & 0 & 0 \\ \ast & \alpha & \gamma \\ \ast & \beta & \delta \end{array} \right) \in GL(3, \mathbb{Z}); \; \alpha + \beta + \gamma + \delta \equiv 0 \; (\text{mod} \; 2) \right\}. $$
Consider $D^2 \times S^1 \times S^1$ to be embedded in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$ or $\mathbb{R}^4 = (E^4 - \{\ast\}) \cup_i (D^2 \times S^1 \times S^1)$ for some point $\ast$ in $\text{Int}E^4$. Then identify $D^2 \times S^1 \times S^1$ with $D^2 \times I_3 \times I_4/ \sim$, where $(x, 0, v) \sim (x, 1, v)$ and $(x, u, 0) \sim (x, u, 1)$ for $x \in D^2$, $u \in I_3 = [0, 1]$ and $v \in I_4 = [0, 1]$. Let us consider a surface link $S$ embedded in $D^2 \times S^1 \times S^1$ such that $S \cap (D^2 \times I_3 \times I_4)$ is a simple braided surface. We will call such a surface link a torus-covering-link.

A torus-covering-link $S$ can be described by a chart on the trivial torus knot, i.e. by a chart $\Gamma_T$ on $D^2 = I_3 \times I_4$ with $\Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\})$ and $\Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4)$. Let us denote the classical braids described by $\Gamma_T \cap (I_3 \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I_4)$ by $\Gamma^v_T$ and $\Gamma^h_T$ respectively. We will call $\Gamma_T$ a torus-covering-chart with boundary braids $\Gamma^v_T$ and $\Gamma^h_T$.

Let $b(\Gamma_T)$ be the number of black vertices in the torus-covering-chart $\Gamma_T$. Then let us consider the case $b(\Gamma_T) = 0$. In this case the torus-covering-link associated with $\Gamma_T$ is determined by the boundary braids $\Gamma^v_T$ and $\Gamma^h_T$. We will call such a $\Gamma_T$ a torus-covering-chart without black vertices and with boundary braids $\Gamma^v_T$ and $\Gamma^h_T$.

Remark. In the case $b(\Gamma_T) = 0$, the boundary braids $\Gamma^v_T$ and $\Gamma^h_T$ are commutative.

By definition, torus-covering-links contain symmetry-spun tori (and spun $T^2$-knots, turned spun $T^2$-knots and torus $T^2$-knots).

As we stated in Theorem 1.6, if there are two surface link charts of the same degree, their associated surface links are equivalent if and only if their charts are C-move equivalent. It follows that if two torus-covering-charts are C-move equivalent, their associated torus-covering-links are equivalent.
A torus-covering-link has no 2-knot component. In particular, if a torus-covering-chart has no black vertices, then each component of the associated torus-covering-link is of genus one.

Let \( \Gamma_T \) be a torus-covering-chart of degree \( m \) and with the trivial boundary braids. Let \( F \) be the surface link associated with \( \Gamma_T \) by assuming \( \Gamma_T \) to be a surface link chart. Then the torus-covering-link associated with the torus-covering-chart \( \Gamma_T \) is obtained from \( F \) by applying \( m \) trivial 1-handle surgeries.

**Example 2.2.** (2.2.1) Let \( \Gamma_T \) be a torus-covering-chart without black vertices and with boundary braids \( \sigma_1^3 \) and \( e \) (the trivial braid), then the torus-covering-knot \( \Gamma_S \) associated with \( \Gamma_T \) is the spun \( T^2 \)-knot of a right-handed trefoil.

(2.2.2) Let \( \Gamma_T \) be a torus-covering-chart without black vertices and with boundary braids \( \sigma_1^3 \) and \( \sigma_1^{-3} \) (or \( \sigma_1^3 \) and \( \sigma_1^{-3} \)), then the torus-covering-knot \( \Gamma_S \) associated with \( \Gamma_T \) is the turned spun \( T^2 \)-knot of a right-handed trefoil.

(2.2.3) Let \( \Gamma_T \) be a torus-covering-chart without black vertices and with boundary braids \( \beta^2 \) and \( \beta \), then the torus-covering-knot \( \Gamma_S \) associated with \( \Gamma_T \) is a symmetry-spun torus.

**Proposition 2.3** (cf. [36]). Torus-covering-links in \( S^4 \) obtained from a torus-covering-chart \( \Gamma_T \) by rotating it by \( n\pi/2 \) (\( n \in \mathbb{Z} \)) are equivalent.

**proof.** It suffices to show the case when rotating \( \Gamma_T \) by \( \pi/2 \).

Let \( \Gamma'_T \) be the torus-covering-chart obtained by rotating \( \Gamma_T \) by \( \pi/2 \), and \( S \) and \( S' \) be the torus-covering-link obtained by \( \Gamma_T \) and \( \Gamma'_T \), respectively. Let us use the notation in Definition 2.1. We can assume the torus-covering-links \( S \) and \( S' \) to be in \( D^2 \times S^1 \times S^1 \), where \( S^1 = \mathbb{R}/2\pi\mathbb{Z} \).

Then there is a diffeomorphism

\[
 f : (D^2 \times S^1 \times S^1) \longrightarrow (D^2 \times S^1 \times S^1) \\
 f(x, t_1, t_2) = (x, -t_2, t_1).
\]

The diffeomorphism \( f|_{\partial D^2 \times S^1 \times S^1} \) can be considered as an orientation-preserving diffeomorphism from \( \partial E^4 \) to \( E^4 \). Then we have

\[
 A^f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in H.
\]
Hence there is an orientation-preserving diffeomorphism $\tilde{f} : E^4 \to E^4$. Let $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$. Then $g : S^4 \to S^4$ such that $g|_{E^4} = \tilde{f}$ and $g|_{D^2 \times S^1 \times S^1} = f$ is an orientation-preserving diffeomorphism which deforms $S$ to $S'$, therefore $S$ and $S'$ are equivalent.

**Remark.** Teragaito proved in [36] the same theorem in the symmetry-spun version. Lemma 7 in [36] corresponds to the $-\pi/2$ rotation. We will consider the *turned torus-covering-links*, which include the turned spun $T^2$-links (cf. [36], [4]).

**Definition 2.4.** Let us use the notation in Definition 2.1. Let $\sigma : \partial E^4 \to \partial E^4$ be a diffeomorphism of a matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then $E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1$ is diffeomorphic to $S^4$.

Let $\Gamma_T$ be a torus-covering-chart and let $S$ be the torus-covering-link associated with $\Gamma_T$ in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$, Then we can consider the torus-covering-link obtained from $S$ by changing the identification map $i$ to $\sigma i$, which we will call the *turned torus-covering-link associated with $(S, \Gamma_T)$* or $S$, and use the notation $\tau(S, \Gamma_T)$ or $\tau(S)$. Moreover, we will denote by $\Gamma_\tau(T)$ the torus-covering-chart associated with $\tau(S)$. That is, we define $\tau(S)$ as follows:

$$(S, E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1 \simeq S^4) = (\tau(S), S^4 = E^4 \cup_i D^2 \times S^1 \times S^1).$$

The turned torus-covering-link $S$ must be in the form associated with the torus-covering-chart $\Gamma_T$, and for two equivalent torus-covering-links $S$ and $S'$ with their associated torus-covering-charts $\Gamma_T$ and $\Gamma'_T$, their turned torus-covering-links $\tau(S, \Gamma_T)$ and $\tau(S', \Gamma'_T)$ may be different (Proposition 2.7).

**Remark.** For a spun $T^2$-knot $S$, $\tau(S)$ is the turned spun $T^2$-knot.

By this definition, we have the following theorems.

**Theorem 2.5.** Let $S$ be the torus-covering-link associated with a torus-covering-chart $\Gamma_T$, and $\tau(S)$ be the turned torus-covering-link obtained from $S$. Then the torus-covering-chart $\Gamma_\tau(T)$ associated with $\tau(S)$ can be described as in Fig. 2.2.
proof. Let $\bar{l}$, $\bar{r}$, $\bar{s}$ and $l$, $r$, $s$ be as in Definition 2.1. Remark that $\bar{r}$ or $\bar{s}$ is a meridian or a longitude on the trivial torus knot. Let $\bar{\rho}$ be a path on the trivial torus knot with $\bar{\rho} = \bar{r} - \bar{s}$, and let $\rho = r - s$. Then for the diffeomorphism $f: \partial E^4 \to \partial E^4$ with $f_\# (l \ s \ r) = (l \ s \ \rho)$,

$$A^f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \sigma^{-1}.$$ 

By the definition,

$$(S, E^4 \cup_{\sigma} D^2 \times S^1 \times S^1 \cong S^4) = (\tau(S), S^4 = E^4 \cup_{\sigma^{-1}(\sigma)} D^2 \times S^1 \times S^1).$$

Hence, regarding the torus-covering-charts as graphs on the trivial torus knot, $\Gamma_{\tau(T)}$ is obtained from $\Gamma_T$ by cutting $\Gamma_T$ by $\rho$. Hence we obtain the torus-covering-chart as in Fig. 2.2.

**Proposition 2.6** (cf. [36], [4]). For any torus-covering-link $S$, $\tau^2(S) = S$.

proof. Because

$$\sigma^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

there is an orientation-preserving diffeomorphism $g: S^4 \to S^4$ with $g(S) = g(\tau^2(S))$. \hfill \Box

**Proposition 2.7.** The map $\tau$ and the $\pi/2$ rotation $f$ are not compatible.
proof. Consider a torus-covering-chart of degree 2 without black vertices and with boundary braids \( e \) and \( \sigma_1^3 \). Its associated torus-covering-knot \( S \) is the spun \( T^2 \)-knot of a right-handed trefoil. Then \( \tau \circ f(S) \) is represented by a torus-covering-chart with boundary braids \( \sigma_1^3 \) and \( e \), which is equivalent to the spun \( T^2 \)-knot of the trefoil. On the other hand, \( f \circ \tau(S) \) is represented by a torus-covering-chart with \( \sigma_1^3 \) and \( \sigma_1^2 \), which is the turned spun \( T^2 \)-knot of the trefoil. By (0.2), the spun \( T^2 \)-knot and the turned spun \( T^2 \)-knot of a non-trivial knot are not equivalent. Hence, \( \tau \) and \( f \) are not compatible. \( \square \)

From now on throughout this paper, we consider torus-covering-links in \( \mathbb{R}^4 \).

Let us consider when torus-covering-links are ribbon. Before stating Theorem 2.8, we give several definitions about classical braids ([12]). Let \( B_n \) be the braid group of degree \( n \), \( \Delta \), Garside’s half twist, and \( D_n \), an \( n \)-punctured disk.

We say that an element \( b \in B_n \) is periodic if the element of \( B_n/\langle \Delta^2 \rangle \) represented by \( b \) is of finite order.

We say an element \( b \in B_n \) is reducible if there exists a nonempty multicurve \( C \) in \( D_n \) (i.e. a system of disjoint simple closed curves in \( D_n \), none of them isotopic to the boundary or enclosing a single puncture) which is stabilized by \( b \), i.e. such that \( b(C) \) is isotopic to \( C \). Note that \( b \) may permute different components of the multicurve \( C \).

The following definition is taken from [3] (see also [18]). To every reducible braid \( b \in B_n \) one can associate a canonical invariant multicurve: its canonical reduction system, which by definition is the collection of all isotopy classes \( c \) of simple closed curves which have the following two properties: first, \( c \) must be stabilized by some power of \( b \), and secondly any simple closed curve which has non-zero geometric intersection number with \( c \) must not be stabilized by any power of \( b \).

**Theorem 2.8.** Let \( \Gamma_T \) be a torus-covering-chart without black vertices and with boundary braids \( a \) and \( b \), and let \( S \) be the associated torus-covering-link. Then if \( S \) is a torus-covering-knot (i.e. with one component) which is a ribbon surface knot of genus one, \( \Gamma_T \) satisfies one of the following conditions. Let us use the phrase that the ribbon torus-covering-knot \( S \) of genus one is in the normal form.

1. The closed braid \( \text{cl}(a) \) is a classical knot and \( b = e \) (the trivial braid), which means \( S \) is a spun \( T^2 \)-knot of the classical knot \( \text{cl}(a) \).

2. \( (i) \) The boundary braids \( a \) and \( b \) have the same canonical reduction
system, and each element of the canonical reduction system of a
and b consists of the same number of punctures of the disk $D^2$.
Let us denote the corresponding tubular braid of a and b by $R(a)$
and $R(b)$ respectively.

(ii) The tubular braid $R(a)$ is the trivial braid of some degree. Let $n$
be the degree. Let $R_1(a), R_2(a), \ldots, R_n(a)$ be parts of the braid a
corresponding to each string of the trivial tubular braid $R(a)$.

(iii) Then the closure of the corresponding tubular braid $\text{cl}(R(b))$
is a knot.

(iv) For each $R_j(a)$, its closure $\text{cl}(R_j(a))$ is a trivial knot and for one
$R_j(a)$, the relation $ab = ba$ determines the other $R_j(a)$’s.

(3) The turned version of (2).

proof. Let $S$ be a torus-covering-knot which is a ribbon surface knot
of genus one. Then $S$ can be written by a torus-covering-chart $\Gamma_T$
without black vertices.

Let $X := \mathbb{R}^4 - S$. Since $S$ is ribbon, there is a smooth simple closed
curve $\alpha$ in $\partial X$ which is null-homologous in $X$. Then there exists a smooth,
properly-embedded disk $z$ in $X$ with $\partial z = \alpha$. We can cut $\Gamma_T$
along some paths $l$ and $l'$ and obtain a new torus-covering-chart $\Gamma'_T$
with boundary braids $a$ and $b$ such that the closed braid $\text{cl}(a)$ is a split sum of the trivial
knot and a link. The trivial knot corresponds to the smooth simple closed
curve $\alpha$. Let us denote by $S'$ the torus-covering-knot obtained from $\Gamma'_T$.
Then by Proposition 2.3 and Proposition 2.6, $S$ is equivalent to either $S'$ or
$\tau(S')$.

If the closed braid $\text{cl}(a)$ is a trivial knot (one component), there is a solid
torus whose boundary is $S$ i.e. $S'$ or $\tau(S')$ (Paste a disk whose boundary
is $\text{cl}(a)$ for every page $\mathbb{R}^4 \times \{\theta\}$, where $\mathbb{R}^4 = \mathbb{R}^4 / \sim$). If $a$ is the
trivial braid, $S$ is a symmetry-spun torus, which is either a spun $T^2$-knot or
a turned spun $T^2$-knot. Because a turned spun $T^2$-knot of a classical non-
trivial knot is not ribbon ((0.2)), $S$ is a spun $T^2$-knot.

If the classical braid $a$ is non-trivial and $\text{cl}(a)$ has more than one component,
a is a reducible braid and non-periodic. By [18], since $a$ and $b$ are
commutative, the classical braid $b$ preserves the canonical reduction system
of $a$, hence $a$ and $b$ have the same canonical reduction system.

Hence, if each element of the canonical reduction system of $a$ and $b$
doesn’t consist of the same number of punctures of the disk $D^2$, $S$ has more
than one component, which is a contradiction.
Let us consider the case when each element of the canonical reduction system of $a$ and $b$ consists of the same number of punctures of the disk $D^2$. Let us denote the corresponding tubular braid of $a$ and $b$ by $R(a)$ and $R(b)$ respectively. The tubular braid $R(a)$ is the trivial braid of some degree. Let $n$ be the degree. Let $R_1(a), R_2(a), \ldots, R_n(a)$ be parts of the braid $a$ corresponding to each string of the trivial tubular braid $R(a)$.

Since $S$ consists of one component, $R(b)$ must be a knot. We can see that for each $R_j(a)$, its closure $\text{cl}(R_j(a))$ is the trivial knot and for one $R_j(a)$, the relation $ab = ba$ determines the other $R_j(a)$’s.

**Example 2.9.** Let $\Gamma_T$ be a torus-covering-chart with boundary braids $\sigma_1\sigma_3$ and $\Delta^2$, where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, which is Garside’s $\Delta$. Then $\Gamma_T$ is in the normal form of (2). Its boundary braids and the associated tubular braids are as in Fig. 2.3 (cf. Proposition 4.5, Theorems 4.6 and 4.9).

**Remark.** Spun $T^2$-knots are ribbon, and torus-covering-knots in the normal form (2) are also ribbon (Proposition 2.10). However, whether torus-covering-knots in the normal form (3) are ribbon or not is not known. We conjecture that it is not ribbon.
Proposition 2.10. Torus-covering-knots in the normal form (2) are ribbon.

proof. Let $S$ be a torus-covering-knot in the normal form (2). We use the notation in Theorem 2.8.

Let $S$ be embedded in $D^2 \times S^1 \times S^1 = D^2 \times I_3 \times I_4 / \sim$, which is naturally embedded in the four space $\mathbb{R}^4$. The four-space $\mathbb{R}^4$ can be considered as $\mathbb{R}^3 \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}_+ \times S^1)/ \sim \times \mathbb{R}$, with $(x_1,0,\theta) \sim (x_1,0,\theta')$ for $x_1 \in \mathbb{R}, \theta,\theta' \in S^1$, and $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$ is embedded in $\mathbb{R}^4$ as follows:

$$(I_1 \times S^1) \times S^1 \times I_2 \subset ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/ \sim \times \mathbb{R},$$

with $I_1 \times S^1 \subset \mathbb{R} \times \mathbb{R}_+$ and $I_2 \subset \mathbb{R}$. The projection $p : I_1 \times S^1 \times S^1 \times I_2 \to I_1 \times S^1 \times S^1$ is a generic projection in the neighborhood of the trivial torus knot associated with the torus covering chart $\Gamma_T$, and the projection $\pi : \mathbb{R}^4 = ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/ \sim \times \mathbb{R} \to \mathbb{R}^3$ with $\pi(x_1,t_1,t_2,x_2) = (x_1,t_1,t_2)$ is a generic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$. Hence $\pi(S)$ is obtained from $\Gamma_T$, that is, by embedding $p(S)$ into $\mathbb{R}^3$ naturally.

Let us divide $I_4$ into $I_k(k = 1,2,\ldots,\nu)$, where $I_k = [t_{k-1},t_k]$ for $0 = t_0 < t_1 < \cdots < t_\nu = 1$. Let $R(b) \subset D^2 \times \{0\} \times I_4(= D^2 \times \{1\} \times I_4)$, the corresponding tubular braid of $b$, be presented as follows:

$$R(b) = \sigma_{t_{i_1}} \cdots \sigma_{t_{i_\nu}}$$

such that $R(b) \cap D^2 \times \{0\} \times I_k = \sigma_{t_k}$ for each $k$. Let $I_k' := (t_{k-1} + \epsilon,t_k - \epsilon)$ for a sufficiently small $\epsilon > 0$.

First, we show that for each $k$, $\pi(S \cap D^2 \times I_3 \times I_k')$ can be deformed to have only ribbon singularities except self intersections (i.e. intersections in the same component) by an ambient isotopy of $\mathbb{R}^4$ rel $D^2 \times S^1 \times (S^1 - I_k)$. It suffices to show when $R(b)$ is degree 3 and $S \cap (D^2 \times I_3 \times I_k') = \sigma_2$. The corresponding tubular braid of $a$, $R(a)$, is the trivial braid. Let $\tilde{R}_j$ $(j = 1,2,3)$ be the closed braids in the form of $\text{cl}(R_j(a))$. Each $\tilde{R}_j$ is the trivial knot and we denote the trivial knot in the form of $\text{cl}(e)$ by $O_j$ $(j = 1,2,3)$, where $e$ is the trivial braid of degree one. Let $\{h_{\nu}^{(j)}\}$ be an ambient isotopy of $B^3$ with $h_{\nu}^{(j)}(\tilde{R}_j) = O_j$, where $B^3$ is a 3-ball which contains $D^2 \times S^1 = I_1 \times I_2 \times I_3/ \sim$ such that $B^3 = I_1' \times I_2' \times I_3'$ with $I_1 \subset I_1'$ and $I_2 \subset I_2'$.

Before deformation, $S \cap (D^2 \times I_3 \times I_k') = \sigma_2$ is as follows.

We are in the 3-ball $B^3$ and proceed along time $I_k = [t_{k-1},t_k] \subset I_4$. There is the closed braid $\tilde{R}_1 \cup \tilde{R}_2 \cup \tilde{R}_3$ in the 3-ball $B^3 \times \{t_{k-1}\}$, where $\cup$
means a split sum. The component \( \widehat{R}_1 \) is innermost and \( \widehat{R}_3 \) is outermost.

Each component can be deformed to the other by an ambient isotopy of its tubular neighborhood \( B' \times S^1 \) in \( B^3 \), where \( B' \) is a 2-disk. As the time proceeds, \( \widehat{R}_2 \) and \( \widehat{R}_3 \) change their places, and \( \widehat{R}_2 \) (resp. \( \widehat{R}_3 \)) is deformed to the form of \( \widehat{R}_3 \) (resp. \( \widehat{R}_2 \)). The surface link diagram \( \pi(S) \) is obtained by seeing the link diagram of \( \widehat{R}_1 \cup \widehat{R}_2 \cup \widehat{R}_3 \) in \( \pi'(B^3) = \pi(I'_1 \times I'_2 \times I'_3) = I'_1 \times I'_3 \) proceeding along the time, where \( \pi' \) is the projection.

We can deform \( S \cap (D^2 \times I_3 \times I'_k) \) as follows. Let \( B^3 := D^2 \times I_2 \), where \( D^2 := I_1 \times I_3 \). Let \( B_{j,i}, B_{j,o} \) (\( j = 1, 2, 3 \)) be 3-balls such that \( B_{1,i} \subset B_{1,o} \subset B_{2,i} \subset B_{2,o} \subset B_{3,i} \subset B_{3,o} \subset B^3 \) and \( B_{j,i} = D_{j,i} \times I_2 \), \( B_{j,o} = D_{j,o} \times I_2 \) with \( D_{1,i} \subset D_{1,o} \subset D_{2,i} \subset D_{2,o} \subset D_{3,i} \subset D_{3,o} \subset D^2 \). Let the closed braid \( \widehat{R}_j \) be in \( B_{j,o} - B_{j,i} \). From \( t_{k-1} \) to \( t_{k-1} + \epsilon' \) the closed braid \( \widehat{R}_1 \cup \widehat{R}_2 \cup \widehat{R}_3 \) is unchanged. Then we unknotted the innermost component \( \widehat{R}_1 \) to the trivial knot \( O_1 \) by the ambient isotopy \( \{ h_u^{(1)} \} \). Move \( O_1 \) into \( B^3 - B_{2,o} \). Since \( O_1 \) is the trivial knot with no crossings, the singular point set consists only of ribbon singularities. Unknotted the innermost \( \widehat{R}_2 \) to the trivial knot \( O_2 \) by the ambient isotopy \( \{ h_u^{(2)} \} \). Move \( O_2 \) into \( B^3 - B_{3,o} \), and take new 3-balls \( B'_{2,i} = D'_{2,i} \times I_2 \) and \( B'_{2,o} = D'_{2,o} \times I_2 \) with \( D_{2,i} \subset D'_{2,o} \subset D^2 - D_{3,o} \) and \( O_2 \subset B'_{2,o} - B'_{2,i} \). Unknotted the innermost \( \widehat{R}_3 \) to the trivial knot \( O_3 \) by the ambient isotopy \( \{ h_u^{(3)} \} \). Now \( \pi'(O_1 \cup O_2 \cup O_3) \) are located parallelly.

Return \( O_2 \) to \( \widehat{R}_3 \) by \( \{ h_1^{(3)} \} \). Move \( O_1 \cup O_3 \) into \( B'_{2,i} \), and take new 3-balls \( B'_{3,i} = D'_{3,i} \times I_2 \) and \( B'_{3,o} = D'_{3,o} \times I_2 \) with \( D_{3,i} \subset D'_{3,o} \subset D^2 - D_{3,o} - O_1 \) and \( O_3 \subset B'_{3,o} - B'_{3,i} \). Return \( O_3 \) to \( \widehat{R}_2 \) by \( \{ h_1^{(2)} \} \). Move \( O_1 \) into \( B'_{3,i} \). Return \( O_1 \) to \( \widehat{R}_1 \) by \( \{ h_1^{(1)} \} \).

In general, unknotted the innermost closed braid and move it outside, and repeat this step to have \( n \) unknots laid parallel to each other. Re-braid the new outermost closed braid and move the other unknots inside it. Repeat this process to have the new closed braids.

Next, we show that \( S \) can be deformed to have only ribbon singularities. Take \( s_{k}^{1} < \ldots < s_{k}^{i} < s_{k}^{i + 1} < \ldots < s_{k}^{n} \) such that \( t_{k-1} < s_{k}^{1} < s_{k}^{i} < t_{k} \) and in \( B^3 \times \{ t \} \) for \( t \in [s_{k}^{i} + \epsilon, s_{k}^{i + 1} - \epsilon] \), the innermost component is in the form of the closed braid \( \widehat{R}_1 \), and for \( t \in [s_{k}^{1}, s_{k}^{i}] \) the innermost component is in the form of the trivial closed braid of degree one \( O_1 \) for sufficiently small \( \epsilon > 0 \). In general, for \( t \in [s_{k}^{i} + \epsilon, s_{k}^{i + 1} - \epsilon] \) the \( i \)-th component is in the form of the \( i \)-th closed braid, and for \( t \in [s_{k}^{i}, s_{k}^{i + 1}] \) the \( i \)-th component is in the form of the trivial closed braid of degree one.
Unknot the innermost component of $S$ in $D^2 \times [s_k, s_{k+1}]$ by unknotted the closed braid $\widehat{R}_1$ in $[s_k, s_{k+1}]$ by an ambient isotopy of $\{h_u(1)\}$. There appear no new singular points. Then Unknot the second innermost component of $S$ in $D^2 \times [s_k^2, s_{k+1}^2]$ by unknotted the closed braid $\widehat{R}_2$ in $[s_k^2, s_{k+1}^2]$ by ambient isotopy of $\{h_u(2)\}$. The new singular points are ribbon, for in $D^2 \times [s_k^2, s_{k+1}^2]$, there is only $O_1 \times [s_k^2, s_{k+1}^2]$. Repeat this process and we have only ribbon singularities in $D^2 \times [s_k^n, s_{k+1}^n]$.

Repeat this step for every $k$, and we have new $S$ whose surface link diagram have only ribbon singularities.

The triple point number of a surface link $F$ is the minimum number of triple points of surface diagrams of $F$. Symmetry-spun $T^2$-links have the triple point number zero (cf. (0.3)). However, there is a torus-covering-knot whose triple point number is positive.

Theorem 2.11. Let $\Gamma_T$ be the torus-covering-chart of degree 4 without black vertices and boundary braids $\sigma_1 \sigma_3 \sigma_2$ and $(\sigma_1 \sigma_2 \sigma_3)^4$. Then the torus-covering-knot $S$ in $\mathbb{R}^4$ obtained from the torus-covering-chart $\Gamma_T$ has positive triple point number, which means torus-covering-knots contain non-pseudo-ribbon knots.

We use tri-coloring by the dihedral quandle of order three $R_3$. We tricolor the torus-covering-chart and show that Mochizuki’s 3-cocycle invariant has not an integer value. Then it has at least four white vertices (cf. [34]).

We use the following facts.

(2.11.1) Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a generic projection. Then a surface diagram of $F$ is the image $\pi(F)$ with additional crossing information at the singularity set. There are two intersecting sheets along each edge, one of which is higher than the other with respect to $\pi$. They are called an over-sheet and an under sheet along the edge, respectively. In order to indicate crossing information, we break the under-sheet into two pieces missing the over sheet. This can be extended around a triple point. The sheets are called a top sheet, a middle sheet, and a bottom sheet from the higher one. Then the surface diagram is presented by a disjoint union of compact surfaces which are called broken sheets. For a surface diagram $D$, we denote by $B(D)$ the set of broken sheets of $D$.

(2.11.2) A set $X$ with a binary operation $* : X \times X \to X$ is called a quandle if it satisfies the following conditions:

1. $a * a = a$, for every $a$ in $X$. 

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2. for any \( b, c \in X \), there exists a unique \( c \in X \) such that \( a = c \ast b \).

3. \((a \ast b) \ast c = (a \ast c) \ast (b \ast c)\), for \( a, b, c \in X \).

We denote by \( R_3 \) the dihedral quandle of order 3, which is the set \( \{0, 1, 2\} \) with the binary operation \( a \ast b = 2b - a \pmod{3} \). A tricoloring for a surface diagram \( D \) is a map \( C : B(D) \to R_3 \) such that

\[ C(H_1) \ast C(H_2) = C(H_1') \]

along every edge of \( D \), where \( H_2 \) is the over-sheet and \( H_1 \) (resp. \( H_1' \)) is the under-sheet such that the normal vector of \( H_2 \) points from (resp. toward) it. Then the color of the edge is the pair \((C(H_1), C(H_2))\). Remark that the color of the edge which ends with a branch point is \((a, a)\) for some \( a \in R_3 \), for \( a \ast a = a \).

(2.11.3) At a triple point of a surface diagram, there exist broken sheets \( J_1, J_2, J_3 \in B(D) \) uniquely (some of which possibly coincide) such that \( J_3 \) is the top sheet and \( J_2 \) is a middle sheet and \( J_3 \) is a bottom sheet such that the normal vector of \( J_2 \) (resp. \( J_3 \)) points from \( J_1 \) (resp. \( J_2 \)). Assume \( D \) is tricolored by \( C \). Then the color of the triple point is the triplet \((C(J_1), C(J_2), C(J_3)) \in R_3 \times R_3 \times R_3 \). The sign of the triple point is positive (resp. negative) if the triplet of the normal vectors of \( J_1, J_2, J_3 \) is right-handed (resp. left-handed).

This corresponds to the following.

(2.11.4) (Proposition 4.43 (3) in [9]) The color of a white vertex representing \( \sigma_i \sigma_j \sigma_i \to \sigma_j \sigma_i \sigma_j \) is \((a, b, c)\), where \( a, b \) and \( c \) are the tricolors of the \( i' \)-th, \((i' + 1)\)-th, and \((i' + 2)\)-th strings of the classical braid \( \sigma_i \sigma_j \sigma_i \), where \( i' = \min \{i, j\} \). The triple point (or white vertex) as above is positive (resp. negative) if \( j > i \) (resp. \( i > j \)), that is, if there is exactly one edge with the largest (resp. smallest) label oriented towards the triple point.

(2.11.5) Mochizuki’s 3-cocycle is a map \( \theta : R_3 \times R_3 \times R_3 \to \mathbb{Z}_3 \) such that

\[ \theta(x, y, z) = t(x-y)(y-z)(z-x) \in \mathbb{Z}_3. \]

(2.11.6) Let \( D \) be a surface diagram of an oriented surface link \( F \) whose triple points are \( \tau_1, \ldots, \tau_s \) with the signs \( \epsilon_i = \epsilon(\tau_i) \). For a tricoloring \( C \) for \( D \), let \((a_i, b_i, c_i) \in R_3 \times R_3 \times R_3 \) be the color of \( \tau_i \) \((i = 1, \ldots, s)\).

Let us define

\[ W_\theta(\tau_i; C) := \theta(a_i, b_i, c_i) \in \mathbb{Z}_3 \]
for each triple point $\tau_i$, and

$$W_{\theta}(C) := \prod_{i=1}^{s} W_{\theta}(\tau_i; C) \in \mathbb{Z}_3$$

for the tricoloring $C$. Since the set of broken sheets of $D$ is finite, so is the set of tricolorings for $D$. This $W_{\theta}(C)$ is called the Boltzman weight. Let $C_1, \ldots, C_n$ be the tricolorings for $D$. Let us define $\Phi_{\theta}(D)$ as follows:

$$\Phi_{\theta}(D) := \sum_{j=1}^{n} W_{\theta}(C_j) \in \mathbb{Z} \langle t | t^3 = 1 \rangle.$$

Since $\Phi_{\theta}(D)$ satisfies certain conditions, it does not depend on the choice of a surface diagram $D$ of $F$. This $\Phi_{\theta}(D)$ is called a Mochizuki’s cocycle invariant of $F$, and we use the notation $\Phi_{\theta}(F)$.

**Remark.** For the cocycle invariant, there is a general theory (cf.[30]).

Proof of Theorem 2.11. Let $S$ be the torus-covering-knot associated with the torus-covering-chart $\Gamma_7$. First, we show that we can compute Mochizuki’s cocycle invariant by tricoloring $\Gamma_7$. Let $S$ be embedded in $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$, which is naturally embedded in $\mathbb{R}^4$. The four-space $\mathbb{R}^4$ can be considered as $\mathbb{R}^3 \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}^+ \times S^1) / \sim \times \mathbb{R}$, with $(x_1, 0, \theta) \sim (x_1, 0, \theta')$ for $x_1 \in \mathbb{R}$, $\theta, \theta' \in S^1$, and $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$ is embedded in $\mathbb{R}^4$ as follows:

$$(I_1 \times S^1) \times S^1 \times I_2 \subset ((\mathbb{R} \times \mathbb{R}^+) \times S^1) / \sim \times \mathbb{R},$$

with $I_1 \times S^1 \subset \mathbb{R} \times \mathbb{R}^+$ and $I_2 \subset \mathbb{R}$. The projection $p : I_1 \times S^1 \times S^1 \times I_2 \to I_1 \times S^1 \times S^1$ is a generic projection in the neighborhood of the trivial torus knot associated with the torus covering chart $\Gamma_7$ and the projection $\pi : \mathbb{R}^4 = ((\mathbb{R} \times \mathbb{R}^+) \times S^1) / \sim \times \mathbb{R} \to \mathbb{R}^3$ with $\pi(x_1, t_1, t_2, x_2) = (x_1, t_1, t_2)$ is a generic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$. Hence $\pi(S)$ is obtained from $\Gamma_7$, that is, by embedding $p(S)$ into $\mathbb{R}^3$ naturally. Therefore it suffices to tricolor the torus-covering-chart $\Gamma_7$ and show that the Mochizuki’s 3-cocycle invariant does not have an integer value.

Part of the torus-covering-chart without black vertices and with boundary braids $\sigma_1^{-1}$ and $(\sigma_1 \sigma_2 \sigma_3)^4$ contain four white vertices. Denote these by $\tau_{11}, \ldots, \tau_{14}$ from left to right (Fig. 2.4).
Tricolor the classical braid $(\sigma_1\sigma_2\sigma_3)^4$ which is on the upper horizontal boundary of the chart as in Fig. 2.5.

By (2.11.4), the color of each triple point is determined by reading the colors along the dotted path in Fig. 2.5, and its sign is determined by (2.11.4).

Similarly, we have white vertices $\tau_{ij}^2$ (i = 2, 3, 4, j = 1, 2, 3, 4), and $\tau_{ij}^3$, .., $\tau_{ij}^4$. Fig. 2.6 shows the white vertices $\tau_{ij}^2$ and their colors when i = 2.
Remark that the color of $\sigma_2^3$ whose start points and end points have the same color of that of Fig. 2.7 is as follows.

The matrix describing the sign and color of the white vertices are as follows, where its $(i, j)$-element is describing $\tau_{ij}^*$, and $\{a, b, c\} = \{0, 1, 2\}$, $\{0\}$, $\{1\}$ or
Hence we have $\Phi_{\Delta}$, by using Corollary 2.12. There is a torus-covering-knot which is not ($-$)-amphicheiral.

For each coloring, we have knots $-\text{ reactors (resp. } \kappa \Pi \text{) and turned spun } T^2 \text{-knots are (-)-amphicheiral.}

3 Deforming torus-covering-links to coverings of the 2-sphere

For a (classical) braid $\beta$, let $\iota_k^l(\beta)$ be the braid obtained from $\beta$ by adding $k$ (resp. $l$) trivial strings before (resp. after) $\beta$, and

$\Pi^m_i = \sigma_{m+1} \sigma_{m+2} \cdots \sigma_{m+i}$, $\Pi^m_i = \sigma_{m-1} \sigma_{m-2} \cdots \sigma_{m-i}$,

$\Delta_m = \Pi^m_{m-1} \Pi^m_{m-2} \cdots \Pi^m_1$, $\Delta'_m = \Pi^m_{m-1} \Pi^m_{m-2} \cdots \Pi^m_1$,

$\Theta_m = \sigma_m \cdot \Pi^m_{m-1} \cdot \Pi^m_{m-2} \cdots \sigma_m \cdot \Pi^m_{m-2} \cdots \sigma_m \cdot \Pi^m_{m-2} \cdots \sigma_m \cdot \Pi^m_{m-2} \cdots \sigma_m$.

Remark. Let $\Delta$ be Garside’s $\Delta$ for the braid group $B_m$. Then $\iota_m^0(\Delta) = \Delta_m$ (cf. [11]).
Theorem 3.1. Let $\Gamma_T$ be a torus-covering-chart of degree $m$ with boundary braids $a$ and $b$. Then the torus-covering-link associated with $\Gamma_T$ can be described by a surface link chart $\Gamma_S$ of degree $2m$ as in Fig. 3.1, where $H_b$ is a chart describing the simple braided surface as follows:

$$
i_0^m(b) \longrightarrow i_0^m(b) \cdot (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \longrightarrow i_0^m(b) \cdot (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m$$

$$\longrightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot i_0^m(\bar{b}^*) \cdot \Theta_m \longrightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Theta_m \cdot i_0^m(\bar{b}^*)$$

$$\longrightarrow (\Delta'_m)^{-1} \cdot \Delta_m^{-1} \cdot \Delta'_m \cdot \Delta_m \cdot i_0^m(\bar{b}^*) \longrightarrow i_0^m(\bar{b}^*),$$

where $\longrightarrow$ means an isotopy transformation and $\rightarrow$ a hyperbolic transformation along bands corresponding to the $m$ $\sigma_m$'s (Fig. 3.2), and $-(H_b)^*$ is the orientation-reversed mirror image of $H_b$, and $\bar{b}^*$ is the braid obtained from the classical braid $b$ by taking its mirror image and reversing all the crossings (Fig. 3.3).

Definition 3.2. Let us call $H_b$ the 1-handle chart of $\Gamma_T$, and its corresponding braided surface the 1-handle braided surface of $\Gamma_T$. 

![Figure 3.1](image-url)
Remark. The surface link chart $\Gamma_S$ is of degree $2m$ and well-defined, for the edges representing $\iota_m^0(a)$ have labels at most $m - 1$ and the edges representing $\iota_m^0(b^*)$ have labels at least $m + 1$. Note the 1-handle chart $H_b$ has $2m$ black vertices.

proof. (Step 1) Let us consider a trivial torus knot $T^2$ in $\mathbb{R}^4$ as the result of 1-handle surgeries of $S_1 \cup S_2$ along $h_1 \cup h_2$, where $S_1$ (resp. $S_2$)
is a 2-sphere in $\mathbb{R}^4$ with a positive (resp. negative) orientation such that $S_1$ contains $S_2$, and $h_1$ (resp. $h_2$) is a 1-handle attaching to the two spheres trivially in a neighborhood of the north (resp. south) pole (Fig. 3.4).

![Fig. 3.4](image)

(Step 2) Deform the two 1-handles and the inner sphere $h_1 \cup h_2 \cup S_2$ by an ambient isotopy of $\mathbb{R}^4$ to make $S_2$ have a positive orientation as in Fig. 3.5, where the 1-handle $h_1$ is as in Fig. 3.6, which has a double point curve with a branch point for each end, and the other 1-handle $h_2$ is the orientation-reversed mirror image of $h_1$.

![Fig. 3.5](image)

![Fig. 3.6](image)
(Step 3) Slide the 1-handle $h_2$ to the neighborhood of the north pole to make both $h_1$ and $h_2$ be in the neighborhood of the north pole, and cut off the two southern hemispheres to obtain the surface braid and the surface link chart of degree 2 as in Fig. 3.7.

(Step 4) Now, consider the trivial torus knot $T^2$ as the torus-covering-link associated with $\Gamma_T$ (of degree $m$) by drawing $\Gamma_T$ on $T^2$ (Fig. 3.8). Let us denote the $m$ 1-handles corresponding to $h_1$ (resp. $h_2$) by $H_1$ (resp. $H_2$). Then $H_1$ can be deformed to the 1-handle braided surface as in Fig. 3.2, and $H_2$ to the orientation-reversed mirror image of $H_1$, and the surface braid will be as in Fig. 3.9. Hence we obtain the surface link chart $\Gamma_S$ of degree $2m$ as in Fig. 3.1.
Fig. 3.9

The braid index of an oriented surface link $F$ is the minimum degree of simple closed surface braids in $\mathbb{R}^4$ which are equivalent to $F$. Kamada showed in [23] that surface links whose braid index is at most three are indeed ribbon, and Shima showed in [35] that the turned spun $T^2$-knot of a non-trivial classical knot is not ribbon (c.f. (0.2)). Hence we obtain the following corollary:

**Corollary 3.3.** Let $S$ be the torus-covering-link associated with a torus-covering-chart of degree $m$. Then the braid index of $S$ is equal or less than $2m$. In particular, the braid index of the turned spun $T^2$-knot of the torus $(2,p)$-knot is four.

**Remark.** Hasegawa in [14] (10, Part3 “Chart description of twist-spun surface-links”) showed that for the turned spun $T^2$-link of a closed $m$-braid, its braid index is at most $3m$.

**Example 3.4.** Let us consider the torus-covering-chart $\Gamma_T$ of Example 2.2.2 (Fig. 3.10).
The torus-covering-chart $\Gamma_T$ describes the turned spun $T^2$-knot of the right-handed trefoil. Its 1-handle chart is as in Fig. 3.11, and the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is as in Fig. 3.12.

![Diagram of the 1-handle chart $H_b$, where $b = \sigma_1^{-3}$](image)

Fig. 3.11. The 1-handle chart $H_b$, where $b = \sigma_1^{-3}$

![Diagram of the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ (degree 4)](image)

Fig. 3.12. The surface link chart $\Gamma_S$ obtained from $\Gamma_T$ (degree 4)

4 Knot Groups and Link Groups

In this section we consider torus-covering-links which are $T^2$-links, that is, torus-covering-links associated with torus-covering-charts without black ver-
Lemma 4.1. Let $\Gamma_T$ be a torus-covering-chart of degree $m$ without black vertices, and with its boundary braids $a$ and $b$. Let $S$ be the torus-covering-link associated with $\Gamma_T$. Then the link group of $S$ is obtained as follows:

$$\pi_1(\mathbb{R}^4 - S) = \langle x_1, \ldots, x_m | x_j = \text{Artin}(a)(x_j) = \text{Artin}(b)(x_j), \ j \in \{1, 2, \ldots, m\} \rangle,$$

where $\text{Artin}(a) : F_m \to F_m$ (resp. $\text{Artin}(b)$) is Artin’s automorphism of the free group $F_m = \langle x_1, \ldots, x_m \rangle$ associated with the $m$-braid $a$ (resp. $b$).

proof. Apply van Kampen’s theorem. \hfill \square

Before studying link groups of torus-covering-links, we enumerate two well-known theorems about classical link groups.

Theorem 4.2 (Theorem 6.3.1 in [27]). A non-trivial abelian subgroup of a classical link group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$.

If a classical knot group has a non-trivial center, then it is a torus knot ([5]). There is a theorem concerning the center of a classical link group as follows:

Theorem 4.3 (Burde and Murasugi’s Theorem ([6])). The statements listed below are equivalent.

1. The group of a classical link $L$ has a non-trivial center.
2. The link group is isomorphic to one of the groups of type (a), (b), or (c).

(a) $\left( \mathbb{Z} \ast \cdots \ast \mathbb{Z} \right) \times \mathbb{Z}$

(b) $\left( \left( \mathbb{Z} \ast \cdots \ast \mathbb{Z} \right) \times \mathbb{Z} \right) \ast_\mathbb{Z} \left( \left( \mathbb{Z} \times \mathbb{Z} \right) \ast \mathbb{Z} \mathbb{Z} \right)$

(c) $\left( \left( \mathbb{Z} \ast \cdots \ast \mathbb{Z} \right) \times \mathbb{Z} \right) \ast_\mathbb{Z} \left( \left( \mathbb{Z} \times \mathbb{Z} \right) \ast_\mathbb{Z} \left( \mathbb{Z} \ast \mathbb{Z} \mathbb{Z} \right) \right)$,

where $m$ is the number of components of $L$, $\mathbb{Z}$ is a free cyclic group, and $\mathbb{Z} = \langle h \rangle$ is a “special” free cyclic group which is the center of the link group except the link group is that of a Hopfian link of type (a). In case (b) the amalgamation concerning the last factor $\mathbb{Z} = \langle q \rangle$ is given by $h = q^\alpha$, where $\alpha > 1$. In case (c) the last factor $(Z \ast Z \mathbb{Z})$ is the group of the torus knot of type $(\alpha, \beta)$.
There are torus-covering-links whose link groups are $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We can see that they are not classical link groups by Theorem 4.2.

**Theorem 4.4.** Let $\Gamma_T$ be the torus covering chart of degree 3 (resp. 4) without black vertices, and with its boundary braids $\Delta^2$ and $\sigma_1^2 \sigma_2^2$ (resp. $\sigma_1^2 \sigma_2^2 \sigma_3^2$), where $\Delta^2 = (\sigma_1 \sigma_2)^3$ (resp. $(\sigma_1 \sigma_2 \sigma_3)^4$). Then the torus covering link $S$ associated with $\Gamma_T$ has the link group $\pi_1(\mathbb{R}^4 - S) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (resp. $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$).

**Proof.** Let us compute the link group $G = \pi_1(\mathbb{R}^4 - S)$ by Lemma 4.1.

(Degree 3 case) Let $x_1, x_2, x_3$ be the generators. Then the relations concerning the boundary braid $\sigma_2$ are

$$x_1 x_2 = x_2 x_1, \quad (4.1)$$

$$x_2 x_3 = x_3 x_2. \quad (4.2)$$

The other relations concerning the other boundary braid $\Delta^2$ are

$$x_1 = (x_1 x_2 x_3) x_1 (x_1 x_2 x_3)^{-1}$$

$$x_2 = (x_1 x_2 x_3) x_2 (x_1 x_2 x_3)^{-1}$$

$$x_3 = (x_1 x_2 x_3) x_3 (x_1 x_2 x_3)^{-1},$$

which are

$$x_1 x_2 x_3 = x_2 x_3 x_1, \quad (4.3)$$

$$x_2 (x_1 x_2 x_3) = (x_1 x_2 x_3) x_2, \quad (4.4)$$

$$x_3 x_1 x_2 = x_1 x_2 x_3. \quad (4.5)$$

By (4.3) and (4.1), we have

$$x_1 x_3 = x_3 x_1,$$

hence

$$G = \langle x_1, x_2, x_3 \mid x_1 x_2 = x_2 x_1, x_2 x_3 = x_3 x_2, x_3 x_1 = x_1 x_3 \rangle = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

(Degree 4 case) Similarly, for generators $x_1, x_2, x_3, x_4$, we have the following relations:

$$x_i x_{i+1} = x_{i+1} x_i$$

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for $i = 1, 2, 3$ and

\[ x_i = (x_1 x_2 x_3 x_4) x_i (x_1 x_2 x_3 x_4)^{-1} \]

for $i = 1, 2, 3, 4$. By using the former three relations, the latter four relations are deformed as follows:

\[
\begin{align*}
    x_1 x_3 x_4 &= x_3 x_4 x_1, \\
    x_2 x_3 x_4 &= x_3 x_4 x_2, \\
    x_3 x_1 x_2 &= x_1 x_2 x_3, \\
    x_4 x_1 x_2 &= x_1 x_2 x_4.
\end{align*}
\]

(4.6)

(4.7)

(4.8)

(4.9)

By $x_2 x_3 = x_3 x_2$, (4.7) is deformed as follows:

\[
\begin{align*}
    x_3 x_2 x_4 &= x_3 x_4 x_2, \\
    x_2 x_4 &= x_4 x_2.
\end{align*}
\]

Similarly, by $x_2 x_3 = x_3 x_2$ and (4.8), we have

\[ x_1 x_3 = x_3 x_1, \]

and by $x_2 x_4 = x_4 x_2$ and (4.9), we have

\[ x_4 x_1 = x_1 x_4. \]

Hence $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Symmetry-spun $T^2$-links have classical link groups (cf. (0.4)). However, there are torus-covering-links whose link groups are not classical link groups (Proposition 4.5 and Theorem 4.6). Moreover, they are ribbon. This means there are torus-covering-links which can be described by surface link charts without white vertices (i.e. their triple point number is zero) but whose associated torus-covering-charts always have white vertices.

**Proposition 4.5.** Let $\Gamma_T$ be the torus-covering-chart of degree 4 without black vertices, and with boundary braids $\Delta^2$ and $\sigma_1 \sigma_3$, where $\Delta = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_1$ (Garside’s $\Delta$). Then the torus-covering-link $S$ associated with $\Gamma_T$ is a $T^2$-link of two components and moreover ribbon but not a spun $T^2$-link.

**proof.** By the definition of torus-covering-links, $S$ is a surface link of two components and each component is of genus one.

First, let us see that $S$ is ribbon. By Theorem 2.3 and Propsition 2.10, we can see that $S$ is ribbon, but here we will apply Theorem 3.1. Note that the torus-covering-chart $\Gamma_T^0$ with boundary braids $\Delta$ and $\sigma_1 \sigma_3$ is as in Fig. 4.1, and the torus-covering-chart $\Gamma_T$ is as in Fig. 4.2.
By Theorem 3.1, the 1-handle chart $H_b$ can be deformed to the one as in Fig. 4.3. We use the notation of oval nests introduced in Section 1. By Fig. 4.3, the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is as follows, where $b = (2132)^2$:

$$\Gamma_S = F_1 \cup F_3 \cup F_5 \cup F_7 \cup F_1 \cup O(4; 3256) \cup O(4; b) \cup O(4; 3256 \cdot b).$$

Since $\Gamma_S$ is a ribbon chart, the torus-covering-link $S$ is ribbon.
Next, we will see the torus-covering-link $S$ cannot be deformed to a spun $T^2$-link of some classical link. Let us compute the link group $\pi_1(\mathbb{R}^4 - S)$. 

Fig. 4.3. The 1-handle chart $H_b$
By Lemma 4.1, $\pi_1(\mathbb{R}^4 - S)$ is generated by four generators $x_1, x_2, x_3, x_4$ and the relations which come from $(\sigma_1\sigma_2\sigma_3)^4$ and from $\sigma_1\sigma_3$. The relations from $(\sigma_1\sigma_2\sigma_3)^4$ are
\[
x_1 = (x_1 x_2 x_3 x_4)x_1(x_1 x_2 x_3 x_4)^{-1},
\]
\[
x_2 = (x_1 x_2 x_3 x_4)x_2(x_1 x_2 x_3 x_4)^{-1},
\]
\[
x_3 = (x_1 x_2 x_3 x_4)x_3(x_1 x_2 x_3 x_4)^{-1},
\]
\[
x_4 = (x_1 x_2 x_3 x_4)x_4(x_1 x_2 x_3 x_4)^{-1}.
\]
The relations from $\sigma_1\sigma_3$ are
\[
x_1 = x_2,
\]
\[
x_3 = x_4.
\]
By denoting $a := x_1 = x_2$ and $b := x_3 = x_4$, we have
\[
\pi_1(\mathbb{R}^4 - S) = \langle a, b \mid ab^2 = b^2a,\ ba^2 = a^2b \rangle.
\]
We show that $\pi_1(\mathbb{R}^4 - S)$ is not a classical link group using Theorem 4.2.

Let $A$ be the subgroup of $\pi_1(\mathbb{R}^4 - S)$ generated by $ab, a^2$ and $b^2$, that is
\[
A = \langle ab, a^2, b^2 \mid ab^2a = b^2a, ba^2 = a^2b \rangle.
\]
Since the following holds, the subgroup $A$ is abelian:
\[
(ab)a^2 = a^3b = a^2(ab),
\]
\[
(ab)b^2 = ab^3 = b^2(ab),
\]
\[
a^2b^2 = ab^2a = b^2a^2.
\]
We can assume $A$ has no torsion element, for a classical link group has no torsion element (Theorem 4.2). Then $ab$, $a^2$ and $b^2$ are mutually independent. If we have
\[
(ab)^{n_1}(a^{2n_2})(b^{2n_3}) = 1
\]
for some integers $n_1$, $n_2$ and $n_3$, then
\[
(ab)^{2m}(a^{-2m})(b^{-2m}) = ((ab)^2a^{-2}b^{-2})^m = 1
\]

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for some \( m \), because the relations do not change the numbers of \( a \)'s and \( b \)'s. Since \( A \) has no torsion element,

\[
(ab)^2a^{-2}b^{-2} = 1 \\
abab = b^2a^2 = a^2b^2 \\
ba = ab \\
aba^{-1}b^{-1} = 1.
\]

Let \( \phi \) be a homomorphism from \( \pi_1(\mathbb{R}^4 - S) \) to the symmetry group of degree three \( S_3 \) defined by \( \phi(a) = (12) \) and \( \phi(b) = (23) \). Then we have

\[
\phi(aba^{-1}b^{-1}) = (12)(23)(12)(23) = (132) \neq 1.
\]

Hence, \( aba^{-1}b^{-1} \neq 1 \) in \( \pi_1(\mathbb{R}^4 - S) \). This is a contradiction.

Hence the torus-covering-link \( S \) is not a spun \( T^2 \)-link.

\[\square\]

Remark. Let \( \Gamma'_T \) be the torus-covering-chart of degree 4 without black vertices, and with boundary braids \((\sigma_2\sigma_1\sigma_3\sigma_2)^2\) and with \( \sigma_1\sigma_3 \). Then the torus-covering-link \( S' \) associated with \( \Gamma'_T \) is the same as \( S \) in Proposition 4.5.

We can generalize this Proposition 4.5 as follows.

Theorem 4.6. Let \( \Gamma_{T,n,\epsilon,\epsilon'} \) be the torus-covering-chart of degree 4 without black vertices, and with the boundary braids \( \Delta^2n \) and \( \sigma_1^\epsilon \sigma_3^\epsilon' \), where \( \Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1 \) (Garside's \( \Delta \)), \( n \) is a positive integer, and \( \epsilon, \epsilon' \in \{+1, -1\} \). Then the torus-covering-link \( S_{n,\epsilon,\epsilon'} \) associated with \( \Gamma_{T,n,\epsilon,\epsilon'} \) is ribbon but not a spun \( T^2 \)-link. Moreover, for a fixed \( n \), the torus-covering-links are equivalent, which we will denote by \( S_n \). Then \( S_n \) and \( S_m \) are different for \( n \neq m \).

Remark. This torus-covering-link \( S_n \) has two components. Each component of \( S_n \) is a trivial torus knot.

Before the proof, we give a theorem about canonical forms of an amalgamated product and a lemma concerning group isomorphisms.

Theorem 4.7 (Theorem 17.2.1 in [13]). In the amalgamated product of groups \( G_i \) (i = 1, 2, \ldots, n) with the amalgamated subgroup \( U \), there is in each class of equivalent words one, and only one, element in canonical form.
This homomorphism $G$ defined. For $g, g$ isomorphism $f$ can be described by the same ribbon chart.

Lemma 4.8. Let $G_1$ and $G_2$ be groups. Suppose there is an isomorphism $f : G_1 \to G_2$. Then for normal subgroups $N_1$ and $N_2$ of $G_1$ and $G_2$ with the isomorphism $f|_{N_1} : N_1 \to N_2$, there is an induced isomorphism $f : G_1/N_1 \to G_2/N_2$, where $f(q_1N_1) = f(q_1)N_2$. In particular, we can take such normal subgroups as the center groups, and the commutator subgroups of $G_1$ and $G_2$.

proof. The induced homomorphism $f : G_1/N_1 \to G_2/N_2$ is well-defined. For $g, g' \in G_1$, if $g \sim g' \in G_1/N_1$, then there is some $n \in N_1$ such that $g' = gx$. Since $f : G_1 \to G_2$ and $f|_{N_1} : N_1 \to N_2$ are homomorphisms, $f(g') = f(g)f(x)$ and $f(x) \in N_2$. Hence $f(g) \sim f(g') \in G_2/N_2$.

Let $f' : G_2/N_2 \to G_1/N_1$ be a homomorphism with $f'(g_2N_2) = f^{-1}(g_2)N_1$. This homomorphism $f'$ is well-defined, for $f : G_1 \to G_2$ and $f|_{N_1} : N_1 \to N_2$ are isomorphisms. Then $f' \circ f = \text{id}_{G_1/N_1}$ and $f \circ f' = \text{id}_{G_2/N_2}$. Hence $f^{-1} = f'$ and $f'$ is an isomorphism.

proof of Theorem 4.6. The proof that $S_{n, \kappa, \kappa'}$ is ribbon is the same as in Proposition 4.5. We can see that for a fixed $n$, the torus-covering-links can be described by the same ribbon chart.

To show that $S_n$ is not a spun $T^2$-link, it suffices to show that the link group $G_n$ of $S_n$ is not a classical link group. The link group $G_n$ of $S_n$ is computed as follows. Let $x_1, \ldots, x_4$ be the generators. The relations concerning the boundary braid $\sigma_1\sigma_3$ are

\[
\begin{align*}
x_1 &= x_2, \\
x_3 &= x_4.
\end{align*}
\]

The other relations concerning the other boundary braid $\Delta^{2n}$ are

\[
\begin{align*}
x_1 &= (x_1x_2x_3x_4)^n x_1 (x_1x_2x_3x_4)^{-n}, \\
x_2 &= (x_1x_2x_3x_4)^n x_2 (x_1x_2x_3x_4)^{-n}, \\
x_3 &= (x_1x_2x_3x_4)^n x_3 (x_1x_2x_3x_4)^{-n}, \\
x_4 &= (x_1x_2x_3x_4)^n x_4 (x_1x_2x_3x_4)^{-n}.
\end{align*}
\]

By denoting $a := x_1 = x_2$ and $b := x_3 = x_4$, we have

\[
\begin{align*}
G_n &= \pi_1(\mathbb{R}^4 - S) \\
&= \langle a, b | (a^2b^2)^n b = b(a^2b^2)^n, (a^2b^2)^n a = a(a^2b^2)^n \rangle.
\end{align*}
\]

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Let us denote the center of $G_n$ by $Z_n$ and $Z'_n := \langle a^2, b^2 | a^2b^2 = b^2a^2 \rangle$ and $Z_n' := \langle h_n \rangle$, where $h_n = (a^2b^2)^n$ and $n > 1$. We show that $Z_n = Z'_n$ for $n \geq 1$. It is easy to see $Z'_n$ is a subgroup of the center $Z_n$. Since every element of $Z'_n$ commutes with each element of $G_n$, $Z'_n$ is a normal subgroup of $G_n$, and we can consider the quotient group $G_n/Z'_n$. It suffices to show that $G_n/Z'_n$ has a trivial center. For $n = 1$,

$$G_1/Z'_1 = \langle a, b | a^2 = b^2 = 1 \rangle = \mathbb{Z}/2 \ast \mathbb{Z}/2,$$

which has a trivial center. For $n > 1$,

$$G'_n := G_n/Z'_n = \langle a, b | (a^2b^2)^n = 1 \rangle = \langle a, b, x | x = a^2b^2, x^n = 1 \rangle = \langle a \rangle \ast_U \langle b, x | x^n = 1 \rangle,$$

where $U = \langle a^2 \rangle = \langle xb^{-2} \rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = xb^{-2}$.

Let $H_1 := \langle a \rangle$ and $H_2 := \langle b, x | x^n = 1 \rangle$. Note that $H_2 = \langle b, x | x^n = 1 \rangle$ is a free product of $\mathbb{Z}$ and $\mathbb{Z}/n$.

Let $h$ be a center element of $G'_n$. Since $H_1 = \langle a \rangle = U \cup Ua$, let us take $\{1, a\}$ as the left coset representatives of $U$ in $H_1$. Choose some set of left coset representatives of $U$ in $H_2$, and denote it by $C$. By Theorem 4.7, $h$ has a canonical form

$$h = u(a)c_1ac_2 \cdots ac_t(a),$$

where $u \in U$ and $c_1, \ldots, c_t \in C - \{1\}$.

Since $ah = ha$, we have

$$ua(a)c_1ac_2 \cdots ac_t(a) = u(a)c_1ac_2 \cdots ac_t(a)a,$$

hence

$$uac_1ac_2 \cdots ac_t = uc_1ac_2 \cdots ac_ta.$$

If $t > 0$, $uac_1ac_2 \cdots ac_t$ and $uc_1ac_2 \cdots ac_ta$ are both in canonical forms but they differ. This contradicts the uniqueness of canonical forms. Hence we have

$$h = u(a) = a^k$$

for an integer $k$. 

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Since \( hb = bh \), we have \( a^k b = ba^k \). If \( k = 2l + 1 \), we have
\[
\begin{align*}
   a^k b &= (a^2)^l ab = uab, \\
   ba^k &= ba^{2l}a = b(xb^{-2})^l a = ca,
\end{align*}
\]
where \( u = a^{2l} \in U \) and \( c = b(xb^{-2})^l \). If \( k = 2l \neq 0 \), we have
\[
\begin{align*}
   a^k b &= (a^2)^l b = ub, \\
   ba^k &= ba^{2l} = b(xb^{-2})^l = c,
\end{align*}
\]
where \( u = a^{2l} \in U \) and \( c = b(xb^{-2})^l \). Since \( l \neq 0 \), we have
\[
(xb^{-2})^{m_1}b \neq (xb^{-2})^{m_2}c = (xb^{-2})^{m_2}b(xb^{-2})^l
\]
in \( X_2 \) for every \( m_1, m_2 \in \mathbb{Z} \), which means \( Ub \neq Uc \) in \( X_2 \). Hence we can choose \( b \) and \( c \) to be different left coset representatives of \( U \) in \( H_2 \). Then the canonical forms of \( a^kb \) and \( ba^k \) are \( uab \) and \( ca \) (if \( k = 2l + 1 \)), or \( ub \) and \( c \) (if \( k = 2l \)), which differ. This contradicts the uniqueness of canonical forms.

Hence \( k = 0 \) and \( h = 1 \), i.e. \( G'_n = G_n/Z'_n \) has a trivial center, which means that \( Z_n = \langle h_n \rangle = \langle (a^2b^2)^n \rangle \).

This \( Z_n = \langle h_n \rangle \), the center group of \( G_n \), is not trivial. There is a homomorphism \( \psi : G_n \to \mathbb{Z} \) with \( \psi(a) = 1 \) and \( \psi(b) = 1 \). Then \( \psi(h_n) = 4n \neq 1 \) in \( \mathbb{Z} \).

We show that \( G_n \) is not a classical link group. We see \( G_n \) is neither of type (a), (b) nor (c) in Theorem 4.3.

We prove this for \( n > 1 \). Note that since the torus-covering-link \( S_n \) has two components, \( m \) in Theorem 4.3 is two.

(Case (a)) If \( G_n \) is of type (a),
\[
G_n = \mathbb{Z} \times \mathbb{Z}.
\]

The group \( G_n \) is commutative.

There is a natural surjective homomorphism
\[
f : G_n \to \mathbb{Z}/2 * \mathbb{Z}/2 = \langle a' \rangle * \langle b' \rangle \quad (4.10)
\]
with \( f(a) = a' \) and \( f(b) = b' \). Since \( \mathbb{Z}/2 * \mathbb{Z}/2 \) is not commutative, \( G_n \) is not commutative, which is a contradiction.

(Case (b)) If \( G_n \) is of type (b),
\[
G_n = (\mathbb{Z} \times \mathbb{Z}) *_\mathbb{Z} \mathbb{Z} = (\langle k \rangle \times \langle h_n \rangle) *_\mathbb{Z} \langle q \rangle, \quad (4.11)
\]
where the amalgamation is given by \( h_n = q^\alpha \) for some integer \( \alpha > 1 \). Here \( h_n = (a^2b^2)^\alpha \) is the generator of the center \( Z_n \) of the group \( G_n \), and \( q \) is the generator of the last factor \( Z \) of (4.11). The generators \( k, h_n \) and \( q \) are written by \( a \)'s and \( b \)'s. There is a natural surjective homomorphism \( f \) as written in (4.10) in Case (a). Let \( k' := f(k), h'_n := f(h_n) \) and \( q' := f(q) \). Since \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \) has a trivial center, \( \langle h'_n \rangle = 1 \), and the amalgamation becomes \( q'^\alpha = 1 \). Hence we have

\[
f(G_n) = (\langle k' \rangle \times \langle h'_n \rangle) \ast_Z \langle q' \rangle = \langle k' \rangle \ast_Z \langle q' \rangle,
\]

where the amalgamation is given by \( q'^\alpha = 1 \).

Since \( \alpha > 1 \) and a non-trivial element of \( \mathbb{Z}/2 \ast \mathbb{Z}/2 \) has order 2 or \( \infty \), we have \( \alpha = 2 \) and \( q'^2 = 1 \). Then the numbers of \( a \)'s and \( b \)'s in \( q' \) are different mod 2, hence the numbers of \( a \)'s and \( b \)'s in \( q \) are different mod 2. However, since the relations of \( G_n \) do not change the numbers of \( a \)'s and \( b \)'s, and \( h_n = q^2 \) and the numbers of \( a \)'s and \( b \)'s of \( h_n \) are the same, \( q \) has the same numbers of \( a \)'s and \( b \)'s. This is a contradiction.

(Case (c)) If \( G_n \) is of type (c),

\[
G = (Z \times \mathbb{Z}) \ast_Z (Z \ast_Z Z),
\]

where \( Z \ast_Z Z \) is a classical knot group of a torus knot of type \((\alpha, \beta)\) for some \( \alpha > 1 \) and \( |\beta| > 1 \), i.e. \( Z \ast_Z Z = \langle x, y | x^\alpha = y^\beta \rangle \). Similarly to Case (b), \( x'^\alpha = 1 \) for \( x' := f(x) \) and we have \( x' \) is of order 2 i.e. \( \alpha = 2 \). Hence \( x' \) consists of different numbers of \( a \)'s and \( b \)'s mod 2, which means \( x \) consists of different numbers of \( a \)'s and \( b \)'s mod 2. However, since \( h_n = x^\alpha = x^2 \), \( x \) consists of the same numbers of \( a \)'s and \( b \)'s. This is a contradiction.

In the case \( n = 1 \), since the center group of \( G_1 \) is \( Z_1 = \langle a^2, b^2 | a^2b^2 = b^2a^2 \rangle \), which has two generators, if \( G_1 \) is a classical link group, \( G_1 \) must be of type (a). The rest of the proof is the same as in Case (a).

Therefore the link group of \( S_n \) \((n > 0)\) is not a classical link group.

Next we show that the torus-covering-links \( S_n \) and \( S_m \) are different for \( n \neq m \). It suffices to show that there is a contradiction if their link groups \( G_n \) and \( G_m \) are isomorphic. It suffices to show when \( n, m > 1 \). If \( G_n \) and \( G_m \) are isomorphic,

\[
G'_n/[G'_n, G'_n] = G'_m/[G'_m, G'_m],
\]

where \( G'_j = G_j/Z_j = \langle a, b | (a^2b^2)^j = 1 \rangle \) and \([G'_j, G'_j]\) is the commutator.
subgroup of $G'_j$ for $j = n, m$. Then we have
\begin{align*}
G''_n : &= G'_n/[G'_n, G'_n] \\
&= \langle a, b \mid (a^2b^2)^n = 1, ab = ba \rangle \\
&= \langle a, b, c \mid (a^2b^2)^n = 1, ab = ba, c = ab \rangle.
\end{align*}
By eliminating $b$ by $b = a^{-1}c$, we have
\begin{align*}
G''_n &= \langle a, c \mid (aca^{-1}c)^n = 1, ac = ca \rangle \\
&= \langle a, c \mid c^2 = 1, ac = ca \rangle \\
&= \mathbb{Z} \times \mathbb{Z}/2n.
\end{align*}
Hence we have $G''_n \neq G''_m$ and $S_n \neq S_m$ for $n \neq m$.

Remark. There is an algorithm whereby one can determine the center of a group with one defining relation ([1], [32]). This algorithm gives an alternative proof of the fact that the center group of $G'_n = \langle a, b \mid (a^2b^2)^n = 1 \rangle$ is trivial.

We can consider the knot version of Theorem 4.6.

**Theorem 4.9.** Let $\Gamma_{T,n,\epsilon,\epsilon'}$ be the torus-covering-chart of degree 4 without black vertices, and with its boundary braids $\Delta^{2n+1}$ and $\sigma_1^2 \sigma_3^4$, where $\Delta = (\sigma_1 \sigma_2 \sigma_3)^4$, $n$ is a positive integer, and $\epsilon, \epsilon' \in \{+1, -1\}$. Then the torus-covering-knot $S_{n,\epsilon,\epsilon'}$ associated with $\Gamma_{T,n,\epsilon,\epsilon'}$ is ribbon but not a spun $T^2$-knot. Moreover, for a fixed $n$, the torus-covering-links are equivalent, which we will denote by $S_n$. Then $S_n$ and $S_m$ are different for $n \neq m$.

**proof.** The proof that $S_{n,\epsilon,\epsilon'}$ is ribbon and it does not depend on the choice of $\epsilon$ and $\epsilon'$ is the same as Proposition 4.5. We show that the knot group $G_n$ of $S_n$ is not classical knot group and $G_n \neq G_m$ for $n \neq m$.

Note that for $n = 0$, $S_0$ is the trivial torus knot.

The knot group $G_n$ is computed as follows: Let $x_1, \ldots, x_4$ be the generators. Then the relations concerning the boundary braid $\sigma_1 \sigma_3$ are
\begin{align*}
x_1 &= x_2, \\
x_3 &= x_4.
\end{align*}
The other relations concerning the other boundary braid $\Delta^{2n+1}$ are
\begin{align*}
x_1 &= (x_1x_2x_3x_4)^n x_1 x_2 x_3 x_4 x_3^{-1} x_2^{-1} x_1^{-1} (x_1 x_2 x_3 x_4)^{-n}, \\
x_2 &= (x_1 x_2 x_3 x_4)^n x_1 x_2 x_3 x_2^{-1} x_1^{-1} (x_1 x_2 x_3 x_4)^{-n}, \\
x_3 &= (x_1 x_2 x_3 x_4)^n x_1 x_2 x_1^{-1} (x_1 x_2 x_3 x_4)^{-n}, \\
x_4 &= (x_1 x_2 x_3 x_4)^n x_1 (x_1 x_2 x_3 x_4)^{-n}.
\end{align*}
By denoting \( a := x_1 = x_2 \) and \( b := x_3 = x_4 \), we have two relations:

\[
\begin{align*}
  a &= (a^2 b^2)^n a^2 b a^{-2} (a^2 b^2)^{-n}, \\
  b &= (a^2 b^2)^n a (a^2 b^2)^{-n},
\end{align*}
\]

the former of which can be deformed to the following:

\[
\begin{align*}
  a(a^2 b^2)^n a^2 &= (a^2 b^2)^n a^2 b \\
  a(a^2 b^2)^{n+1} &= (a^2 b^2)^{n+1} b.
\end{align*}
\]

Hence we have

\[
G_n = \pi_1(\mathbb{R}^4 - S_n)
\]

\[
= \langle a, b \mid b(a^2 b^2)^n = (a^2 b^2)^n a, a(a^2 b^2)^{n+1} = (a^2 b^2)^{n+1} b \rangle.
\]

Let \( Z_n := \langle h_n \rangle \), where \( h_n = (a^2 b^2)^{2n+1} \). Then \( Z_n \) is the center group of \( G_n \).

Since

\[
\begin{align*}
  a h_n &= a(a^2 b^2)^{2n+1} = (a^2 b^2)^{n+1} b (a^2 b^2)^n = (a^2 b^2)^{2n+1} a = h_n a, \\
  b h_n &= b(a^2 b^2)^{2n+1} = (a^2 b^2)^n a (a^2 b^2)^{n+1} = (a^2 b^2)^{2n+1} b = h_n b,
\end{align*}
\]

\( Z_n \) is a subgroup of the center group, and moreover it is a normal subgroup of \( G_n \).

Similar to the proof of Theorem 4.6, it suffices to show that the quotient group \( G'_n = G_n/Z_n \) has a trivial center.

\[
G'_n = \langle a, b \mid b(a^2 b^2)^n = (a^2 b^2)^n a, a(a^2 b^2)^{n+1} = (a^2 b^2)^{n+1} b, (a^2 b^2)^{2n+1} = 1 \rangle
= \langle a, b, x \mid x = a^2 b^2, b x^n = x^n a, ax^{n+1} = x^{n+1} b, x^{2n+1} = 1 \rangle.
\]

By eliminating \( b \) by \( b = x^n ax^{-n} \), we have

\[
G'_n = \langle a, x \mid x = a^2 x^n a^2 x^{-n}, x^{2n+1} + a = ax^{2n+1}, x^{2n+1} = 1 \rangle
= \langle a, x \mid x = a^2 x^n a^2 x^{-n}, x^{2n+1} = 1 \rangle
= \langle a, x \mid x^{2n+1} = (a^2 x^n)^2 = 1 \rangle
= \langle a, x, y \mid x^{2n+1} = 1, y^2 = 1, a^2 x^n = y \rangle
= \langle a \ast_U \langle x, y \mid x^{2n+1} = 1, y^2 = 1 \rangle,\]

where \( U = \langle a^2 \rangle = \langle yx^{n+1} \rangle = \mathbb{Z} \) and the amalgamation is given by \( a^2 = yx^{n+1} = yx^{-n} \). Let \( H_1 := \langle a \rangle \) and \( H_2 := \langle x, y \mid x^{2n+1} = 1, y^2 = 1 \rangle \). Note that \( H_2 = \langle x, y \mid x^{2n+1} = 1, y^2 = 1 \rangle \) is a free product of \( \mathbb{Z}/(2n + 1) \) and \( \mathbb{Z}/2 \).
Let $h$ be a center element of $G'_n$. Since $H_1 = \langle a \rangle = U \cup Ua$, let us take $\{1, a\}$ as the left coset representatives of $U$ in $H_1$. Choose some set of left coset representatives of $U$ in $H_2$, and denote it by $C$. By Theorem 4.7, $h$ has a canonical form

$$h = u(a)c_1ac_2\cdots ac_t(a),$$

where $u \in U$ and $c_1, \ldots, c_t \in C - \{1\}$.

Since $ah = ha$, we have

$$ua(a)c_1ac_2\cdots ac_t(a) = u(a)c_1ac_2\cdots ac_t(a)a,$$

hence

$$uac_1ac_2\cdots ac_t = uc_1ac_2\cdots ac_t a.$$

If $t > 0$, $uac_1ac_2\cdots ac_t$ and $uc_1ac_2\cdots ac_ta$ are both in canonical forms but they differ. This contradicts the uniqueness of canonical forms. Hence we have

$$h = u(a) = a^k$$

for an integer $k$.

Since $hx = xh$, we have $a^kx = xa^k$. If $k = 2l + 1$, we have

$$a^kx = (a^2)^l ax = uax,$$
$$xa^k = axa^{2l} = x(yx^{n+1})^l a = ca,$$

where $u = a^{2l} \in U$ and $c = x(yx^{n+1})^l$. If $k = 2l \neq 0$, we have

$$a^kx = (a^2)^l x = ux,$$
$$xa^k = axa^{2l} = x(yx^{n+1})^l = c,$$

where $u = a^{2l} \in U$ and $c = x(yx^{n+1})^l$. Since $l \neq 0$, we have

$$(yx^{n+1})^{m_1}x \neq (yx^{n+1})^{m_2}c = (yx^{n+1})^{m_2}x(yx^{n+1})^l$$

in $X_2$ for every $m_1, m_2 \in \mathbb{Z}$, which means $Ux \neq Uc$ in $X_2$. Hence we can choose $x$ and $c$ to be different left coset representatives of $U$ in $H_2$. Then the canonical forms of $a^kx$ and $xa^k$ are $uax$ and $ca$ (if $k = 2l + 1$), or $ux$ and $c$ (if $k = 2l$), which differ. This contradicts the uniqueness of canonical forms.
Hence \( k = 0 \) and \( h = 1 \), i.e. \( G_n' = G_n/Z_n' \) has a trivial center, which means that \( Z_n = \langle h_n \rangle = \langle (a^2b^2)^{2n+1} \rangle \).

This center group \( Z_n = \langle h_n \rangle \) of \( G_n \) is not trivial. There is a homomorphism \( \psi : G_n \rightarrow \mathbb{Z} \) with \( \psi(a) = 1 \) and \( \psi(b) = 1 \). Then \( \psi(h_n) = 4(2n+1) \neq 1 \) in \( \mathbb{Z} \).

By Burde and Murasugi’s Theorem (Theorem 4.3), if \( G_n \) is a classical knot group, then \( G_n \) is a torus knot group. Let \( G_{p,q} \) be the \((p,q)\)-torus knot group, i.e. \( G_{p,q} := \langle u, v \mid u^p = v^q \rangle \) for coprime integers \( p \) and \( q \). We can assume that \( 1 < p < q \). Let \( Z_{p,q} := \langle v^p \rangle = \langle v^q \rangle \) be the center group of \( G_{p,q} \). We have

\[
G_{p,q}' = G_{p,q}/Z_{p,q} = \langle u, v \mid u^p = v^q = 1 \rangle = \mathbb{Z}/p \times \mathbb{Z}/q.
\]

Then we have

\[
G_n''' = G_n'/[G_n', G_n'] = \langle a, x \mid x^{2n+1} = 1, a^4x^{2n} = 1, ax = xa \rangle = \langle a, x \mid x^{2n+1} = 1, a^4 = x, ax = xa \rangle = \langle a \mid a^{4(2n+1)} = 1 \rangle = \mathbb{Z}/4(2n+1),
\]

and

\[
G_{p,q}''' = G_{p,q}'/[G_{p,q}', G_{p,q}'] = \langle u, v \mid u^p = v^q = 1, uv = vu \rangle = \mathbb{Z}/p \times \mathbb{Z}/q.
\]

Since \( G_n''' = G_{p,q}''' \), we have \( pq = 4(2n+1) \). Since \( G_n' \) has an element of order \( 2n+1 \), and a non-trivial torsion element of \( G_{p,q}' \) has order \( r \) which is a divisor of \( p \) or \( q \), we can assume that \( 2n+1 \) is a divisor of \( q \). Since \( p, q \) are coprime positive integers, we have \( p = 4 \) and \( q = 2n+1 \).

For any element \( w \) of order 2 in \( G_{4,2n+1}' = \mathbb{Z}/4 \times \mathbb{Z}/(2n+1) \), \( w \) can be written as \( w = w'^2 \) for some element of order 4 in \( G_{4,2n+1}' \). Since \( y \) in \( G_n' \) is of order 2, there is an element \( y' \) with \( y = y'^2 \) in \( G_n' \). Then we have

\[
G_n' = \langle a, x, y, y' \mid x^{2n+1} = 1, y^2 = 1, a^2x^n = y, y = y'^2 \rangle = \langle a, x, y' \mid x^{2n+1} = 1, y'^4 = 1, a^2x^n = y'^2 \rangle.
\]
Let $N_{4,2n+1,v'}$ be a normal subgroup of $G'_{4,2n+1}$ generated by an element $v'$ of order $2n + 1$. Since for any such $v'$, $v' = 1$ iff $v = 1$, the quotient group $G'_{4,2n+1}/N_{4,2n+1,v'}$ does not depend on the choice of $v$. Let us denote $G'_{4,2n+1}/N_{4,2n+1,v'}$ by $G'_{4,2n+1}/N_{4,2n+1}$. Let $N_n$ be a normal subgroup of $G'_n$ generated by $x$, which is an element of order $2n + 1$. Let $f : G'_n \to G'_{4,2n+1}$ be the isomorphism. Then $N_n$ and $N_{4,2n+1,v'}$ is isomorphic by $f|_{N_n}$ for some $v'$. Then by Lemma 4.8 we have $G'_n/N_n \cong G'_{4,2n+1}/N_{4,2n+1}$. Let us compute:

\[
G''_n := G'_n/N_n = \langle a, x, y' \mid x^{2n+1} = 1, y' = 1, a^2 x^n = y'^2, x = 1 \rangle = \langle a, y' \mid y'^4 = 1, a^2 = y'^2 \rangle,
\]

\[
G''_{4,2n+1} := G'_{4,2n+1}/N_{4,2n+1} = \langle u, v \mid u^4 = 1, v^{2n+1} = 1, v = 1 \rangle = \langle u \mid u^4 = 1 \rangle = \mathbb{Z}/4\mathbb{Z}.
\]

Any element of $G''_{4,2n+1}$ has order 1, 2 or 4. However, the order of $ay'$ in $G''_n$ is neither 1, 2 nor 4. Take a homomorphism $\phi : G''_n \to S_3$ with $\phi(a) = (12)$, $\phi(y) = (23)$, where $\mathcal{S}_3$ is the symmetry group of degree 3. This $\phi$ is well-defined. Then we have

\[
\phi(ay') = (12)(23) = (123) \neq e,
\]

\[
\phi((ay')^2) = (12)(23)(12) = (132) \neq e,
\]

\[
\phi((ay')^4) = (132)^4 = (123) \neq e,
\]

in $\mathcal{S}_3$, which means neither $ay'$, $(ay')^2$ nor $(ay')^4$ is trivial in $G'_n$. This is a contradiction.

For $n \neq m$, $S_n \neq S_m$. Since

\[
G''_n = \mathbb{Z}/4(2n + 1) \neq \mathbb{Z}/4(2m + 1) = G''_m
\]

for $n \neq m$, we have $S_n \neq S_m$. 

A classical link group has no torsion element (cf. Theorem 4.2). There is a surface knot whose knot group has torsion elements (cf. [27]). The link group of a symmetry-spun $T^2$-link has no torsion elements (cf. (0.4)). Then how about torus-covering-links?
Theorem 4.10. Let $S$ be the torus-covering-link associated with a torus-covering-chart of degree $m$ without black vertices. Then its link group has no torsion element.

proof. Let $E := D^2 \times S^1 \times S^1 - S$. Let $A$ be a non trivial abelian subgroup of $\pi_1(E)$, and $\tilde{E}$ a covering space over $E$ corresponding to the subgroup $A$. Then it suffices to show the following:

(1) $\pi_1(E) = \pi_1(\mathbb{R}^4 - S)$,
(2) For $i \geq 2$, $\pi_i(E) = 0$,
(3) $H_3(\tilde{E}) \neq \mathbb{Z}_n$ for $|n| \geq 2$.

If the above hold, then since $\pi_i(\tilde{E}) = \pi_i(E) = 0$ for $i \geq 2$, we have $H_q(A) \cong H_q(\tilde{E})$ for any $q$. Then if $A$ is a finite cyclic group $\mathbb{Z}_n$, by [15] $H_3(A) = \mathbb{Z}_n$, which contradicts (3). Hence, $\pi_1(E) = \pi_1(\mathbb{R}^4 - S)$ has no torsion element.

(1) It can be shown by van Kampen’s theorem.

(2) There is a fiber bundle $\pi : E \to S^1$ with the fiber $F = D^2 \times S^1 - \text{cl}(a)$. Then by the exact sequence of a fiber bundle, we obtain $\pi_q(E) = \pi_q(F)$ ($q \geq 2$). There is another fiber bundle $\pi' : F \to S^1$ with the fiber $F'$ is an $m$-punctured disk, which is homotopic to $S^1 \lor S^1 \lor \cdots \lor S^1$. By the exact sequence of a fiber bundle again, we obtain $\pi_q(F') = \pi_q(S^1 \lor S^1 \lor \cdots \lor S^1)$ ($q \geq 2$). Because $\pi_q(S^1) = 0$ ($q \geq 2$), $\pi_q(S^1 \lor S^1 \lor \cdots \lor S^1) = 0$ ($q \geq 2$).

(3) By the Mayer-Vietoris exact sequence, we see that $H_3(E) = \mathbb{Z}^m$. Let $A$ be a subgroup of $\pi_1(E)$ generated by all the torsion elements of order $n$. Then $\tilde{E}$ is locally path-connected regular covering space.

There is a commutative diagram (cf. [17]),

$$
\begin{array}{ccc}
H_3(\tilde{E}) & \xrightarrow{\pi_*} & H_3(E) \\
\kappa \downarrow & & \nearrow \iota \\
E_{0,3}^\infty & & \\
\end{array}
$$

where $\pi_*$ is a homomorphism induced by the projection $\pi : \tilde{E} \to E$, $\kappa$ is a surjection, and $\iota$ is an injection.

Take a torsion element $x$ of $H_3(\tilde{E}) \cong A$. Since a homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}^m$ is a 0-map, $\pi_*(x) = 0$, so $\iota \circ \pi_*(x) = 0$ in $E_{0,3}^\infty$. On the other hand, if every torsion element (that is, every generator) of $H_3(\tilde{E}) \cong A$ goes to zero in $E_{0,3}^\infty$, $E_{0,3}^\infty = 0$, for $\kappa$ is surjective. However, $E_{0,3}^\infty$ cannot be zero, for $\iota : H_3(E) = \mathbb{Z}^m \to E_{0,3}^\infty$ is injective. Therefore $A$ is an emptyset and $H_3(\tilde{E}) \neq \mathbb{Z}_n$ for $|n| \geq 2$. }
Corollary 4.11. Let $F$ be a 2-knot with torsion elements. Then the surface knot of genus one obtained from $F$ by adding a trivial 1-handle is NOT a torus-covering-knot.

Remark. For example, we can take such a 2-knot $F$ as the 5-twist-spun trefoil.

5 Unknotting numbers

It is known that any surface link can be deformed to a trivial surface link by applying a finite number of 1-handle surgeries.

For an oriented surface link, its **unknotting number** is the minimum number of oriented 1-handle surgeries needed to deform it to trivial.

For an oriented surface link, adding a free edge to the surface link chart corresponds to a **nice 1-handle surgery**, which is a kind of an oriented 1-handle surgery. In this section we study unknotting numbers of torus-covering-links.

**Theorem 5.1.** Let $\tau(S, \Gamma_T)$ be the torus-covering-link obtained by turning a torus-covering-link $(S, \Gamma_T)$ once. Then if $S$ is unknotted, $\tau(S, \Gamma_T)$ is also unknotted.

proof. It suffices to show in the case where $S$ has one component, i.e. $S$ is a torus-covering-knot. We consider that the torus-covering-knot $S$ is embedded in $B^3 \times S^1$ and $B^3 \times S^1$ is embedded in $\mathbb{R}^4$ by $\mathbb{R}^4 = S^4 - \{\ast\} = B^3 \times S^1 \cup S^2 \times B^2 - \{\ast\}$, where $\partial B^3 = S^2$ and $\partial B^2 = S^1$ and $i : S^2 \times S^1 \rightarrow S^2 \times S^1$ is the identity map. There is a handlebody whose boundary is $S$, for $S$ is unknotted. Let us denote this handlebody by $H$. Then $\partial H \cap (B^3 \times S^1) = \partial H$. Let $K$ be $H \cap \partial B = H \cap (S^2 \times S^1)$. We can assume that the handlebody $H$ is in $\mathbb{R}^3 \times \{t_0\}$ and $S^2 \times S^1$ is embedded in $\mathbb{R}^4$. Let $f : S^2 \times S^1 \rightarrow \mathbb{R}^4$ be the embedding. For every $p \in S^1$, $f(S^2 \times \{p\}) \subset D^3$, where $D^3$ is a 3-ball. In other words, $S^2 \times \{p\}$ is trivially embedded. If for some $p \in S^1$, $f(S^2 \times \{p\})$ is not trivially embedded in $\mathbb{R}^4$, then $S^2 \times I \subset S^2 \times S^1$ cannot be embedded in $\mathbb{R}^4$, where $I \ni p$ is a sufficiently small interval.

Hence $f(S^2 \times S^1) = S^2 \times f(S^1)$, where $S^2$ is trivially embedded and $f(S^1)$ may be knotted. This knotted $f(S^1)$ can be deformed to the trivially embedded $S^1$ by an ambient isotopy of $\mathbb{R}^4$, but we must consider the following situation: $H \subset \mathbb{R}^3 \times \{t_0\}$, and $\partial H \subset B^3 \times f(S^1)$, where $B^3$ is a 3-ball with $\partial B^3 = S^2$.

Since $\partial H$ is a trivially embedded closed 2-manifold in $B^3 \times f(S^1)$, we see that $H \subset B^3 \times f(S^1)$. Therefore, $H \subset B$. From the construction of
\(\tau(S, \Gamma_T)\), there is a handlebody whose boundary is \(\tau(S, \Gamma_T)\), which means \(\tau(S, \Gamma_T)\) is also unknotted.

**Corollary 5.2.** Let \(S\) be the torus-covering-link associated with a torus-covering-chart \(\Gamma_T\) and \(\tau(S)\) be the turned torus-covering-link of \((S, \Gamma_T)\). Then \(S\) and \(\tau(S)\) have the same unknotting number.

**Proposition 5.3.** Let \(\Gamma_T\) be a torus-covering-chart of degree \(m\) with only free edges. Then the torus-covering-link \(S\) associated with \(\Gamma_T\) is unknotted.

Before the proof, we give a lemma.

**Lemma 5.4.** Let

\[
O_k := O\left(m; \prod_{j=0}^{k-1} (m - 1 \ \downarrow \ m - k + j) \prod_{j=0}^{k-1} (m + 1 \ \uparrow \ m + k - j)\right)
\]

for \(k = 1, 2, \ldots, m - 1\). Then the oval nest \(O_k\) can be deformed to the following:

\[
O_k = O(m; (m - 1 \ \downarrow \ m - k)(m + 1 \ \uparrow \ m + k))
\]

\[
= O(m - k; (m - k + 1 \ \uparrow \ m)(m + 1 \ \downarrow \ m + k))
\]

\[
= O(m + k; (m + k - 1 \ \downarrow \ m)(m - 1 \ \downarrow \ m - k)).
\]

**proof.** First, we show that the sequence of integers \(\prod_{j=0}^{k-1} (m - 1 \ \downarrow \ m - k + j)\) can be deformed by the braid relations to the following:

\[
\prod_{j=0}^{k-1} (m - 1 \ \downarrow \ m - k + j)
\]

\[
= (m - 1 \ \downarrow \ m - k)(m - 1 \ \downarrow \ m - k + 1) \cdots (m - 1)
\]

\[
= (m - k) \cdot (m - k + 1 \ \downarrow \ m - k) \cdots (m - 1 \ \downarrow \ m - k)
\]

\[
= \prod_{j=0}^{k-1} (m - k + j \ \downarrow \ m - k).
\]

(5.1)

This can be seen from the easy equation

\[
(l \ \downarrow \ i_1)(l \ \downarrow \ i_2) = (l - 1 \ \downarrow \ i_2 - 1)(l \ \downarrow \ i_1).
\]

(5.2)
By (5.2), we see that
\[
\prod_{j=0}^{k-1} (m - 1 \searrow m - k + j) = (m - 1 \searrow m - k) \prod_{j=2}^{k} (m - 1 \searrow m - k + j - 1) = \prod_{j=1}^{k-1} (m - 2 \searrow m - k + j - 1) \cdot (m - 1 \searrow m - k) = \cdots = \prod_{j=0}^{k-1} (m - k + j \searrow m - k),
\]
which is (5.1). Similarly, we have another equation
\[
\prod_{j=0}^{k-1} (m + 1 \nearrow m + k - j) = \prod_{j=0}^{k-1} (m + k - j \nearrow m + k). \tag{5.3}
\]
Hence, by (5.2), (5.3) and (1.7.1),. . .,(1.7.6), we have
\[
O_k = O\left(m; \prod_{j=0}^{k-1} (m - 1 \searrow m - k + j) \prod_{j=0}^{k-1} (m + 1 \nearrow m + k - j)\right) = O\left(m; \prod_{j=0}^{k-2} (m - k + j \searrow m - k) \cdot \prod_{j=0}^{k-2} (m + k - j \nearrow m + k)(m - 1 \searrow m - k)(m + 1 \nearrow m + k)\right) = O(m; (m - 1 \searrow m - k)(m + 1 \nearrow m + k)) = O(m - k; (m - k + 1 \nearrow \overline{m})(m + 1 \nearrow m + k)) = O(m + k; (m + k - 1 \searrow \overline{m})(m - 1 \searrow m - k)).
\]

proof of Proposition 5.3. It suffices to show in the case where \(S\) has one component, i.e. \(S\) is a torus-covering-knot. Moreover, we may assume
that the torus-covering-chart $\Gamma_T$ has the least possible free edges. Then we can assume $\Gamma_T$ to be of degree $m$ and

$$\Gamma_T = \bigcup_{k=1}^{m-1} F_k.$$ 

Let us see what the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is like. Let $O_k := O\left(m; \prod_{j=0}^{k-1} (m - 1 \setminus (m - k+j)) \prod_{j=0}^{k-1} (m + 1 \setminus (m + k - j))\right)$, for $k = 1, 2, \ldots, m - 1$ and $O_0 := O(m; \emptyset) = F_m$.

Then the 1-handle chart $H_e$ is deformed to

$$H_e = \bigcup_{k=0}^{m-1} O_k,$$

where $e$ is the trivial braid of degree $m$. Then the surface link chart $\Gamma_S$ is as follows:

$$\Gamma_S = \bigcup_{k=0}^{m-1} O_k \cup \bigcup_{k=0}^{m-1} O_k \cup \bigcup_{k=1}^{m-1} F_k.$$ 

Now, we show that the surface link chart $\Gamma_S$ can be deformed to an unknotted chart. Let $I_k(k = 1, 2, \ldots, m - 1)$ be a chart such that

$$I_k := O_k \cup \bigcup_{j=m-k+1}^{m+k} F_j \cup F_{m-k}.$$ 

We show that

$$I_k = O_k \cup \bigcup_{j=m-k+1}^{m+k} F_j \cup F_{m-k}$$

$$= \bigcup_{j=m-k}^{m+k} F_j$$

for $k = 1, \ldots, m - 1$. 

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By Lemma 5.4 and (1.7.1),..., (1.7.6) we have

\[
I_k = O(m+k; (m+k-1 \setminus \overline{m})(m-1 \setminus m-k))
\]

\[
\cup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k}
\]

\[
= F_{m+k} \cup \bigcup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k}
\]

\[
= \bigcup_{j=m-k}^{m+k} F_j,
\]

which is (5.4). Since \( O_0 = F_m, \Gamma_S \supset I_1 \). Hence by (5.4), \( \Gamma_S \supset I_2 \). Repeat this step and we have

\[
\Gamma_S \supset I_{m-1} = \bigcup_{j=1}^{2m-1} F_j.
\]

Hence by (1.7.5), \( \Gamma_S \) can be deformed to have only free edges, which is an unknotted chart. \( \square \)

**Proposition 5.5.** Let \( \Gamma_T \) be a torus-covering-chart of degree \( m \) which has no white vertices. In other words, \( \Gamma_T \) has only free edges and loops. Then the unknotting number of the torus-covering-link \( S \) associated with \( \Gamma_T \) is at most \( m - 1 \).

**proof.** The torus-covering-chart \( \Gamma_T \) can be deformed to the form of disjoint union of free edges by adding \( (m - 1) \)-handles, \( \bigcup_{j=1}^{m-1} F_j \). Hence, by Proposition 5.3, the unknotting number of \( S \) is at most \( m - 1 \). \( \square \)

**Theorem 5.6.** There is a torus-covering-knot whose unknotting number is \( n \), where \( n \) is any positive integer. Let \( S \) be the spun or turned spun \( T^2 \)-knot of a classical knot \( \beta = \text{cl}(\sigma_1^3\sigma_2^3\cdots\sigma_n^3) \) (degree \( n + 1 \)) with \( n > 0 \). Then the unknotting number of \( S \) is \( n \).

**proof.** The torus-covering-knot \( S \) can be described by a torus-covering-chart \( \Gamma_T \) of degree \( n + 1 \) with neither black vertices nor white vertices. More precisely, \( \Gamma_T \) has no black vertices and its boundary braids are \( \beta \) and \( e \) (the trivial braid of degree \( n + 1 \)), or \( \beta \) and \( \beta \). Then by Proposition 5.5, the unknotting number of \( S \) is at most \( n \). It suffices to show that we must apply at least \( n \) 1-handle surgeries to deform \( S \) to a trivial surface knot.  

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Let us tricolor $S$. It suffices to tricolor the classical knot $\beta$. The classical braid $b = \sigma_1^3 \sigma_2^3 \cdots \sigma_n^3$ can be divided into $n$ blocks $b_i = \sigma_1^3$ ($i = 1, 2, \ldots, n$). Let us denote by $q_i'$ (resp. $q_{i+1}'$) the start point of the $i$-th (resp. $(i+1)$-th) string of $b_i$ and by $q_i''$ (resp. $q_{i+1}''$) the end point of the $i$-th (resp. $(i+1)$-th) string of $b_i$ for $i = 2, \ldots, n$. For some $q$, a point of the diagram of $b$, let $C(q)$ be the color of the edge with $q$ on it.

We consider tricoloring $b$ with $C(q_i') = C(q_i'')$ for every $i$. The diagram of the braid $b_i$ consists of five edges. The four edges of them each contain one of the start points or end points. Let us denote by $p_i$ a point of the fifth edge.

The tricoloring of the braid $b_i$ is determined by two of the three colors $C(q_i'), C(q_{i+1}), C(p_i)$. Hence the tricoloring of $b$ is determined by $C(q_1), C(q_2), \ldots, C(q_{n+1})$, and Mochizuki’s cocycle invariant $\Phi_\theta(S)$ is as follows:

$$\Phi_\theta(S) = 3^{n+1},$$

which means there are $3^{n+1}$ possible ways of tricoloring.

Adding one 1-handle to $S$ means identifying $C(e_1)$ and $C(e_2)$ for some edges $e_1$ and $e_2$ in the diagram of $b$. Let $e_k$ be on the braid $b_{j_k}$ ($k = 1, 2$). If $e_1 = e_2$ or $e_1, e_2 \in \{e(q_i'), e(q_i'')\}$ or $\{e(q_{i+1}), e(q_{i+1}')\}$, then the cocycle invariant does not change. If not, then $C(q_{j_2}) = C(q_{j_2}')$ or $C(q_{j_2+1}) = C(q_{j_2+1}')$ is determined by the 1-handle, and the cocycle invariant becomes $3^n$. We have

$$\Phi_\theta(S') = \Phi_\theta(S) \text{ or } \Phi_\theta(S)/3,$$

where $S'$ is the resulting surface knot.

Hence we must apply at least $n$ 1-handle surgeries to deform $S$ to have Mochizuki’s cocycle invariant 3, which is a necessary condition for the trivial surface knot. $\square$

As a more concrete example, we can give an alternative proof of the fact that the spun (or the turned spun) $T^2$-knot of a torus $(p, q)$-knot has the unknotting number one:

**Proposition 5.7.** Let $S$ (resp.$\tau(S)$) be the spun (resp.turned spun ) $T^2$-knot of a torus $(p, q)$-knot. Then the unknotting number of $S$ and $\tau(S)$ is one.

**Remark.** Kanenobu and Marumoto showed in [26] that the unknotting number of the spun torus $(p, q)$-knot is one, which shows that the unknotting
number of the spun $T^2$-knot of a torus $(p, q)$-knot is also one.

proof. By Theorem 5.1, it suffices to show that the unknotting number of $S$ is one. We use the same notation as in the Proposition 5.3, that is:

\[ O_k = O(p; (p - 1 \searrow p - k)(p + 1 \nearrow p + k)) \]
\[ H_e = \bigcup_{k=0}^{p-1} O_k \]

for $k = 0, 1, \ldots, m - 1$. Note that for $k = 0, (p - 1 \searrow p - k)(p + 1 \nearrow p + k) = \emptyset$ and $O_k = O(p; \emptyset) = F_p$. Moreover, let us define $b$ and $O'_k$ as follows:

\[ b = (1 \nearrow p - 1)^q, \]
\[ O'_k = O(m; (m - 1 \searrow m - k)(m + 1 \nearrow m + k) \cdot b). \]

The oval nest $O'_k$ is obtained from $O_k$ by adding loops describing $b$ around it.

Since the $(p, q)$-torus knot is the closure of a braid $b = (\sigma_1 \cdot \sigma_2 \cdots \sigma_{p-1})^q$, the surface link chart $\Gamma_S$ obtained from $\Gamma_T$ is as follows:

\[ \Gamma_S = \bigcup_{i=0}^{m-1} O_i \cup \bigcup_{i=0}^{m-1} O'_i. \]

Remark that $\Gamma_S$ is a ribbon chart.

We show that $\Gamma_S$ can be deformed to an unknotted chart by adding a free edge.

(Step 1) Let us denote $q$ mod $p$ by $r \in \{0, 1, 2, \ldots, p - 1\}$. Let us define an element $\tau$ of the symmetry group $S_{p-1}$ by

\[ \begin{pmatrix} 1 & 2 & \cdots & r - 1 & r & r + 2 & \cdots & p - 1 \\ p - r + 1 & p - r + 2 & \cdots & p - 1 & p - r & 1 & \cdots & p - r + 1 \end{pmatrix}. \]

Observe that for $i \in \{0, 1, 2, \ldots, p - 1\} \setminus \{r\}$, $\sigma_i \cdot b = b \cdot \sigma_{\tau(i)}$.

(Step 2) We show that

\[ O_{p-k-1} \cup O_{p-k} \cup F_{2p-k} = O_{p-k-1} \cup F_k \cup F_{2p-k}, \]

for $k \in \{0, 1, 2, \ldots, p - 1\} \setminus \{r\}$.
By Lemma 5.4, we have

\[
O_{p-k-1} \cup O_{p-k} \cup F_{2p-k} = O(k + 1; (k + 2 \not\searrow p)(p + 1 \not\searrow 2p - k))
\]

\[
\cup O(k; (k + 1 \not\searrow p)(p + 1 \not\searrow 2p - k)) \cup F_{2p-k}
\]

\[
= O(k + 1; (k + 2 \not\searrow p)(p + 1 \not\searrow 2p - k))
\]

\[
\cup O(k; (k + 1 \not\searrow p)(p + 1 \not\searrow 2p - k)) \cup F_{2p-k}
\]

\[
= O(k + 1; (k + 2 \not\searrow p)(p + 1 \not\searrow 2p - k))
\]

\[
\cup O(k; (k + 2 \not\searrow p)(p + 1 \not\searrow 2p - k)) \cup F_{2p-k}
\]

\[
= O_{p-k-1} \cup F_k \cup F_{2p-k}.
\]

(Step 3) Similarly, we show that

\[
O'_{p-k-1} \cup O'_{p-k} \cup F_{\tau(k)} = O'_{p-k-1} \cup F_{2p-k} \cup F_{\tau(k)},
\]

for \(k \in \{1, 2, \ldots, p-1\} \setminus \{r\}\).

First, remark that

\[
O(k; b) = F_{\tau(k)}
\]

for \(k \in \{1, 2, \ldots, p-1\} \setminus \{r\}\). This is because \(\overline{b} \cdot k \cdot b = \overline{b} \cdot b \cdot \tau(k) = \tau(k)\).

Then by Lemma 5.4 and \(2p - k > p\), we have

\[
O'_{p-k-1} \cup O'_{p-k} \cup F_{\tau(k)}
\]

\[
= O(2p - k - 1; (2p - k - 2 \not\searrow p)(p - 1 \not\searrow k + 1) \cdot b)
\]

\[
\cup O(2p - k; (2p - k - 1 \not\searrow p)(p - 1 \not\searrow k) \cdot b)
\]

\[
\cup O(k; b)
\]

\[
= O'_{p-k-1} \cup O(2p - k; b) \cup O(k; b)
\]

\[
= O'_{p-k-1} \cup F_{2p-k} \cup F_{\tau(k)}.
\]

(Step 4) Let us denote Step 2 as follows:

\[
\phi_l : O_{l-1} \cup O_l \cup F_{p+l} \longrightarrow O_{l-1} \cup F_{p-l} \cup F_{p+l},
\]

and Step 3 as

\[
\psi_l : O'_{l-1} \cup O'_l \cup F_{\tau(p-l)} \longrightarrow O'_{l-1} \cup F_{p+l} \cup F_{\tau(p-l)}.
\]
We introduce several notations to make things easy to see. Let us define $F_l$, $F'_l$ and $F''_l$ as follows:

\[
F_l := F_{p-l}, \\
F'_l := F_{p+l}, \\
F''_l := F_{\tau(p-l)}.
\]

Then Step 2 is written as follows:

\[
\phi_l : O_{l-1} \cup O_l \cup F'_l \rightarrow O_{l-1} \cup F_l \cup F'_l,
\]
for $1 \leq l \leq p-1$, and Step 3 is

\[
\psi_l : O'_{l-1} \cup O'_l \cup F''_l \rightarrow O'_{l-1} \cup F'_l \cup F''_l,
\]
for $1 \leq l \leq p-1$ with $l \neq p-r$. Moreover, for an integer $m$, let us define $\tau_m$ to be

\[
\tau_m := p - \tau^{-m}(r).
\]

Then we have

\[
F_{\tau^{-m}(r)} = F_{\tau^{-m}(r)} = F''_{\tau^{-m}(r)} = F''_{\tau_m},
\]
and

\[
F_{2p-\tau^{-m}(r)} = F_{p+(p-\tau^{-m}(r))} = F_{p-\tau^{-m}(r)} = F'_{\tau_m}.
\]

We show that $\Gamma_S$ can be deformed to an unknotted chart by adding a free edge $F_r$. Let us define charts $I_0, I_1, \ldots, I_{2p-4}$. First, define $I_0$ as follows:

\[
I_0 := \Gamma_S \cup F_r \\
= O_0 \cup O_{\tau_1} \cup \cdots \cup O_{\tau_p} \cup O_{\tau_0} \\
O'_0 \cup O'_{\tau_1} \cup \cdots \cup O'_{\tau_p} \cup O'_{\tau_0} \\
\cup F_r.
\]

For $n \in \{1, 2, \ldots, p-2\}$, let us define $I_{2n}$ as follows:

\[
I_{2n} := O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_p} \cup O_{\tau_0} \\
\cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_p} \\
\cup F_{\tau_1} \cup \cdots \cup F_{\tau_n} \\
\cup F_r
\]

\[
= O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_p} \cup O_{\tau_0} \\
\cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_p} \\
\cup F'_{\tau_1} \cup \cdots \cup F'_{\tau_n} \\
\cup F_r.
\]
For $n \in \{0, 1, 2, \ldots, p - 3\}$, let us define $I_{2n+1}$ as follows:

$$
I_{2n+1} := O_0 \cup O_{\tau_0} \cup O_{\tau_0} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F^{\tau_0} \cup \cdots \cup F^{\tau_{n}} \cup O_{\tau_0} \cup O'_{\tau_{p-2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup F'_{\tau_0} \cup \cdots \cup F'_{\tau_{n}} \cup F'_{\tau_{n+1}} \cup F_r
$$

$$
= O_0 \cup O_{\tau_0} \cup O_{\tau_0} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F''_{\tau_0} \cup \cdots \cup F''_{\tau_{n+1}} \cup O'_{\tau_0} \cup O'_{\tau_{p-2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup F'_{\tau_0} \cup \cdots \cup F'_{\tau_{n}} \cup F'_{\tau_{n+1}} \cup F_r.
$$

We show that $I_{2n+1}$ (resp. $I_{2n+2}$) is obtained from $I_{2n}$ (resp. $I_{2n+1}$) by applying Step 3 (resp. Step 2) for $n = 0, 1, \ldots, p - 3$.

When we have $I_{2n}$, there is an integer $l_0 < \tau_{n+1}$ such that for any $l$ with $0 < l < \tau_{n+1}$, $O_l' \notin I_{2n}$ and $O_{l_0}' \in I_{2n}$. Since $r = \tau^0(r)$ and $F_r = F''_{r-1}$, we have

$$I_{2n} \supset F''_{l_0+1} \cup \cdots \cup F''_{\tau_{n+1}+1} \cup F'_{n+1} \cup \cdots \cup F'_{\tau_{n+1}+1} \cup O'_{\tau_{n+1}}.$$

By applying Steps 3 and its inverses, we can deform $O_{\tau_{n+1}}$ to $F'_{\tau_{n+1}}$, which is $I_{2n+1}$:

$$
\psi_{l_0+1} \circ \cdots \circ \psi_{\tau_{n+1}+1} \circ \psi_{\tau_{n+1}} \circ \phi_{\tau_{n+1}}^{-1} \circ \cdots \circ \psi_{l_0+2}^{-1} \circ \psi_{l_0+1} (I_{2n}) = I_{2n+1}.
$$

Similarly, we can deform $I_{2n+1}$ to $I_{2n+2}$ by applying Steps 2 and its inverses and deforming $O_{\tau_{n+1}}$ to $F''_{\tau_{n+1}}$:

$$
\phi_{l_0+1} \circ \cdots \circ \phi_{\tau_{n+1}+1} \circ \phi_{\tau_{n+1}} \circ \phi_{\tau_{n+1}}^{-1} \circ \cdots \circ \phi_{l_0+2}^{-1} \circ \phi_{l_0+1} (I_{2n+1}) = I_{2n+2}.
$$

Remark that here $l_0$ is the same integer used in deforming $I_{2n}$ to $I_{2n+1}$.

Since $\tau$ has order $p - 1$, by repeating Steps 3 and Steps 2 alternately $p - 2$ times each, we have

$$
I_{2(p-2)} = O_0 \cup O_{\tau_0} \bigcup_{m=1}^{p-2} F^{\tau_m}
$$

$$
= O_0 \cup O_{\tau_0} \cup \left( \bigcup_{m=1}^{p-2} F^{\tau_m} \right)
$$

$$
\cup F_r
$$

$$
= O_0 \cup O_{\tau_0} \cup \left( \bigcup_{m=1}^{p-2} F^{\tau_0} \right)
$$

$$
\bigcup_{k \neq p, 2p-r} F_k.
$$
On the other hand, $O_0 = F_p$. Hence, we have free edges of all labels except $2p - r$, using which we can deform the oval nest

$O_{p-r} = O(2p - r; (2p - r - 1 \setminus p)(p - 1 \setminus r))$

to the free edge $F_{2p-r}$.

Therefore $\Gamma_S \cup F_r$ can be deformed to contain $\bigcup_{k=1}^{2p-1} F_k$, using which we can deform $\Gamma_S \cup F_r$ to have only free edges, which is an unknotted chart. \( \square \)

**Remark.** Proposition 5.7 holds for a classical braid $b$ of degree $m$ with the following conditions (here we consider the prime integer $p$ as an arbitrary positive integer $m$):

- There exists some element $\tau$ of the symmetry group of degree $m - 1$, and
  - except for some $i$ and $\tau(i)$, $\sigma_i \cdot b = b \cdot \sigma_{\tau(i)}$
  - the order of $\tau$ is $m - 1$.

What classical braids satisfy the conditions in the above-mentioned remark?

**Proposition 5.8.** We consider classical braids whose degree are at least two. The classical braids which satisfy the conditions in the above remark are in the following forms:

1. $i_0^q(\Delta_p^{2m}) \cdot i_0^p(\Delta_q^{2n}) \cdot (\Theta_p^q \cdot \Theta_q^p)^l \cdot \Theta_p^q$,
2. $\sigma_1^{2m} \sigma_2^{2n} - \sigma_2 \sigma_1 \sigma_3 \sigma_2^{2l-1}$ or $\sigma_1^{2m} \sigma_2^{2n} \sigma_2 \sigma_1 \sigma_3 \sigma_2^{2l-1}$,

where the degree of the braid is $p + q$ in (1) and 4 in (2) and $m$, $n$ and $l$ are integers and $p$, $q$ are positive integers such that

- $p = 1, q > 1$,
- $p > 1, q = 1$ or
- $p > 1$ and $q > 1$ are coprime.

And

$$(\Delta_j)^2 = (\sigma_1 \sigma_2 \cdots \sigma_{j-1})^j$$

for $j = p$, $q$, which is, Garside’s $\Delta$ for $j$-braids, and

$$\Theta_p^q = \sigma_p \cdot \Pi_{p-1}^p \cdot \Pi_{q-1}^p \cdot \sigma_p \cdot \Pi_{p-2}^p \cdot \Pi_{q-2}^p \cdots \sigma_p \cdot \Pi_{p-q}^p$$
if \( p > q \), and
\[
\Theta_p^q = \sigma_p \cdot \Pi_{p-1}^p \cdot \Pi_{q-1}^p \cdot \sigma_p \cdot \Pi_{q-2}^p \cdot \Pi_{q-2}^p \cdots \sigma_p \cdot \Pi_{q-p}^p
\]
if \( q > p \), where \( \Pi_i^p \) and \( \Pi_i'^{p} \) is the notations introduced in Section 3, which are
\[
\Pi_i^p = \sigma_{p+1} \sigma_{p+2} \cdots \sigma_{p+i},
\]
\[
\Pi_i'^{p} = \sigma_{p-1} \sigma_{p-2} \cdots \sigma_{p-i}.
\]

proof. Let \( b \) be the classical braid of degree \( m \) which satisfies the conditions. If \( \sigma_i b = b \sigma_j \), we use the phrase that the generator \( \sigma_i \) is sent to the generator \( \sigma_j \) through \( b \). By the condition 1, at most one pair of the \( i \)-th and \((i + 1)\)-th strings do not send a generator \( \sigma_i \) to some generator through \( b \).

Let \( \phi(b) \) be an element of the symmetry group of degree \( m \) induced by the classical braid \( b \), and let \( b_\phi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\} \) be a map such that
\[
\phi(b) = \begin{pmatrix}
1 & 2 & \cdots & m \\
\phi(1) & \phi(2) & \cdots & \phi(m)
\end{pmatrix}.
\]
For any \( j \neq i \), the \( j \)-th and \((j + 1)\)-th strings come to some \( k \)-th and \((k + 1)\)-th or \( k \)-th and \((k - 1)\)-th strings through \( b \), i.e. \( \phi(b, j + 1) = (k, k + 1) \) or \((k, k - 1)\). If the first and second strings come to some \( k \)-th and \((k + 1)\)-th (resp. \( k \)-th and \((k - 1)\)-th) strings, then the third string comes to \((k + 2)\)-th (resp. \( k - 2 \)-th) string, and so on. Hence we see that
\[
b_\phi(1, 2, \ldots, i) = \begin{cases}
(k_1, k_1 + 1, \ldots, k_1 + i - 1) \\
(k_1, k_1 - 1, \ldots, k_1 - i + 1)
\end{cases}
\]
and
\[ b_\phi(i + 1, i + 2, \ldots m) = \begin{cases} (k_2, k_2 + 1, \ldots, k_2 + m - i - 1) \\ (k_2, k_2 - 1, \ldots, k_2 - m + i + 1) \end{cases} \]
for some \( k_1 \) and \( k_2 \).

Let \( b_1 \) (resp. \( b_2 \)) be a classical braid consisting of from the first to the \( i \)-th (resp. from the \((i + 1)\)-th to the \( m \)-th) strings of the classical braid \( b \). Let \( C_{b_1} := \{1, 2, \ldots, i\} \) and \( C_{b_2} := \{i + 1, i + 2, \ldots, m\} \). By the condition, in \( b \), \( b_1 \) and \( b_2 \) can be separated by their tubular neighborhoods and \( C_{b_1} \) and \( C_{b_2} \) are contained in the canonical reduction system of \( b \). We have two cases:

\[
\begin{cases}
  b_\phi(C_{b_1}) = C_{b_1}, & b_\phi(C_{b_2}) = C_{b_2}, \\
  b_\phi(C_{b_1}) = C_{b_2}, & b_\phi(C_{b_2}) = C_{b_1}.
\end{cases}
\]

(Case 1) If \( b_\phi(C_{b_1}) = C_{b_1} \) and \( b_\phi(C_{b_2}) = C_{b_2} \), then \( \text{cl}(b) \) has at least two components, which contradicts the condition that \( \text{cl}(b) \) is a knot.

(Case 2) If \( b_\phi(C_{b_1}) = C_{b_2} \) and \( b_\phi(C_{b_2}) = C_{b_1} \), \( b \) is in the following form:

\[
i_{n_2}^i(b_1) \cdot i_{n_1}^0(b_2) \cdot C,
\]

where \( n_1 := i \) and \( n_2 := m - i \), and \( b_j \) is a braid of degree \( n_j \) with \( \sigma_k b_j = b_j \sigma_{\tau_j(k)} \) for some \( \tau_j \) \((j = 1, 2)\) and every \( k \), and \( C = (\Theta_{n_1} \cdot \Theta_{n_2})^i \cdot \Theta_{n_1}^{n_2} \). By regarding each \( b_1 \) and \( b_2 \) as a string, we have the degree-two-braid \( \sigma_{2i+1}^2 \).

Since \( b_j \) \((j = 1, 2)\) sends every generator to some generator,

\[
b_1 = (\Delta_{n_1}^i)^{m_1}, \quad b_2 = (\Delta_{n_2}^i)^{m_2}
\]

for some \( m_1 \) and \( m_2 \), where \( \Delta_i^j \) is a braid of degree \( i \) which sends every generator \( \sigma_j \) to a generator \( \sigma_{i-j} \). For an even \( n = 2\nu \), \((\Delta_i^i)^n \) is a center element, hence \((\Delta_i^i)^n = (\Delta_i^i)^{2\nu}\) for Garside’s \( \Delta_i \). We see when the closure of the braid \( b \) has one component.

Here \( n_1 \) (resp. \( n_2 \)) corresponds to \( p \) (resp. \( q \)), and \( m_1 \) (resp. \( m_2 \)) corresponds to \( m \) (resp. \( n \)) in the statement of Proposition 5.8.

We consider the following cases:

(I) \( n_1 \neq n_2 \) and \( n_1, n_2 \geq 2 \),

(II) \( n_1 = 1 \) and \( n_2 \geq 2 \) or \( n_1 \geq 2 \) and \( n_2 = 1 \),

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(III) \( n_1 = n_2 \),

(IV) \( n_1 = 0 \).

(I) When \( n_1 \neq n_2 \) and \( n_1, n_2 \geq 2 \).

(I-1) If \( m_1 \) and \( m_2 \) are even, \( n_1 \) and \( n_2 \) are coprime.

We have

\[
\phi(b) = \begin{pmatrix}
1 & 2 & \cdots & n_1 & n_1 + 1 & n_1 + 2 & \cdots & n_1 + n_2 \\
n_2 + 1 & n_2 + 2 & \cdots & n_1 + n_2 & 1 & 2 & \cdots & n_2
\end{pmatrix}
\]

and

\[
b_{\phi}(l) = \begin{cases}
l - n_1 & \text{if } j > n_1 \\
l + n_2 & \text{if } j \leq n_1.
\end{cases}
\]

Then \( b_{\phi}^j(1) = 1 - j_1 n_1 + j_2 n_2 \) for some non-negative integers \( j_1 \) and \( j_2 \) with \( j_1 + j_2 = j \). Since \( \text{cl}(b) \) is a knot, \( b_{\phi}^{n_1+n_2}(1) = 1 \) and \( b_{\phi}^j(1) \neq 1 \) for \( j = 1, 2, \ldots, n_1 + n_2 - 1 \). Hence if non-negative integers \( k_1 \) and \( k_2 \) satisfy

\[
b_{\phi}^{n_1+n_2}(1) = 1 - k_1 n_1 + k_2 n_2 = 1,
\]

that is,

\[
k_1 n_1 = k_2 n_2,
\]

then \( k_1 \) and \( k_2 \) must satisfy

\[
k_1 + k_2 = n_1 + n_2.
\]

We can assume that \( n_1 < n_2 \). If \( n_2 = k n_1 \) for some positive integer \( k \), then (5.5) is \( k_1 n_1 = k_2 k n_1 \) and \( k_1 = k, k_2 = 1 \) satisfies (5.5). However,

\[
n_1 + n_2 = (1 + k)n_1 > 1 + k = k_1 + k_2,
\]

which contradicts (5.6). Hence \( n_1 \) and \( n_2 \) are coprime.

If \( n_1 \) and \( n_2 \) are coprime, since \( k_1 n_1 = k_2 n_2 \), we have \( k_1 = n_2 \) and \( k_2 = n_1 \), and \( \text{cl}(b) \) is a knot.

(I-2) There is no case when \( n_1 < n_2 \) and \( m_2 \) is odd or when \( n_1 > n_2 \) and \( m_1 \) is odd.

We can assume that \( n_1 > n_2 \) and \( m_1 \) is odd. Then \( \phi(b) \) is as follows:

\[
\begin{pmatrix}
1 & \cdots & j & \cdots & n_1 & n_1 + 1 & \cdots & n_1 + n_2 \\
n_1 + n_2 & \cdots & n_1 + n_2 - (j - 1) & \cdots & n_2 + 1 & * & \cdots & *
\end{pmatrix}.
\]
Then similarly to (I-1), since $n_2 + 1 \leq n_1$,

$$b_\phi^2(n_1) = b_\phi(n_2 + 1) = n_1 + n_2 - (n_2 + 1 - 1) = n_1.$$ 

This is a contradiction.

(I-3) There is no case when $n_1 < n_2$ and $m_1$ is odd and $m_2$ is even or when $n_1 > n_2$ and $m_1$ is even and $m_2$ is odd.

We can assume that $n_1 < n_2$ and $m_1$ is odd and $m_2$ is even. Then $\phi(b)$ is as follows:

$$
\begin{pmatrix}
1 & 2 & \cdots & n_1 & n_1 + 1 & n_1 + 2 & \cdots & n_1 + n_2 \\
n_1 + n_2 & n_1 + n_2 - 1 & \cdots & n_2 + 1 & 1 & 2 & \cdots & n_2
\end{pmatrix}.
$$

Then similarly to (I-1), we can show that $n_1$ and $n_2$ are coprime.

Let $k$ be an integer with $0 < k < n_1$ determined by $k := n_2 - ln_1$ for some positive integer $l$. Then we have

$$b_\phi^{l+2}(1) = k,$$

$$\phi(k) = n_1 + n_2 - k + 1 = n_1(l + 1) + 1 > n_1,$$

$$b_\phi^{l+1}(n_1(l + 1) + 1) = 1,$$

which means,

$$b_\phi^{2(l+2)}(1) = 1.$$

Since cl$(b)$ is a knot,

$$n_1 + n_2 = 2(l + 2).$$

Hence we have

$$k + (l + 1)n_1 = 2(l + 2),$$

$$k = 4 - n_1 - (n_1 - 2)l.$$

Since $n_1 \geq 3$ and $l \geq 1$, we have

$$k = 4 - n_1 - (n_1 - 2)l \leq 4 - 3 - 1 = 0,$$

which contradicts the assumption that $k > 0$.

(II) When $n_1 = 1$ and $n_2 \geq 2$ or $n_1 \geq 2$ and $n_2 = 1$. We can assume that $n_1 = 1$ and $n_2 \geq 2$. 

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(II-1) If $m_2$ is even, then

$$\phi(b) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n_2 + 1 \\ n_2 + 1 & 1 & 2 & \cdots & n_2 \end{pmatrix}.$$ 

Since $\phi(b)$ has order $n_2 + 1$, $\text{cl}(b)$ is a knot.

(II-2) There is no case when $m_2$ is odd. If $m_2$ is odd, then

$$\phi(b) = \begin{pmatrix} 1 & 2 & \cdots & n_2 & n_2 + 1 \\ n_2 + 1 & n_2 & \cdots & 2 & 1 \end{pmatrix}.$$ 

Then $b_\phi^2(1) = b_\phi(n_2 + 1) = 1$, which is a contradiction.

(III) When $n_1 = n_2$. Let us denote $n := n_1 = n_2$.

(III-1) If $n \geq 2$, then there are two cases: $n = 2$, $m_1$ is even and $m_2$ is odd, or, $n = 2$, $m_1$ is odd and $m_2$ is even.

By (I-1), there is no case when $m_1$ and $m_2$ are even. We can assume $m_1$ is odd. If $m_2$ is even, then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n + 1 & n + 2 & \cdots & 2n \\ 2n & 2n - 1 & \cdots & n + 1 & 1 & 2 & \cdots & n \end{pmatrix}.$$ 

Then we have

$$b_\phi^4(1) = b_\phi^3(2n) = b_\phi^2(n) = b_\phi(n + 1) = 1.$$ 

Hence, if $n > 2$, this is a contradiction. If $n = 2$, then $\phi(b)$ is order 4 and $\text{cl}(b)$ is a knot.

If $m_2$ is odd, then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n + 1 & n + 2 & \cdots & 2n \\ 2n & 2n - 1 & \cdots & n + 1 & n & n + 1 & \cdots & 1 \end{pmatrix}.$$ 

Then we have

$$b_\phi^2(1) = b_\phi(2n) = 1,$$ 

which is a contradiction.

(III-2) If $n = n_1 = n_2 = 1$, then $b = \sigma_1^{2k-1}$. This is a torus $(2, 2k - 1)$-knot.

(IV) If $n_1 = 0$, the classical braid $b$ is that of in (III-2).

If $m_2$ is even, the classical braid $b$ is a pure braid, which is a contradiction.

If $m_2$ is odd, then we have

$$b_\phi^2(1) = b_\phi(n_2) = 1. \quad (5.7)$$ 

If $n_2 = 2$, the classical braid $b$ is that of in (III-2). If $n_2 > 2$, this contradicts (5.7). 

\[ \Box \]
References


[38] T. Yajima, *On a characterization of knot groups of some spheres in $\mathbb{R}^4$*, Osaka J. Math. 6 (1969) 435–446