SURFACE LINKS WHICH ARE COVERINGS OF A TRIVIAL TORUS KNOT

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Abstract

We consider surface links in the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we call torus-covering-links. Torus-covering-links contain spun T^2-knots, turned spun T^2-knots, symmetry-spun tori and torus T^2-knots. In this paper we study the braid indices, the link groups, the unknotting numbers etc. of torus-covering-links.

0 Introduction

Locally flatly embedded closed 2-manifolds in the 4-space $\mathbb{R}^4$ are called *surface links*. It is known that any oriented surface link can be deformed to the closure of a simple surface braid, that is, a simple branched covering of the 2-sphere ([25]).

As surface knots of genus one which can be made from classical knots, there are *spun T^2-knots, turned spun T^2-knots, symmetry-spun tori* and *torus T^2-knots*. Consider $\mathbb{R}^4$ as obtained by rotating $\mathbb{R}^3_+$ around the boundary $\mathbb{R}^2$. Then a *spun T^2-knot* is obtained by rotating a classical knot ([4]), a *turned spun T^2-knot* by turning it once while rotating ([4]), a *symmetry-spun torus* by turning a classical knot with periodicity rationally while rotating ([36]), and a *torus T^2-knot* is a surface knot on the boundary of a neighborhood of a solid torus in $\mathbb{R}^4$ ([20]). Symmetry-spun tori include spun T^2-knots, turned spun T^2-knots and torus T^2-knots. We call the link version of a symmetry-spun torus, a spun T^2-knot, and a turned spun T^2-knot a *symmetry-spun T^2-link*, a *spun T^2-link* and a *turned spun T^2-link* respectively. We enumerate several properties of symmetry-spun T^2-links.

(0.1) If a symmetry-spun T^2-link is ribbon, then it is a spun T^2-link (cf. [36]).

(0.2) The turned spun T^2-link of a non-trivial classical link is not ribbon (cf. [4], [35]).

(0.3) A symmetry-spun T^2-link is pseudo-ribbon and has the triple point number zero.

(0.4) A symmetry-spun T^2-link has a classical link group (cf. [36]).

Now we consider surface links in the 4-sphere or the 4-space which can be deformed to simple branched coverings of a trivial torus knot, which we will define as *torus-covering-links* (see Definition 2.1). Until Proposition 2.7, we consider torus-covering-links in S^4, and from Theorem 2.8 throughout this paper we consider torus-covering-links in $\mathbb{R}^4$. By definition, a torus-covering-link is described by a *torus-covering-chart*, which is a chart on the trivial torus knot. Torus-covering-links include symmetry T^2-links (and spun T^2-links, turned spun T^2-links, and torus T^2-links). A torus-covering-link has no 2-knot component. Each component of a torus-covering-link is of genus at least one.

There are several natural questions.
Is there a surface knot of genus one which is not a torus-covering-knot?
What are the properties of torus-covering-links and knots?
What difference from symmetry-spun T^2-links do torus-covering-links have?

For the first question we show that "*There are surface knots of genus one which are not torus-covering-knots*" (Corollary 4.11).

In Section 2 we define torus-covering-links (Definition 2.1), the *turned torus-covering-links* (Definition 2.4) and the *normal forms* of ribbon torus-covering-knots of genus one (Theorem 2.8). Moreover we show that "*There is a torus-covering-knot with positive triple point number*" (Theorem 2.11 (cf. (0.3)), which means there is a torus-covering-knot which is not pseudo-ribbon.

In Section 3 we deform the torus-covering-link associated with a torus-covering-chart of degree m to the closure of a simple surface braid, from which we obtain its (surface link) chart description, which is of degree $2m$, and give an upper estimate of its braid index. In particular, we see that the turned spun T^2-knot of the torus $(2, p)$-knot has the braid index four.

In Section 4 we consider torus-covering-links which are T^2-links, that is, torus-covering-links whose each component is of genus one. Such torus-covering-links are associated with torus-covering-charts without black vertices. We study link groups of torus-covering-links associated with torus-covering-charts without black vertices and show that "*There are infinitely many torus-covering-links with two components such that each component* *is of genus one and the link groups are not classical link groups*" (Proposition 4.5 and Theorem 4.6 (cf. (0.4) and (0.1))), which means that they are not symmetry-spun T^2-links. Moreover, they are ribbon, which means that there are infinitely many torus-covering-links which are ribbon but not spun T^2-links. We show its knot version as well: "*There are infinitely many torus-covering-knots of genus one and whose knot groups are not classical link groups*" (Theorem 4.9), which means that they are not symmetry-spun T^2-knots. Moreover, they are ribbon, which means that there are infinitely many torus-covering-knots which are ribbon but not spun T^2-knots.

In Section 5 we study the unknotting numbers of torus-covering-links. In particular, we give an alternative proof of the fact that the spun (or the turned spun) T^2-knot of a classical (p, q)-torus knot has the unknotting number one (Proposition 5.7 (cf. [26])).

The author would like to thank professors Takashi Tsuboi and Elmar Vogt for suggesting this problem, and professor Akio Kawauchi for advising this title.

1 Definitions and Preliminaries

Definition 1.1. A locally flatly embedded closed 2-manifold in S^4 or $\mathbb{R}^4$ is called a *surface link*. A surface link with one component is called a *surface knot*. A surface link whose each component is of genus zero (resp. one) is called a 2-*link* (resp. T^2-*link*). In particular a surface knot of genus zero (resp. one) is called a 2-*knot* (resp. T^2-*knot*).

An orientable surface link F is *trivial* (or *unknotted*) if there is an embedded 3-manifold M with $\partial M = F$ such that each component of M is a handlebody.

An oriented surface link F is called *pseudo-ribbon* if there is a surface link diagram of F whose singularity set consists of double points and *ribbon* if F is obtained from a trivial 2-link F_0 by 1-handle surgeries along a finite number of mutually disjoint 1-handles attaching to F_0. By definition, a ribbon surface link is pseudo-ribbon.

Two surface links are *equivalent* if there is an ambient isotopy or an orientation-preserving diffeomorphism of S^4 or $\mathbb{R}^4$ which deforms one to the other.

Definition 1.2. A compact and oriented 2-manifold S embedded properly and locally flatly in $D_1^2 \times D_2^2$ is called a *braided surface* of degree m if S satisfies the following conditions:

(i) $\mathrm{pr}_2|_S : S \longrightarrow D_2^2$ is a branched covering map of degree m,

(ii) ∂S is a closed m-braid in $D_1^2 \times \partial D_2^2$, where D_1^2, D_2^2 are 2-disks, and $\mathrm{pr}_2 : D_1^2 \times D_2^2 \to D_2^2$ is the projection to the second factor.

A braided surface S is called a *surface braid* if ∂S is the trivial closed braid. Moreover, S is called *simple* if every singular index is two.

Two braided surfaces are *equivalent* if there is a fiber-preserving ambient isotopy of $D_1^2 \times D_2^2$ rel $D_1^2 \times \partial D_2^2$ which carries one to the other.

There is a theorem which corresponds to Alexander's theorem for classical oriented links.

Theorem 1.3 (Kamada [25]). *Any oriented surface link can be deformed by an ambient isotopy of $\mathbb{R}^4$ to the closure of a simple surface braid.*

There is a *chart* which represents a simple surface braid.

Definition 1.4. Let m be a positive integer, and Γ be a graph on a 2-disk D_2^2. Then Γ is called a *surface link chart* of degree m if it satisfies the following conditions:

(i) $\Gamma \cap \partial D_2^2 = \emptyset$.

(ii) Every edge is oriented and labeled , and the label is in $\{1, \dots, m-1\}$.

(iii) Every vertex has degree 1, 4, or 6.

(iv) At each vertex of degree 6, there are six edges adhering to which, three consecutive arcs oriented inward and the other three outward, and those six edges are labeled i and $i+1$ alternately for some i.

(v) At each vertex of degree 4, the diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

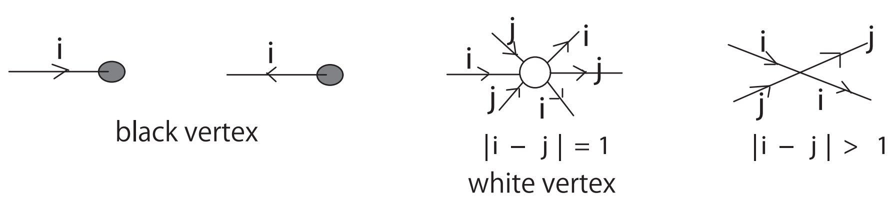

Fig. 1.1

Vertices of degree 1 and 6 are called a *black vertex* and a *white vertex*.

A black vertex (resp. white vertex) of a chart corresponds to a branch point (resp. triple point) of the simple surface braid associated with the chart.

An edge without end points is called a *loop*. An edge whose end points are black vertices is called a *free edge*, and a configuration consisting of a free edge and a finite number of concentric simple loops such that the loops are surrounding the free edge is called an *oval nest*.

An *unknotted chart* is a chart presented by a configuration consisting of free edges. A trivial oriented surface link is presented by an unknotted chart ([25]).

A *ribbon chart* is a chart presented by a configuration consisting of oval nests. A ribbon surface link is presented by a ribbon chart ([25]).

A chart with a boundary represents a simple braided surface.

Definition 1.5. Two charts in D_2^2 of the same degree are *C-move equivalent* if we can deform one to the other by a finite sequence of ambient isotopies of D_2^2 and *chart moves* (or *C-moves*).

Let Γ and Γ' be two charts in D_2^2 of the same degree. Then Γ' is said to be obtained from Γ (or Γ is said to be obtained from Γ') by a *chart move* of type I, II or III, or by a *CI-move*, *CII-move* or *CIII-move* if there exists a 2-disk E in D_2^2 such that the loop ∂E is in general position with respect to Γ and Γ' and $\Gamma \cap (D_2^2 - E) = \Gamma' \cap (D_2^2 - E)$ and the following condition holds:
(CI) There are no black vertices in $\Gamma \cap E$ nor $\Gamma' \cap E$. Any CI-move can be presented by a finite sequence of CI-moves of types (1), (2),. . . , and (7) (cf. [37]). Fig. 1.2 gives CI-moves of types (1), (2), . . . , and (7).

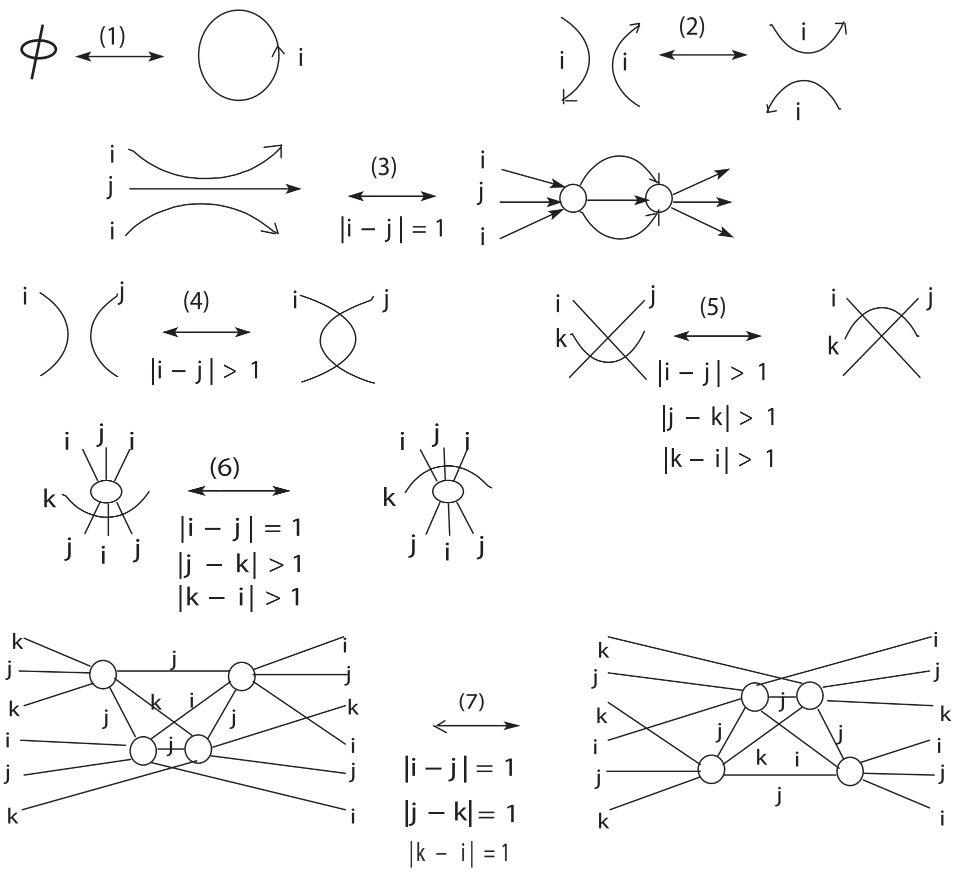

Fig. 1.2. CI-moves of types (1), (2), and (3)

(CII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.3, where $|i - j| > 1$.

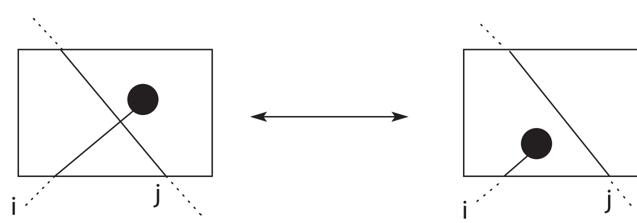

Fig. 1.3. CII-moves

(CIII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 1.4, where $|i - j| = 1$.

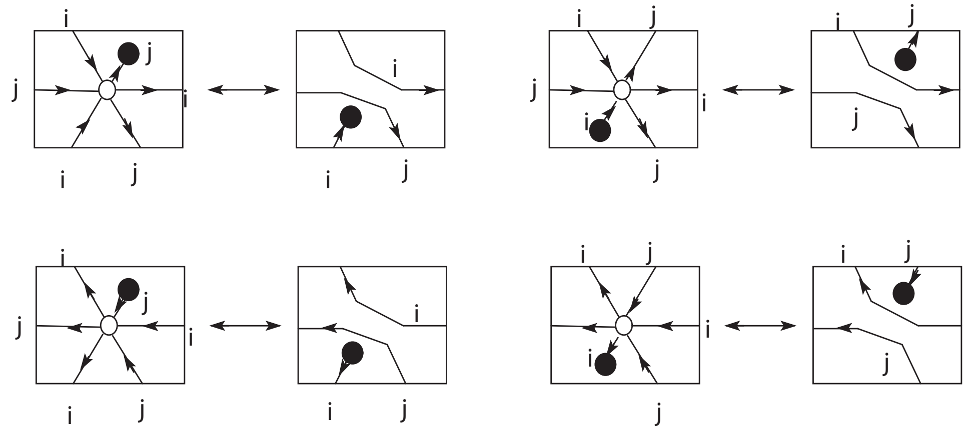

Fig. 1.4. CIII-moves

Theorem 1.6 ([25]). *Two charts of the same degree are C-move equivalent if and only if their associated simple braided surfaces are equivalent.*

Throughout this paper, let us denote the oval nest with a free edge with the label i and its surrounding loops with the labels $i_1, \dots, i_n$ and the orientation $\epsilon_1, \dots, \epsilon_n$ from the free edge outward by $O(i; i_1^* i_2^* \cdots i_n^*)$, where $\epsilon_j = \pm 1$ and $i_j^* = i_j$ (resp. $\overline{i}_j$) if $\epsilon_j = +1$ (resp. -1). In particular, let us denote the free edge $O(i; \emptyset)$ by F_i. For $0 < i < j$, let us denote $i(i+1)\cdots j$ (resp. $\overline{i}(\overline{i+1})\cdots\overline{j}$) by $i \nearrow j$ (resp. $\overline{i} \nearrow \overline{j}$), and for $0 < j < i$, let us denote $i(i-1)\cdots j$ (resp. $\overline{i}(\overline{i-1})\cdots\overline{j}$) by $i \searrow j$ (resp. $\overline{i} \searrow \overline{j}$). Moreover, let us denote a disjoint union of charts by $\cup$.

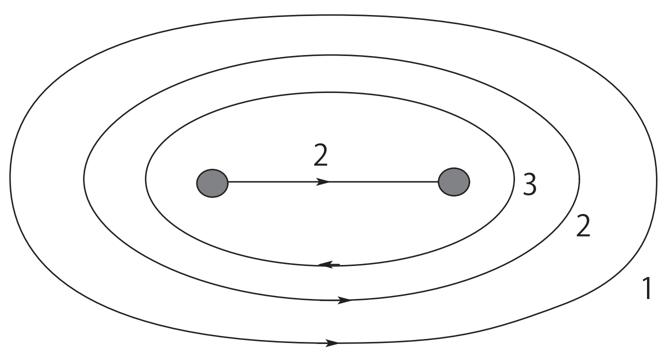

Fig. 1.5. An oval nest $O(2; \overline{3}21)$

Let us define the braid group relations between two sequences of integers as follows:

1. $\emptyset = i \cdot \bar{i} = \bar{i} \cdot i$, for a positive integer i,

2. $i \cdot j = j \cdot i$, for positive integers i, j with $|i - j| > 1$,

3. $i \cdot j \cdot i = j \cdot i \cdot j$, for positive integers i, j with $|i - j| = 1$.

Then for positive integers i, j and two sequences of integers b and b' which are braid group equivalent, the following holds:

(1.7.1) $O(i; b) = O(i; b')$,

(1.7.2) $O(i, i) = O(i; \emptyset) = F_i$ (Fig. 1.6),

(1.7.3) $O(i; j) = O(i; \emptyset) = F_i$, where $|i - j| > 1$ (Fig. 1.7),

(1.7.4) $O(i; j) = O(j; \bar{i})$, where $|i - j| = 1$ (Fig. 1.8),

(1.7.5) $O(i; b \cdot j) \cup F_j = O(i; b) \cup F_j$,

(1.7.6) $O(i; j \cdot b) \cup O(j; b) = O(i; b) \cup O(j; b)$.

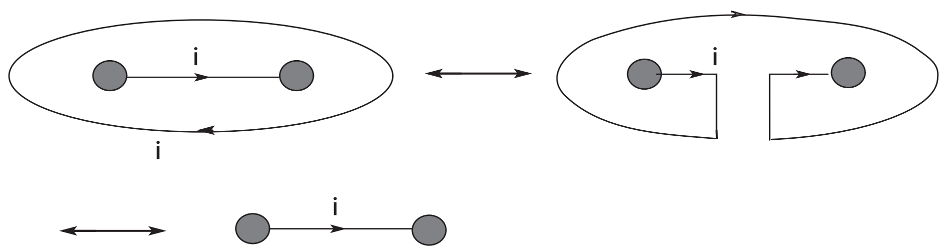

Fig. 1.6

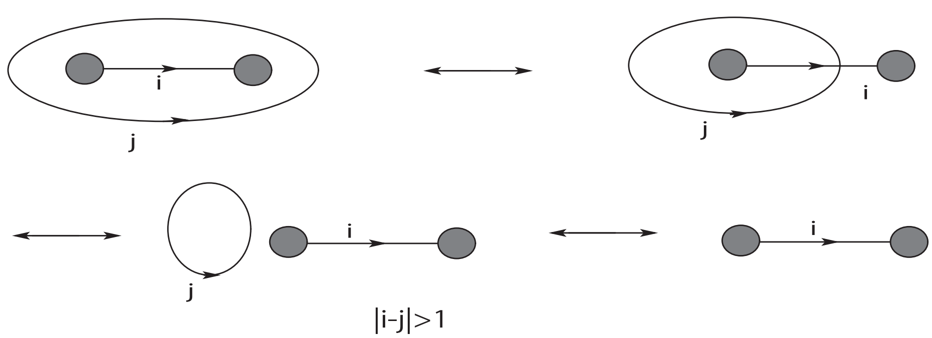

Fig. 1.7

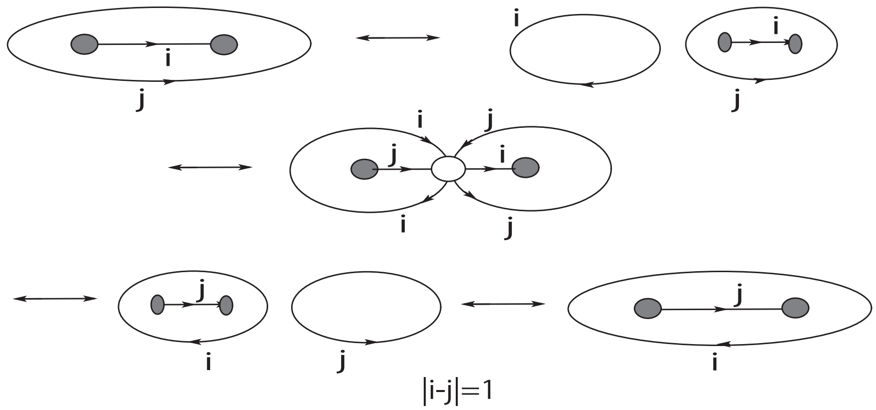

Fig. 1.8

2 Torus-covering-links

In this section, we greatly rely on [36]. Until Proposition 2.7, we consider torus-covering-links in S^4, and from Theorem 2.8 throughout this paper we consider torus-covering-links in $\mathbb{R}^4$.

Definition 2.1. (Torus-covering-links) First, embed $D^2 \times S^1 \times S^1$ into S^4 or $\mathbb{R}^4$ naturally, or more precisely, consider as follows (cf. [36], [4] and [29]). Let $S^1 \times S^1$ be a standardly embedded torus in S^4 and let $D^2 \times S^1 \times S^1$ be a tubular neighborhood of $S^1 \times S^1$ in S^4. We can assume that its framing is canonical, that is, the homomorphism induced by the inclusion map $H_1\left(\{0\} \times S^1 \times S^1;\ \mathbb{Z}\right) \to H_1(\{p\} \times S^1 \times S^1;\ \mathbb{Z}) \to H_1(S^4 - S^1 \times S^1;\ \mathbb{Z})$ where $p \in \partial D^2$, is zero. Let $\bar{l} = \partial D^2 \times 0 \times 0$, $\bar{s} = 0 \times S^1 \times 0$, and $\bar{r} = 0 \times 0 \times S^1$ be curves on $\partial D^2 \times S^1 \times S^1$.

Let $E^4 = \mathrm{cl}(S^4 - D^2 \times S^1 \times S^1)$. Let l, s and r be canonical curves on ∂E^4, which are identified with $\bar{l}$, $\bar{s}$ and $\bar{r}$ under the natural identification map $i = \partial D^2 \times S^1 \times S^1 \to \partial E^4$. Then l, r and s represent a basis of $H_1(\partial E^4; \mathbb{Z})$.

Let $f : \partial E^4 \to E^4$ be a diffeomorphism with $f_\#(\ l \quad s \quad r\ ) = (\ l \quad s \quad r\ )A^f$, where $A^f \in GL(3, \mathbb{Z}) \cong \pi_0 \mathrm{Diff}(\partial E^4)$. Then f can be extended to a diffeomorphism $\tilde{f} : E^4 \to E^4$ if and only if $A^f \in H$, where

$$H = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ * & \alpha & \gamma \\ * & \beta & \delta \end{pmatrix} \in GL(3, \mathbb{Z});\ \alpha + \beta + \gamma + \delta \equiv 0 \pmod 2 \right\}.$$

Consider $D^2 \times S^1 \times S^1$ to be embedded in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$ or $\mathbb{R}^4 = (E^4 - \{*\}) \cup_i (D^2 \times S^1 \times S^1)$ for some point $*$ in $\mathrm{Int}E^4$. Then identify $D^2 \times S^1 \times S^1$ with $D^2 \times I_3 \times I_4/\sim$, where $(x, 0, v) \sim (x, 1, v)$ and $(x, u, 0) \sim (x, u, 1)$ for $x \in D^2$, $u \in I_3 = [0, 1]$ and $v \in I_4 = [0, 1]$.

Let us consider a surface link S embedded in $D^2 \times S^1 \times S^1$ such that $S \cap (D^2 \times I_3 \times I_4)$ is a simple braided surface. We will call such a surface link a *torus-covering-link*.

A torus-covering-link S can be described by a chart on the trivial torus knot, i.e. by a chart Γ_T on $D_2^2 = I_3 \times I_4$ with $\Gamma_T \cap (I_3 \times \{0\}) = \Gamma_T \cap (I_3 \times \{1\})$ and $\Gamma_T \cap (\{0\} \times I_4) = \Gamma_T \cap (\{1\} \times I_4)$. Let us denote the classical braids described by $\Gamma_T \cap (I_3 \times \{0\})$ and $\Gamma_T \cap (\{0\} \times I_4)$ by Γ_T^v and Γ_T^h respectively. We will call Γ_T a *torus-covering-chart with boundary braids* Γ_T^v *and* Γ_T^h.

Let $b(\Gamma_T)$ be the number of black vertices in the torus-covering-chart Γ_T. Then let us consider the case $b(\Gamma_T) = 0$. In this case the torus-covering-link associated with Γ_T is determined by the boundary braids Γ_T^v and Γ_T^h. We will call such a Γ_T a torus-covering-chart *without black vertices and with boundary braids* Γ_T^v *and* Γ_T^h.

Remark. In the case $b(\Gamma_T) = 0$, the boudary braids Γ_T^v and Γ_T^h are commutative.

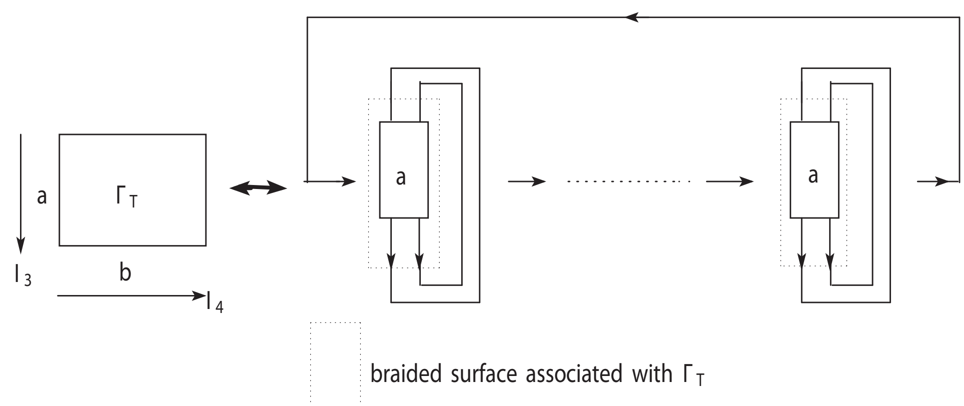

Fig. 2.1. A torus-covering-link

By definition, torus-covering-links contain symmetry-spun tori (and spun T^2-knots, turned spun T^2-knots and torus T^2-knots).

As we stated in Theorem 1.6, if there are two surface link charts of the same degree, their associated surface links are equivalent if and only if their charts are C-move equivalent. It follows that if two torus-covering-charts are C-move equivalent, their associated torus-covering-links are equivalent.

A torus-covering-link has no 2-knot component. In particular, if a torus-covering-chart has no black vertices, then each component of the associated torus-covering-link is of genus one.

Let Γ_T be a torus-covering-chart of degree m and with the trivial boundary braids. Let F be the surface link associated with Γ_T by assuming Γ_T to be a surface link chart. Then the torus-covering-link associated with the torus-covering-chart Γ_T is obtained from F by applying m trivial 1-handle surgeries.

Example 2.2. (2.2.1) Let Γ_T be a torus-covering-chart without black vertices and with boudary braids σ_1^3 and e (the trivial braid), then the torus-covering-knot Γ_S associated with Γ_T is the spun T^2-knot of a right-handed trefoil.

(2.2.2) Let Γ_T be a torus-covering-chart without black vertices and with boudary braids σ_1^3 and σ_1^3 (or σ_1^3 and σ_1^{-3}), then the torus-covering-knot Γ_S associated with Γ_T is the turned spun T^2-knot of a right-handed trefoil.

(2.2.3) Let Γ_T be a torus-covering-chart without black vertices and with boudary braids β^2 and β, then the torus-covering-knot Γ_S associated with Γ_T is a symmetry-spun torus.

Proposition 2.3 (cf. [36]). *Torus-covering-links in S^4 obtained from a torus-covering-chart Γ_T by rotating it by $n\pi/2$ ($n \in \mathbb{Z}$) are equivalent.*

proof. It suffices to show the case when rotating Γ_T by $\pi/2$.

Let Γ'_T be the torus-covering-chart obtained by rotating Γ_T by $\pi/2$, and S and S' be the torus-covering-link obtained by Γ_T and Γ'_T respectively. Let us use the notation in Definition2.1. We can assume the torus-covering-links S and S' to be in $D^2 \times S^1 \times S^1$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Then there is a diffeomorphism

$$f : (D^2 \times S^1 \times S^1) \longrightarrow (D^2 \times S^1 \times S^1)$$
$$f(x, t_1, t_2) = (x, -t_2, t_1).$$

The diffeomorphism $f|_{\partial D^2 \times S^1 \times S^1}$ can be considered as an orientation-preserving diffeomorphism from ∂E^4 to E^4. Then we have

$$A^f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \in H \ .$$

Hence there is an orientation-preserving diffeomorphism $\tilde{f} : E^4 \to E^4$. Let $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$. Then $g : S^4 \to S^4$ such that $g|_{E^4} = \tilde{f}$ and $g|_{D^2 \times S^1 \times S^1} = f$ is an orientation-preserving diffeomorphism which deforms S to S', therefore S and S' are equivalent. $\square$

Remark. Teragaito proved in [36] the same theorem in the symmetry-spun version. Lemma 7 in [36] corresponds to the $-\pi/2$ rotation. We will consider the *turned torus-covering-links*, which include the turned spun T^2-links (cf. [36], [4]).

Definition 2.4. Let us use the notation in Definition 2.1. Let $\sigma : \partial E^4 \to \partial E^4$ be a diffeomorphism of a matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then $E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1$ is diffeomorphic to S^4.

Let Γ_T be a torus-covering-chart and let S be the torus-covering-link associated with Γ_T in $S^4 = E^4 \cup_i D^2 \times S^1 \times S^1$, Then we can consider the torus-covering-link obtained from S by changing the identification map i to σi, which we will call the *turned torus-covering-link associated with* (S, Γ_T) *or* S, and use the notation $\tau(S, \Gamma_T)$ or $\tau(S)$. Moreover, we will denote by $\Gamma_{\tau(T)}$ the torus-covering-chart associated with $\tau(S)$. That is, we define $\tau(S)$ as follows:

$$(S,\ E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1 \cong S^4) = (\tau(S),\ S^4 = E^4 \cup_i D^2 \times S^1 \times S^1).$$

The turned torus-covering-link S must be in the form associated with the torus-covering-chart Γ_T, and for two equivalent torus-covering-links S and S' with their associated torus-covering-charts Γ_T and Γ'_T, their turned torus-covering-links $\tau(S, \Gamma_T)$ and $\tau(S', \Gamma'_T)$ may be different (Proposition 2.7).

Remark. For a spun T^2-knot S, $\tau(S)$ is the turned spun T^2-knot.

By this definition, we have the following theorems.

Theorem 2.5. *Let S be the torus-covering-link associated with a torus-covering-chart Γ_T, and $\tau(S)$ be the turned torus-covering-link obtained from S. Then the torus-covering-chart $\Gamma_{\tau(T)}$ associated with $\tau(S)$ can be described as in Fig. 2.2.*

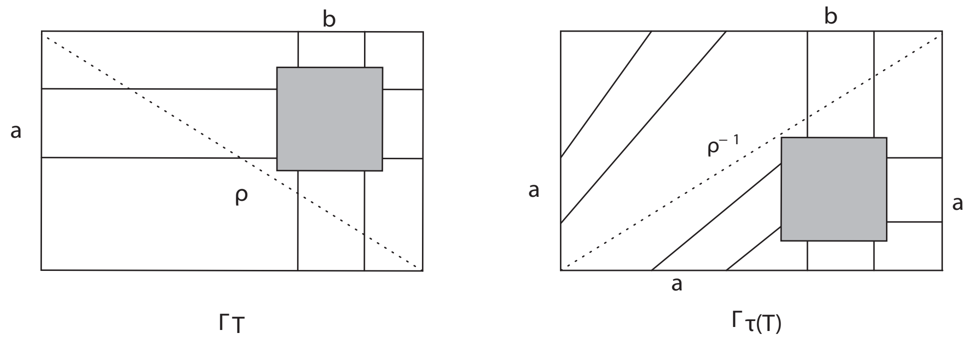

Fig. 2.2

proof. Let $\bar{l},\ \bar{r},\ \bar{s}$ and $l,\ r,\ s$ be as in Definition2.1. Remark that $\bar{r}$ or $\bar{s}$ is a meridian or a longitude on the trivial torus knot. Let $\bar{\rho}$ be a path on the trivial torus knot with $\bar{\rho} = \bar{r} - \bar{s}$, and let $\rho = r - s$. Then for the diffeomorphism $f : \partial E^4 \to \partial E^4$ with $f_\#(\begin{array}{ccc} l & s & r \end{array}) = (\begin{array}{ccc} l & s & \rho \end{array})$,

$$A^f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \sigma^{-1}.$$

By the definition,

$$(S,\ E^4 \cup_{\sigma i} D^2 \times S^1 \times S^1 \cong S^4) = (\tau(S),\ S^4 = E^4 \cup_{\sigma^{-1}(\sigma i)} D^2 \times S^1 \times S^1)\ .$$

Hence, regarding the torus-covering-charts as graphs on the trivial torus knot, $\Gamma_{\tau(T)}$ is obtained from Γ_T by cutting Γ_T by ρ. Hence we obtain the torus-covering-chart as in Fig. 2.2. $\square$

Proposition 2.6 (cf. [36], [4]). *For any torus-covering-link S, $\tau^2(S) = S$.*

proof. Because

$$\sigma^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in H,$$

there is an orientation-preserving diffeomorphism $g : S^4 \to S^4$ with $g(S) = g(\tau^2(S))$. $\square$

Proposition 2.7. *The map τ and the $\pi/2$ rotation f are not compatible.*

proof. Consider a torus-covering-chart of degree 2 without black vertices and with boundary braids e and σ_1^3. Its associated torus-covering-knot S is the spun T^2-knot of a right-handed trefoil. Then $\tau \circ f(S)$ is represented by a torus-covering-chart with boundary braids σ_1^3 and e, which is equivalent to the spun T^2-knot of the trefoil. On the other hand, $f \circ \tau(S)$ is represented by a torus-covering-chart with σ_1^3 and σ_1^3, which is the turned spun T^2-knot of the trefoil. By (0.2), the spun T^2-knot and the turned spun T^2-knot of a non-trivial knot are not equivalent. Hence, τ and f are not compatible. $\square$

From now on throughout this paper, we consider torus-covering-links in $\mathbb{R}^4$.

Let us consider when torus-covering-links are ribbon. Before stating Theorem 2.8, we give several definitions about classical braids ([12]). Let B_n be the braid group of degree n, Δ, Garside's half twist, and D_n, an n-punctured disk.

We say that an element $b \in B_n$ is *periodic* if the element of $B_n/\langle\Delta^2\rangle$ represented by b is of finite order.

We say an element $b \in B_n$ is *reducible* if there exists a nonempty multicurve C in D_n (i.e. a system of disjoint simple closed curves in D_n, none of them isotopic to the boundary or enclosing a single puncture) which is stabilized by b, i.e. such that $b(C)$ is isotopic to C. Note that b may permute different components of the multicurve C.

The following definition is taken from [3] (see also [18]). To every reducible braid $b \in B_n$ one can associate a canonical invariant multicurve: its *canonical reduction system*, which by definition is the collection of all isotopy classes c of simple closed curves which have the following two properties: first, c must be stabilized by some power of b, and secondly any simple closed curve which has non-zero geometric intersection number with c must *not* be stabilized by any power of b.

Theorem 2.8. *Let Γ_T be a torus-covering-chart without black vertices and with boundary braids a and b, and let S be the associated torus-covering-link. Then if S is a torus-covering-knot (i.e. with one component) which is a ribbon surface knot of genus one, Γ_T satisfies one of the following conditions. Let us use the phrase that the ribbon torus-covering-knot S of genus one is in* the normal form*.*

(1) The closed braid $\mathrm{cl}(a)$ *is a classical knot and $b = e$ (the trivial braid), which means S is a spun T^2-knot of the classical knot* $\mathrm{cl}(a)$*.*

(2) (i) The boundary braids a and b have the same canonical reduction

sysetm, and each element of the canonical reduction system of a and b consists of the same number of punctures of the disk D^2. Let us denote the corresponding tubular braid of a and b by $R(a)$ and $R(b)$ respectively.

(ii) The tubular braid $R(a)$ is the trivial braid of some degree. Let n be the degree. Let $R_1(a), R_2(a), \ldots, R_n(a)$ be parts of the braid a corresponding to each string of the trivial tubular braid $R(a)$.

(iii) Then the closure of the corresponding tubular braid $\mathrm{cl}(R(b))$ is a knot.

(iv) For each $R_j(a)$, its closure $\mathrm{cl}(R_j(a))$ is a trivial knot and for one $R_j(a)$, the relation $ab = ba$ determines the other $R_j(a)$'s.

(3) The turned version of (2).

proof. Let S be a torus-covering-knot which is a ribbon surface knot of genus one. Then S can be written by a torus-covering-chart Γ_T without black vertices.

Let $X := \mathbb{R}^4 - S$. Since S is ribbon, there is a smooth simple closed curve α in ∂X which is null-homologous in X. Then there exists a smooth, properly-embedded disk z in X with $\partial z = \alpha$. We can cut Γ_T along some paths l and l' and obtain a new torus-covering-chart Γ'_T with boundary braids a and b such that the closed braid $\mathrm{cl}(a)$ is a split sum of the trivial knot and a link. The trivial knot corresponds to the smooth simple closed curve α. Let us denote by S' the torus-covering-knot obtained from Γ'_T. Then by Proposition 2.3 and Proposition 2.6, S is equivalent to either S' or $\tau(S')$.

If the closed braid $\mathrm{cl}(a)$ is a trivial knot (one component), there is a solid torus whose boundary is S i.e. S' or $\tau(S')$ (Paste a disk whose boundary is $\mathrm{cl}(a)$ for every page $\mathbb{R}^3_+ \times \{\theta\}$, where $\mathbb{R}^4 = \mathbb{R}^3_+ \times S^1/\sim$). If a is the trivial braid, S is a symmetry-spun torus, which is either a spun T^2-knot or a turned spun T^2-knot Because a turned spun T^2-knot of a classical non-trivial knot is not ribbon ((0.2)), S is a spun T^2-knot.

If the classical braid a is non-trivial and $\mathrm{cl}(a)$ has more than one component, a is a reducible braid and non-periodic. By [18], since a and b are commutative, the classical braid b preserves the canonical reduction system of a, hence a and b have the same canonical reduction system.

Hence, if each element of the canonical reduction system of a and b doesn't consist of the same number of punctures of the disk D^2, S has more than one component, which is a contradiction.

Let us consider the case when each element of the canonical reduction system of a and b consists of the same number of punctures of the disk D^2. Let us denote the corresponding tubular braid of a and b by $R(a)$ and $R(b)$ respectively. The tubular braid $R(a)$ is the trivial braid of some degree. Let n be the degree. Let $R_1(a), R_2(a), \ldots, R_n(a)$ be parts of the braid a corresponding to each string of the trivial tubular braid $R(a)$.

Since S consists of one component, $R(b)$ must be a knot. We can see that for each $R_j(a)$, its closure $\mathrm{cl}(R_j(a))$ is the trivial knot and for one $R_j(a)$, the relation $ab = ba$ determines the other $R_j(a)$'s. $\square$

Example 2.9. Let Γ_T be a torus-covering-chart with boundary braids $\sigma_1\sigma_3$ and Δ^2, where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$, which is Garside's Δ. Then Γ_T is in the normal form of (2). Its boundary braids and the associated tubular braids are as in Fig. 2.3 (cf. Proposition 4.5, Theorems 4.6 and 4.9).

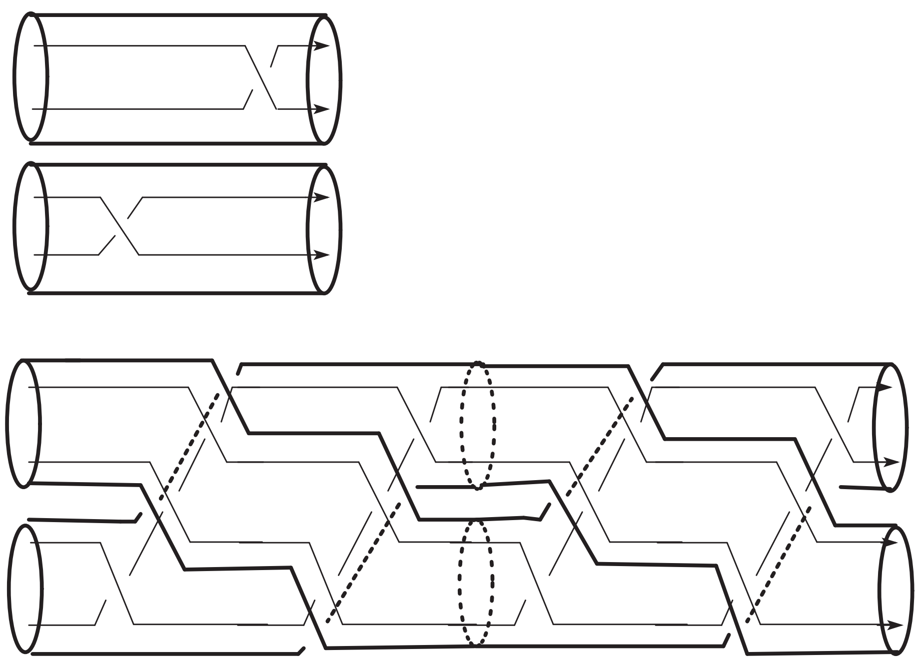

Fig. 2.3

Remark. Spun T^2-knots are ribbon, and torus-covering-knots in the normal form (2) are also ribbon (Propositioon 2.10). However, whether torus-covering-knots in the normal form (3) are ribbon or not is not known. We conjecture that it is not ribbon.

Proposition 2.10. *Torus-covering-knots in the normal form (2) are ribbon.*

proof. Let S be a torus-covering-knot in the normal form (2). We use the notation in Theorem 2.8.

Let S be embedded in $D^2 \times S^1 \times S^1 = D^2 \times I_3 \times I_4/\sim$, which is naturally embedded in the four space $\mathbb{R}^4$. The four-space $\mathbb{R}^4$ can be considered as $\mathbb{R}^3 \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}_+ \times S^1)/\sim \times \mathbb{R}$, with $(x_1, 0, \theta) \sim (x_1, 0, \theta')$ for $x_1 \in \mathbb{R}$, $\theta, \theta' \in S^1$, and $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$ is embedded in $\mathbb{R}^4$ as follows:

$$(I_1 \times S^1) \times S^1 \times I_2 \subset ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/\sim \times \mathbb{R},$$

with $I_1 \times S^1 \subset \mathbb{R} \times \mathbb{R}_+$ and $I_2 \subset \mathbb{R}$. The projection $p : I_1 \times S^1 \times S^1 \times I_2 \to I_1 \times S^1 \times S^1$ is a generic projection in the neighborhood of the trivial torus knot associated with the torus covering chart Γ_T, and the projection $\pi : \mathbb{R}^4 = ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/\sim \times \mathbb{R} \to \mathbb{R}^3$ with $\pi(x_1, t_1, t_2, x_2) = (x_1, t_1, t_2)$ is a generic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$. Hence $\pi(S)$ is obtained from Γ_T, that is, by embedding $p(S)$ into $\mathbb{R}^3$ naturally.

Let us divide I_4 into $I_k (k = 1, 2, \ldots, \nu)$, where $I_k = [t_{k-1}, t_k]$ for $0 = t_0 < t_1 < \cdots < t_\nu = 1$. Let $R(b) \subset D^2 \times \{0\} \times I_4 (= D^2 \times \{1\} \times I_4)$, the corresponding tubular braid of b, be presented as follows:

$$R(b) = \sigma_{i_1}^{\epsilon_1} \cdots \sigma_{i_\nu}^{\epsilon_\nu}$$

such that $R(b) \cap D^2 \times \{0\} \times I_k = \sigma_{i_k}^{\epsilon_k}$ for each k. Let $I'_k := (t_{k-1} + \epsilon, t_k - \epsilon)$ for a sufficiently small $\epsilon > 0$.

First, we show that for each k, $\pi(S \cap D^2 \times I_3 \times I'_k)$ can be deformed to have only ribbon singularities except self intersections (i.e. intersections in the same component) by an ambient isotopy of $\mathbb{R}^4$ rel $D^2 \times S^1 \times (S^1 - I_k)$. It suffices to show when $R(b)$ is degree 3 and $S \cap (D^2 \times I_3 \times I'_k) = \sigma_2$. The corresponding tubular braid of a, $R(a)$, is the trivial braid. Let $\widehat{R_j}$ $(j = 1, 2, 3)$ be the closed braids in the form of $\mathrm{cl}\big(R_j(a)\big)$. Each $\widehat{R_j}$ is the trivial knot and we denote the trivial knot in the form of $\mathrm{cl}(e)$ by O_j $(j = 1, 2, 3)$, where e is the trivial braid of degree one. Let $\{h_u^{(j)}\}$ be an ambient isotopy of B^3 with $h_1^{(j)}(\widehat{R_j}) = O_j$, where B^3 is a 3-ball which contains $D^2 \times S^1 = I_1 \times I_2 \times I_3/\sim$ such that $B^3 = I'_1 \times I'_2 \times I'_3$ with $I_1 \subset I'_1$ and $I_2 \subset I'_2$.

Before deformation, $S \cap (D^2 \times I_3 \times I'_k) = \sigma_2$ is as follows.

We are in the 3-ball B^3 and proceed along time $I_k = [t_{k-1}, t_k] \subset I_4$. There is the closed braid $\widehat{R_1} \cup \widehat{R_2} \cup \widehat{R_3}$ in the 3-ball $B^3 \times \{t_{k-1}\}$, where $\cup$

means a split sum. The component $\widehat{R_1}$ is innermost and $\widehat{R_3}$ is outermost. Each component can be deformed to the other by an ambient isotopy of its tubular neighborhood $B' \times S^1$ in B^3, where B' is a 2-disk. As the time proceeds, $\widehat{R_2}$ and $\widehat{R_3}$ change their places, and $\widehat{R_2}$ (resp. $\widehat{R_3}$) is deformed to the form of $\widehat{R_3}$ (resp. $\widehat{R_2}$). The surface link diagram $\pi(S)$ is obtained by seeing the link diagram of $\widehat{R_1} \cup \widehat{R_2} \cup \widehat{R_3}$ in $\pi'(B^3) = \pi'(I'_1 \times I'_2 \times I'_3) = I'_1 \times I'_3$ proceeding along the time, where π' is the projection.

We can deform $S \cap (D^2 \times I_3 \times I'_k)$ as follows. Let $B^3 := D^2 \times I_2$, where $D^2 := I_1 \times I_3$. Let $B_{j,i}, B_{j,o}$ $(j = 1, 2, 3)$ be 3-balls such that $B_{1,i} \subset B_{1,o} \subset B_{2,i} \subset B_{2.o} \subset B_{3,i} \subset B_{3,o} \subset B^3$ and $B_{j,i} = D_{j,i} \times I_2$, $B_{j,o} = D_{j,o} \times I_2$ with $D_{1,i} \subset D_{1,o} \subset D_{2,i} \subset D_{2,o} \subset D_{3,i} \subset D_{3,o} \subset D^2$. Let the closed braid $\widehat{R_j}$ be in $B_{j,o} - B_{j,o}$. From t_{k-1} to $t_{k-1} + \epsilon'$ the closed braid $\widehat{R_1} \cup \widehat{R_2} \cup \widehat{R_3}$ is unchanged. Then we unknot the innermost component $\widehat{R_1}$ to the trivial knot O_1 by the ambient isotopy $\{h_u^{(1)}\}$. Move O_1 into $B^3 - B_{2,o}$. Since O_1 is the trivial knot with no crossings, the singular point set consists only of ribbon singularities. Unknot the innermost $\widehat{R_2}$ to the trivial knot O_2 by the ambient isotopy $\{h_u^{(2)}\}$. Move O_2 into $B^3 - B_{3,o}$, and take new 3-balls $B'_{2,i} = D'_{2,i} \times I_2$ and $B'_{2,o} = D'_{2,o} \times I_2$ with $D_{2,i} \subset D'_{2,o} \subset D^2 - D_{3,o}$ and $O_2 \subset B'_{2,o} - B'_{2,i}$. Unknot the innermost $\widehat{R_3}$ to the trivial knot O_3 by the ambient isotopy $\{h_u^{(3)}\}$. Now $\pi'(O_1 \cup O_2 \cup O_3)$ are located parallelly. Return O_2 to $\widehat{R_3}$ by $\{h_{1-u}^{(3)}\}$. Move $O_1 \cup O_3$ into $B'_{2,i}$, and take new 3-balls $B'_{3,i} = D'_{3,i} \times I_2$ and $B'_{3,o} = D'_{3,o} \times I_2$ with $D_{3,i} \subset D'_{3,o} \subset D^2 - D_{3,o} - O_1$ and $O_3 \subset B'_{3,o} - B'_{3,i}$. Return O_3 to $\widehat{R_2}$ by $\{h_{1-u}^{(2)}\}$. Move O_1 into $B'_{3,i}$. Return O_1 to $\widehat{R_1}$ by $\{h_{1-u}^{(1)}\}$.

In general, unknot the innermost closed braid and move it outside, and repeat this step to have n unknots laid parallel to each other. Re-braid the new outermost closed braid and move the other unknots inside it. Repeat this process to have the new closed braids.

Next, we show that S can be deformed to have only ribbon singularities. Take $s_k^1 < \ldots < s_k^n < {s'_k}^n < \ldots < {s'_k}^1$ such that $t_{k-1} < s_k^1 < {s'_k}^1 < t_k$ and in $B^3 \times \{t\}$ for $t \in [{s'_k}^1 + \epsilon, s_{k+1}^1 - \epsilon]$, the innermost component is in the form of the closed braid $\widehat{R_1}$, and for $t \in [s_k^1, {s'_k}^1]$ the innermost component is in the form of the trivial closed braid of degree one O_1 for sufficiently small $\epsilon > 0$. . In general, for $t \in [{s'_k}^i + \epsilon, s_{k+1}^i - \epsilon]$ the i-th component is in the form of the i-th closed braid, and for $t \in [s_k^i, {s'_k}^i]$ the i-th component is in the form of the trivial closed braid of degree one.

Unknot the innermost component of S in $D^2 \times [s'^{\,1}_k, s^1_{k+1}]$ to $O_1 \times [s'^{\,1}_k, s^1_{k+1}]$ by unknotting the closed braid $\widehat{R_1}$ in $[s'^{\,1}_k, s^1_{k+1}]$ by an ambient isotopy of $\{h^{(1)}_u\}$. There appear no new singular points. Then Unknot the second innermost component of S in $D^2 \times [s'^{\,2}_k, s^2_{k+1}]$ by unknotting the closed braid $\widehat{R_2}$ in $[s'^{\,2}_k, s^2_{k+1}]$ by an ambient isotopy of $\{h^{(2)}_u\}$. The new singular points are ribbon, for in $D_2 \times [s'^{\,2}_k, s^2_{k+1}]$, there is only $O_1 \times [s'^{\,2}_k, s^2_{k+1}]$. Repeat this process and we have only ribbon singularities in $D^2 \times [s'^{\,n}_k, s^n_{k+1}]$.

Repeat this step for every k, and we have new S whose surface link diagram have only ribbon singularities. □

The triple point number of a surface link F is the minimum number of triple points of surface diagrams of F. Symmetry-spun T^2-links have the triple point number zero (cf. (0.3)). However, there is a torus-covering-knot whose triple point number is positive.

Theorem 2.11. *Let Γ_T be the torus-covering-chart of degree 4 without black vertices and boundary braids $\sigma_1\sigma_2^3\sigma_3$ and $(\sigma_1\sigma_2\sigma_3)^4$. Then the torus-coverig-knot S in $\mathbb{R}^4$ obtaind from the torus-covering-chart Γ_T has positive triple point number, which means torus-covering-knots contain non-pseudo-ribbon knots.*

We use tri-coloring by the dihedral quandle of order three R_3. We tri-color the torus-covering-chart and show that Mochiziki's 3-cocycle invariant has not an integer value. Then it has at least four white vertices (cf. [34]).

We use the following facts.

(2.11.1) Let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be a generic projection. Then a *surface diagram* of F is the image $\pi(F)$ with additional crossing information at the singularity set. There are two intersecting sheets along each edge, one of which is higher than the other with respect to π. They are called an *over-sheet* and an *under sheet* along the edge, respectively. In order to indicate crossing information, we break the under-sheet into two pieces missing the over sheet. This can be extended around a triple point. The sheets are called a *top sheet*, a *middle sheet*, and a *bottom sheet* from the higher one. Then the surface diagram is presented by a disjoint union of compact surfaces which are called *broken sheets*. For a surface diagram D, we denote by $B(D)$ the set of broken sheets of D.

(2.11.2) A set X with a binary operation $* : X \times X \to X$ is called a *quandle* if it satisfies the following conditions:

1. $a * a = a$, for every a in X.

2. for any $b, c \in X$, there exists a unique $c \in X$ such that $a = c * b$.

3. $(a * b) * c = (a * c) * (b * c)$, for $a, b, c \in X$.

We denote by R_3 the dihedral quandle of order 3, which is the set $\{0, 1, 2\}$ with the binary operation $a*b = 2b-a \pmod 3$. A tricoloring for a surface diagram D is a map $C : B(D) \to R_3$ such that

$$C(H_1) * C(H_2) = C(H_1')$$

along every edge of D, where H_2 is the over-sheet and H_1 (resp. H_1') is the under-sheet such that the normal vector of H_2 points from (resp. toward) it. Then the *color* of the edge is the pair $(C(H_1), C(H_2))$. Remark that the color of the edge which ends with a branch point is (a, a) for some $a \in R_3$, for $a * a = a$.

(2.11.3) At a triple point of a surface diagram, there exist broken sheets $J_1, J_2, J_3 \in B(D)$ uniquely (some of which possibly coincide) such that J_3 is the top sheet and J_2 is a middle sheet and J_3 is a bottom sheet such that the normal vector of J_2 (resp. J_3) points from J_1 (resp. J_2). Assume D is tricolored by C. Then the *color of the triple point* is the triplet $(C(J_1), C(J_2), C(J_3)) \in R_3 \times R_3 \times R_3$. The *sign* of the triple point is *positive* (resp. *negative*) if the triplet of the normal vectors of J_1, J_2, J_3 is right-handed (resp. left-handed).

This corresponds to the following.

(2.11.4) (Proposition 4.43 (3) in [9]) The color of a white vertex representing $\sigma_i\sigma_j\sigma_i \to \sigma_j\sigma_i\sigma_j$ is (a, b, c), where a, b and c are the tricolors of the starting points of the i'-th, $(i'+1)$-th , and $(i'+2)$-th strings of the classical braid $\sigma_i\sigma_j\sigma_i$, where $i' = \min\{i, j\}$. The triple point (or white vertex) as above is positive (resp. negative) if $j > i$ (resp. $i > j$), that is, if there is exactly one edge with the largest (resp. smallest) label oriented towards the triple point.

(2.11.5) Mochizuki's 3-cocycle is a map $\theta : R_3 \times R_3 \times R_3 \to \mathbb{Z}_3 = \langle\, t \mid t^3 = 1 \,\rangle$ such that

$$\theta(x, y, z) = t^{(x-y)(y-z)z(x+z)} \in \mathbb{Z}_3.$$

(2.11.6) Let D be a surface diagram of an oriented surface link F whose triple points are $\tau_1, \ldots, \tau_s$ with the signs $\epsilon_i = \epsilon(\tau_i)$. For a tricoloring C for D, let $(a_i, b_i, c_i) \in R_3 \times R_3 \times R_3$ be the color of τ_i $(i = 1, \ldots, s)$.

Let us define

$$W_\theta(\tau_i; C) := \theta(a_i, b_i, c_i)^{\epsilon_i} \in \mathbb{Z}_3$$

for each triple point τ_i, and

$$W_\theta(C) := \prod_{i=1}^{s} W_\theta(\tau_i; C) \in \mathbb{Z}_3$$

for the tricoloring C. Since the set of broken sheets of D is finite, so is the set of tricolorings for D. This $W_\theta(C)$ is called the *Boltzman weight*. Let $C_1, \ldots, C_n$ be the tricolorings for D. Let us define $\Phi_\theta(D)$ as follows:

$$\Phi_\theta(D) := \sum_{j=1}^{n} W_\theta(C_j) \in \mathbb{Z}\langle t \mid t^3 = 1\rangle.$$

Since $\Phi_\theta(D)$ satisfies certain conditions, it does not depend on the choice of a surface diagram D of F. This $\Phi_\theta(D)$ is called a *Mochizuki's cocycle invariant* of F, and we use the notation $\Phi_\theta(F)$.

Remark. For the cocycle invariant, there is a general theory (cf.[30]).

proof of Theorem 2.11. Let S be the torus-covering-knot associated with the torus-covering-chart Γ_T. First, we show that we can compute Mochizuki's cocycle invariant by tricoloring Γ_T. Let S be embedded in $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$, which is naturally embedded in $\mathbb{R}^4$. The four-space $\mathbb{R}^4$ can be considered as $\mathbb{R}^3 \times \mathbb{R} = (\mathbb{R} \times \mathbb{R}_+ \times S^1)/\sim \times \mathbb{R}$, with $(x_1, 0, \theta) \sim (x_1, 0, \theta')$ for $x_1 \in \mathbb{R}$, $\theta, \theta' \in S^1$, and $D^2 \times S^1 \times S^1 = I_1 \times I_2 \times S^1 \times S^1$ is embedded in $\mathbb{R}^4$ as follows:

$$(I_1 \times S^1) \times S^1 \times I_2 \subset ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/\sim \times \mathbb{R},$$

with $I_1 \times S^1 \subset \mathbb{R} \times \mathbb{R}_+$ and $I_2 \subset \mathbb{R}$. The projection $p : I_1 \times S^1 \times S^1 \times I_2 \to I_1 \times S^1 \times S^1$ is a generic projection in the neighborhood of the trivial torus knot associated with the torus covering chart Γ_T and the projection $\pi : \mathbb{R}^4 = ((\mathbb{R} \times \mathbb{R}_+) \times S^1)/\sim \times \mathbb{R} \to \mathbb{R}^3$ with $\pi(x_1, t_1, t_2, x_2) = (x_1, t_1, t_2)$ is a generic projection from $\mathbb{R}^4$ to $\mathbb{R}^3$. Hence $\pi(S)$ is obtained from Γ_T, that is, by embedding $p(S)$ into $\mathbb{R}^3$ naturally. Therefore it suffices to tricolor the torus-covering-chart Γ_T and show that the Mochizuki's 3-cocycle invariant does not have an integer value.

Part of the torus-covering-chart without black vertices and with boundary braids σ_1^{-1} and $(\sigma_1\sigma_2\sigma_3)^4$ contain four white vertices. Denote these by $\tau_{11}^1, \ldots, \tau_{14}^1$ from left to right (Fig. 2.4).

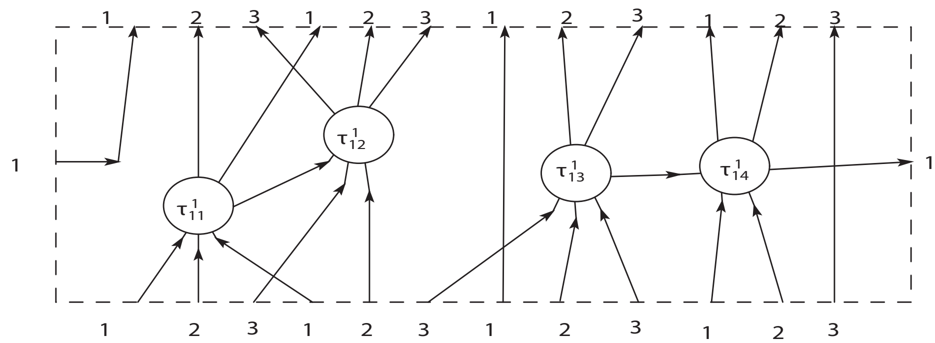

Fig. 2.4

Tricolor the classical braid $(\sigma_1\sigma_2\sigma_3)^4$ which is on the upper horizontal boundary of the chart as in Fig. 2.5.

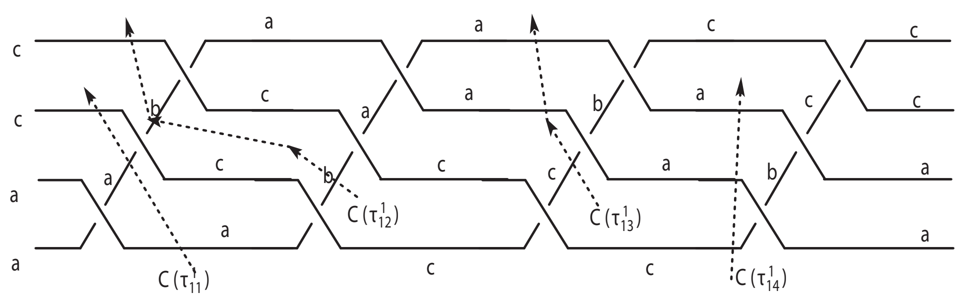

Fig. 2.5

By (2.11.4), the color of each triple point is determined by reading the colors along the dotted path in Fig. 2.5, and its sign is determined by (2.11.4).

Similarly, we have white vertices τ^2_{ij} ($i = 2,3,4,\ j = 1,2,3,4$), and $\tau^3_{51},\ldots,\tau^3_{54}$. Fig. 2.6 shows the white vertices τ^2_{ij} and their colors when $i = 2$.

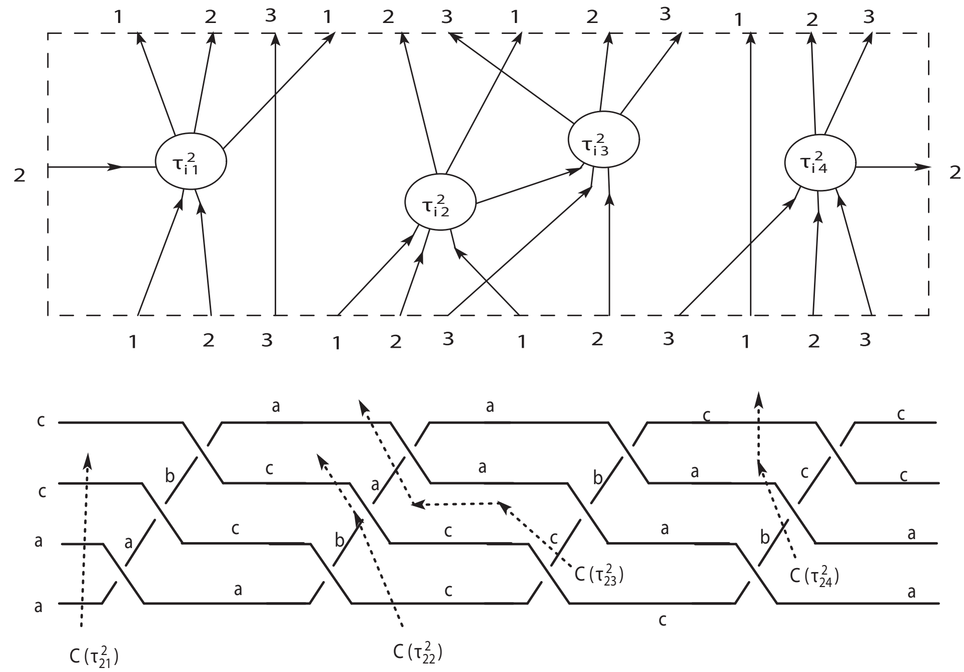

Fig. 2.6

Remark that the color of σ_2^3 whose start points and end points have the same color of that of Fig. 2.7 is as follows.

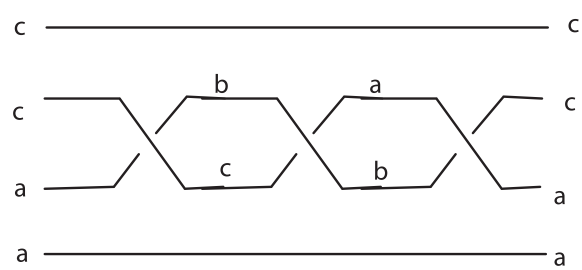

Fig. 2.7

The matrix describing the sign and color of the white vertices are as follows, where its (i, j)-element is describing τ_{ij}^*, and $\{a, b, c\} = \{0, 1, 2\}$, $\{0\}$, $\{1\}$ or

$\{2\}$, which are all the possible tricolorings:

$$\begin{pmatrix} +(a,a,c) & +(b,b,c) & -(c,a,a) & -(c,a,a) \\ -(a,a,c) & +(c,b,c) & +(c,a,a) & -(b,a,c) \\ -(a,c,b) & +(b,a,c) & +(a,b,a) & -(b,c,b) \\ -(a,b,a) & +(a,c,c) & +(b,c,a) & -(b,b,a) \\ -(a,c,c) & -(a,c,c) & +(c,c,a) & +(b,b,a) \end{pmatrix}.$$

Let us compute the Boltzman weight $W_\theta(C)$:

$$\begin{aligned} W_\theta(C) &= \theta(c,b,c) \cdot \theta(a,c,b)^{-1} \cdot \theta(b,c,b)^{-1} \cdot \theta(b,c,a) \\ &= t^{(c-b)(b-c)c \cdot 2c}\ t^{-(a-c)(c-b)b(a+b)}\ t^{-(b-c)(c-b)b \cdot 2b}\ t^{(b-c)(c-a)a(a+b)} \\ &= t^{2(b-c)^3(b+c)}\ t^{-(b+a)(b-a)(a-c)(c-b)}. \end{aligned}$$

For each coloring, we have

$$W_\theta(C) = \begin{cases} t^2 & \{a,b,c\} = \{0,1,2\} \\ 1 & \{a,b,c\} = \{0\},\{1\},\{2\}. \end{cases}$$

Hence we have $\Phi_\theta(S) = 3 + 6t^2 \notin \mathbb{Z}$ in $\mathbb{Z}[t]/\langle t^3 = 1 \rangle$. □

A surface knot F is called *(-)-amphicheiral* if $F = -F^*$. Similarly to Theorem 2.11, by using $-\Gamma_T$ and $-\Gamma_T^*$,we can see that for torus-covering-knots $-S$ and $-S^*$, $\Phi_\theta(-S) = \Phi_\theta(-S^*) = 3 + 6t \neq \Phi_\theta(S)$. Therfore we have

Corollary 2.12. *There is a torus-covering-knot which is not (-)-amphicheiral.*

Remark. Spun T^2-knots and turned spun T^2-knots are (-)-amphicheiral.

3 Deforming torus-covering-links to coverings of the 2-sphere

For a (classical) braid β, let $\iota_k^l(\beta)$ be the braid obtained from β by adding k (resp. l) trivial strings before(resp.after) β, and
$\Pi_i^m = \sigma_{m+1}\sigma_{m+2}\cdots\sigma_{m+i}$, $\Pi_i'^m = \sigma_{m-1}\sigma_{m-2}\cdots\sigma_{m-i}$,
$\Delta_m = \Pi_{m-1}^m \Pi_{m-2}^m \cdots \Pi_1^m$, $\Delta_m' = \Pi_{m-1}'^m \Pi_{m-2}'^m \cdots \Pi_1'^m$,
$\Theta_m = \sigma_m \cdot \Pi_{m-1}'^m \cdot \Pi_{m-1}^m \cdot \sigma_m \cdot \Pi_{m-2}'^m \cdot \Pi_{m-2}^m \cdots \sigma_m \cdot \Pi_1'^m \cdot \Pi_1^m \cdot \sigma_m$.

Remark. Let Δ be Garside's Δ for the braid group B_m. Then $\iota_m^0(\Delta) = \Delta_m$ (cf. [11]).

Theorem 3.1. *Let Γ_T be a torus-covering-chart of degree m with boudary braids a and b. Then the torus-covering-link associated with Γ_T can be described by a surface link chart Γ_S of degree $2m$ as in Fig. 3.1, where H_b is a chart describing the simple braided surface as follows:*

$$
\begin{aligned}
\iota_0^m(b) &\longrightarrow \iota_0^m(b)\cdot(\Delta_m')^{-1}\cdot\Delta_m^{-1}\cdot\Delta_m'\cdot\Delta_m \overset{\cdot}{\longrightarrow} \iota_0^m(b)\cdot(\Delta_m')^{-1}\cdot\Delta_m^{-1}\cdot\Theta_m\\
&\longrightarrow (\Delta_m')^{-1}\cdot\Delta_m^{-1}\cdot\iota_0^m(\bar{b}^*)\cdot\Theta_m \longrightarrow (\Delta_m')^{-1}\cdot\Delta_m^{-1}\cdot\Theta_m\cdot\iota_m^0(\bar{b}^*)\\
&\overset{\cdot}{\longrightarrow} (\Delta_m')^{-1}\cdot\Delta_m^{-1}\cdot\Delta_m'\cdot\Delta_m\cdot\iota_m^0(\bar{b}^*) \longrightarrow \iota_m^0(\bar{b}^*),
\end{aligned}
$$

where $\longrightarrow$ means an isotopy transformation and $\overset{\cdot}{\longrightarrow}$ a hyperbolic transformation along bands corresponding to the m σ_m's (Fig. 3.2), and $-(H_b)^*$ is the orientation-reversed mirror image of H_b, and $\bar{b}^*$ is the braid obtained from the classical braid b by taking its mirror image and reversing all the crossings (Fig. 3.3).

Definition 3.2. Let us call H_b the *1-handle chart* of Γ_T, and its corresponding braided surface the *1-handle braided surface* of Γ_T.

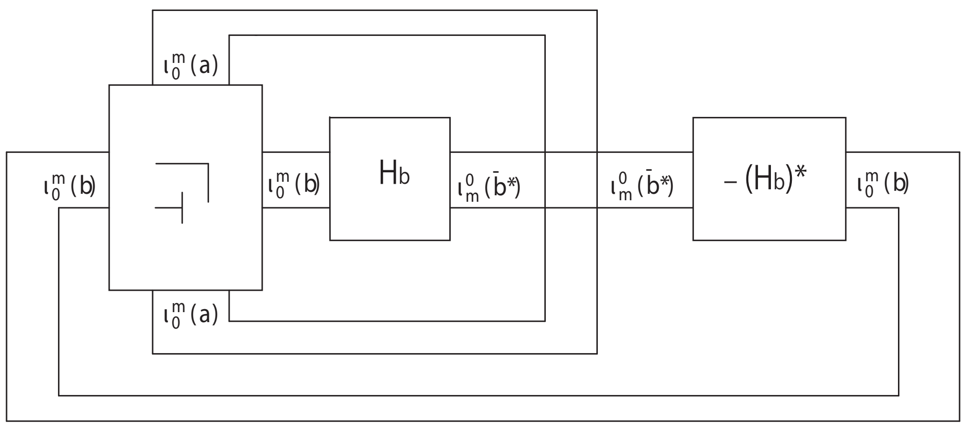

Fig. 3.1. The surface link chart Γ_S

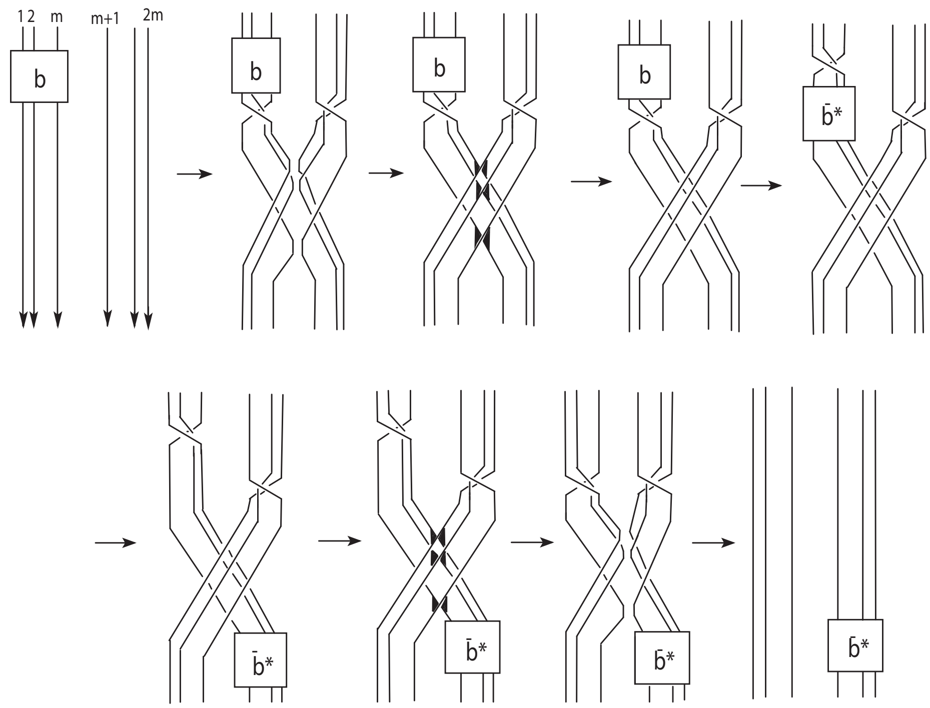

Fig. 3.2. The 1-handle braided surface of Γ_T

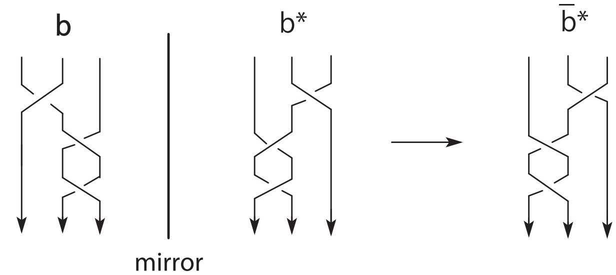

Fig. 3.3. The classical braid $\bar{b}^*$

Remark. The surface link chart Γ_S is of degree $2m$ and well-defined, for the edges representing $\iota_0^m(a)$ have labels at most $m-1$ and the edges representing $\iota_m^0(\bar{b}^*)$ have labels at least $m+1$. Note the 1-handle chart H_b has $2m$ black vertices.

proof. (Step 1) Let us consider a trivial torus knot T^2 in $\mathbb{R}^4$ as the result of 1-handle surgeries of $S_1 \cup S_2$ along $h_1 \cup h_2$, where S_1 (resp. S_2)

is a 2-sphere in $\mathbb{R}^4$ with a positive (resp.negative) orientation such that S_1 contains S_2, and h_1 (resp. h_2) is a 1-handle attaching to the two spheres trivially in a neighborhood of the north (resp. south) pole (Fig. 3.4).

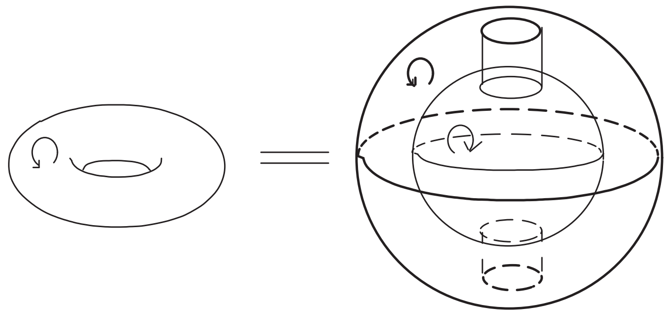

Fig. 3.4

(Step 2) Deform the two 1-handles and the inner sphere $h_1 \cup h_2 \cup S_2$ by an ambient isotopy of $\mathbb{R}^4$ to make S_2 have a positive orientation as in Fig. 3.5, where the 1-handle h_1 is as in Fig. 3.6, which has a double point curve with a branch point for each end, and the other 1-handle h_2 is the orientation-reversed mirror image of h_1.

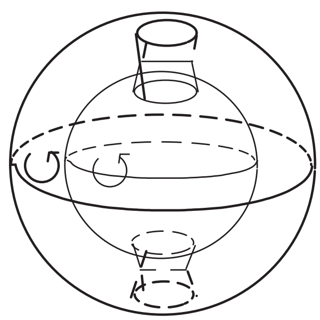

Fig. 3.5

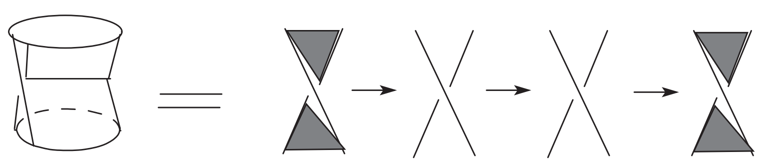

Fig. 3.6

(Step 3) Slide the 1-handle h_2 to the neighborhood of the north pole to make both h_1 and h_2 be in the neighborhood of the north pole, and cut off the two southern hemispheres to obtain the surface braid and the surface link chart of degree 2 as in Fig. 3.7.

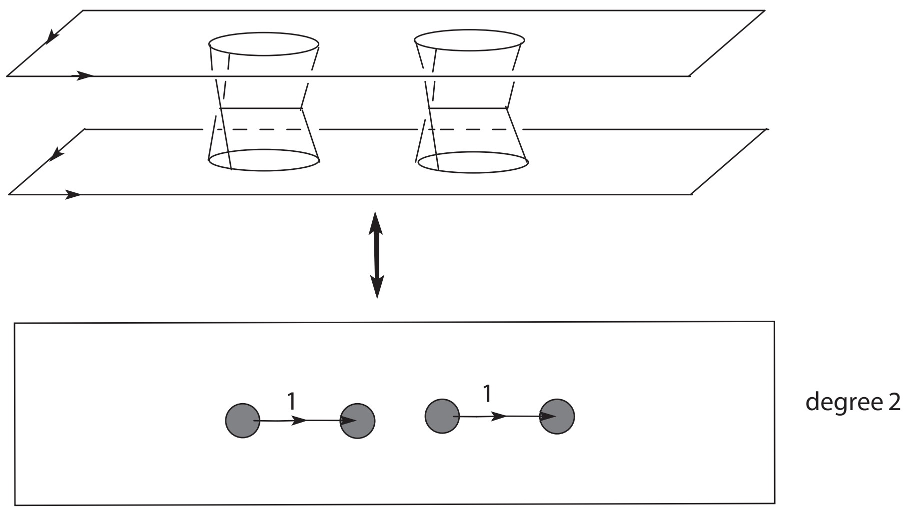

Fig. 3.7

(Step 4) Now, consider the trivial torus knot T^2 as the torus-covering-link associated with Γ_T (of degree m) by drawing Γ_T on T^2 (Fig. 3.8). Let us denote the m 1-handles corresponding to h_1 (resp. h_2) by H_1 (resp. H_2). Then H_1 can be deformed to the 1-handle braided surface as in Fig. 3.2, and H_2 to the orientation-reversed mirror image of H_1, and the surface braid will be as in Fig. 3.9. Hence we obtain the surface link chart Γ_S of degree $2m$ as in Fig. 3.1.

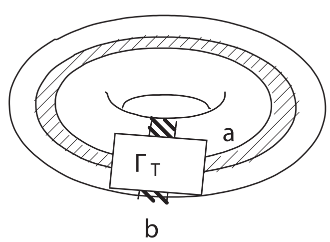

Fig. 3.8

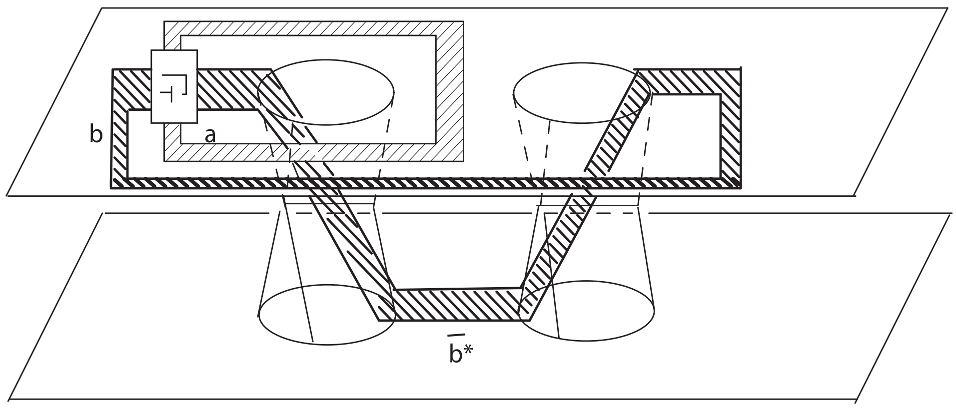

Fig. 3.9

□

The *braid index* of an oriented surface link F is the minimum degree of simple closed surface braids in $\mathbb{R}^4$ which are equivalent to F. Kamada showed in [23] that surface links whose braid index is at most three are indeed ribbon, and Shima showed in [35] that the turned spun T^2-knot of a non-trivial classical knot is not ribbon (c.f. (0.2)). Hence we obtain the following corollary:

Corollary 3.3. *Let S be the torus-covering-link associated with a torus-covering-chart of degree m. Then the braid index of S is equal or less than $2m$. In particular, the braid index of the turned spun T^2-knot of the torus $(2,p)$-knot is four.*

Remark. Hasegawa in [14] (10, Part3 "Chart description of twist-spun surface-links") showed that for the turned spun T^2-link of a closed m-braid, its braid index is at most $3m$.

Example 3.4. Let us consider the torus-covering-chart Γ_T of Example 2.2.2 (Fig. 3.10).

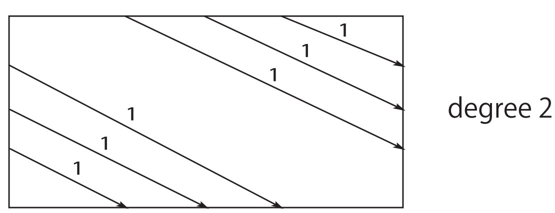

Fig. 3.10. The torus-covering-chart Γ_T

The torus-covering-chart Γ_T describes the turned spun T^2-knot of the right-handed trefoil. Its 1-handle chart is as in Fig. 3.11, and the surface link chart Γ_S obtained from Γ_T is as in Fig. 3.12.

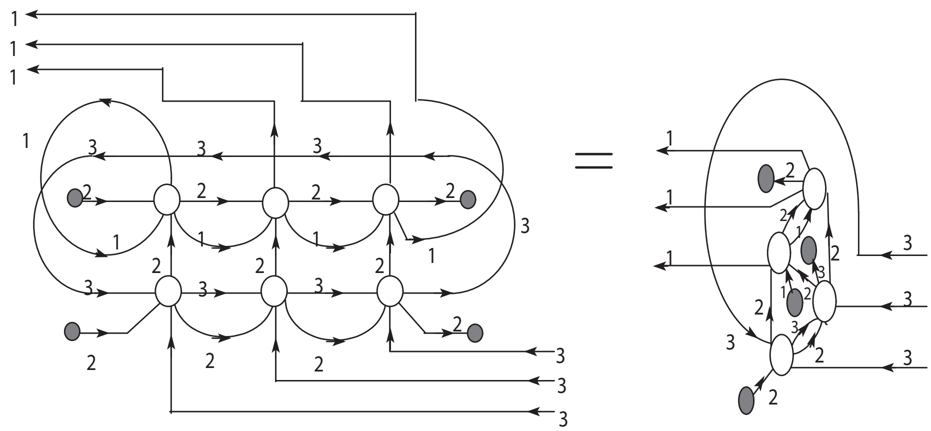

Fig. 3.11. The 1-handle chart H_b, where $b = \sigma_1^{-3}$

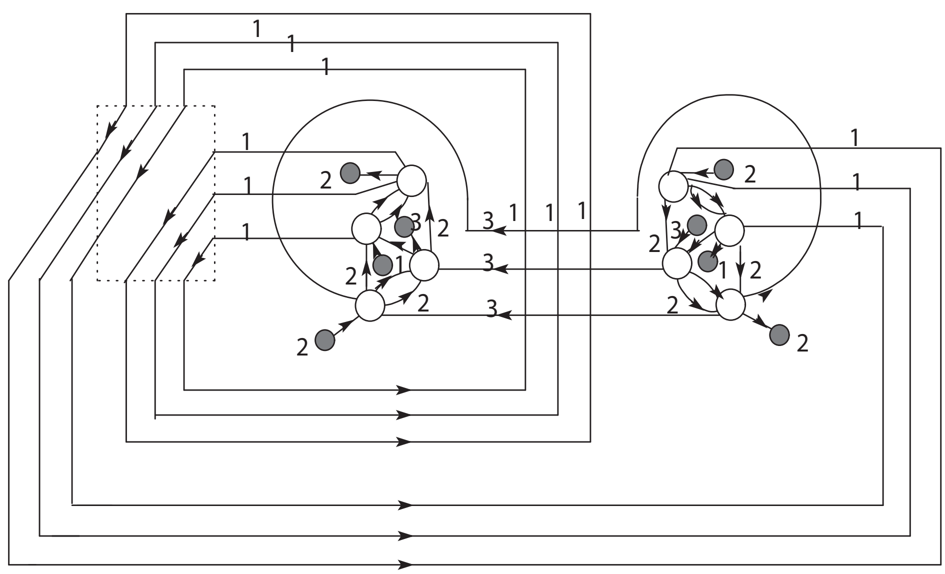

Fig. 3.12. The surface link chart Γ_S obtained from Γ_T (degree 4)

4 Knot Groups and Link Groups

In this section we consider torus-covering-links which are T^2-links, that is, torus-covering-links associated with torus-covering-charts without black vertices.

Lemma 4.1. *Let Γ_T be a torus-covering-chart of degree m without black vertices, and with its boundary braids a and b. Let S be the torus-covering-link associated with Γ_T. Then the link group of S is obtained as follows:*

$$\pi_1(\mathbb{R}^4 - S) = \langle x_1, \ldots, x_m \; | x_j = \mathrm{Artin}(a)(x_j) = \mathrm{Artin}(b)(x_j), \; j \in \{1, 2, \ldots, m\}\rangle,$$

where $\mathrm{Artin}(a) : F_m \to F_m$ *(resp.* $\mathrm{Artin}(b)$*) is Artin's automorphism of the free group* $F_m = \langle x_1, \ldots, x_m\rangle$ *associated with the m-braid a (resp. b).*

proof. Apply van Kampen's theorem. □

Before studying link groups of torus-covering-links, we enumerate two well-known theorems about classical link groups.

Theorem 4.2 (Theorem 6.3.1 in [27]). *A non-trivial abelian subgroup of a classical link group is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \bigoplus \mathbb{Z}$.*

If a classical knot group has a non-trivial center, then it is a torus knot ([5]). There is a theorem concerning the center of a classical link group as follows:

Theorem 4.3 (Burde and Murasigi's Theorem ([6])). *The statements listed below are equivalent.*

(1) The group of a classical link L has a non-trivial center.

(2) The link group is isomorphic to one of the groups of type (a), (b), or (c).

(a) $(\underbrace{Z * \cdots * Z}_{m-1}) \times \mathbb{Z}$

(b) $((\underbrace{Z * \cdots * Z}_{m-2}) \times \mathbb{Z}) *_{\mathbb{Z}} ((Z \times \mathbb{Z}) *_Z Z)$

(c) $((\underbrace{Z * \cdots * Z}_{m-2}) \times \mathbb{Z}) *_{\mathbb{Z}} \big((\mathbb{Z} \times Z) *_{\mathbb{Z}} (Z *_Z Z)\big),$

where m is the number of components of L, Z is a free cyclic group, and $\mathbb{Z} = \langle h\rangle$ is a "special" free cyclic group which is the center of the link group except the link group is that of a Hopfian link of type (a). In case (b) the amalgamation concerning the last factor $Z = \langle q\rangle$ is given by $h = q^\alpha$, where $\alpha > 1$. In case (c) the last factor $(Z *_Z Z)$ is the group of the torus knot of type (α, β).

There are torus-covering-links whose link groups are $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. We can see that they are not classical link groups by Theorem 4.2.

Theorem 4.4. *Let Γ_T be the torus covering chart of degree 3 (resp. 4) without black vertices, and with its boundary braids Δ^2 and $\sigma_1^2\sigma_2^2$ (resp. $\sigma_1^2\sigma_2^2\sigma_3^2$), where $\Delta^2 = (\sigma_1\sigma_2)^3$ (resp. $(\sigma_1\sigma_2\sigma_3)^4$). Then the torus covering link S associated with Γ_T has the link group $\pi_1(\mathbb{R}^4 - S) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ (resp. $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$).*

proof. Let us compute the link group $G = \pi_1(\mathbb{R}^4 - S)$ by Lemma 4.1.

(Degree 3 case) Let x_1, x_2, x_3 be the generators. Then the relations concerning the boundary braid $\sigma_1^2\sigma_2^2$ are

$$\begin{aligned} x_1x_2 &= x_2x_1, & (4.1) \\ x_2x_3 &= x_3x_2. & (4.2) \end{aligned}$$

The other relations concerning the other boundary braid Δ^2 are

$$\begin{aligned} x_1 &= (x_1x_2x_3)x_1(x_1x_2x_3)^{-1} \\ x_2 &= (x_1x_2x_3)x_2(x_1x_2x_3)^{-1} \\ x_3 &= (x_1x_2x_3)x_3(x_1x_2x_3)^{-1}, \end{aligned}$$

which are

$$\begin{aligned} x_1x_2x_3 &= x_2x_3x_1, & (4.3) \\ x_2(x_1x_2x_3) &= (x_1x_2x_3)x_2, & (4.4) \\ x_3x_1x_2 &= x_1x_2x_3. & (4.5) \end{aligned}$$

By (4.3) and (4.1), we have

$$x_1x_3 = x_3x_1,$$

hence

$$\begin{aligned} G &= \langle x_1,\ x_2,\ x_3 \mid x_1x_2 = x_2x_1,\ x_2x_3 = x_3x_2,\ x_3x_1 = x_1x_3 \rangle \\ &= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

(Degree 4 case) Similarly, for generators x_1, x_2, x_3, x_4, we have the following relations:

$$x_ix_{i+1} = x_{i+1}x_i$$

for $i = 1, 2, 3$ and

$$x_i = (x_1x_2x_3x_4)x_i(x_1x_2x_3x_4)^{-1}$$

for $i = 1, 2, 3, 4$. By using the former three relations, the latter four relations are deformed as follows:

$$\begin{aligned} x_1x_3x_4 &= x_3x_4x_1, && (4.6)\\ x_2x_3x_4 &= x_3x_4x_2, && (4.7)\\ x_3x_1x_2 &= x_1x_2x_3, && (4.8)\\ x_4x_1x_2 &= x_1x_2x_4. && (4.9) \end{aligned}$$

By $x_2x_3 = x_3x_2$, (4.7) is deformed as follows:

$$\begin{aligned} x_3x_2x_4 &= x_3x_4x_2,\\ x_2x_4 &= x_4x_2. \end{aligned}$$

Similarly, by $x_2x_3 = x_3x_2$ and (4.8), we have

$$x_1x_3 = x_3x_1,$$

and by $x_2x_4 = x_4x_2$ and (4.9), we have

$$x_4x_1 = x_1x_4.$$

Hence $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. □

Symmetry-spun T^2-links have classical link groups (cf. (0.4)). However, there are torus-covering-links whose link groups are not classical link groups (Proposition 4.5 and Theorem 4.6). Moreover, they are ribbon. This means there are torus-covering-links which can be described by surface link charts without white vertices (i.e. their triple point number is zero) but whose associated torus-covering-charts always have white vertices.

Proposition 4.5. *Let Γ_T be the torus-covering-chart of degree 4 without black vertices, and with boudary braids Δ^2 and $\sigma_1\sigma_3$, where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ (Garside's Δ). Then the torus-covering-link S associated with Γ_T is a T^2-link of two components and moreover ribbon but not a spun T^2-link.*

proof. By the definition of torus-covering-links, S is a surface link of two components and each component is of genus one.

First, let us see that S is ribbon. By Theorem 2.3 and Propsition 2.10, we can see that S is ribbon, but here we will apply Theorem 3.1. Note that the torus-covering-chart Γ_T^0 with boundary braids Δ and $\sigma_1\sigma_3$ is as in Fig. 4.1, and the torus-covering-chart $Gamma_T$ is as in Fig. 4.2.

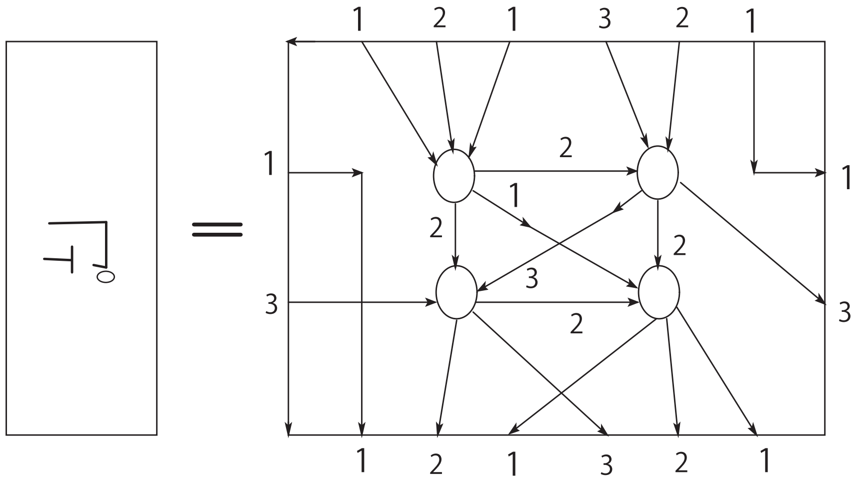

Fig. 4.1. The torus-covering-chart Γ_T^0

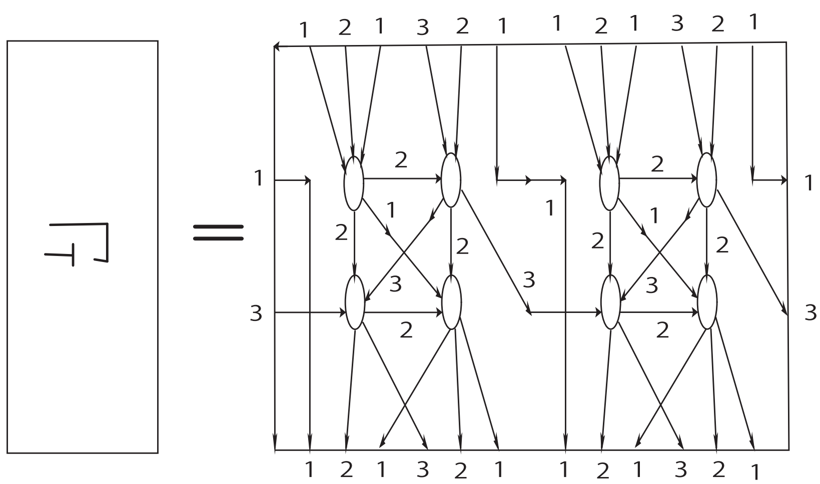

Fig. 4.2. The torus-covering-chart Γ_T

By Theorem 3.1, the 1-handle chart H_b can be deformed to the one as in Fig. 4.3. We use the notation of oval nests introduced in Section 1. By Fig. 4.3, the surface link chart Γ_S obtained from Γ_T is as follows, where $b = (2132)^2$:

$$\begin{aligned} \Gamma_S &= F_1 \cup F_3 \cup F_5 \cup F_7 \\ &\quad \cup F_4 \cup O(4; 3256) \cup O(4; b) \cup O(4; 3256 \cdot b). \end{aligned}$$

Since Γ_S is a ribbon chart, the torus-covering-link S is ribbon.

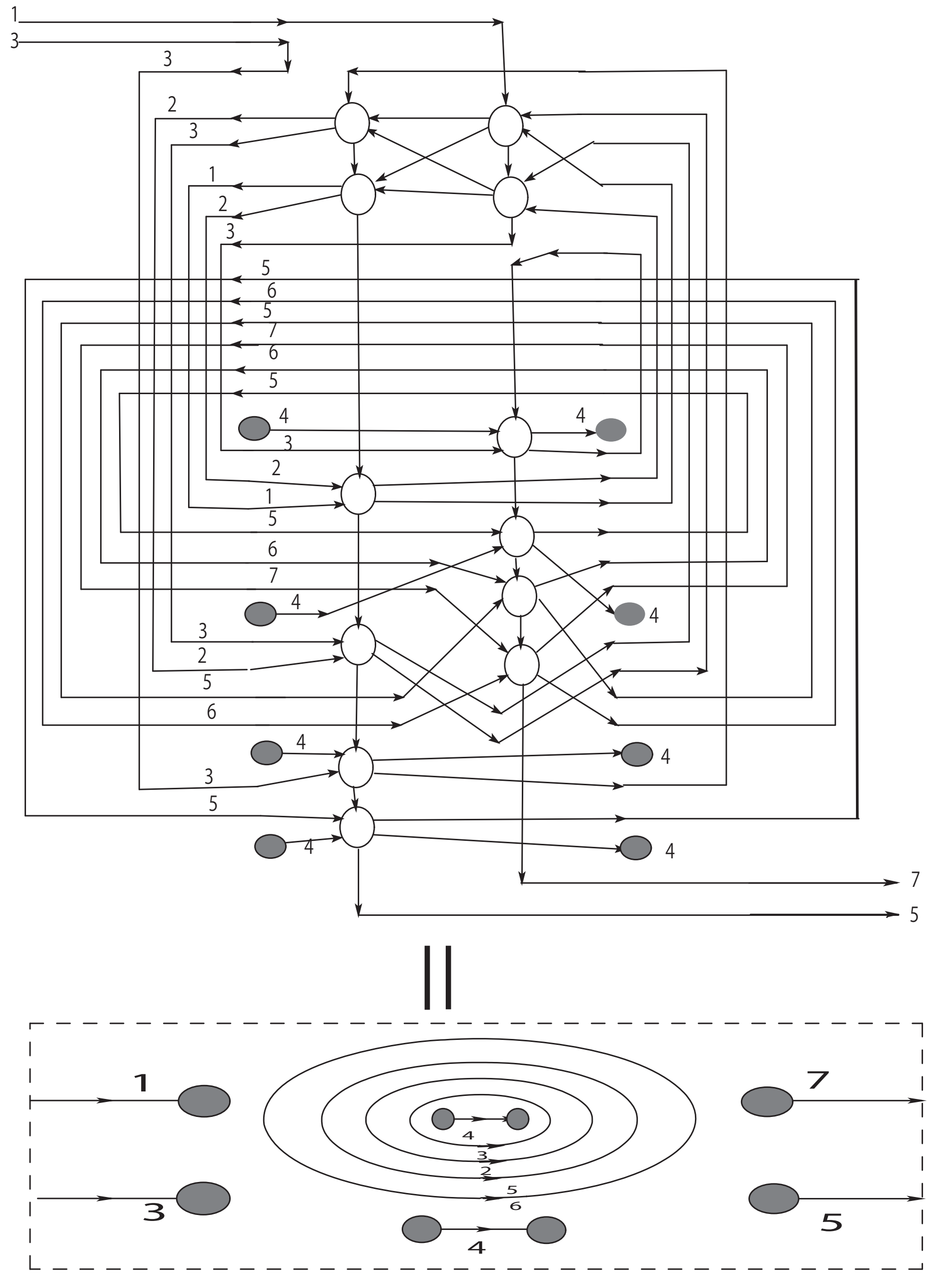

Fig. 4.3. The 1-handle chart H_b

Next, we will see the torus-covering-link S cannot be deformed to a spun T^2-link of some classical link. Let us compute the link group $\pi_1(\mathbb{R}^4 - S)$.

By Lemma 4.1, $\pi_1(\mathbb{R}^4 - S)$ is generated by four generators x_1, x_2, x_3, x_4 and the relations which come from $(\sigma_1\sigma_2\sigma_3)^4$ and from $\sigma_1\sigma_3$. The relations from $(\sigma_1\sigma_2\sigma_3)^4$ are

$$\begin{array}{rcl} x_1 & = & (x_1x_2x_3x_4)x_1(x_1x_2x_3x_4)^{-1}, \\ x_2 & = & (x_1x_2x_3x_4)x_2(x_1x_2x_3x_4)^{-1}, \\ x_3 & = & (x_1x_2x_3x_4)x_3(x_1x_2x_3x_4)^{-1}, \\ x_4 & = & (x_1x_2x_3x_4)x_4(x_1x_2x_3x_4)^{-1}. \end{array}$$

The relations from $\sigma_1\sigma_3$ are

$$\begin{array}{rcl} x_1 & = & x_2, \\ x_3 & = & x_4. \end{array}$$

By denoting $a := x_1 = x_2$ and $b := x_3 = x_4$, we have

$$\pi_1(\mathbb{R}^4 - S) = \langle a,\ b\ | ab^2 = b^2a,\ ba^2 = a^2b\rangle.$$

We show that $\pi_1(\mathbb{R}^4 - S)$ is not a classical link group using Theorem 4.2.

Let A be the subgroup of $\pi_1(\mathbb{R}^4 - S)$ generated by ab, a^2 and b^2, that is

$$A = \langle ab,\ a^2,\ b^2\ | ab^2a = b^2a,\ ba^2 = a^2b\rangle.$$

Since the following holds, the subgroup A is abelian:

$$\begin{array}{rcl} (ab)a^2 & = & a^3b = a^2(ab), \\ (ab)b^2 & = & ab^3 = b^2(ab), \\ a^2b^2 & = & ab^2a = b^2a^2. \end{array}$$

We can assume A has no torsion element, for a classical link group has no torsion element (Theorem 4.2). Then $ab,\ a^2$ and b^2 are mutually independent. If we have

$$(ab)^{n_1}(a^{2n_2})(b^{2n_3}) = 1$$

for some integers $n_1,\ n_2$ and n_3, then

$$(ab)^{2m}(a^{-2m})(b^{-2m}) = ((ab)^2a^{-2}b^{-2})^m = 1$$

for some m, because the relations do not change the numbers of a's and b's. Since A has no torsion element,

$$\begin{array}{rcl} (ab)^2a^{-2}b^{-2} & = & 1 \\ abab & = & b^2a^2 = a^2b^2 \\ ba & = & ab \\ aba^{-1}b^{-1} & = & 1. \end{array}$$

Let ϕ be a homomorphism from $\pi_1(\mathbb{R}^4 - S)$ to the symmetry group of degree three $\mathbb{S}_3$ defined by $\phi(a) = (12)$ and $\phi(b) = (23)$. Then we have

$$\phi(aba^{-1}b^{-1}) = (12)(23)(12)(23) = (132) \neq 1.$$

Hence, $aba^{-1}b^{-1} \neq 1$ in $\pi_1(\mathbb{R}^4 - S)$. This is a contradiction.

Therefore the abelian subgroup A is isomorphic to $\mathbb{Z} \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}$, which contradicts Theorem 4.2.

Hence the torus-covering-link S is not a spun T^2-link. $\square$

Remark. Let Γ'_T be the torus-covering-chart of degree 4 without black vertices, and with boundary braids $(\sigma_2\sigma_1\sigma_3\sigma_2)^2$ and with $\sigma_1\sigma_3$. Then the torus-covering-link S' associated with Γ'_T is the same as S in Proposition 4.5.

We can generalize this Proposition 4.5 as follows.

Theorem 4.6. *Let $\Gamma_{T,n,\epsilon,\epsilon'}$ be the torus-covering-chart of degree 4 without black vertices, and with the boundary braids Δ^{2n} and $\sigma_1^{\epsilon}\sigma_3^{\epsilon'}$, where $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ (Garside's Δ), n is a positive integer, and $\epsilon, \epsilon' \in \{+1, -1\}$. Then the torus-covering-link $S_{n,\epsilon,\epsilon'}$ associated with $\Gamma_{T,n,\epsilon,\epsilon'}$ is ribbon but not a spun T^2-link. Moreover, for a fixed n, the torus-covering-links are equivalent, which we will denote by S_n. Then S_n and S_m are different for $n \neq m$.*

Remark. This torus-covering-link S_n has two components. Each component of S_n is a trivial torus knot.

Before the proof, we give a theorem about canonical forms of an amalgamated product and a lemma concerning group isomorphisms.

Theorem 4.7 (Theorem 17.2.1 in [13]). *In the amalgamated product of groups G_i $(i = 1, 2, \ldots, n)$ with the amalgamated subgroup U, there is in each class of equivalent words one, and only one, element in canonical form*

$f = uz_1z_2\cdots z_t$. Here $u \in U$ and the z_j $(j = 1, 2, \dots, t)$ are left coset representatives x_{ik} of U in the G_i with $x_{ik} \neq 1$, taken from an arbitrarily prefixed selection of left coset representatives. For each $j \in \{1, 2, \dots, t-1\}$, z_j and z_{j+1} belong to different G_i's.

Lemma 4.8. *Let G_1 and G_2 be groups. Suppose there is an isomorphism $f : G_1 \to G_2$. Then for normal subgroups N_1 and N_2 of G_1 and G_2 with the isomorphism $f|_{N_1} : N_1 \to N_2$, there is an induced isomorphism $\underline{f} : G_1/N_1 \to G_2/N_2$, where $\underline{f}(g_1N_1) = f(g_1)N_2$. In particular, we can take such normal subgroups as the center groups, and the commutator subgroups of G_1 and G_2.*

proof. The induced homomorphism $\underline{f} : G_1/N_1 \to G_2/N_2$ is well-defined. For $g, g' \in G_1$, if $g \sim g' \in G_1/N_1$, then there is some $n \in N_1$ such that $g' = gx$. Since $f : G_1 \to G_2$ and $f|_{N_1} : N_1 \to N_2$ are homomorphisms, $f(g') = f(g)f(x)$ and $f(x) \in N_2$. Hence $f(g) \sim f(g') \in G_2/N_2$.

Let $\underline{f'} : G_2/N_2 \to G_1/N_1$ be a homomorphism with $\underline{f'}(g_2N_2) = f^{-1}(g_2)N_1$. This homomorphism $\underline{f'}$ is well-defined, for $f : G_1 \to G_2$ and $f|_{N_1} : N_1 \to N_2$ are isomorphisms. Then $\underline{f'} \circ \underline{f} = \mathrm{id}_{G_1/N_1}$ and $\underline{f} \circ \underline{f'} = \mathrm{id}_{G_2/N_2}$. Hence $\underline{f}^{-1} = \underline{f'}$ and $\underline{f'}$ is an isomorphism. □

proof of Theorem 4.6. The proof that $S_{n,\epsilon,\epsilon'}$ is ribbon is the same as in Proposition 4.5. We can see that for a fixed n, the torus-covering-links can be described by the same ribbon chart.

To show that S_n is not a spun T^2-link, it suffices to show that the link group G_n of S_n is not a classical link group. The link group G_n of S_n is computed as follows. Let $x_1, \dots, x_4$ be the generators. The relations concerning the boundary braid $\sigma_1\sigma_3$ are

$$\begin{aligned} x_1 &= x_2, \\ x_3 &= x_4. \end{aligned}$$

The other relations concerning the other boundary braid Δ^{2n} are

$$\begin{aligned} x_1 &= (x_1x_2x_3x_4)^n x_1 (x_1x_2x_3x_4)^{-n}, \\ x_2 &= (x_1x_2x_3x_4)^n x_2 (x_1x_2x_3x_4)^{-n}, \\ x_3 &= (x_1x_2x_3x_4)^n x_3 (x_1x_2x_3x_4)^{-n}, \\ x_4 &= (x_1x_2x_3x_4)^n x_4 (x_1x_2x_3x_4)^{-n}. \end{aligned}$$

By denoting $a := x_1 = x_2$ and $b := x_3 = x_4$, we have

$$\begin{aligned} G_n &= \pi_1(\mathbb{R}^4 - S) \\ &= \langle a,\ b \,|(a^2b^2)^n b = b(a^2b^2)^n,\ (a^2b^2)^n a = a(a^2b^2)^n \rangle. \end{aligned}$$

Let us denote the center of G_n by Z_n and $Z'_1 := \langle a^2,\ b^2\ | a^2b^2 = b^2a^2\rangle$ and $Z'_n := \langle h_n\rangle$, where $h_n = (a^2b^2)^n$ and $n > 1$. We show that $Z_n = Z'_n$ for $n \geq 1$. It is easy to see Z'_n is a subgroup of the center Z_n. Since every element of Z'_n commutes with each element of G_n, Z'_n is a normal subgroup of G_n, and we can consider the quotient group G_n/Z'_n. It suffices to show that G_n/Z'_n has a trivial center. For $n = 1$,

$$G_1/Z'_1 = \langle a,\ b\ | a^2 = b^2 = 1\rangle = \mathbb{Z}/2 * \mathbb{Z}/2,$$

which has a trivial center. For $n > 1$,

$$\begin{aligned} G'_n &:= G_n/Z'_n \\ &= \langle a,\ b\ |(a^2b^2)^n = 1\rangle \\ &= \langle a,\ b,\ x\ |x = a^2b^2,\ x^n = 1\rangle \\ &= \langle a\rangle *_U \langle b,\ x\ |x^n = 1\rangle, \end{aligned}$$

where $U = \langle a^2\rangle = \langle xb^{-2}\rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = xb^{-2}$. Let $H_1 := \langle a\rangle$ and $H_2 := \langle b,\ x\ |x^n = 1\rangle$. Note that $H_2 = \langle b,\ x\ |x^n = 1\rangle$ is a free product of $\mathbb{Z}$ and $\mathbb{Z}/n$.

Let h be a center element of G'_n. Since $H_1 = \langle a\rangle = U \cup Ua$, let us take $\{1, a\}$ as the left coset representatives of U in H_1. Choose some set of left coset representatives of U in H_2, and denote it by C. By Theorem 4.7, h has a canonical form

$$h = u(a)c_1ac_2 \cdots ac_t(a),$$

where $u \in U$ and $c_1, \dots , c_t \in C - \{1\}$.

Since $ah = ha$, we have

$$ua(a)c_1ac_2 \cdots ac_t(a) = u(a)c_1ac_2 \cdots ac_t(a)a,$$

hence

$$uac_1ac_2 \cdots ac_t = uc_1ac_2 \cdots ac_ta.$$

If $t > 0$, $uac_1ac_2 \cdots ac_t$ and $uc_1ac_2 \cdots ac_ta$ are both in canonical forms but they differ. This contradicts the uniqueness of canonical forms. Hence we have

$$h = u(a) = a^k$$

for an integer k.

Since $hb = bh$, we have $a^k b = ba^k$. If $k = 2l + 1$, we have

$$\begin{aligned} a^k b &= (a^2)^l ab = uab, \\ ba^k &= ba^{2l}a = b(xb^{-2})^l a = ca, \end{aligned}$$

where $u = a^{2l} \in U$ and $c = b(xb^{-2})^l$. If $k = 2l \neq 0$, we have

$$\begin{aligned} a^k b &= (a^2)^l b = ub, \\ ba^k &= ba^{2l} = b(xb^{-2})^l = c, \end{aligned}$$

where $u = a^{2l} \in U$ and $c = b(xb^{-2})^l$. Since $l \neq 0$, we have

$$(xb^{-2})^{m_1} b \neq (xb^{-2})^{m_2} c = (xb^{-2})^{m_2} b(xb^{-2})^l$$

in X_2 for every $m_1, m_2 \in \mathbb{Z}$, which means $Ub \neq Uc$ in X_2. Hence we can choose b and c to be different left coset representatives of U in H_2. Then the canonical forms of $a^k b$ and ba^k are uab and ca (if $k = 2l+1$), or ub and c (if $k = 2l$), which differ. This contradicts the uniqueness of canonical forms.

Hence $k = 0$ and $h = 1$, i.e. $G'_n = G_n/Z'_n$ has a trivial center, which means that $Z_n = \langle h_n \rangle = \langle (a^2b^2)^n \rangle$.

This $Z_n = \langle h_n \rangle$, the center group of G_n, is not trivial. There is a homomorphism $\psi : G_n \to \mathbb{Z}$ with $\psi(a) = 1$ and $\psi(b) = 1$. Then $\psi(h_n) = 4n \neq 1$ in $\mathbb{Z}$.

We show that G_n is not a classical link group. We see G_n is neither of type (a), (b) nor (c) in Theorem 4.3.

We prove this for $n > 1$. Note that since the torus-covering-link S_n has two components, m in Theorem 4.3 is two.

(Case (a)) If G_n is of type (a),

$$G_n = Z \times \mathbb{Z}.$$

The group G_n is commutative.

There is a natural surjective homomorphism

$$f : G_n \to \mathbb{Z}/2 * \mathbb{Z}/2 = \langle a' \rangle * \langle b' \rangle \tag{4.10}$$

with $f(a) = a'$ and $f(b) = b'$. Since $\mathbb{Z}/2 * \mathbb{Z}/2$ is not commutative, G_n is not commutative, which is a contradiction.

(Case (b)) If G_n is of type (b),

$$G_n = (Z \times \mathbb{Z}) *_Z Z = (\langle k \rangle \times \langle h_n \rangle) *_Z \langle q \rangle, \tag{4.11}$$

where the amalgamation is given by $h_n = q^\alpha$ for some integer $\alpha > 1$. Here $h_n = (a^2b^2)^n$ is the generator of the center Z_n of the group G_n, and q is the generator of the last factor Z of (4.11). The generators k, h_n and q are written by a's and b's. There is a natural surjective homomorphism f as written in (4.10) in Case (a). Let $k' := f(k)$, $h'_n := f(h_n)$ and $q' := f(q)$. Since $\mathbb{Z}/2 * \mathbb{Z}/2$ has a trivial center, $\langle h'_n \rangle = 1$, and the amalgamation becomes $q'^\alpha = 1$. Hence we have

$$\begin{aligned} f(G_n) &= (\langle k' \rangle \times \langle h'_n \rangle) *_Z \langle q' \rangle \\ &= \langle k' \rangle *_Z \langle q' \rangle, \end{aligned}$$

where the amalgamation is givem by $q'^\alpha = 1$.

Since $\alpha > 1$ and a non-trivial element of $\mathbb{Z}/2 * \mathbb{Z}/2$ has order 2 or ∞, we have $\alpha = 2$ and $q'^{\,2} = 1$. Then the numbers of a''s and b''s in q' are different mod 2, hence the numbers of a's and b's in q are different mod 2. However, since the relations of G_n do not change the numbers of a's and b's, and $h_n = q^2$ and the numbers of a's and b's of h_n are the same, q has the same numbers of a's and b's. This is a contradiction.

(Case (c)) If G_n is of type (c),

$$G = (Z \times \mathbb{Z}) *_\mathbb{Z} (Z *_Z Z),$$

where $Z *_Z Z$ is a classical knot group of a torus knot of type (α, β) for some $\alpha > 1$ and $|\beta| > 1$, i.e. $Z *_Z Z = \langle x,\ y\ | x^\alpha = y^\beta \rangle$. Similarly to Case (b), $x'^\alpha = 1$ for $x' := f(x)$ and we have x' is of order 2 i.e. $\alpha = 2$. Hence x' consists of different numbers of a''s and b''s mod 2, which means x consists of different numbers of a's and b's mod 2. However, since $h_n = x^\alpha = x^2$, x consists of the same numbers of a's and b's. This is a contradiction.

In the case $n = 1$, since the center group of G_1 is $Z_1 = \langle a^2,\ b^2\ | a^2b^2 = b^2a^2 \rangle$, which has two generators, if G_1 is a classical link group, G_1 must be of type (a). The rest of the proof is the same as in Case (a).

Therefore the link group of S_n $(n > 0)$ is not a classical link group.

Next we show that the torus-covering-links S_n and S_m are different for $n \neq m$. It suffices to show that there is a contradiction if their link groups G_n and G_m are isomorphic. It suffices to show when $n,\ m > 1$. If G_n and G_m are isomorphic,

$$G'_n/[G'_n, G'_n] = G'_m/[G'_m, G'_m],$$

where $G'_j = G_j/Z_j = \langle a,\ b\ | (a^2b^2)^j = 1 \rangle$ and $[G'_j, G'_j]$ is the commutator subgroup of G'_j for $j = n, m$. Then we have

$$\begin{aligned} G''_n : &= G'_n/[G'_n, G'_n] \\ &= \langle a,\ b\ |(a^2b^2)^n = 1,\ ab = ba\rangle \\ &= \langle a,\ b,\ c\ |(a^2b^2)^n = 1,\ ab = ba,\ c = ab\rangle. \end{aligned}$$

By eliminating b by $b = a^{-1}c$, we have

$$\begin{aligned} G''_n &= \langle a,\ c\ |(aca^{-1}c)^n = 1,\ ac = ca\rangle \\ &= \langle a,\ c\ |c^{2n} = 1,\ ac = ca\rangle \\ &= \mathbb{Z} \times \mathbb{Z}/2n. \end{aligned}$$

Hence we have $G''_n \neq G''_m$ and $S_n \neq S_m$ for $n \neq m$. □

Remark. There is an algorithm whereby one can determine the center of a group with one defining relation ([1], [32]). This algorithm gives an alternative proof of the fact that the center group of $G'_n = \langle a,\ b\ |(a^2b^2)^n = 1\rangle$ is trivial.

We can consider the knot version of Theorem 4.6.

Theorem 4.9. *Let $\Gamma_{T,n,\epsilon,\epsilon'}$ be the torus-covering-chart of degree 4 without black vertices, and with its boundary braids Δ^{2n+1} and $\sigma_1^{\epsilon}\sigma_3^{\epsilon'}$, where $\Delta = (\sigma_1\sigma_2\sigma_3)^4$, n is a positive integer, and $\epsilon, \epsilon' \in \{+1, -1\}$. Then the torus-covering-knot $S_{n,\epsilon,\epsilon'}$ associated with $\Gamma_{T,n,\epsilon,\epsilon'}$ is ribbon but not a spun T^2-knot. Moreover, for a fixed n, the torus-covering-links are equivalent, which we will denote by S_n. Then S_n and S_m are different for $n \neq m$.*

proof. The proof that $S_{n,\epsilon,\epsilon'}$ is ribbon and it does not depend on the choice of ϵ and ϵ' is the same as Proposition 4.5. We show that the knot group G_n of S_n is not classical knot group and $G_n \neq G_m$ for $n \neq m$.

Note that for $n = 0$, S_0 is the trivial torus knot.

The knot group G_n is computed as follows: Let $x_1, \ldots, x_4$ be the generators. Then the relations concerning the boundary braid $\sigma_1\sigma_3$ are

$$\begin{aligned} x_1 &= x_2, \\ x_3 &= x_4. \end{aligned}$$

The other relations concerning the other boundary braid Δ^{2n+1} are

$$\begin{aligned} x_1 &= (x_1x_2x_3x_4)^n x_1x_2x_3x_4x_3^{-1}x_2^{-1}x_1^{-1}(x_1x_2x_3x_4)^{-n}, \\ x_2 &= (x_1x_2x_3x_4)^n x_1x_2x_3x_2^{-1}x_1^{-1}(x_1x_2x_3x_4)^{-n}, \\ x_3 &= (x_1x_2x_3x_4)^n x_1x_2x_1^{-1}(x_1x_2x_3x_4)^{-n}, \\ x_4 &= (x_1x_2x_3x_4)^n x_1(x_1x_2x_3x_4)^{-n}. \end{aligned}$$

By denoting $a := x_1 = x_2$ and $b := x_3 = x_4$, we have two relations:

$$\begin{array}{rcl} a & = & (a^2b^2)^n a^2 b a^{-2} (a^2b^2)^{-n}, \\ b & = & (a^2b^2)^n a (a^2b^2)^{-n}, \end{array}$$

the former of which can be deformed to the following:

$$\begin{array}{rcl} a(a^2b^2)^n a^2 & = & (a^2b^2)^n a^2 b \\ a(a^2b^2)^{n+1} & = & (a^2b^2)^{n+1} b. \end{array}$$

Hence we have

$$\begin{array}{rcl} G_n & = & \pi_1(\mathbb{R}^4 - S_n) \\ & = & \langle a,\ b\ |b(a^2b^2)^n = (a^2b^2)^n a,\ a(a^2b^2)^{n+1} = (a^2b^2)^{n+1} b \rangle. \end{array}$$

Let $Z_n := \langle h_n \rangle$, where $h_n = (a^2b^2)^{2n+1}$. Then Z_n is the center group of G_n.

Since

$$\begin{array}{rcl} ah_n & = & a(a^2b^2)^{2n+1} = (a^2b^2)^{n+1} b (a^2b^2)^n = (a^2b^2)^{2n+1} a = h_n a, \\ bh_n & = & b(a^2b^2)^{2n+1} = (a^2b^2)^n a (a^2b^2)^{n+1} = (a^2b^2)^{2n+1} b = h_n b, \end{array}$$

Z_n is a subgroup of the center group, and moreover it is a normal subgroup of G_n.

Similar to the proof of Theorem 4.6, it suffices to show that the quotient group $G'_n = G_n/Z_n$ has a trivial center.

$$\begin{array}{rcl} G'_n & = & \langle a,\ b\ |b(a^2b^2)^n = (a^2b^2)^n a,\ a(a^2b^2)^{n+1} = (a^2b^2)^{n+1} b,\ (a^2b^2)^{2n+1} = 1 \rangle \\ & = & \langle a,\ b,\ x\ |x = a^2b^2,\ bx^n = x^n a,\ ax^{n+1} = x^{n+1} b,\ x^{2n+1} = 1 \rangle. \end{array}$$

By eliminating b by $b = x^n a x^{-n}$, we have

$$\begin{array}{rcl} G'_n & = & \langle a,\ x\ |x = a^2 x^n a^2 x^{-n},\ x^{2n+1} a = a x^{2n+1},\ x^{2n+1} = 1 \rangle \\ & = & \langle a,\ x\ |x = a^2 x^n a^2 x^{-n},\ x^{2n+1} = 1 \rangle \\ & = & \langle a,\ x\ |x^{2n+1} = (a^2 x^n)^2 = 1 \rangle \\ & = & \langle a,\ x,\ y\ |x^{2n+1} = 1,\ y^2 = 1,\ a^2 x^n = y \rangle \\ & = & \langle a \rangle *_U \langle x,\ y\ |x^{2n+1} = 1,\ y^2 = 1 \rangle, \end{array}$$

where $U = \langle a^2 \rangle = \langle y x^{n+1} \rangle = \mathbb{Z}$ and the amalgamation is given by $a^2 = yx^{n+1} = yx^{-n}$. Let $H_1 := \langle a \rangle$ and $H_2 := \langle x,\ y\ |x^{2n+1} = 1,\ y^2 = 1 \rangle$. Note that $H_2 = \langle x,\ y\ |x^{2n+1} = 1,\ y^2 = 1 \rangle$ is a free product of $\mathbb{Z}/(2n+1)$ and $\mathbb{Z}/2$.

Let h be a center element of G'_n. Since $H_1 = \langle a \rangle = U \cup Ua$, let us take $\{1, a\}$ as the left coset representatives of U in H_1. Choose some set of left coset representatives of U in H_2, and denote it by C. By Theorem 4.7, h has a canonical form

$$h = u(a)c_1ac_2 \cdots ac_t(a),$$

where $u \in U$ and $c_1, \ldots, c_t \in C - \{1\}$.

Since $ah = ha$, we have

$$ua(a)c_1ac_2 \cdots ac_t(a) = u(a)c_1ac_2 \cdots ac_t(a)a,$$

hence

$$uac_1ac_2 \cdots ac_t = uc_1ac_2 \cdots ac_ta.$$

If $t > 0$, $uac_1ac_2 \cdots ac_t$ and $uc_1ac_2 \cdots ac_ta$ are both in canonical forms but they differ. This contradicts the uniqueness of canonical forms. Hence we have

$$h = u(a) = a^k$$

for an integer k.

Since $hx = xh$, we have $a^kx = xa^k$. If $k = 2l + 1$, we have

$$\begin{aligned} a^kx &= (a^2)^l ax = uax, \\ xa^k &= xa^{2l}a = x(yx^{n+1})^l a = ca, \end{aligned}$$

where $u = a^{2l} \in U$ and $c = x(yx^{n+1})^l$. If $k = 2l \neq 0$, we have

$$\begin{aligned} a^kx &= (a^2)^l x = ux, \\ xa^k &= xa^{2l} = x(yx^{n+1})^l = c, \end{aligned}$$

where $u = a^{2l} \in U$ and $c = x(yx^{n+1})^l$. Since $l \neq 0$, we have

$$(yx^{n+1})^{m_1}x \neq (yx^{n+1})^{m_2}c = (yx^{n+1})^{m_2}x(yx^{n+1})^l$$

in X_2 for every $m_1, m_2 \in \mathbb{Z}$, which means $Ux \neq Uc$ in X_2. Hence we can choose x and c to be different left coset representatives of U in H_2. Then the canonical forms of a^kx and xa^k are uax and ca (if $k = 2l + 1$), or ux and c (if $k = 2l$), which differ. This contradicts the uniqueness of canonical forms.

Hence $k = 0$ and $h = 1$, i.e. $G'_n = G_n/Z'_n$ has a trivial center, which means that $Z_n = \langle h_n \rangle = \langle (a^2b^2)^{2n+1} \rangle$.

This center group $Z_n = \langle h_n \rangle$ of G_n is not trivial. There is a homomorphism $\psi : G_n \to \mathbb{Z}$ with $\psi(a) = 1$ and $\psi(b) = 1$. Then $\psi(h_n) = 4(2n+1) \neq 1$ in $\mathbb{Z}$.

By Burde and Murasugi's Theorem (Theorem 4.3), if G_n is a classical knot group, then G_n is a torus knot group. Let $G_{p,q}$ be the (p,q)-torus knot group, i.e. $G_{p,q} := \langle u,\ v \ | u^p = v^q \rangle$ for coprime integers p and q. We can assume that $1 < p < q$. Let $Z_{p,q} := \langle u^p \rangle = \langle v^q \rangle$ be the center group of $G_{p,q}$. We have

$$\begin{aligned} G'_{p,q} &= G_{p,q}/Z_{p,q} \\ &= \langle u,\ v \ | u^p = v^q = 1 \rangle \\ &= \mathbb{Z}/p * \mathbb{Z}/q. \end{aligned}$$

Then we have

$$\begin{aligned} G''_n : &= G'_n/[G'_n, G'_n] \\ &= \langle a,\ x \ | x^{2n+1} = 1,\ a^4x^{2n} = 1,\ ax = xa \rangle \\ &= \langle a,\ x \ | x^{2n+1} = 1,\ a^4 = x,\ ax = xa \rangle \\ &= \langle a \ | a^{4(2n+1)} = 1 \rangle \\ &= \mathbb{Z}/4(2n+1), \end{aligned}$$

and

$$\begin{aligned} G''_{p,q} : &= G'_{p,q}/[G'_{p,q}, G'_{p,q}] \\ &= \langle u,\ v \ | u^p = v^q = 1,\ uv = vu \rangle \\ &= \mathbb{Z}/p \times \mathbb{Z}/q. \end{aligned}$$

Since $G''_n = G''_{p,q}$, we have $pq = 4(2n+1)$. Since G'_n has an element of order $2n+1$, and a non-trivial torsion element of $G'_{p,q}$ has order r which is a devisor of p or q, we can assume that $2n+1$ is a devisor of q. Since p, q are coprime positive integers, we have $p = 4$ and $q = 2n+1$.

For any element w of order 2 in $G'_{4,2n+1} = \mathbb{Z}/4 * \mathbb{Z}/(2n+1)$, w can be written as $w = w'^{\,2}$ for some element of order 4 in $G'_{4,2n+1}$. Since y in G'_n is of order 2, there is an element y' with $y = y'^{\,2}$ in G'_n. Then we have

$$\begin{aligned} G'_n &= \langle a,\ x,\ y,\ y' \ | x^{2n+1} = 1, y^2 = 1,\ a^2x^n = y,\ y = y'^{\,2} \rangle \\ &= \langle a,\ x,\ y' \ | x^{2n+1} = 1,\ y'^{\,4} = 1,\ a^2x^n = y'^{\,2} \rangle. \end{aligned}$$

Let $N_{4,2n+1,v'}$ be a normal subgroup of $G'_{4,2n+1}$ generated by an element v' of order $2n+1$. Since for any such v', $v' = 1$ iff $v = 1$, the quotient group $G'_{4,2n+1}/N_{4,2n+1,v'}$ does not depend on the choice of $v\prime$. Let us denote $G'_{4,2n+1}/N_{4,2n+1,v'}$ by $G'_{4,2n+1}/N_{4,2n+1}$. Let N_n be a normal subgroup of G'_n generated by x, which is an element of order $2n+1$. Let $f : G'_n \to G'_{4,2n+1}$ be the isomorphism. Then N_n and $N_{4,2n+1,v'}$ is isomorphic by $f|_{N_n}$ for some v'. Then by Lemma 4.8 we have $G'_n/N_n \cong G'_{4,2n+1}/N_{4,2n+1}$. Let us compute:

$$
\begin{array}{rcl}
G''_n & := & G'_n/N_n \\
& = & \langle a,\ x,\ y' \ | x^{2n+1} = 1,\ y'^{\,4} = 1,\ a^2 x^n = y'^{\,2},\ x = 1 \rangle \\
& = & \langle a,\ y' \ | y'^{\,4} = 1,\ a^2 = y'^{\,2} \rangle,
\end{array}
$$

$$
\begin{array}{rcl}
G''_{4,2n+1} & := & G'_{4,2n+1}/N_{4,2n+1} \\
& = & \langle u,\ v \ | u^4 = 1,\ v^{2n+1} = 1,\ v = 1 \rangle \\
& = & \langle u \ | u^4 = 1 \rangle \\
& = & \mathbb{Z}/4\mathbb{Z}.
\end{array}
$$

Any element of $G''_{4,2n+1}$ has order 1, 2 or 4. However, the order of ay' in G''_n is neither 1, 2 nor 4. Take a homomorphism $\phi : G''_n \to S_3$ with $\phi(a) = (12)$, $\phi(y) = (23)$, where $\mathfrak{S}_3$ is the symmetry group of degree 3. This ϕ is well-defined. Then we have

$$
\begin{array}{rcl}
\phi(ay') & = & (12)(23) = (123) \neq e, \\
\phi((ay')^2) & = & (12)(23)(12)(23) = (132) \neq e, \\
\phi((ay')^4) & = & (123)^4 = (123) \neq e,
\end{array}
$$

in $\mathfrak{S}_3$, which means neither ay', $(ay')^2$ nor $(ay')^4$ is trivial in G''_n. This is a contradiction.

For $n \neq m$, $S_n \neq S_m$. Since

$$G''_n = \mathbb{Z}/4(2n+1) \neq \mathbb{Z}/4(2m+1) = G''_m$$

for $n \neq m$, we have $S_n \neq S_m$. □

A classical link group has no torsion element (cf. Theorem 4.2). There is a surface knot whose knot group has torsion elements (cf. [27]). The link group of a symmetry-spun T^2-link has no torsion elements (cf. (0.4)). Then how about torus-covering-links?

Theorem 4.10. *Let S be the torus-covering-link associated with a torus-covering-chart of degree m without black vertices. Then its link group has no torsion element.*

proof. Let $E := D^2 \times S^1 \times S^1 - S$. Let A be a non trivial abelian subgroup of $\pi_1(E)$, and $\tilde{E}$ a covering space over E corresponding to the subgroup A. Then it suffices to show the following:

(1) $\pi_1(E) = \pi_1(\mathbb{R}^4 - S)$,

(2) For $i \geq 2$, $\pi_i(E) = 0$,

(3) $H_3(\tilde{E}) \neq \mathbb{Z}_n$ for $|n| \geq 2$.

If the above hold, then since $\pi_i(\tilde{E}) = \pi_i(E) = 0$ for $i \geq 2$, we have $H_q(A) \cong H_q(\tilde{E})$ for any q. Then if A is a finite cyclic group $\mathbb{Z}_n$, by [15] $H_3(A) = \mathbb{Z}_n$, which contradicts (3). Hence, $\pi_1(E) = \pi_1(\mathbb{R}^4 - S)$ has no torsion element.

(1) It can be shown by van Kampen's theorem.

(2) There is a fiber bundle $\pi : E \to S^1$ with the fiber $F = D^2 \times S^1 - \mathrm{cl}(a)$. Then by the exact sequence of a fiber bundle, we obtain $\pi_q(E) = \pi_q(F)$ ($q \geq 2$). There is another fiber bundle $\pi' : F \to S^1$ with the fiber F' is an m-punctured disk, which is homotopic to $S^1 \vee S^1 \vee \cdots \vee S^1$. By the exact sequence of a fiber bundle again, we obtain $\pi_q(F) = \pi_q(S^1 \vee S^1 \vee \cdots \vee S^1)$ ($q \geq 2$). Because $\pi_q(S^1) = 0$ ($q \geq 2$), $\pi_q(S^1 \vee S^1 \vee \cdots \vee S^1) = 0$ ($q \geq 2$).

(3) By the Mayer-Vietoris exact sequence, we see that $H_3(E) = \mathbb{Z}^m$. Let A be a subgroup of $\pi_1(E)$ generated by all the torsion elements of order n. Then $\tilde{E}$ is locally path-connected regular covering space.

There is a commutative diagram (cf. [17]),

$$\begin{array}{ccc} H_3(\tilde{E}) & \xrightarrow{\pi_*} & H_3(E) \\ \kappa \searrow & & \swarrow \iota \\ & E^{\infty}_{0,3} & \end{array}$$

where π_* is a homomorphism induced by the projection $\pi : \tilde{E} \to E$, κ is a surjection, and ι is an injection.

Take a torsion element x of $H_3(\tilde{E}) \cong A$. Since a homomorphisim from $\mathbb{Z}_n$ to $\mathbb{Z}^m$ is a 0-map, $\pi_*(x) = 0$, so $\iota \circ \pi_*(x) = 0$ in $E^{\infty}_{0,3}$. On the other hand, if every torsion element (that is, every generator) of $H_3(\tilde{E}) \cong A$ goes to zero in $E^{\infty}_{0,3}$, $E^{\infty}_{0,3} = 0$, for κ is surjective. However, $E^{\infty}_{0,3}$ cannot be zero, for $\iota : H_3(E) = \mathbb{Z}^m \to E^{\infty}_{0,3}$ is injective. Therefore A is an emptyset and $H_3(\tilde{E}) \neq \mathbb{Z}_n$ for $|n| \geq 2$. $\square$

Corollary 4.11. *Let F be a 2-knot with torsion elements. Then the surface knot of genus one obtained from F by adding a trivial 1-handle is NOT a torus-covering-knot.*

Remark. For example, we can take such a 2-knot F as the 5-twist-spun trefoil.

5 Unknotting numbers

It is known that any surface link can be deformed to a trivial surface link by applying a finite number of 1-handle surgeries.

For an oriented surface link, its *unknotting number* is the minimum number of oriented 1-handle surgeries needed to deform it to trivial.

For an oriented surface link, adding a free edge to the surface link chart corresponds to a *nice 1-handle surgery*, which is a kind of an oriented 1-handle surgery. In this section we study unknotting numbers of torus-covering-links.

Theorem 5.1. *Let $\tau(S, \Gamma_T)$ be the torus-covering-link obtained by turning a torus-covering-link (S, Γ_T) once. Then if S is unknotted, $\tau(S, \Gamma_T)$ is also unknotted.*

proof. It suffices to show in the case where S has one component, i.e. S is a torus-covering-knot. We consider that the torus-covering-knot S is embedded in $B^3 \times S^1$ and $B^3 \times S^1$ is embedded in $\mathbb{R}^4$ by $\mathbb{R}^4 = S^4 - \{*\} = B^3 \times S^1 \cup_i S^2 \times B^2 - \{*\}$, where $\partial B^3 = S^2$ and $\partial B^2 = S^1$ and $i : S^2 \times S^1 \to S^2 \times S^1$ is the identity map. There is a handlebody whose boundary is S, for S is unknotted. Let us denote this handlebody by H. Then $\partial H \cap (B^3 \times S^1) = \partial H$. Let K be $H \cap \partial B = H \cap (S^2 \times S^1)$. We can assume that the handlebody H is in $\mathbb{R}^3 \times \{t_0\}$ and $S^2 \times S^1$ is embedded in $\mathbb{R}^4$. Let $f : S^2 \times S^1 \to \mathbb{R}^4$ be the embedding. For every $p \in S^1$, $f(S^2 \times \{p\}) \subset D^3$, where D^3 is a 3-ball. In other words, $S^2 \times \{p\}$ is trivially embedded. If for some $p \in S^1$, $f(S^2 \times \{p\})$ is not trivially embedded in $\mathbb{R}^4$, then $S^2 \times I \subset S^2 \times S^1$ cannot be embedded in $\mathbb{R}^4$, where $I \ni p$ is a sufficiently small interval.

Hence $f(S^2 \times S^1) = S^2 \times f(S^1)$, where S^2 is trivially embedded and $f(S^1)$ may be knotted. This knotted $f(S^1)$ can be deformed to the trivially embedded S^1 by an ambient isotopy of $\mathbb{R}^4$, but we must consider the following situation: $H \subset \mathbb{R}^3 \times \{t_0\}$, and $\partial H \subset B^3 \times f(S^1)$, where B^3 is a 3-ball with $\partial B^3 = S^2$.

Since ∂H is a trivially embedded closed 2-manifold in $B^3 \times f(S^1)$, we see that $H \subset B^3 \times f(S^1)$. Therefore, $H \subset B$. From the construction of

$\tau(S, \Gamma_T)$, there is a handlebody whose boundary is $\tau(S, \Gamma_T)$, which means $\tau(S, \Gamma_T)$ is also unknotted. $\square$

Corollary 5.2. *Let S be the torus-covering-link associated with a torus-covering-chart Γ_T and $\tau(S)$ be the turned torus-covering-link of (S, Γ_T). Then S and $\tau(S)$ have the same unknotting number.*

Proposition 5.3. *Let Γ_T be a torus-covering-chart of degree m with only free edges. Then the torus-covering-link S associated with Γ_T is unknotted.*

Before the proof, we give a lemma.

Lemma 5.4. *Let*

$$O_k := O\Big(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\Big)$$

for $k = 1, 2, \ldots, m-1$. Then the oval nest O_k can be deformed to the following:

$$\begin{aligned} O_k &= O\big(m; (m-1 \searrow m-k)(m+1 \nearrow m+k)\big) \\ &= O\big(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)\big) \\ &= O\big(m+k; (\overline{m+k-1} \searrow \overline{m})(m-1 \searrow m-k)\big). \end{aligned}$$

proof. First, we show that the sequence of integers $\prod_{j=0}^{k-1}(m-1 \searrow m-k+j)$ can be deformed by the braid relations to the following:

$$\begin{aligned} & \prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \\ = \; & (m-1 \searrow m-k)(m-1 \searrow m-k+1) \cdots (m-1) \\ = \; & (m-k) \cdot (m-k+1 \searrow m-k) \cdots (m-1 \searrow m-k) \\ = \; & \prod_{j=0}^{k-1}(m-k+j \searrow m-k). \end{aligned} \tag{5.1}$$

This can be seen from the easy equation

$$(l \searrow i_1)(l \searrow i_2) = (l-1 \searrow i_2 - 1)(l \searrow i_1). \tag{5.2}$$

By (5.2), we see that

$$\begin{aligned}
&\prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \\
= \;& (m-1 \searrow m-k)\prod_{j=2}^{k}(m-1 \searrow m-k+j-1) \\
= \;& \prod_{j=1}^{k-1}(m-2 \searrow m-k+j-1)\cdot(m-1 \searrow m-k) \\
= \;& \cdots \\
= \;& \prod_{j=0}^{k-1}(m-k+j \searrow m-k),
\end{aligned}$$

which is (5.1). Similarly, we have another equation

$$\prod_{j=0}^{k-1}(m+1 \nearrow m+k-j) = \prod_{j=0}^{k-1}(m+k-j \nearrow m+k). \qquad (5.3)$$

Hence, by (5.2), (5.3) and (1.7.1),. . . ,(1.7.6), we have

$$\begin{aligned}
&O_k \\
= \;& O\Big(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j)\prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\Big) \\
= \;& O\Big(m; \prod_{j=0}^{k-2}(m-k+j \searrow m-k)\cdot \\
& \prod_{j=0}^{k-2}(m+k-j \nearrow m+k)(m-1 \searrow m-k)(m+1 \nearrow m+k)\Big) \\
= \;& O\big(m; (m-1 \searrow m-k)(m+1 \nearrow m+k)\big) \\
= \;& O\big(m-k; (\overline{m-k+1} \nearrow \overline{m})(m+1 \nearrow m+k)\big) \\
= \;& O\big(m+k; (\overline{m+k-1} \searrow \overline{m})(m-1 \searrow m-k)\big).
\end{aligned}$$

□

proof of Proposition 5.3. It suffices to show in the case where S has one component, i.e. S is a torus-covering-knot. Moreover, we may assume that the torus-covering-chart Γ_T has the least possible free edges. Then we can assume Γ_T to be of degree m and

$$\Gamma_T = \bigcup_{k=1}^{m-1} F_k.$$

Let us see what the surface link chart Γ_S obtained from Γ_T is like. Let

$$O_k := O\Big(m; \prod_{j=0}^{k-1}(m-1 \searrow m-k+j) \prod_{j=0}^{k-1}(m+1 \nearrow m+k-j)\Big),$$

for $k = 1, 2, \ldots, m-1$ and

$$O_0 := O(m; \emptyset) = F_m.$$

Then the 1-handle chart H_e is deformed to

$$H_e = \bigcup_{k=0}^{m-1} O_k,$$

where e is the trivial braid of degree m. Then the surface link chart Γ_S is as follows:

$$\Gamma_S = \bigcup_{k=0}^{m-1} O_k \cup \bigcup_{k=0}^{m-1} O_k \cup \bigcup_{k=1}^{m-1} F_k.$$

Now, we show that the surface link chart Γ_S can be deformed to an unknotted chart. Let $I_k (k = 1, 2, \ldots, m-1)$ be a chart such that

$$I_k := O_k \cup \bigcup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k}.$$

We show that

$$\begin{aligned} I_k &= O_k \cup \bigcup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k} \\ &= \bigcup_{j=m-k}^{m+k} F_j \end{aligned} \tag{5.4}$$

for $k = 1, \ldots, m-1$.

By Lemma 5.4 and (1.7.1),. . . , (1.7.6) we have

$$\begin{aligned} I_k &= O\big(m+k; (\overline{m+k-1} \searrow \overline{m})(m-1 \searrow m-k)\big) \\ &\quad \cup \bigcup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k} \\ &= F_{m+k} \cup \bigcup_{j=m-k+1}^{m+k-1} F_j \cup F_{m-k} \\ &= \bigcup_{j=m-k}^{m+k} F_j, \end{aligned}$$

which is (5.4). Since $O_0 = F_m$, $\Gamma_S \supset I_1$. Hence by (5.4), $\Gamma_S \supset I_2$. Repeat this step and we have

$$\Gamma_S \supset I_{m-1} = \bigcup_{j=1}^{2m-1} F_j.$$

Hence by (1.7.5), Γ_S can be deformed to have only free edges, which is an unknotted chart. □

Proposition 5.5. *Let Γ_T be a torus-covering-chart of degree m which has no white vertices. In other words, Γ_T has only free edges and loops. Then the unknotting number of the torus-covering-link S associated with Γ_T is at most $m-1$.*

proof. The torus-covering-chart Γ_T can be deformed to the form of disjoint union of free edges by adding $(m-1)$-1-handles, $\bigcup_{j=1}^{m-1} F_j$. Hence, by Proposition 5.3, the unknotting number of S is at most $m-1$. □

Theorem 5.6. *There is a torus-covering-knot whose unknotting number is n, where n is any positive integer. Let S be the spun or turned spun T^2-knot of a classical knot $\beta = \mathrm{cl}(\sigma_1^3\sigma_2^3 \cdots \sigma_n^3)$ (degree $n+1$) with $n > 0$. Then the unknotting number of S is n.*

proof. The torus-covering-knot S can be described by a torus-covering-chart Γ_T of degree $n+1$ with neither black vertices nor white vertices. More precisely, Γ_T has no black vertices and its boundary braids are β and e (the trivial braid of degree $n+1$), or β and β. Then by Proposition 5.5, the unknotting number of S is at most n. It suffices to show that we must apply at least n 1-handle surgeries to deform S to a trivial surface knot.

Let us tricolor S. It suffices to tricolor the classical knot β. The classical braid $b = \sigma_1^3\sigma_2^3 \cdots \sigma_n^3$ can be devided into n blocks $b_i = \sigma_i^3$ $(i = 1, 2, \ldots, n)$. Let us denote by q_i' (resp. q_{i+1}) the start point of the i-th (resp. $(i+1)$-th) string of b_i and by q_i'' (resp. q_{i+1}') the end point of the i-th (resp. $(i+1)$-th) string of b_i for $i = 2, \ldots, n$. For some q, a point of the diagram of b, let $C(q)$ be the color of the edge with q on it.

We consider tricoloring b with $C(q_i) = C(q_i'')$ for every i. The diagram of the braid b_i consists of five edges. The four edges of them each contain one of the start points or end points. Let us denote by p_i a point of the fifth edge.

The tricoloring of the braid b_i is determined by two of the three colors $C(q_i') = C((q_i'')$, $C(q_{i+1}) = C(q_{i+1}')$ and $C(p_i)$. Hence the tricoloring of b is determined by $C(q_1), C(q_2), \ldots, C(q_{n+1})$, and Mochizuki's cocycle invariant $\Phi_\theta(S)$ is as follows:

$$\Phi_\theta(S) = 3^{n+1},$$

which means there are 3^{n+1} possible ways of tricoloring.

Adding one 1-handle to S means identifying $C(e_1)$ and $C(e_2)$ for some edges e_1 and e_2 in the diagram of b. Let e_k be on the braid b_{j_k} $(k = 1, 2)$. If $e_1 = e_2$ or $e_1, e_2 \in \{e(q_i'), e(q_i'')\}$ or $\{e(q_{i+1}), e(q_{i+1}')\}$, then the cocycle invariant does not change. If not, then $C(q_{j_2}) = C(q_{j_2}')$ or $C(q_{j_2+1}) = C(q_{j_2+1}')$ is determined by the 1-handle, and the cocycle invariant becomes 3^n. We have

$$\Phi_\theta(S') = \Phi_\theta(S) \quad \text{or} \quad \Phi_\theta(S)/3,$$

where S' is the resulting surface knot.

Hence we must apply at least n 1-handle surgeries to deform S to have Mochizuki's cocycle invariant 3, which is a necessary condition for the trivial surface knot. $\square$

As a more concrete example, we can give an alternative proof of the fact that the spun (or the turned spun) T^2-knot of a torus (p,q)-knot has the unknotting number one:

Proposition 5.7. *Let S (resp.$\tau(S)$) be the spun (resp.turned spun) T^2-knot of a torus (p,q)-knot. Then the unknotting number of S and $\tau(S)$ is one.*

Remark. Kanenobu and Marumoto showed in [26] that the unknotting number of the spun torus (p,q)-knot is one, which shows that the unknotting number of the spun T^2-knot of a torus (p, q)-knot is also one.

proof. By Theorem 5.1, it suffices to show that the unknotting number of S is one. We use the same notation as in the Proposition 5.3, that is;

$$O_k = O\big(p; (p-1 \searrow p-k)(p+1 \nearrow p+k)\big),$$
$$H_e = \bigcup_{k=0}^{p-1} O_k$$

for $k = 0, 1, \dots, m-1$. Note that for $k = 0$, $(p-1 \searrow p-k)(p+1 \nearrow p+k) = \emptyset$ and $O_k = O(p; \emptyset) = F_p$. Moreover, let us define b and O'_k as follows:

$$b = (1 \nearrow p-1)^q,$$
$$O'_k = O\big(m; (m-1 \searrow m-k)(m+1 \nearrow m+k) \cdot b\big).$$

The oval nest O'_k is obtained from O_k by adding loops describing b around it.

Since the (p, q)-torus knot is the closure of a braid $b = (\sigma_1 \cdot \sigma_2 \cdots \sigma_{p-1})^q$, the surface link chart Γ_S obtained from Γ_T is as follows:

$$\Gamma_S = \bigcup_{i=0}^{m-1} O_i \cup \bigcup_{i=0}^{m-1} O'_i.$$

Remark that Γ_S is a ribbon chart.

We show that Γ_S can be deformed to an unknotted chart by adding a free edge.

(Step 1) Let us denote q mod p by $r \in \{0, 1, 2, \dots, p-1\}$. Let us define an element τ of the symmetry group $\mathfrak{S}_{p-1}$ by

$$\begin{pmatrix} 1 & 2 & \cdots & r-1 & r & r+1 & r+2 & \cdots & p-1 \\ p-r+1 & p-r+2 & \cdots & p-1 & p-r & 1 & 2 & \cdots & p-r-1 \end{pmatrix}.$$

Observe that for $i \in \{0, 1, 2, \dots, p-1\} \backslash \{r\}$, $\quad \sigma_i \cdot b = b \cdot \sigma_{\tau(i)}$.

(Step 2) We show that

$$O_{p-k-1} \cup O_{p-k} \cup F_{2p-k} = O_{p-k-1} \cup F_k \cup F_{2p-k},$$

for $k \in \{0, 1, 2, \dots, p-1\} \backslash \{r\}$.

By Lemma 5.4, we have

$$\begin{aligned}
& O_{p-k-1} \cup O_{p-k} \cup F_{2p-k} \\
= \ & O\big(k+1; (\overline{k+2} \nearrow \overline{p})(p+1 \nearrow 2p-k-1)\big) \\
& \cup O\big(k; (\overline{k+1} \nearrow \overline{p})(p+1 \nearrow 2p-k)\big) \cup F_{2p-k} \\
= \ & O\big(k+1; (\overline{k+2} \nearrow \overline{p})(p+1 \nearrow 2p-k-1)\big) \\
& \cup O\big(k; (\overline{k+1} \nearrow \overline{p})(p+1 \nearrow 2p-k-1)\big) \cup F_{2p-k} \\
= \ & O\big(k+1; (\overline{k+2} \nearrow \overline{p})(p+1 \nearrow 2p-k-1)\big) \\
& \cup O\big(k; (\overline{k+2} \nearrow \overline{p})(p+1 \nearrow 2p-k-1)\big) \cup F_{2p-k} \\
= \ & O_{p-k-1} \cup F_k \cup F_{2p-k}.
\end{aligned}$$

(Step 3) Similarly, we show that

$$O'_{p-k-1} \cup O'_{p-k} \cup F_{\tau(k)} = O'_{p-k-1} \cup F_{2p-k} \cup F_{\tau(k)},$$

for $k \in \{1, 2, \dots, p-1\} \backslash \{r\}$.

First, remark that

$$O(k; b) = F_{\tau(k)}$$

for $k \in \{1, 2, \dots, p-1\} \backslash \{r\}$. This is because $\overline{b} \cdot k \cdot b = \overline{b} \cdot b \cdot \tau(k) = \tau(k)$. Then by Lemma 5.4 and $2p - k > p$, we have

$$\begin{aligned}
& O'_{p-k-1} \cup O'_{p-k} \cup F_{\tau(k)} \\
= \ & O\big(2p-k-1; (\overline{2p-k-2} \searrow \overline{p})(p-1 \searrow k+1) \cdot b\big) \\
& \cup O\big(2p-k; (\overline{2p-k-1} \searrow \overline{p})(p-1 \searrow k) \cdot b\big) \\
& \cup O\,(k; b) \\
= \ & O'_{p-k-1} \cup O(2p-k; b) \cup O\,(k; b) \\
= \ & O'_{p-k-1} \cup F_{2p-k} \cup F_{\tau(k)}.
\end{aligned}$$

(Step 4) Let us denote Step 2 as follows:

$$\phi_l \ : \ O_{l-1} \cup O_l \cup F_{p+l} \longrightarrow O_{l-1} \cup F_{p-l} \cup F_{p+l},$$

and Step 3 as

$$\psi_l \ : \ O'_{l-1} \cup O'_l \cup F_{\tau(p-l)} \longrightarrow O'_{l-1} \cup F_{p+l} \cup F_{\tau(p-l)}.$$

We introduce several notations to make things easy to see. Let us define F^l, $F'^{\,l}$ and $F''^{\,l}$ as follows:

$$\begin{aligned}
&F^l := F_{p-l},\\
&F'^{\,l} := F_{p+l},\\
&F''^{\,l} := F_{\tau(p-l)}.
\end{aligned}$$

Then Step 2 is written as follows:

$$\phi_l \ : \ O_{l-1} \cup O_l \cup F'^{\,l} \longrightarrow O_{l-1} \cup F^l \cup F'^{\,l},$$

for $1 \leq l \leq p-1$, and Step 3 is

$$\psi_l \ : \ O'_{l-1} \cup O'_l \cup F''^{\,l} \longrightarrow O'_{l-1} \cup F'^{\,l} \cup F''^{\,l},$$

for $1 \leq l \leq p-1$ with $l \neq p-r$. Moreover, for an integer m, let us define τ_m to be

$$\tau_m := p - \tau^{-m}(r).$$

Then we have

$$\begin{aligned}
F_{\tau^{-m}(r)} &= F_{\tau(\tau^{-(m+1)}(r))} = F''^{\,p-\tau^{-(m+1)}(r)} = F''^{\,\tau_{m+1}}\\
&= F_{p-(p-\tau^{-m}(r))} = F^{p-\tau^{-m}(r)} = F^{\tau_m},
\end{aligned}$$

and

$$F_{2p-\tau^{-m}(r)} = F_{p+(p-\tau^{-m}(r))} = F'^{\,p-\tau^{-m}(r)} = F'^{\,\tau_m}.$$

We show that Γ_S can be deformed to an unknotted chart by adding a free edge F_r. Let us define charts $I_0, I_1, \ldots$, and I_{2p-4}. First, define I_0 as follows:

$$\begin{aligned}
I_0 \ &:= \ \Gamma_S \cup F_r\\
&= \ O_0 \cup O_{\tau_1} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0}\\
& O'_0 \cup O'_{\tau_1} \cup \cdots \cup O'_{\tau_{p-2}} \cup O'_{\tau_0}\\
& \cup F_r.
\end{aligned}$$

For $n \in \{1, 2, \ldots , p-2\}$, let us define I_{2n} as follows:

$$\begin{aligned}
I_{2n} \ := \ & O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F^{\tau_1} \cup \cdots \cup F^{\tau_n}\\
& \cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup O'_{\tau_0} \cup F'^{\,\tau_1} \cup \cdots \cup F'^{\,\tau_n}\\
& \cup F_r\\
= \ & O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F''^{\,\tau_2} \cup \cdots \cup F''^{\,\tau_{n+1}}\\
& \cup O'_0 \cup O'_{\tau_{n+1}} \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup O'_{\tau_0} \cup F'^{\,\tau_1} \cup \cdots \cup F'^{\,\tau_n}\\
& \cup F_r.
\end{aligned}$$

For $n \in \{0, 1, 2, \dots, p-3\}$, let us define I_{2n+1} as follows:

$$\begin{aligned} I_{2n+1} &:= O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F^{\tau_1} \cup \cdots \cup F^{\tau_n} \\ & \quad \cup O'_0 \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup O'_{\tau_0} \cup F'^{\ \tau_1} \cup \cdots \cup F'^{\ \tau_n} \cup F'^{\ \tau_{n+1}} \\ & \quad \cup F_r \\ &= O_0 \cup O_{\tau_{n+1}} \cup O_{\tau_{n+2}} \cup \cdots \cup O_{\tau_{p-2}} \cup O_{\tau_0} \cup F''^{\ \tau_2} \cup \cdots \cup F''^{\ \tau_{n+1}} \\ & \quad \cup O'_0 \cup O'_{\tau_{n+2}} \cup \cdots \cup O'_{\tau_{p-2}} \cup O'_{\tau_0} \cup F'^{\ \tau_1} \cup \cdots \cup F'^{\ \tau_{n+1}} \\ & \quad \cup F_r. \end{aligned}$$

We show that I_{2n+1} (resp. I_{2n+2}) is obtained from I_{2n} (resp. I_{2n+1}) by applying Step 3 (resp. Step 2) for $n = 0, 1, \dots, p-3$.

When we have I_{2n}, there is an integer $l_0 < \tau_{n+1}$ such that for any l with $l_0 < l < \tau_{n+1}$, $O'_l \notin I_{2n}$ and $O'_{l_0} \in I_{2n}$. Since $r = \tau^0(r)$ and $F_r = F''^{\ p-\tau^{-1}(r)} = F''^{\ \tau_1}$, we have

$$\begin{aligned} I_{2n} \quad \supset \quad & F''^{\ l_0+1} \cup \cdots \cup F''^{\ \tau_{n+1}-1} \cup F''^{\ \tau_{n+1}} \\ & \cup O'_{l_0} \cup F'^{\ l_0+1} \cup \cdots \cup F'^{\ \tau_{n+1}-1} \cup O'_{\tau_{n+1}}. \end{aligned}$$

By applying Steps 3 and its inverses, we can deform $O'_{\tau_{n+1}}$ to $F'_{\tau_{n+1}}$, which is I_{2n+1}:

$$\psi_{l_0+1} \circ \cdots \circ \psi_{\tau_{n+1}-1} \circ \psi_{\tau_{n+1}} \circ \psi^{-1}_{\tau_{n+1}-1} \circ \cdots \circ \psi^{-1}_{l_0+2} \circ \psi_{l_0+1}(I_{2n}) = I_{2n+1}.$$

Similarly, we can deform I_{2n+1} to I_{2n+2} by applying Steps 2 and its inverses and deforming $O_{\tau_{n+1}}$ to $F^{\tau_{n+1}}$:

$$\phi_{l_0+1} \circ \cdots \circ \phi_{\tau_{n+1}-1} \circ \phi_{\tau_{n+1}} \circ \phi^{-1}_{\tau_{n+1}-1} \circ \cdots \circ \phi^{-1}_{l_0+2} \circ \phi_{l_0+1}(I_{2n+1}) = I_{2n+2}.$$

Remark that here l_0 is the same integer used in deforming I_{2n} to I_{2n+1}.

Since τ has order $p-1$, by repeating Steps 3 and Steps 2 alternately $p-2$ times each, we have

$$\begin{aligned} I_{2(p-2)} &= O_0 \cup O_{\tau_0} \bigcup_{m=1}^{p-2} F^{\tau_m} \\ & \quad \cup O'_0 \cup O'_{\tau_0} \bigcup_{m=1}^{p-2} F'^{\ \tau_m} \\ & \quad \cup F_r \\ &= O_0 \cup O_{p-r} \cup O'_0 \cup O'_{p-r} \bigcup_{k \neq p,\ 2p-r} F_k. \end{aligned}$$

On the other hand, $O_0 = F_p$. Hence, we have free edges of all labels except $2p-r$, using which we can deform the oval nest

$$O_{p-r} = O\big(2p-r; (\overline{2p-r-1} \searrow \overline{p})(p-1 \searrow r)\big)$$

to the free edge F_{2p-r}.

Therefore $\Gamma_S \cup F_r$ can be deformed to contain $\bigcup_{k=1}^{2p-1} F_k$, using which we can deform $\Gamma_S \cup F_r$ to have only free edges, which is an unknotted chart. $\square$

Remark. Proposition 5.7 holds for a classical braid b of degree m with the following conditions (here we consider the prime integer p as an arbitrary positive integer m);

There exists some element τ of the symmetry group of degree $m-1$, and

1. except for some i and $\tau(i)$, $\sigma_i \cdot b = b \cdot \sigma_{\tau(i)}$
2. the order of τ is $m-1$.

What classical braids satisfy the conditions in the above-mentioned remark?

Proposition 5.8. *We consider classical braids whose degree are at least two. The classical braids which satisfy the conditions in the above remark are in the following forms:*

(1) $\iota_0^q(\Delta_p^{2m}) \cdot \iota_p^0(\Delta_q^{2n}) \cdot (\Theta_p^q \cdot \Theta_q^p)^l \cdot \Theta_p^q$,

(2) $\sigma_1^{2m}\sigma_3^{2n-1}(\sigma_2\sigma_1\sigma_3\sigma_2)^{2l-1}$ *or* $\sigma_1^{2m-1}\sigma_3^{2n}(\sigma_2\sigma_1\sigma_3\sigma_2)^{2l-1}$,

where the degree of the braid is $p+q$ *in (1) and* 4 *in (2) and* m, n *and* l *are integers and* p, q *are positive integers such that*

1. $p = 1, q > 1$,
2. $p > 1, q = 1$ *or*
3. $p > 1$ *and* $q > 1$ *are coprime.*

And

$$(\Delta_j)^2 = (\sigma_1\sigma_2\cdots\sigma_{j-1})^j$$

for $j = p,\ q$, *which is, Garside's* Δ *for* j*-braids, and*

$$\Theta_p^q = \sigma_p \cdot \Pi'^p_{p-1} \cdot \Pi^p_{q-1} \cdot \sigma_p \cdot \Pi'^p_{p-2} \cdot \Pi^p_{q-2} \cdots \sigma_p \cdot \Pi'^p_{p-q}$$

if $p > q$, and

$$\Theta_p^q = \sigma_p \cdot \Pi'^p_{p-1} \cdot \Pi^p_{q-1} \cdot \sigma_p \cdot \Pi'^p_{p-2} \cdot \Pi^p_{q-2} \cdots \sigma_p \cdot \Pi^p_{q-p}$$

if $q > p$, where Π_i^p and Π'^p_i is the notations introduced in Section 3, which are

$$\begin{aligned} \Pi_i^p &= \sigma_{p+1}\sigma_{p+2}\cdots\sigma_{p+i}, \\ \Pi'^p_i &= \sigma_{p-1}\sigma_{p-2}\cdots\sigma_{p-i}. \end{aligned}$$

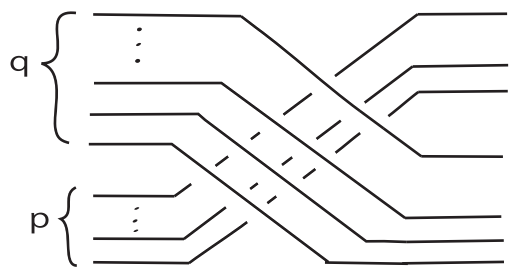

Fig. 5.1. Θ_p^q

proof. Let b be the classical braid of degree m which satisfies the conditions. If $\sigma_i b = b\sigma_j$, we use the phrase that *the generator σ_i is sent to the generator σ_j through b*. By the condition 1, at most one pair of the i-th and $(i+1)$-th strings do not send a generator σ_i to some generator through b.

Let $\phi(b)$ be an element of the symmetry group of degree m induced by the classical braid b, and let $b_\phi : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ be a map such that

$$\phi(b) = \begin{pmatrix} 1 & 2 & \cdots & m \\ b_\phi(1) & b_\phi(2) & \cdots & b_\phi(m) \end{pmatrix}.$$

For any $j \neq i$, the j-th and $(j+1)$-th strings come to some k-th and $(k+1)$-th or k-th and $(k-1)$-th strings through b, i.e. $\phi_b(j, j+1) = (k, k+1)$ or $(k, k-1)$. If the first and second strings come to some k-th and $(k+1)$-th (resp. k-th and $(k-1)$-th) strings, then the third string comes to $(k+2)$-th (resp. $(k-2)$-th) string, and so on. Hence we see that

$$b_\phi(1, 2, \ldots i) = \begin{cases} (k_1, k_1+1, \ldots, k_1+i-1) \\ (k_1, k_1-1, \ldots, k_1-i+1) \end{cases}$$

and

$$b_\phi(i+1, i+2, \dots m) = \begin{cases} (k_2, k_2+1, \dots, k_2+m-i-1) \\ (k_2, k_2-1, \dots, k_2-m+i+1) \end{cases}$$

for some k_1 and k_2.

Let b_1 (resp. b_2) be a classical braid consisting of from the first to the i-th (resp. from the $(i+1)$-th to the m-th) strings of the classical braid b. Let $C_{b_1} := \{1, 2, \dots, i\}$ and $C_{b_2} := \{i+1, i+2, \dots, m\}$. By the condition, in b, b_1 and b_2 can be separated by their tubular neighborhoods and C_{b_1} and C_{b_2} are contained in the canonical reduction system of b. We have two cases:

$$\begin{cases} b_\phi(C_{b_1}) = C_{b_1}, \ b_\phi(C_{b_2}) = C_{b_2}, \\ b_\phi(C_{b_1}) = C_{b_2}, \ b_\phi(C_{b_2}) = C_{b_1}. \end{cases}$$

(Case 1) If $b_\phi(C_{b_1}) = C_{b_1}$ and $b_\phi(C_{b_2}) = C_{b_2}$, then $\mathrm{cl}(b)$ has at least two components, which contradicts the condition that $\mathrm{cl}(b)$ is a knot.

(Case 2) If $b_\phi(C_{b_1}) = C_{b_2}$ and $b_\phi(C_{b_2}) = C_{b_1}$, b is in the following form:

$$\iota_0^{n_2}(b_1) \cdot \iota_{n_1}^0(b_2) \cdot C,$$

where $n_1 := i$ and $n_2 := m - i$, and b_j is a braid of degree n_j with $\sigma_k b_j = b_j \sigma_{\tau_j(k)}$ for some τ_j $(j = 1,\ 2)$ and every k, and $C = (\Theta_{n_1}^{n_2} \cdot \Theta_{n_2}^{n_1})^l \cdot \Theta_{n_1}^{n_2}$. By regarding each b_1 and b_2 as a string, we have the degree-two-braid σ_1^{2l+1}.

Since b_j $(j = 1,\ 2)$ sends every generator to some generator,

$$\begin{aligned} b_1 &= (\Delta'_{n_1})^{m_1}, \\ b_2 &= (\Delta'_{n_2})^{m_2} \end{aligned}$$

for some m_1 and m_2, where Δ'_i is a braid of degree i which sends every generator σ_j to a generator σ_{i-j}. For an even $n = 2\nu$, $(\Delta'_i)^n$ is a center element, hence $(\Delta'_i)^n = (\Delta_i)^{2\nu}$ for Garside's Δ_i. We see when the closure of the braid b has one component.

Here n_1 (resp. n_2) corresponds to p (resp. q), and m_1 (resp. m_2) corresponds to m (resp. n) in the statement of Proposition 5.8.

We consider the following cases:

(I) $n_1 \neq n_2$ and $n_1, n_2 \geq 2$,

(II) $n_1 = 1$ and $n_2 \geq 2$ or $n_1 \geq 2$ and $n_2 = 1$,

(III) $n_1 = n_2$,

(IV) $n_1 = 0$.

(I) When $n_1 \neq n_2$ and $n_1, n_2 \geq 2$.

(I-1) If m_1 and m_2 are even, n_1 and n_2 are coprime.

We have

$$\phi(b) = \begin{pmatrix} 1 & 2 & \cdots & n_1 & n_1+1 & n_1+2 & \cdots & n_1+n_2 \\ n_2+1 & n_2+2 & \cdots & n_1+n_2 & 1 & 2 & \cdots & n_2 \end{pmatrix}$$

and

$$b_\phi(l) = \begin{cases} l - n_1 & \text{if } j > n_1 \\ l + n_2 & \text{if } j \leq n_1. \end{cases}$$

Then $b_\phi^j(1) = 1 - j_1 n_1 + j_2 n_2$ for some non-negative integers j_1 and j_2 with $j_1 + j_2 = j$. Since $\mathrm{cl}(b)$ is a knot, $b_\phi^{n_1+n_2}(1) = 1$ and $b_\phi^j(1) \neq 1$ for $j = 1, 2, \ldots, n_1 + n_2 - 1$. Hence if non-negative integers k_1 and k_2 satisfy

$$b_\phi^{n_1+n_2}(1) = 1 - k_1 n_1 + k_2 n_2 = 1,$$

that is,

$$k_1 n_1 = k_2 n_2, \tag{5.5}$$

then k_1 and k_2 must satisfy

$$k_1 + k_2 = n_1 + n_2. \tag{5.6}$$

We can assume that $n_1 < n_2$. If $n_2 = kn_1$ for some positive integer k, then (5.5) is $k_1 n_1 = k_2 k n_1$ and $k_1 = k, k_2 = 1$ satisfies (5.5). However,

$$n_1 + n_2 = (1 + k)n_1 > 1 + k = k_1 + k_2,$$

which contradicts (5.6). Hence n_1 and n_2 are coprime.

If n_1 and n_2 are coprime, since $k_1 n_1 = k_2 n_2$, we have $k_1 = n_2$ and $k_2 = n_1$, and $\mathrm{cl}(b)$ is a knot.

(I-2) There is no case when $n_1 < n_2$ and m_2 is odd or when $n_1 > n_2$ and m_1 is odd.

We can assume that $n_1 > n_2$ and m_1 is odd. Then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & \cdots & j & \cdots & n_1 & n_1+1 & \cdots & n_1+n_2 \\ n_1+n_2 & \cdots & n_1+n_2-(j-1) & \cdots & n_2+1 & * & \cdots & * \end{pmatrix}.$$

Then similarly to (I-1), since $n_2 + 1 \leq n_1$,

$$b_\phi^2(n_1) = b_\phi(n_2+1) = n_1 + n_2 - (n_2 + 1 - 1) = n_1.$$

This is a contradiction.

(I-3) There is no case when $n_1 < n_2$ and m_1 is odd and m_2 is even or when $n_1 > n_2$ and m_1 is even and m_2 is odd.

We can assume that $n_1 < n_2$ and m_1 is odd and m_2 is even. Then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n_1 & n_1+1 & n_1+2 & \cdots & n_1+n_2 \\ n_1+n_2 & n_1+n_2-1 & \cdots & n_2+1 & 1 & 2 & \cdots & n_2 \end{pmatrix}.$$

Then similarly to (I-1), we can show that n_1 and n_2 are coprime.

Let k be an integer with $0 < k < n_1$ determined by $k := n_2 - ln_1$ for some positive integer l. Then we have

$$\begin{aligned}
&b_\phi^{l+2}(1) = k, \\
&b_\phi(k) = n_1 + n_2 - k + 1 = n_1(l+1) + 1 > n_1, \\
&b_\phi^{l+1}(n_1(l+1)+1) = 1,
\end{aligned}$$

which means,

$$b_\phi^{2(l+2)}(1) = 1.$$

Since $\mathrm{cl}(b)$ is a knot,

$$n_1 + n_2 = 2(l+2).$$

Hence we have

$$\begin{aligned}
k + (l+1)n_1 &= 2(l+2), \\
k &= 4 - n_1 - (n_1 - 2)l.
\end{aligned}$$

Since $n_1 \geq 3$ and $l \geq 1$, we have

$$k = 4 - n_1 - (n_1 - 2)l \leq 4 - 3 - 1 = 0,$$

which contradicts the assumption that $k > 0$.

(II) When $n_1 = 1$ and $n_2 \geq 2$ or $n_1 \geq 2$ and $n_2 = 1$. We can assume that $n_1 = 1$ and $n_2 \geq 2$.

(II-1) If m_2 is even, then

$$\phi(b) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n_2+1 \\ n_2+1 & 1 & 2 & \cdots & n_2 \end{pmatrix}.$$

Since $\phi(b)$ has order n_2+1, $\mathrm{cl}(b)$ is a knot.

(II-2) There is no case when m_2 is odd. If m_2 is odd, then

$$\phi(b) = \begin{pmatrix} 1 & 2 & \cdots & n_2 & n_2+1 \\ n_2+1 & n_2 & \cdots & 2 & 1 \end{pmatrix}.$$

Then $b_\phi^2(1) = b_\phi(n_2+1) = 1$, which is a contradiction.

(III) When $n_1 = n_2$. Let us denote $n := n_1 = n_2$.

(III-1) If $n \geq 2$, then there are two cases: $n = 2$, m_1 is even and m_2 is odd, or, $n = 2$, m_1 is odd and m_2 is even.

By (I-1), there is no case when m_1 and m_2 are even. We can assume m_1 is odd. If m_2 is even, then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 2n & 2n-1 & \cdots & n+1 & 1 & 2 & \cdots & n \end{pmatrix}.$$

Then we have

$$b_\phi^4(1) = b_\phi^3(2n) = b_\phi^2(n) = b_\phi(n+1) = 1.$$

Hence, if $n > 2$, this is a contradiction. If $n = 2$, then $\phi(b)$ is order 4 and $\mathrm{cl}(b)$ is a knot.

If m_2 is odd, then $\phi(b)$ is as follows:

$$\begin{pmatrix} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n \\ 2n & 2n-1 & \cdots & n+1 & n & n+1 & \cdots & 1 \end{pmatrix}.$$

Then we have

$$b_\phi^2(1) = b_\phi(2n) = 1,$$

which is a contradiction.

(III-2) If $n = n_1 = n_2 = 1$, then $b = \sigma_1^{2k-1}$. This is a torus $(2, 2k-1)$-knot.

(IV) If $n_1 = 0$, the classical braid b is that of in (III-2).

If m_2 is even, the classical braid b is a pure braid, which is a contradiction.

If m_2 is odd, then we have

$$b_\phi^2(1) = b_\phi(n_2) = 1. \tag{5.7}$$

If $n_2 = 2$, the classical braid b is that of in (III-2). If $n_2 > 2$, this contradicts (5.7). $\square$

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