On the existence of homomorphisms between principal series of complex semisimple Lie groups

（複素半単純リー群の主系列表現の間の準同型の存在について）

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ABSTRACT. We determine when there exists a nonzero homomorphism between principal series representations of a complex semisimple Lie group. We also determine the condition for the existence of nonzero homomorphisms between twisted Verma modules.

§1. Introduction

Let $G$ be a complex semisimple Lie group. Then the principal series representations of $G$ are defined and play an important role in the representation theory of $G$. One of a fundamental problem about principal series is a description of the space of homomorphisms between such representations (cf. [Zel75, p. 720, II]). In this paper, we determine when there exists a nonzero homomorphism between principal series representations of a complex semisimple Lie group. We also determines the existence of homomorphisms between twisted Verma modules. This gives a generalization of results of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71].

We state our main results. Let $g$ be the Lie algebra of $G$, $h$ its Cartan subalgebra, $\Delta$ the root system for $(g, h)$ and $W$ the Weyl group of $\Delta$. By the Killing form we identify $g$ with $g^* = \text{Hom}_C(g, C)$. Then the Killing form also defines a non-degenerate bilinear form on $g^*$. We denote this form by $\langle \cdot , \cdot \rangle$. For $\alpha \in h^*$, put $\alpha = 2\alpha / \langle \alpha, \alpha \rangle$ and $s_{\alpha}(\lambda) = \lambda - \langle \alpha, \lambda \rangle \alpha$. Take a positive system $\Delta^+ \subset \Delta$. Then $\Delta^+$ determines a Borel subalgebra $b$. Let $n = [b, b]$. Let $O$ be the Bernstein-Gelfand-Gelfand category [BGG76, Definition 1] for $(g, b)$ and $M(\lambda)$ the Verma module with highest weight $\lambda - \rho$ for $\lambda \in h^*$ where $\rho$ is the half sum of positive roots. Fix an involution $\sigma$ of $g$ such that $\sigma|_h = -\text{id}_h$. The category $O$ has a dualizing functor $\delta$ defined by $\delta M = \text{Hom}_C(M, C)_{h\text{-finite}}$ where the action is given by $(X f)(m) = f(-\sigma(X)m)$. Put $\mathfrak{t} = \{ (X, \sigma(X)) \mid X \in g \} \subset g \oplus g$. For $M, N \in O$, we define the $g \oplus g$-module $L(M, N) = \text{Hom}_C(M, N)_{t\text{-finite}}$ where the action is given by $((X, Y)f)(m) = \sigma(X)f(-Ym)$. Then under some identification $g \otimes_R C \simeq g \oplus g$, the principal representations of $G$ are $L(\lambda, \mu) = L(M(-\mu), \delta M(-\lambda))$. This is an object of $\mathcal{H}$ where $\mathcal{H}$ is a category of Harish-Chandra modules.

For $\lambda \in h^*$, let $\Delta_\lambda$ be the integral root system of $\lambda$, $W_\lambda$ the Weyl group of $\Delta_\lambda$. Let $P$ be the integral weight lattice of $\Delta$. Then it is well-known that $W_\lambda = \{ w \in W \mid w\lambda - \lambda \in P \}$. Let $w_\lambda$ be the longest element of $W_\lambda$. Put $\Delta_\lambda^+ = \Delta^+ \cap \Delta_\lambda$. Then $\Delta_\lambda^+$ determines the set of simple roots $\Pi_\lambda$. Put $S_\lambda = \{ s_\alpha \mid \alpha \in \Pi_\lambda \}$ and $W_\lambda^0 = \{ w \in W_\lambda \mid w\lambda = \lambda \}$. For $w \in W_\lambda$, let $\ell_\lambda(w)$ be a length of $w$ as an element of $W_\lambda$.
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For a sequence of simple roots $\alpha_1, \ldots, \alpha_l \in \Pi$ and $\mu \in \mathfrak{h}^*$, we define a subset $A_{(s_{\alpha_1}, \ldots, s_{\alpha_l})}(\mu)$ of $\mathfrak{h}^*$ as follows. Put $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ for $i = 1, \ldots, l$. For $\mu \in \mathfrak{h}^*$, put

$$A_{(s_{\alpha_1}, \ldots, s_{\alpha_l})}(\mu) = \left\{ \mu' \in \mathfrak{h}^* \mid \text{for some } 1 \leq i_1 < \cdots < i_r \leq l, \mu' = s_{\beta_{i_1}} \cdots s_{\beta_{i_r}} \mu \text{ and } \langle \beta_{i_k}, s_{\beta_{i_k-1}} \cdots s_{\beta_{i_1}} \mu \rangle \in \mathbb{Z}_{<0} \text{ for all } k = 1, \ldots, r \right\}$$

For a reduced expression $w = s_1 \cdots s_l \in W$, it will be proved that the set $A_{(s_1, \ldots, s_l)}(\mu)$ is independent of the choice of a reduced expression (Lemma 2.3). We write $A_w(\mu)$ instead of $A_{(s_1, \ldots, s_l)}(\mu)$.

Now we state the main theorems of this paper.

**Theorem 1.1.** Let $\lambda \in \mathfrak{h}^*$, $\mu_1, \mu_2 \in \lambda + P$ and $w, w' \in W_{\lambda}$. Assume that $\lambda$ is dominant, i.e., $\langle \bar{\alpha}, \lambda \rangle \not\in \mathbb{Z}_{<0}$ for all $\alpha \in \Delta^+$. Then $\text{Hom}_G(L(M(w_1 \lambda), \delta M(\mu_1)), L(M(w_2 \lambda), \delta M(\mu_2))) \neq 0$ if and only if $w_1^{-1}w_1 A_{w_1 w_1^{-1} w_1 \lambda} \cap W_{\lambda}^0 w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset$.

Moreover, if $\text{Hom}_G(L(M(w_1 \lambda), \delta M(\mu_1)), L(M(w_2 \lambda), \delta M(\mu_2))) = 0$, then for all $k \in \mathbb{Z}_{\geq 0}$, we have $\text{Ext}^k_G(L(M(w_1 \lambda), \delta M(\mu_1)), L(M(w_2 \lambda), \delta M(\mu_2))) = 0$.

We can determine when there exists a nonzero homomorphisms between principal series representations of $G$ from Theorem 1.1 (see Lemma 3.2).

Let $T_w$ be the twisting functor for $w \in W$ [AL03, 6.2] and $w_0$ the longest element of $W$ (see also Arkhipov [Ark04, Definition 2.3.4]).

**Theorem 1.2.** We have $\text{Hom}_G(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) \neq 0$ if and only if $w_1 A_{w_1^{-1}}(\mu_1) \cap w_2 w_0 A_{w_0 w_2^{-1}} w_0(\mu_2) \neq \emptyset$.

Moreover, if $\text{Hom}_G(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) = 0$, then $\text{Ext}^k_G(T_{w_1} M(\mu_1), T_{w_2} M(\mu_2)) = 0$ for all $k \in \mathbb{Z}_{\geq 0}$.

The proof of this theorem gives a new proof of the famous result of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71] about homomorphisms between Verma modules.

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§2. General theory

We use the notation in Section 1. It is easy to prove the following lemma. We omit the proof.

**Lemma 2.1.** Let $s_1, \ldots, s_l, s'_1, \ldots, s'_l \in S_{\lambda}$ be simple reflections. Put $w = s_1 \cdots s_l$. Then we have $A_{(s_1, \ldots, s_l, s'_1, \ldots, s'_l)}(\mu) = \bigcup_{\mu' \in A_{(s_1, \ldots, s_l)}(\mu)} w A_{(s'_1, \ldots, s'_l)}(w^{-1} \mu')$.

Fix a dominant $\lambda \in \mathfrak{h}^*$. Let $C$ be an abelian category with enough injective objects, $\mathcal{D} \subset \mathfrak{h}^*$ a $W_\lambda$-stable subset. Let $\{M_\lambda(w, \mu) \mid w \in W, \mu \in \mathcal{D}\}$ be objects of $C$ such that the following conditions are satisfied:

(A1) For $w \in W_\lambda$ and $w' \in W_\lambda^0$, $M_\lambda(ww', \mu) \simeq M_\lambda(w, \mu)$. 

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(A2) For $\alpha \in \Pi_\lambda$ such that $s_\alpha w > w$, if $\langle \alpha, \mu \rangle \not\in \mathbb{Z}_{<0}$ then we have $M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, s_\alpha \mu)$.

(A3) For $\alpha \in \Pi_\lambda$ such that $s_\alpha w > w$, $\langle \alpha, \lambda \rangle \neq 0$ and $\langle \alpha, \mu \rangle \in \mathbb{Z}_{<0}$ there exists an exact sequence $0 \rightarrow M_\lambda(w, \mu) \rightarrow M_\lambda(s_\alpha w, \mu) \rightarrow M_\lambda(w, s_\alpha \mu) \rightarrow M_\lambda(w, \mu) \rightarrow 0$.

(A4) We have $\text{Hom}_C(M_\lambda(w, \mu'), M_\lambda(e, \mu)) \neq 0$ if and only if $\mu \in W^0_\lambda w_\lambda \mu'$.

(A5) We have $\text{Ext}_C^k(M_\lambda(w, \mu'), M_\lambda(e, \mu)) = 0$ for $k > 0$.

**Lemma 2.2.** Let $\alpha \in \Pi_\lambda$, $w \in W_\lambda$, $\mu \in D$. Assume that $\langle \alpha, \lambda \rangle = 0$. Then we have $M_\lambda(w, \mu) \simeq M_\lambda(w, s_\alpha \mu) \simeq M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(s_\alpha w, s_\alpha \mu)$.

**Proof.** If necessary, replacing $w$ by $s_\alpha w$, we may assume that $s_\alpha w < w$. By applying the condition (A1) as $w' = s_\alpha^{-1} w$, we get $M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, \mu)$ and $M_\lambda(s_\alpha w, s_\alpha \mu) \simeq M_\lambda(w, s_\alpha \mu)$. If $\langle \alpha, \mu \rangle \geq 0$, then $M_\lambda(w, \mu) \simeq M_\lambda(s_\alpha w, s_\alpha \mu)$ by the condition (A2). If $\langle \alpha, \mu \rangle \leq 0$, then $M_\lambda(w, s_\alpha \mu) \simeq M_\lambda(s_\alpha w, \mu)$ by the condition (A2). Hence we have $M_\lambda(w, \mu) \simeq M_\lambda(s_\alpha w, \mu) \simeq M_\lambda(w, s_\alpha \mu)$ for all $\mu$. \qed

**Lemma 2.3.** Let $w_1 \in W$, $w_2 = s_1 \cdots s_t$ be a reduced expression and $\mu_1, \mu_2 \in D$. Then the following conditions are equivalent.

1. $\text{Hom}_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0$.

2. There exists $k \in \mathbb{Z}_{\geq 0}$ such that $\text{Ext}_C^k(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0$.

3. $\mu_1 \in w_1 W^0_\lambda w_2^{-1} A(s_1, \ldots, s_t)(\mu_2)$.

**Proof.** Obviously, (1) implies (2). We prove the lemma by induction on $\ell_\lambda(w_2)$. If $w_2 = e$, then the lemma follows from the conditions (A4) and (A5).

Assume that $\ell_\lambda(w_2) > 0$. Take $\alpha \in \Pi_\lambda$ such that $s_1 = s_\alpha$. First assume that $\langle \alpha, w_2 \lambda \rangle = 0$. Then we have $W^0_\lambda(s_\alpha w_2)^{-1} A(s_2, \ldots, s_t)(\mu_2) = W^0_\lambda(s_\alpha w_2)^{-1} A(s_2, \ldots, s_t)(s_\alpha \mu_2)$ by Lemma 2.2 and induction hypothesis. By the definition, we have $A(s_\alpha)(\mu_2) = \{\mu_2\}$ or $A(s_\alpha)(\mu_2) = \{\mu_2, s_\alpha \mu_2\}$. Therefore $W^0_\lambda w_2^{-1} A(s_1, \ldots, s_t)(\mu_2) = W^0_\lambda(s_\alpha w_2)^{-1} A(s_2, \ldots, s_t)(s_\alpha \mu_2)$ by Lemma 2.1. This implies the lemma in the case of $\langle \alpha, w_2 \lambda \rangle = 0$.

In the rest of this proof, we assume that $\langle \alpha, w_2 \lambda \rangle \neq 0$. Assume that $\langle \alpha, \mu_2 \rangle \not\in \mathbb{Z}_{<0}$, then, by the condition (A2), $M_\lambda(w_2, \mu_2) \simeq M_\lambda(s_\alpha w_2, s_\alpha \mu_2)$. Since $A(s_\alpha)(\mu_2) = \{\mu_2\}$, we have $w_2^{-1} A(s_1, \ldots, s_t)(\mu_2) = (s_\alpha w_2)^{-1} A(s_2, \ldots, s_t)(s_\alpha \mu_2)$ by Lemma 2.1. Hence (1)–(3) are equivalent in this case.

Finally assume that $\langle \alpha, \mu_2 \rangle \in \mathbb{Z}_{<0}$. Then we have $A(s_\alpha)(\mu_2) = \{\mu_2, s_\alpha \mu_2\}$. This implies that $w_2^{-1} A(s_1, \ldots, s_t)(\mu_2) = (s_\alpha w_2)^{-1} A(s_2, \ldots, s_t)(s_\alpha \mu_2)$ by Lemma 2.1. By the induction hypothesis, $\mu_1 \not\in w_1 W^0_\lambda w_2^{-1} A(s_1, \ldots, s_t)(\mu_2)$ if and only if

$$\text{Hom}_C(M_\lambda(w_1, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = \text{Hom}_C(M_\lambda(w_1, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)) = 0.$$ 

From the condition (A3), we have an exact sequence

$$0 \rightarrow M_\lambda(s_\alpha w_2, \mu_2) \rightarrow M_\lambda(w_2, \mu_2) \rightarrow M_\lambda(s_\alpha w_2, s_\alpha \mu_2) \rightarrow M_\lambda(s_\alpha w_2, \mu_2) \rightarrow 0. \quad (2.1)$$

Since the functor $\text{Hom}_C$ is left-exact, we have an exact sequence

$$0 \rightarrow \text{Hom}_C(M_\lambda(w_1, \mu), M_\lambda(s_\alpha w_2, \mu_2)) \rightarrow \text{Hom}_C(M_\lambda(w_1, \mu), M_\lambda(w_2, \mu_2)) \rightarrow \text{Hom}_C(M_\lambda(w_1, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)).$$
If (3) does not hold, \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = 0 \). Hence we have \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) = 0 \), i.e., (1) does not hold. Therefore, (1) implies (3).

Now assume that (1) does not hold, i.e., \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) = 0 \). We prove \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = 0 \) and, for all \( k \in \mathbb{Z}_{\geq 0} \), \( \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) = 0 \). These imply the lemma.

By the exact sequence
\[
0 \to \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) \to \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)),
\]
we have \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = 0 \). Hence we have \( \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = 0 \) for all \( k \in \mathbb{Z}_{\geq 0} \) by induction hypothesis. Put \( L = \text{Ker}(M_\lambda(s_\alpha w_2, s_\alpha \mu_2) \to M_\lambda(s_\alpha w_2, \mu_2)) \).

From an exact sequence (2.1), we have exact sequences
\[
0 \to M_\lambda(s_\alpha w_2, \mu_2) \to M_\lambda(w_\lambda, \mu_2) \to L \to 0
\]
and
\[
0 \to L \to M_\lambda(s_\alpha w_2, s_\alpha \mu_2) \to M_\lambda(s_\alpha w_2, \mu_2) \to 0.
\]
Using \( \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, \mu_2)) = 0 \) and the long exact sequences induced from there sequences, we have
\[
\text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) \simeq \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), L) \simeq \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)).
\]
In particular, \( \text{Hom}_C(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)) = 0 \). By induction hypothesis, we have \( \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(s_\alpha w_2, s_\alpha \mu_2)) = 0 \) for all \( k \in \mathbb{Z}_{\geq 0} \). Hence we have \( \text{Ext}_C^k(M_\lambda(w_\lambda, \mu), M_\lambda(w_2, \mu_2)) = 0 \) for all \( k \in \mathbb{Z}_{\geq 0} \).

If for some abelian category \( C \) and some regular \( \lambda \) there exist objects which satisfy the conditions (A1–5), then the set \( A_{(s_1, \ldots, s_l)}(\mu) \) is independent of the choice of a reduced expression by Lemma 2.3. In the rest of this section, we assume it (It will be proved in Section 3). Put \( A_{w_\lambda \mu}(\mu) = A_{(s_1, \ldots, s_l)}(\mu) \).

**Theorem 2.4.** Let \( w_1, w_2 \in W_\lambda \) and \( \mu_1, \mu_2 \in \mathcal{D} \). The following conditions are equivalent.

1. \( \text{Hom}_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0 \).
2. There exists \( k \in \mathbb{Z}_{\geq 0} \) such that \( \text{Ext}_C^k(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \neq 0 \).
3. \( w_1^{-1}w_\lambda A_{w_\lambda w_1}(w_\lambda \mu_1) \cap W_\lambda w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset \).

**Proof.** We prove by backward induction on \( \ell_\lambda(w_1) \). If \( w_1 = w_\lambda \), then from Lemma 2.3, (1)–(3) are equivalent. We use the similar argument in the proof of Lemma 2.3.

Take \( \alpha \in \Pi_\lambda \) such that \( s_\alpha w_1 > w_1 \). Put \( \beta = -w_\lambda(\alpha) \in \Pi_\lambda \). We have \( A_{w_\lambda w_1}(w_\lambda \mu_1) = \bigcup_{\mu \in A_{s_\beta}(w_\lambda \mu_1)} s_\beta A_{w_\lambda s_\beta w_1}(s_{\beta \mu_0}) \) by Lemma 2.1. First assume that \( \langle \alpha, w_1 \lambda \rangle = 0 \). Then by Lemma 2.2, we have \( M_\lambda(w_1, \mu_1) \simeq M_\lambda(s_\alpha w_1, \mu_1) \simeq M_\lambda(s_\alpha w_1, s_\alpha \mu_1) \). This implies the lemma.

In the rest of this proof, we assume that \( \langle \alpha, w_1 \lambda \rangle \neq 0 \). First assume that \( \langle \alpha, \mu_1 \rangle \notin \mathbb{Z}_{\geq 0} \), then by the condition (A2), \( M_\lambda(w_1, \mu_1) \simeq M_\lambda(s_\alpha w_1, s_\alpha \mu_1) \). Since \( A_{s_\alpha}(w_\lambda \mu_1) = \{w_\lambda \mu_1\} \), \( A_{w_\lambda w_1}(w_\lambda \mu_1) = A_{w_\lambda s_\alpha w_1}(w_\lambda s_\alpha \mu_0) \). Hence we have the lemma.
Finally, we assume that \((\alpha, \mu_1) \in \mathbb{Z}_{>0}\). We have \(w_1^{-1} w_\lambda A(w_\lambda \mu_1) \cap W_\lambda^0 w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset\) if and only if \((s_{\alpha} w_1)^{-1} w_\lambda A_{w_\lambda s_{\alpha} w_1}(w_\lambda \mu_1) \cap W_\lambda^0 w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset\) or \((s_{\alpha} w_1)^{-1} w_\lambda A_{w_\lambda s_{\alpha} w_1}(w_\lambda s_{\alpha} \mu_1) \cap W_\lambda^0 w_2^{-1} A_{w_2}(\mu_2) \neq \emptyset\) since \(A_{s_{\alpha}}(w_\lambda \mu_1) = \{w_\lambda \mu_1, w_\lambda s_{\alpha} \mu_1\}\). By the condition (A2), we have \(M_\lambda(w_1, s_{\alpha} \mu_1) \simeq M_\lambda(s_{\alpha} w_1, \mu_1)\). Hence, there exists an exact sequence \(0 \rightarrow M_\lambda(s_{\alpha} w_1, \mu_1) \rightarrow M_\lambda(s_{\alpha} w_1, s_{\alpha} \mu_1) \rightarrow M_\lambda(w_1, \mu_1) \rightarrow M_\lambda(s_{\alpha} w_1, \mu_1) \rightarrow 0\) by the condition (A3). Therefore (1) implies (3).

Now assume (1) does not hold, i.e., \(\text{Hom}_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) = 0\). We prove that \(\text{Hom}_C(M_\lambda(s_{\alpha} w_1, \mu_1), M_\lambda(w_2, \mu_2)) = \text{Hom}_C(M_\lambda(s_{\alpha} w_1, s_{\alpha} \mu_1), M_\lambda(w_2, \mu_2)) = 0\) and, for all \(k \in \mathbb{Z}_{>0}\), \(\text{Ext}^k_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) = 0\). Since we have an exact sequence
\[
0 \rightarrow \text{Hom}_C(M_\lambda(s_{\alpha} w_1, \mu_1), M_\lambda(w_2, \mu_2)) \rightarrow \text{Hom}_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)),
\]
we have \(\text{Hom}_C(M_\lambda(s_{\alpha} w_1, \mu_1), M_\lambda(w_2, \mu_2)) = 0\). Hence, by induction hypothesis, we have that \(\text{Ext}^k_C(M_\lambda(s_{\alpha} w_1, \mu_1), M_\lambda(w_2, \mu_2)) = 0\). Therefore we have
\[
\text{Ext}^k_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \simeq \text{Ext}^k_C(M_\lambda(s_{\alpha} w_1, s_{\alpha} \mu_1), M_\lambda(w_2, \mu_2)).
\]
In particular, \(\text{Hom}_C(M_\lambda(s_{\alpha} w_1, s_{\alpha} \mu_1), M_\lambda(w_2, \mu_2)) = 0\). Hence we have
\[
\text{Ext}^k_C(M_\lambda(w_1, \mu_1), M_\lambda(w_2, \mu_2)) \simeq \text{Ext}^k_C(M_\lambda(s_{\alpha} w_1, s_{\alpha} \mu_1), M_\lambda(w_2, \mu_2)) = 0.
\]

\[\square\]

§3. Proof of the main theorems

In this section, we prove Theorem 1.1 and Theorem 1.2 using the result of Section 2. First we consider the twisted Verma modules. Fix a regular dominant integral element \(\lambda\). Put \(C = O\), \(D = \mathfrak{h}^*\). Set \(M_\lambda(w, \mu) = T_{w^{-1} w_\lambda} M(w_0 \mu)\).

**Lemma 3.1.** The modules \(\{M_\lambda(w, \mu)\}\) satisfy the conditions (A1–5).

**Proof.** The condition (A1) is obvious since \(\lambda\) is regular. The conditions (A2) and (A3) are [AL03, Proposition 6.3]. Since \(T_{w_\mu} M(w_0 \mu) \simeq \delta M(\mu)\) [AL03, Corollary 5.1], we have \(\text{Hom}_O(M_\lambda(w, \mu), M_\lambda(\varepsilon, \mu)) = \text{Hom}_O(M(w_0 \mu), \delta M(\mu))\). Since \(\text{Hom}_O(M(w_0 \mu), \delta M(\mu)) \neq 0\) if and only if \(w_0 \mu = \mu\), we have (A4). Moreover, we have \(\bigoplus_{\mu' \in \mathfrak{h}^*} \text{Ext}^k_O(M(w_0 \mu'), \delta M(\mu)) \simeq H^k(\mathfrak{n}, \delta M(\mu)) \simeq H_k(\overline{\mathfrak{n}}, M(\mu)) = 0\) where \(\overline{\mathfrak{n}}\) is the nilradical of the opposite Borel subalgebra of \(\mathfrak{h}\). Hence we have (A5). \[\square\]

From Lemma 3.1 and Theorem 2.4, we have Theorem 1.2.

Next, we consider the principal series representations of \(G\). This is a full-subcategory of \(g \oplus g\)-modules. We also regard \(\mathcal{H}\) as a full-subcategory of \(g\)-bimodules.

**Lemma 3.2.** Let \(\lambda, \mu \in \mathfrak{h}^*\) such that \(\lambda - \mu \in \mathcal{P}\), \(w \in \mathcal{W}\). Put \(\Delta^- = -\Delta^+\) and \(\Delta^-_\lambda = -\Delta^+_\lambda\).

1. There exists \(w' \in \mathcal{W}_\lambda\) such that \(\Delta^+ \cap (w' w^{-1})^{-1} \Delta^- \cap w \Delta^-_\lambda = \emptyset\).

2. Take \(w'\) as in (1). Then we have \(L(M(w_\lambda), \delta M(w \mu)) \simeq L(M(w' \lambda), \delta M(w' \mu))\)
Proof. (1) Since $w^{-1} \Delta^+ \cap \Delta_\lambda$ is a positive system of $\Delta_\lambda$, there exists $w' \in W_\lambda$ such that $w^{-1} \Delta^+ \cap \Delta_\lambda = (w')^{-1} \Delta^+ \cap \Delta_\lambda$. Since $(w')^{-1} \Delta^- \cap \Delta_\lambda = (w')^{-1} (\Delta^- \cap \Delta_\lambda) = (w')^{-1} \Delta^- \cap \Delta_\lambda$, we have $\Delta^+ \cap (w'w^{-1})^{-1} \Delta^- \cap \Delta_\lambda = w(w^{-1} \Delta^+ \cap (w')^{-1} \Delta^- \cap \Delta_\lambda) = w(w^{-1} \Delta^+ \cap (w')^{-1} \Delta^- \cap \Delta_\lambda) = \emptyset$.

(2) By the condition of $w'$, for all $\alpha \in \Delta^+ \cap (w'w^{-1})^{-1} \Delta^-$ we have $\langle \alpha, -w\lambda \rangle \notin \mathbb{Z}$. Hence by [Duf77, 4.8. Proposition] we have $L(M(w\lambda), \delta M(w\mu)) \simeq L(M(w'\lambda), \delta M(w'\mu))$. $\square$

By Lemma 3.2, it is sufficient to study $\text{Hom}_\mathcal{X}(L(M(w'\lambda), \delta M(\mu')), L(M(w\lambda), \delta M(\mu)))$ for dominant $\lambda$ and $w, w' \in W_\lambda$. Moreover, we may assume $\mu \in W_\lambda \mu'$ since $L(M(w\lambda), \delta M(\mu)) = 0$ unless $w\lambda - \mu \in \mathcal{P}$. Fix such a $\lambda$ and put $M_\lambda(w, \mu) = L(M(w\lambda), \delta M(\mu))$ for $\mu \in \lambda + \mathcal{P}$ and $w \in W_\lambda$. Put $D = \lambda + \mathcal{P}$. For $\alpha \in \Pi_\lambda$, let $C_\alpha$ be Joseph’s Enriques functor [Jos82]. Recall that $M \in \mathcal{O}$ is called $\alpha$-free if the canonical map $M \to C_\alpha M$ is injective.

**Lemma 3.3.** Let $\mu \in \lambda + \mathcal{P}$ and $\alpha \in \Pi_\lambda$.

1. If $N \in \mathcal{O}$ is $\alpha$-free and $\langle \alpha, \mu \rangle \in \mathbb{Z}_{\leq 0}$ then $L(M_\lambda(s_\alpha \mu), C_\alpha N) \simeq L(M_\lambda(\mu), N).

2. Let $w \in W_\lambda$. If $\langle \alpha, w\lambda \rangle \in \mathbb{Z}_{\leq 0}$ and $\langle \alpha, \mu \rangle \in \mathbb{Z}_{\leq 0}$, then $L(M(s_\alpha w\lambda), M(s_\alpha \mu)) \simeq L(M(w\lambda), M(\mu))$.

3. Let $w \in W_\lambda$. If $\langle \alpha, w\lambda \rangle \in \mathbb{Z}_{\leq 0}$ and $\langle \alpha, \mu \rangle \in \mathbb{Z}_{\geq 0}$, then $L(M(s_\alpha w\lambda), \delta M(s_\alpha \mu)) \simeq L(M(w\lambda), \delta M(\mu))$.

4. We have $L(M(w\lambda), \delta M(\mu)) \simeq L(M(\lambda), M(w\lambda \mu))$.

5. The modules $\{M_\lambda(w, \mu)\}$ satisfy the conditions (A1–5).

**Proof.** (1) Put $M = M(\mu)$ and $M' = M(s_\alpha \mu)$ in [Jos82, 3.8. Lemma]. Then we get (1).

(2) Take $N = M(\mu)$ in (1) and use [Jos82, 2.5. Lemma].

(3) Let $\lambda \in \lambda + \mathcal{P}$ be a regular element such that $\lambda$ is dominant. Then by [Jos83, 2.5. Lemma], we have $C_\alpha \delta M(\lambda) \simeq \delta M(s_\alpha \lambda)$. For $g \oplus g$-module $N$, let $N^\lambda$ be a $g \oplus g$-module where the action is twisted by $(X, Y) \mapsto (Y, X)$. Using [Jos82, 2.8], we have $L(M(\lambda), C_\alpha \delta M(\mu)) \simeq L(M(s_\alpha \lambda), \delta M(\mu)) \simeq L(M(\mu), \delta M(s_\alpha \mu))^\lambda \simeq L(M(\mu), C_\alpha \delta M(\lambda))^\eta \simeq L(M(\lambda), \delta M(\mu))^\eta \simeq L(M(\lambda), \delta M(s_\alpha \mu))$ by (1). Therefore we have $C_\alpha \delta M(\mu) \simeq \delta M(s_\alpha \mu)$. We get (3) by (1).

(4) Take $w \in W_\lambda$ such that $\langle \beta, w\mu \rangle \in \mathbb{Z}_{\leq 0}$ for all $\beta \in \Delta_\lambda^+$. Put $\mu_0 = w\mu$. Let $w = s_{\alpha_1} \cdots s_{\alpha_1}$ be a reduced expression. Then we have $\langle \alpha_i, s_{\alpha_1} \cdots s_{\alpha_1} \mu \rangle \in \mathbb{Z}_{\geq 0}$ and $\langle \alpha_i, s_{\alpha_1} \cdots s_{\alpha_1} w\lambda \rangle \in \mathbb{Z}_{\leq 0}$. Hence by (3), we have $L(M(w\lambda), \delta M(\mu)) \simeq L(M(ww\lambda), \delta M(\mu_0))$. Take a reduced expression of $ww\lambda$ and use (2), then we have $L(M(ww\lambda), M(\mu_0)) \simeq L(M(\lambda), M(w\lambda \mu))$ by the same argument. Since $\delta M(\mu_0) \simeq M(\mu_0)$, we have (4).

(5) The condition (A1) is obvious. The condition (A2) follows from (2) and (A3) from [Jos82, 4.7. Corollary]. To prove (A4) and (A5), we may assume that $\mu' \in W_\lambda \mu$. Let $\mu_1 \in W_\lambda \mu$ such that $\langle \beta, \mu_1 \rangle \geq 0$ for all $\beta \in \Delta_\lambda^+$. Take $w, w' \in W_\lambda$ such that $\mu' = w' \mu_1$ and $\mu = w \mu_1$. Then by the argument in (4), we have $L(M(w\lambda), \delta M(\mu)) \simeq L(M(w\lambda(w')^{-1} w\lambda_1), \delta M(w \mu_1))$ and $L(M(\lambda), \delta M(\mu)) \simeq L(M(w \lambda^{-1} \lambda), \delta M(\mu_1))$. We prove (A4) and (A5). First we assume that $\mu_1$
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is regular. Then by (4), we have

\begin{align*}
\text{Ext}^k_H(M_\lambda(w, \mu'), M_\lambda(e, \mu)) \\
\cong \text{Ext}^k_H(L(M(w, \lambda^{-1}w, \lambda), \delta M(w, \lambda)), L(M(w^{-1}\lambda), \delta M(\mu))) \\
\cong \text{Ext}^k_H(L(M(w, \lambda^{-1}w, \lambda), \delta M(w, \lambda)), L(M(w^{-1}\lambda), \delta M(\mu)))^\sigma \\
\cong \text{Ext}^k_H(L(M(w, \lambda), \delta M(w^{-1}w, \lambda)), L(M(\mu), \delta M(w^{-1}\lambda))) \\
\cong \text{Ext}^k_H(L(M(\mu), M((w')^{-1}w, \lambda)), L(M(\mu), \delta M(w^{-1}\lambda)))
\end{align*}

By the Bernstein-Gelfand-Joseph-Enright equivalence [BG80, 5.9. Theorem], this space is isomorphic to \(\text{Ext}^k(H(M((w')^{-1}w, \lambda), \delta M(w^{-1}\lambda))\). Hence the proof is done in this case (see the proof of Lemma 3.3).

We prove (A4) and (A5) for general \(\mu_1\). Take a regular element \(\mu_2 \in \mu_1 + \mathcal{P}\) such that for all \(\beta \in \Delta_\chi\). Let \(T^\mu_\mu\) be the translation functor of \(\mathcal{O}\) and \(L^\mu_\mu\) the translation functor of \(\mathcal{H}\) with respect to the left \(g\)-action. Then we have \(L^\mu_\mu L(M, N) = L(M, T^\mu_\mu N)\) for \(M, N \in \mathcal{O}\). Since \(T^\mu_\mu\) commutes with \(\delta\), we have

\begin{align*}
\text{Ext}^k_H(M_\lambda(w, \mu'), M_\lambda(e, \mu)) \\
= \text{Ext}^k_H(L(M(w, \lambda), T^\mu_\mu M(w, \mu_2)), L(M(\lambda), T^\mu_\mu \delta M(w, \mu_2))) \\
= \text{Ext}^k_H(L(M(w, \lambda), T^\mu_\mu M(w, \mu_2)), L(M(\lambda), T^\mu_\mu \delta M(w, \mu_2))) \\
= \text{Ext}^k_H(L(M(w, \lambda), M(w', \mu_2)), L(M(\lambda), T^\mu_\mu \delta M(w, \mu_2))) \\
= \text{Ext}^k_H(L(M(w, \lambda), M(w', \mu_2)), L(M(\lambda), \delta T^\mu_\mu T^\mu_\mu M(w, \mu_2)))
\end{align*}

The module \(T^\mu_\mu T^\mu_\mu M(w, \mu_2)\) has a filtration \(0 = M_\lambda \subset M_1 \subset \cdots \subset M_r = T^\mu_\mu T^\mu_\mu M(w, \mu_2)\) such that \(\{M_i/M_{i-1} \mid 1 \leq i \leq r\} = \{M(vw, \mu_2) \mid v \in \mathcal{W}_\mu\}\) [Jan79, 2.3 Satz (b)]. Since \(\lambda\) is dominant, \(L(M(\lambda), \cdot)\) is an exact functor. Hence we have an exact sequence \(0 \to L(M(\lambda), M) \to L(M(\lambda), M_{i-1}) \to L(M(\lambda), M_i/M_{i-1}) \to 0\). Using the long exact sequence and the result in regular case, we have \(\text{Ext}^k(H(M_\lambda, \mu'), M_\lambda(e, \mu)) = 0\) for \(k > 0\). Moreover, by the vanishing of the Ext-groups,

\[\dim \text{Hom}_H(M_\lambda(w, \mu'), M_\lambda(e, \mu)) = \sum_{v \in \mathcal{W}_\mu} \dim \text{Hom}_H(L(M(w_\lambda, \lambda)), L(M(\lambda), \delta M(v, \mu_2))).\]

From this formula, we have \(\text{Hom}_H(M_\lambda(w, \mu'), M_\lambda(e, \mu)) \neq 0\) if and only if \(w' \in \mathcal{W}_\mu\) and \(v \in \mathcal{W}_\mu\) for some \(v \in \mathcal{W}_\mu\). This condition is equivalent to \(\mu' = \mu_1 \in \mathcal{W}_\mu\) and \(w_\lambda w_\lambda w \mu_1 = w_\lambda w_\lambda w \mu_1\).

From Lemma 3.3 and Theorem 2.4, we have Theorem 1.1.

References


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