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Bifurcation from flat-layered solutions to reaction diffusion systems in two space dimensions

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Abstract. Bifurcation from equilibrium solutions to reaction diffusion systems is considered in a two-dimensional domain. This solution has an internal transition layer that forms a flat interface. If the length of the interface in the tangential direction is small enough, the equilibrium solution is stable, but it is unstable if the length is larger than some critical value. In this paper, it is shown that bifurcation occurs at this critical length. We construct the bifurcating solutions and discuss their stability. Numerical results suggest that the bifurcation is subcritical.

1. Introduction

Formation of spatial patterns in various two-phase systems is largely governed by what is going on around the *interface* between the phases. Because of this, it is important to study the property of interfaces. In this paper, we deal with two-phase systems in the framework of reaction diffusion equations, in which the interfaces appear to have an internal structure. Such an internal structure, though confined in a very thin region, plays an important role in determining the property of the interface.

To be more precise, we study the following system of nonlinear partial differential equations of parabolic type:

$$\text{(1.1a)} \qquad \left.\begin{aligned} \tau u_t &= \varepsilon\Delta u + \frac{1}{\varepsilon}f(u,v) \\ v_t &= D\Delta v + g(u,v) \end{aligned}\right. \qquad \text{in } \Omega(\ell),\ t>0,$$

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with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial\Omega(\ell),\ t > 0. \tag{1.1b}$$

The assumptions on f and g will be given at the end of this section. Both D and τ are some positive constants, while $\varepsilon > 0$ is a small parameter. Here Δ is the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega(\ell)$. The domain $\Omega(\ell)$ is a rectangle in the x-y plane:

$$\Omega(\ell) = (0, 1) \times (0, \ell), \tag{1.2}$$

where $\ell > 0$ is an arbitrary given number. The general theory of semilinear parabolic equations shows that for every bounded $u_0(x)$ and $v_0(x)$, the solution to (1.1) under the initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) \qquad \text{in } \Omega(\ell)$$

is unique and exists globally in time (see Henry [6], Smoller [16]). We study the bifurcation of equilibrium solutions from the flat-layered solution defined below.

If D, τ satisfy the assumptions as in Theorem 2.1 in §2, and if ε is sufficiently small, there exists a nontrivial equilibrium solution $(\bar{u}(x), \bar{v}(x))$ to

$$\begin{gathered} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u, v), \quad v_t = D v_{xx} + g(u, v) \qquad \text{in } I,\ t > 0, \\ u_x = 0 = v_x \qquad \text{on } \partial I,\ t > 0, \end{gathered} \tag{1.3}$$

where $I = (0, 1)$. This solution has an internal transition layer as in Fig. 1. The construction is given in Nishiura and Fujii [10] via the singular perturbation method (see also Fife [4], Ito [7], Mimura, Tabata and Hosono [9], and Sakamoto [13]). The stability of $(\bar{u}, \bar{v})$ is proved by studying the linearized eigenvalue problem in [10]; see also Nishiura and Mimura [11].

By setting

$$\begin{aligned} \bar{u}(x, y) &= \bar{u}(x) \\ \bar{v}(x, y) &= \bar{v}(x) \end{aligned} \qquad \text{for all } x \in (0, 1),\ y \in (0, \ell), \tag{1.4}$$

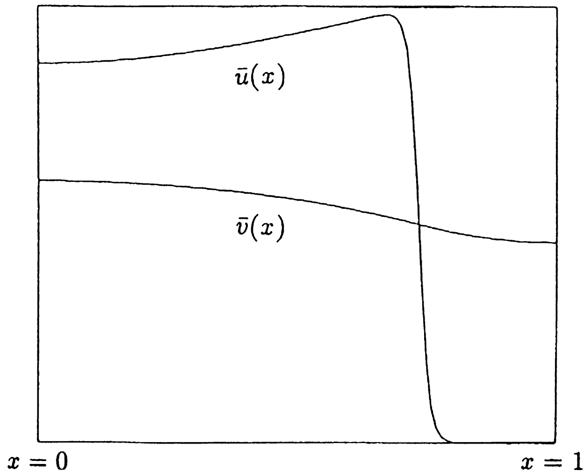

Fig. 1. The graph of $(\bar{u}, \bar{v})$.

we get an equilibrium solution to (1.1). This solution has an internal transition layer that forms a flat interface in $\Omega(\ell)$. The smaller we take ε, the sharper the layer becomes. There exists some critical length $\ell_c(\varepsilon)$ where the solution (1.4) changes stability. This phenomenon was first observed by Ohta, Mimura and Kobayashi [12] for a prototype model in which f is a discontinuous piecewise linear function of a simple form. Analysis of the full model (1.1) was done by Nishiura and myself [17]. More precisely, we have the following:

THEOREM ([17], Stability criterion for flat-layered solutions). *There exists $\varepsilon_1 = \varepsilon_1(f, g, D, \tau)$ such that, for any $\varepsilon \in (0, \varepsilon_1)$, the equilibrium solution $(\bar{u}, \bar{v})$ is asymptotically stable if $\ell < \ell_c(\varepsilon)$, and unstable if $\ell > \ell_c(\varepsilon)$. Here $\ell_c(\varepsilon)$ is some positive-valued function of ε with*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1/2} \ell_c(\varepsilon) = \pi \widehat{\zeta}_0(0)^{-1/2} > 0, \tag{1.5}$$

where $\widehat{\zeta}_0(0) > 0$ is given in (2.9).

From the point of view of the bifurcation theory (see [5] for instance), one may suspect that bifurcation occurs at $\ell = \ell_c(\varepsilon)$, where the flat-layered solution $(\bar{u}, \bar{v})$ becomes unstable. To show that this is truly the case, one needs to further analyze the linearized eigenvalue problems at $\ell = \ell_c(\varepsilon)$. However the analysis of the linearized problems is rather complicated, especially in two or more space dimensions rather than in one space dimension. In the linearized eigenvalue problems of one space dimension, the eigenfunctions have useful limiting forms as $\varepsilon \downarrow 0$. This fact is essential for showing the occurrence of a bifurcation in [11] or [8] for small $\varepsilon > 0$. In two or more space dimensional linearized problems, we generally cannot extract useful information from the limiting forms of the eigenfunctions. limiting forms as $\varepsilon \downarrow 0$. In this paper, we shall apply the general theory of Crandall and Rabinowitz ([2], [3]) to the present problem to prove the occurrence of a bifurcation. For this purpose, one needs to study the eigenfunctions associated with the zero eigenvalue in the linearized problems. These eigenfunctions turn out to have no useful limiting forms as $\varepsilon \downarrow 0$. Therefore we must study precisely the forms of these eigenfunctions for small but positive ε.

Our main results are the following:

THEOREM 1.1. *Assume that* $\varepsilon > 0$ *is sufficiently small for* f, g, D *and* τ. *Then there exists an open interval* $\mathcal{I} = (s_1, s_2)$ *containing* 0 *such that, for every* $s \in \mathcal{I} \setminus \{0\}$, *a bifurcating solution* $(\tilde{u}(s), \tilde{v}(s))$ *to* (1.1) *exists when* $\ell = \ell(s)$. *Here* $(\tilde{u}(s), \tilde{v}(s))$ *is given by* (4.10) *and satisfies* $(\tilde{u}(0), \tilde{v}(0)) = (\bar{u}, \bar{v})$, *and* $\ell(s)$ *is some real-valued smooth function in* $\mathcal{I}$ *with* $\ell(0) = \ell_c(\varepsilon)$. *There exist no equilibrium solutions to* (1.1) *other than* $(\bar{u}, \bar{v})$ *and* $\{(\tilde{u}(s), \tilde{v}(s)); s \in \mathcal{I}\}$ *near* $\ell = \ell_c(\varepsilon)$, $(u, v) = (\bar{u}, \bar{v})$ *in an appropriate function space to be specified in Proposition* 4.1.

For more details, see Proposition 4.1 in §4. The bifurcation equation, which we shall write as $G(s, \ell) = 0$, is also given in §4 by (4.14). $G(s, \ell)$ is a real-valued function defined in some neighborhood of $(0, \ell_c(\varepsilon))$, and is smooth with respect to s and ℓ. Using $G_{sss}(0, \ell_c(\varepsilon))$ given by (4.24), we have the following:

COROLLARY 1.1. *If* $G_{sss}(0, \ell_c(\varepsilon)) > 0$, *the bifurcating solution* $(\tilde{u}(s), \tilde{v}(s))$ *is stable, and* $\ell(s) > \ell_c(\varepsilon)$ *for every* $s \in \mathcal{I} \backslash \{0\}$ *(supercritical bifurcation, see Fig. 2. (a)). And if* $G_{sss}(0, \ell_c(\varepsilon)) < 0$, *the bifurcating solution* $(\tilde{u}(s), \tilde{v}(s))$ *is unstable, and* $\ell(s) < \ell_c(\varepsilon)$ *for every* $s \in \mathcal{I} \backslash \{0\}$ *(subcritical bifurcation, see Fig. 2. (b)).*

REMARK 1.1. The right-hand side of (4.24) contains the eigenfunc-

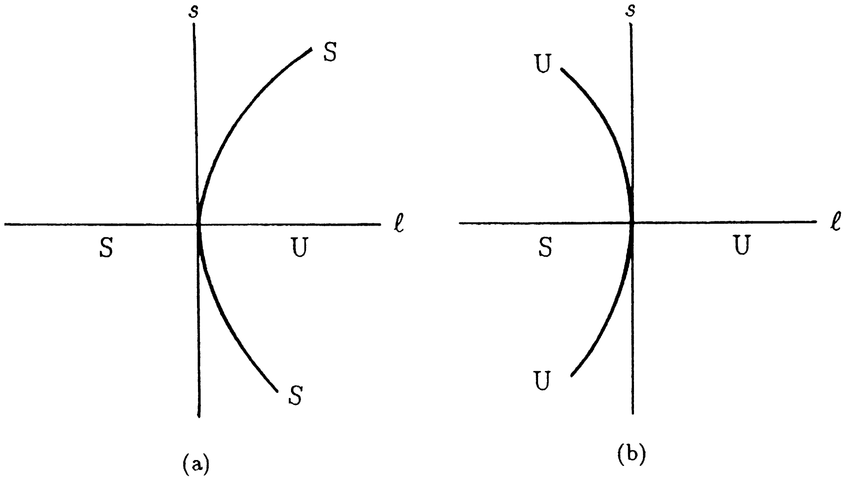

Fig. 2

tions W_1 and W_1^* given by (3.1) and (3.20), respectively. We see that the terms of (3.2) and (3.21) that contain $\overline{\kappa}(\varepsilon)$ $(= \pi/\ell_c(\varepsilon))$ vanish in the limit of $\varepsilon \downarrow 0$ by using Lemma 2.3 and (2.34). Hence one needs to further study (4.24) for small but positive ε. The direction of the bifurcation is yet to be determined.

Let $\varepsilon \in (0, \varepsilon_1)$ be arbitrarily fixed, and ℓ be slightly larger than $\ell_c(\varepsilon)$. Then $(\bar{u}, \bar{v})$ is an unstable equilibrium solution, and in view of Remark 3.2, the linearized eigenvalue problem (2.16) has a unique positive eigenvalue, and the corresponding wavelength of the associated eigenfunction is ℓ. All other eigenvalues of (2.16) have negative real parts.

Figure 3 shows the evolution of randomly perturbed flat interfaces for the data

$$\begin{gathered} f(u,v) = u^2(\tfrac{3}{2} - u) - \tfrac{1}{2}uv, \quad g(u,v) = \tfrac{3}{4}uv - \tfrac{1}{10}v - \tfrac{2}{5}v^2, \\ D = 0.4, \quad \tau = 1, \quad \varepsilon = 0.0075. \end{gathered} \tag{1.6}$$

The interfaces are described in terms of the contour lines of $u - h_0(v)$, where h_0 is to be defined later in this section.

In Fig. 3. (b), the interface changes its shape largely as it evolves. From

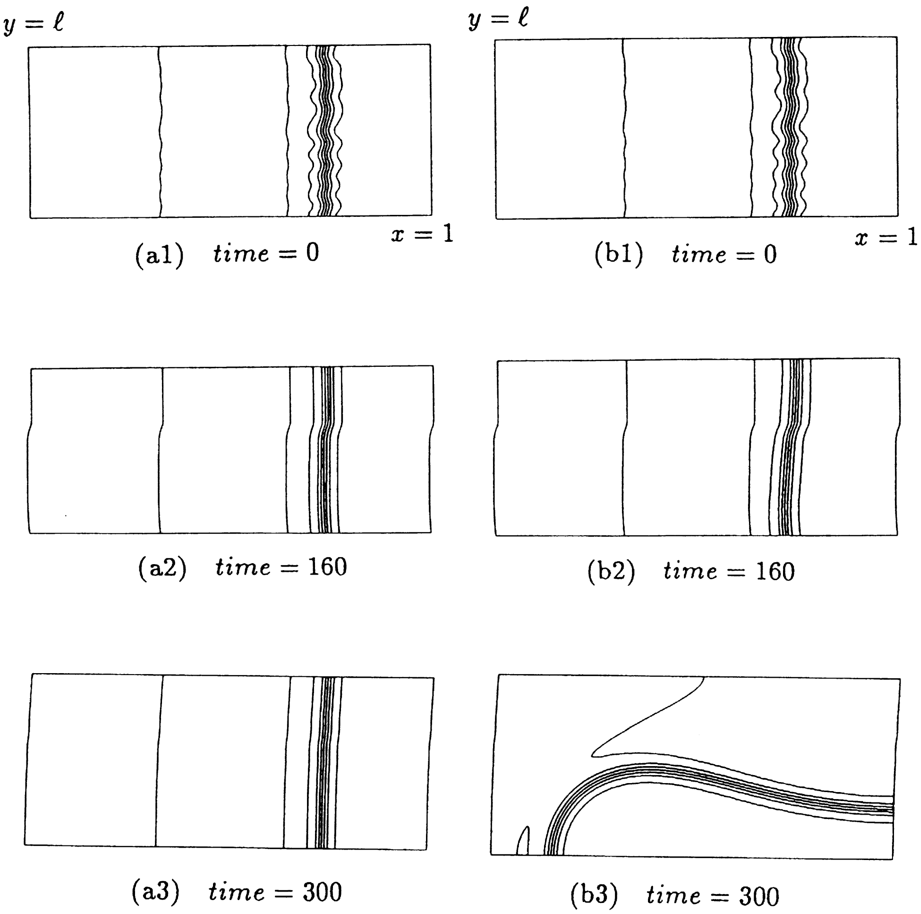

Fig. 3. The evolution of perturbed interfaces from originally flat ones: (a) $\ell = 0.43$; (b) $\ell = 0.45$.

(1.5), we have

$$\ell_c(\varepsilon) = \pi\widehat{\zeta}_0(0)^{-1/2}\varepsilon^{1/2} + o(\varepsilon^{1/2}),$$

for which the principal term is numerically calculated to be 0.35 in [17], but the residual term is not estimated. Figure 3 suggests that

$$0.43 < \ell_c(\varepsilon) < 0.45$$

for (1.6). Moreover one may speculate from Fig. 3 that the bifurcation in

Theorem 1.1 is subcritical. A picture similar to Fig. 3 has been obtained in [12], where the function g is varied rather than the length parameter.

This paper is organized as follows. §2 is a preliminary section, in which we define notations used in subsequent sections and summarize known results. In §3 we study the zero eigenvalue and the associated eigenspace when $\ell = \ell_c(\varepsilon)$. In §4 we give the proof of Theorem 1.1 and Corollary 1.1.

Now we state the standing assumptions for f and g throughout this paper.

(A1) f, g are smooth functions of u, v defined on some open set $\mathcal{O}$ in $\mathbb{R}^2$.

(A2) The nullcline $\{(u,v) \in \mathcal{O}; f(u,v) = 0\}$ is S-shaped and consists of three curves defined by

$$\begin{aligned} C_0 &= \{(h_0(v), v); v \in (\underline{v}, \overline{v})\}, \\ C_- &= \{(h_-(v), v); v > \underline{v}\}, \\ C_+ &= \{(h_+(v), v); v < \overline{v}\}. \end{aligned}$$

Here $h_0(v)$, $h_-(v)$, $h_+(v)$ are continuous functions with $h_-(v) < h_0(v) < h_+(v)$ for any $v \in (\underline{v}, \overline{v})$.

(A3) Let $J(v) \stackrel{\text{def}}{=} \int_{h_-(v)}^{h_+(v)} f(s,v)\,ds$, then there exists a unique $v^* \in (\underline{v}, \overline{v})$ such that $J(v^*) = 0$, $J'(v^*) < 0$.

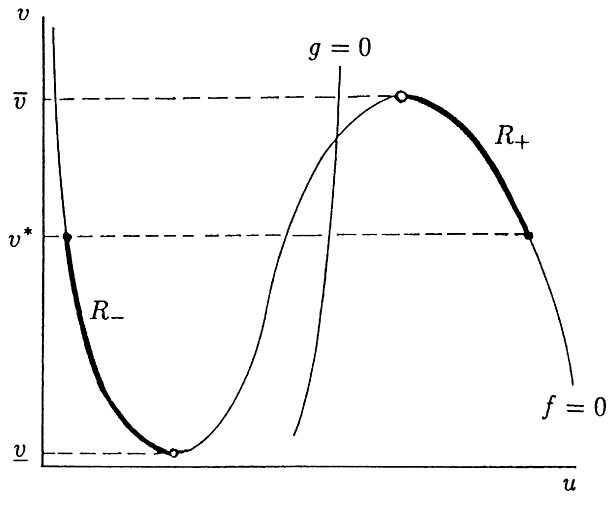

Fig. 4

(A4) The nullcline of g intersects transversally with that of f. Let the intersection point on C_i, if it exists, be denoted by $P_i = (h_i(v_i), v_i)$, for $i = -, 0, +$. Then we assume that $v_- < v^* < v_+$.

(A5)

(a) $f_u < 0$ on $R_- \cup R_+$, where $R_\pm$ are defined by

$$R_- = \{(h_-(v), v); v_- < v \leq v^*\}, \quad R_+ = \{(h_+(v), v); v^* \leq v < v_+\};$$

(b) $g|_{R_-} < 0 < g|_{R_+}$;

(c) $(f_u g_v - f_v g_u)|_{R_- \cup R_+} > 0$, $g_v|_{R_- \cup R_+} \leq 0$.

It should be noted that, in order to satisfy (A1)$\sim$(A5), it is not necessary to assume that f and g intersect in the way shown in Fig. 4.

2. Preliminaries

We first summarize known results for the existence and stability of the one-dimensional equilibrium solutions to (1.3) with a single layer.

THEOREM 2.1 ([10], [11]). *Assume that f, g satisfy* (A1) $\sim$ (A5). *Then for each sufficiently large D (say $D > D_*$), there exists $\tau_* > 0$ such that, for any fixed $\tau \in (\tau_*, \infty)$,* (1.3) *has a stable equilibrium solution $(\bar{u}(x), \bar{v}(x))$ for any $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 = \varepsilon_0(f, g, D, \tau) > 0$. The set $\{(\bar{u}(x), \bar{v}(x)); \varepsilon \in (0, \varepsilon_0)\}$ is bounded in $C(\bar{I}) \times C^2(\bar{I})$. Moreover there exists $x^* \in I$ and a monotone decreasing function $V \in C^1(\bar{I})$ such that*

$$\begin{aligned} &\bar{u} \to U && \text{in} \quad C([0, x^* - \sigma] \cup [x^* + \sigma, 1]), \\ &\bar{v} \to V && \text{in} \quad C^1(\bar{I}), \end{aligned} \tag{2.1}$$

as $\varepsilon \downarrow 0$, where σ is an arbitrary positive number and

$$U(x) \overset{\text{def}}{=} \begin{cases} h_+(V(x)) & \text{if} \quad x \in [0, x^*], \\ h_-(V(x)) & \text{if} \quad x \in (x^*, 1]. \end{cases} \tag{2.2}$$

PROOF. See Theorem 1.1 in [10] for the existence of $(\bar{u}(x), \bar{v}(x))$ (see also [4], [7], [9] and [13]). For the stability, see [11] and [10]. □

REMARK 2.1. One can choose the constant D_* in Theorem 2.1 in such a way that the function $V(x)$ does not exist if $D < D_*$. For more details, see Proposition 1.1 in [10]. An explicit expression of τ_* is given by (2.31).

For the subsequent discussions, it is useful to know the spectral behavior of the following Sturm-Liouville problem:

$$L(\varepsilon)\phi = \zeta\phi \quad \text{in } I, \qquad \phi_x = 0 \quad \text{at } x = 0, 1, \tag{2.3}$$

where $L(\varepsilon)$ is a self-adjoint operator defined by

$$L(\varepsilon) \stackrel{\text{def}}{=} \varepsilon^2 \frac{d^2}{dx^2} + f_u(\bar{u}(x), \bar{v}(x)). \tag{2.4}$$

Let $\{\phi_i(\varepsilon)\}_{i\geq 0}$ be the complete orthonormal system in $L^2(I)$ consisting of the eigenfunctions of $L(\varepsilon)$, and let $\{\zeta_i(\varepsilon)\}_{i\geq 0}$ be the associated eigenvalues. The eigenvalues $\{\zeta_i(\varepsilon)\}_{i\geq 0}$ are all real and simple ($\zeta_0(\varepsilon) > \zeta_1(\varepsilon) > \zeta_2(\varepsilon) > \cdots$).

LEMMA 2.1 ([10; Lemmas 1.4, 2.3]).

(1) *There exists $\zeta_* = \zeta_*(f, g) > 0$ such that*

$$\zeta_i(\varepsilon) < -\zeta_* < 0 < \zeta_0(\varepsilon) \quad (i = 1, 2, 3, \cdots) \tag{2.5}$$

holds for any $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is the same as in Theorem 2.1. *Moreover $\widehat{\zeta}_0(\varepsilon) \stackrel{\text{def}}{=} \varepsilon^{-1}\zeta_0(\varepsilon)$ converges to $\widehat{\zeta}_0(0) > 0$ as $\varepsilon \downarrow 0$. Here $\widehat{\zeta}_0(0)$ is given by* (2.9b).

(2) *Let h_1, h_2 be defined by*

$$h_1(x, \varepsilon) \stackrel{\text{def}}{=} -\varepsilon^{-\frac{1}{2}} f_v(\bar{u}(x), \bar{v}(x))\phi_0(x, \varepsilon), \tag{2.6a}$$

$$h_2(x, \varepsilon) \stackrel{\text{def}}{=} \varepsilon^{-\frac{1}{2}} g_u(\bar{u}(x), \bar{v}(x))\phi_0(x, \varepsilon). \tag{2.6b}$$

Then they satisfy

$$h_i(x, \varepsilon) \underset{\varepsilon\downarrow 0}{\longrightarrow} c_i^*\delta(x - x^*) \quad \textit{in } (H^1(I))', \tag{2.7}$$

where $\delta(x - x^*)$ is Dirac's δ-function at x^*, and

$$c_1^* \stackrel{\text{def}}{=} -\gamma J'(v^*), \qquad c_2^* \stackrel{\text{def}}{=} \gamma\{g(h_+(v^*), v^*) - g(h_-(v^*), v^*)\}. \tag{2.8}$$

Here $(H^1(I))'$ denotes the dual space of $H^1(I)$, and γ is a positive constant given by (2.9a).

REMARK 2.2. In [10], γ and $\widehat{\zeta}_0(0)$ in Lemma 2.1 are explicitly given as follows. Let $\Phi(\xi)$ be a monotone increasing function in $C^\infty(\mathbb{R})$ defined as the unique solution to

$$\Phi_{\xi\xi}(\xi) + f(\Phi(\xi), v^*) = 0, \quad \Phi(0) = h_0(v^*), \quad \Phi(\pm\infty) = h_\pm(v^*).$$

Then

$$\gamma \stackrel{\text{def}}{=} \|d\Phi/d\xi\|_{L^2(I)}^{-1}, \tag{2.9a}$$

$$\widehat{\zeta}_0(0) \stackrel{\text{def}}{=} -\gamma^2 J'(v^*) D^{-1} \int_0^{x^*} g(U(x), V(x)) dx > 0. \tag{2.9b}$$

Let us now go back to the two-dimensional problem (1.1). We set

$$\begin{aligned} u(x,y,t) &= \hat{u}(x,\eta,t) + \bar{u}(x) \\ v(x,y,t) &= \hat{v}(x,\eta,t) + \bar{v}(x) \end{aligned} \tag{2.10}$$

for $x \in (0,1)$, $y \in (0,\ell)$, where $\eta = \ell^{-1}y$. From the definition of $(\bar{u},\bar{v})$,

$$\varepsilon^2 \bar{u}_{xx} + f(\bar{u},\bar{v}) = 0, \quad D\bar{v}_{xx} + g(\bar{u},\bar{v}) = 0. \tag{2.11}$$

Substituting (2.10) into (1.1), and using (2.11), we obtain

$$\begin{aligned} \varepsilon\tau\hat{u}_t &= \varepsilon^2 \left(\frac{\partial^2}{\partial x^2} + \ell^{-2} \frac{\partial^2}{\partial \eta^2} \right) \hat{u} - f(\bar{u},\bar{v}) + f(\hat{u}+\bar{u}, \hat{v}+\bar{v}), \\ \hat{v}_t &= D \left(\frac{\partial^2}{\partial x^2} + \ell^{-2} \frac{\partial^2}{\partial \eta^2} \right) \hat{v} - g(\bar{u},\bar{v}) + g(\hat{u}+\bar{u}, \hat{v}+\bar{v}). \end{aligned} \tag{2.12}$$

From now on, we denote η, $\hat{u}(x,\eta,t)$, $\hat{v}(x,\eta,t)$ by the letters y, $u(x,y,t)$, $v(x,y,t)$, respectively. We denote the square domain $(0,1)^2$ by Ω. Then (2.12) is written as

$$\begin{aligned} \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} &= \mathcal{F}(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) \quad \text{in } \Omega,\ t>0, \\ \frac{\partial u}{\partial \nu} &= 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega,\ t>0, \end{aligned} \tag{2.13}$$

where $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$. Here

$$\mathcal{F}(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tau}\{\varepsilon^2(\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2})u + f(u+\bar{u}, v+\bar{v}) - f(\bar{u},\bar{v})\} \\ D(\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2})v + g(u+\bar{u}, v+\bar{v}) - g(\bar{u},\bar{v}) \end{pmatrix},$$

where $\hat{\tau}$ denotes $(\varepsilon\tau)^{-1}$. We put

$$\mathcal{Y} = L^2(\Omega) \times L^2(\Omega), \quad \mathcal{X} = \mathcal{D} \times \mathcal{D}, \tag{2.14}$$

where $\mathcal{D}$ denotes the domain of the associated Laplace operator Δ in $L^2(\Omega)$ under the Neumann boundary conditions. It holds that $\mathcal{D} \subset H^2(\Omega)$. $\mathcal{F}$ is a smooth mapping from $\mathbb{R}_+ \times \mathcal{X}$ to $\mathcal{Y}$.

The flat-layered equilibrium solution (1.4) now corresponds to the solution $(u,v) \equiv (0,0)$ to the time-independent equation

$$\mathcal{F}(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 = \frac{\partial v}{\partial \nu} \quad \text{on } \partial\Omega. \tag{2.15}$$

We consider the linearized eigenvalue problem of (2.13) at $(u,v) \equiv (0,0)$:

$$\lambda \begin{pmatrix} w \\ z \end{pmatrix} = \mathcal{L}(\ell) \begin{pmatrix} w \\ z \end{pmatrix} \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial \nu} = 0 = \frac{\partial z}{\partial \nu} \quad \text{on } \partial\Omega, \tag{2.16}$$

where

$$\mathcal{L}(\ell) \stackrel{\text{def}}{=} \mathcal{F}_{\binom{u}{v}}\,(\ell, \begin{pmatrix} 0 \\ 0 \end{pmatrix}). \tag{2.17}$$

The linear operator $\mathcal{L}(\ell)$ is expressed in the form

$$\mathcal{L}(\ell) = \begin{pmatrix} \hat{\tau}\{\varepsilon^2(\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2}) + f_u\} & \hat{\tau} f_v \\ g_u & D\{\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2}\} + g_v \end{pmatrix}.$$

Here we denote $f_u(\bar{u},\bar{v})$ by f_u and so on. This abbreviation will be used hereafter.

It is convenient to use a complete orthonormal system $\{Y_m\}_{m=0}^{\infty}$ in $L^2(I)$. Here

$$Y_m(y) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{for } m = 0, \\ \sqrt{2}\cos(m\pi y) & \text{for } m > 0. \end{cases}$$

For (w, z) in (2.16), we put

$$\begin{pmatrix} w_m \\ z_m \end{pmatrix} = P_m \begin{pmatrix} w \\ z \end{pmatrix}, \tag{2.18}$$

where

$$P_m \begin{pmatrix} w \\ z \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \int_0^1 w(x,y)Y_m(y)dy, \\ \int_0^1 z(x,y)Y_m(y)dy, \end{pmatrix}. \tag{2.19}$$

Then (w, z) is expanded as follows.

$$w(x,y) = \sum_{m=0}^{\infty} w_m(x)Y_m(y), \quad z(x,y) = \sum_{m=0}^{\infty} z_m(x)Y_m(y) \tag{2.20}$$

in $L^2(\Omega)$. This decomposes (2.16) into countably many eigenvalue problems in $I = (0, 1)$:

$$\begin{aligned} &\lambda \begin{pmatrix} w_m \\ z_m \end{pmatrix} = \mathcal{L}_m(\ell) \begin{pmatrix} w_m \\ z_m \end{pmatrix} \quad \text{in } I, \\ &\frac{dw_m}{dx} = 0 = \frac{dz_m}{dx} \quad \text{at } x = 0, 1, \end{aligned} \tag{2.21}$$

where

$$\mathcal{L}_m(\ell) \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tau}\{\varepsilon^2(\frac{d^2}{dx^2} - \frac{m^2\pi^2}{\ell^2}) + f_u\} & \hat{\tau} f_v \\ g_u & D(\frac{d^2}{dx^2} - \frac{m^2\pi^2}{\ell^2}) + g_v \end{pmatrix}.$$

REMARK 2.3. If (2.21) has an eigenvalue $\lambda \in \mathbb{C}$ and an eigenfunction $(w_m(x), z_m(x))$, then $(w_m(x)Y_m(y), w_m(x)Y_m(x))$ satisfies (2.16) with the same λ. Conversely, if (2.16) has an eigenvalue $\lambda \in \mathbb{C}$ and an eigenfunction (w, z), then $(w_m(x), z_m(x))$ given by (2.18) satisfies (2.21) with the same λ for any $m \in \bar{\mathbb{N}}$. Moreover $(w_m(x), z_m(x))$ is non-trivial for some $m \in \bar{\mathbb{N}}$.

We put

$$\kappa = \frac{m\pi}{\ell}. \tag{2.22}$$

Though κ takes only discrete values for any fixed $\ell > 0$, it is often useful for computational purposes to treat κ like a continuous variable in $[0, \infty)$.

LEMMA 2.2 ([17; Proposition 3.1]). *There exists $\lambda_* = \lambda_*(f, g, D) > 0$, and a positive function $\bar{\delta}(\varepsilon)$, with $\bar{\delta}(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, such that, if $\lambda \in \mathbb{C}$ with $Re\,\lambda \geq -\lambda_*$ is an eigenvalue of* (2.21) *for some $m \in \bar{\mathbb{N}}$, then it holds that*

$$|\varepsilon^2\kappa^2 + \varepsilon\tau\lambda| < \bar{\delta}(\varepsilon), \tag{2.23}$$

where κ is as in (2.22).

It suffices to study only those eigenvalues of (2.21) or (2.16) that have non-negative real parts, because the other ones have nothing to do with the stability properties of the equilibrium solution. In particular, it suffices to deal with the eigenvalues in

$$\mathbb{C}_{\lambda_*} \stackrel{\text{def}}{=} \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda \geq -\lambda_*\}.$$

Then Lemma 2.2 allows us to assume (2.23), and also assume $\lambda_* < \zeta_*/2$ without loss of generality, where ζ_* is as in Lemma 2.1. It should be noted that $\bar{\delta}(\varepsilon)$ is independent of (κ, λ). From (2.5) and (2.23), it holds that for small ε

$$|\zeta_i(\varepsilon) - \varepsilon^2\kappa^2 - \varepsilon\tau\lambda| \geq \zeta_*/2 > 0 \quad (i = 1, 2, \dots), \tag{2.24}$$

which guarantees the existence and boundedness of (2.25) below.

In what follows, we state some basic facts about the eigenvalue problem (2.21). Let E be the projection in $L^2(I)$ onto the subspace $\operatorname{span}\{\phi_i(\varepsilon)\}_{i\geq 1}$. We introduce operators from $L^2(I)$ to $L^2(I)$ as follows:

$$\begin{aligned}
P(\varepsilon, \kappa, \lambda) &\stackrel{\text{def}}{=} (L(\varepsilon) - \varepsilon^2\kappa^2 - \varepsilon\tau\lambda)^{-1}\{E(-f_v\,\cdot)\},\\
Q(\varepsilon, \kappa, \lambda) &\stackrel{\text{def}}{=} (L(\varepsilon) - \varepsilon^2\kappa^2 - \varepsilon\tau\lambda)^{-1}\{E(g_u\,\cdot)\},\\
R(\varepsilon, \kappa, \lambda) &\stackrel{\text{def}}{=} -g_v - g_u P(\varepsilon, \kappa, \lambda),\\
S(\varepsilon, \kappa, \lambda) &\stackrel{\text{def}}{=} -g_v + f_v Q(\varepsilon, \kappa, \lambda),
\end{aligned} \tag{2.25}$$

when $\varepsilon > 0$. And when $\varepsilon = 0$, we set

$$P(0, \kappa, \lambda) \stackrel{\text{def}}{=} -\, f_v/f_u|_{u=U(x),v=V(x)}\,, \tag{2.26a}$$

$$Q(0, \kappa, \lambda) \stackrel{\text{def}}{=} g_u/f_u|_{u=U(x),v=V(x)}\,, \tag{2.26b}$$

$$R(0, \kappa, \lambda) \stackrel{\text{def}}{=} (f_u g_v - f_v g_u)/(-f_u)|_{u=U(x),v=V(x)}\,, \tag{2.26c}$$

$$S(0, \kappa, \lambda) \stackrel{\text{def}}{=} (f_u g_v - f_v g_u)/(-f_u)|_{u=U(x),v=V(x)}\,. \tag{2.26d}$$

Note that the right-hand sides of (2.26) are independent of (κ, λ), and that those of (2.26c), (2.26d) are a strictly positive function on I, by virtue of (2.2) and (A5) in §1. Next we define bilinear forms on $H^1(I)$ as follows:

$$B(\varepsilon,\kappa,\lambda)(z^1,z^2) \stackrel{\text{def}}{=} D\left(z_x^1, z_x^2\right)_{L^2(I)} + \left(\left(R + D\kappa^2 + \lambda\right) z^1, z^2\right)_{L^2(I)},$$
$$B^*(\varepsilon,\kappa,\lambda)(z^1,z^2) \stackrel{\text{def}}{=} D\left(z_x^1, z_x^2\right)_{L^2(I)} + \left(\left(S + D\kappa^2 + \lambda\right) z^1, z^2\right)_{L^2(I)},$$

for $z^1, z^2 \in H^1(I)$. Here we denote $R(\varepsilon,\kappa,\lambda)$, $S(\varepsilon,\kappa,\lambda)$ by R, S, respectively. We also define operators from $H^1(I)$ to $(H^1(I))'$ by

$$T(\varepsilon,\kappa,\lambda)z \stackrel{\text{def}}{=} -Dz_{xx} + (R(\varepsilon,\kappa,\lambda) + D\kappa^2 + \lambda)z,$$
$$T^*(\varepsilon,\kappa,\lambda)z \stackrel{\text{def}}{=} -Dz_{xx} + (S(\varepsilon,\kappa,\lambda) + D\kappa^2 + \lambda)z,$$

for $z \in H^1(I)$. By applying the Lax-Milgram theorem, we have the following:

Lemma 2.3. *There exist $\varepsilon_* = \varepsilon_*(f,g,D) > 0$, $\delta_* = \delta_*(f,g,D) > 0$ $(0 < \delta_* < \zeta_*/2)$ such that (1) $\sim$ (5) hold true for any $\varepsilon \in [0,\varepsilon_*)$, $\kappa \in [0,\infty)$ and $\lambda \in \mathbb{C}_{\lambda_*}$ that satisfy*

$$|\varepsilon^2\kappa^2 + \varepsilon\tau\lambda| < \delta_*. \tag{2.27}$$

(1) *$P(\varepsilon,\kappa,\lambda)$, $Q(\varepsilon,\kappa,\lambda)$, $R(\varepsilon,\kappa,\lambda)$, $S(\varepsilon,\kappa,\lambda)$ are uniformly bounded linear operators in $L^2(I)$ for $(\varepsilon,\kappa,\lambda)$.*
(2) *$B(\varepsilon,\kappa,\lambda)$, $B^*(\varepsilon,\kappa,\lambda)$ are bounded and coercive bilinear forms on $H^1(I)$.*
(3) *$T(\varepsilon,\kappa,\lambda)$, $T^*(\varepsilon,\kappa,\lambda)$ have the inverses $K(\varepsilon,\kappa,\lambda)$, $K^*(\varepsilon,\kappa,\lambda)$ that are uniformly bounded linear operators from $(H^1(I))'$ to $H^1(I)$ for $(\varepsilon,\kappa,\lambda)$, respectively.*
(4) *If (2.23) is assumed in addition, then*

$$\begin{matrix} K(\varepsilon,\kappa,\lambda) \to K(0,\kappa,\lambda) \\ K^*(\varepsilon,\kappa,\lambda) \to K(0,\kappa,\lambda) \end{matrix} \quad \textit{in} \quad \mathcal{B}((H^1(I))', H^1(I)),$$

as $\varepsilon \downarrow 0$. *The convergence is uniform for* (κ, λ) *on any compact subset of* $[0, \infty) \times \mathbb{C}_{\lambda_*}$. *Here* $\mathcal{B}((H^1(I))', H^1(I))$ *denotes the space of bounded linear operators from* $(H^1(I))'$ *to* $H^1(I)$.

(5) *There exists* $M_1 = M_1(f, g, D) > 0$ *such that, for any* $\kappa > 0$,

$$\|K(\varepsilon, \kappa, \lambda)\|_{\mathcal{B}((H^1(I))', L^\infty(I))} < M_1 \cdot \kappa^{-1/2}, \tag{2.28a}$$

$$\|K^*(\varepsilon, \kappa, \lambda)\|_{\mathcal{B}((H^1(I))', L^\infty(I))} < M_1 \cdot \kappa^{-1/2}, \tag{2.28b}$$

$$|(K(\varepsilon, \kappa, \lambda)^2 h_2(\varepsilon), h_1(\varepsilon))_{L^2(I)}| < M_1 \cdot \kappa^{-5/2}. \tag{2.29}$$

Proof. We have (1) from Theorem 2.1 and (2.24). The proof of (2), (3) and (4) can be carried out by the same argument as in Lemma 3.1 in [10]. For the proof of (2.28a), (2.29), see Lemmas 4.3, 4.4 and 4.5 in [17]. The proof of (2.28b) can be done just as in the proof of Lemmas 4.3, 4.4 in [17]. □

We consider a Sturm-Liouville operator $T_0 \overset{\text{def}}{=} T(0, 0, 0)$ subject to the Neumann boundary condition. Let $\{\gamma_n\}_{n=0}^\infty$ and $\{\psi_n\}_{n=0}^\infty$ be the eigenvalues of T_0 and the complete orthonormal system consisting of the associated eigenfunctions, respectively. It follows from the general theory of Sturm-Liouville problems (see [1]) that $\{\psi_n\}$ is bounded in $C(\bar{I})$, that $\gamma_n = O(n^2)$ (as $n \uparrow \infty$), and that

$$K(0, \kappa, \lambda)\delta(x - x_*) = \sum_{n=0}^{\infty} \frac{\psi_n(x^*)\psi_n(x)}{\gamma_n + D\kappa^2 + \lambda} \qquad \text{in } C(\bar{I}). \tag{2.30}$$

An explicit expression of the quantity τ_* in Theorem 2.1 can now be given by

$$\tau_* \overset{\text{def}}{=} c_1^* c_2^* \|K(0, 0, -\lambda_*)\delta(x - x_*)\|_{L^2(I)}^2 = \sum_{n=0}^{\infty} \frac{c_1^* c_2^* |\psi_n(x^*)|^2}{(\gamma_n - \lambda_*)^2}. \tag{2.31}$$

The second equality of (2.31) follows from (2.30). Our standing hypothesis for τ is

$$\tau_* < \tau. \tag{2.32}$$

The following proposition is fundamental in regard to the eigenvalue problem (2.21). Let $\varepsilon_1 = \varepsilon_1(f, g, D, \tau)$ be sufficiently small, then we have the following:

PROPOSITION 2.1 ([17]). *Under the same assumptions on D and τ as in Theorem* 2.1, *assertions* (1), (2) *and* (3) *below hold true for any fixed* $\varepsilon \in (0, \varepsilon_1)$*:*

(1) *Every eigenvalue* $\lambda \in \mathbb{C}_{\lambda_*}$ *of* (2.21) *is a real number in* $(-\lambda_*, M_2)$, *where* $M_2 = M_2(f, g, D) > 0$ *is independent of* ε, m *and* ℓ.

(2) *There exist* $\underline{\kappa}(\varepsilon)$, $\overline{\kappa}(\varepsilon)$ $(0 < 2\underline{\kappa}(\varepsilon) < \overline{\kappa}(\varepsilon) < \infty)$ *such that* (2.21) *has a non-negative eigenvalue if and only if*

$$\kappa \overset{\text{def}}{=} \frac{m\pi}{\ell} \in [\underline{\kappa}(\varepsilon), \overline{\kappa}(\varepsilon)]. \tag{2.33}$$

Moreover the following limits exist:

$$\underline{\kappa}(0) \overset{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \underline{\kappa}(\varepsilon) \in (0, \infty), \qquad \lim_{\varepsilon \downarrow 0} \varepsilon \overline{\kappa}(\varepsilon)^2 = \widehat{\zeta}_0(0). \tag{2.34}$$

(3) *For each* κ *satisfying* (2.33), *the problem* (2.21) *has a unique eigenvalue that lies in* $\mathbb{C}_{\lambda_*}$. *This eigenvalue, denoted by* $\widetilde{\lambda}(\varepsilon, \kappa)$, *is simple, and the associated eigenfunction is given by*

$$\begin{pmatrix} w_m(x) \\ z_m(x) \end{pmatrix} = \begin{pmatrix} \varepsilon^{-\frac{1}{2}} \phi_0(\varepsilon) + (PK)(\varepsilon, \kappa, \lambda) h_2(\varepsilon) \\ K(\varepsilon, \kappa, \lambda) h_2(\varepsilon), \end{pmatrix}. \tag{2.35}$$

Furthermore the real-valued function $\kappa \mapsto \widetilde{\lambda}(\varepsilon, \kappa)$ *satisfies*

(i) $\widetilde{\lambda}(\varepsilon, \cdot\,) \in C^\infty[\underline{\kappa}(\varepsilon), \overline{\kappa}(\varepsilon)]$;
(ii) $\widetilde{\lambda}(\varepsilon, \underline{\kappa}(\varepsilon)) = 0 = \widetilde{\lambda}(\varepsilon, \overline{\kappa}(\varepsilon))$;
(iii) $\widetilde{\lambda}(\varepsilon, \kappa) > 0$ *for any* $\kappa \in (\underline{\kappa}(\varepsilon), \overline{\kappa}(\varepsilon))$.

PROOF. See Theorem 4.1, Proposition 4.1 and Lemma 3.2 in [17] for the proof. □

REMARK 2.4. The critical length $\ell_c(\varepsilon)$ mentioned in §1 is given by

$$\ell_c(\varepsilon) = \pi \overline{\kappa}(\varepsilon)^{-1}. \tag{2.36}$$

In fact, if $\ell < \ell_c(\varepsilon)$, then there does not exist a positive integer m satisfying (2.33). Therefore all the eigenvalues of (2.16) have negative real parts.

Proposition 2.1, in particular, tells us that it suffices to consider only real eigenvalues of (2.21) or (2.16) as far as the stability of the flat-layered equilibrium solution is concerned.

3. The simplicity of the zero eigenvalue when $\ell = \ell_c(\varepsilon)$

In this section we show that the largest eigenvalue of (2.16) is zero if $\ell = \ell_c(\varepsilon)$; moreover zero is an algebraically simple eigenvalue of this linearized eigenvalue problem. This fact is necessary for using the methods of Crandall and Rabinowitz ([2], [3]) in §4.

We can easily check the geometric simplicity of zero and the fact that all other eigenvalues are negative (if $\ell = \ell_c(\varepsilon)$), but we must go further on. The following proposition shows that the algebraic and geometric multiplicities of a non-negative eigenvalue coincide with each other.

Proposition 3.1. *Suppose that D and τ satisfy the assumptions as in Theorem* 2.1*, and that ε is sufficiently small. For arbitrary $\ell > 0$ and an arbitrary non-negative eigenvalue λ of* (2.16)*, it holds that*

$$\mathcal{N}(\lambda - \mathcal{L}(\ell)) = \mathcal{N}((\lambda - \mathcal{L}(\ell))^2),$$

where $\mathcal{N}$ denotes the null space.

We prove this proposition later in this section. Proposition 3.1 will be used to prove Corollary 3.1 below, the main concern of this section, but it also has another important implication concerning the dynamical behavior of interfaces: namely, if we add a small perturbation to the equilibrium state (that is, the flat-layered solution), this perturbation will grow in proportion to $\exp(\lambda t)$ for some non-negative eigenvalue λ, and never will it grow in proportion to $t\exp(\lambda t), t^2\exp(\lambda t), \cdots$. This fact is particularly important when λ is the largest eigenvalue (say $\lambda_{\max}$) of the linearized eigenvalue problem (2.16). Because if the given small perturbation is a random one, it will grow in proposition to $\exp(\lambda_{\max} t)$.

Corollary 3.1. *When $\ell = \ell_c(\varepsilon)$, zero is an eigenvalue of* (2.16) *with $\mathcal{N}(\mathcal{L}(\ell_c(\varepsilon))) = span\{W_1\}$. Here W_1 is defined by*

$$W_1 \overset{\text{def}}{=} \begin{pmatrix} w_1(x)Y_1(y) \\ z_1(x)Y_1(y) \end{pmatrix}, \tag{3.1}$$

where

$$\begin{pmatrix} w_1(x) \\ z_1(x) \end{pmatrix} = \begin{pmatrix} \varepsilon^{-\frac{1}{2}}\phi_0(\varepsilon) + (PK)(\varepsilon, \overline{\kappa}(\varepsilon), 0)h_2(\varepsilon) \\ K(\varepsilon, \overline{\kappa}(\varepsilon), 0)h_2(\varepsilon), \end{pmatrix}. \tag{3.2}$$

The range $\mathcal{R}(\mathcal{L}(\ell_c(\varepsilon)))$ is a closed subspace with codimension 1 *and*

$$W_1 \notin \mathcal{R}(\mathcal{L}(\ell_c(\varepsilon))). \tag{3.3}$$

All other eigenvalues have strictly negative real parts.

In order to show the assertions stated above, we make some preparations. Let us introduce the adjoint operators

$$\mathcal{L}^*(\ell) \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tau}\{\varepsilon^2(\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2}) + f_u\} & g_u \\ \hat{\tau} f_v & D(\frac{\partial^2}{\partial x^2} + \ell^{-2}\frac{\partial^2}{\partial y^2}) + g_v \end{pmatrix},$$
$$\mathcal{L}_m^*(\ell) \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tau}(\varepsilon^2\frac{d^2}{dx^2} + f_u - \varepsilon^2\frac{m^2\pi^2}{\ell^2}) & g_u \\ \hat{\tau} f_v & D\frac{d^2}{dx^2} + g_v - D\frac{m^2\pi^2}{\ell^2} \end{pmatrix}$$

of $\mathcal{L}(\ell)$, $\mathcal{L}_m(\ell)$, respectively.

REMARK 3.1. Remark 2.3 remains valid if we replace $\mathcal{L}(\ell)$ by $\mathcal{L}^*(\ell)$, and $\mathcal{L}_m(\ell)$ by $\mathcal{L}_m^*(\ell)$.

If λ is a real eigenvalue of $\mathcal{L}_m(\ell)$, it is also an eigenvalue of $\mathcal{L}_m^*(\ell)$, and we can express the associated eigenfunction as follows.

LEMMA 3.1. *For any fixed $\varepsilon \in (0, \varepsilon_1)$, $\lambda \in [0, \infty)$ is an eigenvalue of $\mathcal{L}_m^*(\ell)$ if and only if $\lambda = \widetilde{\lambda}(\varepsilon, \kappa)$ with $\kappa = m\pi/\ell$. The associated eigenfunction is given by*

$$\begin{pmatrix} w_m^*(x) \\ z_m^*(x) \end{pmatrix} = \begin{pmatrix} -\varepsilon\tau\{\varepsilon^{-\frac{1}{2}}\phi_0(\varepsilon) + (QK^*)(\varepsilon, \kappa, \lambda)h_1(\varepsilon)\} \\ K^*(\varepsilon, \kappa, \lambda)h_1(\varepsilon) \end{pmatrix}. \tag{3.4}$$

PROOF. The proof can be carried out in just the same way as in Lemma 3.2 of [17] if we replace the *singular dispersion relation*

$$\widehat{\zeta}_0(\varepsilon) - \varepsilon\kappa^2 - \tau\lambda - (K(\varepsilon, \kappa, \lambda)h_2(\varepsilon), h_1(\varepsilon))_{L^2(I)} = 0 \tag{3.5}$$

thereof by

$$\widehat{\zeta}_0(\varepsilon) - \varepsilon\kappa^2 - \tau\lambda - (K^*(\varepsilon, \kappa, \lambda)h_1(\varepsilon), h_2(\varepsilon))_{L^2(I)} = 0. \tag{3.6}$$

It should be noted that (3.5) is equivalent to $\lambda = \widetilde{\lambda}(\varepsilon, \kappa)$ (see Proposition 4.3 in [17]). It suffices to show the equivalence of (3.5) and (3.6). Since λ is a real number, we have

$$(K(\varepsilon, \kappa, \lambda)h_2, h_1)_{L^2(I)} = (K^*(\varepsilon, \kappa, \lambda)h_1, h_2)_{L^2(I)} \tag{3.7}$$

for any $h_1, h_2 \in L^2(I)$. The equivalence follows immediately. $\square$

Proof of Proposition 3.1. Let $(\tilde{w}, \tilde{z}) \in \mathcal{X}$ satisfy

$$(\lambda - \mathcal{L}(\ell))^2 \begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

We will show that $(\tilde{w}, \tilde{z})$ belongs to $\mathcal{N}(\lambda - \mathcal{L}(\ell))$. By applying P_m on both sides and using $P_m \mathcal{L}(\ell) = \mathcal{L}_m(\ell) P_m$, we obtain for each $m \in \bar{\mathbb{N}}$

$$(\lambda - \mathcal{L}_m(\ell))^2 P_m \begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.9}$$

From this equality and Proposition 2.1, we find

$$(\lambda - \mathcal{L}_m(\ell)) P_m \begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix} = c_m \begin{pmatrix} w_m(x) \\ z_m(x) \end{pmatrix} \tag{3.10}$$

with some $c_m \in \mathbb{R}$. Here (w_m, z_m) is given by (2.35). Since $\mathcal{L}_m(\ell)$ has compact resolvent, $\mathcal{R}(\lambda - \mathcal{L}_m(\ell))$ is a closed subspace, and the closed range theorem is available. Then we have

$$\mathcal{R}(\lambda - \mathcal{L}_m(\ell)) = \mathcal{N}(\lambda - \mathcal{L}_m^*(\ell))^{\perp}. \tag{3.11}$$

From (3.10) and (3.11), we see that the right-hand side of (3.10) belongs to $\mathcal{N}(\lambda - \mathcal{L}_m^*(\ell))^{\perp}$.

It suffices to show

$$0 \neq \left(\begin{pmatrix} w_m^* \\ z_m^* \end{pmatrix}, \begin{pmatrix} w_m \\ z_m \end{pmatrix} \right)_{L^2(I) \times L^2(I)} \qquad \text{for any } m \in \bar{\mathbb{N}}. \tag{3.12}$$

Here (w_m^*, z_m^*) is as in (3.4). In fact, we see from (3.12) and (3.10) that $c_m = 0$, and that

$$P_m(\lambda - \mathcal{L}(\ell)) \begin{pmatrix} \tilde{w} \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.13}$$

for any $m \in \bar{\mathbb{N}}$. This means that $(\tilde{w}, \tilde{z})$ belongs to $\mathcal{N}(\lambda - \mathcal{L}(\ell))$.

Let us show (3.12). The right-hand side of (3.12) is equal to

$$(w_m, w_m^*)_{L^2(I)} + (z_m, z_m^*)_{L^2(I)}.$$

From (2.35), (3.4) and Lemma 2.3,

$$(w_m, w_m^*)_{L^2(I)} = -\tau\{1 + O(\varepsilon^{\frac{1}{2}})\} \quad \text{as } \varepsilon \downarrow 0. \tag{3.14}$$

The convergence of the second term is uniform with respect to (κ, λ), where $\kappa = m\pi/\ell$. On the other hand, since

$$(z_m, z_m^*)_{L^2(I)} = (K(\varepsilon, \kappa, \lambda)h_2(\varepsilon), K^*(\varepsilon, \kappa, \lambda)h_1(\varepsilon))_{L^2(I)},$$

we obtain

$$(z_m, z_m^*)_{L^2(I)} \leq \tau_* \qquad \text{for any } m \in \bar{\mathbb{N}}. \tag{3.15}$$

Indeed, if not, we have $\{(\bar{\varepsilon}_n, \kappa_n)\}$ such that $\bar{\varepsilon}_n \to 0$ as $n \uparrow \infty$, and

$$(K(\bar{\varepsilon}_n, \kappa_n, \lambda_n)h_2(\bar{\varepsilon}_n), K^*(\bar{\varepsilon}_n, \kappa_n, \lambda_n)h_1(\bar{\varepsilon}_n))_{L^2(I)} > \tau_*. \tag{3.16}$$

In the case where $\kappa_n \uparrow \infty$, we have a contradiction from (2.28). In the case where $\{\kappa_n\}$ is bounded, we can assume without loss of generality that $\kappa_n \to \kappa_0$ and $\lambda_n \to \lambda_0$. Recall that $\{\lambda_n\}$ remains bounded by virtue of Proposition 2.1. Here $\kappa_0 \geq 0$ and $\lambda_0 \geq -\lambda_*$. Then the left-hand side of (3.16) converges to

$$\sum_{n=0}^{\infty} \frac{c_1^* c_2^* |\psi_n(x^*)|^2}{(\gamma_n + D\kappa_0^2 + \lambda_0)^2}.$$

This fact combined with (3.16) contradicts (2.31). Thus we find that (3.15) holds. Combining (3.14) and (3.15), and using $\tau_* < \tau$, we obtain (3.12) for small $\varepsilon > 0$. We have completed the proof of Proposition 3.1. □

PROOF OF COROLLARY 3.1. From (2.36) and $\underline{\kappa}(\varepsilon) > 0$, we have

$$\left.\frac{m\pi}{\ell_c(\varepsilon)}\right|_{m=1} = \overline{\kappa}(\varepsilon), \qquad \left.\frac{m\pi}{\ell_c(\varepsilon)}\right|_{m\neq 1} \notin [\underline{\kappa}(\varepsilon), \overline{\kappa}(\varepsilon)]. \tag{3.17}$$

From (3.17) and Proposition 2.1, zero is the largest real eigenvalue of (2.21) when $m = 1$, and when $m \neq 1$ any real eigenvalue of (2.21) is negative. Hence (2.16) has zero as the largest real eigenvalue with $\mathcal{N}(\mathcal{L}(\ell_c(\varepsilon))) =$

span$\{W_1\}$. From Proposition 2.1, all other eigenvalues have negative real parts.

Since $\mathcal{L}(\ell_c(\varepsilon))$ has compact resolvent, $\mathcal{R}(\ell_c(\varepsilon), \varepsilon)$ is a closed subspace with

$$\mathcal{R}(\mathcal{L}(\ell_c(\varepsilon))) = \mathcal{N}(\mathcal{L}^*(\ell_c(\varepsilon)))^{\perp}, \tag{3.18}$$

by virtue of the closed range theorem. From Lemma 3.1 and (3.18), the codimension of $\mathcal{R}(\mathcal{L}(\ell_c(\varepsilon)))$ is one. It remains to prove (3.3). Using (3.12) with $m = 1$, we have

$$0 \neq \left(\begin{pmatrix} w_1^*(x) Y_1(y) \\ z_1^*(x) Y_1(y) \end{pmatrix}, \begin{pmatrix} w_1(x) Y_1(y) \\ z_1(x) Y_1(y) \end{pmatrix} \right)_{\mathcal{Y}}. \tag{3.19}$$

This, together with (3.18), implies (3.3). The proof of Corollary 3.1 is completed. □

Remark 3.2. Let ℓ be slightly larger than $\ell_c(\varepsilon)$, then from (3.17) and the property of $\widetilde{\lambda}(\varepsilon, \cdot\,)$, it holds that

$$\widetilde{\lambda}\left(\varepsilon, \frac{m\pi}{\ell}\right)\Big|_{m=1} > 0, \quad \widetilde{\lambda}\left(\varepsilon, \frac{m\pi}{\ell}\right)\Big|_{m\neq 1} < 0.$$

From Proposition 2.1 and Remark 2.3, (2.16) has a unique positive eigenvalue with the associated eigenfunction $(w_1 Y_1, z_1 Y_1)$, and all other eigenvalues have negative real parts. Here (w_1, z_1) is the same as in (2.35) when $m = 1$.

Remark 3.3 When $\ell = \ell_c(\varepsilon)$, we get $\mathcal{N}(\mathcal{L}^*(\ell_c(\varepsilon))) = \mathrm{span}\{W_1^*\}$ from (3.17), Proposition 2.1, Lemma 3.1 and Remark 3.1. Here W_1^* is defined by

$$W_1^* \stackrel{\text{def}}{=} \begin{pmatrix} w_1^*(x) Y_1(y) \\ z_1^*(x) Y_1(y) \end{pmatrix}, \tag{3.20}$$

where

$$\begin{pmatrix} w_1^*(x) \\ z_1^*(x) \end{pmatrix} = \begin{pmatrix} -\varepsilon\tau\{\varepsilon^{-\frac{1}{2}}\phi_0(\varepsilon) + (QK^*)(\varepsilon, \overline{\kappa}(\varepsilon), 0) h_1(\varepsilon)\} \\ K^*(\varepsilon, \overline{\kappa}(\varepsilon), 0) h_1(\varepsilon) \end{pmatrix}. \tag{3.21}$$

4. Bifurcation of equilibrium solutions with a non-flat layer

In this section we apply the theorems of Crandall and Rabinowitz to (2.15), and prove the theorem in §1.

LEMMA 4.1. *Suppose ε is sufficiently small. Then*

$$\mathcal{F}_{\ell\binom{u}{v}}(\ell_c(\varepsilon), \begin{pmatrix} 0 \\ 0 \end{pmatrix})W_1 \notin \mathcal{R}(\mathcal{L}(\ell_c(\varepsilon))). \tag{4.1}$$

Here W_1 is as in (3.1).

PROOF. From the definition of $\mathcal{F}$, we have

$$\mathcal{F}_{\ell\binom{u}{v}}(\ell_c(\varepsilon), \begin{pmatrix} 0 \\ 0 \end{pmatrix}) = -2\ell_c(\varepsilon)^{-3} \begin{pmatrix} \frac{\varepsilon}{\tau}\frac{\partial^2}{\partial y^2} & 0 \\ 0 & D\frac{\partial^2}{\partial y^2} \end{pmatrix} \tag{4.2}$$

and thus

$$(\text{the left-hand side of (4.1)}) = 2\pi^2\ell_c(\varepsilon)^{-3} \begin{pmatrix} \frac{\varepsilon}{\tau}w_1Y_1 \\ Dz_1Y_1 \end{pmatrix}. \tag{4.3}$$

Because we have (3.18), it suffices to show

$$\left(\begin{pmatrix} w_1^*Y_1 \\ z_1^*Y_1 \end{pmatrix}, \begin{pmatrix} \frac{\varepsilon}{\tau}w_1Y_1 \\ Dz_1Y_1 \end{pmatrix} \right)_{\mathcal{Y}} \neq 0, \tag{4.4}$$

where (w_1^*, z_1^*) is the same as in (3.21). The left-hand side of (4.4) is equal to

$$\frac{\varepsilon}{\tau}(w_1, w_1^*)_{L^2(I)} + D(z_1, z_1^*)_{L^2(I)}. \tag{4.5}$$

From (3.14), the first term of (4.5) is equal to $-\varepsilon\{1 + O(\varepsilon^{\frac{1}{2}})\}$. From (3.2) and (3.21), the second term of (4.5) is equal to

$$\begin{aligned} &D(K(\varepsilon,\overline{\kappa}(\varepsilon),0)h_2(\varepsilon), K^*(\varepsilon,\overline{\kappa}(\varepsilon),0)h_1(\varepsilon))_{L^2(I)} \\ = \;& D(K(\varepsilon,\overline{\kappa}(\varepsilon),0)^2h_2(\varepsilon), h_1(\varepsilon))_{L^2(I)}. \end{aligned} \tag{4.6}$$

Using (2.29) and (2.34), we have

$$|(\text{the right-hand side of (4.6)})| = O(\varepsilon^{\frac{5}{4}}).$$

Then we find

$$\left(\begin{pmatrix} w_1^*Y_1 \\ z_1^*Y_1 \end{pmatrix}, \begin{pmatrix} \frac{\varepsilon}{\tau}w_1Y_1 \\ Dz_1Y_1 \end{pmatrix} \right)_{Y} = -\varepsilon\{1 + O(\varepsilon^{\frac{1}{4}})\} < 0. \tag{4.7}$$

This completes the proof. □

REMARK 4.1 A subtler calculation shows that the right-hand side of (4.6) is $O(\varepsilon^{\frac{3}{2}})$. For details, the reader can refer to §4 of [17].

Combining Corollary 3.1 and Lemma 4.1, we see that the hypotheses of Theorem 1.7 of [2] and Theorem 1.16 of [3] are satisfied (see also Theorem 4.2 of [15]). By applying Theorem 1.7 of [2], we obtain the result below, from which Theorem 1.1 follows at once.

PROPOSITION 4.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be the real Banach spaces as in* (2.14). *Let $\mathcal{X}_2$ be the orthogonal complement of $\mathcal{X}_1 = \mathcal{N}(\mathcal{L}(\ell_c(\varepsilon)))$ in $\mathcal{X}$ with respect to the inner product of $L^2(\Omega) \times L^2(\Omega)$. Then there exist an open interval $\mathcal{I} = (s_1, s_2)$ containing* 0 *and smooth functions*

$$\ell(s) : \mathcal{I} \to \mathbb{R},$$
$$(\varphi(s), \psi(s)) : \mathcal{I} \to \mathcal{X}_2,$$

with $\ell(0) = \ell_c(\varepsilon)$, $(\varphi(0), \psi(0)) = (0, 0)$ such that

$$\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} \stackrel{\text{def}}{=} sW_1 + s \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix}, \tag{4.8}$$

satisfies

$$\mathcal{F}(\ell(s), \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.9}$$

Here W_1 is given by (3.1). *Moreover $\mathcal{F}^{-1}(\{0\})$ consists precisely of*

$$\left\{(\ell, \begin{pmatrix} 0 \\ 0 \end{pmatrix})\right\} \cup \left\{(\ell(s), \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}); s \in \mathcal{I}\right\}$$

in some neighborhood of $\ell = \ell_c(\varepsilon)$ and $(u, v) \equiv (0, 0)$ in $\mathcal{X}$.

PROOF OF THEOREM 1.1. By setting

$$\begin{pmatrix} \tilde{u}(x,y,s) \\ \tilde{v}(x,y,s) \end{pmatrix} = \begin{pmatrix} \bar{u}(x) \\ \bar{v}(x) \end{pmatrix} + s \begin{pmatrix} w_1(x) Y_1(y/\ell) \\ z_1(x) Y_1(y/\ell) \end{pmatrix} + s \begin{pmatrix} \varphi(x, y/\ell, s) \\ \psi(x, y/\ell, s) \end{pmatrix}, \tag{4.10}$$

we have Theorem 1.1 immediately from Proposition 4.1. □

The bifurcation equation can be obtained by the method of Ljapunov and Schmidt. We decompose $\mathcal{Y}$ as

$$\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2,$$

where

$$\mathcal{Y}_1 \stackrel{\text{def}}{=} \mathcal{R}(\mathcal{L}(\ell_c(\varepsilon))),$$
$$\mathcal{Y}_2 \stackrel{\text{def}}{=} (\text{the orthogonal complement of } \mathcal{Y}_1 \text{ in } \mathcal{Y}).$$

From (3.18) and Remark 3.3, $\mathcal{Y}_2$ is equal to $\mathcal{N}(\mathcal{L}^*(\ell_c(\varepsilon))) = \operatorname{span}\{W_1^*\}$, where W_1^* is given by (3.20). Let $\mathcal{P}_1$, $\mathcal{P}_2$ be the projections onto $\mathcal{Y}_1$, $\mathcal{Y}_2$, respectively. Equation (2.15) can be split into

$$\mathcal{P}_1\mathcal{F}(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{4.11a}$$

$$\mathcal{P}_2\mathcal{F}(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.11b}$$

We define a mapping $\mathcal{F}_1$ from a neighborhood of $(\ell_c(\varepsilon), 0, 0)$ in $\mathbb{R} \times \mathcal{X}_1 \times \mathcal{X}_2$ to $\mathcal{Y}_1$ by

$$\mathcal{F}_1(\ell, x_1, x_2) \stackrel{\text{def}}{=} \mathcal{P}_1\mathcal{F}(\ell, x_1 + x_2).$$

Then

$$(\mathcal{F}_1)_{x_2}(\ell_c(\varepsilon), 0, 0) = \mathcal{P}_1\mathcal{F}_{\binom{u}{v}}(\ell_c(\varepsilon), \begin{pmatrix} 0 \\ 0 \end{pmatrix})$$

is a bijective mapping from $\mathcal{X}_2$ onto $\mathcal{Y}_1$. By applying the usual implicit function theorem, we find a unique $x_2 = \chi_2(\ell, x_1)$ that satisfies

$$\mathcal{P}_1\mathcal{F}(\ell, sW_1 + \chi_2(\ell, sW_1)) = 0 \tag{4.12}$$

in some neighborhood of $(s, \ell) = (0, \ell_c(\varepsilon))$. We put $x_1 = sW_1$, where W_1 is as in (3.1). Here χ_2 is a smooth mapping from some neighborhood of $(\ell_c(\varepsilon), 0)$ in $\mathbb{R}_+ \times \mathcal{X}_1$ into $\mathcal{X}_2$. Since

$$\mathcal{F}(\ell, \begin{pmatrix} 0 \\ 0 \end{pmatrix}) \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{for } \ell > 0,$$

we see from the uniqueness of χ_2 that

$$\chi_2(\ell, 0) \equiv 0. \tag{4.13}$$

Substituting $x_2 = \chi_2(\ell, x_1)$ into (4.11b), we obtain

$$G(s, \ell) = 0, \tag{4.14}$$

where

$$G(s, \ell) \stackrel{\text{def}}{=} (W_1^*, \mathcal{F}(\ell, sW_1 + \chi_2(\ell, sW_1)))_{\mathcal{Y}}\,. \tag{4.15}$$

Here W_1 and W_1^* are given by (3.1) and (3.20), respectively. In some neighborhood of the bifurcation point

$$(\ell, \begin{pmatrix} u \\ v \end{pmatrix}) = (\ell_c(\varepsilon), \begin{pmatrix} 0 \\ 0 \end{pmatrix}),$$

the solutions to (2.15) are in one-to-one correspondence with the solutions to a single scalar equation (4.15). Equation (4.15) is called the *bifurcation equation.* From (4.13), we obtain

$$G(0, \ell) \equiv 0. \tag{4.16}$$

We have

$$s \begin{pmatrix} \varphi(s) \\ \psi(s) \end{pmatrix} = \chi_2 \left(\ell(s), s \begin{pmatrix} w_1 Y_1 \\ z_1 Y_1 \end{pmatrix} \right). \tag{4.17}$$

We make a remark on the bifurcation given in Proposition 4.1 from the point of view of the classification in Sattinger[14]. Putting

$$r \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} u(x, 1-y) \\ v(x, 1-y) \end{pmatrix}, \tag{4.18}$$

we have

$$\mathcal{F}(\ell, r \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}) = r\mathcal{F}(\ell, \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}). \tag{4.19}$$

Using this fact we have two solutions

$$\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}, \quad r \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$$

of (2.15) for the same $\ell = \ell(s)$. These two solutions are really distinct because $\mathcal{X}_1$-components differ for $s \neq 0$. Thus the bifurcation is supercritical or subcritical if $\ell^{(n)}(0) \neq 0$ for some $n \geq 1$.

PROOF OF COROLLARY 1.1. We have seen that either of

$$(\ell(s), r\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}), \quad (\ell(-s), \begin{pmatrix} u(-s) \\ v(-s) \end{pmatrix}) \tag{4.20}$$

satisfies (2.15). Both of these have the same $\mathcal{X}_1$-component

$$-s\begin{pmatrix} w_1 Y_1 \\ z_1 Y_1 \end{pmatrix}.$$

Because

$$\left\{ (\ell(s), \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}); s \in \mathcal{I} \right\}$$

can be parameterized by the $\mathcal{X}_1$-component, the fact stated above implies that

$$\ell(-s) = \ell(s),$$
$$r\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \begin{pmatrix} u(-s) \\ v(-s) \end{pmatrix},$$

which immediately leads to

$$\ell_s(0) = 0. \tag{4.21}$$

From now on, we denote $\chi_2(\ell, sW_1)$ by $\chi_2(\ell, s)$ for simplicity. Differentiating (4.12) twice by s and putting $s = 0$, we have

$$(\chi_2)_s(\ell_c(\varepsilon), 0) = 0, \tag{4.22}$$
$$(\chi_2)_{ss}(\ell_c(\varepsilon), 0) = -\mathcal{L}(\ell_c(\varepsilon))^{-1}\mathcal{P}_1(d^2\mathcal{F})(W_1, W_1), \tag{4.23}$$

where

$$(d^2\mathcal{F})(W_1, W_1) \stackrel{\text{def}}{=} \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\mathcal{F}(\ell_c(\varepsilon), t_1 W_1 + t_2 W_1)\Big|_{t_1 = t_2 = 0}.$$

Differentiating (4.15) three times by s, and using (4.22) and (4.23) , we obtain

$$\begin{aligned} G_{sss}(0, \ell_c(\varepsilon)) = \big(W_1^*, &(d^3\mathcal{F})(W_1, W_1, W_1) \\ &-3(d^2\mathcal{F})(W_1, \mathcal{L}(\ell_c(\varepsilon))^{-1}\mathcal{P}_1(d^2\mathcal{F})(W_1, W_1))\big)_{\mathcal{Y}}. \end{aligned} \tag{4.24}$$

Here W_1 and W_1^* are given by (3.1) and (3.20), respectively. We have

$$(d^3\mathcal{F})(W_1, W_1, W_1) \overset{\text{def}}{=} \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\frac{\partial}{\partial t_3}\mathcal{F}(\ell_c(\varepsilon), t_1 W_1 + t_2 W_1 + t_3 W_1)\Big|_{t_1=t_2=t_3=0}.$$

In the following, we assume that the right-hand side of (4.24) is not 0.

Let us differentiate

$$G(s, \ell(s)) = 0,$$

three times by s, and put $s = 0$. Then using (4.16) and (4.21), we obtain

$$G_{sss}(0, \ell(0)) + 3G_{s\ell}(0, \ell(0))\ell_{ss}(0) = 0. \tag{4.25}$$

If

$$G_{s\ell}(0, \ell(0)) < 0, \tag{4.26}$$

then the desired result follows from Theorem 1.16 of [3]. We differentiate (4.15) by s, then by ℓ, and put $s = 0$. Then using (4.13), (3.18) and (4.22), we get

$$\begin{aligned} G_{s\ell}(0, \ell(0)) &= -2\ell_c(\varepsilon)^{-3}\left(\begin{pmatrix} w_1^* Y_1 \\ z_1^* Y_1 \end{pmatrix}, \begin{pmatrix} \frac{\varepsilon}{\tau}\frac{\partial^2}{\partial y^2}(w_1 Y_1) \\ D\frac{\partial^2}{\partial y^2}(z_1 Y_1) \end{pmatrix}\right)_{\mathcal{Y}} \\ &= 2\ell_c(\varepsilon)^{-3}\left(\begin{pmatrix} w_1^* Y_1 \\ z_1^* Y_1 \end{pmatrix}, \begin{pmatrix} \frac{\varepsilon}{\tau}(w_1 Y_1) \\ D(z_1 Y_1) \end{pmatrix}\right)_{\mathcal{Y}}. \end{aligned}$$

Combining this equality and (4.7), we obtain (4.26). We have thus completed the proof of Corollary 1.1. □

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