On Gevrey Singularities of Solutions of Equations with Non-symplectic Characteristics

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Abstract

Using the theory of the second microlocalization, we prove a theorem on propagation of Gevrey singularities for a class of Grušin type operators whose characteristic sets have natural bicharacteristic foliations.

§ 1. Notation and statement of the result

If X is an open set of $\boldsymbol{R}^N$ and $\nu \geq 1$, the Gevrey class of order ν, which we denote by $G^\nu(X)$, is the set of all $u \in C^\infty(X)$ such that for every compact set $K \subset X$ there is a constant C_K with

$$|\partial_x^\alpha u(x)| \leq C_K^{|\alpha|+1}(\alpha\,!)^\nu \quad x \in K,$$

for all multi-indices $\alpha \in \boldsymbol{N}^N$.

If $u \in \mathcal{D}'(X)$ we denote by $WF_\nu(u)$ the G^ν wave front set of u introduced by Hörmander [8]. Then u is in G^ν in the complement of $\pi(WF_\nu(u))$, where π denotes the projection of $T^*(X)$ to X.

Let Σ be the submanifold in $T^*(\boldsymbol{R}^N)\backslash 0$ of codimension $2d+d'$ given by

$$\Sigma = \{(x, \xi) \in T^*(\boldsymbol{R}^N)\backslash 0\,;\ x_1 = \cdots = x_d = 0,\ \xi_1 = \cdots = \xi_{d+d'} = 0\}\,,$$

where $0<d<d+d'<N$. With this Σ we set

$$\boldsymbol{R}_x^N = \boldsymbol{R}_t^d \times \boldsymbol{R}_y^n = \boldsymbol{R}_t^d \times \boldsymbol{R}_{y'}^{d'} \times \boldsymbol{R}_{y''}^{d''} \quad (d+n=N,\ d'+d''=n)$$

and denote by $\xi=(\tau, \eta)=(\tau, \eta', \eta'')$ the dual variables of $x=(t, y)=(t, y', y'') \in \boldsymbol{R}_t^d \times \boldsymbol{R}_{y'}^{d'} \times \boldsymbol{R}_{y''}^{d''}$. (In this coordinate $\Sigma = \{(t, y, \tau, \eta', \eta'')\,;\ |t|=|\tau|=|\eta'|=0,\ \eta'' \neq 0\}$).

For a fixed integer $h \geq 1$ we shall consider a differential operator of order m with polynomial coefficients of the form:

$$(1.1) \qquad P = p(t, D_t, D_y) = \sum_{\substack{|\alpha|+|\beta| \leq m \\ |\gamma| = |\alpha|+|\beta'|+(1+h)|\beta''|-m}} a_{\alpha\beta\gamma} t^\gamma D_y^\beta D_t^\alpha\,,$$

where $(\alpha, \beta, \gamma)=(\alpha, \beta', \beta'', \gamma) \in \boldsymbol{N}^d \times \boldsymbol{N}^{d'} \times \boldsymbol{N}^{d''} \times \boldsymbol{N}^d$ and $(D_t, D_y)=(-i\partial_t, -i\partial_y)$. Then the symbol $p(t, \tau, \eta)$ has the following quasi-homogeneity:

$$(1.2) \qquad p(t/\lambda^\rho, \lambda^\rho\tau, \lambda^\rho\eta', \lambda\eta'') = \lambda^{\rho m} p(t, \tau, \eta', \eta''),\ \lambda>0$$

with $\rho=1/(1+h)$.

Let p_0 denote the principal symbol given by

(1.3) $$p_0(t, \tau, \eta)=\sum_{\substack{|\alpha|+|\beta|=m \\ |\gamma|=h|\beta''|}} a_{\alpha\beta\gamma}t^{\gamma}\eta^{\beta}\tau^{\alpha}.$$

For a point $(\mathring{x}, \mathring{\xi})=(0, \mathring{y}; 0, 0, \mathring{\eta}'')\in\Sigma$ ($|\mathring{\eta}''|\neq 0$) we suppose:

(H-1) There exists a constant $c>0$ such that

$$|p_0(t, \tau, \eta', \mathring{\eta}'')|\geq c(|\tau|+|\eta'|+|t|^h)^m,\ (t, \tau, \eta')\in \boldsymbol{R}^d\times\boldsymbol{R}^d\times\boldsymbol{R}^{d'}.$$

The operators of this form have been studied by Grušin in [4], [5], [6] in the C^∞ theory and for $\nu\geq 1+h$ we have proved in [16] G^ν microhypoellipticity of P under the additional hypothesis:

(H-2) For all $\eta'\in\boldsymbol{R}^{d'}$, Ker $p(t, D_t, \eta', \mathring{\eta}'')\cap\varphi(\boldsymbol{R}_t^d)=\{0\}$.

The purpose of this paper is to study G^ν singularities for $1\leq\nu<1+h$.

Let us introduce the submanifold $\Lambda\supset\Sigma$ given by

$$\Lambda=\{(t, y; \tau, \eta', \eta'')\in T^*(\boldsymbol{R}^N)\backslash 0;\ \eta'=0\}.$$

Then in the canonical way Λ defines a bicharacteristic foliation in Σ as well as in Λ; that is, each leaf Γ_0 is an integral submanifold of dimension d' of the vector fields generated by $\{\partial_{y_1}, \cdots, \partial_{y_{d'}}\}$. (Note that $T_\rho(\Gamma_0)=T_\rho(\Sigma)\cap T_\rho(\Sigma)^\perp$ for all $\rho\in\Gamma_0$).

We get the following theorem on the propagation of singularities.

THEOREM 1.1. *Let Γ_0 be the bicharacteristic leaf passing through $(\mathring{x}, \mathring{\xi})\in\Sigma$ defined as above and W be an open set containing $(\mathring{x}, \mathring{\xi})$ such that $\Gamma_0\cap W$ is connected. Suppose that P is an operator of the form* (1.1) *satisfying* (H-1) *for $(\mathring{x}, \mathring{\xi})$ and that $1\leq\nu<1+h$. If $u\in\mathcal{D}'(\boldsymbol{R}^N)$ and $WF_\nu(Pu)\cap\Gamma_0\cap W=\phi$ then either $\Gamma_0\cap W\cap WF_\nu(u)=\phi$ or $\Gamma_0\cap W\subset WF_\nu(u)$.*

REMARK 1.2. When $h=1$ and $\nu=1$ this is a spacial case of Theorem 2 in Grigis-Schapira-Sjöstrand [3]. See also Sjöstrand [17], [18] and Hasegawa [7] in this connexion.

In [16] we have construced the left parametrix of P under the hypothesis (H-1) and (H-2). However, in Theorem 1.1 (H-2) is not assumed for P. Thus we need to prepare the theory of the second microlocalizarion; which will be given in Section 2. Then we shall prove Theorem 1.1 using the results in [16] and in Section 2.

§2. Second microlocalization in Gevrey class

In the analytic theory, the notion of second microlocalization along an involutive submanifold was introduced by Kashiwara-Kawai [9]. In [17], Sjöstrand defined the second wave front set along the Lagrangian submanifold. Then Lebeau extended the notion to the isotropic submanifold in [13].

Recently, Esser introduced a modified notion of the second microlocalization in order to obtain propagation property for Gevrey singularities. Here we prepare several notation on this theory following Esser [2]. (See also Kishida [10]).

Let us introduce the Fourier-Bros-Iagolnitzer transform (F.B.I. tr.):

$$T^{(1)}f(z, \lambda)=\int e^{-\lambda(z-x)^2/2} f(x)dx\,, \quad (f\in\mathcal{E}'(\boldsymbol{R}^N)) \tag{2.1}$$

associated to κ: $T^*(\boldsymbol{R}^N)\backslash 0\ni(x, \xi)\longmapsto x-i\xi\in \boldsymbol{C}_z^N$. Then $T^{(1)}f$ is defined on $\boldsymbol{C}_z^N\times\boldsymbol{R}_\lambda^+$, holomorphic with respect to z and bounded by $Ce^{\lambda|\mathrm{Im}z|^2/2}(1+\lambda)^k$ for $\lambda\geq 1$. For $f\in\mathcal{E}'(\boldsymbol{R}^N)$ we have $(\mathring{x}, \mathring{\xi})\notin WF_\nu(f)$ if and only if there are constants C, $c>0$ such that

$$|T^{(1)}f(z, \lambda)|\leq Ce^{\lambda|\mathrm{Im}z|^2/2-c\lambda^{1/\nu}} \text{ for } |z-(\mathring{x}-i\mathring{\xi})|<c\,. \tag{2.2}$$

Let Λ be the involutive submanifold of $T^*(\boldsymbol{R}^N)$:

$$\Lambda=\{(x, \xi)\in T^*(\boldsymbol{R}^N)\,;\ \xi_1=\cdots=\xi_{d'}=0\}\ (1\leq d'<N)\,,$$

and Γ_0 be the bicharacteristic leaf passing through $(\mathring{x}, \mathring{\xi})\in\Lambda$. Then Λ and Γ_0 can be identified with $\kappa(\Lambda)=\{z\in\boldsymbol{C}^N\,;\ \mathrm{Im}z'=0\}$ and $\kappa(\Gamma_0)=\{z\in\boldsymbol{C}^N\,;\ \mathrm{Im}z'=0,\ z''=\mathring{x}''-i\mathring{\xi}''\}$ respectively, where $z=(z', z'')\in\boldsymbol{C}^{d'}\times\boldsymbol{C}^{N-d'}$.

Let $\varphi_\Lambda(z)=|\mathrm{Im}z''|^2/2$; which is the pluri-subharmonic function canonically associated to Λ. If Ω is a neighborhood of $\mathring{z}\in\kappa(\Lambda)$, we denote by $H_\Lambda^{\nu,loc}(\Omega)$ the space of holomorphic functions $u(z, \lambda)$ in Ω with a parameter $\lambda>0$ such that for all $K\subset\subset\Omega$ and $\varepsilon>0$ there exists $C_{K,\varepsilon}$ with the estimate:

$$|u(z, \lambda)|\leq C_{K,\varepsilon}e^{\lambda\varphi_\Lambda+\varepsilon\lambda^{1/\nu}} \text{ for } z\in K. \tag{2.3}$$

For $\mathring{z}\in\Lambda$ we also use the notation: $u\in H_{\Lambda,\mathring{z}}^\nu$ if there is a neighborhood $\omega_{\mathring{z}}$ of $\mathring{z}$ such that $u\in H_\Lambda^{\nu,loc}(\omega_{\mathring{z}})$.

If $u\in H_\Lambda^{\nu,loc}(\Omega)$ we denote by $S_\Lambda^\nu(u)$ the subset in $\Omega\cap\kappa(\Lambda)$ defined by:

(2.4) $\mathring{z}\notin S_\Lambda^\nu(u)$ if and only if there exist a neighborhood $\omega_{\mathring{z}}$ of $\mathring{z}$ and constants C, $c>0$ such that

$$|u(z, \lambda)|\leq Ce^{\lambda\varphi_\Lambda-c\lambda^{1/\nu}} \text{ for } z\in\omega_{\mathring{z}}\,.$$

Then we have the following two lemmas:

LEMMA 2.1. *Let Γ_0 be a bicharacteristic leaf in Λ and ω be a connected open set in Γ_0 containing $(\mathring{x}, \mathring{\xi})$. If $u\in H_{\Lambda,z}^\nu$ for all $z\in\kappa(\omega)$ and $\mathring{x}-i\mathring{\xi}\notin S_\Lambda^\nu(u)$ then $\kappa(\omega)\cap S_\Lambda^\nu(u)=\phi$.*

LEMMA 2.2. *Let $(\mathring{x}, \mathring{\xi})\in\Lambda$, $f\in\mathcal{E}'(\boldsymbol{R}^N)$. If $(\mathring{x}, \mathring{\xi})\notin WF_\nu(f)$ and $T^{(1)}f\in H_{\Lambda,\mathring{x}-i\mathring{\xi}}^\nu$ then $\mathring{x}-i\mathring{\xi}\notin S_\Lambda^\nu(T^{(1)}f)$.*

Following Laubin-Esser [11], we introduce the F.B.I. tr. of second kind along Λ:

$$T_\Lambda^{(2)} f(w,\ \mu,\ \lambda)=\int e^{-\lambda(w''-x'')^2/2-\lambda\mu(w'-x')^2/2} f(x)dx \quad (f\in\mathcal{E}'(\boldsymbol{R}^N)). \tag{2.5}$$

Then $T_\Lambda^{(2)} f(w,\ \mu,\ \lambda)$ is a holomorphic function with respect to $w\in\boldsymbol{C}^N$ with the bound:

$$|T_\Lambda^{(2)} f(w,\ \mu,\ \lambda)|\leq Ce^{\lambda|\mathrm{Im}\,w''|^2/2+\lambda\mu|\mathrm{Im}\,w'|^2/2}(1+\lambda)^k$$

It was shown in [1] and [13] that the relation between $T^{(1)}f$ and $T_\Lambda^{(2)}f$ is as follows:

$$T_\Lambda^{(2)} f(w,\ \mu,\ \lambda)=\left(\frac{\lambda}{2\pi(1-\mu)}\right)^{d'/2}\int_{\boldsymbol{R}^{d'}} e^{-\lambda\rho(w'-x')^2/2}\, T^{(1)}f(x',\ w'',\ \lambda)dx', \tag{2.6}$$

where $\rho=\mu/(1-\mu)$ and

$$\begin{aligned} &T^{(1)}f(z,\ \lambda) \\ &=\frac{1}{2}\left(\frac{1}{2\pi\lambda}\right)^{d'/2}\int_{\boldsymbol{R}^{d'}} e^{-\lambda R|\xi'|/2}\left(1-i\frac{\langle\xi',\ \nabla'\rangle}{\lambda|\xi'|^2}\right) T_\Lambda^{(2)}f\left(z'-i\frac{R\xi'}{|\xi'|},\ z'',\ \mu,\ \lambda\right)\frac{Rd\xi'}{R+|\xi'|}, \end{aligned} \tag{2.7}$$

where $R>0$ is in our disposal and $\mu=|\xi'|/(R+|\xi'|)$.

DEFINITION 2.3. If $1\leq\nu<+\infty$ and $f\in\mathcal{E}'(\boldsymbol{R}^N)$, *the second wave front set along* Λ of f, denoted by $WF_{\Lambda,\nu}^{(2)}(f)$, is the subset in $T_\Lambda(T^*(\boldsymbol{R}^N)\backslash 0)$ defined by the following condition:

$$(\mathring{x},\ 0,\ \mathring{\xi}'';\ \mathring{\sigma}')\notin WF_{\Lambda,\nu}^{(2)}(f) \tag{2.8}$$

if and only if there exist C, $c>0$, $\mu_0\in]0,\ 1[$ and a decreasing function $o(\lambda)$ with $\lim_{\lambda\to+\infty} o(\lambda)=0$ such that

$$|T_\Lambda^{(2)} f(w,\ \mu,\ \lambda)|\leq Ce^{\lambda|\mathrm{Im}\,w''|^2/2+\lambda\mu|\mathrm{Im}\,w'|^2/2-c\lambda\mu} \tag{2.9}$$

for

$$0<\mu<\mu_0,\ \lambda\mu>o(\lambda)\lambda^{1/\nu},\ |w'-(\mathring{x}'-i\mathring{\sigma}')|+|w''-(\mathring{x}''-i\mathring{\xi}'')|<c. \tag{2.10}$$

We have

LEMMA 2.4. *Let* $(\mathring{x},\ \mathring{\xi})\in\Lambda$ *and* $f\in\mathcal{E}'(\boldsymbol{R}^N)$. *Then* $T^{(1)}f\in H_{\Lambda,\mathring{x}-i\mathring{\xi}}^{\nu}$ *if and only if* $\pi_\Lambda^{-1}(\mathring{x},\ \mathring{\xi})\cap WF_{\Lambda,\nu}^{(2)}(f)=\phi$, *where* π_Λ: $T_\Lambda(T^*(\boldsymbol{R}^N)\backslash 0)\longrightarrow\Lambda$ *is the canonical projection.*

Our definitions of H_Λ^ν and $WF_{\Lambda,\nu}^{(2)}$ are slightly different to those in Esser [2]. However, Lemma 2.1, 2.2 and 2.4 are proved by almost words to words translation of the proofs of Lemma 3.3, 3.1 and Theorem 2.2 in [2] respectively.

At last, we introduce the space of the partially holomorphic Gevrey functions in $\Omega\subset\boldsymbol{R}^{d'}\times\boldsymbol{R}^{N-d'}$ as follows: $f\in G^\nu\mathcal{A}_{x'}(\Omega)$ if and only if for every compact set $K\subset\subset\Omega$ there is a constant C such that

(2.11) $$|\partial_{x'}^{\alpha'}\partial_{x''}^{\alpha''}f(x)|\leq C^{|\alpha|+1}\alpha'!(\alpha''!)^{\nu} \text{ for } x\in K.$$

By deforming the dx' contour in the definition of $T^{(1)}f$ into a complex domain one can easily show the following

LEMMA 2.5. *If* $f\in\mathcal{E}'(\boldsymbol{R}^N)\cap G^{\nu}\mathcal{A}_{x'}(\Omega)$ *then* $T^{(1)}f\in H^{\nu'}_{1,z}$ *for every* $z\in\kappa(\pi^{-1}(\Omega)\cap\Lambda)$ *and every* $1\leq\nu'<+\infty$.

§ 3. Proof of Theorem 1.1

As in [16] Section 2, we may suppose $\mathring{x}=0$, $\mathring{\xi}=(0, 0, \mathring{\eta})=(0, \ldots, 0, 1)$ and set $Q=(P^*P)^k$ with $2km\geq d+1$. Since (H-2) is not assumed for P, here we shall introduce the pseudo-differential oprator $R=r(D_x)$ with the symbol:

(3.1) $$r(\xi)=|\eta_n|^{2km/(1+h)}\exp\left(-\frac{|\eta'|^{2l(1+h)}}{|\eta_n|^{2l}}\right),$$

where l is a positive integer to be fixed later.

Consider the operator $Q+R$. Then it satisfies (H-2) because Q is a non negative self-adjoint operator. We also note that though not being polynomial, r is holomorphic with the uniform bound $O(|\xi|^{2km/(1+h)})$ in a small quasi-homogeneous neighborhood of ξ of the form:

$$\{(\eta', \eta'')\in\boldsymbol{C}^{d'}\times\boldsymbol{C}^{d''}\,;\ |\mathrm{Im}\,\eta'|<\varepsilon(|\mathrm{Re}\,\eta_n|^{1/(1+h)}+|\mathrm{Re}\,\eta'|),$$
$$|\eta''/\mathrm{Re}\,\eta_n-\mathring{\eta}''|<\varepsilon\}.$$

With this fact we can apply the results in Section 2 in [16] to $Q+R$. Then we obtain the pseudo-differential operator $K_g=k_g(t, D_t, D_y)$ such that

(3.2) $$K_g^*(Q+R)=g(D_x).$$

Here the symbol k_g has the form $k(t, \xi)g(\xi)$ with k defined in a fixed conic neighborhood V_0 of ξ and $g\in C^\infty(\boldsymbol{R}^N)$ such that $\mathrm{supp}(g)\subset V_0$ and that

(3.3) $$|\partial_\xi^\alpha g(\xi)|\leq C^{|\alpha|+1}\left(\frac{|\alpha|}{|\xi|}\right)^{\rho|\alpha|}$$

for $|\alpha|\leq|\xi|$. Moreover k_g satisfies the estimate:

(3.4) $$|\partial_\eta^\beta\partial_\tau^{\alpha_-}\partial_t^{\alpha_+}k_g(t, \tau, \eta)|\leq C^{|\alpha|+|\beta|+1}(1+|t|)^{|\beta|}(|\alpha_+|^{1-\rho}|\xi|^\rho)^{|\alpha_+|}$$
$$\left(\frac{|\alpha_-|}{|\xi|}\right)^{\rho|\alpha_-|}\left(\frac{|\beta|}{|\xi|^\rho+|\eta'|}+\left(\frac{|\beta'|}{|\xi|}\right)^\rho\right)^{|\beta'|}\left(\frac{|\beta''|}{|\xi|}\right)^{\rho|\beta''|}$$

for $|\alpha_-|+|\beta|\leq|\xi|$, where $\xi=(\tau, \eta)=(\tau, \eta', \eta'')\in\boldsymbol{R}^N$, $(\alpha, \beta)=(\alpha_+, \alpha_-, \beta', \beta'')\in\boldsymbol{N}^d\times\boldsymbol{N}^d\times\boldsymbol{N}^{d'}\times\boldsymbol{N}^{d''}$.

By Proposition 3.2 and 3.6 in [16] we have

(3.5) $$WF_A(K_g)\subset\{(t, y, t, w\,;\ \tau, \eta, -\tau, -\eta)\in T^*(\boldsymbol{R}^{2N})\backslash 0\,;\ y''=w'', (\tau, \eta)\in\bar{V}_0\},$$

(3.6) $K_g(t, y, s, w) \in G^{1+h}\mathcal{A}_{y', w'}((\mathbf{R}^N \times \mathbf{R}^N) \backslash \mathrm{diag}(\mathbf{R}^N))$.

Here we identify the operator K_g with its distribution kernel and $WF_A = WF_1$ denotes the analytic wave front set.

If $(\mathring{x}, \mathring{\xi}) = (0; 0, 0, \mathring{\eta}'') \in \Sigma$ then the bicharacteristic leaf is $\Gamma_0 = \{(0, y', 0; 0, 0, \mathring{\eta}''); y' \in \mathbf{R}^{d'}\}$. For any compact set $F \subset \pi(\Gamma_0 \cap W)$ there exist a neighborhood $U \subset\subset O_R = \{x \in \mathbf{R}^N; |x| < R\}$ of F and a conic neighborhood V of $\mathring{\xi}$ such that

(3.7) $$WF_\nu(Pu) \cap \bar{U} \times (\bar{V} \backslash 0) = \phi,$$

where $\bar{U}$, $\bar{V}$ denote the closures of U, V respectively.

After replacing u by φu with a suitable $\varphi \in C_0^\infty(O_R)$ we can suppose $u \in \mathcal{E}'(O_R)$ with no influence on (3.7).

We fix two conic neighborhoods V_1, V_2 of ξ with $V_1 \subset\subset V_2 \subset\subset V \cap V_0$ and take a cut off function g given by Lemma 3.1 in Métivier [14] such that $g(\xi) = 1$ if $\xi \in V_1$ and $|\xi| \geq 2$, $\mathrm{supp}\, g \subset V_2$ and satisfies (3.3).

Now we write for $u \in \mathcal{E}'(O_R)$

(3.8) $$g(D_x)u = K_g^* Qu + K_g^* Ru$$
$$= K_g^* Qu + RK_g^* u$$

We shall apply the theory of second microlocalization along the involutive submanifold:

$$\Lambda = \{(t, y; \tau, \eta', \eta'') \in T^*(\mathbf{R}^N) \backslash 0; \eta' = 0\}.$$

Hereafter, we shall use the notation in Section 2 with

$$x' = y', \ x'' = (t, y'') \quad \text{and} \quad \xi' = \eta', \ \xi'' = (\tau, \eta'').$$

First we study $RK_g^* u$. Now choose l so that $(1+h) - (1/2l) > \nu$. Then

(3.9) $$|\eta'|^{2l(1+h)}/\eta_n^{2l} \geq |\eta'| \quad \text{for} \quad |\eta'| \geq \eta_n^{1/\nu'}, \ \eta_n > 0,$$

where $\nu' = (1+h) - (1/2l)$.

On the other hand, writing

$$(K_g^* u)^\wedge(\xi) = \int e^{-ix\xi} \overline{{}_{\mathrm{t}}k_g(t, \xi)} u(x) dx$$

we see that $(K_g^* u)^\wedge \in C^\infty(\mathbf{R}^N)$, $\mathrm{supp}(K_g^* u) \subset V_2$ and

(3.10) $$|(K_g^* u)^\wedge(\xi)| \leq C(1+|\xi|)^k$$

with some C, k real.

LEMMA 3.1. $WF_\Lambda^{(2)}(RK_g^* u) \cap \pi_\Lambda^{-1}(\Gamma_0) = \phi$.

Proof. Consider

(3.11) $T_\Lambda^{(2)}(RK_g^* u)(w, \mu, \lambda)$

$$=\int e^{-\lambda(w''-x'')^2/2-\lambda\mu(w'-x')^2/2+ix\xi}r(\xi)(K_g^* u)^\wedge(\xi)dxd\xi.$$

First we show that for any $\delta>0$ the contribution from $|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|\geq\delta\lambda\mu$ are dominated by

$$C\lambda^{k'}e^{-\lambda|\mathrm{Im}\,w''|^2/2+\lambda\mu|\mathrm{Im}\,w|^2/2-c\lambda\mu} \tag{3.12}$$

for $\lambda\mu\geq 1$ with some $c>0$.

To see this we deform the dx contour. The exponent of the integrand has the real part

$$\begin{aligned}(3.13)\qquad &-\frac{\lambda}{2}|\mathrm{Re}\,(w''-x'')|^2-\frac{\lambda}{2}\mu|\mathrm{Re}\,(w'-x')|^2\\ &+\frac{\lambda}{2}|\mathrm{Im}\,(w'-x')|^2+\frac{\lambda}{2}\mu|\mathrm{Im}\,(w'-x')|^2-\xi\,\mathrm{Im}\,x\\ &=-\frac{\lambda}{2}|\mathrm{Re}\,(w''-x'')|^2-\frac{\lambda}{2}\mu|\mathrm{Re}\,(w'-x')|^2+\frac{\lambda}{2}|\mathrm{Im}\,w''|^2+\frac{\lambda}{2}\mu|\mathrm{Im}\,w'|^2\\ &-(\lambda\,\mathrm{Im}\,w''+\xi'')\,\mathrm{Im}\,x''-(\lambda\mu\,\mathrm{Im}\,w'+\xi')\,\mathrm{Im}\,x'-\frac{\lambda}{2}|\mathrm{Im}\,x''|^2-\frac{\lambda}{2}\mu|\mathrm{Im}\,x'|^2.\end{aligned}$$

Now deforming the contour to

$$\mathrm{Im}\,x=\frac{s(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu^2\xi''+\lambda\mu^2\,\mathrm{Im}\,w'')}{|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|}\quad(s=\delta/3)$$

we have

$$\begin{aligned}(3.14)\qquad &-(\lambda\,\mathrm{Im}\,w''+\xi'')\,\mathrm{Im}\,x''-(\lambda\mu\,\mathrm{Im}\,w'+\xi')\,\mathrm{Im}\,x'-\frac{\lambda}{2}|\mathrm{Im}\,x''|^2-\frac{\lambda}{2}\mu|\mathrm{Im}\,x'|^2.\\ &\leq-(\delta/3)|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|+(\delta^2/9)\lambda\mu\\ &\leq-(\delta^2/9)|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|-(\delta^2/9)\,\lambda\mu\end{aligned}$$

for $|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|\geq\delta\lambda\mu$. By (3.13) and (3.14), the integral over $|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu w'')|\geq\delta\lambda\mu$, $\mathrm{Re}\,x\in\boldsymbol{R}^N$ is dominated by

$$C(1+\lambda)^{k'}\,e^{-\lambda|\mathrm{Im}\,w''|^2/2-\lambda\mu|\mathrm{Im}\,w'|^2/2-\delta^2\lambda\mu/9} \tag{3.15}$$

if $\lambda\mu\geq 1$.

Next we consider the contribution from $|(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|\leq\delta\lambda\mu$. Now suppose that $\lambda\mu\geq 2\lambda^{1/\nu'}$. If $\delta>0$ is sufficiently small then on the set

(3.16) $\{(\xi',\ \xi'');\ |(\xi'+\lambda\mu\,\mathrm{Im}\,w',\ \mu\xi''+\lambda\mu\,\mathrm{Im}\,w'')|\leq\delta\lambda\mu$, for some w such that $1-\delta\leq|\mathrm{Im}\,w'|\leq 1+\delta$ and $|\mathrm{Im}\,w''+\xi''|\leq\delta\}$

we have $|\xi'|=|\eta'|\geq|\eta_n|^{1/\nu'}$. Using (3.9), we can then obtain the estimate

$$C(1+\lambda)^{k''}e^{-\lambda|\mathrm{Im}w''|^2/2-\lambda\mu|\mathrm{Im}w'|^2/2-\lambda\mu/2} \tag{3.17}$$

for its contribution when $\lambda\mu\geq 2\lambda^{1/\nu'}$. From (3.15) and (3.17) we conclude that $\pi_\Lambda^{-1}(\Gamma_0)\cap WF_\Lambda^{(2)}(RK_g^* u)=\phi$, because $\nu'>\nu$. □

Note that Lemma 3.1 implies

$$T^{(1)}(RK_g^* u)\in H_{\Lambda,z}^{\nu} \quad \text{for all } z\in\kappa(\Gamma_0) \tag{3.18}$$

in view of Lemma 2.4.

Next we consider $K_g^* Qu$. Let $\tilde{g}$ be another cut off function satisfying (3.3) with supp $\tilde{g}\subset V$ and $\tilde{g}(\xi)=1$ for $\xi\in\tilde{V}_1$, $|\xi|\geq 2$, where $\tilde{V}_1$ is an open cone such that $V_2\subset\subset\tilde{V}_1\subset\subset V$.

Noticing that $WF_\nu(Qu)\subset WF_\nu(Pu)$, we then get by (3.7)

$$WF_\nu(\tilde{g}(D_x)Qu)\subset WF_\nu(Pu)\cap(\boldsymbol{R}^N\times\bar{V})\subset\pi^{-1}(O_R\backslash U), \tag{3.19}$$

$$WF_\nu(1-\tilde{g}(D_x))Qu)\subset WF_\nu(Pu)\cap(\boldsymbol{R}^N\times(\boldsymbol{R}^N\backslash\tilde{V}_1))\subset O_R\times(\boldsymbol{R}^N\backslash\bar{V}_2). \tag{3.20}$$

Hence we can write

$$Qu=\chi_{F_\varepsilon}\tilde{g}(D_x)Qu+\chi_{O_R}(1-\chi_{F_\varepsilon})\tilde{g}(D_x)Qu+\chi_{O_R}(1-\tilde{g}(D_x))Qu \tag{3.21}$$
$$(\overrightarrow{=} v_1+v_2+v_3).$$

Here χ_B denotes the characteristic function of each set B and

$$F_\varepsilon=\{(x',x'')\in\boldsymbol{R}^N;\ (x',0)\in F,\ |x''|\leq\varepsilon\}$$

with $\varepsilon>0$ so small that $F_\varepsilon\subset U$.

In the following we assume further that

(3.22) F is convex with an analytic boundary in $\pi(\Gamma_0)$,

By (3.19) we see that

$$WF_\nu(v_1)\subset\{(x,\xi);\ (x',\xi')\in T_{\partial F}^*(\pi(\Gamma_0)),\ |x''|<\varepsilon,\ \xi''=0\}\cup\pi^{-1}(\{x;\ |x''|\geq\varepsilon\}).$$

Hence by (3.5)

$$K_g^* v_1\in G^\nu(\mathrm{Int}(F_\varepsilon)), \tag{3.23}$$

where $\mathrm{Int}(F_\varepsilon)$ denotes the interior of F_ε.

Since $\mathrm{supp}(v_2)\subset\bar{O}_R\backslash F_\varepsilon$, it follows by (3.6)

$$K_g^* v_2\in G^{1+h}\mathcal{A}_{x'}(\mathrm{Int}(F_\varepsilon)). \tag{3.24}$$

Thus by Lemma 2.5,

$$T^{(1)}(K_g^* v_2)\in H_{\Lambda,z}^{\nu} \quad \text{for all } z\in\kappa(\pi^{-1}(\mathrm{Int}(F_\varepsilon)\cap\Lambda)). \tag{3.25}$$

In view of (3.20),

$$WF_\nu(v_3)\subset O_R\times(\boldsymbol{R}^N\backslash\bar{V}_1)\cup T^*_{\dot{O}_R}(\boldsymbol{R}^N).$$

Again by (3.5) this yields

$$K_g^* v_3\in G^\nu(\mathrm{Int}(F_\varepsilon)). \tag{3.26}$$

Consequently, by (3.18) and (3.23)-(3.26), we have

$$g(D_x)u=u_1+u_2, \tag{3.27}$$

where

$$u_1=K_g^*(v_1+v_3)\in G^\nu(\mathrm{Int}(F_\varepsilon))$$

and

$$u_2=K_g^* v_2+RK_g^* u$$

with

$$T^{(1)}(u_2)\in H^\nu_{A,z}\quad\text{for all}\quad z\in\kappa(\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0).$$

Now we apply Lemma 2.1, 2.2 and obtain

(3.28) If $(x,\ \xi)\in\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0$ and $(x,\ \xi)\notin WF_\nu(u_2)$

then $\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0\cap WF_\nu(u_2)=\phi$.

Because $g\equiv1$ in the neighborhood V_1 of $\mathring{\xi}$,

$$WF_\nu(u_2)\cap\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0=WF_\nu(u)\cap\pi^{-1}(\mathrm{Int}(F_\varepsilon))\cap\Gamma_0.$$

Therefore (3.21) implies Theorem 1.1 for $\widetilde{W}=\pi^{-1}(\mathrm{Int}(F_\varepsilon))$.

Since any compact set in $\Gamma_0\cap W$ can be covered by a finite number of such $\widetilde{W}$'s we have actually proved Theorem 1.1.

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