The Plancherel Formula for the Universal Covering Group of $SL(2, \boldsymbol{R})$

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(Introduced by M. Sugiura)
(Received September 17, 1979)

Introduction

Pukanszky proved the Plancherel formula for the universal covering group G of $SL(2, \boldsymbol{R})$ in [1]. Here we shall prove the formula by a different method.

Let us explain our method in more details. Let $u_{\mu\mu}(g, s)$ be the matrix coefficient of the principal series representations of G (cf. § 1), and f be a C^∞-function on G with compact support which satisfies $f(kgk^{-1})=f(g)$ for any $k\in K$, where K is a subgroup of G defined in § 1.

First we calculate the μ-th Fourier transform

$$F(\mu, s)=\int_G f(g)u_{\mu\mu}(g, s)dg$$

Since f can be regarded as a function of two variables (θ, t), the transform $f\to F$ is composed of the following three transformations: the Fourier transform with respect to θ, and the Abel transform and the Fourier transform with respect to t. The Abel transform of a function on G is defined as follows:

$$\Phi(t)=e^{t/2}\int_{-\infty}^{\infty} f(a_t n_\xi)d\xi$$

where $g=k_\theta a_t n_\xi$ is the Iwasawa decomposition of g. $\Phi(t)$ can be expressed by the Tchebycheff function $T_{2\mu}$. The key point of our proof is the inverse transform formula (Theorem 2.3.1) of the Abel transform. From the theorem, we obtain the inverse transform $F\to f$.

In § 3 we shall give two proofs of the Plancherel formula. One is analogous to the proof of R. Takahashi for $SL(2, \boldsymbol{R})$ in [2]. Another is more elementary and more direct (cf. Proposition 3.1.2).

The author would like to express his sincere gratitude to Prof. T. Shintani for the kind advice and suggestions.

§ 1. Preliminaries.

In this section we describe the universal covering group of $SL(2, \boldsymbol{R})$ and its representations, and state their properties without proof. The proof can be found in the references [1], [3], [4].

1.1 Description of the universal covering group of $SL(2, \boldsymbol{R})$.

Let G_0 be the group of $SU(1,1)$ consisting all 2×2 matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \text{ satisfing } |\alpha|^2-|\beta|^2=1.$$

As is well known, G_0 is conjugate to $SL(2, \boldsymbol{R})$ in $GL(2, \boldsymbol{C})$. The group G_0 may be parametrized as follows. As in Bargmann [3], put $\beta/\alpha=\gamma$ and $\arg\alpha=\omega\in \boldsymbol{R}/2\pi\boldsymbol{Z}$ then

$$G_0=\{(\gamma, \omega): |\gamma|<1, \omega\in\boldsymbol{R}/2\pi\boldsymbol{Z}\}$$

and the map $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}\leftrightarrow(\gamma, \omega)$ is a diffeomorphism. The group operation in G_0 can be expressed in terms of γ and ω. If $a_0=(\gamma, \omega)$ and $a_0'=(\gamma', \omega')$ are in G_0, then for $a_0a_0'=a_0''=(\gamma'', \omega'')$ we get

$$\gamma''=(\gamma e^{-2i\omega'}+\gamma')(1+\gamma\bar{\gamma}'e^{-2i\omega'})^{-1} \tag{1.1.1}$$

$$\omega''=\omega+\omega'+\frac{1}{2i}\log(1+\gamma\bar{\gamma}'e^{-2i\omega'})(1+\bar{\gamma}\gamma'e^{2i\omega})^{-1} \tag{1.1.2}$$

where $\log z$ is defined by its principal value and ω'' is taken mod 2π.

Let G be the universal covering group of G_0. G can be parametrized by γ and ω as follows:

$$G=\{(\gamma, \omega): |\gamma|<1, -\infty<\omega<+\infty\}.$$

The multiplication on G is given by equation (1.1.1) and (1.1.2) except that ω'' is no longer defined mod 2π. The canonical covering map is defined by

$$\Phi(\gamma, \omega)=(\gamma, \omega(\text{mod } 2\pi)) \tag{1.1.3}$$

Put $k_\theta=(0, \theta/2)$, $a_t=(\text{th}\, t/2, 0)$ and $n_\xi=\left(-\frac{i(\xi/2)}{1+i\xi/2}, \arg\left(1+\frac{i\xi}{2}\right)\right)$. The elements k_θ, a_t, n_ξ are in G.

We use the following subgroups:

$$K=\{k_\theta: \theta\in\boldsymbol{R}\},\ A=\{a_t: t\in\boldsymbol{R}\}, \text{ and } N=\{n_\xi; \xi\in\boldsymbol{R}\}.$$

Each element g in G can be expressed uniquely by $g=k_\theta a_t n_\xi$ (Iwasawa decomposition), where θ, t and ξ are real numbers. And we have another expression $g=k_\theta a_t k_\phi$ (Cartan decomposition) where $\theta\in\boldsymbol{R}$, $t\geqq 0$, $0\leqq\phi<2\pi$. For $g\notin K$ this expression is unique.

1.2 Principal continuous series of representations of G

Let us define an action of G on the unit circle $\boldsymbol{T}$. For $a=(\gamma, \omega)$ in G, $a\cdot e^{i\theta}=\Phi(a)\cdot e^{i\theta}=\dfrac{\alpha e^{i\theta}+\beta}{\bar{\beta}e^{i\theta}+\bar{\alpha}}$, where $\Phi(a)=(\gamma, \omega(\mathrm{mod}\, 2\pi))=\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$.

Put $H_0=L^2(\boldsymbol{T})$, the Hilbert space of functions on the unit circle which are square integrable with respect to the normalized Haar measure of $\boldsymbol{T}$.

Definition 1.2.1. For $f\in H_0$, and $a^{-1}=(\gamma, \omega)\in G$, a linear operator $U_h(a, s)$ on H_0 is defined by

$$[U_h(a, s)f](e^{i\theta})=e^{-2i\omega h}\left(\frac{1+e^{i\theta}\bar{\gamma}}{1+e^{-i\theta}\gamma}\right)|e^{i\theta}\bar{\beta}+\bar{\alpha}|^{-2s}f(a^{-1}\cdot e^{i\theta}),$$

where h and s are complex numbers and $\Phi(a^{-1})=\begin{pmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$.

The following is well known (cf. [4]).

Proposition 1.2.1. *Suppose $h\in \boldsymbol{C}$ and $s\in \boldsymbol{C}$.*

(1) *The map $a\longmapsto U_h(a, s)$ is a continuous representation of G on H_0.*

(2) *This representation is unitary if and only if $Re\ s=1/2$ and $h\in \boldsymbol{R}$.*

(3) *When $Re\ s=1/2$ and $h\in \boldsymbol{R}$, the representations $a\longmapsto U_h(a, s)$ and $a\longmapsto U_{h+1}(a, s)$ are unitarily equivalent so that each unitary representation is equivalent to one in the range $-1/2<h\leqq 1/2$.*

(4) *The representation $a\longmapsto U_h(a, s)$ of G on H_0 are irreducible for s satisfying $Re\ s=1/2$, and $-1/2<h\leqq 1/2$ excluding the case $s=1/2$ and $h=1/2$.*

1.3 Discrete series of representations of G.

Let D be the unit disk in the complex plane; $D=\{z\in \boldsymbol{C}: |z|<1\}$. For $h>1/2$, we define a Hilbert space $H_h(D)$ of holomorphic functions on D with the inner product

$$(f, g)_h=\frac{2h-1}{\pi}\int_D f(z)\overline{g(z)}(1-|z|^2)^{2h-2}dxdy. \tag{1.3.1}$$

The functions

$$\phi_{n,h}(z)=\left[\frac{\Gamma(2h+n)}{\Gamma(2h)\Gamma(n+1)}\right]^{1/2}z^n, \quad (n\in \boldsymbol{Z}\ n\geqq 0) \tag{1.3.2}$$

form an orthonormal basis in $H_h(D)$. When $h=1/2$, the space $H_h(D)$ can be defined by the inner product

$$(f, g)_{1/2}=\lim_{h\to 1/2+0}\left(\frac{2h-1}{\pi}\int_D f(z)\overline{g(z)}(1-|z|^2)^{2h-2}dxdy\right) \tag{1.3.3}$$

The functions

(1.3.4) $$\phi_{n,1/2}(z)=z^n \quad (n\in \boldsymbol{Z}, n\geqq 0)$$

form an orthonormal basis in $H_{1/2}(D)$. Now take $h\leqq -1/2$. The Hilbert space $H_h(D)$ is made up of the complex conjugates of the functions in $H_{-h}(D)$ and the inner products are defined as in (1.3.1) and (1.3.3) with $|h|$ in place of h. The orthonormal basis in this case is given by

(1.3.5) $$\phi_{n,h}(z)=\left[\frac{\Gamma(2|h|+n)}{\Gamma(2|h|)\Gamma(1+n)}\right]^{1/2}(\bar{z}), \quad n\in \boldsymbol{Z},\ n\geqq 0.$$

We now define a representation $a\longmapsto U^+(a, h)$ of G on $H_h(D)$, $h\geqq 1/2$ by

(1.3.6) $$[U^+(a, h)f](z)=e^{2i\omega h}(1-|\gamma|^2)^h(1+\bar{\gamma}z)^{-2h}f\left(\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}\right),$$

where $a^{-1}=(\gamma, \omega)$ and $\Phi(a^{-1})=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$.

PROPOSITION 1.3.1. *For $h\geqq 1/2$, the representations $a\longmapsto U^+(a, h)$ of G on $H_h(D)$ are unitary and irreducible.*

We shall call the representations $a\longmapsto U^+(a, h)$, $(h\in \boldsymbol{R}, h\geqq 1/2)$ the positive discrete series of representations of G.

The discrete series $a\longmapsto U^-(a, h), h\leqq -1/2$ are defined similarly. The representation spaces are the spaces $H_h(D), h\leqq -1/2$. For $h\leqq -1/2$, we define

(1.3.7) $$[U^-(a, h)f](z)=e^{2i\omega h}(1-|\gamma|^2)^{-h}(1+\bar{\gamma}z)^{2h}f\left(\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}\right),$$

where $a^{-1}=(\gamma, \omega)$.

The following proposition is known also (cf. [4]).

PROPOSITION 1.3.2. *For $h\leqq -1/2$, the representations $a\longmapsto U^-(a, h)$ of G on $H_h(D)$ are irreducible and unitary.*

We shall call the representations $a\longmapsto U^-(a, h)$, $(h\in \boldsymbol{R}, h\leqq -1/2)$ the negative discrete series of representations of G.

Remark. For $-1/2<h<1/2$, the representation $U^{\pm}(a, h)$ can be defined. However we shall not use these representations. So we omit the description of them which can be found in Sally [4].

1.4. Matrix coefficients of the irreducible representations of G.

For $f, g\in L^2(\boldsymbol{T})$, put $(f, g)=\int_{\boldsymbol{T}} f(z)\overline{g(z)}dz$. The functions $\phi_n(z)=z^n(|z|=1, n\in \boldsymbol{Z})$ form an orthonormal basis in $L^2(\boldsymbol{T})$.

We now define the matrix coefficients $u_{m-h\ n-h}(g, s)$ of the operator $U_h(g, s)$

by the following equality.

(1.4.1) $$u_{m-h\ n-h}(g, s)=(U_h(g, s)\phi_n, \phi_m),$$

where $-1/2<h\leqq 1/2$ and $s\in C$. We get the following lemma from the equations;

$$[U_h(k_\theta, s)\phi_n](z)=e^{-i(n-h)\theta}\phi_n(z).$$

LEMMA 1.4.1. *We have* $u_{\mu\nu}(k_\theta g k_\phi)=\overline{\chi_\mu(k_\theta)}u_{\mu\nu}(g, s)\overline{\chi_\nu(k_\phi)}$, *where* $\chi_\mu(\mu\in \boldsymbol{R})$ *are the characters of* K *defined by* $\chi_\mu(k_\theta)=e^{i\mu\theta}$.

The properties of $u_{\mu\nu}$ will be summarized in the following lemma.

LEMMA 1.4.2. *We have the following equalities.*

(1) $$u_{\mu\mu}(a_t, s)=(1-\lambda^2)^s F(s+\mu, s-\mu\,; 1\,; \lambda^2),$$

where F *is the hypergeometric function and* $\lambda=\mathrm{th}\, t/2$.

(2) $$u_{\mu\mu}(g, s)=u_{\mu\mu}(g, 1-s).$$

Proof. The proof of (1) can be found in Sally [4]. So we give the proof of (2).

From the formula $F(a, b\,; c\,; x)=(1-x)^{c-a-b}F(c-a, c-b\,; c\,; x)$ (cf. [6]) and (1), we get $u_{\mu\mu}(a_t, s)=u_{\mu\mu}(a_t, 1-s)$. Since $G=KAK$, Lemma 1.4.1 asserts that for any g of G we obtain $u_{\mu\mu}(g, s)=u_{\mu\mu}(g, 1-s)$. Q. E. D.

Put $v^{\pm}_{mn}(g, l)=(U^{\pm}(g, l)\phi^{\pm}_{n,l}(z), \phi^{\pm}_{m,l}(z))_l$. Namely $v^{\pm}_{mn}(g, l)$ are the matrix coefficients of the discrete series representations of G.

The proof of the following lemma can be found in Sally [4].

LEMMA 1.4.3. *We have* $v^{\pm}_{mm}(a_t, l)=(l-\lambda^2)^{|l|}F(m+2|l|, -m\,; 1\,; \lambda^2)$ *where* $\lambda=\mathrm{th}\, t/2$.

From Lemma 1.4.2 and Lemma 1.4.3, we have

$$u_{m+l\ m+l}(a_t, l)=v^{+}_{mm}(g, l).$$

Since $v^{+}_{mm}(k_\theta g k_\phi, l)=e^{-i(m+l)\theta}v^{+}_{mm}(g, l)e^{-i(m+l)\phi}$, we get (1) in the following lemma.

LEMMA 1.4.3. *Let* m *be a positive integer, then we have*

(1) *for* $l\geqq 1/2$, $u_{m+l\ m+l}(g, l)=v^{+}_{mm}(g, l)$

(2) *for* $l\leqq -1/2$, $u_{-m+l\ -m+l}(g, -l)=v^{-}_{mm}(g, l)$.

The proof of (2) is similar to the proof of (1).

§ 2. The Fourier transform on G.

2.1 The family of functions $D_0(G)$.

We denote by $D(G)$ the set of all C-valued indefinitely differentiable functions on G with compact support.

LEMMA 2.1.1. *For any f in $D(G)$,*

$$f(g)=\sum_{n\in Z}\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta g k_\theta^{-1})\overline{\chi_n(k_\theta)}d\theta$$

Proof. Put $F(\phi)=f(k_\phi g k_\phi^{-1})$, then F is a periodic function of period 2π. In fact since $k_{2\pi}$ is an element of the center of G,

$$\begin{aligned}F(\phi+2\pi)&=f(k_{\phi+2\pi}gk_{\phi+2\pi}^{-1})\\&=f(k_\phi g k_\phi^{-1})\\&=F(\phi).\end{aligned}$$

So by the Fourier expansion,

$$F(\phi)=\sum_{n\in Z}\frac{1}{2\pi}\int_0^{2\pi}F(\theta)e^{-in\theta}d\theta e^{in\phi}.$$

Put $\phi=0$, then we get

$$f(g)=\sum_{n\in Z}\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta g k_\theta^{-1})\overline{\chi_n(k_\theta)}d\theta. \quad \text{Q.E.D.}$$

LEMMA 2.1.2 *Put*

$$f_n(g)=\frac{1}{2\pi}\int_0^{2\pi}f(k_\theta g k_\theta^{-1})\overline{\chi_n(k_\theta)}d\theta.$$

Then $f_n(gk)=\overline{\chi_n(k)}f_n(kg)$ for any $k\in K$. The proof is easy.

DEFINITION 2.1.1 Put $D_n(G)=\{f\in D(G): f(gk)=\overline{\chi_n(k)}f(kg)\}$

LEMMA 2.1.3. *If f belongs to $D_n(G)$ and $n\neq\mu-\nu$, then $\int_G f(g)u_{\mu\nu}(g,s)dg=0$ where dg is the Haar measure of G.*

Proof. Put $I=\int_G f(g)u_{\mu\nu}(g,s)dg$. Since G is unimodular,

$$\begin{aligned}I&=\int_G f(k_\theta g k_\theta^{-1})u_{\mu\nu}(k_\theta g k_\theta^{-1},s)dg\\&=e^{i\{(\mu-\nu)-n\}\theta}\int_G f(g)u_{\mu\nu}(g,s)dg\\&=e^{i\{(\mu-\nu)-n\}\theta}I.\end{aligned}$$

By the assumption $e^{i\{(\mu-\nu)-n\}\theta}-1$ is not identically zero, so we have $I=0$. Q.E.D.

REMARK 1. If n is not equal to zero, then $f_n(e)=0$ where e is the identity element. Hence for any f in $D(G)$, we have

$$f(e)=f_0(e).$$

REMARK 2. It is easy to verify that $D_0(G)$ is a commutative algebra with respect to the convolution product.

2.2. Spherical functions on G

Let us start with the following definition.

DEFINITION 2.2.1. For $\nu\in\boldsymbol{R}, s\in\boldsymbol{C}$ we set

(2.2.1) $$\alpha_{\nu,s}(g)=\overline{\chi_\nu(k_\theta)}e^{-st} \text{ where } g=k_\theta a_t n_\xi$$

(2.2.2) $$\zeta_{\nu,s}(g)=\frac{1}{2\pi}\int_0^{2\pi}\alpha_{\nu,s}(k_\theta{}^{-1}gk_\theta)d\theta$$

LEMMA 2.2.1 *The following equality holds*:

(1) $\zeta_{\nu,s}(k_\theta g k_\phi)=\overline{\chi_\nu(k_{\theta+\phi})}\zeta_{\nu,s}(g)$

(2) *If Ω is the Casimir operator of G, then*

$$\Omega\zeta_{\nu,s}=s(1-s)\zeta_{\nu,s}$$

Proof. From the definition, $\alpha_{\nu,s}(k_\theta g)=\overline{\chi_\nu(k_\theta)}\alpha_{\nu,s}(g)$. Hence

$$\begin{aligned}\zeta_{\nu,s}(k_\theta g k_\phi)&=\frac{1}{2\pi}\int_0^{2\pi}\alpha_{\nu,s}(k_\psi^{-1}k_\theta g k_\phi k_\psi)d\psi\\&=\frac{1}{2\pi}\int_0^{2\pi}\alpha_{\nu,s}(k_{\theta+\phi}k_{\psi+\phi}^{-1}gk_{\phi+\psi})d\psi\\&=\chi_\nu(k_{\theta+\phi})\frac{1}{2\pi}\int_0^{2\pi}\alpha_{\nu,s}(k_\psi^{-1}gk_\psi)d\psi\\&=\chi_\nu(k_{\theta+\phi})\zeta_{\nu,s}(g).\end{aligned}$$

We used the fact $k_{2\pi}$ is in the center of G. This shows (1).

For any g in G, $g=k_\theta a_t n_\xi$. In terms of this coordinate system (θ, t, ξ)

$$\Omega=-\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial t}+2e^{-t}\frac{\partial^2}{\partial\xi\partial\theta}-e^{-2t}\frac{\partial}{\partial\xi^2}, \quad \text{(cf. [2])}$$

So we get

$$\Omega\alpha_{\nu,s}=s(1-s)\alpha_{\nu,s}.$$

Hence $$\Omega\zeta_{\nu,s}=s(1-s)\zeta_{\nu,s}. \qquad \text{Q.E.D.}$$

LEMMA 2.2.2 *The following equality holds*:

$$\zeta_{\nu,s}(a_t)=\mathrm{ch}^{-2s}\frac{t}{2}F\left(s+\nu,\ s-\nu\,;1\,;\mathrm{th}^2\frac{t}{2}\right)$$

Proof. For any $g\in G$, $g=k_\theta a_t k_\psi$, $\theta\in \boldsymbol{R}$, $t>0$, $0\leqq\psi\leqq 2\pi$. In terms of this coordinate system (cf. [2]),

$$\Omega=-\frac{1}{\mathrm{sh}^2 t}\left(\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial\psi^2}\right)+2\frac{\mathrm{ch}\,t}{\mathrm{sh}^2 t}\frac{\partial^2}{\partial\theta\partial\psi}-\frac{\mathrm{ch}\,t}{\mathrm{ch}\,t}\frac{\partial}{\partial t}-\frac{\partial^2}{\partial t^2}.$$

From the equality and Lemma 2.2.1, easy computation shows

$$\left[\frac{d^2}{dt^2}+\frac{\mathrm{ch}\,t}{\mathrm{ch}\,t}\frac{d}{dt}+\left(s(1-s)+\frac{2\nu^2}{\mathrm{sh}^2 t}(\mathrm{ch}\,t-1)\right)\right]\zeta_{\nu,s}(a_t)=0$$

We put $X=\mathrm{th}^2\frac{t}{2}$ and define $Z(X)$ by the equation:

$$(1-X)^s Z(X)=\zeta_{\nu,s}(a_t).$$

Then $Z(X)$ satisfies

$$X(1-X)Z''(X)+(1-(2s+1)X)Z'(X)+(\nu^2-s^2)Z(X)=0.$$

Since $Z(X)$ is analytic around $X=0$ and $Z(0)=\zeta_{\nu,s}(e)=1$, we get

$$Z(X)=F(s+\nu,\ s-\nu\,;1\,;X)$$

Hence we obtain the lemma. Q.E.D.

LEMMA 2.2.3. *The following relations hold*:

(1) $\zeta_{\mu,s}(g)=u_{\mu\mu}(g,s)$ (2) $\zeta_{m+l,\,l}(g)=v^+_{mm}(g,l)$ (3) $\zeta_{-m+l,\,-l}(g,l)=v^-_{\bar{m}m}(g,l)$

Proof. Lemma 1.4.1, Lemma 1.4.2, Lemma 1.4.3, Lemma 2.2.1 and Lemma 2.2.2 show (1), (2) and (3) immediately.

2.3. The Fourier transform on G and its inverse transform.

The purpose of this section is to find an explicit formula of the integral transform of the functions $f\in D_0(G)$.

The transform $\zeta_{\nu,s}(f)$ is expressed as follows:

$$\zeta_{\nu,s}(f)=\int_G f(g)\zeta_{\nu,s}(g)dg$$

$$=\frac{1}{2\pi}\int_G f(g)\int_0^{2\pi}\alpha_{\nu,s}(k_\theta g k_\theta^{-1})dg d\theta$$

$$=\frac{1}{2\pi}\int_G\int_0^{2\pi} f(k_\theta^{-1}gk_\theta)\alpha_{\nu,s}(g)d\theta dg$$

$$=\int_G f(g)\alpha_{\nu,s}(g)dg$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\theta a_t n_\xi)\overline{\chi_\nu(k_\theta)}e^{-st}e^t d\theta dt d\xi$$

because $dg=e^t d\theta dt d\xi$ for $g=k_\theta a_t n_\xi$. Here, we put

$$F_f(\nu, t)=e^{t/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\theta a_t n_\xi)\overline{\chi_\nu(k_\theta)}d\theta d\xi$$

Then

(2.3.1) $$\zeta_{\nu,s}(f)=\int_{-\infty}^{\infty} F_f(\nu, t)e^{(1/2-s)t}dt$$

If $a_t n_\xi=k_{\phi'}a_{t'}k_{\psi'}$, then

(2.3.2) $$\operatorname{ch} t'=\operatorname{ch} t+e^t\frac{\xi^2}{2},$$ (2.3.3) $$e^{i\frac{\psi'+\phi'}{2}}=\frac{\operatorname{ch}^2\frac{t}{2}+i\frac{\xi}{2}e^{t/2}}{\sqrt{\operatorname{ch}^2\frac{t}{2}+e^t\frac{\xi^2}{4}}}$$ (cf. [2])

$$F_f(\nu, t)=e^{t/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\theta k_{\phi'}a_{t'}k_{\psi'})\overline{\chi_\nu(k_\theta)}d\theta d\xi$$

$$=e^{t/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_\theta a_{t'})e^{-i\nu(\theta-\phi'-\psi')}d\theta d\xi$$

Put $f(k_\theta a_t)=f(\theta, t)$, then $f(\theta, t)=f(\theta, -t)$. In fact since f belongs to $D_0(G)$,

$$f(k_\theta a_t)=f(k_\theta k_\pi a_t k_{-\pi})=f(k_\theta a_{-t})$$

Therefore as in R. Takahashi [2], we can write

$$f(\theta, t)=f[\theta, \operatorname{ch} t],$$

where $f[\theta, x]$ is an indefinitely differentiable function on $\boldsymbol{R}\times[1, \infty)$ (cf. [8] pp. 350). So by (2.3.2) and (2.3.3),

$$F_f(\nu, t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left[\theta, \operatorname{ch} t+\frac{\xi^2}{2}\right]T_{2\nu}\left(\frac{\operatorname{ch}^2\frac{t}{2}}{\sqrt{\operatorname{ch}^2\frac{t}{2}+\frac{\xi^2}{4}}}\right)e^{-i\nu\theta}d\theta d\xi$$

where T_α is the Tchebycheff function of the first kind defined by $T_\alpha(z)=$

$F\left(\alpha, -\alpha; \frac{1}{2}; \frac{1-z}{2}\right)$. In case of $\alpha \in \boldsymbol{Z}$, $\alpha \geqq 0$, T_α is a Tchebycheff polynomial. In the same way as Tchebycheff polynomial, T_α satisfies

$$T_\alpha(\cos\theta) = \cos\alpha\theta \text{ and } T_\alpha(\operatorname{ch} t) = \operatorname{ch}\alpha t. \quad \text{(cf. [7]).}$$

Now from the explicit formula (2.3.4), $F_f(\nu, t) = F_f(\nu, -t)$. So we can put

$$F_f(\nu, t) = F_f[\nu, \operatorname{ch} t] = F_f[\nu, x].$$

Summing up these calculations, we get the following proposition.

PROPOSITION 2.3.1. *If f belongs to $D_0(G)$, then*

$$\zeta_{\nu,s}(f) = \int_{-\infty}^{\infty} F_f[\nu, \operatorname{ch} t] e^{(1/2-s)t} dt$$

where

$$F_f[\nu, \operatorname{ch} t] = 2\int_{-\infty}^{\infty}\int_0^{\infty} f\left[\theta, \operatorname{ch} t + \frac{\xi^2}{2}\right] T_{2\nu}\left(\sqrt{\frac{\operatorname{ch} t + 1}{\operatorname{ch} t + \xi^2/2 + 1}}\right) e^{-i\nu\theta} d\theta d\xi.$$

THEOREM 2.3.1. *Let f and g be C^∞-functions on $[1, \infty)$ with compact supports. Put*

$$(2.3.5) \qquad {}^{\sharp}f[x] = 2\int_0^{\infty} f\left[x + \frac{\xi^2}{2}\right] T_{2\nu}\left(\sqrt{\frac{x+1}{x+1+\xi^2/2}}\right) d\xi$$

$$(2.3.6) \qquad {}^{\natural}g[x] = -\frac{1}{\pi}\int_0^{\infty} g'\left[x + \frac{\eta^2}{2}\right] T_{2\nu}\left(\sqrt{\frac{x+1+\eta^2/2}{x+1}}\right) d\eta$$

Then ${}^{\natural}({}^{\sharp}f) = f$ and ${}^{\sharp}({}^{\natural}g) = g$.

Proof. It is easy to verify that ${}^{\sharp}f$ and ${}^{\natural}g$ are C^∞-functions with compact supports because T_α is a real analytic function on $(-1, \infty)$. We prove that ${}^{\sharp}({}^{\natural}g) = g$. The equality ${}^{\natural}({}^{\sharp}f) = f$ can be proved in a similar way. In (2.3.5) put $x + \xi^2/2 = y$, then

$$f[x] = \sqrt{2}\int_x^{\infty} f[y] T_{2\nu}\left(\sqrt{\frac{y+1}{x+1}}\right) \frac{dy}{\sqrt{y-x}}$$

In (2.3.6) put $x + \eta^2/2 = y$, then

$$g[x] = -\frac{1}{\pi}\int_x^{\infty} g'[y] T_{2\nu}\left(\sqrt{\frac{y+1}{x+1}}\right) \frac{dy}{\sqrt{2(y-x)}}$$

Put $\tilde{g} = {}^{\sharp}({}^{\natural}g)$, $G[x] = g[x-1]$, ${}^{\natural}G[X] = {}^{\natural}g[x-1]$ and $\tilde{G}[x] = \tilde{g}[x-1]$. Then

$$(2.3.7) \qquad {}^{\natural}G[x] = -\frac{1}{\sqrt{2}\pi}\int_x^{\infty} G'[y] T_{2\nu}\left(\sqrt{\frac{y}{x}}\right) \frac{dy}{\sqrt{y-x}}$$

(2.3.8) $$\tilde{G}[x]=\sqrt{2}\int_x^{\infty} {}^{\natural}G[y]T_{2\nu}\left(\sqrt{\frac{x}{y}}\right)\frac{dy}{\sqrt{y-x}}$$

Now we extend the derivative $G'[x]$ to a continuous and bounded function on $[1, \infty)$, and we extend ${}^{\natural}G[x]$ to a continuous function on $(0, \infty)$ defined by (2.3.7) and also extend $\tilde{G}[x]$ to a continuous function on $(0, \infty)$ defined by (2.3.8).

Let us consider the Mellin transform of $\tilde{G}[x]$.

$$\begin{aligned}\tilde{\mathcal{G}}(s)&=\int_0^{\infty}\tilde{G}[x]x^{s-1}dx\\
&=\int_0^{\infty}\sqrt{2}\int_x^{\infty}{}^{\natural}G[y]T_{2\nu}\left(\sqrt{\frac{x}{y}}\right)\frac{dy}{\sqrt{y-x}}\,x^{s-1}dx\\
&=\sqrt{2}\int_0^{\infty}{}^{\natural}G[y]dy\int_0^{y}T_{2\nu}\left(\sqrt{\frac{x}{y}}\right)\frac{x^{s-1}}{\sqrt{y-x}}dx\\
&=\sqrt{2}\int_0^{\infty}{}^{\natural}G[y]dy\int_0^{1}T_{2\nu}(\sqrt{t})\frac{y^{s-1}t^{s-1}y}{\sqrt{y(1-t)}}dt\\
&=\sqrt{2}\int_0^{\infty}{}^{\natural}G[y]y^{s-1/2}dy\int_0^{1}T_{2\nu}(\sqrt{t})\frac{t^{s-1}}{\sqrt{1-t}}dt\\
&=2C_1{}^{\natural}\mathcal{G}\left(s+\frac{1}{2}\right)\end{aligned}$$

where

$C_1=\int_0^1 T_{2\nu}(\sqrt{t})\frac{t^{s-1}}{\sqrt{1-t}}dt$ and ${}^{\natural}\mathcal{G}$ is the Mellin transform of ${}^{\natural}G$.

$$\begin{aligned}{}^{\natural}\mathcal{G}\left(s+\frac{1}{2}\right)&=\int_0^{\infty}{}^{\natural}G[y]y^{(s+1/2)-1}dy\\
&=-\frac{1}{\sqrt{2}\,\pi}\int_0^{\infty}\int_0^{\infty}G'[z]T_{2\nu}\left(\sqrt{\frac{z}{y}}\right)\frac{dz}{\sqrt{z-y}}\,y^{s-1/2}dy\\
&=-\frac{1}{\sqrt{2}\,\pi}\int_0^{\infty}G'[z]dz\int_0^{z}T_{2\nu}\left(\sqrt{\frac{z}{y}}\right)\frac{y^{s-1/2}}{\sqrt{z-y}}dy\\
&=\frac{s}{\sqrt{2}\,\pi}\int_0^{\infty}G[z]z^{s-1}dz\int_0^{1}T_{2\nu}\left(\sqrt{\frac{1}{t}}\right)\frac{t^{s-1/2}}{\sqrt{1-t}}dt\\
&=\frac{s}{\sqrt{2}\,\pi}\,\mathcal{G}(s)C_2\end{aligned}$$

where

$C_2=\int_0^1 T_2\left(\sqrt{\frac{1}{t}}\right)\frac{t^{s-1/2}}{\sqrt{1-t}}dt$ and $\mathcal{G}$ is the Mellin transform of $G[x]$.

So we get

$$\tilde{\mathcal{G}}(s)=\frac{s}{\pi}\mathcal{G}(s)C_1C_2.$$

Putting $\sqrt{t}=\cos\theta$ (cf. [6] pp. 9),

$$C_1=2\int_0^{\pi/2}\cos 2\nu\theta\cos^{2s-1}\theta d\theta=2^{1-2s}\frac{\pi\Gamma(s)}{\Gamma\left(s+\frac{1+2\nu}{2}\right)\Gamma\left(s+\frac{1-2\nu}{2}\right)}$$

Putting $\frac{1}{\sqrt{t}}=\mathrm{ch}\,x$ (cf. [6] pp. 10)

$$C_2=2\int_0^{\infty}\mathrm{ch}\,2\nu x\,\mathrm{ch}^{-1-2s}x dx=2^{2s-1}\frac{\Gamma\left(s+\frac{2\nu+1}{2}\right)\Gamma\left(s+\frac{1-2\nu}{2}\right)}{s\Gamma(2s)}$$

Therefore $C_1C_2=\pi/s$. It follows $\tilde{\mathcal{G}}(s)=\mathcal{G}(s)$. So by the inverse Mellin transform, we get

$$\tilde{G}[x]=G[x]. \qquad \text{Q.E.D.}$$

Remark. In case of $\nu=0$, the theorem is well known (cf. [9]). In case of $\nu=n\in \boldsymbol{Z}$, the theorem is proved by T. Shintani (unpublished).

The proof of the Plancherel formula is based on the following proposition.

PROPOSITION 2.3.2. *If* $f\in D_0(G)$, *then*

$$f(e)=-\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\partial}{\partial x}F_f\left[\nu,\, x+\frac{\eta^2}{2}\right]_{x=1}T_{2\nu}\left(\sqrt{1+\frac{\eta^2}{4}}\right)d\eta d\nu$$

Proof. By Proposition 2.3.1

$$F_f[\nu,\, x]=2\int_{-\infty}^{\infty}\int_0^{\infty}f\left[\theta,\, x+\frac{\xi^2}{2}\right]T_{2\nu}\left(\sqrt{\frac{x+1}{x+1+\xi^2/2}}\right)d\xi e^{-i\nu\theta}d\theta$$

So by Theorem 2.3.1 and the Fourier inverse transform,

$$f[\theta,\, x]=-\frac{1}{2\pi^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{\partial}{\partial x}F_f\left[\nu,\, x+\frac{\eta^2}{2}\right]T_{2\nu}\left(\sqrt{\frac{x+\eta^2/2+1}{x+1}}\right)d\eta e^{i\nu\theta}d\nu$$

Putting $\theta=0$ and $x=1$, we get the above proposition.

§ 3. The Plancherel formula for G.

3.1. The Plancherel formula in $D_0(G)$.

We start with the following lemma.

LEMMA 3.1.1

$$\textit{Put } Z(s\,;\nu, f)=\left(s-\frac{1}{2}\right)\frac{\sin\pi(s-\nu-1/2)}{\cos\pi(s-\nu-1/2)}\zeta_{\nu,s}(f),$$

where $f\in D_0(G)$ *and* $\nu\in\mathbf{R}$. *Then* $Z(s\,;\nu, f)$ *is a meromorphic function on the complex s-plane and if* $a<b$,

$$\lim_{|\mathrm{Im}\, s|\to\infty} Z(s\,;\nu, f)=0$$

The convergence is uniform on the strip $a\leqq \mathrm{Re}\ s\leqq b$

The proof of this lemma can be given by the similar method as in Lemma 7 of [2]. So we omit the proof.

LEMMA 3.1.2 *We have the following relation*:

$$-i\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,\nu+1/2+i\mu}(f)\ \mathrm{th}\ \pi\mu d\mu$$

$$=\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\nu\ \mathrm{Re\ th}\ \pi(\nu+i\mu)d\mu+2\sum_{\substack{1/2\leq p\leq|\nu|\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f).$$

Proof. We may assume $\nu>0$ since $\zeta_{\nu,s}(g)=\zeta_{-\nu,s}(g)$. In case of $\nu-[\nu]=1/2$, where [ ] is the Gauss symbol, we integrate $Z(s\,;\nu, f)$ along the rectangle Γ_ν having vertices $\pm iT$, $\nu+1/2\pm iT$ counterclockwise. We have

$$\int_{T}^{-T}Z\left(\frac{1}{2}+i\mu\,;\nu, f\right)id\mu+\int_{1/2}^{1/2+\nu}Z(\sigma-iT\,;\nu, f)d\sigma+\int_{-T}^{T}Z\left(\nu+\frac{1}{2}+i\mu\,;\nu, f\right)id\mu$$

$$+\int_{\nu+1/2}^{1/2}Z(\sigma+iT\,;\nu, f)d\sigma=2\pi i\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(-\frac{1}{\pi}\left(p-\frac{1}{2}\right)\right)\zeta_{\nu,p}(f).$$

Letting $T\to\infty$, by Lemma 3.1.1, we get

$$-\int_{-\infty}^{\infty}Z\left(\frac{1}{2}+i\mu\,;\nu, f\right)id\mu+\int_{-\infty}^{\infty}Z\left(\nu+\frac{1}{2}+i\mu\,;\nu, f\right)id\mu=-2i\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f).$$

Therefore,

$$\int_{-\infty}^{\infty}\mu\zeta_{\nu,1/2+i\mu}(f)\frac{\sin\pi(-\nu+i\mu)}{\cos\pi(-\nu+i\mu)}d\mu-\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,\nu+1/2+i\mu}(f)\ \mathrm{th}\ \pi\mu d\mu$$

$$=-2i\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f).$$

Put

$$I=\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\frac{\sin\pi(-\nu+i\mu)}{\cos\pi(-\nu+i\mu)}d\mu.$$

Then

$$I=i\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\,\mathrm{th}\,\pi(\mu+i\nu)\mu d\mu.$$

Since $\zeta_{\nu,1-s}(f)=\zeta_{\nu,s}(f)$, we have also

$$I=i\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\,\mathrm{th}\,\pi(\mu-i\nu)\mu d\mu.$$

Hence

$$I=i\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\,\mathrm{Re}\ \mathrm{th}\,\pi(\mu+i\nu)\mu d\mu.$$

Consequently we obtain

$$\text{(3.1.1)}\qquad -i\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,1/2+\nu+i\mu}(f)\,\mathrm{th}\,\pi\mu d\mu$$
$$=\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\,\mathrm{Re}\ \mathrm{th}\,\pi(\mu+i\nu)d\mu+2\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f).$$

In case of $\nu-[\nu]=1/2$, by slight modification of the path of integral as in [2], we get

$$-i\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,1/2+\nu+i\mu}(f)\,\mathrm{th}\,\pi\mu d\mu$$
$$=\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\coth\pi\mu d\mu+2\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f).$$

But this equality is no other than the equality which is obtained by putting $\nu=1/2+n$ $(n=0,1,2,\cdots)$ in (3.1.1). So we complete the proof.

LEMMA 3.1.3. *If f is a C^{∞}-function with compact support such that $f(0)=0$, then*

$$\int_{-\infty}^{\infty}f(t)\frac{dt}{\mathrm{sh}\frac{t}{2}}=i\int_{-\infty}^{\infty}\hat{f}(\tau)\,\mathrm{th}\,\pi\tau d\tau$$

$$\textit{where}\quad \hat{f}(\tau)=\int_{-\infty}^{\infty}f(t)e^{-it\tau}dt$$

The proof can be found in [8] (pp. 341).

LEMMA 3.1.4. *If $f\in D_0(G)$, then*

$$f(e)=\frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,\nu+1/2+i\mu}(f)\,\mathrm{th}\,\pi\mu d\mu d\nu.$$

Proof. Put

$$I=-\frac{i}{4\pi}\int_{-\infty}^{\infty}(\nu+i\mu)\xi_{\nu,\nu+1/2+i\mu}(f)\,\mathrm{th}\,\pi\mu d\mu.$$

Then

$$I=-\frac{i}{4\pi}\int_{-\infty}^{\infty}\mathrm{th}\,\pi\mu d\mu\int_{-\infty}^{\infty}F_f(\nu, t)(\nu+i\mu)e^{-(\nu+i\mu)t}dt$$

$$=-\frac{i}{4\pi}\int_{-\infty}^{\infty}\mathrm{th}\,\pi\mu d\mu\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial t}F_f(\nu, t)\right)\mathrm{ch}\,\nu t e^{-i\mu t}dt$$

$$=-\frac{1}{4\pi}\int\left(\frac{\partial}{\partial t}F_f(\nu, t)\right)\frac{\mathrm{ch}\,\nu t}{\mathrm{sh}\frac{t}{2}}dt \qquad \text{(Lemma 3.1.3)}$$

$$=-\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}F_f[\nu, \mathrm{ch}\,t]2\,\mathrm{ch}\frac{t}{2}\mathrm{ch}\,\nu t dt, \quad (x=\mathrm{ch}\,t).$$

Put $\xi=2\,\mathrm{sh}\frac{t}{2}$. Then we have $1+(\xi^2/2)=\mathrm{ch}\,t=x$ and

$$I=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}F_f\left[\nu, 1+\frac{\xi^2}{2}\right]T_{2\nu}\left(\sqrt{1+\frac{\xi^2}{2}}\right)d\xi.$$

By Proposition 2.3.2, we get

$$f(e)=\frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\nu+i\mu)\zeta_{\nu,\nu+1/2+i\mu}(f)\,\mathrm{th}\,\pi\mu d\mu d\nu. \qquad \text{Q.E.D.}$$

Lemma 3.1.2 and Lemma 3.1.4 prove the following proposition.

PROPOSITION 3.1.1. *For any function f in $D_0(G)$, we have*

$$f(e)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\,\mathrm{Re}\,\mathrm{th}\,\pi(\mu+i\nu)d\mu d\nu$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\sum_{\substack{1/2\le p\le|\nu|\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f)d\nu.$$

We shall give a more direct proof of Proposition 3.1.1. First let us prove the following proposition.

PROPOSITION 3.1.2. *If g is a C^∞-function on $[1, \infty)$ with compact support, then*

(3.1.2) $$-\frac{1}{\pi}\int_0^\infty g'\left[x+\frac{\eta^2}{2}\right]T_{2\nu}\left(\sqrt{\frac{x+1+\eta^2/2}{x+1}}\right)d\eta$$

$$=\frac{1}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^\infty\int_{-\infty}^\infty g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{-i\mu t}\mu\,\mathrm{Re}\,\mathrm{th}\,\pi(\mu+i\nu)dtd\mu$$

$$+\frac{1}{\pi\sqrt{2(x+1)}}\sum_{\substack{1/2\leq p\leq|\nu|\\ \nu-p\in\boldsymbol{Z}}}\left(p-\frac{1}{2}\right)\int_{-\infty}^\infty g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{(1/2-p)t}dt.$$

Proof. We may assume $\nu>0$, since $T_\alpha(z)=T_{-\alpha}(z)$ and $\mathrm{Re}\,\mathrm{th}\,\pi(\mu+i\nu)=\mathrm{Re}\,\mathrm{th}\,\pi(\mu-i\nu)$. Put $\sqrt{\frac{x+1+\eta^2/2}{x+1}}=\mathrm{ch}\frac{t}{2}$ and let p_0 be the real number satisfying $-\frac{1}{2}\leqq p_0<\frac{1}{2}$ and $\nu-p_0\in\boldsymbol{Z}$. Then

$$-\frac{1}{\pi}\int_0^\infty g'\left[x+\frac{\eta^2}{2}\right]T_{2\nu}\left(\sqrt{\frac{x+1+\eta^2/2}{x+1}}\right)d\eta$$

$$=-\sqrt{\frac{x+1}{2}}\int_0^\infty g'\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\mathrm{ch}\,\nu t\,\mathrm{ch}\frac{t}{2}dt$$

$$=-\frac{1}{\pi}\frac{1}{2\sqrt{2(x+1)}}\int_0^\infty\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\frac{\mathrm{ch}\,\nu t}{\mathrm{sh}\frac{t}{2}}dt$$

$$=-\frac{1}{\pi}\frac{1}{2\sqrt{2(x+1)}}\int_0^\infty\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\frac{e^{\nu t}}{\mathrm{sh}\frac{t}{2}}dt$$

(3.1.3) $$=-\frac{1}{2\pi\sqrt{2(x+1)}}\int_0^\infty\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\frac{e^{p_0t}}{\mathrm{sh}\frac{t}{2}}dt$$

$$-\frac{1}{2\pi\sqrt{2(x+1)}}\int_0^\infty\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\frac{e^{\nu t}-e^{p_0t}}{\mathrm{sh}\frac{t}{2}}dt.$$

Let us denote the first term by I_1 and the second term by I_2. Then by Lemma 3.1.3,

$$I_1=-\frac{i}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{p_0t}e^{-i\mu t}\mathrm{th}\,\pi\mu dtd\mu$$

Since $\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]$ is a C^∞-function with compact support, its Fourier transform is a rapidly decreasing entire function. We denote it by $F(z)$. Then

$$I_1=-\frac{i}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^\infty F(\mu-ip_0)\mathrm{th}\,\pi\mu d\mu$$

Since $-\frac{1}{2}\leqq p<\frac{1}{2}<\frac{\pi}{2}$ and F is rapidly decreasing,

$$I_1=-\frac{i}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^{\infty}F(\mu)\,\mathrm{th}\,\pi(\mu+ip_0)d\mu.$$

Hence

$$I_1=\frac{1}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{-i\mu t}\mu\,\mathrm{th}\,\pi(\mu+ip_0)dtd\mu$$

By (3.1.3), p_0 can be replaced by $-p_0$, hence we have

$$I_1=\frac{1}{2\pi\sqrt{2(x+1)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{-i\mu t}\mu\,\mathrm{Re}\ \mathrm{th}\,\pi(\mu+ip_0)dtd\mu$$

We have $\mathrm{Re}\ \mathrm{th}\,\pi(\mu+i\nu)=\mathrm{Re}\ \mathrm{th}\,\pi(\mu+ip_0)$, because there exists an integer n such that $p_0+n=\nu$. So I_1 is equal to the first term in the right isde of (3.1.2).

$$I_2=-\frac{1}{\pi\sqrt{2(x+1)}}\int\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\frac{e^{\nu t}-e^{p_0 t}}{e^{t/2}-e^{-t/2}}dt$$

$$=-\frac{1}{\pi\sqrt{2(x+1)}}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}e^{(p-1/2)t}dt$$

$$=\frac{1}{\pi\sqrt{2(x+1)}}\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\int_{-\infty}^{\infty}g\left[(x+1)\,\mathrm{ch}^2\frac{t}{2}-1\right]e^{(1/2-p)t}dt\,.$$

This completes the proof.

Now we give another proof of Proposition 3.1.1. Substituting $F_f[\nu, x]$ for $g[x]$ in (3.1.2) and putting $x=1$ we get

$$-\frac{1}{\pi}\int_0^{\infty}\frac{\partial}{\partial x}F_f\left[\nu, x+\frac{\eta^2}{2}\right]_{x=1}T_{2\nu}\left(\sqrt{1+\frac{\eta^2}{4}}\right)d\eta$$

$$=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_f[\nu, \mathrm{ch}\,t]e^{-i\mu t}\mu\ \mathrm{Re}\ \mathrm{th}\,\pi(\mu+i\nu)dtd\mu$$

$$+\frac{1}{2\pi}\sum_{\substack{1/2\leq p\leq\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\int_{-\infty}^{\infty}F_f[\nu, \mathrm{ch}\,t]e^{(1/2-p)t}dt$$

Note that $\zeta_{\nu,s}(a_t)=\int_{-\infty}^{\infty}F_f[\nu, \mathrm{ch}\,t]e^{(1/2-s)t}dt.$ By Proposition 2.3.2, we get

$$f(e)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\,\mathrm{Re}\ \mathrm{th}\,\pi(\mu+i\nu)dtd\mu$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\sum_{\substack{1/2\le p\le\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f)d\nu$$

This shows Proposition 3.1.1.

3.2. The Plancherel formula for G.

We first note that $\zeta_{\nu,s}(f)=\zeta_{\nu,s}(f_0)$ and $f(e)=f_0(e)$ for $f\in D(G)$. So Proposition 3.1.1 holds for any function in $D(G)$.

We put

$$U_h(f, s)=\int_G U_h(g, s)f(g)dg \text{ and } U^{\pm}(f, l)=\int_G U^{\pm}(g, l)f(g)dg$$

where f is a C^∞-function on G with compact support. Then $U_h(f, s)$ and $U^{\pm}(f, l)$ are, as is well known, of trace class. We have

$$\mathrm{Tr}(U_h(f, s))=\sum_{n=-\infty}\zeta_{n-h,s}(f),\ \mathrm{Tr}(U^+(f, l))=\sum_{m=0}^{\infty}\zeta_{m+l,l}(f) \text{ and}$$

$$\mathrm{Tr}(U^-(f, l))=\sum_{m=0}^{\infty}\zeta_{-m+l,-l}(f).$$

Hence

$$\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\zeta_{\nu,1/2+i\mu}(f)\mu\,\mathrm{Re}\,\mathrm{th}\,\pi(\mu+i\nu)d\mu d\nu$$

$$=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-1/2}^{1/2}\sum_{n=-\infty}^{\infty}\zeta_{n-h,1/2+i\mu}(f)\,\mathrm{Re}\,\mathrm{th}\,\pi(\mu+ih)dhd\mu$$

$$=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{\substack{1/2\le p\le|\nu|\\ |\nu|-p\in Z}}\mathrm{Tr}\left(U_h\left(f, \frac{1}{2}+i\mu\right)\right)\mathrm{Re}\,\mathrm{th}\,\pi(\mu+ih)dhd\mu$$

And

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\sum_{\substack{1/2\le p\le|\nu|\\ |\nu|-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f)d\nu$$

$$=\frac{1}{4\pi^2}\left[\int_0^{\infty}\sum_{\substack{1/2\le p\le\nu\\ \nu-p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f)d\nu+\int_{-\infty}^{\infty}\sum_{\substack{1/2\le p\le-\nu\\ \nu+p\in Z}}\left(p-\frac{1}{2}\right)\zeta_{\nu,p}(f)d\nu\right]$$

$$=\frac{1}{4\pi^2}\left[\int_{1/2}^{\infty}\sum_{n=0}^{\infty}\left(p-\frac{1}{2}\right)\zeta_{n+p,p}(f)dp+\int_{1/2}^{\infty}\sum_{n=0}^{\infty}\left(p-\frac{1}{2}\right)\zeta_{-n-p,p}(f)dp\right]$$

$$=\frac{1}{4\pi^2}\int_{1/2}^{\infty}\left(p-\frac{1}{2}\right)\mathrm{Tr}(U^+(f, p)-U^-(f, -p))dp.$$

Hence we get the following theorem (the Plancherel formula for G).

THEOREM 3.2.1. *If* $f \in D(G)$, *then*

$$f(e)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}\int_{-1/2}^{1/2}\mathrm{Tr}\left(U_h\left(f, \frac{1}{2}+i\mu\right)\right)\mathrm{Re\ th}\,\pi(\mu+ih)dhd\mu$$

$$+\frac{1}{4\pi^2}\int_{1/2}^{\infty}\left(p-\frac{1}{2}\right)\mathrm{Tr}(U^+(f,p)+U^-(f, -p))dp.$$

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