On the Structure of Spinor Groups with Positive Index

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In this paper, we study the structure of the spinor group Spin(V), where V is a quadratic space with positive index over a field K of characteristic $\neq 2$. We will show that Spin(V) is generated by two subgroups which are isomorphic to a certain subspace of V, when dim $V \geqq 3$.

1. Preliminaries.

Assume that K be an arbitrary field of characteristic $\neq 2$. Let V be an n-dimensional non-degenerate quadratic space over the field K with quadratic form Q. We denote also by $Q(x, y)$ the symmetric bilinear form which determines the quadratic form Q, that is, for any $x \in V$, we have $Q(x)=Q(x, x)$. We take an orthogonal base $\{e_1, \cdots, e_n\}$ of V: $Q(e_i, e_j)=\alpha_i \delta_{ij}$, with $\alpha_i \in K^\times$ and $1 \leqq i, j \leqq n$.

In a vector space V, we denote by $<x_1, \cdots, x_r>$ the subspace of V spanned by the vectors $x_1, \cdots, x_r$.

We assume that the index of V is positive. So we may take an orthogonal base $\{e_1, \cdots, e_n\}$ such that $Q(e_1)=1$ and $Q(e_2)=-1$. Put

$$(1) \qquad a=e_1+e_2, \qquad b=2^{-1}(e_1-e_2),$$

then $\{a, b\}$ is a hyperbolic pair in V (i.e., $Q(a)=Q(b)=0$ and $Q(a, b)=1$), and we have a hyperbolic plane $H=<a, b>=<e_1, e_2>$ in V. We denote by U the orthogonal complement of H. Thus, $U=<e_3, \cdots, e_n>$. It is clear that Q induces a non-degenerate quadratic form on U.

Let $C(V)$ be the Clifford algebra of V which contains V canonically, and let $C_0(V)$ be the even Clifford algebra of V. We denote by J the main involution of $C(V)$, that is, $J(x_1 \cdots x_r)=x_r \cdots x_1$, for $x_i \in V$. From the definition, we have, $x^2=Q(x)$ and $xy+yx=2Q(x, y)$, in $C(V)$, where $x, y \in V$. For an invertible element X of $C(V)$, we denote by ϕ_X the inner automorphism of $C(V)$ defined by X, that is, for $Y \in C(V)$, $\phi_X(Y)=XYX^{-1}$. The Clifford group $\Gamma(V)$ of V is defined by

$$\Gamma(V)=\{X \in C(V)^\times : \ \phi_X(V) \subset V\}.$$

For $X \in \Gamma(V)$, the restriction of ϕ_X to V is an automorphism of the quadratic space V which we will denote by φ_X or $\varphi(X)$. Clearly the mapping $\varphi: X \to \varphi(X)$ is a homomorphism of $\Gamma(V)$ into the orthogonal group $O(V)$ of V, which we call the vector representation of $\Gamma(V)$.

We note that, for a non-singular vector x, we have $x \in \Gamma(V)$, and $\varphi_x = -\tau_x$, where τ_x denotes the symmetry defined by x. Thus we have

$$xyx = -Q(x).\,\tau_x y, \tag{2}$$

for a non-singular vector x of V and any vector y of V.

The even Clifford group $\Gamma_0(V)$ is defined by $\Gamma_0(V) = \Gamma(V) \cap C_0(V)$. Then, for an element X of $\Gamma_0(V)$, we have $\varphi_X \in SO(V)$. The kernel of φ in $\Gamma_0(V)$ is the intersection of $\Gamma_0(V)$ and the center of $C(V)$ which is the multiplicative group $K^\times$ of K. Note that the field K is contained in the center of $C(V)$ canonically.

For any $X \in C(V)$, we define the norm $\nu(X)$ by $\nu(X) = X \cdot J(X)$. It is well-known that the mapping ν induces a homomorphism of $\Gamma_0(V)$ into the multiplicative group $K^\times$ of K. The spinor group Spin(V) of V is defined as the kernel of this homomorphism, that is,

$$\mathrm{Spin}(V) = \{X \in \Gamma_0(V) : \nu(X) = 1\}.$$

In our case (ind$(V) > 0$), the homomorphism ν is surjective. (For these facts, see E. Artin [1].)

2. Elementary Subgroups.

From now on, we assume that $n = \dim V \geqq 3$. We have fixed a hyperbolic pair $\{a, b\}$ in V by (1). For any $x \in U = \langle e_3, \cdots, e_n \rangle$, we put

$$E(x) = 1 + ax, \tag{3}$$

$$F(x) = 1 + bx, \tag{4}$$

then we have the following two propositions.

PROPOSITION 1.

(a) *$E(x) \in$ Spin(V) and $E(x)E(y) = E(x+y)$ for $x, y \in U$.*

(b) *For $v \in V$, we have*

$$\varphi_{E(x)}(v) = v + 2Q(x, v)a - 2Q(a, v)Q(x)a - 2Q(a, v)x. \tag{5}$$

(c) *$\mathfrak{E} = \{E(x) : x \in U\}$ is a subgroup of* Spin(V) *which is isomorphic to the additive group of U.*

Proof. Clearly $E(x) \in C_0(V)$. The norm of $E(x)$ is given by

$$\nu(E(x))=(1+ax)(1-ax)=1-axax=1.$$

Calculating the inner automorphism $\varphi_{E(x)}$, we have

$$\varphi_{E(x)}(v)=(1+ax)v(1-ax)=v+2Q(x,v)a-2Q(a,v)Q(x)a-2Q(a,v)x,$$

for any $v\in V$. Thus, $\varphi_{E(x)}(v)\in V$. This shows the first part of (a) and (b). Clearly

$$E(x)E(y)=1+ax+ay+axay=1+a(x+y)=E(x+y).$$

Thus all are proved.

The same holds for $F(x)$, that is,

PROPOSITION 2.
(a) *$F(x)\in\mathrm{Spin}(V)$ and $F(x)F(y)=F(x+y)$ for $x, y\in U$.*
(b) *For $v\in V$, we have*

(6) $$\varphi_{F(x)}(v)=v+2Q(x,v)b-2Q(b,v)Q(x)b-2Q(b,v)x.$$

(c) *$\mathfrak{F}=\{F(x): x\in U\}$ is a subgroup of Spin (V) which is isomorphic to the additive group of U.*

We call these subgroups $\mathfrak{E}$ and $\mathfrak{F}$ the elementary subgroups of Spin (V) defined by the hyperbolic pair $\{a, b\}$. Note that, in Eichler's notation [2], p. 13, we have

$$\varphi(E(x))=E^{a}_{-2x} \text{ and } \varphi(F(x))=E^{b}_{-2x}.$$

LEMMA 1. *We put $\varepsilon=e_1e_2$, then $\varepsilon^2=1$, and*
(a) *$\varepsilon x=x\varepsilon$, for any $x\in U$.*
(b) *$-\varepsilon a=a\varepsilon=a$ and $\varepsilon b=-b\varepsilon=b$.*
(c) *$ab=1-\varepsilon$ and $ba=1-\varepsilon$.*

Proof. These can be proved by direct calculations.

Putting, for $\alpha\in K^{\times}$,

(7) $$P(\alpha)=2^{-1}\{(1+\alpha)+(1-\alpha)\varepsilon\},$$

we have

PROPOSITION 3.
(a) *$P(\alpha)\in\Gamma_0(V)$ and $\nu(P(\alpha))=\alpha$.*
(b) *$P(\alpha)P(\beta)=P(\alpha\beta)$.*

Proof. For $\alpha\in K^{\times}$, $a+b$ and $a+\alpha b$ are non-singular vectors, and we have $(a+b)(a+\alpha b)=2P(\alpha)$. Thus $P(\alpha)\in\Gamma_0(V)$. The other statements can be proved by direct calculations.

We put $\mathfrak{P}=\{P(\alpha): \alpha\in K^{\times}\}$, then $\mathfrak{P}$ is a subgroup of $\Gamma_0(V)$ which is isomorphic to $K^\times$. Prop. 3. shows that $\Gamma_0(V)$ is a semi-direct product of Spin (V) and $\mathfrak{P}$ in which Spin (V) is normal.

From Lemma 1, it follows that $P(\alpha)a=\alpha a$, $aP(\alpha)=a$, $P(\alpha)b=b$, $bP(\alpha)=\alpha b$, and $P(\alpha)x=xP(\alpha)$ for any $x\in U$. So we have $\varphi_{P(\alpha)}(a)=\alpha a$, $\varphi_{P(\alpha)}(b)=\alpha^{-1}b$, and $\varphi_{P(\alpha)}(x)=x$ for any $x\in U$. Also we have

$$P(\alpha)E(x)P(\alpha)^{-1}=E(\alpha x), \tag{8}$$
$$P(\alpha)F(x)P(\alpha)^{-1}=F(\alpha^{-1}x). \tag{9}$$

That is, $\mathfrak{P}$ normalises $\mathfrak{E}$ and $\mathfrak{F}$, respectively.

Now, for $\alpha\in K^\times$, we put

$$\Lambda(\alpha)=\alpha^{-1}P(\alpha^2). \tag{10}$$

Then we have

PROPOSITION 4.

(a) *$\Lambda(\alpha)\in$ Spin (V) and $\Lambda(\alpha)\Lambda(\beta)=\Lambda(\alpha\beta)$.*

(b) *$\mathfrak{L}=\{\Lambda(\alpha):\alpha\in K^\times\}$ is a subgroup of* Spin (V) *which is isomorphic to $K^\times$. Especially $\Lambda(-1)=-1$.*

Proof. These are clear from the definition of $\Lambda(\alpha)$.

Remark. If dim $V=2$, it is easy to see that Spin $(V)=\mathfrak{L}$.

3. The Structure of spinor group.

Let U be the orthogonal complement of $H=\langle a, b\rangle=\langle e_1, e_2\rangle$. We denote by $U^\times$ the set of all non-singular vectors in U. For $x\in U^\times$, we put $\xi=(2Q(x))^{-1}$, and put

$$W(x)=E(x)F(\xi x)E(x). \tag{11}$$

From (8) and (9), we have

$$P(\alpha)W(x)P(\alpha)^{-1}=W(\alpha x). \tag{12}$$

LEMMA 2. *For $x\in U^\times$, we have*

$$W(x)=(a+\xi b)x, \tag{13}$$

where $\xi=(2Q(x))^{-1}$.

Proof. In the algebra $C(V)$, we have

$$W(x)=(1+ax)(1+\xi bx)(1+ax)$$
$$=1+2ax+\xi bx-\xi Q(x)(ba+ab)-\xi Q(x)abax$$

$$=ax+\xi bx=(a+\xi b)x.$$

Note that $aba=2a$. q. e. d.

From Lemma 2, it follows $\varphi_{W(x)}=\tau_{a+\xi b}\cdot\tau_x$, for $x\in U^{\times}$. This shows that $\varphi_{W(x)}(U)=U$ and $\varphi_{W(x)}(H)=H$. The restriction of $\varphi_{W(x)}$ to U is the symmetry in U defined by the non-singular vector x which is nothing but the restriction of the symmetry τ_x in V to the subspace U.

PROPOSITION 5. *For $x, y\in U^{\times}$, we have*

$$W(x)W(y)=-Q(x)^{-1}P(Q(x)/Q(y))xy. \tag{14}$$

Proof: From Lemma 2, we have

$$W(x)W(y)=-(a+\xi b)(a+\eta b)xy$$

where $\xi=(2Q(x))^{-1}$ and $\eta=(2Q(y))^{-1}$. It is easy to show that $(a+\xi b)(a+\eta b)=2\xi P(\eta/\xi)$. q. e. d.

COROLLARY 1. *For $x\in U^{\times}$ and $\lambda\in K^{\times}$, we have*

$$W(x)^{-1}=W(-x)=-W(x). \tag{15}$$

$$W(\lambda x)W(x)^{-1}=A(\lambda) \text{ and } W(x)^{-1}W(\lambda x)=A(\lambda^{-1}). \tag{16}$$

Proof. These can be deduced from (14) and the definition of $W(x)$.

COROLLARY 2. *If we denote by G the subgroup of* Spin(V) *generated by $\mathfrak{E}$ and $\mathfrak{F}$, then $\mathfrak{L}$ is contained in G.*

Remark. Interchanging E and F, we put $W'(x)=F(x)E(\xi x)F(x)$, then we have $W'(x)W(x)^{-1}=A(\xi)$, where $\xi=(2Q(x))^{-1}$.

PROPOSITION 6. *For $x\in U^{\times}$, we have*

$$W(x)E(y)W(x)^{-1}=F(-\xi\cdot\tau_x y), \tag{17}$$

$$W(x)F(y)W(x)^{-1}=E(-\xi^{-1}\cdot\tau_x y), \tag{18}$$

where $\xi=(2Q(x))^{-1}$ and τ_x is the symmetry in V defined by x.

Proof. We put $c=a+\xi b$, then

$$W(x)E(y)W(x)^{-1}=-cx(1+ay)cx=-(cxcx+cxaycx)$$
$$=Q(c)Q(x)+cac\cdot xyx.$$

As $Q(c)=2\xi=Q(x)^{-1}$, we have, from (2),

$$W(x)E(y)W(x)^{-1}=1+\tau_c a\cdot\tau_x y=1-\xi b\cdot\tau_x y=F(-\xi\cdot\tau_x y).$$

Also (18) can be proved in the similar way. q. e. d.

DEFINITION. We denote by $\mathfrak{W}$ the subgroup of Spin(V) generated by $W(x)$ with $x\in U^{\times}$ and by $\mathfrak{U}$ the subgroup of Spin(V) generated by $W(x)W(y)^{-1}$ with

$x, y \in U^{\times}$. It follows from (16) that $\mathfrak{L}$ is contained in $\mathfrak{U}$.

It is already shown that $\varphi_{W(x)}(U)=U$. So, for any $X\in\mathfrak{W}$, the restriction ω_X of φ_X to U is an automorphism of the quadratic space U. Thus we have a homomorphism ω of $\mathfrak{W}$ into $O(U)$. Clearly ω is surjective, and the restriction of ω to $\mathfrak{U}$ is a homomorphism of $\mathfrak{U}$ onto $SO(U)$.

PROPOSITION 7. *The kernel of the homomorphism ω is the subgroup $\mathfrak{L}$, and $\mathfrak{L}$ is central in $\mathfrak{U}$.*

Proof. From Cor. 1 to Prop. 5, we have

$$W(x)A(\lambda)W(x)^{-1}=A(\lambda^{-1}).$$

Thus $\mathfrak{L}$ is central in $\mathfrak{U}$, but is not central in $\mathfrak{W}$. As $\mathfrak{U}$ has index 2 in $\mathfrak{W}$, it suffices to show that the kernel of ω in $\mathfrak{U}$ is $\mathfrak{L}$. Assume that $X\in\mathfrak{U}$ be in the kernel. We write

$$X=W(x_1)W(x_2)\cdots\cdots W(x_{2h}),$$

with $x_i\in U^{\times}$. By Prop. 5, we can write $X=\alpha P(\beta)x_1x_2\cdots x_{2h}$, with certain scalars α and β. As $\alpha P(\beta)$ commutes with every vector of U, the element $x_1\cdots x_{2h}$ of $C_0(U)$ should be central in the algebra $C(U)$ of U. That is, $x_1\cdots x_{2h}$ is a scalar. Thus $X=\alpha' P(\beta)$. From $1=\nu(X)=\alpha'^2\cdot\beta$, it follows that $X\in\mathfrak{L}$. q. e. d.

The special orthogonal group $SO(H)$ of the hyperbolic plane $H=\langle a, b\rangle$ is isomorphic to the multiplicative group $K^{\times}$ of K in the following way. Let $\sigma\in SO(H)$, then $\sigma(a)$ is a scalar multiple of a. We denote this scalar by π_σ. Then π is an isomorphism of $SO(H)$ onto $K^{\times}$.

For $X\in\mathfrak{U}$, the restriction π_X of φ_X to the hyperbolic plane H induces an element of $SO(H)$. We also denote by π_X the corresponding scalar of this restriction. Thus we have a homomorphism π of $\mathfrak{U}$ into $K^{\times}$. From (14), we have $\pi(W(x)W(y))=Q(x)/Q(y)$. Thus the image of π is a subgroup of $K^{\times}$ generated by $Q(x)/Q(y)$ with $x, y\in\mathfrak{U}^{\times}$. Also it is clear that the kernel of π is isomorphic to the spinor group Spin (U) of U.

4. A generator system for the spinor group Spin (V).

We will show that Spin (V) is generated by the two elementary subgroups $\mathfrak{E}$ and $\mathfrak{F}$.

LEMMA 3. *For $T\in$ Spin (V), if $\varphi_T(a)=\beta^{-1}a$ and $\varphi_T(b)=\beta b$, with $\beta\in K^{\times}$, then T is contained in $\mathfrak{U}$.*

Proof. We put $\tau=\varphi_T$. As $\tau(H)=H$, we have $\tau(U)=U$. Thus τ induces an element of $SO(U)$ which we will denote by τ'. By the theorem of Cartan-Dieudonné, there exist even number of non-singular vectors $z_i (1\leqq i\leqq 2h)$ in U

such that $\tau'=\tau_{z_1}\cdots\cdots\tau_{z_{2h}}$, where τ_z means the symmetry in U defined by z, in this case. The spinor norm of τ is a class of $K^\times/(K^\times)^2$ which contains the scalar $\beta^{-1}Q(z_1)\cdots Q(z_{2h})$. As $\tau=\varphi_T$ with $T\in\mathrm{Spin}(V)$, this scalar is equal to a square μ^2 with $\mu\in K^\times$. We put

$$T_1=W(z_1)\cdots W(z_{2h}).A(\lambda).$$

Clearly we have $\omega_{T_1}=\tau'=\omega_T$. Taking the scalar λ suitably, we can show that $\pi(T_1)=\pi(T)$. More exactly, it suffices to put

$$\lambda=\mu(Q(z_1)Q(z_3)\cdots Q(z_{2h-1}))^{-1}.$$

Thus $\varphi_T=\varphi_{T_1}$, that is, T and T_1 differ only by the factor ±1. As $-1=A(-1)\in\mathfrak{U}$, $T=\pm T_1$ is contained in $\mathfrak{U}$. q. e. d.

For an element $T\in\mathrm{Spin}(V)$, we put $\beta(T)=Q(a,\varphi_T(b))$, and we define a subset $\mathfrak{X}$ of $\mathrm{Spin}(V)$ in the following way;

$$\mathfrak{X}=\{T\in\mathrm{Spin}(V):\beta(T)\neq0\}.$$

PROPOSITION 8. *Notation being as above, we have* $\mathfrak{X}=\mathfrak{E}\mathfrak{F}\mathfrak{U}$, *and for* $T\in\mathfrak{X}$, *the decomposition* $T=E(x)F(y)X$, *with* $X\in\mathfrak{U}$, *is unique.*

Proof. It is clear that $T=E(x)F(y)X$ is contained in $\mathfrak{X}$, because

$$\varphi_T(b)=\varphi_{E(x)}(\pi_X{}^{-1}\cdot b)=\pi_X{}^{-1}(b-2Q(x)a-2x),$$

that is, $\beta(T)=\pi_X{}^{-1}$. Now we prove the uniqueness of the decomposition. Put $E(x)F(y)X=E(x_1)F(y_1)X_1$, with $X, X_1\in\mathfrak{U}$, then we have

$$E(x-x_1)F(y)=F(y_1)X_1X^{-1}=X_1X^{-1}F(y_2),$$

where y_2 is a vector in U determined by (17) and (18). That is, $E(x-x_1)F(y-y_2)=X_1X^{-1}$. Consider the operation on b of the vector representation of both sides of the above formula, then we have

$$\varphi_{E(x_3)F(y_3)}(b)=b-2Q(x_3)a-2x_3,\ \text{and}\ \varphi_Y(b)=\pi_Y{}^{-1}\cdot b,$$

where $x_3=x-x_1$, $y_3=y-y_2$ and $Y=X_1X^{-1}$. Thus we have $x_3=0$, that is, $x=x_1$. As $\mathfrak{F}\cap\mathfrak{U}=\{1\}$, the equality $F(y)X=F(y_1)X_1$ means that $y=y_1$ and $X=X_1$. Note that every element $F(y)\neq1$ does not leave invariant the subspace U. (cf. Prop. 2).

Finally, assume that $T\in\mathfrak{X}$, then we can write

$$\varphi_T(b)=\alpha a+\beta b+x,$$

where $\beta=\beta(T)\neq0$, and $x\in U$. As $Q(\varphi_T(b))=0$, we have $2\alpha\beta+Q(x)=0$. We put

$$T'=E((2\beta)^{-1}x)T,$$

then we have $\varphi_{T'}(b)=\beta b$. Writing $\varphi_{T'}(a)=\gamma a+\delta b+y$, we have $\gamma=\beta^{-1}$ and $2\gamma\delta+Q(y)=0$. Put

$$T''=F(2^{-1}\beta y)T',$$

then we have $\varphi_{T''}(a)=\beta^{-1}a$ and $\varphi_{T''}(b)=\beta b$. From Lemma 3, it follows that $T''\in\mathfrak{U}$. That is,

$$T=E(-(2\beta)^{-1}x)F(-2^{-1}\beta y)T'',$$

with $T''\in\mathfrak{U}$. q. e. d.

PROPOSITION 9.

$$\mathrm{Spin}(V)=\bigcup_{x\in U}F(x)\mathfrak{X}F(x)^{-1}.$$

Proof. For $T\in\mathrm{Spin}(V)$, if $\beta(T)=0$, then we can write $\varphi_T(b)=\alpha a+y$. Putting $T^*=F(x)TF(-x)$, we will show that $T^*\in\mathfrak{X}$ for some $x\in U$. As $\varphi_{T^*}(b)=\varphi_{F(x)}(\alpha a+y)$, we have $\beta(T^*)=2(-\alpha Q(x)+Q(x,y))$. Changing x by λx, if necessary, we can show that there exists a vector λx such that $T^*\in\mathfrak{X}$. Note that $|K|\geqq 3$ from the assumption. q. e. d.

These proofs are analogy of Eichler's one [2].

As $\mathfrak{U}$ normalises the subgroup $\mathfrak{F}$, Prop. 8 and 9 shows that

$$\mathrm{Spin}(V)=\mathfrak{F}\mathfrak{E}\mathfrak{F}\mathfrak{U}. \tag{19}$$

Remark 1. From (19), we can explain the cellular decomposition of the quadratic surface in the projective space $P(V)$ of V defined by the equation $Q(x)=0$.

Remark 2. If $\mathrm{ind}(U)=0$, then the subgroups $\mathfrak{F}\mathfrak{U}$ and $\mathfrak{W}$ make the B-N pair for the group $\mathrm{Spin}(V)$ in the sense of Tits. But this is already known, because $\mathfrak{F}\mathfrak{U}$ is a minimal K-parabolic subgroup of $\mathrm{Spin}(V)$ [3]. If $\mathrm{ind}(U)>0$, we could not seek any B-N pair in $\mathrm{Spin}(V)$.

References

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