Functional Studies of Automata (II)

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In the preceding paper [1], we have introduced several notions related to the "functional completeness."

We shall now study the relationship between these notions.

1. (*k*)-completeness and ∼-completeness

We consider here the whole set $\mathcal{E}(k)^*$ of elementary functions defined over $(k)^*$.

DEFINITION 1.

$$DK(a, b)=\{F\in \mathcal{E}(k)^*;\ \exists f\in K(a, b),\ D^{-1}F=f^*\}$$

where

$$K(a, b)=\{f\in \Omega(k);\ f(a, \cdots, a)=f(b, \cdots, b)\}$$

Remark.

$$K(a, b)\circ K(a, b)\subseteq K(a, b)$$

LEMMA 1. Let $\mathcal{F}$ be a subset of $\mathcal{E}(k)^*$.

1) If $\mathcal{F}$ is ∼-complete, then $\mathcal{F}$ is strongly (k)-complete.
2) $DK(a, b)$ is strongly (k)-complete while it is *not* ∼-complete.

Proof. 1) is evident.

2) Let S be the spectrum of $\mathcal{F}$.

Obviously,

$$S=(\phi, K(a, b), \phi, \phi, \cdots)$$

and

$$\bigcup_{i=0}^{\infty} (\tilde{S})_i=K(a, b)$$

as it is easily verified. Therefore $DK(a, b)$ is not ∼-complete.

Now let us consider an arbitrary mapping h in $\Omega(k)$ having p variables. We define a mapping g as follows.

$$g(x_1, \cdots, x_p, y, z)=\begin{cases} h(x_1, \cdots, x_p) & \text{if } y\neq z \\ 0 & \text{if } y=z \end{cases}$$

Since $g \in K(a, b)$,

$$D\cdot g^* \in DK(a, b)$$

Let O, E be the mappings in $\mathcal{E}(k)^*$ defined as follows.
For any $u \in (k)^*$ and any $t \geqq 0$,

$$O(u)(t)=0 \text{ and } E(u)(t)=1.$$

Obviously,

$$O,\ E \in DK(a, b)$$

and therefore

$$F=D\cdot g^*(P_1^p \times \cdots \times P_p^p \times O\cdot P_1^p \times E\cdot P_1^p) \in \overline{DK(a, b)},$$

Besides,

$$\begin{aligned} &D^{-1}\cdot F(u_1, \cdots, u_p)(t) \\ &=g(u_1(t), \cdots, u_p(t), 0, 1) \\ &=h(u_1(t), \cdots, u_p(t)) \end{aligned}$$

Thus we have

$$D^{-1}\cdot F=h^*$$

Since h is arbitrary, $DK(a, b)$ is strongly (k)-complete.

DEFINITION 2.

1) $C(a, b)=\{f \in \Omega(k);\ \forall x_1, \cdots, x_n \in \{a, b\},$
$f(x_1, \cdots, x_n)=f(a, a, \cdots, a)\}$

2) $DC(a, b)=\{D\cdot f^*;\ f \in C(a, b)\}$

Remark. If $F \in DC(a, b)$, then

$$F(u_1, \cdots, u_n)(0)=0$$

for any $u_i \in (k)^*$.

LEMMA 2. Let $\mathcal{F}$ be a subset of $\mathcal{E}(k)^*$.

1) If $\mathcal{F}$ is strongly (k)-complete, it is (k)-complete.

2) $DC(0, 1)$ is (k)-complete while it is *not* strongly (k)-complete, provided that $k \geqq 3$.

Proof. 1) is evident.

2) Suppose that

$$F \in \overline{DC(0, 1)}$$

and that

$$D^{-N}\cdot F=f^* \qquad \text{.....................(1)}$$

for some $f\in\Omega(k)$ and some $N\geqq 1$.

A. F can be written in the following form.

$$F=D\cdot f_0^*(G_1\times\cdots\times G_p) \qquad \text{.....................(2)}$$

where

$$f_0\in C(0,1),\ G_1,\ \cdots,\ \ G_p\in\overline{DC(0,1)}\cup\wp$$

Therefore

$$D^{-n}\cdot F=f_0^*(D^{-n+1}\cdot G_1\times\cdots\times D^{-n+1}\cdot G_p) \qquad \text{.....................(3)}$$

Remark. $D^{-n}\cdot f^*=f^*D^{-n}$ for $n\geqq 0$.

$R_p^{-1}\cdot f^*=f^*R_p^{-1}$ for $p\geqq 1$.

$D^{-n}\cdot D^m=D^{-(n-m)}$ if n, $(n-m)\geqq 0$.

$R_p^{-1}\cdot R_q^{-1}=R_{pq}^{-1}$ for p, $q\geqq 1$.

Now let us consider the sequence v defined as follows

$$v=f^*(u_1,\ \cdots,\ u_n)$$

Obviously,

$$v=D^{-N}\cdot F(u_1,\ \cdots,\ u_n)$$
$$=f_0^*(D^{-N+1}\cdot G_1\times\cdots\times D^{-N+1}\cdot G_p)(u_1,\ \cdots,\ u_n) \qquad \text{.....................(4)}$$

If $G_i=P_j^n$, then we can substitute $D^{-N+1}G_i$ in (4) by $D^{-N+1}u_j$.

If $G_i\in\overline{DC(0,1)}$, then G_i can be written in such a form as (2) and $D^{-N+1}G_i$ can therefore be substituted by

$$Dg_i^*(G_1'\times\cdots\times G_q') \qquad \text{.....................(5)}$$

or

$$g_i^*(D^{-N+2}\cdot G_1'\times\cdots\times D^{-N+2}\cdot G_q') \qquad \text{.....................(6)}$$

for some g_i in $C(0,1)$ and some $G_1',\ \cdots,\ G_q'$ in $\overline{DC(0,1)}\cup\wp$, according to

$$N=1 \qquad \text{.....................(5')}$$

or

$$N\geqq 2. \qquad \text{.....................(6')}$$

In repeating such substitution, we shall obtain the following representation of v:

$$v=h^*(D^{-N}u_1,\ \cdots,\ D^{-1}u_n,\ u_1,\ \cdots,\ u_n,$$
$$DH_1(u_1,\ \cdots,\ u_n),\ \cdots,\ DH_s(u_1,\ \cdots,\ u_n))$$

where

$$h \in \overline{C(0,1)},$$

$$H_1, \cdots, H_s \in \overline{DC(0,1)} \cup \wp.$$

Evidently,

$$v(0) = f(u_1(0), \cdots, u_n(0))$$
$$= h(u_1(N), \cdots, u_n(1), u_1(0), \cdots, n_n(0), 0, \cdots, 0)$$

since $D \cdot u(0) = 0$ by the definition of D.

The value

$$v(0) = f(u_1(0), \cdots, u_n(0))$$

depends only on $u_i(0)$'s. So the value of the function h is independent of its first Nn variables $u_1(N), \cdots, u_n(1)$ which can be considered as free variables independent of $u_i(0)$'s.

This independence has an important consequence: in the precess of substitution explained before, we can replace $D^{-s}P_j^N$ by any function without affecting the value $v(0)$.

Now suppose that all functions of the form

$$D^{-s}P_j^N, \quad s \neq 0$$

have been replaced by

$$D^{-s}(D \cdot h_0 * \cdot P_j^N)^s$$

where h_0 is an arbitrarily fixed function with one variable in $C(0, 1)$. Then we shall obtain the following relation:

$$v(0) = [h'*(u_1, \cdots, u_n, DH_1'(u_1, \cdots, n_n), \cdots,$$
$$DH_r'(u_1 \cdots, u_n))](0)$$

where

$$h' \in C(0, 1)^N,$$

$$H_1', \cdots, H_r' \in \overline{DC(0,1)} \cup \wp$$

Remark. $\quad h' \in C(0, 1)^N \subseteq C(0, 1)$

Suppose that

$$x_1, \cdots, x_n \in \{0, 1\}.$$

Then

$$f(x_1, \cdots, x_n) = h'(x_1, \cdots, x_n, 0, \cdots, 0)$$
$$= h'(0, \cdots, 0, 0, \cdots, 0)$$

since $h' \in C(0, 1)$. Thus we can conclude that any function f represented strongly by a function F in $\overline{DC(0,1)}$ belongs to $C(0, 1)$.

$DC(0, 1)$ is therefore *not* strongly (k)-complete.

B. *The* (k)*-completeness of* $DC(0,1)$ *can be shown in the following manner, provided that* $k \geqq 3$.

Let f be an arbitrary function having n variables. We consider functions h, g defined as follows.

$$h(x_1, \cdots, x_n, y) = \begin{cases} f(x_1, \cdots, x_n) & \text{if } y=2 \\ 0 & \text{otherwise} \end{cases}$$

$$g(x) = 2 \quad \text{for any } x.$$

Evidently,

$$h,\ g \in C(0,1), \qquad Dh^*,\ Dg^* \in DC(0,1).$$

Now for any $u_1, \cdots, u_n \in (k)^*$,

$$\begin{aligned} & D^{-2}Dh^*(P_1^n \times \cdots \times P_n^n \times Dg^* P_1^n) \\ &= h^*(D^{-1}P_1^n \times \cdots \times D^{-1}P_n^n \times g^* P_1^n) \\ &= f^* D^{-1}. \end{aligned}$$

Therefore f is represented by

$$h^*(P_1^n \times \cdots \times P_n^n \times Dg^* P_1^n) \in \overline{DC(0,1)}$$

with index $(1, 1, 1)$.

Since f is arbitrary, $DC(0,1)$ is (k)-complete.

LEMMA 3. Le $\mathscr{F}$ be a subset of $\mathscr{E}(k)^*$.

1) If $\mathscr{F}$ is (k)-complete, then it is weakly (k)-complete.

2) Let S be the subset of $\Omega(k)^*$ defined as follows:

$$S = \{R_2 \cdot f^* \cdot R_2^{-1};\ f \in \Omega(k)\}$$

S is weakly (k)-complete, while it is *not* (k)-complete. The proof is immediate.

THEOREM 1. Let $\mathscr{F}$ be a subset of $\mathscr{E}(k)^*$.

1) $\mathscr{F}$ is $\sim$-complete

$\Longrightarrow$ $\mathscr{F}$ is strongly (k)-complete

$\Longrightarrow$ $\mathscr{F}$ is (k)-complete

$\Longrightarrow$ $\mathscr{F}$ is weakly (k)-complete

2) Provided that $k \geqq 3$, $\mathscr{F}$ is weakly (k)-completete

$\not\Longrightarrow$ $\mathscr{F}$ is (k)-complete

$\not\Longrightarrow$ $\mathscr{F}$ is strongly (k)-complete

$\not\Longrightarrow$ $\mathscr{F}$ is $\sim$-complete.

2. (*k*)-completeness and (*k*)-universality

Here we consider the set $\mathcal{A}(k)^*$ of all admissible functions defined over (k).*

THEOREM 2.

1) If a subset $\mathcal{F}$ of $\mathcal{A}(k)^*$ is weakly (k)-complete, then it is (k)-universal.

2) There exists a subset $\mathcal{F}$ of $\mathcal{A}(k)^*$ which is *not* weakly (k)-complete although it is (k)-universal.

Proof. 1) is evident since

$$\bar{\mathcal{F}} \subseteq [\mathcal{F}]$$

(see the proposition 1 in [1], page 28.)

2) Let F be a mapping defined as follows.

$$F(u, v, w)(t) = \mathrm{Max}\{u(t-1), v(t-1)\} \oplus 1$$

if both of the following conditions are satisfied.

a) $t \geqq 1$

b) $w(i) = F(u, v, w)(i)$ for all i less than t.

Otherwise,

$$F(u, v, w)(t) = 0.$$

Evidently,

$$F \in \mathcal{A}(k)^* \quad \text{although} \quad F \notin \mathcal{E}(k)^*$$

Now let us consider a mapping G:

$$G = [F]_{2, h}$$

where 2 indicates that G has two variables and h is the conector defined as follows (see the figure bellow.)

i	j	$h(i, j)$
0	0	3
1	1	1
1	2	2
1	3	3

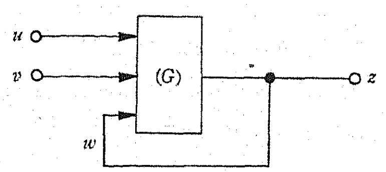

Figure 1.

We can verify easily that

$$G(u, v)(t)=\mathrm{Webb}\,(u(t-1), v(t-1))$$

for any $t \geqq 1$, where

$$\mathrm{Webb}\,(m, n)=\mathrm{Max}\,\{m, n\} \oplus 1.$$

As it is well known, the mapping Webb is a "Sheffer function", i.e.,

$$\overline{\{\mathrm{Webb}\}}=\Omega(k).$$

Theorefore $\{G, D\}$ is (k)-complete and $\{F, D\}$ is (k)-universal.

Now we shall show that $\{F, D\}$ is *not* weakly (k)-complete.

We consider a mapping H with one variable obtained from $\{F, D\}$ by loop-free composition. H can therefore be written in the form:

$$H=D^s \cdot F(G_1 \times G_2 \times G_3) \qquad \text{......................(1)}$$

where s is an integer and G_1, G_2, G_3 are mappings in

$$\overline{\{F, D\}} \cup \mathfrak{F}$$

We assume that the expression (1) is in a sens minimal: more precisely, we assume that

$$G_1,\ G_2,\ G_3 \neq F(G_1 \times G_2 \times G_3)$$

Remark. If $G_1=F(G_1 \times G_2 \times G_3)$, then $H=D^s G_1$.

In consequence, there exists a sequence u such that

$$F(G_1 \times G_2 \times G_3)(u)(t_0) \neq G_3(u)(t_0)$$

for some t_0. Then by the definition of F,

$$D^s \cdot (G_1 \times G_2 \times G_3)(u)(t)=0$$

for all $t > t_0+s$.

It is now obvious that H can *not* represent weakly any function f with one variable in $\Omega(k)$ except the constant function whose value is always equal to 0.

This completes the proof of the theorem.

3. (k)-univeasality and universality

The notion of weak representability is too broad to discuss the construction of automata. We shall therefore introduce another variant of representability.

DEFINITION 3. A function G is said to be *synchronously representable* (or *s-representable*) by a function F if

$$R_p^{-1} D^{-(pd+c)} F = G R_p^{-1} D^{-c}$$

for some non-negative integer c and some positive integers p and d.

Remark. We assume here that the delay is an integer multiple of p.

DEFINITION 4. Let $\mathfrak{F}$ be a subset of $\mathcal{A}(k)^*$.

1) $\mathfrak{F}$ is said to be *strongly (k)-universal* if for every function h in $\Omega(k)$ there exists a function H in $[\mathfrak{F}]$ which represents synchronously h^*.

2) $\mathfrak{F}$ is said to be *strongly universal* if for every automaton A there exists a triple of integers of the form

$$(p, pd, c)$$

which satisfies the following condition.

(*): "Let G be the output sequence function of A. Let r be the number of output of A. Then

$$P_1^r G, \cdots, P_r^r G$$

are s-representable by functions

$$F_1, \cdots, F_r$$

in $[\mathfrak{F}]$ each of which has the common index (p, pd, c)."

LEMMA 4. Let $\mathfrak{F}$ be a subset of $\mathcal{A}(k)^*$.

1) $\mathfrak{F}$ is (k)-complete

$$\Longrightarrow \mathfrak{F} \text{ is strongly } (k)\text{-universal}$$

$$\Longrightarrow \mathfrak{F} \text{ is } (k)\text{-universal.}$$

2) $\mathfrak{F}$ is strongly universal

$$\Longrightarrow \mathfrak{F} \text{ is universal}$$

$$\Longrightarrow \mathfrak{F} \text{ is } (k)\text{-universal.}$$

The proof is immediate.

Remark. Any function f in $\Omega(k)$ can be taken as the output sequence function of a one-state automaton with one output.

DEFINITION 5.

$$\left.\begin{array}{l} E(u)(t)=1 \\ O(u)(t)=0 \end{array}\right\} \text{for any } u \in (k)^* \text{ and any } t \geqq 0$$

$$T(u)(0)=1, \ T(u)(t)=0 \quad \text{for } t \neq 0.$$

E, O, T are functions with one variable whose values are independent of the argument u. In what follows we shall identify $E \cdot P_i^N$, $O \cdot P_i^N$, $T \cdot P_i^N$ with E, O, T, respectively.

THEOREM 3. Let $\mathfrak{F}$ be a subset of $\mathcal{A}(k)^*$ containing the set:

$$B=\{E, O\} \cup \{D^n \cdot T;\ n \geqq 0\}$$

Then the following conditions are equivalent.

1) $\mathcal{F}$ is strongly universal.

2) $\mathcal{F}$ is strongly (k)-universal.

Proof. "1)$\Rightarrow$2)" is obvious (see the remark just before the definition 5.) The proof of thd converse is rather complicated.

A. *Let*

$$A=((k)^m, (k)^n, (k)^r, (0), f, g)$$

be an arbitrary (k)*-automaton with n-input and r-output.*

Let G be the output sequence function of A.
We denote:

$$A_i=P_i^r G,\ f_i=P_i^m f,\ g_j=P_i^r g.$$

B. *We define a function h with* $(2m+n+r+1)$ *variables as follows.*

$$h(x_1, \cdots, x_m, y_1, \cdots, y_n, a_1, \cdots, a_m, b_1, \cdots, b_r, c)$$
$$=\begin{cases} f_i(x_1, \cdots, x_m, y_1, \cdots, y_n) \\ \quad \text{if } a_1=\cdots=a_1=1,\ a_{i+1}=\cdots a_m=0 \text{ and} \\ \quad\quad b_1=\cdots\cdots=b_n=c=0, \\ g_j(x_1, \cdots, x_m, y_1, \cdots, y_n) \\ \quad \text{if } a_1=\cdots\cdots=a_m=c=0,\ b_1=\cdots=b_j=1 \\ \quad\quad \text{and } b_{j+1}=\cdots=b_r=0, \\ 0 \quad\quad \text{otherwise.} \end{cases}$$

Remark. If $c=1$, then $h=0$.

C. *Suppose that a subset* $\mathcal{F}$ *of* $\mathcal{A}(k)^*$ *is strongly* (k)*-universal.*

There exists then a function H in $[\mathcal{F}]$ which represents synchronously h^*, i.e.,

$$R_p^{-1} D^{-(pd+c)} H = h^* R_p^{-1} D^{-c} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(1)$$

for some p, d and c. (Note that p, $d \geqq 1$, $c \geqq 0$).
We construct from H the following functions.

$$F_i=H(P_1^{m+n} \times \cdots \times P_{m+n}^{m+n} \times \overbrace{E \times E \times \cdots \times E}^{i} \times O \times O \times \cdots \times O \times D^c T)$$
$$H_j=H(P_1^{m+n} \times \cdots \times P_{m+n}^{m+n} \times \underbrace{O \times \cdots \times O}_{m} \times \underbrace{E \times \cdots \times E}_{j} \times O \times \cdots \times O)$$

Obviously,

$$F_i,\ H_j \in \mathfrak{F} \subseteq [\mathfrak{F}]$$

and

$$R_p^{-1}D^{-(pd+c)}H_j = g_j{}^*R_p^{-1}D^{-c} \qquad \cdots\cdots(2)$$

Now let us consider the value of F_i.

$$\begin{aligned}&[R_p^{-1}D^{-(pd+c)}F_i(S_1, \cdots, S_m, U_1, \cdots, U_n)](t)\\ =&[h^*R_p^{-1}D^{-c}(S_1, \cdots, S_m, U_1, \cdots, U_n, 1, \cdots, 1, 0, \cdots, 0, D^cT)](t)\\ =&\begin{cases}0 & \text{if } R_p^{-1}D^{-c}D^cT(t)=1,\\ [f^*R_p^{-1}D^{-c}(S_1, \cdots, S_m, U_1, \cdots, U_n)](t), & \text{otherwise.}\end{cases}\end{aligned} \qquad \cdots\cdots(3)$$

Remark 1. $R_p^{-1}D^{-c}D^cT(t)=1 \Longleftrightarrow t=0.$

2. $$D^{-1}R_p^{-1}D^{-c}F_i = D^{-1}f^*R_p^{-1}D^{-c} \qquad \cdots\cdots(4)$$

D. *We compose now function A_i' with n variables in the following manner.*

$$A_i' = [H_j, F_1, \cdots, F_m]_{n,q}$$

where q is the connector defined as follows.

$$q(i, j) = \begin{cases} n+1 & \text{for } i=j=0,\\ n+j+1 & \text{for } j \leqq m\\ j-m & \text{for } j>m\end{cases}$$

Therefore

$$A_j'(U_1, \cdots, U_n) = H_j(S_1, \cdots, S_m, U_1, \cdots, U_n) \qquad \cdots\cdots(5)$$

where

$$S_i = F_i(S_1, \cdots, S_m, U_1, \cdots, U_n) \qquad \cdots\cdots(6)$$

In the following we shall show that A_j' represents synchronously A_j with index (p', d', c'), where

$$p'=pd,\ d'=1,\ c'=pd+c.$$

E. *The goal of this paragraph is the following equality.*

$$R_{pd}^{-1}D^{-(2pd+c)}A_j' = A_jR_{pd}^{-1}D^{-(pd+c)} \qquad \cdots\cdots(7)$$

Let $U_1, \cdots, U_n$ be arbitrary (k)-sequences.
We denote:

$$u_i = R_{pd}^{-1}D^{-(pd+c)}U_i \qquad \cdots\cdots(8)$$

$$v = A_j(u_1, \cdots, u_n) \qquad \cdots\cdots(9)$$

Then

$$v = g_j{}^*(s_1, \cdots, s_m, u_1, \cdots, u_n) \qquad \cdots\cdots(10)$$

where $s_1, \cdots, s_m$ are the sequences determined by tqe following equations.

$$s_1(0)=\cdots\cdots=s_m(0)=0 \qquad \cdots\cdots(11)$$

$$D^{-1}s_i=f_i^*(s_1, \cdots, s_m, u_1, \cdots, u_n) \qquad \cdots\cdots(12)$$

(see the definition 5 in [1].)

On the other hand,

$$R_{pd}^{-1}D^{-(2pd+c)}A_j'(U_1, \cdots, U_n)$$
$$=R_d^{-1}D^{-d}R_p^{-1}D^{-(pd+c)}A_j'(U_1, \cdots, U_n) \qquad (D^{-d}R_p^{-1}=R_p^{-1}D^{-pd})$$
$$=R_d^{-1}D^{-d}g^*R_p^{-1}D^{-c}(S_1, \cdots, S_m, U_1, \cdots, U_n) \quad \text{(see (5) and (2).)}$$
$$=g^*R_{pd}^{-1}D^{(-pd+c)}(S_1, \cdots, S_m, U_1, \cdots, U_n)$$
$$=g^*(s_1', \cdots, s_m', u_1, \cdots, u_n) \qquad \cdots\cdots(13)$$

where

$$s_i'=R_{pd}^{-1}D^{-(pd+c)}S_i \qquad \cdots\cdots(14)$$

We shall now verify that

$$s_i'=s_i$$

a/ $s_i'(0)=R_p^{-1}D^{-(pd+c)}F_i(S_1, \cdots, U_n)](0)=0=s(0).$

(see (3) and Remark 1 in the paragraph C.)

b/ By (4) and (6),

$$D^{-1}s_i'=D^{-1}R_{pd}^{-1}D^{-(pd+c)}S_i=D^{-1}R_d^{-1}R_p^{-1}D^{-(pd+c)}S_i$$
$$=D^{-1}R_d^{-1}f^*R_p^{-1}(S_1, \cdots, S_m, U_1, \cdots, U_u)$$
$$=f^*R_{pd}^{-1}D^{-(pd+c)}(S_1, \cdots, S_m, U_1, \cdots, U_n)$$
$$=f^*(s_1', \cdots, s_m', u_1, \cdots, u_n)$$

Thus s_i' satisfies the same equations as s_i. Since the equations of the form (11)-(12) have unique solution, we have

$$s_i'=s_i.$$

By (5), (10) and (13), we obtain the desired equality (7).

F. (*Conclusion*) *A_j is s-representable by A_j' with index $(pd, pd, pd+c)$.*

Since the index is independent of j, the condition (*) in the definition 4, 2) is satisfied and therefore $\mathcal{F}$ is strongly universal.

4. Open problems

The following problems still remain to be solved.

1) Suppose that a subset $\mathcal{F}$ of $\mathcal{E}(2)^*$ (or $\mathcal{A}(2)^*$) is (2)-complete.

Is the set $\mathscr{F}$ strongly (2)-complete?

2) Suppose that a subset $\mathscr{F}$ of $\mathscr{E}(k)^*$ is (k)-universal.
Is the set $\mathscr{F}$ weakly (k)-complete? (This not the case for $\mathscr{F} \subseteq \mathscr{A}(k)^*$)

3) Suppose that a subset $\mathscr{F}$ of $\mathscr{A}(k)^*$ is weakly (k)-complete.
Is the set $\mathscr{F}$ strongly (k)-universal?

4) Suppose that a subset $\mathscr{F}$ of $\mathscr{A}(k)^*$ is (k)-univeral.
Is the set $\mathscr{F}$ universal?

5) To obtain the following equivalence, what condition(s) should be imposed on the set $\mathscr{F}$?
$\mathscr{F}$ is $\sim$-complete $\Longleftrightarrow$ $\mathscr{F}$ is strongly (k)-complete.

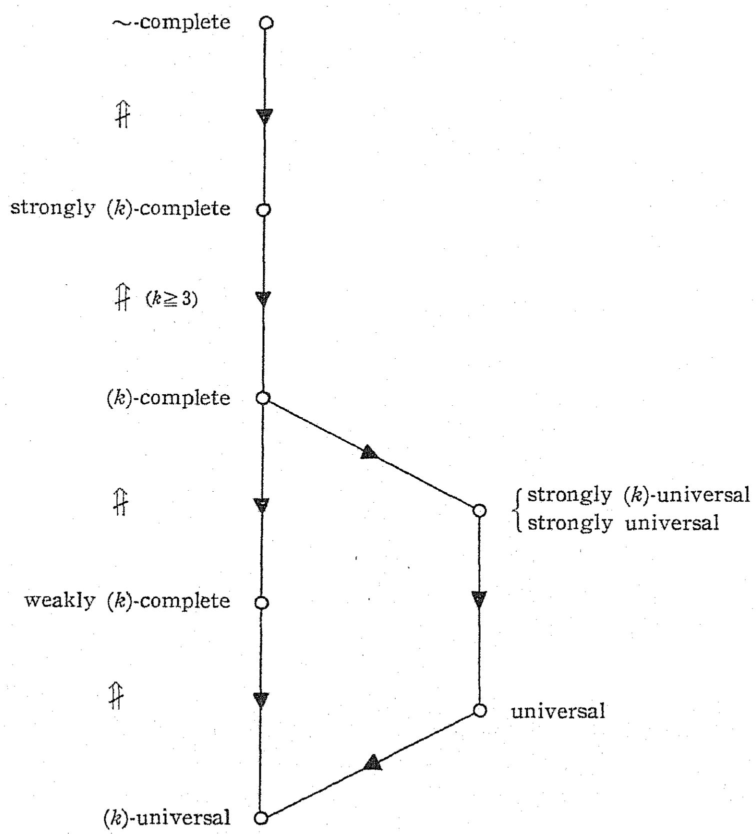

Figure 2. Each arrow represents logical implication.

Reference

[1] NOZAKI, A., Functional Studies of Automata (I), Sci. Pap. of College of General Education, Univ. of Tokyo, **20**, pp. 21-36.

Errata in [1]

pp. 25–26, Definition 4.

a/ We denoted in [1] a composed function by

$$[F_1, \cdots, F_s]_h$$

However, as in this paper (II), the number n of the variables of the composed function should have been explicitly specified:

$$[F_1, \cdots, F_s]_{n, h}$$

b/ We should have assumed that

$$h(0, 0)=n+s$$

or, at least,

$$h(0, 0)>n$$

(If not, $(P_i^n)^* \in [\mathfrak{F}]$. Therefore Lemma 1 becomes invalid.)

page 28, Lemma 3, 1): The following condition must be assumed.

$$G \circ \wp \subseteq G$$

page 34, line 19: The right hand of the definition of the set K should be read as follows.

$$\{D \cdot f^*;\ f \in \Omega(2),\ f(0, \cdots, 0)=f(1, \cdots, 1)\}$$