SHc Description of Minimal Models and Triality

その他のタイトル	SHc代数によるミニマル模型とトライアリティの記述
学位授与年月日	2016-03-24
URL	http://doi.org/10.15083/00073300

学位論文

SH^c Description of Minimal Models and Triality
(SH^c代数によるミニマル模型とトライアリティの記述)

平成 27 年 12 月博士 (理学) 申請

東京大学大学院理学系研究科
物理学専攻
仲村　智

Abstract

In the efforts to prove the 4-dimensional supersymmetric gauge theory/2-dimensional conformal field theory correspondence, a new series of bases for representation spaces of $\mathscr{W}$-algebras was found. In the case of the $\mathscr{W}_N$-algebra, such a basis has an interpretation as the set of all fixed points in the moduli space of $\mathrm{U}(N)$-instantons on a 4-dimensional spacetime. Using the new basis, one can construct an action of a nonlinear algebra SH^c explicitly which is found to be equivalent to a representation of the $\mathscr{W}_N$-algebra. The explicitness opens a way to prove the correspondence by rather simple computations and has led to proofs of several versions of the conjecture.

In this thesis, we study 2-dimensional conformal field theories using the new type algebra SH^c in anticipation that the new basis simplifies known properties and gives new structures behind them. We reconsider minimal models, the level-rank duality and the triality relation in particular and describe them in terms of the explicit action of SH^c.

We prove that, for each minimal model representation, there is a corresponding irreducible representation of SH^c. We obtain a basis of its representation space thanks to its explicit construction and find that it satisfies the N-Burge condition. The SH^c descriptions of minimal model representations then reveal that there is a partially ordered set structure behind the level-rank duality. A minimal model representation space is spanned by the above basis consisting of some N-tuple Young diagrams. Shuffling their rows by following a single rule, we can map the representation to its level-rank dual representation spanned by some $M(\neq N)$-tuple Young diagrams. It suggests that we should change how to label the rows and leads to the notion of a P-partition over a partially ordered set, an integer partition compatible to the partial order. The shuffling means that we see a single P-partition in two different multiple Young diagrams. The theory of P-partitions reproduces a connection between the Rogers–Ramanujan identities and the Lee–Yang singularity. There is another mapping between representations of SH^c. The map is obtained from the fact that the transposition of a Young diagram is also a Young diagram. Combining it with the level-rank duality, we obtain a triality relation of SH^c. This triality is analogous to the triality relation of another algebra $\mathscr{W}_\infty[\mu]$.

This thesis is based on the paper [1].

Contents

Chapter 1

Introduction

This thesis is based on our article [1], which presents more direct link between a 2-dimensional conformal field theory and a new method using a nonlinear algebra SH^c [†1] which appeared in [2] to prove a version of the correspondence between 4-dimensional gauge theories and 2-dimensional conformal field theories, called the Alday–Gaiotto–Tachikawa correspondence [3]. We will describe a special class of conformal field theories, called minimal models of $\mathscr{W}_N$-algebras, in terms of the new algebra SH^c and will reveal that such minimal model representations of SH^c have a triality relation. Before stating our motivation and results, we start this introduction by a rough explanation where the targets we will describe are positioned. We refer to a great review [4] and a nice textbook [5] for readers who want to understand this area in detail.

Belavin, Polyakov and Zamolodchikov showed in [6] the importance of the symmetry in 2-dimensional conformal field theories and initiated a lot of articles on solving physical models algebraically. There are many applications of the conformal symmetry to a broad area of physics, including particle physics, string theory, general relativity, condensed matter physics, and statistical mechanics. It is notable that the conformal invariance contains the scale transformation and then this algebraic approach shines light on the study of the second-order phase transition.

The BPZ work has been collecting so much attention mainly because the conformal symmetry becomes infinite-dimensional in the 2-dimensional spacetime and then it reduces the theory itself to a quasi solvable model. Even for a general D-dimension, the conformal symmetry can be considered as an extended algebra $\mathfrak{so}(D,2)$ of the Lorentz algebra $\mathfrak{so}(D-1,1)$ and then imposes more constraints on the energy spectrum of particles. We have the Virasoro algebra for $D=2$, which is generated by an infinite number of elements $(L_n)_{n\in\mathbb{Z}}$ and a central charge c subject to the following commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\,. \tag{1.0.1}$$

This is an infinite-dimensional Lie algebra and then gives rise to many conservation laws. These laws allow us to focus only on a smaller number of fields, called primary fields, to investigate the whole theory. In other words, one can determine a large part of physical quantities from a consideration with the representation theory of the Virasoro algebra. A primary field

[†1] The algebra has a lengthy name: a central extension of a certain limit $n \to \infty$ of a degenerate double affine Hecke algebra of $\mathrm{GL}(n,\mathbb{C})$. We abbreviate the name to SH^c and postpone its definition until Chapter 2.

corresponds to a highest weight state $|h\rangle$ of the Virasoro algebra Vir_c (1.0.1):

$$L_0\,|h\rangle = h\,|h\rangle\ ,\ L_n\,|h\rangle = 0\,,\ ^{\forall}n>0\,, \tag{1.0.2}$$

and other operators belong to its descendant states. Returning each descendant state to its highest primary state, one can factor a given correlation function into a Virasoro part and a model-dependent part. The former factor is determined by the representation theory of the Virasoro algebra, and the conformal field theory is characterized only by correlations among primary fields.

A conformal field theory contains an infinite number of primary fields in general since operators are closed under multiplication. Choosing the central charge c and the set $\{h_i\}$ of conformal dimensions appropriately, however, one can close the operator algebra expansion (OPE) by a finite primary fields. Such a special class of conformal field theories with finite number of primary fields, called *minimal models*, has been studied extensively since it describes the critical exponents of the known models, including the Ising model [6], restricted solid-on-solid (RSOS) models [7], and so on. The finite number of primary fields correspond to the so-called local scaling operators of such models. The conformal algebra provides a strong tool to investigate critical behaviors by searching solutions from minimal models.

The author of [8] realized that there exists a nonlinear symmetry behind some conformal field theories. The nonlinear algebra $\mathscr{W}_3$ is given by adding a spin-3 current $W_3(z)$ to a Virasoro algebra, but does *not* form a Lie algebra. A primary field with respect to $\mathscr{W}_3$ now corresponds to a highest weight state parameterized by two components $\Lambda_{1,2}$ like

$$\begin{aligned} &L_0\,|\Lambda_1,\Lambda_2\rangle = w_2(\vec{\Lambda})\,|\Lambda_1,\Lambda_2\rangle\ ,\ W_{3,0}\,|\Lambda_1,\Lambda_2\rangle = w_3(\vec{\Lambda})\,|\Lambda_1,\Lambda_2\rangle\ ,\\ &L_n\,|\Lambda_1,\Lambda_2\rangle = W_{3,n}\,|\Lambda_1,\Lambda_2\rangle = 0\,,\ ^{\forall}n>0\,. \end{aligned} \tag{1.0.3}$$

A similar reasoning motivates the representation theory of the nonlinear algebra $\mathscr{W}_3$ and its minimal models describe the critical behaviour of the tricritical Ising model [9] and the 3-state Potts model [10]. This work was generalized in [11] to the algebra $\mathscr{W}_N$ by adding a spin-i current to a Virasoro algebra for each $3 \le i \le N$. Its minimal models correspond to RSOS models [12, 13]. Such an extended Virasoro algebra containing higher spin currents is usually called a W-algebra, which is named after that 'W' is the next character to the capital letter of 'Virasoro'. There are several ways to present W-algebras. The coset construction [14, 15, 16] and the quantized Drinfeld–Sokolov reduction[17] are the examples. In this sense, the above construction in [8, 11] is called the free field realizations or the Feigin–Fuchs construction[18].

A certain series of minimal models was known to present a duality relation [19, 20, 21], called the *level-rank duality*. The duality concerns a conformal field theory with symmetry $\mathscr{W}_N$ whose coset construction is given as $\mathrm{SU}(N)_k \times \mathrm{SU}(N)_1/\mathrm{SU}(N)_{k+1}$ with rational level $k = -N + N/M$ for a pair of mutually prime positive numbers (N, M). The replacement $N \longleftrightarrow M$ gives another conformal field theory with symmetry $\mathscr{W}_M$ and level $l = -M + M/N$. These two minimal models are called level-rank dual to each other since they relate nontrivially. For example, they share the same central charge c of the Virasoro subalgebra and the same q-dimensions of their primary fields. Note that a rank-N W-algebra $\mathscr{W}_N$ is *not* a subalgebra of another W-algebra $\mathscr{W}_{>N}$ with larger rank in general. The duality relates two quite different conformal field theories by interchanging their ranks and levels. There is various manifestation of the level-rank duality in terms of quantum groups[22], conformal embeddings[23, 24], Wess–Zumino–Witten theories[25, 26].

In the context of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, the level-rank duality identifies the symmetry of the higher spin theory on the bulk AdS_3 with that of the boundary conformal field theory [27, 28]. The proposed symmetry $\mathscr{W}_\infty[\mu]$ is given by a large-N limit of $\mathscr{W}_N$ where the 't Hooft coupling parameter μ is held invariant. In the bulk theory, the 't Hooft parameter is fixed to $\mu = N/(N+k)$, while the boundary symmetry is $\mathscr{W}_\infty[\mu = N] = \mathscr{W}_N$. The duality relation connects them for $N/(N+k) = M$ and can be considered as redundancy of the parameter μ in $\mathscr{W}_\infty[\mu]$. The parameter is still redundant since the replacement $k \longleftrightarrow \hat{k} = -2N-k-1$ gives rise to an equivalence between two coset models with the same rank N. Therefore three points in the parameter space should be regarded as the same class, which is called the *triality* in $\mathscr{W}_\infty[\mu]$.

We revisit minimal models, their level-rank duality and the triality relation in this thesis, from a rather new viewpoint which was developed in the efforts to prove the AGT conjecture. In 2009, the authors of [3] revealed a connection of 4-dimensional supersymmetric gauge theories to 2-dimensional conformal field theories. They considered several compactifications of 6-dimensional M5-branes in M-theory, and conjectured that the Virasoro factor of a 4-point correlation function is identical[†2] to the instanton partition function of a 4-dimensional supersymmetric gauge theory with gauge group SU(2). On the one hand, one can compute the Virasoro factor inductively by using the representation theory of the Virasoro algebra. On the other hand, the instanton partition function was shown computable by Nekrasov and Okounkov[29, 30] and is reduced to a summation over fixed points in the moduli space of U(2)-instantons under localization. The conjecture claims a strange coincidence of the two computable quantities of different origin. This AGT conjecture was soon generalized by Wyllard to SU(N), which connects the $\mathscr{W}_N$ factor of a correlation function to the U(N)-instanton partition function [31].

Numerous studies appeared to prove the coincidence[32, 33, 34, 35, 36, 37] and then led to a new basis for a representation space of a $\mathscr{W}$-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$, the tensor product of $\mathscr{W}_N$ and a Heisenberg algebra. The new basis describes the set of all the fixed points in the moduli space of U(N)-instantons and the AGT conjecture claims in part that the $\mathscr{W}$-algebra does act on its span. In fact, Schiffmann and Vasserot showed that there is an action of $\mathscr{W}_{N+\mathrm{U}(1)}$ on the vector space of gauge theoretical origin, by constructing a representation of a nonlinear algebra SH^c on the space and by proving that there is a correspondence between representations of SH^c and those of $\mathscr{W}_{N+\mathrm{U}(1)}$ [2]. The new basis has played an intermediate role connecting the two computable quantities and led to proofs of the conjecture for several gauge theories[36, 37, 38, 39]. What makes the new basis so powerful to analyze the AGT conjecture is that we can write the action of SH^c explicitly. This explicit form makes it easy to construct recursion relations which help us to solve the conjecture.

We take the explicit expression with the algebra SH^c more seriously, and move our focus from the AGT conjecture to the equivalence of SH^c and $\mathscr{W}_{N+\mathrm{U}(1)}$ itself. In other words, *we intend to apply* SH^c *to studying* 2*-dimensional conformal field theories.* We think that our intent is not meaningless for the following expectations:

1. Conformal field theories have been studied for a long time and then the equivalence provides ample properties to the representation theory of SH^c. It may help researchers to study an analog algebra and its representations.

2. We may write down various properties of conformal field theories explicitly by using the action of SH^c on the span of the new basis. The explicit form may reveal a new structure behind those properties as well as 4-dimensional gauge theories.

The latter is the starting point of our research. We want to rewrite known aspects of confor-

[†2] Up to the so-called U(1) factor.

mal field theories in terms of the new basis, and find less-known structures. We mention in advance that we will see a partially ordered set structure and a shuffling operation behind the level-rank duality. There are related works [40, 41, 42] on searching structures, where their authors found the so-called Burge structure behind minimal models through a gauge theoretical consideration, but not through SH^c.

The authors of [2] considered a general representation of SH^c which corresponds to the irreducible action of $\mathscr{W}_{N+\mathrm{U}(1)}$ on the Fock space of free N bosons, or the Verma space since they focused on the proof of a version of the AGT conjecture. Moving our focus to the representation theory of $\mathscr{W}$-algebra, some representations of $\mathscr{W}_{N+\mathrm{U}(1)}$ are not of this non-degenerate type. It is natural to ask what happens to SH^c when it concerns other degenerate representations. In this thesis, we tackle this problem and focus on one of the most important class of degenerate representations originating from primary fields of minimal models. We perform a description of such a minimal model representation in the representation theory of the new type algebra SH^c. We can construct a series of irreducible representations of SH^c corresponding to minimal model representations. We also obtain an explicit basis of the space of a minimal model representation using the new basis of gauge theoretical origin. The basis enables us to get a state-to-state correspondence between two minimal model representation which are level-rank dual to each other and to see a partially ordered set behind the level-rank duality. We have mentioned that there is a triality relation in $\mathscr{W}_\infty[\mu]$ which contains the level-rank duality in part. It is crucial that the algebra $\mathscr{W}_\infty[\mu]$ expresses the $\mathscr{W}$-algebra $\mathscr{W}_N = \mathscr{W}_\infty[\mu = N]$ for any N. The algebra SH^c also present such a universality in a sense that it expresses representations of $\mathscr{W}_{N+\mathrm{U}(1)}$ for any N. We show that there exists an analogous triality in SH^c.

1.0.1 Main results

Let us describe our results in this thesis. Let $L^{(N)}$ be a vector space with a basis $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}^{(N)}}$ whose label $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)})$ runs over the set $\mathscr{Y}^{(N)}$ of all the N-tuple Young diagrams. The set $\mathscr{Y}^{(N)}$ is none other than the set of all the fixed points in the moduli space of $\mathrm{U}(N)$-instantons [29, 30, 43]. This basis also becomes the simultaneous eigenvectors with respect to the set of commuting operators $D_{0,l+1} \in \mathrm{SH}^c$, $l \geq 0$. A series of irreducible actions of SH^c on $L^{(N)}$ exists, which is parameterized by $N+2$ values $a_1, a_2, \cdots, a_N, \varepsilon_1$ and ε_2 with no linear relation in $\mathbb{Z}$. These parameters are of the 4-dimensional $\mathscr{N} = 2$ supersymmetric pure Yang–Mills theory on the Ω-background spacetime. We denote the represented algebra in $\mathrm{End}(L^{(N)})$ by $\mathrm{SH}^{(N)}$. It was shown in [2] that there is an "equivalence" between $\mathrm{SH}^{(N)}$ and a $\mathscr{W}$-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ defined as the tensor product of $\mathscr{W}_N$ and a Heisenberg algebra. This equivalence identifies the representation theory of $\mathscr{W}_{N+\mathrm{U}(1)}$ with that of $\mathrm{SH}^{(N)}$. They also proved that the induced action of $\mathscr{W}_{N+\mathrm{U}(1)}$ on $L^{(N)}$ can be identified with an irreducible representation on the Fock space of N free bosons if we keep the $N+2$ parameters generic.

We use this representation and the basis to describe minimal model representations whose existence itself is expected by the above equivalence. Such a description can be obtained by fixing parameters to

$$
\begin{aligned}
&\varepsilon_1 = q\,,\ \varepsilon_2 = -p\,,\\
&a_1 - a_N = q n'_N - p n_N\,, a_{j+1} - a_j = q n'_j - p n_j\,,\ 1 \leq j \leq N-1\,,\\
&\sum_{i=1}^{N} a_i = \frac{1}{2} N(N-1)(q-p)\,,
\end{aligned}
\tag{1.0.4}
$$

where (p,q) is a pair of mutually prime positive numbers and n_i, n_i' $(1 \le i \le N)$ are positive integers satisfying

$$\sum_{j=1}^{N} n_j = q\,, \sum_{j=1}^{N} n_j' = p\,. \tag{1.0.5}$$

These integers are well-known to become a label of a minimal model representation of the $\mathscr{W}_N$-algebra[11]. We showed that SH^c also acts on a subspace $L_B^{(N)}$ of $L^{(N)}$ after fixing the parameters as above:

Theorem 1.1. *With the notations above,* SH^c *acts irreducibly on a subspace* $L_B^{(N)}$ *of* $L^{(N)}$ *whose basis is given by all the* N*-tuple* $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)}) \in \mathscr{Y}^{(N)}$ *satisfying*

$$\lambda^{(i+1)}(l + n_i - 1) - \lambda^{(i)}(l) \le n_i' - 1\,, \tag{1.0.6}$$

for each $1 \le i \le N$, $l \ge 1$*, where* $\lambda^{(N+1)} = \lambda^{(1)}$. ■

The lowest weight state $|\emptyset\rangle$, labeled by an empty N-tuple, corresponds to the highest weight state of the minimal model representation of $\mathscr{W}_{N+\mathrm{U}(1)}$. Then two representations should be identified with each other by the equivalence. In this sense, we obtain a series of description of minimal model representations in terms of SH^c. This irreducible action is a main result of this thesis, while the above series of inequalities themselves were already obtained in [42] and were called the N-Burge condition. What we find is that there is an action of SH^c even after we specialize parameters to fit minimal models. We will prove it in Section 3.2. The above series of inequalities themselves were obtained in [42] and were called the N-Burge condition. We reached the condition in our paper [1] from another viewpoint and may say that this construction gives an extended meaning of the condition as the basis of the representation space of the nonlinear algebra.

We obtain a new basis for each minimal model representation as a byproduct of the theorem. We use this basis to study the level-rank duality. The level-rank duality concerns a restricted class of minimal model representations with

$$p = N, q = N + M, n_i' = 1\,, 1 \le i \le N\,, \tag{1.0.7}$$

where M is a positive integer such that N and M are mutually prime. For such a datum $\langle N, N+M, (n_i)_{i=1,\cdots,N}\rangle$, we construct a dual datum $\langle M, N+M, (m_i)_{i=1,\cdots,M}\rangle$ describing a minimal model representation of $\mathscr{W}_{M+\mathrm{U}(1)}$ by the same way as [21]. We can prove the following statements:

Theorem 1.2. *Let us denote by* $L_B^{(N)}$, $L_B^{(M)}$ *the corresponding representation space of* $\mathscr{W}_{N+\mathrm{U}(1)}$*, resp.* $\mathscr{W}_{M+\mathrm{U}(1)}$*, and by* $\mathrm{SH}^{(N)} \subset \mathrm{End}(L_B^{(N)})$*, resp.* $\mathrm{SH}^{(M)} \subset \mathrm{End}(L_B^{(M)})$*, the represented algebra.*

1. *The two represented algebras* $\mathrm{SH}^{(N)}$ *and* $\mathrm{SH}^{(M)}$ *are isomorphic to each other after a certain rescale.*

2. *There is a linear isomorphism between* $L_B^{(N)}$ *and* $L_B^{(M)}$ *which commutes any action of zero-mode operators* $D_{0,l+1} \in \mathrm{SH}^c$, $l \ge 0$.

■

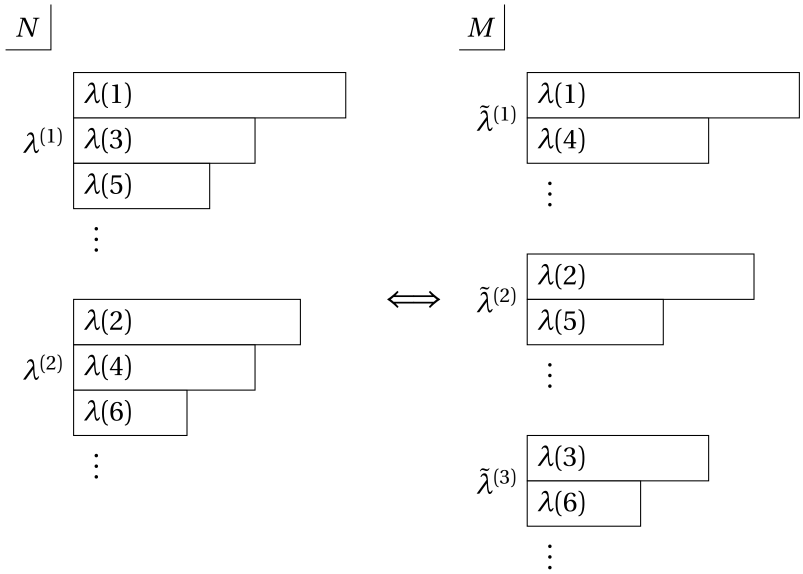


Figure 1.1: An N-tuple Young diagram λ is mapped to an M-tuple $\tilde{\lambda}$ for $N=2, M=3$. They share the same set X which labels every row in multiple Young diagrams.

Therefore such two representations can be considered as the same one. Note that $L_B^{(M)}$ is spanned by M-tuple Young diagrams satisfying the M-Burge condition while $L_B^{(N)}$ is by N-tuples meeting the N-Burge condition. The latter statement is shown by that we have a combinational operation which transforms an N-tuple to an M-tuple like Figure 1.1. The key point is that the linear isomorphism is induced by shuffling each row in Young diagrams. They share the labeling set X and the action of $D_{0,l+1} \in \mathrm{SH}^c$ on $|\lambda\rangle$ is determined not by that λ is expressed as an N-tuple Young diagram but by the map $\lambda = \tilde{\lambda} : X \longrightarrow \mathbb{Z}_{\geq 0},\ x \longmapsto \lambda(x)$. We also see that a multiple Young diagram meeting the N-Burge condition is transformed into another multiple "Young" diagram, which gives a constraint on maps $X \longrightarrow \mathbb{Z}_{\geq 0}$. In fact, the basis of the representation space is identified with the set of all P-partitions[44] over a partially ordered set X. The same partially ordered set is shared by two minimal model representations which are level-rank dual to each other. We prove this partially ordered set structure as well as the state-to-state correspondence behind the level-rank duality in Section 3.3, which is the second main result of this thesis. *We think that this shuffling operation is the most original result in this thesis.* As an application of these P-partitions, we can obtain the Rogers–Ramanujan identities[45] from the Lee–Yang singularity ($N=2, M=3$), while such a connection between them was already mentioned in [46].

The transposition of a Young diagram is also a Young diagram. The transposition induces an automorphism in SH^c. This algebra map and the level-rank duality form a triality of SH^c in a sense that three algebras are identified. This is the last main result of this thesis. This triality is analogous to that of another universal algebra $\mathcal{W}_\infty[\mu]$. We show the triality of SH^c in Section 3.5.

1.0.2 The organization of this thesis

We finish this chapter by mentioning the organization of this thesis. Our main study is given in Chapter 3. Chapter 2 is devoted to reviewing the algebra SH^c and its actions in order to help

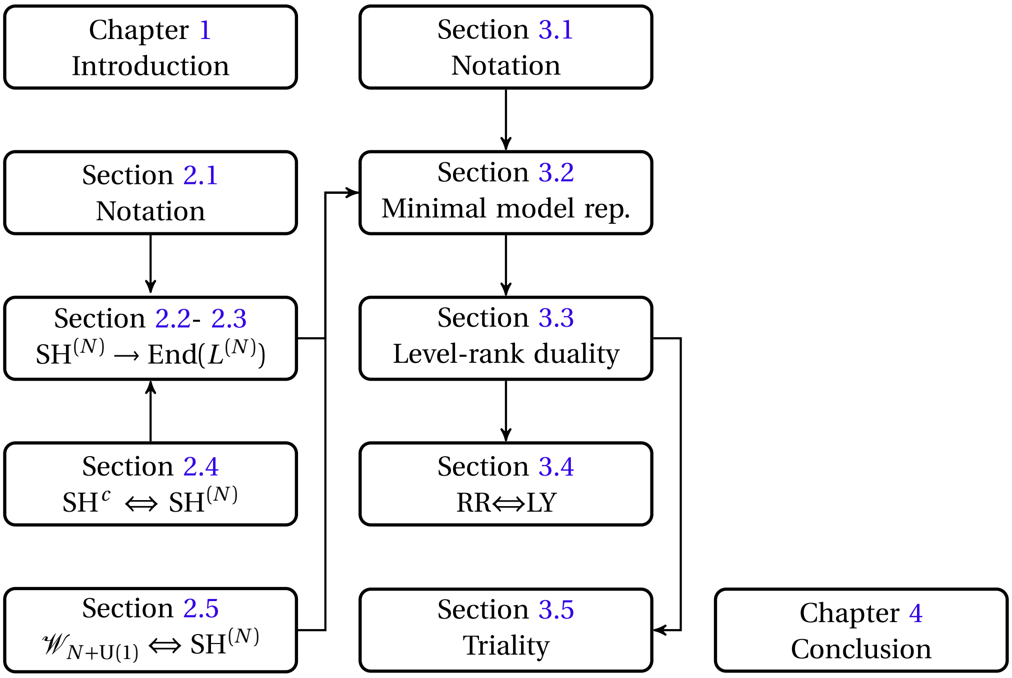


Figure 1.2: The flow of this thesis

the reader to understand the next main chapter more easily. We draw Figure 1.2 to show the flow of this thesis.

In the first half (Section 2.1-2.4) of Chapter 2, we explain the explicit action of SH^c on the vector space $L^{(N)}$ and present some combinational computations. We need a careful explanation of the explicit action as well as of the less-known algebra SH^c since our main analysis depends heavily on the action on it. We construct an irreducible minimal model representation by restricting the original action on $L^{(N)}$ to a smaller one on a certain subspace of $L^{(N)}$. In the construction, we will focus on how each generator of SH^c is expressed by a matrix with respect to the basis for $L^{(N)}$ of gauge theoretical origin and on which matrix elements vanish or diverge. We first explain the vector space $L^{(N)}$ and its gauge theoretical origin in Section 2.2. We associate the vector space with the above basis there. We then introduce in Section 2.3 an action of SH^c by assigning a matrix expression for each generator of the algebra. We also confirm some commutation relations in SH^c in that section. Note that we rather introduce the represented algebra $\mathrm{SH}^{(N)} \subset \mathrm{End}(L^{(N)})$ than SH^c itself since we repeatedly use such matrices and perform computations with them. Such computations are expressed using a combinational aspect of Young diagrams. We fix some notions with Young diagrams in Section 2.1. Section 2.4 is a supplement to this first half. We present the original algebra SH^c in Section 2.4 by giving its generators and defining relations. The goal of this section is to tell the reader that the commutation relations confirmed in Section 2.3 are sufficient to determine the algebra and the matrix-expressed algebra $\mathrm{SH}^{(N)}$ does not lose the generality of the abstract algebra SH^c.

The latter half (Section 2.5) of Chapter 2 is devoted to introducing an equivalence between $\mathrm{SH}^{(N)}$ and the $\mathscr{W}$-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$. In [2], they showed the equivalence of their representation theories. While our motivation is based on that result, we do not need the detailed proof of the equivalence. We rather motivate the reader to accept the equivalence by telling that the two algebras share the same Virasoro–Heisenberg subalgebra, and only give a user guide to use the equivalence. In Section 2.5, we first fix our notation with the $\mathscr{W}$-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ and then introduces the equivalence by following the above procedure.

In Chapter 3, we discuss our applications of SH^c to minimal model representations (Section 3.2), the level-rank duality (Section 3.3) and a triality relation (Section 3.5) . We first fix our notations with minimal model representations of the $\mathscr{W}$-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ in Section 3.1. Then we show in 3.2 our description of minimal model representations as follows. We first state the resultant representation space $L_B^{(N)}$ and then we check that SH^c acts irreducibly on this space. As mentioned above, we consider the matrix coefficients of the represented algebra $\mathrm{SH}^{(N)}$ and see that it suffices to focus on the east boundary of Young diagrams to evaluate whether the coefficients vanish or diverge. We give some propositions and lemmas to express the vanishment condition and the divergent condition. As a result, we obtain an appropriate action on $L_B^{(N)}$. Next we move to Section 3.3 and consider the level-rank duality as follows. We first introduce another labeling set $X_B^{(N)}$ for rows of N-tuple Young diagrams and see that each element in the basis for $L_B^{(N)}$ is rephrased as a P-partition over $X_B^{(N)}$ with a certain partial ordering. We then show that two minimal model representations share the same partially ordered set $X_B^{(N)}$. We construct such a dual representation by using a disk expression which appeared in [21]. We then rephrase the action of $\mathrm{SH}^{(N)}$ in terms of the shared set $X_B^{(N)}$ and finish our description of the level-rank duality. Section 3.4 is a supplement to Section 3.3. We revisit a connection between the Rogers–Ramanujan identities and the Lee–Yang singularity from the theory of P-partitions. At last, we consider a triality relation in Section 3.5. We will see that the level-rank duality and the transposition operation of Young diagrams give an identification of three minimal model representations.

We conclude this thesis by Chapter 4 with giving some future prospects of our results.

Chapter 2

The algebra SH^c and its actions

In [2], Schiffmann and Vasserot constructed a representation of a nonlinear algebra SH^c on an infinite-dimensional graded vector space $L^{(N)}$, containing the set of all the N-tuple Young diagrams as a basis. The basis of $L^{(N)}$ was shown in [43] to correspond to the set of all the fixed points with respect to a torus action in the instanton moduli space of the pure $\mathcal{N}=2$ supersymmetric Yang–Mills theory with gauge group $\mathrm{U}(N)$ on the Ω-deformed spacetime $\mathbb{R}^4$. The action of SH^c then means that an infinite-dimensional symmetry exists behind the equivariant cohomology of the moduli space. The authors of [2] proved that the representation of SH^c is equivalent to that of a W-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ given as the tensor product of an extended conformal algebra $\mathscr{W}_N$ and the Heisenberg algebra. The induced action of the W-algebra on $L^{(N)}$ shows a version of the AGT conjecture.

In this chapter, we review the representation of SH^c on $L^{(N)}$. We summarize a necessary part of the topic in order for readers to get familiar with our upcoming discussions which depends heavily on the new type algebra SH^c. We mainly focus on the matrix coefficients of the representation with respect to the basis of $L^{(N)}$. These quantities will play the most important role with our research on the triality among minimal models of the W-algebra. We will tune some parameters in the representation, whose origin is in a 4-dimensional supersymmetric gauge theory, to 2-dimensional conformal field theories in the next chapter. We will focus on the situation when they vanish or diverge after a specialization of the parameters.

We construct the representation on $L^{(N)}$ explicitly and perform some computations with them in Section 2.2 and 2.3. We prepare our notations with Young diagrams in Section 2.1 since the matrix coefficients are associated with combinational aspects of those diagrams. We also check some commutation relations in SH^c in Section 2.3, which are known to be enough to express the algebra itself by generators and defining relations, explained in Section 2.4. The representation of SH^c has a potential for further studies on conformal field theories because one can dive into a certain representation of the W-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ by simple computations associated with the equivalent representation of SH^c. We introduce a user guide to use the correspondence between representations of SH^c and of the W-algebra in Section 2.5. We give proofs for some propositions with combinational computations for our later use, but we only introduce the other propositions without proofs when we do not need their detailed proofs.

We should note that we rather introduce an algebra $\mathrm{SH}^{(N)}$ than the algebra SH^c itself. The algebra SH^c was originally introduced in [2] as a central extension of a certain limit $n\to\infty$ of a degenerate double affine Hecke algebra of $\mathrm{GL}(n,\mathbb{C})$ and was shown in [47] to be expressed by generators and defining relations. The original construction is crucial to justify our user guide connecting SH^c and the W-algebra. We present the relation between $\mathrm{SH}^{(N)}$ and SH^c

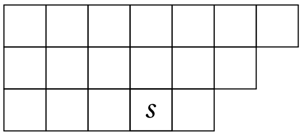
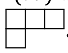


Figure 2.1: A Young diagram $Y = (7,6,5)$ and a box s

in Section 2.4 but we do not give a detailed explanation of SH^c. The reason why we choose a path with $\mathrm{SH}^{(N)}$, not with the original SH^c, is that our study depends rather on the user guide and on the explicit form of the representation of $\mathrm{SH}^{(N)}$ on $L^{(N)}$ than on the detailed proof.

2.1 Notation

We first fix our notations with Young diagrams. A map $Y : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ is called a partition when it satisfies $Y(n_1) \geq Y(n_2)$ for $n_1 \leq n_2$ and there is a number $n \geq 1$ such that $Y(n) = 0$. We identify a partition Y with a Young diagram whose nth row has $Y(n)$ boxes for $n \geq 1$. For example, a partition $Y = (3,1,0,\cdots)$ is expressed as a Young diagram . Given two Young diagrams Y_1 and Y_2, we write

$$Y_1 \subset Y_2 \Leftrightarrow Y_1(n) \leq Y_2(n), {}^\forall n \geq 1\,. \tag{2.1.1}$$

We denote by $|Y| = \sum_{n\geq 1} Y(n)$ the total number of boxes in a Young diagram Y and call it the level of Y. Given a Young diagram Y and a box s in Y, we denote by $x(s), y(s), a_Y(s), l_Y(s)$ the number of boxes in Y lying strictly to the west, resp. north, resp. east, resp. south, of the box s. For example, we have $x(s) = 3, y(s) = 2, a_Y(s) = 1, l_Y(s) = 0$ for the case in Figure 2.1.

We denote by $\mathscr{Y}^{(N)}$ the set of all the N-tuples $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)})$ of Young diagrams. Given two N-tuples $\lambda_1, \lambda_2 \in \mathscr{Y}^{(N)}$, we write

$$\lambda_1 \subset \lambda_2 \Leftrightarrow \lambda_1^{(i)} \subset \lambda_2^{(i)},\ 1 \leq {}^\forall i \leq N\,. \tag{2.1.2}$$

We define $|\lambda| = \sum_{i=1}^N |\lambda^{(i)}|$ for $\lambda \in \mathscr{Y}^{(N)}$ and call it the level of λ. We express the level decomposition of $\mathscr{Y}^{(N)}$ by $\mathscr{Y}^{(N)} = \sqcup_{n\geq 0} \mathscr{Y}_n^{(N)}$. Given an N-tuple $\lambda \in \mathscr{Y}^{(N)}$ and $1 \leq i \leq N$, we write $a_\lambda^{(i)} = a_{\lambda^{(i)}}, l_\lambda^{(i)} = l_{\lambda^{(i)}}$ and canonically define $x(s), y(s)$ and possibly negative numbers $a_\lambda^{(i)}(s)$, $l_\lambda^{(i)}(s)$ for any point s in the lattice where boxes are arranged.

2.2 The vector space $L^{(N)}$

We first define a vector space $L^{(N)}$ for a given positive number N. It has the basis $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}^{(N)}}$ and some structures which originate from the instanton counting of 4-dimensional $\mathscr{N} = 2$ supersymmetric gauge theories. We start by a rough explanation of the instanton counting to give a physical meaning of $L^{(N)}$, and then define the vector space at the last of this section.

2.2.1 Instanton counting

We first briefly recall a gauge theoretical background of the instanton counting and extract some basic results needed for our calculations from [43, 29, 30]. We refer to a nice review [48] on the instanton counting and to references therein for the reader who wants to understand the topic in detail.

Nekrasov and Okounkov considered in [29, 30] a certain class of 4-dimensional $\mathcal{N}=2$ supersymmetric gauge theories on the Ω-deformed spacetime[†1] $\mathbb{R}^4$. Here the Ω-deformation is a background gauge field in the spacetime with respect to the spacetime symmetry SO(4), and is characterized by two parameters $\varepsilon_1, \varepsilon_2$ where $(e^{i\varepsilon_1}, e^{i\varepsilon_2}) \in \mathrm{U}(1)^2 \subset \mathrm{SO}(4)$ acts on $\mathbb{C}^2 \simeq \mathbb{R}^4$ canonically. Let us focus only on a theory in the class, the Ω-deformed $\mathcal{N}=2$ supersymmetric pure super-Yang-Mills theory with gauge group $\mathrm{U}(N)$ and on its partition function at a supersymmetric vacuum. Such a vacuum state gives a boundary condition of the path integral enforcing the adjoint scalar in the theory to be a constant diagonal matrix $\mathrm{diag}(a_1, \cdots, a_N) \in \mathfrak{u}(1)^N \subset \mathfrak{u}(N)$ at the infinities of the spacetime.

They showed that the partition function of the theory is reduced, by use of the so-called SUSY localization, to an integral over the (regularized) moduli space of $\mathrm{U}(N)$-instantons. In particular, its nonperturbative correction, called the instanton partition function, is none other than the equivariant Euler class of the moduli space with respect to the action of the torus $\tilde{D} = T \times D$. Here we denote by $T = \mathrm{U}(1)^2$ the torus associated to the Ω-deformation and by $D = \mathrm{U}(1)^N$ the torus associated with the diagonal framing $\vec{a} = (a_1, \cdots, a_N)$ at the infinities. The instanton partition function $Z_{\mathrm{inst}}(\vec{a}, \varepsilon_1, \varepsilon_2)$ is then reduced, by use of an equivariant localization, to an integral over the fixed points in the moduli space with respect to the torus action. If any fixed point is isolated, the integral is interpreted as a sum of reciprocals of Euler characters

$$Z_{\mathrm{inst}}(\vec{a}, \varepsilon_1, \varepsilon_2) = \sum_{n \geq 0} q^n \sum_{p} \frac{1}{\mathrm{eu}\left(T_p \mathcal{M}_n\right)}, \tag{2.2.1}$$

where $\mathcal{M} = \sqcup_{n\geq 0} \mathcal{M}_n$ is the moduli space of instantons, which is decomposed by its instanton number n, a point p runs over any fixed point in $\mathcal{M}_n$, and a coefficient q is a parameter associated with the gauge coupling constant of the theory. Here $\mathrm{eu}(E)$ means the Euler character of a given vector space E with respect to the action of the torus $\tilde{D}$, which means that if $\tilde{D}$ acts on E by

$$\begin{aligned}(e^{i\varepsilon_1}, e^{i\varepsilon_2}, e^{ia_1}, \cdots, e^{ia_N}) \in \tilde{D} \mapsto \mathrm{diag}(e^{iw_1}, \cdots, e^{iw_{\dim E}}) \in \mathrm{Aut}(E), \\ w_1, \cdots, w_{\dim E} \in \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2 + \sum_{j=1}^{N} \mathbb{Z} a_j,\end{aligned} \tag{2.2.2}$$

then its Euler character is given by

$$\mathrm{eu}(E) = \prod_{f=1}^{\dim E} w_f. \tag{2.2.3}$$

There are mathematical facts about the fixed points and their Euler classes, summarized as follows [43, 30]:

Proposition 2.1 ([43]). *1. Any fixed point in $\mathcal{M}$ with respect to the action of the torus $\tilde{D}$ is isolated. (Then the instanton partition function can be expressed as a sum (2.2.1).)*

[†1] The spacetime is also deformed to a noncommutative one by a Poisson tensor Θ [30]. In their calculations with the partition functions, these deformations justify the applications of two localization methods, a SUSY localization and an equivariant localization, to noncompact spaces by regularizing their infiniteness.

2. *The set $\mathscr{M}_n^0$ of all the fixed points in $\mathscr{M}_n$ is identified with the set $\mathscr{Y}_n^{(N)}$ of all the N-tuple Young diagrams with level n.*

3. *Given a fixed point $\lambda = (\lambda^{(1)}, \cdots, \lambda^{(N)}) \in \mathscr{Y}_n^{(N)} \simeq \mathscr{M}_n^0$, then the tangent space $T_\lambda \mathscr{M}_n$ is identified with the middle cohomology T_λ of an exact sequence*

$$\begin{aligned} &0 \longrightarrow \mathrm{End}(V_\lambda) \\ &\longrightarrow t_1 \otimes \mathrm{End}(V_\lambda) \oplus t_2 \otimes \mathrm{End}(V_\lambda) \oplus \mathrm{Hom}(W, V_\lambda) \oplus t_1 \otimes t_2 \otimes \mathrm{Hom}(V_\lambda, W) \\ &\longrightarrow t_1 \otimes t_2 \otimes \mathrm{End}(V_\lambda) \longrightarrow 0 \,, \end{aligned} \tag{2.2.4}$$

where

$$V_\lambda = \bigoplus_{j=1}^{N} \left(\bigoplus_{s \in \lambda^{(j)}} \chi_j \otimes t_1^{-\otimes x(s)} \otimes t_2^{-\otimes y(s)} \right), \; W = \bigoplus_{j=1}^{N} \chi_j \,. \tag{2.2.5}$$

Here we denote by t_1, t_2 and χ_j for $1 \le j \le N$, 1-dimensional vector spaces where the action of $\tilde{D}$ to the $(N+2)$-dimensional vector space $t_1 \oplus t_2 \oplus \chi_1 \oplus \cdots \oplus \chi_N$ is canonical.

■

Note that the Atiyah–Drinfeld–Hitchin–Manin (ADHM) construction [49] parameterizes the moduli space $\mathscr{M}$ as

$$\mathscr{M}_n = \mathscr{N}_n / \mathrm{GL}(n, \mathbb{C}), \tag{2.2.6}$$

$$\mathscr{N}_n = \left\{ (B_1, B_2, I, J) \middle| \begin{array}{c} B_1, B_2 \in \mathrm{End}(\mathbb{C}^n), I \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^n), J \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N) \\ \mu_c \equiv [B_1, B_2] + IJ = 0, \text{ stable} \end{array} \right\}. \tag{2.2.7}$$

Here the canonical action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$ induces its action on $\mathscr{N}_n$. The stability condition in (2.2.7) is a result of the regularization by the deformation of the theory, but we do not need its detail. The action of the torus $\tilde{D}$ on $\mathscr{M}_n$ is then expressed by its action on the ADHM matrices (B_1, B_2, I, J) in $\mathscr{N}_n$,

$$B_1 \in t_1^{\oplus n^2}, \; B_2 \in t_2^{\oplus n^2}, \; I \in (\chi_1^{-1} \oplus \cdots \oplus \chi_N^{-1})^{\oplus n}, \; J \in \big(t_1 \otimes t_2 \otimes (\chi_1 \oplus \cdots \oplus \chi_N)\big)^{\oplus n}. \tag{2.2.8}$$

An element in $\mathscr{M}_n$ is a fixed point if and only if the torus action to its representative ADHM matrices in $\mathscr{N}_n$ is absorbed by an action of $\mathrm{GL}(n, \mathbb{C})$. This parametrization explains where the sequence (2.2.4) comes from. The second arrow in (2.2.4) represents the differential of the $\mathrm{GL}(n)$-action to $\mathscr{N}_n$ at a fixed point λ, and the third represents the differential $d\mu_{c,\lambda}$ at λ.

Henceforth, we often omit $E_1 \oplus E_2$, $E_1 \otimes E_2$ and E/F by $E_1 + E_2$, resp. $E_1 E_2$ and resp. $E - F$, for given vector spaces E_1, E_2, or E, F such that F is the subspace of E, $F \subset E$. Note that these "±" abbreviations are compatible with their Euler characters:

$$\mathrm{eu}(E_1 + E_2) = \mathrm{eu}(E_1)\mathrm{eu}(E_2), \; \mathrm{eu}(E - F) = \frac{\mathrm{eu}(E)}{\mathrm{eu}(F)}. \tag{2.2.9}$$

The middle cohomology $T_\lambda = T_\lambda \mathscr{M}_n$ of the sequence (2.2.4) for a given $\lambda \in \mathscr{Y}_n^{(N)}$ is then written as[†2]

$$T_\lambda = -(1 - t_1)(1 - t_2) V_\lambda^* \otimes V_\lambda + W^* \otimes V_\lambda + t_1 t_2 V_\lambda^* \otimes W \,, \tag{2.2.10}$$

[†2]For a vector space E, we denote its dual vector space by E^*. Note that if the torus $\tilde{D}$ acts on E, it also acts on E^* by the dual representation. This means that $t_1^* = t_1^{-1}$, $t_2^* = t_2^{-1}$, and $\chi_j^* = \chi_j^{-1}$ for $1 \le j \le N$.

and is shown in [43] to be simplified to

$$T_\lambda = \sum_{j,k=1}^{N} \chi_j^{-1}\chi_k \left(\sum_{s\in\lambda^{(j)}} t_1^{a_\lambda^{(j)}(s)+1} t_2^{-l_\lambda^{(k)}(s)} + \sum_{s\in\lambda^{(k)}} t_1^{-a_\lambda^{(k)}(s)} t_2^{l_\lambda^{(j)}(s)+1} \right). \tag{2.2.11}$$

This reduction is a corollary of the following proposition.

Proposition 2.2 ([43])**.** *Let (Y_1, Y_2) be a pair of Young diagrams and set $V_{Y_i} = \sum_{s\in Y_i} t_1^{-x(s)} t_2^{-y(s)}$ for $i = 1,2$. Then we have*

$$-(1-t_1)(1-t_2)V_{Y_1}^* \otimes V_{Y_2} + V_{Y_2} + t_1 t_2 V_{Y_1}^* = \sum_{s\in Y_1} t_1^{a_{Y_1}(s)+1} t_2^{-l_{Y_2}(s)} + \sum_{s\in Y_2} t_1^{-a_{Y_2}(s)} t_2^{l_{Y_1}(s)+1}. \tag{2.2.12}$$

■

Combining the above results, one can obtain the following summation formula of the instanton partition function,

$$\begin{aligned} Z_{\text{inst}}(\vec{a},\varepsilon_1,\varepsilon_2) &= \sum_{\lambda\in\mathscr{Y}(N)} \frac{q^{|\lambda|}}{\text{eu}(T_\lambda)}, \\ \text{eu}(T_\lambda) &= \prod_{j,k=1}^{N} \left[\prod_{s\in\lambda^{(j)}} \left(-a_j + a_k + (a_\lambda^{(j)}(s)+1)\varepsilon_1 - l_\lambda^{(k)}(s)\varepsilon_2\right) \right. \\ &\qquad \left. \times \prod_{s\in\lambda^{(k)}} \left(-a_j + a_k - a_\lambda^{(k)}(s)\varepsilon_1 + (l_\lambda^{(j)}(s)+1)\varepsilon_2\right) \right]. \end{aligned} \tag{2.2.13}$$

Note that the vector space T_λ contains no trivial representation, $1 \not\subset T_\lambda$, if there is no linear relation in $\mathbb{Z}$ among parameters $\varepsilon_{1,2}, a_{1,\cdots,N}$. It is clear from the fact that the fix point λ is isolated. A trivial subrepresentation may exist only when $j = k$ in (2.2.11). In this case, however, we have $a_\lambda^{(j)}(s)+1, l_\lambda^{(j)}(s)+1 > 0$ for $s \in \lambda^{(j)}$, which shows $1 \not\subset T_\lambda$.

In this chapter, we assume the linear independence among these parameters, while we focus on a certain specialization of the parameters later.

Here we explicitly determine some Euler characters in order for the reader to get familiar with such a potentially less-known quantity. Let us consider a contribution of a Young diagram $\square\square$ to the U(1)-instanton partition function. Assigning numbers as $\boxed{1}\boxed{2}$, we obtain

$$a_{\square\square}(1) = 1,\ l_{\square\square}(1) = 0,\ a_{\square\square}(2) = 0,\ l_{\square\square}(2) = 0. \tag{2.2.14}$$

Then we have

$$\begin{aligned} \text{eu}(T_{\square\square}) =& (-a_1 + a_1 + (1+1)\varepsilon_1 - 0\cdot\varepsilon_2)(-a_1 + a_1 - 1\cdot\varepsilon_1 + (0+1)\varepsilon_2) \\ &\times (-a_1 + a_1 + (0+1)\varepsilon_1 - 0\cdot\varepsilon_2)(-a_1 + a_1 - 0\cdot\varepsilon_1 + (0+1)\varepsilon_2) \\ =& 2\varepsilon_1(-\varepsilon_1+\varepsilon_2)\varepsilon_1\varepsilon_2. \end{aligned} \tag{2.2.15}$$

We also have

$$\text{eu}(T_{\square}) = \varepsilon_1\varepsilon_2,\ \text{eu}(T_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}) = 2\varepsilon_2(\varepsilon_1-\varepsilon_2)\varepsilon_1\varepsilon_2. \tag{2.2.16}$$

Therefore we obtain the U(1)-instanton partition function up to order q^2:

$$Z_{\text{inst}}^{\text{U}(1)}(a_1,\varepsilon_1,\varepsilon_2) = 1 + \frac{q}{\varepsilon_1\varepsilon_2} + \frac{q^2}{2\varepsilon_1^2\varepsilon_2^2} + \mathscr{O}(q^3). \tag{2.2.17}$$

2.2.2 The vector space $L^{(N)}$

We now impose two structures to a vector space $L^{(N)}$ which has the set $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}^{(N)}}$ of vectors as its basis. The first structure is a grading $L^{(N)}=\oplus_{n\geq 0}L_n^{(N)}$ by degree associated with the level over $\mathscr{Y}^{(N)}$. The second is a paring for $L^{(N)}$ defined by

$$\langle\mu|\lambda\rangle=\delta_{\mu,\lambda}\mathrm{eu}(T_\lambda),\ \lambda,\mu\in\mathscr{Y}^{(N)}. \tag{2.2.18}$$

Note that, denoting by $\mathbf{H}$ the operator on $L^{(N)}$ which counts the degree of $|\lambda\rangle\in L^{(N)}$,

$$\mathbf{H}|\lambda\rangle=|\lambda||\lambda\rangle=n|\lambda\rangle\ ,\ \lambda\in\mathscr{Y}_n^{(N)}, \tag{2.2.19}$$

the formula (2.2.13) is rewritten as

$$Z_{\mathrm{inst}}(\vec{a},\varepsilon_1,\varepsilon_2)=\langle G|q^{\mathbf{H}}|G\rangle\ ,\ |G\rangle=\sum_{\lambda\in\mathscr{Y}^{(N)}}\mathrm{eu}^{-1}(T_\lambda)|\lambda\rangle\ . \tag{2.2.20}$$

2.3 The algebra $\mathrm{SH}^{(N)}$

Here we define the representation of the algebra $\mathrm{SH}^{(N)}$ on $L^{(N)}$. We write down some computations explicitly in order to prepare for our main analysis.

2.3.1 The vector space $N_{\sigma,\lambda}$

For a given pair of N-tuple Young diagrams $\lambda,\sigma\in\mathscr{Y}^{(N)}$ satisfying $\sigma\subset\lambda$ and $|\lambda|=|\sigma|+1$, we define a vector space $N_{\sigma,\lambda}=N_{\lambda,\sigma}$ by

$$N_{\sigma,\lambda}=-(1-t_1)(1-t_2)V_\lambda^*\otimes V_\sigma+W^*\otimes V_\sigma+t_1t_2V_\lambda^*\otimes W-t_1t_2, \tag{2.3.1}$$

which is similar to the definition of T_λ in (2.2.10). By a similar way, we can simplify $N_{\sigma,\lambda}$ to

$$N_{\sigma,\lambda}=\sum_{j,k=1}^{N}\chi_j^{-1}\chi_k\left(\sum_{s\in\lambda^{(j)}}t_1^{a_\lambda^{(j)}(s)+1}t_2^{-l_\sigma^{(k)}(s)}+\sum_{s\in\sigma^{(k)}}t_1^{-a_\sigma^{(k)}(s)}t_2^{l_\lambda^{(j)}(s)+1}\right)-t_1t_2. \tag{2.3.2}$$

Let s be the unique box in λ not contained in σ. The component associated with this box s in $\lambda^{(j)}$ and $j=k$ in the above equation becomes

$$\chi_j^{-1}\chi_j t_1^{a_\lambda^{(j)}(s)+1}t_2^{-l_\sigma^{(j)}(s)}=t_1t_2, \tag{2.3.3}$$

and then cancel the last term $-t_1t_2$. Therefore (2.3.1) indeed defines a vector space. We can show that $1\not\subset N_{\lambda,\sigma}$ by a similar reasoning for T_λ.

2.3.2 The operators $D_{0,l+1}$, $D_{\pm1,l}$ and the algebra $\mathrm{SH}^{(N)}$

Given a vector space E where the torus $\tilde{D}$ acts by weights $(w_f)_{f=1}^{\dim E}$, we define the character $c_1(E)$ of the vector space E by

$$c_1(E)=\sum_{f=1}^{\dim E}w_f, \tag{2.3.4}$$

while the Euler character is $\mathrm{eu}(E)=\prod_{f=1}^{\dim E} w_f$. Note that we have

$$c_1(t_1)=\varepsilon_1\,,\ c_1(t_2)=\varepsilon_2\,,\ c_1(\chi_i)=a_i\,,\ 1\le i\le N\,. \tag{2.3.5}$$

For $l\geq 0$, we define three operators $D_{0,l+1}, D_{\pm1,l}\in \mathrm{End}(L^{(N)})$, in degrees ±1 and 0 respectively, by the following actions; for $\lambda\in\mathscr{Y}^{(N)}$,

$$\begin{aligned}
D_{0,l+1}\,|\lambda\rangle &= \sum_{j=1}^{N}\sum_{s\in\lambda^{(j)}}\left(\frac{c_1(\chi_j^{-1}t_1^{x(s)}t_2^{y(s)})}{\varepsilon_1}\right)^l|\lambda\rangle\,,\\
D_{1,l}\,|\lambda\rangle &= \varepsilon_1\varepsilon_2\sum_{\pi\supset\lambda}\left(\frac{c_1(V_\pi^*-V_\lambda^*)}{\varepsilon_1}\right)^l \mathrm{eu}(N_{\lambda,\pi}-T_\pi)|\pi\rangle\,,\\
D_{-1,l}\,|\lambda\rangle &= (-1)^{N-1}\sum_{\sigma\subset\lambda}\left(\frac{c_1(V_\lambda^*-V_\sigma^*)}{\varepsilon_1}\right)^l \mathrm{eu}(N_{\sigma,\lambda}-T_\sigma)|\sigma\rangle\,,
\end{aligned} \tag{2.3.6}$$

where $\pi,\sigma\in\mathscr{Y}^{(N)}$ runs over $|\pi|=|\lambda|+1$ and $|\sigma|=|\lambda|-1$, respectively. We should note that the Euler classes of $N_{\lambda,\pi}$, T_π do not vanish when the parameters $\varepsilon_{1,2}$ and $a_{1,\cdots,N}$ are kept generic and then the action of SH$^{(N)}$ on $L^{(N)}$ is well-defined. The algebra SH$^{(N)}$ is none other than what the above operators generate:

Definition 2.1. SH$^{(N)}$ is the graded subalgebra of End$(L^{(N)})$ generated by $D_{0,l+1}, D_{\pm1,l}$ for $l\geq 0$. ■

The action is compatible to the pairing of $L^{(N)}$ in the following sense. For $\lambda,\pi\in\mathscr{Y}^{(N)}$ satisfying $\pi\supset\lambda, |\pi|=|\lambda|+1$, the two coefficients $\langle\lambda|\,D_{-1,l}\,|\pi\rangle$ and $\langle\pi|\,D_{1,l}\,|\lambda\rangle$ relates to each other by

$$\langle\lambda|\,D_{-1,l}\,|\pi\rangle=(-1)^{N-1}\left(\frac{c_1(V_\pi^*-V_\lambda^*)}{\varepsilon_1}\right)^l \mathrm{eu}(N_{\lambda,\pi})=\frac{(-1)^{N-1}}{\varepsilon_1\varepsilon_2}\langle\pi|\,D_{1,l}\,|\lambda\rangle\,. \tag{2.3.7}$$

We will focus on matrix coefficients in (2.3.6) with respect to the basis of $L^{(N)}$ in the next main chapter. It is useful for us to simplify the coefficients. To express them, we first define two vector spaces I_λ, J_λ for a given N-tuple $\lambda\in\mathscr{Y}^{(N)}$ by

$$I_\lambda=\sum_{\substack{\pi\supset\lambda\\|\pi|=|\lambda|+1}}(V_\pi-V_\lambda),\ J_\lambda=\sum_{\substack{\sigma\subset\lambda\\|\sigma|=|\lambda|-1}}(V_\lambda-V_\sigma). \tag{2.3.8}$$

For $\lambda,\pi\in\mathscr{Y}^{(N)}$ satisfying $\pi\supset\lambda, |\pi|=|\lambda|+1$, we have

$$N_{\lambda,\pi}-T_\pi=((1-t_1)(1-t_2)V_\pi^*-W^*)\otimes(V_\pi-V_\lambda)-t_1t_2=(t_1t_2J_\pi^*-I_\pi^*)\otimes(V_\pi-V_\lambda)-t_1t_2\,. \tag{2.3.9}$$

The last transformation can be seen from a consideration with the Figure 2.2. Here each colored box represents a 1-dimensional vector space and its weight of the torus action is expressed by its position. A red box $\square$, a blue box $\square$, contribute to the Euler class by $c_1(\square)^{+1}$, resp. $c_1(\square)^{-1}$. A red box $\square$ and a blue box $\square$ cancel each other out, $\square+\square=0$, when they lie in the same position.

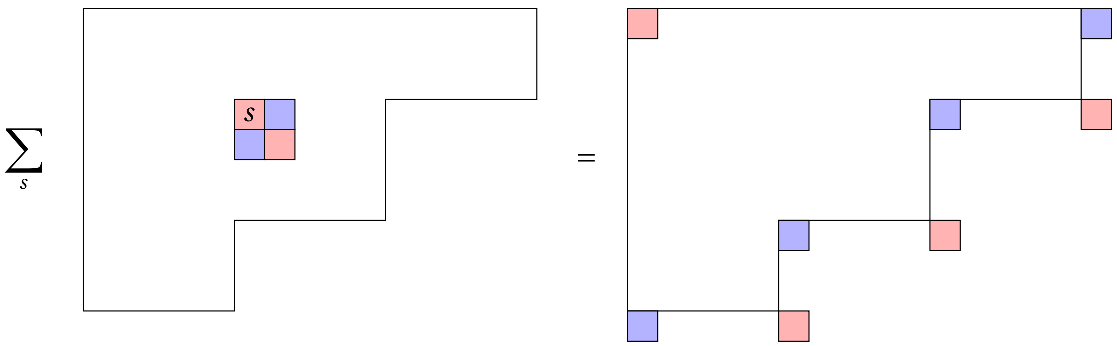


Figure 2.2: Diagrammatic expression of the equation $(1-t_1)(1-t_2)V_\pi^* = t_1 t_2 J_\pi^* - I_\pi^* + W^*$. The summation is taken over all the box s in the above Young diagram.

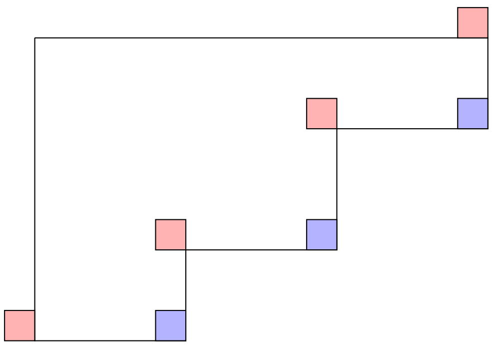


Figure 2.3: Diagrammatic expression of $t_1^{-1} t_2^{-1} I_\sigma^* - J_\sigma^*$.

Note that any colored box in Figure 2.2 lies in the outside of the N-tuple Young diagram π. It means that both vector spaces $t_1 t_2 J_\pi^* \otimes (V_\pi - V_\lambda)$ and $I_\pi^* \otimes (V_\pi - V_\lambda)$ contain no trivial representation, which leads both Euler classes not to vanish.

We also have

$$N_{\lambda,\sigma} - T_\sigma = (t_1 t_2 I_\sigma - J_\sigma) \otimes (V_\lambda^* - V_\sigma^*) - t_1 t_2 , \tag{2.3.10}$$

for $\lambda, \sigma \in \mathscr{Y}^{(N)}$, $|\lambda| = |\sigma| + 1$. Note that, for a given vector space E, the characters of the dual vector space E^* are given by

$$c_1(E^*) = -c_1(E) , \ \mathrm{eu}(E^*) = (-1)^{\dim E} \mathrm{eu}(E) . \tag{2.3.11}$$

We see that

$$\dim I_\sigma - \dim J_\sigma = N , \tag{2.3.12}$$

and then we have

$$\mathrm{eu}((t_1 t_2 I_\sigma - J_\sigma) \otimes (V_\lambda^* - V_\sigma^*)) = (-1)^N \mathrm{eu}((t_1^{-1} t_2^{-1} I_\sigma^* - J_\sigma^*) \otimes (V_\lambda - V_\sigma)) . \tag{2.3.13}$$

The corresponding figure is expressed in Figure 2.3. We see that $1 \not\subset t_1^{-1} t_2^{-1} I_\sigma^* \otimes (V_\lambda - V_\sigma)$ and $1 \not\subset J_\sigma^* \otimes (V_\lambda - V_\sigma)$ by a reasoning similar to the previous case. Then we have the simplified version; for $l \geq 0$ and $\lambda \in \mathscr{Y}^{(N)}$,

$$\begin{aligned}
D_{0,l+1}|\lambda\rangle &= \sum_{j=1}^{N} \sum_{s\in\lambda^{(j)}} \left(\frac{c_1(\chi_j^{-1} t_1^{x(s)} t_2^{y(s)})}{\varepsilon_1} \right)^l |\lambda\rangle \,, \\
D_{1,l}|\lambda\rangle &= \frac{\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2} \sum_{\substack{\pi\supset\lambda \\ |\pi|=|\lambda|+1}} \left(\frac{c_1(V_\pi^* - V_\lambda^*)}{\varepsilon_1} \right)^l \mathrm{eu}((t_1 t_2 J_\pi^* - I_\pi^*)\otimes(V_\pi - V_\lambda))|\pi\rangle \,, \\
D_{-1,l}|\lambda\rangle &= -\frac{1}{\varepsilon_1+\varepsilon_2} \sum_{\substack{\sigma\subset\lambda \\ |\sigma|=|\lambda|-1}} \left(\frac{c_1(V_\lambda^* - V_\sigma^*)}{\varepsilon_1} \right)^l \mathrm{eu}((t_1^{-1} t_2^{-1} I_\sigma^* - J_\sigma^*)\otimes(V_\lambda - V_\sigma))|\sigma\rangle \,.
\end{aligned} \tag{2.3.14}$$

The following formulae may be useful:

$$\begin{aligned}
(t_1 t_2 J_\pi^* - I_\pi^* - t_1 t_2 J_\lambda^* + I_\lambda^*)\otimes(V_\pi - V_\lambda) &= (1-t_1)(1-t_2)\,, \\
(t_1 t_2 J_\sigma^* - I_\sigma^* - t_1 t_2 J_\lambda^* + I_\lambda^*)\otimes(V_\lambda - V_\sigma) &= -(1-t_1)(1-t_2)\,.
\end{aligned} \tag{2.3.15}$$

We can rewrite (2.3.14) as follows:

$$\begin{aligned}
D_{1,l}|\lambda\rangle &= \sum_{\substack{\pi\supset\lambda \\ |\pi|=|\lambda|+1}} \left(\frac{c_1(V_\pi^* - V_\lambda^*)}{\varepsilon_1} \right)^l \mathrm{eu}((t_1 t_2 J_\lambda^* - I_\lambda^*)\otimes(V_\pi - V_\lambda)+1)|\pi\rangle \,, \\
D_{-1,l}|\lambda\rangle &= \frac{1}{\varepsilon_1\varepsilon_2} \sum_{\substack{\sigma\subset\lambda \\ |\sigma|=|\lambda|-1}} \left(\frac{c_1(V_\lambda^* - V_\sigma^*)}{\varepsilon_1} \right)^l \mathrm{eu}((t_1^{-1} t_2^{-1} I_\lambda^* - J_\lambda^*)\otimes(V_\lambda - V_\sigma)+1)|\sigma\rangle \,.
\end{aligned} \tag{2.3.16}$$

2.3.3 Commutation relations among $D_{0,l+1}, D_{1,l}$, and $D_{-1,l}$: I

A first step to investigate Lie algebras and their representations is to seek commutation relations in the algebras. Following it, let us consider commutation relations in SH$^{(N)}$.

It is clear from (2.3.6) that

$$[D_{0,l+1}, D_{0,k+1}] = 0\,,\ l,k \geq 0\,. \tag{2.3.17}$$

The basis $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}^{(N)}}$ is none other than the set of simultaneous eigenvectors with respect to commuting operators $D_{0,l+1}$, $l \geq 0$. It is also immediate from (2.3.6) that we have

$$\begin{aligned}
[D_{0,l+1}, D_{1,k}] &= D_{1,l+k}\,, \qquad l,k \geq 0\,, \\
[D_{-1,l}, D_{0,k+1}] &= D_{-1,l+k}\,, \qquad l,k \geq 0\,.
\end{aligned} \tag{2.3.18}$$

The above relations are like what we see with a Lie algebra, in which each commutator is given by a linear function of the underlining vector space of the Lie algebra. It is not the case, however, with the commutation relation between $D_{1,l}$ and $D_{-1,k}$.

Proposition 2.3.

$$[D_{-1,k}, D_{1,l}] = E_{k+l}\,,\ l,k \geq 0\,, \tag{2.3.19}$$

where the elements E_{k+l} are expressed by the set of commuting operators $D_{0,l+1}$ and are determined through the formula

$$1+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{l\geq 0}E_l\zeta^{l+1}=\exp\left(\sum_{l\geq 0}(-1)^{l+1}\frac{p_l(a_1,\ldots,a_N)}{\varepsilon_1^l}\phi_l(\zeta)\right)\exp\left(\sum_{l\geq 0}D_{0,l+1}\varphi_l(\zeta)\right). \quad (2.3.20)$$

Here

$$p_0(a_1,\ldots,a_N)=N\,,\ p_l(a_1,\ldots,a_N)=\sum_{j=1}^{N}a_j^l\,,\ l\geq 1\,, \quad (2.3.21)$$

are the power sum polynomials and

$$\phi_l(\zeta)=\zeta^l G_l\left(1+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\zeta\right),\ l\geq 0\,, \quad (2.3.22)$$

$$\varphi_l(\zeta)=\sum_{z=-1,-\frac{\varepsilon_2}{\varepsilon_1},1+\frac{\varepsilon_2}{\varepsilon_1}}\left(\zeta^l G_l(1+z\zeta)-\zeta^l G_l(1-z\zeta)\right),\ l\geq 0\,, \quad (2.3.23)$$

$$G_0(\zeta)=-\log(\zeta)\,,\ G_l(\zeta)=\frac{\zeta^{-l}-1}{l}\,,\ l\geq 1\,. \quad (2.3.24)$$

■

Here we present the proof of this proposition to obtain an intermediate result (2.3.39) used in section 2.5. First we prepare the following lemma.

Lemma 2.1. *For $l,k\geq 0$ and $\lambda\in\mathcal{Y}^{(N)}$, there is a coefficient $C_\lambda^{(l+k)}$ satisfying*

$$[D_{-1,k},D_{1.l}]|\lambda\rangle=C_\lambda^{(l+k)}|\lambda\rangle\,. \quad (2.3.25)$$

In other words, the commutator $[D_{-1,k},D_{1.l}]$ is diagonal with respect to the basis $\{|\lambda\rangle\}_{\lambda\in\mathcal{Y}^{(N)}}$. ■

Proof. Let $\lambda,\mu\in\mathcal{Y}^{(N)}$ be a pair of N-tuple Young diagrams which have the same level, $|\lambda|=|\mu|$, but are not identical, $\lambda\neq\mu$. Assume that there are two N-tuples $\pi,\sigma\in\mathcal{Y}^{(N)}$ such that

$$|\sigma|=|\lambda|-1,|\pi|=|\lambda|+1,\ \sigma\subset\lambda\subset\pi,\ \sigma\subset\mu\subset\pi\,. \quad (2.3.26)$$

The N-tuple π and σ are unique in this case. Note that we have

$$V_\lambda+V_\mu=V_\pi+V_\sigma\,, \quad (2.3.27)$$

and then $N_{\lambda,\pi}+N_{\mu,\pi}-T_\pi=N_{\lambda,\sigma}+N_{\mu,\sigma}-T_\sigma$ since

$$\begin{aligned}N_{\lambda,\pi}+N_{\mu,\pi}-T_\pi&=(-(1-t_1)(1-t_2)V_\pi^*+W^*)\otimes(V_\lambda+V_\mu-V_\pi)+t_1t_2V_\pi^*-2t_1t_2\\&=(-(1-t_1)(1-t_2)V_\pi^*+W^*)\otimes V_\sigma+t_1t_2V_\pi^*-2t_1t_2\,,\\N_{\lambda,\sigma}+N_{\mu,\sigma}-T_\sigma&=(-(1-t_1)(1-t_2)V_\sigma+t_1t_2W)\otimes(V_\lambda^*+V_\mu^*-V_\sigma^*)+W^*\otimes V_\sigma^*-2t_1t_2\\&=(-(1-t_1)(1-t_2)V_\pi^*+W^*)\otimes V_\sigma+t_1t_2V_\pi^*-2t_1t_2\,.\end{aligned} \quad (2.3.28)$$

Then $[D_{-1,k},D_{1,l}]$ is diagonal with respect to the basis $\{|\lambda\rangle\}_{\lambda\in\mathcal{Y}^{(N)}}$. □

Proof of the proposition. The coefficient $C_\lambda^{(l+k)}$ in the above lemma is given by

$$\begin{aligned} C_\lambda^{(l+k)} =& -\frac{\varepsilon_1\varepsilon_2}{(\varepsilon_1+\varepsilon_2)^2} \\ &\times\left[\sum_{\substack{\pi\supset\lambda\\|\pi|=|\lambda|+1}}\left(\frac{c_1(V_\pi^*-V_\lambda^*)}{\varepsilon_1}\right)^{l+k}\mathrm{eu}((t_1t_2J_\pi^*-I_\pi^*+t_1^{-1}t_2^{-1}I_\lambda^*-J_\lambda^*)\otimes(V_\pi-V_\lambda))\right. \\ &\left.-\sum_{\substack{\sigma\subset\lambda\\|\sigma|=|\lambda|-1}}\left(\frac{c_1(V_\lambda^*-V_\sigma^*)}{\varepsilon_1}\right)^{l+k}\mathrm{eu}((t_1t_2J_\lambda^*-I_\lambda^*+t_1^{-1}t_2^{-1}I_\sigma^*-J_\sigma^*)\otimes(V_\lambda-V_\sigma))\right] \end{aligned} \tag{2.3.29}$$

Here we decompose vector spaces I_λ and J_λ into 1-dimensional subspaces by the torus action and denote by $c_1(I_\lambda^*)=\sum_{i\in I}A_i$ and $c_1(J_\lambda^*)=\sum_{j\in J}B_j$ the corresponding summations of characters over the subspaces, respectively. Using the formula (2.3.15), we have

$$\begin{aligned} C_\lambda(\tilde\zeta)\equiv\sum_{n\geq0}C_\lambda^{(n)}\varepsilon_1^n\tilde\zeta^n =& -\sum_{j\in J}\frac{1}{1-\tilde\zeta B_j}\prod_{i\in I}\frac{B_j-A_i+\varepsilon_1+\varepsilon_2}{B_j-A_i}\prod_{k\in J\setminus\{j\}}\frac{B_j-B_k-\varepsilon_1-\varepsilon_2}{B_j-B_k} \\ &+\sum_{i\in I}\frac{1}{1-\tilde\zeta A_i}\prod_{k\in I\setminus\{i\}}\frac{A_i-A_k+\varepsilon_1+\varepsilon_2}{A_i-A_k}\prod_{j\in J}\frac{A_i-B_j-\varepsilon_1-\varepsilon_2}{A_i-B_j} \\ =&\frac{1}{\tilde\zeta(\varepsilon_1+\varepsilon_2)}\left[\prod_{i\in I}\frac{1-\tilde\zeta(A_i-\varepsilon_1-\varepsilon_2)}{1-\tilde\zeta A_i}\prod_{j\in J}\frac{1-\tilde\zeta(B_j+\varepsilon_1+\varepsilon_2)}{1-\tilde\zeta B_j}-1\right]. \end{aligned} \tag{2.3.30}$$

The last transformation can be deduced from contour integrations with respect to the parameter $\tilde\zeta$. Then we have

$$\begin{aligned} 1+\tilde\zeta(\varepsilon_1+\varepsilon_2)C_\lambda(\tilde\zeta)=&\prod_{i=1}^N\frac{1-\tilde\zeta(-a_i-\varepsilon_1-\varepsilon_2)}{1-\tilde\zeta(-a_i)} \\ &\times\prod_{s\in\lambda}\frac{(1-\tilde\zeta c_1(t_1t_2s))(1-\tilde\zeta c_1(t_1^{-1}s))(1-\tilde\zeta c_1(t_2^{-1}s))}{(1-\tilde\zeta c_1(t_1^{-1}t_2^{-1}s))(1-\tilde\zeta c_1(t_1s))(1-\tilde\zeta c_1(t_2s))}, \end{aligned} \tag{2.3.31}$$

where each s represents a 1-dimensional vector space $\chi_j^{-1}t_1^{x(s)}t_2^{y(s)}\subset V_\lambda^*$.Here we rewrite the product over I and J by the product over λ. This transformation can be understood by a diagrammatic operation like Figure 2.4. Replacing $\tilde\zeta$ by ζ/ε_1, we obtain

$$\begin{aligned} &\left[1+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{l\geq0}\zeta^{l+1}E_l\right]|\lambda\rangle \\ =&\left[\prod_{i=1}^N\frac{1-\frac{\zeta}{\varepsilon_1}(-a_i-\varepsilon_1-\varepsilon_2)}{1-\frac{\zeta}{\varepsilon_1}(-a_i)}\prod_{s\in\lambda}\frac{(1-\frac{\zeta}{\varepsilon_1}c_1(t_1t_2s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_1^{-1}s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_2^{-1}s))}{(1-\frac{\zeta}{\varepsilon_1}c_1(t_1^{-1}t_2^{-1}s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_1s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_2s))}\right]|\lambda\rangle\,. \end{aligned} \tag{2.3.32}$$

The formula

$$G_0(1+z\zeta)=-\log(1+z\zeta)=\sum_{l\geq1}(-1)^l\frac{z^l\zeta^l}{l}, \tag{2.3.33}$$

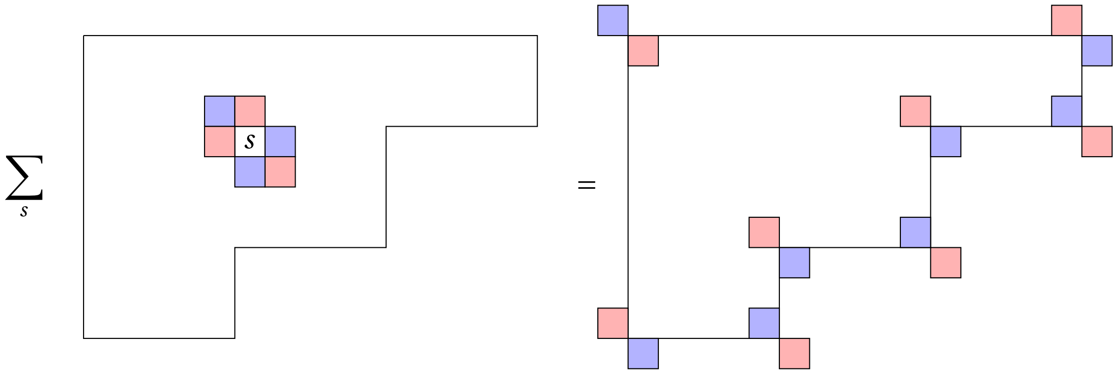


Figure 2.4: Diagrammatic expression of the transformation (2.3.31). The summation is taken over all the box s in the above Young diagram.

leads to

$$\begin{aligned}\sum_{l\geq 1} w^l \zeta^l G_l(1+z\zeta) &= \sum_{l\geq 1}\frac{1}{l}\left(\left(\frac{w\zeta}{1+z\zeta}\right)^l - w^l\zeta^l\right)\\ &= G_0(1+(z-w)\zeta) - G_0(1+z\zeta) - G_0(1-w\zeta)\,,\end{aligned} \tag{2.3.34}$$

and then we have

$$\sum_{l\geq 0} w^l\zeta^l G_l(1+z\zeta) = \log\left(\frac{1-\zeta w}{1-\zeta(w-z)}\right). \tag{2.3.35}$$

This formula shows

$$\begin{aligned}\sum_{l\geq 0}(-1)^{l+1}\left(\frac{w}{\varepsilon_1}\right)^l \phi_l(\zeta) &= \log\left(\frac{1-\frac{\zeta}{\varepsilon_1}(-w-\varepsilon_1-\varepsilon_2)}{1-\frac{\zeta}{\varepsilon_1}(-w)}\right),\\ \sum_{l\geq 0}\left(\frac{w}{\varepsilon_1}\right)^l \varphi_l(\zeta) &= \sum_{z=-\varepsilon_1,-\varepsilon_2,\varepsilon_1+\varepsilon_2}\log\left(\frac{1-\frac{\zeta}{\varepsilon_1}(w+z)}{1-\frac{\zeta}{\varepsilon_1}(w-z)}\right),\end{aligned} \tag{2.3.36}$$

and then we finally obtain

$$\begin{aligned}\exp\left(\sum_{l\geq 0}(-1)^{l+1}\frac{p_l(a_1,\ldots,a_N)}{\varepsilon_1^l}\phi_l(\zeta)\right) &= \prod_{i=1}^{N}\frac{1-\frac{\zeta}{\varepsilon_1}(-a_i-\varepsilon_1-\varepsilon_2)}{1-\frac{\zeta}{\varepsilon_1}(-a_i)}\,,\\ \exp\left(\sum_{l\geq 0}D_{0,l+1}\varphi_l(\zeta)\right)|\lambda\rangle &= \left[\prod_{s\in\lambda}\frac{(1-\frac{\zeta}{\varepsilon_1}c_1(t_1t_2s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_1^{-1}s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_2^{-1}s))}{(1-\frac{\zeta}{\varepsilon_1}c_1(t_1^{-1}t_2^{-1}s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_1s))(1-\frac{\zeta}{\varepsilon_1}c_1(t_2s))}\right]|\lambda\rangle\,.\end{aligned} \tag{2.3.37}$$

□

As a summary, we have the following commutation relations among $D_{0,l+1}, D_{1,l}$, and $D_{-1,l}$.

Proposition 2.4. *We have*

$$\begin{aligned}[D_{0,l+1}, D_{0,k+1}] &= 0\,,\\ [D_{0,l+1}, D_{1,k}] &= D_{1,l+k}\,,\\ [D_{-1,l}, D_{0,k+1}] &= D_{-1,l+k}\,,\\ [D_{-1,k}, D_{1,l}] &= E_{k+l}\,,\end{aligned} \tag{2.3.38}$$

for $l,k\geq 0$, where E_l, $l\geq 0$, is expressed by $D_{0,k+1}$, $k\geq 0$, through the formula (2.3.20).

In the above proof, we also obtain the following formula (2.3.30), which is useful to obtain eigenvalues of E_l, given by

$$\left[1+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{l\geq 0}E_l\zeta^{l+1}\right]|\lambda\rangle=\left[\prod_{s\subset I^*_\lambda}\frac{1-\frac{\zeta}{\varepsilon_1}c_1(t_1^{-1}t_2^{-1}s)}{1-\frac{\zeta}{\varepsilon_1}c_1(s)}\prod_{s\subset J^*_\lambda}\frac{1-\frac{\zeta}{\varepsilon_1}c_1(t_1t_2s)}{1-\frac{\zeta}{\varepsilon_1}c_1(s)}\right]|\lambda\rangle\,, \tag{2.3.39}$$

where each s represents a 1-dimensional subspace $\chi_j^{-1}t_1^{x(s)}t_2^{y(s)}$ in I^*_λ and in J^*_λ, respectively.

We repeat that E_l is expressed generally by a polynomial of $D_{0,k+1}$ and not by a linear function. In this sense, the algebra SH$^{(N)}$ is called nonlinear.

2.3.4 Commutation relations among $D_{0,l+1}$, $D_{1,l}$, and $D_{-1,l}$: II

There exist other relations in SH$^{(N)}$, given as follows.

Proposition 2.5. *1. We have*

$$[D_{1,0},[D_{1,0},D_{1,1}]]=[D_{-1,0},[D_{-1,0},D_{-1,1}]]=0\,, \tag{2.3.40}$$

2. For $l,k\geq 0$, we have

$$\begin{aligned}&3[D_{1,l+2},D_{1,k+1}]-3[D_{1,l+1},D_{1,k+2}]-[D_{1,l+3},D_{1,k}]+[D_{1,l},D_{1,k+3}]+[D_{1,l+1},D_{1,k}]\\&-[D_{1,l},D_{1,k+1}]+\frac{\varepsilon_2}{\varepsilon_1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\left(-D_{1,l}D_{1,k}-D_{1,k}D_{1,l}+[D_{1,l+1},D_{1,k}]-[D_{1,l},D_{1,k+1}]\right)=0\,,\\[1ex]&3[D_{-1,l+2},D_{-1,k+1}]-3[D_{-1,l+1},D_{-1,k+2}]-[D_{-1,l+3},D_{-1,k}]+[D_{-1,l},D_{-1,k+3}]\\&+[D_{-1,l+1},D_{-1,k}]-[D_{-1,l},D_{-1,k+1}]\\&+\frac{\varepsilon_2}{\varepsilon_1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\left(D_{-1,l}D_{-1,k}+D_{-1,k}D_{-1,l}+[D_{-1,l+1},D_{-1,k}]-[D_{-1,l},D_{-1,k+1}]\right)=0\,,\end{aligned} \tag{2.3.41}$$

■

Note that the latter relations (2.3.41) are summarized as follows. Define

$$D_{\pm 1}(z)=\sum_{l\geq 0}D_{\pm 1,l}z^{-l}\,, \tag{2.3.42}$$

and set

$$\kappa(\zeta)=(\zeta+1)\left(\zeta+\frac{\varepsilon_2}{\varepsilon_1}\right)\left(\zeta-\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right)\,. \tag{2.3.43}$$

Then the left parts of the relations (2.3.41) are the coefficient in front of $z^{-l}w^{-k}$ in

$$\begin{gathered}\kappa(w-z)D_1(z)D_1(w)+\kappa(z-w)D_1(w)D_1(z)\,,\\\kappa(w-z)D_{-1}(w)D_{-1}(z)+\kappa(z-w)D_{-1}(z)D_{-1}(w)\,,\end{gathered} \tag{2.3.44}$$

respectively, for $l,k\geq 0$. We should note that the above series contain $z^{-l}w^{-k}$ parts with $l<0$ or $k<0$ and we do *not* say that the series themselves vanish. We only say that (2.3.41) corresponds to the coefficients in front of the convergent parts as $z,w\to\infty$.

The $D_{1,l}$ part of the above proposition is proven by machinery computations after the following lemma:

Lemma 2.2. *For $\lambda \in \mathscr{Y}^{(N)}$ and $l_1, l_2, \cdots, l_n$, we have*

$$D_{1,l_n} D_{1,l_{n-1}} \cdots D_{1,l_1} |\lambda\rangle = (\varepsilon_1\varepsilon_2)^n \sum_{\substack{\pi \supset \lambda \\ |\pi| = |\lambda| + n}} \left(\sum_{w \in \mathfrak{S}_n} \mathrm{eu}\left(-(1-t_1)(1-t_2) \sum_{i>j}^{n} s_{w(i)} s_{w(j)}^{-1} \right) \prod_{i=1}^{n} \left(\frac{c_1\left(s_{w(i)}\right)}{\varepsilon_1} \right)^{l_i} \right) \times \mathrm{eu}\Big(((1-t_1)(1-t_2)V_\pi^* - W^*) \otimes (V_\pi - V_\lambda) - n t_1 t_2 \Big) |\pi\rangle \,, \tag{2.3.45}$$

where $V_\pi^* - V_\lambda^* = \sum_{i=1}^n s_i$ represents a decomposition of $V_\pi^* - V_\lambda^*$ by the torus action.

We give a proof in appendix A. The $D_{-1,l}$ part can be shown similarly. The corresponding lemma is as follows:

Lemma 2.3. *For $\lambda \in \mathscr{Y}^{(N)}$ and $l_1, l_2, \cdots, l_n$, we have*

$$D_{-1,l_1} D_{-1,l_2} \cdots D_{-1,l_n} |\lambda\rangle = (-1)^n \sum_{\substack{\sigma \subset \lambda \\ |\sigma| = |\lambda| - n}} \left(\sum_{w \in \mathfrak{S}_n} \mathrm{eu}\left(-(1-t_1)(1-t_2) \sum_{i>j}^{n} s_{w(i)} s_{w(j)}^{-1} \right) \prod_{i=1}^{n} \left(\frac{c_1\left(s_{w(i)}\right)}{\varepsilon_1} \right)^{l_i} \right) \times \mathrm{eu}\Big((t_1^{-1} t_2^{-1} W^* - (1-t_1^{-1})(1-t_2^{-1}) V_\sigma^*) \otimes (V_\lambda - V_\sigma) - n t_1 t_2 \Big) |\sigma\rangle \,, \tag{2.3.46}$$

where $V_\lambda^* - V_\sigma^* = \sum_{i=1}^n s_i$ represents a decomposition of $V_\lambda^* - V_\sigma^*$ by the torus action.

2.4 The algebra SH^c

Here we introduce an abstract algebra SH^c which has the commutation relations which appeared in section 2.3 as the whole defining relations.

2.4.1 The algebra SH^c

Definition 2.2. The algebra SH^c is an algebra generated by elements $\{\tilde{D}_{0,l+1} \mid l \geq 0\}$, $\{\tilde{D}_{\pm 1,l} \mid l \geq 0\}$ and $\{\tilde{c}_l \mid l \geq 0\}$, subject to the following set of relations:

$$\left[\tilde{D}_{0,l+1}, \tilde{D}_{0,k+1}\right] = 0\,,\ \left[\tilde{D}_{0,l+1}, \tilde{D}_{1,k}\right] = \tilde{D}_{1,l+k}\,,\ \left[\tilde{D}_{0,l+1}, \tilde{D}_{-1,k}\right] = -\tilde{D}_{-1,l+k}\,, \left[\tilde{D}_{-1,k}, \tilde{D}_{1,l}\right] = \tilde{E}_{k+l}\,, \tag{2.4.1}$$

$$\left[\tilde{D}_{1,0}, \left[\tilde{D}_{1,0}, \tilde{D}_{1,1}\right]\right] = \left[\tilde{D}_{-1,0}, \left[\tilde{D}_{-1,0}, \tilde{D}_{-1,1}\right]\right] = 0\,, \tag{2.4.2}$$

and

$$\begin{aligned} &3[\tilde{D}_{1,l+2}, \tilde{D}_{1,k+1}] - 3[\tilde{D}_{1,l+1}, \tilde{D}_{1,k+2}] - [\tilde{D}_{1,l+3}, \tilde{D}_{1,k}] + [\tilde{D}_{1,l}, \tilde{D}_{1,k+3}] + [\tilde{D}_{1,l+1}, \tilde{D}_{1,k}] \\ &- [\tilde{D}_{1,l}, \tilde{D}_{1,k+1}] + \frac{\varepsilon_2}{\varepsilon_1}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\Big(-\tilde{D}_{1,l}\tilde{D}_{1,k} - \tilde{D}_{1,k}\tilde{D}_{1,l} + [\tilde{D}_{1,l+1}, \tilde{D}_{1,k}] - [\tilde{D}_{1,l}, \tilde{D}_{1,k+1}]\Big) = 0\,, \\ &3[\tilde{D}_{-1,l+2}, \tilde{D}_{-1,k+1}] - 3[\tilde{D}_{-1,l+1}, \tilde{D}_{-1,k+2}] - [\tilde{D}_{-1,l+3}, \tilde{D}_{-1,k}] + [\tilde{D}_{-1,l}, \tilde{D}_{-1,k+3}] \\ &+ [\tilde{D}_{-1,l+1}, \tilde{D}_{-1,k}] - [\tilde{D}_{-1,l}, \tilde{D}_{-1,k+1}] \\ &+ \frac{\varepsilon_2}{\varepsilon_1}\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right)\Big(\tilde{D}_{-1,l}\tilde{D}_{-1,k} + \tilde{D}_{-1,k}\tilde{D}_{-1,l} + [\tilde{D}_{-1,l+1}, \tilde{D}_{-1,k}] - [\tilde{D}_{-1,l}, \tilde{D}_{-1,k+1}]\Big) = 0\,, \end{aligned} \tag{2.4.3}$$

for $l, k \geq 0$, where the operator $\tilde{E}_l$ is defined through the equation

$$1+\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{l\geq 0}\tilde{E}_l\zeta^{l+1}=\exp\left(\sum_{l\geq 0}(-1)^{l+1}\tilde{c}_l\phi_l(\zeta)\right)\exp\left(\sum_{l\geq 0}\tilde{D}_{0,l+1}\varphi_l(\zeta)\right). \tag{2.4.4}$$

■

The algebra SH^c was originally introduced as a central extension of the algebra SH defined as a certain limit $n\to\infty$ of the degenerate double affine Hecke algebra SH$_n$ of GL$(n,\mathbb{C})$ in [2], and was shown to be equivalent to the above "generator-and-relation" definition [47]. The original construction is so powerful that one can equip SH^c with a Hopf algebra structure. The comultiplication map is crucial to embed SH^c into the free field system of N bosons and to show an equivalence between a certain class of representations of SH$^{(N)}$ and of a W-algebra.

2.4.2 Relation to SH$^{(N)}$

It is clear that the assignment

$$\mathrm{SH}^c\to\mathrm{SH}^{(N)},\ \tilde{D}_x\mapsto D_x,\ \tilde{c}_l\mapsto p_l(a_1,\cdots,a_N)/\varepsilon_1^l, \tag{2.4.5}$$

defines a representation of SH^c on $L^{(N)}$. Moreover, denoting by SH$^{c(N)}$ the specialization[†3] of SH^c to $\tilde{c}_l=p_l(a_1,\cdots,a_N)/\varepsilon_1^l$, $l\geq 0$, the following correspondence is known:

Proposition 2.6 ([2, 47])**.** *After the specialization to $\tilde{c}_l=p_l(a_1,\cdots,a_N)/\varepsilon_1^l$ for $l\geq 0$, the assignment $\tilde{D}_x\mapsto D_x$ induces an isomorphism of algebras*

$$\mathrm{SH}^{c(N)}\simeq\mathrm{SH}^{(N)}. \tag{2.4.6}$$

■

This proposition means that the commutation relations obtained above are sufficient to determine the algebra SH$^{(N)}$.

2.5 Relation between SH$^{(N)}$ and W-algebra

In this section, we introduce the relation of SH$^{(N)}$ with a W-algebra. It is based on that SH$^{(N)}$ has a free field representation of N bosons. This representation gives an embedding into a W-algebra consisting of spin $1,2,\cdots,N$ currents and leads to an equivalence between representations of two algebras. It means that there is an action of the W-algebra on $L^{(N)}$, which is a version of the AGT conjecture. Conversely, a representation of the W-algebra can be understood from some explicit calculations with the action of SH$^{(N)}$ on $L^{(N)}$.

We start by fixing notations with W-algebras. We next see an equivalence between the Virasoro–Heisenberg subalgebra of both algebras, and then introduce a certain equivalence between the whole algebras. We should note that we only give a user guide how to use the equivalence. See [2] for the readers who need its detailed proof.

[†3] In other words, we define SH$^{c(N)}$ as the image of the representation $\tilde{D}_x\mapsto\tilde{D}_x$, $\tilde{c}_l\mapsto p_l(a_1,\cdots,a_N)/\varepsilon_1^l$.

2.5.1 The W-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ in the system of free N bosons

Let us recall the free field system of N bosons and fix our notations associated with a W-algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ by following [4, 8, 11]. Let $\vec{\varphi}=(\varphi^{(1)},\cdots,\varphi^{(N)})$ be an N-component free massless scalar chiral field on the 2-dimensional flat spacetime. The correlation function of this scalar field is given by

$$\langle\varphi^{(i)}(z)\varphi^{(j)}(z')\rangle=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\delta_{ij}\log(z-z')\,. \tag{2.5.1}$$

Then a set of operators

$$\partial\varphi^{(i)}(z)=\sum_{l\in\mathbb{Z}}\alpha_l^{(i)}z^{-l-1}\,,\ 1\le i\le N\,, \tag{2.5.2}$$

satisfies the following commutation relations

$$[\alpha_l^{(i)},\alpha_k^{(j)}]=l\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\delta_{ij}\delta_{l+k,0}\,,\ l,k\in\mathbb{Z},\ 1\le i,j\le N\,. \tag{2.5.3}$$

We denote by $\mathscr{H}^{(N)}$ the algebra generated by all such $\alpha_l^{(i)}$. For a given vector $\vec{\Lambda}=(\Lambda^{(1)},\cdots,\Lambda^{(N)})$, the Fock space $M_{\vec{\Lambda}}^{(N)}$ is defined as the representation space of the algebra $\mathscr{H}^{(N)}$ with the vacuum state, an element $|\vec{\Lambda}\rangle\in M_{\vec{\Lambda}}^{(N)}$ satisfying

$$\begin{aligned}\alpha_l^{(i)}|\vec{\Lambda}\rangle&=0\,, && l>0\,,\\ \alpha_0^{(i)}|\vec{\Lambda}\rangle&=\Lambda^{(i)}|\vec{\Lambda}\rangle\,, && 1\le i\le N\,.\end{aligned} \tag{2.5.4}$$

We see that all the elements

$$\begin{aligned}(\alpha_{l_{1,1}}^{(1)}\cdots\alpha_{l_{1,i_1}}^{(1)})(\alpha_{l_{2,1}}^{(2)}\cdots\alpha_{l_{2,i_2}}^{(2)})\cdots(\alpha_{l_{N,1}}^{(N)}\cdots\alpha_{l_{N,i_N}}^{(N)})|\vec{\Lambda}\rangle\,,\ l_{j,1}\le\cdots\le l_{j,i_j}<0\,,\\ i_j\ge0,1\le j\le N\,,\end{aligned} \tag{2.5.5}$$

forms a basis of $M_{\vec{\Lambda}}^{(N)}$. It is clear that the action of $\mathscr{H}^{(N)}$ on $M_{\vec{\Lambda}}^{(N)}$ is irreducible. We set $\alpha_l^{(i)}$ in degree l for each $1\le i\le N$. The Fock space is then graded by the degree.

We can embed a Virasoro algebra into the free field system as follows. Consider a Virasoro algebra with the basis H_l, $l\in\mathbb{Z}$ and commutation relations

$$[H_l,H_k]=(l-k)H_{l+k}+\frac{C_N}{12}l(l^2-1)\delta_{l+k,0}\,, \tag{2.5.6}$$

with the central charge C_N is given by

$$C_N=N-12\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)Q^2\vec{\rho}^2=1+(N-1)\left(1-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2N(N+1)\right). \tag{2.5.7}$$

Here we write

$$Q=-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right), \tag{2.5.8}$$

and denote by $\vec{\rho}$ the Weyl vector of $\mathfrak{su}(N)$,

$$\vec{\rho}=\left(\frac{N-1}{2},\ \frac{N-3}{2},\ \cdots,\ -\frac{N-3}{2},\ -\frac{N-1}{2}\right)=\sum_{i=1}^{N}\frac{N+1-2i}{2}\vec{e}_i\,, \tag{2.5.9}$$

where $(\vec{e}_j)_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$. The Virasoro algebra acts on $M_{\vec{\Lambda}}^{(N)}$ as

$$H(z)=\sum_{l\in\mathbb{Z}}H_l z^{-l-2}=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left[\frac{1}{2}\sum_{i=1}^{N}:\partial\varphi^{(i)}(z)^2:+Q\vec{\rho}\cdot\partial^2\vec{\varphi}(z)\right], \tag{2.5.10}$$

where denote by : : the normal ordering of creation operators $\alpha_l^{(i)}$ with $l\le 0$, and annihilation ones with $l>0$. We often call the expression (2.5.10) the free field representation of the Virasoro algebra, which means that the field $H(z)$ defining the Virasoro algebra is expressed in terms of $\mathscr{H}^{(N)}$ and the normal ordering.

As an extension of the Virasoro algebra, we construct a W-algebra in End($M_{\vec{\Lambda}}^{(N)}$) as follows. For $1\le i\le N$, let $\vec{h}_i$ be the ith weight of the defining representation of $\mathfrak{su}(N)$, which is given explicitly as

$$\vec{h}_i=\vec{e}_i-\frac{1}{N}\sum_{j=1}^{N}e_j\,,\ 1\le i\le N\,. \tag{2.5.11}$$

Note that we have

$$\sum_{i=1}^{N}\vec{h}_i=0\,,\ \vec{h}_i\cdot\vec{h}_j=\delta_{ij}-\frac{1}{N}\,,\ \sum_{i=1}^{N}i\vec{h}_i=-\vec{\rho}\,,\vec{\rho}\cdot\vec{h}_i=\frac{N+1-2i}{2}\,, \tag{2.5.12}$$

for $1\le i,j\le N$. We define fields $W_j(z)=\sum_{l\in\mathbb{Z}}W_{j,l}z^{-l-j}$ for $2\le j\le N$ by the so-called quantum Miura transformation

$$\begin{aligned}&\sum_{j=2}^{N}W_j(z)(Q\partial)^{N-j}\\&\quad=-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left[:(Q\partial+\vec{h}_1\cdot\partial\vec{\varphi})(Q\partial+\vec{h}_2\cdot\partial\vec{\varphi})\cdots(Q\partial+\vec{h}_N\cdot\partial\vec{\varphi}):-(Q\partial)^N\right],\end{aligned} \tag{2.5.13}$$

and also define a field $\alpha(z)=\sum_{l\in\mathbb{Z}}\alpha_l z^{-l-1}$ by

$$\alpha(z)=\sum_{i=1}^{N}\alpha^{(i)}(z)\,. \tag{2.5.14}$$

In particular, we have

$$\begin{aligned}W_2&=-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left[\sum_{i<j}^{N}:(\vec{h}_i\cdot\partial\vec{\varphi})(\vec{h}_j\cdot\partial\vec{\varphi}):+Q\sum_{i=1}^{N}(i-1)\vec{h}_i\partial^2\vec{\varphi}\right]\\&=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left[\frac{1}{2}\sum_{i=1}^{N}:\partial\varphi^{(i)2}:+Q\vec{\rho}\cdot\partial^2\vec{\varphi}-\frac{1}{2N}:\alpha^2:\right].\end{aligned} \tag{2.5.15}$$

We see that $W_{i,l}, \alpha_l$ are in degree l. We denote by $\mathscr{W}_{N+\mathrm{U}(1)}$ the graded subalgebra in End$(M_{\vec{\Lambda}}^{(N)})$ generated by the fields $W_2, W_3, \cdots, W_N$ and α.

We have

$$H(z) = W_2(z) + \frac{1}{2N}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right) : \alpha(z)^2 :, \tag{2.5.16}$$

and then the algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ contains the Virasoro subalgebra generated by the field $H(z)$. Note that the field $\alpha(z)$ generates a Heisenberg algebra

$$[\alpha_l, \alpha_k] = l \tilde{K}_N \delta_{l+k,0}, \, l, k \in \mathbb{Z}, \tag{2.5.17}$$

where the level K_N is given by

$$K_N = N\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}. \tag{2.5.18}$$

The commutation relations between the two fields $H(z)$ and $\alpha(z)$ are given by

$$[H_l, \alpha_k] = -k\alpha_{k+l}, \; l, k \in \mathbb{Z}. \tag{2.5.19}$$

Therefore the algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ contains the Virasoro–Heisenberg subalgebra Vir $\otimes$ U$(1)_N$ generated by the fields $H(z)$ and $\alpha(z)$.

The vacuum state $|\vec{\Lambda}\rangle$ also defines the action of the graded algebra $\mathscr{W}_{N+\mathrm{U}(1)}$ on a vector space defined by

$$V_{\vec{\Lambda}}^{(N)} = \mathscr{W}_{N+\mathrm{U}(1)} |\vec{\Lambda}\rangle \,. \tag{2.5.20}$$

We give a gradation of $V_{\vec{\Lambda}}^{(N)}$ by setting the vacuum state in degree 0. Note that operators α_l and $W_{j,l}$ for $j \geq 2$, with $l > 0$ kill the vacuum state since annihilation operators appear at their right ends. In other words, the degree function in $V_{\vec{\Lambda}}^{(N)}$ is bounded above. Such a graded vector space whose degree is bounded above is called an admissible vector space. A similar consideration indicates that the vacuum vector becomes a simultaneous eigenvector of the operators α_0 and $W_{j,0}$ for $j \geq 2$. We denote the eigenvalues, called the weights of $V_{\vec{\Lambda}}^{(N)}$, by $w_i(\vec{\Lambda})$ for $1 \leq i \leq N$:

$$\begin{aligned} \alpha_0 |\vec{\Lambda}\rangle &= w_1(\vec{\Lambda})|\vec{\Lambda}\rangle \,, \\ W_{j,0} |\vec{\Lambda}\rangle &= w_j(\vec{\Lambda})|\vec{\Lambda}\rangle \,, \; j \geq 2 \,. \end{aligned} \tag{2.5.21}$$

The relation (2.5.13) leads to

$$\sum_{j=2}^{N} w_j(\vec{\Lambda}) z^{-j} (Q\partial)^{N-j} = -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left[(Q\partial + z^{-1}\vec{h}_1 \cdot \vec{\Lambda}) \cdots (Q\partial + z^{-1}\vec{h}_N \cdot \vec{\Lambda}) - (Q\partial)^N\right], \tag{2.5.22}$$

and its action to z^k gives

$$\sum_{j \geq N-k}^{N} \frac{k!}{(k+j-N)!} Q^{N-j} w_j(\vec{\Lambda}) = -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right) \prod_{i=1}^{N} ((k+i-N)Q + \vec{h}_i \cdot \vec{\Lambda}). \tag{2.5.23}$$

As a result, we have

$$\begin{aligned} w_1(\vec{\Lambda}) &= \sum_{i=1}^{N} \Lambda^{(i)}, \\ w_j(\vec{\Lambda}) &= -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right) \sum_{i_1<i_2<\cdots<i_j}^{N} \prod_{m=1}^{j} (\vec{h}_{i_m}\cdot\vec{\Lambda} - Q(j-m)), \, j \geq 2. \end{aligned} \tag{2.5.24}$$

We summarize the above properties:

Proposition 2.7. *The representation of $\mathscr{W}_{N+\mathrm{U}(1)}$ on $V_{\vec{\Lambda}}^{(N)}$ satisfies*

$$\begin{aligned} &\alpha_l\,|\vec{\Lambda}\rangle = W_{j,l}\,|\vec{\Lambda}\rangle = 0\,, l>0\,, \\ &\alpha_0\,|\vec{\Lambda}\rangle = w_1(\vec{\Lambda})|\vec{\Lambda}\rangle\;,\; W_{j,0}|\vec{\Lambda}\rangle = w_j(\vec{\Lambda})|\vec{\Lambda}\rangle\;,\; j\geq 2 \\ &w_1(\vec{\Lambda}) = \sum_{i=1}^{N}\Lambda^{(i)},\; w_j(\vec{\Lambda}) = -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{i_1<i_2<\cdots<i_j}^{N}\prod_{m=1}^{j}(\vec{h}_{i_m}\cdot\vec{\Lambda}-Q(j-m)),\; j\geq 2. \end{aligned} \tag{2.5.25}$$

■

The free field expression (2.5.10) of the field $H(z)$ shows that

$$H_0\,|\Lambda\rangle = \Delta(\vec{\Lambda},Q)\,|\Lambda\rangle\;,\;\Delta(\vec{\Lambda},Q) = \frac{1}{2}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\vec{\Lambda}\cdot(\vec{\Lambda}-2Q\vec{\rho}). \tag{2.5.26}$$

2.5.2 The Virasoro–Heisenberg subalgebra in SH^c

Before we introduce the free field representation of SH$^{(N)}$, we first focus on a Virasoro–Heisenberg subalgebra in SH^c. For $r,l\geq 1$, we recursively define a set of operators in SH^c

$$\begin{aligned} \tilde{D}_{r+1,0} &= \frac{1}{r}[\tilde{D}_{1,1},\tilde{D}_{r,0}],\; \tilde{D}_{-r-1,0} = \frac{1}{r}[\tilde{D}_{-r,0},\tilde{D}_{-1,1}], \qquad r\geq 1\,, \\ \tilde{D}_{r,l} &= [\tilde{D}_{0,l+1},\tilde{D}_{r,0}],\; \tilde{D}_{-r,l} = [\tilde{D}_{-r,0},\tilde{D}_{0,l+1}], \qquad r,l\geq 1\,, \end{aligned} \tag{2.5.27}$$

and also define for $l\geq 1$

$$\begin{aligned} \tilde{\alpha}_l &= \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-l}\tilde{D}_{-l,0}\,,\; \tilde{\alpha}_{-l} = \tilde{D}_{l,0}\,,\; \tilde{\alpha}_0 = \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\tilde{E}_1\,, \\ \tilde{H}_l &= \frac{1}{l}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-l}\tilde{D}_{-l,1} + \frac{1-l}{2}\tilde{c}_0\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\tilde{\alpha}_l\,, \\ \tilde{H}_{-l} &= \frac{1}{l}\tilde{D}_{l,1} + \frac{1-l}{2}\tilde{c}_0\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\tilde{\alpha}_{-l}\,, \\ \tilde{H}_0 &= \frac{1}{2}[\tilde{H}_1,\tilde{H}_{-1}] = \frac{1}{2}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\tilde{E}_2\,. \end{aligned} \tag{2.5.28}$$

Note that the formula (2.4.4) leads to

$$\begin{aligned} \tilde{E}_0 &= \tilde{c}_0\,, \\ \tilde{E}_1 &= -\tilde{c}_1 + \frac{1}{2}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\tilde{c}_0(\tilde{c}_0-1), \\ \tilde{E}_2 &= \tilde{c}_2 + \left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\tilde{c}_1(1-\tilde{c}_0) + \frac{1}{6}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2\tilde{c}_0(\tilde{c}_0-1)(\tilde{c}_0-2) + 2\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\tilde{D}_{0,1}\,. \end{aligned} \tag{2.5.29}$$

These elements are known to form a Virasoro–Heisenberg subalgebra.

Proposition 2.8 ([38])**.** *We have the following commutation relations*

$$\begin{aligned}
&[\tilde{\alpha}_l,\tilde{\alpha}_k]=l\tilde{K}\delta_{l,-k}\,,\\
&[\tilde{H}_l,\tilde{\alpha}_k]=-k\tilde{\alpha}_{k+l}\,,\\
&[\tilde{H}_l,\tilde{H}_k]=(l-k)\tilde{H}_{l-k}+\frac{\tilde{C}}{12}(l^3-l)\delta_{l,-k}\,,
\end{aligned}\tag{2.5.30}$$

for $l,k\in\mathbb{Z}$, where

$$\begin{aligned}
&\tilde{K}=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\tilde{c}_0\,,\\
&\tilde{C}=1+(\tilde{c}_0-1)\left(1-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2\tilde{c}_0(\tilde{c}_0+1)\right).
\end{aligned}\tag{2.5.31}$$

■

The map $\mathrm{SH}^c\to\mathrm{SH}^{(N)}$ induces a Virasoro–Heisenberg subalgebra in $\mathrm{SH}^{(N)}$ whose central charge $\tilde{C}_N$ and level $\tilde{K}_N$ are given by

$$\begin{aligned}
&\tilde{C}_N=1+(N-1)\left(1-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2N(N+1)\right)=C_N\,,\\
&\tilde{K}_N=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}N=K_N\,.
\end{aligned}\tag{2.5.32}$$

These values are none other than what we have obtained from the Virasoro–Heisenberg subalgebra $\mathrm{Vir}\otimes\mathrm{U}(1)_N$ in $\mathscr{W}_{N+\mathrm{U}(1)}$. The assignment

$$\tilde{\alpha}_l\mapsto\alpha_l\,,\tilde{H}_l\mapsto H_l\,,\ l\in\mathbb{Z}\,.\tag{2.5.33}$$

then gives a free field representation of the Virasoro–Heisenberg subalgebra in $\mathrm{SH}^{(N)}$. In particular, the map takes $D_{\pm1,0}\in\mathrm{SH}^{(N)}$ to

$$D_{1,0}\mapsto\alpha_{-1}=\sum_{i=1}^N\alpha_{-1}^{(i)}\,,\ D_{-1,0}\mapsto\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\alpha_1=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{i=1}^N\alpha_1^{(i)}\,.\tag{2.5.34}$$

Note that the formula (2.5.29) tells the eigenvalues of the vector $|\emptyset\rangle\in L^{(N)}$ is expressed by

$$\begin{aligned}
&\tilde{\alpha}_0|\emptyset\rangle=\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left[-\sum_{j=1}^N\frac{a_j}{\varepsilon_1}+\frac{N(N-1)}{2}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right]|\emptyset\rangle\,,\\
&\tilde{H}_0|\emptyset\rangle=\frac{1}{2}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left[\sum_{j=1}^N\left(\frac{a_j}{\varepsilon_1}\right)^2-(N-1)\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{j=1}^N\frac{a_j}{\varepsilon_1}+\frac{N(N-1)(N-2)}{6}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2\right]|\emptyset\rangle\,.
\end{aligned}\tag{2.5.35}$$

These eigenvalues are identical to (2.5.24) and (2.5.26) when we fix

$$\Lambda^{(i)}=-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left[\frac{a_i}{\varepsilon_1}-(i-1)\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right]=\frac{a_i}{\varepsilon_2}-Q(i-1),\ 1\le i\le N\,.\tag{2.5.36}$$

Under the identification (2.5.36), the two modules of the Virasoro–Heisenberg algebra generated by $|\vec{\Lambda}\rangle$ and $|\emptyset\rangle\in L^{(N)}$ are isomorphic by the map $|\emptyset\rangle\mapsto|\vec{\Lambda}\rangle$ and the assignment (2.5.33). We draw Figure 2.5 to summarize the above identifications. Here we denote by $V^{(N)}$ the vector space $V_{\vec{\Lambda}}^{(N)}$ at (2.5.36).

$$\begin{array}{ccccccc} \mathrm{SH}^{(N)} & \supset \mathrm{Vir}\otimes \mathrm{U}(1)_N & \curvearrowright & |\emptyset\rangle & \in & L^{(N)} \\ & \wr\!\mid & & \wr\!\mid & & \\ \mathscr{W}_{N+\mathrm{U}(1)} & \supset \mathrm{Vir}\otimes \mathrm{U}(1)_N & \curvearrowright & |\vec{\Lambda}\rangle & \in & V^{(N)} \end{array}$$

Figure 2.5: The identification of two Virasoro–Heisenberg subalgebras. The vector $\vec{\Lambda}$ is fixed to (2.5.36).

2.5.3 The free field representation of SH$^{(N)}$

We have seen the isomorphism between two modules of the Virasoro–Heisenberg subalgebra generated by $|\vec{\Lambda}\rangle$ and $|\emptyset\rangle$. It was shown that one can extend it to the equivalence between the admissible modules[†4] over the whole algebras.

Note that the extention is specified by the free field representation of $D_{0,2} \in \mathrm{SH}^{(N)}$ since the algebra SH$^{(N)}$ is generated by $D_{\pm1,0}$ and $D_{0,2}$. We assign

$$D_{0,2} = \sum_{i=1}^{N} D_{0,2}^{(i)} + \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{i<j}^{N}\sum_{l\geq 1} l\,\alpha_{-l}^{(j)}\alpha_l^{(i)}, \tag{2.5.37}$$

where the elements $D_{0,2}^{(i)}$ for $1 \leq i \leq N$ are given by

$$\begin{aligned} D_{0,2}^{(i)} = \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\Bigg[&\sum_{l\geq 1}\left(-\frac{a_i}{\varepsilon_1} + \frac{l-1}{2}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right)\alpha_{-l}^{(i)}\alpha_l^{(i)} \\ &+ \frac{1}{2}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)\sum_{l,k\geq 1}\left(\alpha_{-l-k}^{(i)}\alpha_l^{(i)}\alpha_k^{(i)} + \alpha_{-l}^{(i)}\alpha_{-k}^{(i)}\alpha_{l+k}^{(i)}\right)\Bigg]. \end{aligned} \tag{2.5.38}$$

It is quite important that the assignments (2.5.34), (2.5.37) actually give an embedding of graded algebras from SH$^{(N)}$ *to* $\mathscr{W}_{N+\mathrm{U}(1)}$*, while the sign of the degree is flipped!* Then the admissible module $V^{(N)}$ over $\mathscr{W}_{N+\mathrm{U}(1)}$ can be considered as an admissible module over SH$^{(N)}$. Moreover, this embedding is *essentially surjective*, which means that the admissible module over SH$^{(N)}$ can be also considered as an admissible module over $\mathscr{W}_{N+\mathrm{U}(1)}$. The mapping between admissible representations of two algebras has the following properties:

Proposition 2.9 ([2]). 1. *The free field embedding of* SH$^{(N)}$ *into* $\mathscr{W}_{N+\mathrm{U}(1)}$ *induces a one-to-one correspondence between admissible modules over* SH$^{(N)}$ *and over* $\mathscr{W}_{N+\mathrm{U}(1)}$.

2. *The admissible module* $L^{(N)}$ *over* SH$^{(N)}$ *corresponds to a representation of* $\mathscr{W}_{N+\mathrm{U}(1)}$ *on* $L^{(N)}$*, where the lowest weight vector* $|\emptyset\rangle \in L^{(N)}$ *is identified with the highest weight vector* $|\vec{\Lambda}\rangle \in V^{(N)}$*, with the identification (2.5.36):*

$$\Lambda^{(i)} = -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left[\frac{a_i}{\varepsilon_1} - (i-1)\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right], \ 1 \leq i \leq N. \tag{2.5.39}$$

■

We draw Figure 2.6 to summarize the result, showing an extension of Figure 2.5 to the whole algebras.

[†4] A graded module over SH$^{(N)}$ is called admissible when its degree is bounded below. The vector space $L^{(N)}$ is admissible.

$$\begin{array}{ccccc} \mathrm{SH}^{(N)} & \curvearrowright & |\emptyset\rangle & \in & L^{(N)} \\ \text{essentially} \wr\! | & & \wr\! | & & \cup \\ \mathscr{W}_{N+\mathrm{U}(1)} & \curvearrowright & |\vec{\Lambda}\rangle & \in & V^{(N)} \end{array}$$

Figure 2.6: The equivalence between admissible representations of two algebras

If there is no linear relation among the parameters $\varepsilon_{1,2}, a_{1,\cdots,N}$ in $\mathbb{Z}$, infinitely many operators $D_{1,l}, D_{0,l+1}$ are sufficient to give $\mathrm{SH}^{(N)}|\emptyset\rangle = L^{(N)}$, which means that the representation is irreducible. In this case, the embedding of $\mathrm{SH}^{(N)}$ into $\mathscr{W}_{N+\mathrm{U}(1)}$ leads to the action of $\mathscr{W}_{N+\mathrm{U}(1)}$ on $L^{(N)}$ is also irreducible. Then we have $V^{(N)} = L^{(N)}$ since $V^{(N)}$ is irreducible by definition. Let us consider the q-dimension of $V^{(N)}$ defined by

$$\dim_q V^{(N)} = \mathrm{Tr}_{V^{(N)}} q^{\mathbf{H}}, \tag{2.5.40}$$

where the operator $\mathbf{H}$ is what we have defined in (2.2.19). It counts

$$\mathbf{H} v = -\deg(v) v, \tag{2.5.41}$$

for each homogeneous vector $v \in V^{(N)}$. If $V^{(N)} = L^{(N)}$, the q-dimension is none other than

$$\dim_q V^{(N)} = \dim_q L^{(N)} = \left(\prod_{m=1}^{\infty} \frac{1}{1-q^m} \right)^N, \tag{2.5.42}$$

since the basis of $L^{(N)}$ is the set of all the N-tuple Young diagrams. The dimension is also identical to that of the Fock space $M_{\vec{\Lambda}}^{(N)}$. This means that the irreducible module $V^{(N)}$ is not properly smaller than $M^{(N)}$, the Fock space $M_{\vec{\Lambda}}^{(N)}$ at (2.5.39).

We comment on the redundancy among the parameters a_i at the last of this chapter. A permutation of the indices i $(1 \le i \le N)$ in a_i induces an intertwining map from the representation of $\mathrm{SH}^{(N)}$ on $L^{(N)}$ to another one. The map associated with a permutation $\sigma \in \mathfrak{S}_N$ means that we may identify $\vec{\Lambda}$ and $\vec{a}$ by

$$\Lambda^{(i)} = -\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1} \left[\frac{a_{\sigma(i)}}{\varepsilon_1} - (i-1)\left(1 + \frac{\varepsilon_2}{\varepsilon_1}\right) \right], \quad 1 \le i \le N, \tag{2.5.43}$$

rather than by (2.5.39). We will use this redundancy for cyclic permutations.

Chapter 3

SH^c description of Minimal Model and Triality

We have considered the action of $\mathrm{SH}^{(N)}$ when the parameters are kept generic, and have noted that it corresponds to the irreducible action of $\mathcal{W}_{N+\mathrm{U}(1)}$ on the Fock space of N free bosons. We move to considering an application of $\mathrm{SH}^{(N)}$ to minimal model representations of the W-algebra $\mathcal{W}_{N+\mathrm{U}(1)}$. We need to fix the vector $\vec{\Lambda}$ at a rational point to obtain a minimal model representation. The identification (2.5.39) then suggests that we should specialize the parameters $\varepsilon_{1,2}, a_{1,\cdots,N}$ to certain rational numbers, while we have repeatedly mentioned in the previous chapter that we were considering the case when the parameters have no linear relation in $\mathbb{Z}$. We expect that $\mathrm{SH}^{(N)}$ act on a proper subspace of $L^{(N)}$ since each minimal model representation has a proper subspace of the Fock space as its representation space. The matrix coefficients is then expected to vanish when we raise or lower a vector in the subspace to another vector which is not in.

We should be now careful with whether the matrix coefficients in (2.3.6) diverge or vanish. We must avoid the divergence because it makes the action of $\mathrm{SH}^{(N)}$ on $L^{(N)}$ ill-defined, and we focus on when the coefficients vanish in order to obtain an irreducible representation. We will see that the vanishment condition is none other than the so-called N-Burge condition in [42]. After fixing our notations with minimal model representations in Section 3.1, we will construct in Section 3.2 an irreducible representation of $\mathrm{SH}^{(N)}$ whose corresponding vector $\vec{\Lambda}$ is of a minimal model representation. This construction mainly owes to inequalities arising from the N-Burge condition on $\mathcal{Y}^{(N)}$. We will start our main analysis by imposing the N-Burge condition on N-tuple Young diagrams and then evaluate their matrix coefficients. As a result, we obtain a basis of the representation space explicitly, while the existence of the representation is clear from the equivalence between $\mathrm{SH}^{(N)}$ and $\mathcal{W}_{N+\mathrm{U}(1)}$. The explicit construction of the minimal model representation of the algebra $\mathrm{SH}^{(N)}$ is one of the main results in this thesis.

We next focus on a duality in a special class of minimal model representations by using the above representations, as was performed in [1]. The duality was claimed in [20, 21] and has been called the level-rank duality in minimal models. We revisit this duality from the viewpoint of the new algebra $\mathrm{SH}^{(N)}$ and obtain some new results. Our results are summarized as follows.

- We realize the level-rank duality as an algebra isomorphism between $\mathrm{SH}^{(N)}$ and $\mathrm{SH}^{(M)}$ both of which have specialized parameters. Such two algebras share the same representation space, where an N-tuple Young diagram is identified with an M-tuple Young diagram by shuffling their rows.

- The N-Burge condition is found to impose on an N-tuple Young diagram to be a so-called P-partition over a partially ordered set. The dual M-Burge condition leads to the same partially ordered set. Our realization shows where the level-rank duality stems from.
- The level-rank duality is combined with a trivial $\mathbb{Z}_2$-symmetry and forms a triality $\mathfrak{S}_3$ in minimal model representations of SH^c.

It is now clear that two level-rank-dual representations have the same q-dimension, while the coincidence of the q-dimensions itself was already known in [20, 21]. The new point in the first assertion is that we show not only that the two graded spaces have the same q-dimension but also that there is a state-to-state correspondence between them in the whole degree. The second assertion reveals that there is a partially ordered structure behind the level-rank duality. This shows a connection between the Rogers–Ramanujan identities and the Lee–Yang singularity, which was expected in [46]. The third assertion is about a triality in minimal models. Such a triality has been known for another algebra $W_\infty[\mu]$ [28]. The algebra encodes the W-algebra $\mathscr{W}_N$ for any positive integer N since $W_\infty[\mu]$ becomes to $\mathscr{W}_N$ after fixing to $\mu = N$. The algebra SH^c also has such a universal property which is recognized as the assignment $\mathrm{SH}^c \to \mathrm{SH}^{(N)}$. We show that there also exists a triality in the analogous universal algebra SH^c. The trivial $\mathbb{Z}_2$-symmetry comes from the fact that the transposition of a Young diagram is also a Young diagram. We manifest the level-rank duality in Section 3.3 and then give the triality in Section 3.5. We also express the connection between the Rogers–Ramanujan identities and the Lee–Yang singularity in Section 3.4.

Our new results are the SH^c-description of minimal model representations in Section 3.2, that of the level-rank duality in Section 3.3 and that of the triality in Section 3.5. They are based on the paper [1].

3.1 Minimal models of $\mathscr{W}_{N+\mathrm{U}(1)}$

We first fix our notations with minimal models. We have considered when the parameters $\varepsilon_{1,2}, a_{1,\cdots,N}$ are kept general. The identification (2.5.39) then makes the vector $\vec{\Lambda}$ generic so that $\mathscr{W}_{N+\mathrm{U}(1)}$ acts irreducibly on the Fock space $M^{(N)}_{\vec{\Lambda}}$. Not all of representations of $\mathscr{W}_{N+\mathrm{U}(1)}$ belong to this non-degenerate type [6, 8, 11]. The irreducible space $V^{(N)}_{\vec{\Lambda}}$ may become different from the whole Fock space, $V^{(N)}_{\vec{\Lambda}} \subsetneq M^{(N)}_{\vec{\Lambda}}$, after we fix the vector $\vec{\Lambda}$ to a certain point. We concentrate on the so-called strongly degenerate type where all the component $\vec{\Lambda}$ are fixed to be

$$\vec{\Lambda} = \sum_{j=1}^{N-1} \left(\left(-\frac{\varepsilon_2}{\varepsilon_1} \right)^{-1} n'_j - n_j \right) \vec{\omega}_j + Q\vec{\rho}\,, \tag{3.1.1}$$

where n_j, n'_j are positive integers and the vectors $\vec{\omega}_j$ are the fundamental weights of $\mathfrak{su}(N)$, which are given explicitly by

$$\vec{\omega}_i = \left(1 - \frac{i}{N} \right) \sum_{j\le i} \vec{e}_j - \frac{i}{N} \sum_{j>i}^{N} \vec{e}_j\,,\ 1 \le i \le N-1\,. \tag{3.1.2}$$

This special vector is known to make the action of $\mathscr{W}_{N+\mathrm{U}(1)}$ on $M^{(N)}_{\vec{\Lambda}}$ reducible. It is because there exist at least $N-1$ degenerate vectors $|\Phi_j\rangle \in M^{(N)}_{\vec{\Lambda}} (1 \le j \le N-1)$ such that

$$\begin{aligned} \alpha_l\,|\Phi_j\rangle &= 0,\ l>0\,, \\ W_{k,l}\,|\Phi_j\rangle &= 0\,,\ l>0\,, k>1\,. \end{aligned} \tag{3.1.3}$$

Each vector $|\Phi_j\rangle$ has the degree $n_j n'_j > 0$ and generates a proper submodule $\mathscr{W}_{N+\mathrm{U}(1)}|\Phi_j\rangle \not\ni |\vec{\Lambda}\rangle$. The $\mathscr{W}_N$ side in (3.1.3) was derived in terms of screening operators in [11]. A similar discussion and the traceless property $\sum_{i=1}^{N} \Lambda^{(i)} = 0$ then lead to the other Heisenberg side.

We further restrict ourselves to when the remaining parameters $\varepsilon_{1,2}$ are fixed to

$$\varepsilon_1 = q\,, \varepsilon_2 = -p\,, \left(Q = -\frac{q-p}{p}\right), \tag{3.1.4}$$

where $p, q \ge N$ are positive integers and mutually prime, and to a finite number of completely degenerate representations labeled by positive integers n_i, n'_i satisfying

$$\sum_{i=1}^{N-1} n_i < q\,, \sum_{i=1}^{N-1} n'_i < p\,. \tag{3.1.5}$$

These representations are often called minimal model representations. In [11], it was shown that, for fixed p, q, such representations are closed under the operator product expansion (OPE) or the so-called fusion rule and that they give a minimal model, a conformal field theory consisting of finite primary fields. Such minimal models have been intensively studied because they describe the critical limit of many 2-dimensional models including the Ising models[6] and RSOS models[7, 12, 13].

Note that there is $\mathbb{Z}_N$-redundancy among integers n_i, n'_i parameterizing minimal model representations. We can see from (2.5.26) that the weights (2.5.24) are invariant under a shift of a traceless vector $\vec{\Lambda}$ to a traceless $\vec{\Lambda}'$ satisfying

$$\vec{h}_i \cdot \vec{\Lambda} + Q\,i = \vec{h}_{\sigma(i)} \cdot \vec{\Lambda}' + Q\sigma(i), 1 \le i \le N\,, \tag{3.1.6}$$

for a permutation $\sigma \in \mathfrak{S}_N$. Two completely degenerate representations with the same weights are clearly equivalent and we should identify them. For the cyclic shift $\sigma^{\text{cyc}} \in \mathfrak{S}_N$, which maps $\sigma^{\text{cyc}}(i) = i+1$ for $i < N$ and $\sigma^{\text{cyc}}(N) = 1$, a minimal model representation with $n_i^{(\text{old})}, n_i^{\prime(\text{old})}$ is taken to another one with $n_i^{(\text{new})}, n_i^{\prime(\text{new})}$, where

$$\begin{aligned} n_i^{(\text{new})} &= n_{i-1}^{(\text{old})}\,,\ n_i^{\prime(\text{new})} = n_{i-1}^{\prime(\text{old})}\,,\ 2 \le i \le N-1\,, \\ n_1^{(\text{new})} &= q - \sum_{j=1}^{N-1} n_j^{(\text{old})}\,,\ n_1^{\prime(\text{new})} = p - \sum_{j=1}^{N-1} n_j^{\prime(\text{old})}\,. \end{aligned} \tag{3.1.7}$$

Iterated shifts by σ^{cyc} generate the $\mathbb{Z}_N$-redundancy. The redundancy leads us to think it natural to introduce two numbers n_N, n'_N, for each label (n_i, n'_i) of minimal model representations, by

$$\sum_{j=1}^{N} n_j = q\,, \sum_{j=1}^{N} n'_j = p\,. \tag{3.1.8}$$

The cyclic group $\mathbb{Z}_N$ then acts on N pairs $(n_i, n'_i)_{i=1}^N$ of positive integers by cyclic shifts of their indices. Henceforth, we mean by an N-datum a collection of pairs (p, q) and $(n_i, n'_i)_{i=1}^N$ satisfying (3.1.8).

3.2 SH^c descriptions of minimal model representations

Let us go back to SH$^{(N)}$. The identification (2.5.39) suggests that a minimal model representation of $\mathscr{W}_{N+\mathrm{U}(1)}$ with an N-datum $\langle (p,q),(n_i,n_i')_{1\le i\le N}\rangle$ gives an equivalent representation of SH$^{(N)}$ where the parameters $a_{1,\cdots,N}$ are fixed to

$$\begin{aligned} a_i &= -\sum_{j=1}^{N-1}(\varepsilon_1 n_j' + \varepsilon_2 n_j)\vec{e}_i\cdot\vec{\omega}_j + (\varepsilon_1+\varepsilon_2)(\vec{e}_i\cdot\vec{\rho}+i-1) \\ &= -\sum_{j=1}^{N-1}(q n_j' - p n_j)\vec{e}_i\cdot\vec{\omega}_j + \frac{1}{2}(N-1)(q-p), \end{aligned} \tag{3.2.1}$$

for $1\le i\le N$. We can rewrite them by

$$\begin{aligned} & a_1 - a_N = q n_N' - p n_N\,,\, a_{j+1}-a_j = q n_j' - p n_j\,,\ 1\le j\le N-1\,, \\ & \sum_{i=1}^{N} a_i = \frac{1}{2}N(N-1)(q-p)\,, \end{aligned} \tag{3.2.2}$$

which shows that the $\mathbb{Z}_N$-redundancy generates the same shift of the index i in a_i as that in n_i. The above identification are none other than linear relations in $\mathbb{Z}$ among the parameters $\varepsilon_{1,2}$ and $a_{1,\cdots,N}$. The algebra SH^c is then expected not to act on the whole $L^{(N)}$ since the matrix coefficients in (2.3.6) may diverge or vanish.

In this section, we rather construct a certain lowest weight representation of SH^c generating from the degree 0 state $|\emptyset\rangle$ by performing the following program. (I) Firstly, we separate the whole space $L^{(N)} = L_1^{(N)}\oplus L_2^{(N)}$ into two spaces $L_1^{(N)}$ and $L_2^{(N)}$ satisfying the following four conditions.

- The $(N+2)$-dimensional torus $\tilde{D}$ acts on $L_1^{(N)}$ and $L_2^{(N)}$ separately.
- $|\emptyset\rangle \in L_1^{(N)}$.
- The naive actions (2.3.6) of $D_{\pm1,l}$ on $L^{(N)}$ have the same triangular form with respect to the decomposition $L_1^{(N)}\oplus L_2^{(N)}$;

$$D_{\pm1,l} = \begin{pmatrix} D_{\pm1,l,11} & D_{\pm1,l,12} \\ D_{\pm1,l,21} & D_{\pm1,l,22}\end{pmatrix} = \begin{pmatrix} \text{finite} & 0 \\ \text{finite} & * \end{pmatrix}. \tag{3.2.3}$$

  In other words, we have well-defined linear maps

$$D_{\pm1,l}|_{L_1^{(N)}} = {}^t(D_{\pm1,l,11}, D_{\pm1,l,21}) : L_1^{(N)} \to L_1^{(N)}\oplus L_2^{(N)}, \tag{3.2.4}$$

  and the zero map $D_{\pm1,l,12}$ does not create any nonzero vector in $L_1^{(N)}$ from a finite vector in $L_2^{(N)}$.

We should note that we allow $D_{\pm1,l,22}$ to contain a divergent value in their matrix expressions. The quadratic relations (2.4.1), (2.4.3) in SH^c automatically hold for $D_{0,l+1,11}, D_{\pm1,l,11}\in \mathrm{End}(L_1^{(N)})$ for

$$D_{r_1,l_1,12}D_{r_2,l_2,21} = 0\,,\ r_1,r_2\in\{0,\pm1\}\,, \tag{3.2.5}$$

which means that there is no contribution to the quadratic relations from $L_2^{(N)}$. In particular, the equation (2.3.29) is justified even after specializing parameters. (II) Secondly, we check whether the cubic relations (2.4.2) in SH^c hold for $D_{\pm1,l,11}$. It suffices to show

$$D_{r,l_1,22}D_{r,l_2,21} = \text{finite}\,,\ r = \pm1\,. \tag{3.2.6}$$

We will show the cubic relations hold for a certain separation $L^{(N)} = L_B^{(N)} \oplus L_*^{(N)}$. (III) At last, we check the irreducibility of the induced action of SH^c on $L_B^{(N)}$.

The most nontrivial step in the above program is to find out a separation of $\mathscr{Y}^{(N)}$ which induces such an appropriate separation $L^{(N)} = L_B^{(N)} \oplus L_*^{(N)}$. We will obtain the selection rule in $\mathscr{Y}^{(N)}$ by investigating the coefficients of (2.3.6).

3.2.1 The vector space $L_B^{(N)}$

We first state the rule of the appropriate separation $L^{(N)} = L_B^{(N)} \oplus L_*^{(N)}$, and then start to show its appropriateness:

Definition 3.1 (the vector space $L_B^{(N)}$)**.** Let $\langle (p,q),(n_i,n_i')_{1\le i\le N}\rangle$ be an N-datum.

1. An N-tuple $\lambda \in \mathscr{Y}^{(N)}$ is said to meet *the N-Burge condition* associated with the N-datum when it satisfies

$$\lambda^{(i+1)}(l+n_i-1)-\lambda^{(i)}(l) \le n_i'-1\,, \tag{3.2.7}$$

   for each $1 \le i \le N, l \ge 1$, where we use a cyclic identification $\lambda^{(N+1)} = \lambda^{(1)}$ if $i = N$. We denote by

$$\mathscr{Y}_B^{(N)} = \mathscr{Y}_B^{(N)}((n_i,n_i')_{1\le i\le N}; p,q) \subset \mathscr{Y}^{(N)}\,, \tag{3.2.8}$$

   the set of all the elements in $\mathscr{Y}^{(N)}$ meeting the N-Burge condition.

2. $L_B^{(N)}$ is the subspace of $L^{(N)}$ spanned by the subset $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}_B^{(N)}}$ of the basis for $L^{(N)}$, and $L_*^{(N)}$ is the span of the others $\{|\lambda\rangle\}_{\lambda\in\mathscr{Y}^{(N)}\setminus\mathscr{Y}_B^{(N)}}$.

■

The torus $\tilde{D}$ clearly acts on $L_B^{(N)}$ and $L_*^{(N)}$ separately. It also clear that $|\emptyset\rangle \in L_B^{(N)}$.

The N-Burge condition was introduced in [42] to remove null vectors when computing minimal model conformal blocks, after some of its authors obtained the Burge condition [50] for the Virasoro case ($N = 2$) in [40, 41] from the viewpoint of the AGT correspondence. The null condition for $|\lambda\rangle$ ($\lambda \in \mathscr{Y}^{(N)}$) is none other than $\langle\lambda|\lambda\rangle = \mathrm{eu}(T_\lambda) = 0$ in our notation. The authors derived the N-Burge condition as the condition whether a fixed point in the instanton moduli space gives a divergent contribution to the instanton partition function after specializing parameters to minimal models. Therefore it is quite natural for our program to relate to the N-Burge condition. Our construction can be seen as an extension of such a null condition with the pairing of $L^{(N)}$ to that with the action of SH$^{(N)}$, and gives a new meaning of the N-Burge condition from the viewpoint of the algebra SH^c. We will revisit the null condition at the last of this section.

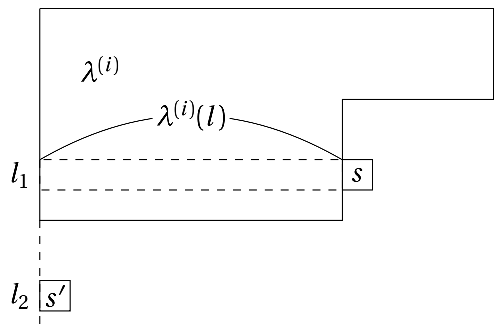


Figure 3.1: Two east-end boxes s, s' with respect to $\lambda^{(i)}$

3.2.2 The difference of two east-end boxes

To see what makes the separation $L^{(N)} = L_B^{(N)} \oplus L_*^{(N)}$ appropriate, we first recall which coefficients we have to focus on. We write again the resultant formula (2.3.16) for $D_{\pm 1,0}$; for $\lambda \in \mathscr{Y}^{(N)}$, we have

$$\begin{aligned} D_{1,0}|\lambda\rangle &= \sum_{\substack{\pi \supset \lambda \\ |\pi| = |\lambda|+1}} \mathrm{eu}((t_1 t_2 J_\lambda^* - I_\lambda^*) \otimes (V_\pi - V_\lambda) + 1)|\pi\rangle\,, \\ D_{-1,0}|\lambda\rangle &= \frac{1}{\varepsilon_1 \varepsilon_2} \sum_{\substack{\sigma \subset \lambda \\ |\sigma| = |\lambda|-1}} \mathrm{eu}((t_1^{-1} t_2^{-1} I_\lambda^* - J_\lambda^*) \otimes (V_\lambda - V_\sigma) + 1)|\sigma\rangle\,. \end{aligned} \tag{3.2.9}$$

where

$$V_\lambda = \sum_{j=1}^{N} \sum_{s \in \lambda^{(j)}} \chi_j t_1^{-x(s)} t_2^{-y(s)}, \ I_\lambda = \sum_{\substack{\pi \supset \lambda \\ |\pi| = |\lambda|+1}} (V_\pi - V_\lambda), \ J_\lambda = \sum_{\substack{\sigma \subset \lambda \\ |\sigma| = |\lambda|-1}} (V_\lambda - V_\sigma). \tag{3.2.10}$$

What we want to check is whether two distinct boxes[†1] in I_λ^* or $t_1 t_2 J_\lambda^*$ have the same character after the fixings (3.1.4) and (3.2.1) of parameters. Seeing from Figure 2.2 and 2.3, it is sufficient to compare boxes adjacent to the east boundaries of Young diagrams. For an N-tuple $\lambda \in \mathscr{Y}^{(N)}$, such a box s is labeled by two parameters (i, l), $1 \le i \le N$, $l \ge 1$, where its character is given by

$$c_1(s) = -a_i + x(s)\varepsilon_1 + y(s)\varepsilon_2\,, \ x(s) = \lambda^{(i)}(l)\,, \ y(s) = l - 1\,. \tag{3.2.11}$$

This is the box at the lth row and the $(\lambda^{(i)}(l)+1)$-th column in the ith lattice where $\lambda^{(i)}$ is assigned (see Figure 3.1). We allow l to be arbitrary and then such a box s may be at the south away from the diagram $\lambda^{(i)}$. We call it *the east-end box along λ labeled by (i, l)*. Let us denote by $X^{(N)}$ the set of all such labelings:

$$X^{(N)} = \{(i, l) | 1 \le i \le N, l \ge 1\}\,. \tag{3.2.12}$$

[†1] Here and henceforth, we regard a box as a corresponding 1-dimensional vector space.

We now take two east-end boxes s_1, s_2 along λ labeled by $(i_1, l_1), (i_2, l_2) \in X^{(N)}$, respectively. We assume that $i_1 \geq i_2$. The difference of their characters becomes

$$\begin{aligned} c_1(s_2) - c_1(s_1) &= a_{i_1} - a_{i_2} - q(\lambda^{(i_1)}(l_1) - \lambda^{(i_2)}(l_2)) + p(l_1 - l_2) \\ &= -q\left(\lambda^{(i_1)}(l_1) - \lambda^{(i_2)}(l_2) - \sum_{j=i_2}^{i_1-1} n'_j\right) + p\left(l_1 - l_2 - \sum_{j=i_2}^{i_1-1} n_j\right), \end{aligned} \tag{3.2.13}$$

Since p, q are mutually prime, we have

$$c_1(s_2) - c_1(s_1) = 0 \Leftrightarrow \lambda^{(i_1)}(l_1) - \lambda^{(i_2)}(l_2) - \sum_{j=i_2}^{i_1-1} n'_j = pr\,, l_1 - l_2 - \sum_{j=i_2}^{i_1-1} n_j = qr\,,\ ^{\exists} r \in \mathbb{Z}\,. \tag{3.2.14}$$

3.2.3 Program I: the quadratic relations

The above equivalence (3.2.14) shows a role of the N-Burge condition.

Proposition 3.1. *For two distinct east-end boxes s_1, s_2 along $\lambda \in \mathscr{Y}_B^{(N)}$, we have $c_1(s_1) \neq c_1(s_2)$.* ■

Lemma 3.1. *For $\lambda \in \mathscr{Y}_B^{(N)}$ and $(i_1, l_1), (i_2, l_2) \in X^{(N)}$ with $i_1 \geq i_2$, we have the following inequalities:*

1. *For $r \geq 0$, we have*

$$\lambda^{(i_1)}\left(l_2 + \sum_{j=i_2}^{i_1-1} n_j + qr\right) - \lambda^{(i_2)}(l_2) - \sum_{j=i_2}^{i_1-1} n'_j - pr \leq -Nr - (i_1 - i_2) \leq 0\,. \tag{3.2.15}$$

2. *For $r < 0$, we have*

$$\lambda^{(i_1)}(l_1) - \lambda^{(i_2)}\left(l_1 + q(-r) - \sum_{j=i_2}^{i_1-1} n_j\right) - \sum_{j=i_2}^{i_1-1} n'_j - pr \geq N(-r-1) + (N - (i_1 - i_2)) > 0\,. \tag{3.2.16}$$

■

Proof. 1. For $r \geq 0$, we have

$$\begin{aligned} \lambda^{(i_1)}\left(l_2 + \sum_{j=i_2}^{i_1-1} n_j + qr\right) &\leq \lambda^{(i_1-1)}\left(l_2 + 1 + \sum_{j=i_2}^{i_1-2} n_j + qr\right) + n'_{i_1-1} - 1 \leq \cdots \\ &\leq \lambda^{(i_2)}\left(l_2 + (i_1 - i_2) + r\sum_{j=1}^{N} n_j\right) + \sum_{j=i_2}^{i_1-1} n'_j - (i_1 - i_2) \\ &\leq \lambda^{(i_2)}(l_2) + \sum_{j=i_2}^{i_1-1} n'_j + (p - N)r - (i_1 - i_2)\,. \end{aligned} \tag{3.2.17}$$

2. For $r<0$, we have

$$\begin{aligned}
\lambda^{(i_2)}\left(l_1+q(-r)-\sum_{j=i_2}^{i_1-1}n_j\right) &\le \lambda^{(1)}\left(l_1+(i_2-1)+q(-r)-\sum_{j=1}^{i_1-1}n_j\right)+\sum_{j=1}^{i_2-1}n'_j-(i_2-1)\\
&=\lambda^{(1)}\left(l_1+(i_2-1)+q(-r-1)+\sum_{j=i_1}^{N}n_j\right)+p-\sum_{j=i_2}^{N}n'_j-(i_2-1)\\
&\le\lambda^{(N)}\left(l_1+i_2+q(-r-1)+\sum_{j=i_1}^{N-1}n_j\right)+p-\sum_{j=i_2}^{N-1}n'_j-i_2\\
&\le\lambda^{(i_1)}(l_1+(N-(i_1-i_2))+q(-r-1))+p-\sum_{j=i_2}^{i_1-1}n'_j-(N-(i_1-i_2))\\
&\le\lambda^{(i_1)}(l_1)+p(-r)-\sum_{j=i_2}^{i_1-1}n'_j-N(-r-1)-(N-(i_1-i_2)).
\end{aligned} \tag{3.2.18}$$

□

Proof of the proposition. Let $\lambda\in\mathscr{Y}_B^{(N)}$ and $(i_1,l_1),(i_2,l_2)\in X^{(N)}$ be labels of two east-end boxes s_1,s_2 along λ, respectively. We may assume that $i_1\ge i_2$ by permutating the boxes. We assume that $c_1(s_1)=c_1(s_2)$. For (3.2.14), there is an integer r satisfying

$$\lambda^{(i_1)}(l_1)-\lambda^{(i_2)}(l_2)-\sum_{j=i_2}^{i_1-1}n'_j=pr\,,\, l_1-l_2-\sum_{j=i_2}^{i_1-1}n_j=qr\,, \tag{3.2.19}$$

The inequality (3.2.16) means that $r\ge0$. The inequality (3.2.15) then shows $r=0$ and $i_1=i_2$, which leads to $l_1=l_2$ for the above equations. Therefore the two boxes s_1 and s_2 are identical. □

This proposition leads to the following property for each element of $\mathscr{Y}_B^{(N)}$:

Corollary 3.1. *For $\lambda\in\mathscr{Y}_B^{(N)}$,*

1. *any two distinct boxes in I_λ^* have different characters,*
2. *any two distinct boxes in J_λ^* have different characters, and*
3. *the coefficients in (3.2.9)*

$$\mathrm{eu}((t_1t_2J_\lambda^*-I_\lambda^*)\otimes(V_\pi-V_\lambda)+1),\ \mathrm{eu}((t_1^{-1}t_2^{-1}I_\lambda^*-J_\lambda^*)\otimes(V_\lambda-V_\sigma)+1) \tag{3.2.20}$$

do not diverge for any $\sigma,\pi\in\mathscr{Y}^{(N)}$ such that $\sigma\subset\lambda\subset\pi,|\sigma|+1=|\lambda|=|\pi|-1$. In other words, each linear map $D_{\pm1,l}|_{L_B^{(N)}}$ from $L_B^{(N)}$ to $L^{(N)}$ is well-defined.

■

The N-Burge condition also makes $D_{\pm1,l,12}$ to be zero. We first focus on $D_{-1,l,12}$:

Proposition 3.2. *For* $\lambda \in \mathscr{Y}_B^{(N)}$ *and* $\pi \in \mathscr{Y}^{(N)} \setminus \mathscr{Y}_B^{(N)}$ *such that* $\lambda \subset \pi, |\pi| = |\lambda| + 1$, *we have*

$$\mathrm{eu}((t_1^{-1} t_2^{-1} I_\pi^* - J_\pi^*) \otimes (V_\pi - V_\lambda) + 1) = 0\,. \tag{3.2.21}$$

■

The following lemma shows the proposition:

Lemma 3.2. *With the notations above we have*

1. $\mathrm{eu}(t_1^{-1} t_2^{-1} I_\pi^* \otimes (V_\pi - V_\lambda)) = 0\,,$
2. *any two distinct east-end boxes along* π *have different characters, and*
3. $\mathrm{eu}(J_\pi^* \otimes (V_\pi - V_\lambda) - 1) \neq 0\,.$

■

Proof. 1. Let $(i, l) \in X^{(N)}$ be the label of the extra east-end box $\hat{s} \in \pi \setminus \lambda$ along λ. The extra box $\hat{s}$ should break some inequalities in the N-Burge condition, but the only representative is

$$\text{“}\pi^{(i)}(l) - \pi^{(i-1)}(l - (n_{i-1} - 1)) \leq n'_{i-1} - 1\,, (i-1, l - (n_{i-1} - 1)) \in X^{(N)}\text{”}\,, \tag{3.2.22}$$

where $i - 1$ should be treated as N if $i = 1$. Since λ satisfies the N-Burge condition, we should have instead[†2]

$$\lambda^{(i)}(l) - \lambda^{(i-1)}(l - (n_{i-1} - 1)) = \pi^{(i)}(l) - \pi^{(i-1)}(l - (n_{i-1} - 1)) - 1 = n'_{i-1} - 1\,. \tag{3.2.23}$$

Note that $\hat{s} \subset I_\lambda^*$, which gives an inequality

$$\lambda^{(i)}(l) < \lambda^{(i)}(l-1), \tag{3.2.24}$$

where $\lambda^{(i)}(0) = \infty$ if $l = 1$. We also have an inequality in the N-Burge condition

$$\lambda^{(i)}(l-1) - \lambda^{(i-1)}(l - 1 - (n_{i-1} - 1)) \leq n'_{i-1} - 1\,. \tag{3.2.25}$$

We then obtain

$$\lambda^{(i-1)}(l - (n_{i-1} - 1)) < \lambda^{(i-1)}(l - 1 - (n_{i-1} - 1))\,, \tag{3.2.26}$$

which means that the east-end box $\check{s} (\neq \hat{s})$ along λ labeled by $(i-1, l - (n_{i-1} - 1))$ is in $I_\pi^* \cap I_\lambda^*$. The difference between the two boxes with respect to the character is

$$\begin{aligned} c_1(\check{s}) - c_1(\hat{s}) &= -q \left(\lambda^{(i)}(l) - \lambda^{(i-1)}(l - (n_{i-1} - 1)) - n'_{i-1} \right) + p\,(l - (l - (n_{i-1} - 1)) - n_{i-1}) \\ &= q - p\,. \end{aligned} \tag{3.2.27}$$

Therefore we have obtained two boxes $\hat{s} = V_\pi^* / V_\lambda^*$ and $t_1^{-1} t_2^{-1} \check{s} \subset t_1^{-1} t_2^{-1} I_\pi^*$ satisfying

$$c_1(t_1^{-1} t_2^{-1} \check{s}) - c_1(\hat{s}) = 0\,, \tag{3.2.28}$$

which proves $\mathrm{eu}(t_1^{-1} t_2^{-1} I_\pi^* \otimes (V_\pi - V_\lambda)) = 0$.

[†2] Here we use $N > 1$. In fact, the existence of $\pi \in \mathscr{Y}^{(N)} \setminus \mathscr{Y}_B^{(N)}$ leads to $N > 1$. When $N = 1$, any Young diagram satisfies the 1-Burge condition, which means $\mathscr{Y}_B^{(1)} = \mathscr{Y}^{(1)}$.

2. Let s_1 and s_2 be two distinct east-end boxes along π labeled by $(i_1,l_1),(i_2,l_2)\in X^{(N)}$. We may assume that $i_1\geq i_2$ and that either s_1 or s_2 is identical to $\hat{s}$ since the other cases are shown by the previous proposition with $\lambda\in\mathscr{Y}_B^{(N)}$. Recall that

$$c_1(s_2)-c_1(s_1)=0\Leftrightarrow \pi^{(i_1)}(l_1)-\pi^{(i_2)}(l_2)-\sum_{j=i_2}^{i_1-1}n'_j=pr\,,l_1-l_2-\sum_{j=i_2}^{i_1-1}n_j=qr\,,\ ^{\exists}r\in\mathbb{Z}. \tag{3.2.29}$$

If $s_1=\hat{s}$, the inequalities (3.2.15) and (3.2.16) for λ show that

$$c_1(s_2)-c_1(\hat{s})=0\Rightarrow -1\leq -Nr-(i-i_2)\leq 0\,,\ ^{\exists}r\in\mathbb{Z}_{\geq 0}\,. \tag{3.2.30}$$

We then have

$$\begin{aligned}c_1(s_2)-c_1(\hat{s})=0&\Rightarrow (i_2,l_2)=(i-1,l-n_{i-1})\,,\ \pi^{(i)}(l)-\pi^{(i-1)}(l_2)=n'_{i-1}\\&\Rightarrow \pi^{(i-1)}(l-(n_{i-1}-1))=\pi^{(i-1)}(l-1-(n_{i-1}-1))\,,\end{aligned} \tag{3.2.31}$$

under $s_1\neq s_2$ and $N>1$. The inequality (3.2.26) gives, however,

$$\pi^{(i-1)}(l-(n_{i-1}-1))-\pi^{(i-1)}(l-1-(n_{i-1}-1))<0\,, \tag{3.2.32}$$

which proves that $c_1(s_2)\neq c_1(\hat{s})$ if $s_1=\hat{s}$. We have $c_1(s_1)\neq c_1(\hat{s})$ if $s_2=\hat{s}$ similarly, for

$$\begin{aligned}c_1(\hat{s})-c_1(s_1)=0&\Rightarrow 1\geq N(-r-1)+(N-(i_1-i))>0\,,\ ^{\exists}r\in\mathbb{Z}_{<0}\,,\\&\Rightarrow i=1\,,\ (i_1,l_1)=(N,l-n_N)\,,\ \pi^{(i)}(l)-\pi^{(N)}(l_1)=n'_N\,.\end{aligned} \tag{3.2.33}$$

3. Immediate from the second statement. □

We next move to $D_{1,l}$:

Proposition 3.3. *For $\lambda\in\mathscr{Y}_B^{(N)}$ and $\sigma\in\mathscr{Y}^{(N)}\setminus\mathscr{Y}_B^{(N)}$ such that $\sigma\subset\lambda, |\sigma|=|\lambda|-1$, we have*

$$\mathrm{eu}((t_1t_2J^*_\sigma-I^*_\sigma)\otimes(V_\lambda-V_\sigma)+1)=0\,. \tag{3.2.34}$$

■

This is immediate from the following lemma:

Lemma 3.3. *With the notations above we have*

1. $\mathrm{eu}(t_1t_2J^*_\sigma\otimes(V_\lambda-V_\sigma))=0\,,$
2. *any two distinct east-end boxes along σ have different characters, and*
3. $\mathrm{eu}(I^*_\sigma\otimes(V_\lambda-V_\sigma)-1)\neq 0.$

■

Proof of lemma. 1. Let $(i,l)\in X^{(N)}$ be the label of the extra east-end box $\hat{s}\in\lambda\backslash\sigma$ along σ. Then a similar reasoning which appeared in the proof with the case $\lambda\in\mathscr{Y}_B^{(N)},\pi\in\mathscr{Y}^{(N)}\setminus\mathscr{Y}_B^{(N)}$ gives

$$\lambda^{(i+1)}(l+(n_i-1))-\lambda^{(i)}(l)=n'_i-1\,, \tag{3.2.35}$$

and

$$\lambda^{(i+1)}(l+1+(n_i-1))-\lambda^{(i+1)}(l+(n_i-1))<0\,. \tag{3.2.36}$$

The latter shows that the east-end box $\check{s}(\neq \hat{s})$ along σ labeled by $(i+1,l+(n_i-1))\in X^{(N)}$ is contained in J^*_σ and that

$$c_1(\hat{s})-c_1(t_1t_2\check{s})=0\,. \tag{3.2.37}$$

This proves that $\mathrm{eu}(t_1t_2J^*_\sigma\otimes(V_\lambda-V_\sigma))=0$.

2. A similar reasoning shows that two distinct east-end boxes along σ have the same character only if the two are $\hat{s}$ and $t_2\check{s}$. We have, however,

$$c_1(\hat{s})-c_1(t_2\check{s})=q\neq 0\,, \tag{3.2.38}$$

and then obtain the claim.

3. Immediate from the second. □

We have confirmed that the operators $D_{\pm1,l}$ have the same triangular form with respect to the decomposition $L^{(N)}_B\oplus L^{(N)}_*$,

$$D_{\pm1,l}=\begin{pmatrix} D_{\pm1,l,11} & D_{\pm1,l,12}\\ D_{\pm1,l,21} & D_{\pm1,l,22}\end{pmatrix}=\begin{pmatrix}\text{finite} & 0\\ \text{finite} & *\end{pmatrix}, \tag{3.2.39}$$

and then the quadratic relations in SH^c hold for the $(1,1)$-components $D_{\pm1,l,11},D_{0,l+1,11}\in\mathrm{End}(L^{(N)}_B)$.

3.2.4 Program II: the cubic relations

We next prove the cubic relation (2.4.2) for $D_{\pm1,l,11}\in\mathrm{End}(L^{(N)}_B)$. It suffices to show the following:

Lemma 3.4. *With the notations above, we have*

$$D_{r,l_1,22}D_{r,l_2,21}=\text{finite}\,,\ \ r=\pm1\,. \tag{3.2.40}$$

■

Proof. For $\lambda\in\mathscr{Y}^{(N)}_B$ and $\pi\in\mathscr{Y}^{(N)}\setminus\mathscr{Y}^{(N)}_B$ such that $\lambda\subset\pi,|\pi|=|\lambda|+1$, we proved in Lemma 3.2 that any two distinct east-end boxes along π have different characters. A similar reasoning with $\lambda\in\mathscr{Y}^{(N)}_B$ shows a linear map $D_{1,l_1,22}|_{\mathrm{Im}D_{1,l,21}}$ is well-defined. The $r=-1$ case can be proven similarly. □

Therefore the assignments

$$\begin{aligned}
&\tilde{D}_x\in\mathrm{SH}^c\mapsto D_{x,11}\in\mathrm{End}(L^{(N)}_B),\ x=(0,l+1),(\pm1,l),l\geq 0\,,\\
&\varepsilon_1=q\,,\ \varepsilon_2=-p\,,\ \tilde{c}_l\mapsto p_l(a_1,\cdots,a_N)/\varepsilon_1^l\,,\ l\geq0\,,\\
&a_i=-\sum_{j=1}^{N-1}(qn'_j-pn_j)\vec{e}_i\cdot\vec{\omega}_j+\frac{1}{2}(N-1)(q-p),\ 1\leq i\leq N\,,
\end{aligned} \tag{3.2.41}$$

induces a representation of SH^c on $L^{(N)}_B$.

3.2.5 Program III: the irreducibility

What remains in our program is to show that the induced action is irreducible. We need the following lemma:

Lemma 3.5. *For $\lambda \in \mathscr{Y}_B^{(N)}$, we have*

1. $\mathrm{eu}(t_1 t_2 J_\lambda^* \otimes (V_\pi - V_\lambda)) \neq 0$ *for $\pi \in \mathscr{Y}^{(N)}$ such that $\lambda \subset \pi, |\pi| = |\lambda| + 1$, and*
2. $\mathrm{eu}(t_1^{-1} t_2^{-1} I_\lambda^* \otimes (V_\lambda - V_\sigma)) \neq 0$ *for $\sigma \in \mathscr{Y}^{(N)}$ such that $\sigma \subset \lambda, |\sigma| = |\lambda| - 1$.*

■

The above lemma and the fact that any two distinct boxes in I_λ^* for $\lambda \in \mathscr{Y}_B^{(N)}$ have different characters tell us that, for $\lambda, \pi \in \mathscr{Y}_B^{(N)}$ such that $\lambda \subset \pi, |\pi| = |\lambda| + 1$, there exists a linear combination $\sum_{l \geq 0} k_l D_{1,l} (k_l \in \mathbb{C})$ which satisfies $(\sum_{l \geq 0} k_l D_{1,l}) |\lambda\rangle = |\pi\rangle$. We then see the irreducibility of the action by using this property inductively with the degree in $L_B^{(N)}$.

We prove the following stronger form:

Lemma 3.6. *Let $\lambda \in \mathscr{Y}_B^{(N)}$ and s_1, s_2 be two east-end boxes along λ. If either $s_1 \subset I_\lambda^*$ or $s_2 \subset J_\lambda^*$ holds, we have $c_1(t_2 s_2) - c_1(s_1) \neq 0$.*

Proof. Fix an N-tuple $\lambda \in \mathscr{Y}_B^{(N)}$. Let s_1 and s_2 be two arbitrary east-end boxes along λ labeled by $(i_1, l_1), (i_2, l_2) \in X^{(N)}$. It is sufficient to show $c_1(t_2 s_2) \neq c_1(s_1)$ whenever $s_1 \subset I_\lambda$. Cycling the indices in the N-datum by the $\mathbb{Z}_N$-redundancy if necessary, we may assume that $i_1 \geq i_2$. We have

$$c_1(t_2 s_2) - c_1(s_1) = 0 \Leftrightarrow \lambda^{(i_1)}(l_1) - \lambda^{(i_2)}(l_2) - \sum_{j=i_2}^{i_1-1} n'_j = pr\,, l_1 - l_2 - 1 - \sum_{j=i_2}^{i_1-1} n_j = qr\,, \ {}^\exists r \in \mathbb{Z}. \quad (3.2.42)$$

The inequalities (3.2.16) and (3.2.15) give

$$\lambda^{(i_1)}\left(l_2 + \sum_{j=i_2}^{i_1-1} n_j + qr + 1 \right) - \lambda^{(i_2)}(l_2) - \sum_{j=i_2}^{i_1-1} n'_j - pr \leq -Nr - (i_1 - i_2), \quad (3.2.43)$$

for $r \geq 0$, and

$$\lambda^{(i_1)}(l_1) - \lambda^{(i_2)}\left(l_1 - 1 - \sum_{j=i_2}^{i_1-1} n_j - qr \right) - \sum_{j=i_2}^{i_1-1} n'_j - pr \geq N(-r-1) + (N - (i_1 - i_2)) > 0 \quad (3.2.44)$$

for $r < 0$. Then we have

$$c_1(t_2 s_2) - c_1(s_1) = 0 \Leftrightarrow i_1 = i_2, l_1 = l_2 + 1, \lambda^{(i_1)}(l_1) = \lambda^{(i_2)}(l_2). \quad (3.2.45)$$

If we take $s_1 \subset I_\lambda$ or $s_2 \subset J_\lambda^*$, we have $\lambda^{(i_1)}(l_1 - 1) > \lambda^{(i_1)}(l_1)$ or $\lambda^{(i_2)}(l_2) > \lambda^{(i_2)}(l_2 + 1)$, and then $c_1(t_2 s_2) \neq c_1(s_1)$. □

We finally obtain an irreducible action of SH^c associated with an N-datum:

Theorem 3.1. *Let $p, q \geq N$ be mutually prime positive integers and $\langle (p,q), (n_i, n'_i)_{1 \leq i \leq N} \rangle$ be an N-datum (3.1.8). Denoting by $L_B^{(N)}$ the subspace of $L^{(N)}$ associated with its N-Burge condition. Then SH^c acts irreducibly on $L_B^{(N)}$ by (3.2.41).* ■

3.2.6 Null vectors

We revisit the null condition in [42], as mentioned just after we introduced the N-Burge condition.

Proposition 3.4. *For $\lambda \in \mathscr{Y}_B^{(N)}$, we have*

1. $\langle \lambda|\lambda\rangle = \mathrm{eu}(T_\lambda) \neq 0$,
2. $\langle \sigma|\sigma\rangle = 0$ *for* $\sigma \in \mathscr{Y}^{(N)} \setminus \mathscr{Y}_B^{(N)}$ *such that* $\sigma \subset \lambda$ *and* $|\sigma| = |\lambda| - 1$, *and*
3. $\langle \pi|\pi\rangle = 0$ *for* $\pi \in \mathscr{Y}^{(N)} \setminus \mathscr{Y}_B^{(N)}$ *such that* $\lambda \subset \pi$ *and* $|\pi| = |\lambda| + 1$.

■

Proof. 1. For $\lambda \in \mathscr{Y}_B^{(N)}$, there is an N-tuple $\tilde{\sigma} \in \mathscr{Y}_B^{(N)}$ such that $\tilde{\sigma} \subset \lambda$ and $|\tilde{\sigma}| = |\lambda| - 1$. We have

$$\langle \lambda|\lambda\rangle = c\,\langle \lambda|D_{1,0}|\tilde{\sigma}\rangle = \frac{(-1)^N c}{pq}\,\langle \tilde{\sigma}|D_{-1,0}|\lambda\rangle = c'\,\langle \tilde{\sigma}|\tilde{\sigma}\rangle\,, \tag{3.2.46}$$

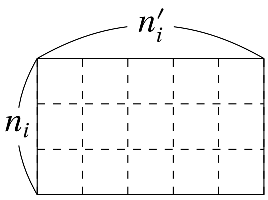
for nonzero coefficients c, c'. An induction on the degree in $L_B^{(N)}$ shows $\langle \lambda|\lambda\rangle = c''\,\langle \emptyset|\emptyset\rangle = c''$ for a certain $c'' \neq 0$.

2. and 3. For such σ and π, we have

$$\begin{aligned} \langle \sigma|\sigma\rangle &= c\,\langle \sigma|D_{-1,0}|\lambda\rangle = c'\,\langle \lambda|D_{1,0}|\sigma\rangle = 0\,,\\ \langle \pi|\pi\rangle &= c''\,\langle \pi|D_{1,0}|\lambda\rangle = c'''\,\langle \lambda|D_{1,0}|\pi\rangle = 0\,, \end{aligned} \tag{3.2.47}$$

for some parameters c, c', c'' and c'''. □

Null vectors surround $L_B^{(N)}$. We can construct a null vector $|\Phi_i\rangle$ in degree $n_i n_i'$ for each $1 \le i \le N$ as a rectangle Young diagram

$\Phi_i^{(i+1)} =$  $,\ \Phi_i^{(j)} = \emptyset,\ j \neq i+1\,,$ (3.2.48)

where we treat $i+1$ as 1 when $i = N$.

3.3 SH^c description of the level-rank duality

The level-rank duality concerns an N-datum $\langle (p,q),(n_i,n_i')_{1\le i\le N}\rangle$ satisfying

$$p = N, q = N + M\,, \tag{3.3.1}$$

for a pair (N, M) of coprime positive integers. We first re-express our notions for such a N-datum. The equation (3.1.8) implies that $n_i' = 1$ for any $1 \le i \le N$ and then the corresponding irreducible representation SH^c is labeled by a restricted datum $\langle (N, N+M), (n_i, 1)_{i=1,\cdots,N}\rangle$. We rather introduce integers $\tilde{n}_i \in \mathbb{Z}_{\geq 0} (1 \leq i \leq N)$ by

$$\begin{aligned} \tilde{n}_i &= n_i - 1\,,\ 1 \leq i \leq N\,, \\ \sum_{i=1}^{N} \tilde{n}_i &= M\,, \end{aligned} \tag{3.3.2}$$

and call a collection $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ a *restricted* N*-datum*. The irreducible representation of SH^c associated with such a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ has

$$\lambda \in \mathscr{Y}_B^{(N)} \Leftrightarrow \lambda^{(i+1)}(l + \tilde{n}_i) \leq \lambda^{(i)}(l), 1 \leq i \leq N\,, l \geq 1\,, \tag{3.3.3}$$

as its N-Burge condition for $\lambda \in \mathscr{Y}^{(N)}$ and the parameters $\varepsilon_{1,2}$ and $a_{1,\cdots,N}$ which are fixed to the following integers:

$$\begin{aligned} &\varepsilon_1 = q = N + M\,, \varepsilon_2 = -p = -N\,, \\ &a_i = -\sum_{j=1}^{N-1} (M - N\tilde{n}_j) \vec{e}_i \cdot \vec{\omega}_j + \frac{1}{2} M(N-1) \\ &\quad = M(i-1) + N \sum_{j=1}^{N-1} \tilde{n}_j \vec{e}_i \cdot \vec{\omega}_j \in \mathbb{Z},\ 1 \leq i \leq N\,. \end{aligned} \tag{3.3.4}$$

The latter parameters looks clear if we rewrite them as

$$\begin{aligned} a_{j+1} - a_j &= M - N\tilde{n}_j\,,\ 1 \leq j \leq N\,, \\ \sum_{i=1}^{N} a_i &= \frac{1}{2} NM(N-1). \end{aligned} \tag{3.3.5}$$

The level-rank duality claims that there exists a dual restricted M-datum for each restricted N-datum, and both has the same property in a certain sense. They share the same Virasoro subalgebra in $\mathscr{W}_N$ and in $\mathscr{W}_M$, and their representation spaces have the same q-dimension[†3][20, 21]. What we are going to show is that we can construct an algebra isomorphism between such two level-rank-dual representations by using the SH^c-descriptions we have just proved.

3.3.1 The labeling set $X_B^{(N)}$

The level-rank duality claims in part a connection $\mathscr{Y}_B^{(N)}$ with $\mathscr{Y}_B^{(M)}$ for $M \neq N$. This suggests that there is another interpretation of $\mathscr{Y}_B^{(N)}$, which is originally defined as the set of all the N-tuple Young diagrams satisfying the N-Burge condition. Recall that an element λ of $\mathscr{Y}_B^{(N)}$ is a map from $X^{(N)}$ to $\mathbb{Z}_{\geq 0}$ which meets not only the N-Burge condition but also the "N-Young" condition

$$\lambda^{(i)}(l+1) \leq \lambda^{(i)}(l),\ l \geq 1,\ 1 \leq i \leq N\,. \tag{3.3.6}$$

[†3] We use the symbol q in two ways. We have denoted by q an integer in a N-datum as well as a formal parameter in the q-dimension of a graded vector space. We hope that such an overlap will not create any confusion.

We want to merge the two conditions into a suitable one to see the level-rank duality manifestly. We first rewrite the labeling set $X^{(N)} = \{(i,k)|1 \le i \le N, k \ge 1\}$ and then re-express the N-Burge condition (3.3.3) as well as the N-Young condition.

Fix a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$. We assign an integer for each element of $X^{(N)}$ by

$$\mathscr{X} = \mathscr{X}\,(N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}) : X^{(N)} \to \mathbb{Z},\ (i,l) \mapsto a_i + N(l-1). \tag{3.3.7}$$

The map relates to each east-end box s along $\lambda \in \mathscr{Y}_B^{(N)}$ labeled by $(i,l) \in X^{(N)}$, by

$$c_1(s) = -\mathscr{X}(i,l) + (N+M)\lambda^{(i)}(l). \tag{3.3.8}$$

Comparing integers for two elements in $X^{(N)}$ associated with the N-Burge condition or with the N-Young condition, we obtain

$$\begin{aligned}\mathscr{X}(i+1, l+\tilde{n}_i) &= \mathscr{X}(i,l) + M\,,\\ \mathscr{X}(i, l+1) &= \mathscr{X}(i,l) + N\,.\end{aligned} \tag{3.3.9}$$

This map gives another interpretation of $X^{(N)}$:

Lemma 3.7. *With the notations above, we have*

1. $x+N, x+M \in \mathscr{X}(X^{(N)})$ *if* $x \in \mathscr{X}(X^{(N)})$, *and*
2. $\mathscr{X}$ *is injective.*

■

Proof. 1. Clear from (3.3.9).

2. We have $a_{j+1} - a_j = M - N\tilde{n}_j$ for $1 \le j \le N$, which means that $a_i - a_j (i \ne j)$ can not be divided by N. □

Definition 3.2. For a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$, we define a set $X_B^{(N)}$ of integers by

$$X_B^{(N)} = X_B^{(N)}(N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}) = \mathscr{X}(X^{(N)}) \subset \mathbb{Z}. \tag{3.3.10}$$

■

We may consider $\lambda \in \mathscr{Y}^{(N)}$ as a map

$$\lambda : X_B^{(N)} \to \mathbb{Z}_{\ge 0},\ \lambda(x) = \lambda^{(i)}(l),\ x = \mathscr{X}(i,l). \tag{3.3.11}$$

The N-Burge condition and the "Young-diagram" condition are then re-expressed as

$$\begin{aligned}\lambda^{(i+1)}(l+\tilde{n}_i) \le \lambda^{(i)}(l) &\Leftrightarrow \lambda(x+M) \le \lambda(x),\\ \lambda^{(i)}(l+1) \le \lambda^{(i)}(l) &\Leftrightarrow \lambda(x+N) \le \lambda(x).\end{aligned} \tag{3.3.12}$$

Therefore we obtain another interpretation of $\mathscr{Y}_B^{(N)}$ associated with a restricted N-datum:

$$\mathscr{Y}_B^{(N)} = \left\{\lambda : X_B^{(N)} \to \mathbb{Z}_{\ge 0} \,\middle|\, \lambda(x+M), \lambda(x+N) \le \lambda(x) \text{ for all } x \in X_B^{(N)}\right\}. \tag{3.3.13}$$

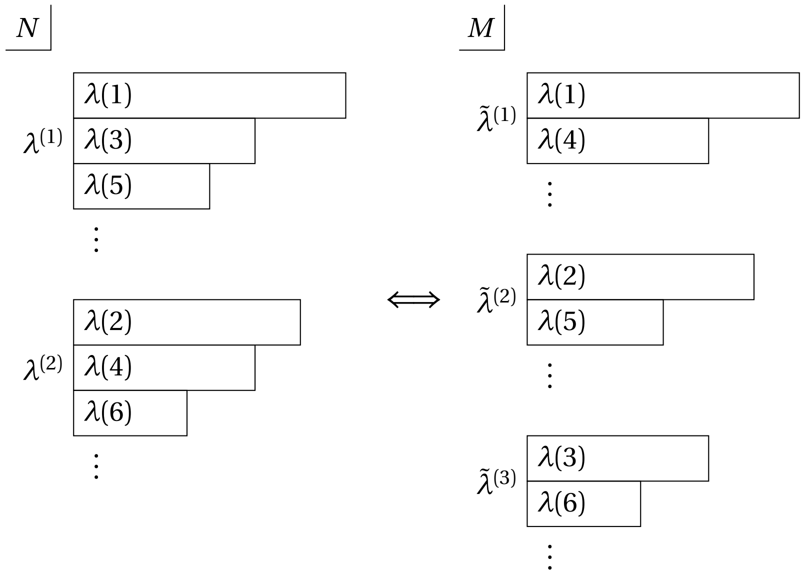


Figure 3.2: Two multiple Young diagrams λ and $\tilde{\lambda}$ related by shuffling rows. ($N=2, M=3$). We take $N=2, M=3$ and the set of labels $X_B^{(N)} = X_B^{(M)} = \{1, 2, \cdots\}$ for example.

3.3.2 The level-rank dual labeling set $X_B^{(M)} = X_B^{(N)}$

Fix a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ and an element $\lambda \in \mathscr{Y}_B^{(N)}$. Denoting by $s^{(x)}$ an east-end box along λ labeled by $x \in X_B^{(N)}$, we have

$$c_1(s^{(x)}) = -x + (N+M)\lambda(x). \tag{3.3.14}$$

We then obtain

$$D_{0,l+1}|\lambda\rangle = \sum_{x \in X_B^{(N)}} \prod_{r=0}^{\lambda(x)-1} \left(\frac{-x + r(N+M)}{N+M}\right)^l |\lambda\rangle\,, \tag{3.3.15}$$

Then if we construct a restricted M-datum $\langle M, N+M, (\tilde{m}_i)_{i=1,\cdots,M}\rangle$ which has the same labeling set $X_B^{(M)} = X_B^{(N)}$ as that of the N-datum, the two representation spaces $L_B^{(N)}$ and $L_B^{(M)}$ have the same spectrum with respect to the infinitely many commuting operators $D_{0,l+1} (l \geq 0)$. The element $\lambda \in \mathscr{Y}_B^{(N)}$ can be identified with an M-tuple Young diagram $\tilde{\lambda}$ as well as an N-tuple Young diagram by shuffling rows. We express such a shuffling in Figure 3.2. This shuffling induces a linear map

$$\mathscr{S}_1 : L_B^{(N)} \longrightarrow L_B^{(M)}\,,\ |\lambda\rangle \longmapsto |\tilde{\lambda}\rangle\,, \tag{3.3.16}$$

which commutes, at least, with any $D_{0,l+1}$.

We claim that such a dual restricted M-datum exists for any restricted N-datum. We prepare a disk expression[†4] before we prove it. Let $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ be a restricted n-datum. We encode the datum into a properly decreasing positive integer sequence $(x_i)_{i=1}^N$ by setting

$$x_1 = N+M\,,\ x_i - x_{i+1} = n_i = \tilde{n}_i + 1\,,\ 1 \leq i < N\,. \tag{3.3.17}$$

[†4] This disk was originally introduced in [20, 21].

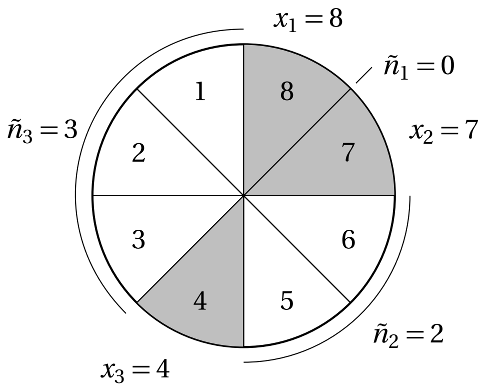


Figure 3.3: The disk expression of the sequence $(x_i)_{i=1}^N$. Here we take $N = 3, M = 5$ and $(n_1, n_2, n_3) = (1,3,4)$ for example.

Note that we have $x_N = n_N > 0$ and then we may consider $(x_i)_{i=1}^N \subset \mathbb{Z}_{N+M}$. We associate such a sequence with a disk divided into $(N+M)$ parts where the x_i-th piece is drawn black for each $1 \le i \le N$, while the other pieces are kept white. See an example of such a disk expression in Figure 3.3. We see that , for $1 \le i \le N$, the integer $\tilde{n}_i$ appears in the disk expression as the number of white pieces from the i-th black piece to the $(i+1)$-th one in the clockwise direction. (We repeat that $i+1$ should be treated as 1 if $i = N$.) Note that the $\mathbb{Z}_N$-redundancy in the N-datum is none other than the freedom that we may re-take a sequence by rotating the disk and selecting a black piece from N choices as the representative of the first element x_1.

Let us reverse the color in the disk. Each white piece becomes a black piece, and vise versa. Taking one new black piece as the representative of the first element $y_1 = N+M$, we obtain a new properly decreasing sequence $(y_i)_{i=1}^M$ of positive integers. Then we define M integers $(m_i)_{i=1}^M$ by

$$y_i - y_{i+1} = m_{M+1-i}\,,\ 1 \le i < M\,,\ m_1 = y_M\,. \tag{3.3.18}$$

and $(\tilde{m}_i)_{i=1}^M$ by

$$\tilde{m}_i = m_i - 1\,,\ 1 \le i \le M\,. \tag{3.3.19}$$

We draw Figure 3.4 which expresses a dual sequence with respect to Figure 3.3. Since we have

$$\sum_{i=1}^M m_i = y_1 = N+M\,, \tag{3.3.20}$$

then a triple $\langle M, N+M, (\tilde{m}_i)_{i=1}^M \rangle$ becomes a restricted M-datum. We call it *the dual restricted M-datum* for the original N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1}^N \rangle$. We note that, while there are M black pieces in the dual disk and then M choices for our first pick, such arbitrariness is identified with the group $\mathbb{Z}_M$ of cyclic shifts of the indices i in $\tilde{m}_i$. We may neglect the cyclic shifts since the resultant datum gives the same representation of SH$^{(M)}$.

The dual datum is what we want to construct! To prove it, we focus on nonzero elements in the two data. We see that the nonzero elements in $(\tilde{n}_i)$, $(\tilde{m}_i)$, express the ratios of each connected component of white, resp. black, pieces in the original disk associated (x_i) (see Figure 3.5). Let us denote by $\{v(f)\}_{f=1}^S$ the properly increasing sequence of positive integers

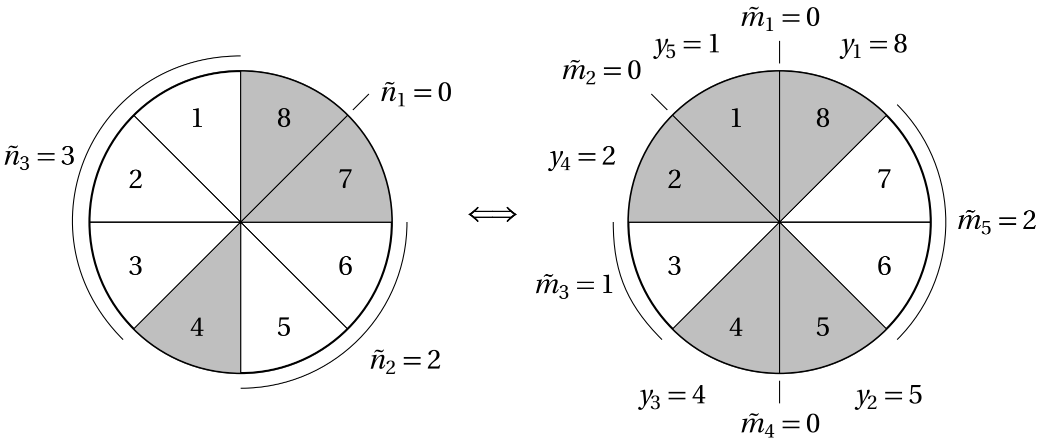


Figure 3.4: The dual of the disk in Figure 3.3.

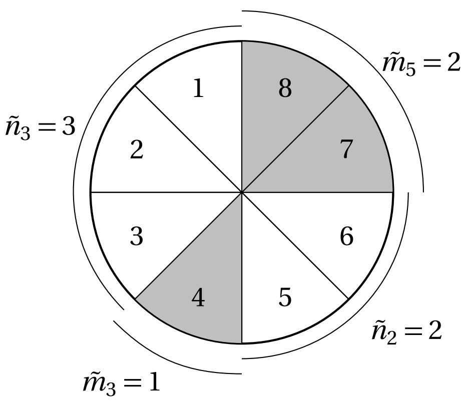


Figure 3.5: Nonzero elements in $(\tilde{n}_i)$ and $(\tilde{m}_i)$.

representing nonzero elements in $(\tilde{n}_i)$, $\tilde{n}_{v(f)} > 0$, and by $\{w(f)\}_{f=1}^{S}$ the properly decreasing sequence determined by $\tilde{m}_{w(f)} > 0$. Here we denote by S the number of the white components in the disk. Using the $\mathbb{Z}_M$-degeneracy, we may choose the white component with ratio $\tilde{n}_{v(f)}$ appear just after the black component with ratio $\tilde{m}_{w(f)}$ in the clockwise direction, as in Figure 3.5. This construction immediately provides the following property with the two data:

Lemma 3.8. *With the notations above, we have*

$$\tilde{n}_{v(f)} = w(f) - w(f+1), \quad \tilde{m}_{w(f)} = v(f) - v(f-1), \tag{3.3.21}$$

for $1 \le f \le S$*, where* $v(0) = w(S+1) = 0$. ■

We start to prove that the above two data $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ and $\langle M, N+M, (\tilde{m}_i)_{i=1,\cdots,M}\rangle$ have the same labeling set, $X_B^{(M)} = X_B^{(N)}$. We first consider the generators of the labeling set $X_B^{(N)}$:

Lemma 3.9. *With the notations above, the labeling set* $X_B^{(N)}$ *is generated by* $(a_{v(f)+1})_{f=1}^{L}$ *and the recursive relation,* $x \in X_B^{(N)} \Rightarrow x+M, x+N \in X_B^{(N)}$. ■

Proof. It is clear that $X_B^{(N)}$ is generated by $(a_i)_{i=1}^{N}$. We have $a_{i+1} = a_i + M$ if $\tilde{n}_i = 0$, and then $X_B^{(N)}$ is generated by the elements a_{i+1} with $\tilde{n}_i > 0$. □

Applying the above lemma to the dual, we see that $X_B^{(M)}$ is generated by $(\tilde{a}_{w(f)+1})_{f=1}^{L}$. Here we denote by $(\tilde{a}_i)_{i=1}^{M}$ the parameters (a_i) for the datum $\langle M, N+M, (\tilde{m}_i)_{i=1}^{M}\rangle$.

We are ready to prove the claim:

Proposition 3.5. *Let N and M be mutually prime positive integers. For each restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$, there is a restricted M-datum $\langle M, N+M, (\tilde{m}_i)_{i=1,\cdots,M}\rangle$ which has the same labeling set:* $X_B^{(M)} = X_B^{(N)}$. ■

Proof. Let us take an arbitrary restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$, and set $\langle M, N+M, (\tilde{m}_i)_{i=1,\cdots,M}\rangle$ as the dual restricted M-datum. Two labeling sets $X_B^{(N)}$ and $X_B^{(M)}$ share one generator since we have

$$\begin{aligned}
a_{v(S)+1} &= -M(N-v(S)) + a_1 = M\,v(S) - \sum_{f=1}^{S} v(f)\tilde{n}_{v(f)} = MN - \sum_{f=1}^{S} v(f)(w(f) - w(f+1)) \\
&= MN - \sum_{f=1}^{S} w(f)(v(f) - v(f-1)), \\
\tilde{a}_{w(1)+1} &= -N(M-w(1)) + \tilde{a}_1 = N\,w(1) - \sum_{f=1}^{S} w(f)\tilde{m}_{w(f)} = MN - \sum_{f=1}^{S} w(f)(v(f) - v(f-1)) \\
&= a_{v(S)+1}\,.
\end{aligned} \tag{3.3.22}$$

The cyclic relations

$$\begin{aligned}
&a_{v(f)+1} - a_{v(f-1)+1} = M(v(f) - v(f-1)) - N\tilde{n}_{v(f)} = M\tilde{m}_{w(f)} - N\tilde{n}_{v(f)}\,, 1 < f \le S\,, \\
&a_{v(1)+1} - a_{v(S)+1} = -M\sum_{f>1}^{S} \tilde{m}_{w(f)} + N\sum_{f>1}^{S} \tilde{n}_{v(f)} = M\tilde{m}_{w(1)} - N\tilde{n}_{v(1)}\,,
\end{aligned} \tag{3.3.23}$$

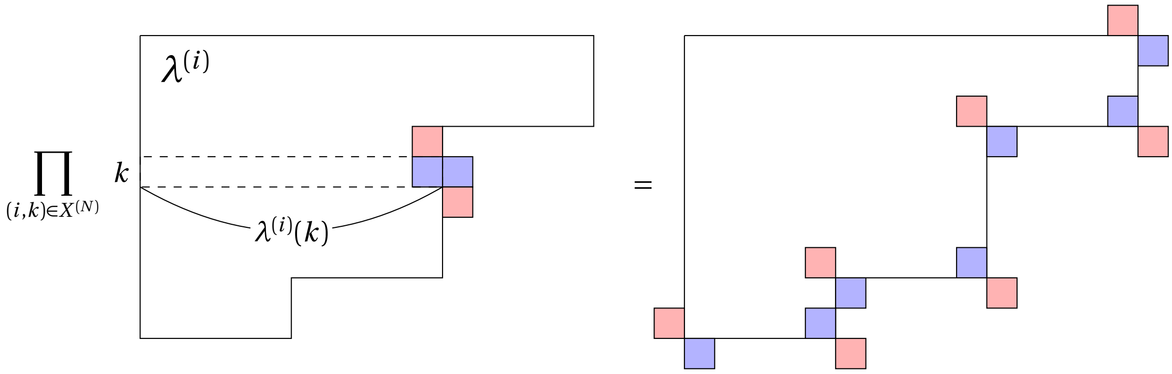


Figure 3.6: Diagrammatic expression of the rewriting for E_l.

and

$$\begin{aligned}
&\tilde{a}_{w(f+1)+1}-\tilde{a}_{w(f)+1}=-N(w(f)-w(f+1))+M\tilde{m}_{w(f)}=M\tilde{m}_{w(f)}-N\tilde{n}_{v(f)},1\le f<S,\\
&\tilde{a}_{w(1)+1}-\tilde{a}_{w(S)+1}=N\sum_{f=1}^{S-1}\tilde{n}_{v(f)}-M\sum_{f=1}^{S-1}\tilde{m}_{w(f)}=M\tilde{m}_{v(S)}-N\tilde{n}_{v(S)},
\end{aligned} \tag{3.3.24}$$

shows they are identical. □

Conversely, we also have a dual restricted N-datum for each restricted M-datum by reversing the clockwise direction along the disk. This realizes an isomorphism between the set of all the restricted N-datum and that of all the restricted M-datum for any pair (N,M) of coprime integers.

3.3.3 The level-rank dual mapping $\mathscr{S}_1$

We have seen that, for each restricted N-datum, there is a dual restricted M-datum which gives the same labeling set $X_B^{(M)}=X_B^{(N)}$, and then they have the same set of eigenvalues with respect to the infinitely many commuting operators $D_{0,l+1}$. Such eigenvalues are invariant under shuffling an N-tuple into an M-tuple.

We move to other degree-0 operators $E_l(l\ge 0)$ in (2.3.20). We fix a restricted N-datum $\langle N,N+M,(\tilde{n}_i)_{i=1,\cdots,N}\rangle$. We see from Figure 2.4 that we can rewrite their actions to $|\lambda\rangle\in L_B^{(N)}$ in a symmetric form. For $\lambda\in\mathscr{Y}_B^{(N)}$, we have

$$\begin{aligned}
&\left[1+\frac{M}{N+M}\sum_{l\ge0}\zeta^{l+1}E_l\right]|\lambda\rangle\\
&=\left[\prod_{x\in X_B^{(N)}}\frac{\left(1+\frac{\zeta}{N+M}(x+M-(N+M)\lambda(x))\right)\left(1+\frac{\zeta}{N+M}(x+N-(N+M)\lambda(x))\right)}{\left(1+\frac{\zeta}{N+M}(x-(N+M)\lambda(x))\right)\left(1+\frac{\zeta}{N+M}(x+N+M-(N+M)\lambda(x))\right)}\right]|\lambda\rangle\,.
\end{aligned} \tag{3.3.25}$$

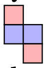
We draw Figure 3.6 to express the above rewriting. The right side of the above equation is actually symmetric. In other words, denoting by $E'_l(l\ge0)$ the dual side operators, two operators $ME_l\in\mathrm{SH}^{(N)}\subset\mathrm{End}(L_B^{(N)})$ and $NE'_l\in\mathrm{SH}^{(M)}\subset\mathrm{End}(L_B^{(M)})$ act on their representation spaces identically. For any degree-0 operator in SH^c, the vacuum vector in $L_B^{(N)}$ and that in $L_B^{(M)}$ have the same eigenvalue after rescaling $\sqrt{M}D^{(N)}_{\pm1,l} = \sqrt{N}D^{(M)}_{\pm1,l}, D^{(N)}_{0,l+1} = D^{(M)}_{0,l+1}$ [†5]. Two algebras $\mathrm{SH}^{(N)}$ and $\mathrm{SH}^{(M)}$ also share the same function $\kappa(\zeta)$ which was introduced in (2.3.43) to express a lengthy quadratic commutation relation. This function is indeed symmetric under the replacement;

$$\kappa(\zeta) = (\zeta+1)\left(\zeta+\frac{\varepsilon_2}{\varepsilon_1}\right)\left(\zeta-\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right) = (\zeta+1)\left(\zeta-\frac{N}{N+M}\right)\left(\zeta-\frac{M}{N+M}\right). \tag{3.3.26}$$

They share the whole defining relations[†6] after the rescaling, and then the two representations are of the same algebra. In other words, the assignments

$$\mathscr{S}_1(D_{\pm1,l}) = \left(\frac{N}{M}\right)^{\frac{1}{2}} D_{\pm1,l}\,,\ \mathscr{S}_1(D_{0,l+1}) = D_{0,l+1}\,,\ \mathscr{S}_1(E_l) = \left(\frac{N}{M}\right)^{-1} E_l\,, \tag{3.3.27}$$

induce an algebra map from $\mathrm{SH}^{(N)}$ to $\mathrm{SH}^{(M)}$.

One can manifest the duality by taking “geometric means" of ladder operators. We should note that a colored tetromino  is symmetric under the replace $N \longleftrightarrow M$, and then plays a fundamental role with the level-rank duality. We can see from Figure 2.2 and Figure 2.3 that we can construct the tetromino from the geometric means of the coefficients in $D_{1,l}$ and $D_{-1,l}$. This observation leads us to introduce their “geometric means" $\mathscr{D}_{\pm1,l}$ $(l \geq 0)$ by

$$\begin{aligned}
\mathscr{D}_{1,l}\,|\lambda\rangle \equiv{}& \frac{(-\varepsilon_1\varepsilon_2)^{\frac{1}{2}}}{\varepsilon_1+\varepsilon_2} \sum_{\substack{\pi\supset\lambda \\ |\pi|=|\lambda|+1}} \left(\frac{c_1\left(V_\pi^*-V_\lambda^*\right)}{\varepsilon_1}\right)^l \\
&\times \mathrm{eu}^{\frac{1}{2}}\left(((1-t_1)(1-t_2)V_\pi^*-W^*+t_1^{-1}t_2^{-1}W^*-(1-t_1^{-1})(1-t_2^{-1})V_\lambda^*)\otimes(V_\pi-V_\lambda)\right)|\pi\rangle \\
={}& \left(\frac{-1}{\varepsilon_1+\varepsilon_2}\right)^{\frac{1}{2}} \sum_{\substack{\pi\supset\lambda \\ |\pi|=|\lambda|+1}} \left(\frac{c_1\left(V_\pi^*-V_\lambda^*\right)}{\varepsilon_1}\right)^l \\
&\times \mathrm{eu}^{\frac{1}{2}}\left(((1-t_1)(1-t_2)V_\lambda^*-W^*+t_1^{-1}t_2^{-1}W^*-(1-t_1^{-1})(1-t_2^{-1})V_\lambda^*)\otimes(V_\pi-V_\lambda)+1\right)|\pi\rangle\,,
\end{aligned} \tag{3.3.28}$$

and

$$\begin{aligned}
\mathscr{D}_{-1,l}\,|\lambda\rangle \equiv{}& \frac{(-\varepsilon_1\varepsilon_2)^{\frac{1}{2}}}{\varepsilon_1+\varepsilon_2} \sum_{\substack{\sigma\subset\lambda \\ |\sigma|=|\lambda|-1}} \left(\frac{c_1\left(V_\lambda^*-V_\sigma^*\right)}{\varepsilon_1}\right)^l \\
&\times \mathrm{eu}^{\frac{1}{2}}\left(((1-t_1)(1-t_2)V_\lambda^*-W^*+t_1^{-1}t_2^{-1}W^*-(1-t_1^{-1})(1-t_2^{-1})V_\sigma^*)\otimes(V_\lambda-V_\sigma)\right)|\sigma\rangle \\
={}& \left(\frac{1}{\varepsilon_1+\varepsilon_2}\right)^{\frac{1}{2}} \sum_{\substack{\sigma\subset\lambda \\ |\sigma|=|\lambda|-1}} \left(\frac{c_1\left(V_\lambda^*-V_\sigma^*\right)}{\varepsilon_1}\right)^l \\
&\times \mathrm{eu}^{\frac{1}{2}}\left(((1-t_1)(1-t_2)V_\lambda^*-W^*+t_1^{-1}t_2^{-1}W^*-(1-t_1^{-1})(1-t_2^{-1})V_\lambda^*)\otimes(V_\lambda-V_\sigma)+1\right)|\sigma\rangle\,,
\end{aligned} \tag{3.3.29}$$

[†5] We regard $D_x^{(N)} \in \mathrm{SH}^{(N)}$ and so on.

[†6] There are no other relations among $D_{0,l+1}, D_{\pm1,l}$ which are not generated from the defining relations. It can be proven by a similar reasoning in [2, 47]. Each variable in the polynomials appearing in their proofs ranges a subset of $\mathbb{R}$ for our cases, but this does not matter since the subset contains an infinite number of elements.

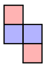
after rewritten forms of $D_{\pm1,l}$:

$$\begin{aligned} D_{1,l}\,|\lambda\rangle &= \varepsilon_1\varepsilon_2 \sum_{\substack{\pi\supset\lambda\\ |\pi|=|\lambda|+1}} \left(\frac{c_1\left(V_\pi^*-V_\lambda^*\right)}{\varepsilon_1}\right)^l \mathrm{eu}\Big(((1-t_1)(1-t_2)V_\pi^*-W^*)\otimes(V_\pi-V_\lambda)-t_1t_2\Big)|\pi\rangle\,,\\ D_{-1,l}\,|\lambda\rangle &= -\sum_{\substack{\sigma\subset\lambda\\ |\sigma|=|\lambda|-n}} \left(\frac{c_1\left(V_\lambda^*-V_\sigma^*\right)}{\varepsilon_1}\right)^l \mathrm{eu}\Big((t_1^{-1}t_2^{-1}W^*-(1-t_1^{-1})(1-t_2^{-1})V_\sigma^*)\otimes(V_\lambda-V_\sigma)-t_1t_2\Big)|\sigma\rangle\,. \end{aligned} \tag{3.3.30}$$

These geometric means were originally introduced in [38], and we can obtain similar iterated forms in Lemma 2.2 and 2.3. Three classes of operators $\mathscr{D}_{\pm1,l}$ and of $\mathscr{D}_{0,l+1}\equiv D_{0,l+1}$ then generate another SH$^{(N)}$ with the same nonlinear terms $\mathscr{E}_l=E_l$. This new algebra acts irreducibly on the same space $L_B^{(N)}$ even when we specialize parameters to minimal model representations since we have discussed the coefficients in $D_{1,l}$ and $D_{-1,l}$ in a completely parallel manner.

For a restricted N-datum $\langle N,N+M,(\tilde{n}_i)_{i=1,\cdots,N}\rangle$, we can reduce the coefficients with $\mathscr{D}_{\pm1,l}$, by a similar way in Figure 3.6, to

$$\begin{aligned} \mathscr{D}_{1,l}\,|\lambda\rangle &= \left(\frac{-1}{M}\right)^{\frac12} \sum_{\substack{x\in X_B^{(N)}\\ \lambda(x)<\lambda(x-M)\\ \lambda(x)<\lambda(x-N)}} \left(\frac{c_1(s^{(x)})}{N+M}\right)^l\\ &\quad\times\prod_{\substack{y\in X_B^{(N)}\\ y\neq x}}\left[\frac{c_1(t_2s^{(y)})-c_1(s^{(x)})}{c_1(s^{(y)})-c_1(s^{(x)})}\,\frac{c_1(t_1^{-1}t_2^{-1}s^{(y)})-c_1(s^{(x)})}{c_1(t_1^{-1}s^{(y)})-c_1(s^{(x)})}\right]^{\frac12}|\lambda\oplus s^{(x)}\rangle\,,\\ \mathscr{D}_{-1,l}\,|\lambda\rangle &= \left(\frac{1}{M}\right)^{\frac12} \sum_{\substack{x\in X_B^{(N)}\\ \lambda(x)>\lambda(x+M)\\ \lambda(x)>\lambda(x+N)}} \left(\frac{c_1(t_1^{-1}s^{(x)})}{N+M}\right)^l\\ &\quad\times\prod_{\substack{y\in X_B^{(N)}\\ y\neq x}}\left[\frac{c_1(t_2s^{(y)})-c_1(t_1^{-1}s^{(x)})}{c_1(s^{(y)})-c_1(t_1^{-1}s^{(x)})}\,\frac{c_1(t_1^{-1}t_2^{-1}s^{(y)})-c_1(t_1^{-1}s^{(x)})}{c_1(t_1^{-1}s^{(y)})-c_1(t_1^{-1}s^{(x)})}\right]^{\frac12}|\lambda\ominus t_1^{-1}s^{(x)}\rangle\,, \end{aligned} \tag{3.3.31}$$

for $\lambda\in\mathscr{Y}_B^{(N)}$, where we mean by $\lambda\oplus s^{(x)}$, $\lambda\ominus t_1^{-1}s^{(x)}$ a diagram λ plus a box $s^{(x)}$, resp. minus a box $t_1^{-1}s^{(x)}$. The tetromino  appears as the factors in the above products, which makes $\sqrt{M}\mathscr{D}_{\pm1,l}$ symmetric under the replacement $N\longleftrightarrow M$.

As a result, we have a manifestation of the level-rank duality as an equivalence between two representations of the nonlinear algebra SH^c:

Theorem 3.2. *Let (N,M) be a pair of mutually prime positive integers, $\langle N,N+M,(\tilde{n}_i)_{i=1,\cdots,N}\rangle$ be a restricted N-datum and $\langle M,N+M,(\tilde{m}_i)_{i=1,\cdots,M}\rangle$ be its dual restricted M-datum.*

1. *The two represented algebras $\mathrm{SH}^{(N)}\subset\mathrm{End}(L_B^{(N)})$, $\mathrm{SH}^{(M)}\subset\mathrm{End}(L_B^{(M)})$ of SH^c are isomorphic to each other with the rescaling $\sqrt{M}D_{\pm1,l}^{(N)}=\sqrt{N}D_{\pm1,l}^{(M)}$, $D_{0,l+1}^{(N)}=D_{0,l+1}^{(M)}$ and $M\,E_l^{(N)}=N\,E_l^{(M)}$.*
2. *The two representations associated with the restricted data have the same spectrum with respect to degree-0 operators $D_{0,l+1}$ and E_l under the rescaling isomorphism.*

3. *Denote by $\mathscr{S}_1 : L_B^{(N)} \to L_B^{(M)}$ the linear map (3.3.16), $|\lambda\rangle \mapsto |\tilde{\lambda}\rangle$, associated with the shuffling of $\mathscr{Y}_B^{(N)}$. Then we have the following relations for $\lambda \in \mathscr{Y}_B^{(N)}$:*

$$
\begin{aligned}
&\mathscr{S}_1(\mathscr{D}_{0,l+1}|\lambda\rangle) = \mathscr{D}_{0,l+1}(\mathscr{S}_1|\lambda\rangle) = \mathscr{D}_{0,l+1}|\tilde{\lambda}\rangle\ ,\ l \geq 0\,,\\
&\mathscr{S}_1(\sqrt{M}\mathscr{D}_{\pm1,l}|\lambda\rangle) = \sqrt{N}\mathscr{D}_{\pm1,l}(\mathscr{S}_1|\lambda\rangle) = \sqrt{N}\mathscr{D}_{\pm1,l}|\tilde{\lambda}\rangle\ ,\ l \geq 0\,,\\
&\mathscr{S}_1(M\mathscr{E}_l|\lambda\rangle) = N\mathscr{E}_l(\mathscr{S}_1|\lambda\rangle) = N\mathscr{E}_l|\tilde{\lambda}\rangle\ ,\ l \geq 0\,,
\end{aligned}
\tag{3.3.32}
$$

where $\mathscr{D}_x$ are the geometric means (3.3.28) of the algebra SH^c*, and $\mathscr{E}_{l+k} = [\mathscr{D}_{-1,k}, \mathscr{D}_{1,l}]$.*

■

One may normalize $\mathscr{S}_1$ by a certain diagonal map to obtain $\langle\lambda|{}^t\mathscr{S}_1\mathscr{S}_1|\lambda\rangle = \langle\tilde{\lambda}|\tilde{\lambda}\rangle$.

Now we compare their Virasoro–Heisenberg subalgebras. On the N-side, we have zero modes

$$
\begin{aligned}
\alpha_0 &= \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1} E_1 = \left(\frac{N+M}{NM}\right) M E_1\,,\\
H_0 &= \frac{1}{2}\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1} E_2 = \frac{1}{2}\left(\frac{N+M}{NM}\right) M E_2\,,
\end{aligned}
\tag{3.3.33}
$$

and the central charges

$$
\begin{aligned}
K_N &= \left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1} N = N+M\,,\\
C_N &= 1+(N-1)\left(1-\left(-\frac{\varepsilon_2}{\varepsilon_1}\right)^{-1}\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)^2 N(N+1)\right) = 1-\frac{(N-1)(M-1)(N+M+NM)}{N+M}\,,
\end{aligned}
\tag{3.3.34}
$$

which are all symmetric under the replacement $N \longleftrightarrow M$ and the rescaling. Therefore the two vacuum states have the same conformal dimension as well as the Heisenberg weight, as was shown in [20, 21]. We can also intertwine nonzero modes α_l, H_l with certain constant rescaling factors.

3.4 On the Lee–Yang singularity and the Rogers-Ramanujan identity

It is now clear that the vector space $L_B^{(N)}$ for a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ has the same q-dimension as that of its dual, which was originally shown in [21]. We revisit this coincidence from the viewpoint of SH^c.

3.4.1 A partial ordering $<_{X_B^{(N)}}$ in $X_B^{(N)}$

Fix a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ and set $X = X_B^{(N)}$ just for an abbreviation. Recall that the N-Burge and N-Young conditions are reduced to a class of inequalities on a map $\lambda : X \to \mathbb{Z}_{\geq 0}$,

$$
\lambda(x+M) \leq \lambda(x),\ \lambda(x+N) \leq \lambda(x),\ x \in X \subset \mathbb{Z}. \tag{3.4.1}
$$

Drawing an analogy with Young diagrams, it is natural to associate a partial ordering $<_X$ with X, which is generated by relations

$$x <_X x+M\,,\, x <_X x+N\,,\, x \in X\,, \tag{3.4.2}$$

and then the above condition for λ is rewritten as

$$x \geq_X y \Rightarrow \lambda(x) \leq \lambda(y)\,,\ x, y \in X\,. \tag{3.4.3}$$

Such a partition λ is known for mathematicians as a P-partition over the partially ordered set $(X, <_X)$, and its combinatorics is described in a famous book [44]. Denoting by $\mathscr{A}(X)$ the set of all the P-partitions over X, its *generating function*

$$F_X(q) \equiv \sum_{\lambda \in \mathscr{A}(X)} q^{|\lambda|}\,, \tag{3.4.4}$$

is none other than the q-dimension $\dim_q L_B^{(N)}$ of the vector space $L_B^{(N)}$.

Such a P-partition is also known as a cylindric partition. This notion was introduced in [51] and its generating function was given in [52] as a product form:

Proposition 3.6 ([52])**.** *With the notations above, we have*

$$F_X(q) = \frac{(q^{N+M}; q^{N+M})_\infty^{N-1} \prod_{i<j} (q^{\sum_{l=i}^{j-1} n_l}; q^{N+M})_\infty \prod_{i>j} (q^{N+M-\sum_{l=j}^{i-1} n_l}; q^{N+M})_\infty}{(q;q)_\infty^N}\,, \tag{3.4.5}$$

where $(a;q)_m \equiv \prod_{l=0}^{m-1}(1-aq^l)$ *is the* q*-Pochhammer symbol.* ∎

The above formula is none other than what is obtained as the character formula for (restricted) minimal models of W-algebras [21, 46, 13]. We refer to [53] and references therein for readers who want to see connections between W-algebras and cylindric partitions. The level-rank duality shows that we obtain the same generating function when we concern the dual datum, which was proved in [21]. From the viewpoint of SH^c, the coincidence stems from the fact that a partially ordered set is shared by two representations which are level-rank-dual to each other.

3.4.2 The Lee–Yang singularity and the Rogers-Ramanujan identity

The partially ordered set plays a fundamental role with a connection between the Lee–Yang singularity and the Rogers-Ramanujan identity. Here we revisit the connection, which was first noted in [46], from a viewpoint of P-partitions.

There are only two inequivalent restricted 2-data for $(N, M) = (2, 3)$,

$$\langle 2, 5, (1,2)\rangle\,,\ \langle 2, 5, (3,0)\rangle\,. \tag{3.4.6}$$

The minimal model associated with $(N, M) = (2, 3)$ is often called the Lee–Yang singularity. We have $a_1 = 2, a_2 = 1$ for the datum $\langle 2, 5, (1,2)\rangle$ and then its labeling set becomes

$$X_1 = \mathbb{Z}_{>0}\,. \tag{3.4.7}$$

We also have $a_1 = 0, a_2 = 3$ and

$$X_2 = \mathbb{Z}_{\geq 0} \setminus \{1\}\,, \tag{3.4.8}$$

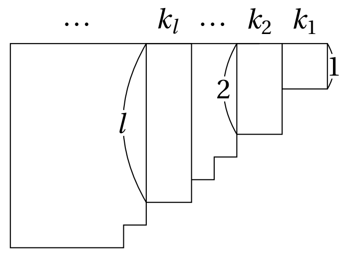


Figure 3.7: A rectangle decomposition of a Young diagram

as its labeling set for the datum $\langle 2,5,(3,0)\rangle$. We have

$$\begin{aligned} F_{X_1}(q) &= \frac{(q^5;q^5)_\infty (q^{n_1};q^5)_\infty (q^{n_2};q^5)_\infty}{(q;q)_\infty^2} \\ &= \frac{1}{(q;q)_\infty (q^1;q^5)_\infty (q^4;q^5)_\infty}, \\ F_{X_2}(q) &= \frac{1}{(q;q)_\infty (q^2;q^5)_\infty (q^3;q^5)_\infty}. \end{aligned} \tag{3.4.9}$$

Note that we can rewrite them into sum forms (times $(q;q)_\infty^{-1}$) by using the Rogers–Ramanujan identities [45];

$$\begin{aligned} \sum_{k=0}^\infty \frac{q^{k^2}}{(q;q)_k} &= \frac{1}{(q^1;q^5)_\infty (q^4;q^5)_\infty}, \\ \sum_{k=0}^\infty \frac{q^{k(k+1)}}{(q;q)_k} &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \end{aligned} \tag{3.4.10}$$

We can derive these sum forms by using a P-partition counting method, which is described in the book [44]. The key point for this method is the identity associated with Young diagrams,

$$\sum_{\lambda \in \mathscr{Y}^{(1)}} q^{|\lambda|} = \prod_{l=1}^\infty \sum_{k_l=0}^\infty q^{l k_l} = \frac{1}{(q;q)_\infty}, \tag{3.4.11}$$

which is obtained from a consideration with a rectangle decomposition of a Young diagram in Figure 3.7.

A toy example

Consider a partially ordered set $P = \{p_1, p_2, p_3\}$ of three points as an example, whose ordering is given by $p_1 <_P p_2$ and $p_1 <_P p_3$. Its Hasse diagram[†7] is represented as Figure 3.8. Each P-partition λ can be considered as a Young diagram with respect to a total ordering $<_T$ of P such that

$$x <_P y \Rightarrow x <_T y, \tag{3.4.12}$$

[†7] Given a partially ordered set P, its Hasse diagram is defined by assigning a point for each $x \in P$ and drawing a line between two points representing $x, y \in P$ if $x <_P y$ and there is no element $z \in P$ such that $x <_P z <_P y$. For our cases, we assign a line between x and $x+M$ and another one between x and $x+N$ for each $x \in X$.

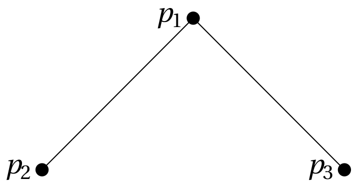


Figure 3.8: The Hasse diagram of $P = \{p_1, p_2, p_3\}$

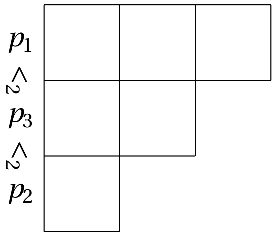


Figure 3.9: Each P-partition is a Young diagram with respect to a total ordering.

and

$$x <_T y \Rightarrow \lambda(x) \geq \lambda(y). \tag{3.4.13}$$

For example, for the P-partition $(p_1, p_2, p_3) \mapsto (3,1,2)$, we linearize P by a total order $p_1 <_2 p_3 <_2 p_2$ and then the corresponding diagram is expressed by a Young diagram appearing in Figure 3.9. There are just two total orderings $p_1 <_1 p_2 <_1 p_3$ and $p_1 <_2 p_3 <_2 p_2$.

We want to obtain a one-to-one correspondence between a P-partition and a pair consisting of a Young diagram and a total ordering compatible with the partial ordering of P. Note that, however, both of the two total orderings induce a Young diagram for a P-partition $(p_1, p_2, p_3) \mapsto (2,1,1)$. This example tells us that we should *not* count all pairs. To count without overlaps, we fix one total ordering, say $p_1 <_1 p_2 <_1 p_3$, as the canonical one, and we associate such a P-partition with the pair of this canonical ordering and the corresponding Young diagram. As a result, we separate $\mathscr{A}(P)$ into the two set S_1, S_2:

$$\begin{aligned} &\mathscr{A}(P) = S_1 \sqcup S_2, \\ &\quad S_1 \equiv \{\lambda \in \mathscr{A}(P) \mid \lambda(p_1) \geq \lambda(p_2) \geq \lambda(p_3)\}, \\ &\quad S_2 \equiv \{\lambda \in \mathscr{A}(P) \mid \lambda(p_1) \geq \lambda(p_3) > \lambda(p_2)\}. \end{aligned} \tag{3.4.14}$$

We have

$$\sum_{\lambda \in S_1} q^{|\lambda|} = \frac{1}{(q;q)_3}. \tag{3.4.15}$$

Through a consideration in Figure 3.10, we also have

$$\sum_{\lambda \in S_2} q^{|\lambda|} = q^2 \sum_{\lambda \in S_1} q^{|\lambda|} = \frac{q^2}{(q;q)_3}. \tag{3.4.16}$$

Then we have $Z_P(q) = (1+q^2)/(q;q)_3$. Note that the last factor q^2 comes from where the total ordering $p_1 <_2 p_3 <_2 p_2$ is different from the canonical one $p_1 <_1 p_2 <_1 p_3$. Let us denote by $T(i)$

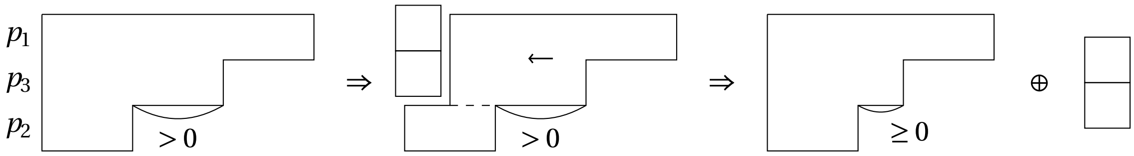


Figure 3.10: Modification of the condition $\lambda(p_3) > \lambda(p_2)$ in S_2

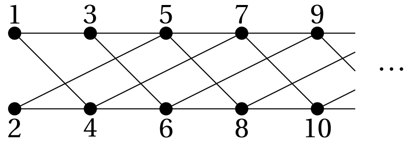


Figure 3.11: The Hasse diagram of X_1

the i-th lowest element in P with respect to a compatible total ordering T and set

$$D_T = \{\, i \in \mathbb{Z}_{>0} \mid T(i) >_C T(i+1)\}\,. \tag{3.4.17}$$

where we denote by $<_C$ the canonical total ordering. Then we have

$$\sum_{\lambda \in S_i} q^{|\lambda|} = \frac{\prod_{j\in D_i} q^j}{(q;q)_{|P|}}\,, \tag{3.4.18}$$

for each $i = 1, 2$, where $|P| = 3$ is the cardinality of P.

We move to our case with a general partially ordered set $X = X_B^{(N)}$, which is bounded below with finite local minima and whose Hasse diagram is connected. We denote by $\mathscr{L}(X)$ the set[†8] of all the total orderings compatible to the partial ordering. We take the total ordering induced from the injection $X_B^{(N)} \subset \mathbb{Z}$ as the canonical one and define D_T as described above for each compatible total ordering $T \in \mathscr{L}(X)$. We have the following counting formula:[†9]

Proposition 3.7 ([44])**.** *With the notations above, we have*

$$F_X(q) = \sum_{\lambda \in \mathscr{A}(X)} q^{|\lambda|} = \frac{\sum_{\pi\in\mathscr{L}(X)} \prod_{j\in D_\pi} q^j}{(q;q)_{|X|}}, \tag{3.4.19}$$

where $|X|$ is the cardinality of X.

It is difficult to determine the Jordan–Hölder set in general, but we can do that for the Lee–Yang singularity. The Hasse diagram of X_1 is given by Figure 3.11. We see that, for any element T of $\mathscr{L}(X_1)$, the compatibility leads to

$$T(1) = 1, \quad \text{or} \quad (T(1), T(2)) = (2, 1)\,. \tag{3.4.20}$$

We have $1 \notin D_T$ for the first case and $1 \in D_T, 2 \notin D_T$ for the latter. Note that we have the same Hasse diagram even after cutting one element $1 \in X_1$ or two $1, 2 \in X_1$. Therefore a similar analysis goes well for higher elements of X, and then we can identify $\mathscr{L}(X)$ with a set of integers

$$\{1 \leq t_1 < t_2 < \cdots < t_k \mid k \in \mathbb{Z}_{\geq 0},\, t_i - t_{i-1} \geq 2\}, \tag{3.4.21}$$

[†8]The set $\mathscr{L}(X)$ is often called the Jordan–Hölder set of X in mathematics.

[†9]The corresponding statement in the book [44] is for partially ordered sets with *finite* cardinality. However, it can be enlarged for our infinite cases since boxes are piled up from the bottom.

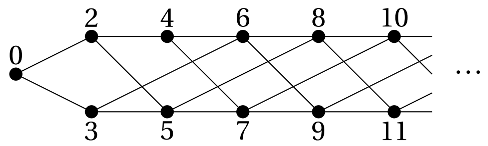


Figure 3.12: The Hasse diagram of X_2

which is identical to the corresponding D_T. The above set is none other than an example of Gordon's integer partitions[54, 55]. As a result, we have

$$\begin{aligned} F_{X_1}(q) &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \sum_{\substack{1\le t_1<\cdots<t_k \\ t_i-t_{i-1}\ge 2}} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \sum_{0\le t_1\le\cdots\le t_k} q^{k+\sum_{j=0}^{k-1} 2j+\sum_{j=1}^k t_j} \\ &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} q^{k^2} \sum_{0\le t_1\le\cdots\le t_k} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q;q)_k}. \end{aligned} \tag{3.4.22}$$

Similarly, the Hasse diagram of X_2 is given in Figure 3.12. We have $T(1)=0$ for any $T\in\mathscr{L}(X_2)$ and obtain the Hasse diagram of X_1 after cutting $0\in X_2$. Then the Jordan–Hölder set is identified with

$$\{1<t_1<t_2<\cdots<t_k \,|\, k\in\mathbb{Z}_{\ge 0}, t_i-t_{i-1}\ge 2\}, \tag{3.4.23}$$

which gives

$$\begin{aligned} Z_{X_2}(q) &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \sum_{\substack{1< t_1<\cdots<t_k \\ t_i-t_{i-1}\ge 2}} q^{\sum_{j=1}^k t_j} \\ &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \sum_{0\le t_1\le\cdots\le t_k} q^{2k+\sum_{j=0}^{k-1} 2j+\sum_{j=1}^k t_j} \\ &= \frac{1}{(q;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q;q)_k}. \end{aligned} \tag{3.4.24}$$

Equating the results from the P-partition counting with the product form with W-algebras, we can re-derive the Rogers–Ramanujan identities.

Note that the factor $(q;q)_\infty$ in the denominators in (3.4.22), (3.4.24) does not appear when we concern $\mathscr{W}_N$-algebra. It then seems to come from the fact that SH^c correspond to the tensor product of the $\mathscr{W}_N$-algebra and the Heisenberg algebra.

3.5 The triality in SH^c

Here we construct another algebra map $\mathscr{S}_2$ associated with the transposition of Young diagrams. Two maps $\mathscr{S}_1$ and $\mathscr{S}_2$ then give an identification of three minimal model representa-

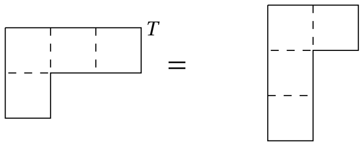


Figure 3.13: The transposition of a Young diagram

tions. This is an analog of the triality in $\mathscr{W}_\infty[\mu]$[28].

3.5.1 The transpose mapping $\mathscr{S}_2$

Note that the transposition Y^T of a Young diagram Y is also a Young diagram (see Figure 3.13). One can extend the transposition to an isomorphism $\mathscr{S}_2$ between representations of SH$^{(N)}$ as follows. We write $\lambda^T = (\lambda^{(1)T}, \cdots, \lambda^{(N)T})$ for an N-tuple $\lambda \in \mathscr{Y}^{(N)}$ and define a linear map $\mathscr{S}_2 : L^{(N)} \longrightarrow L^{(N)}$ by setting $\mathscr{S}_2|\lambda\rangle = |\lambda^T\rangle$. We assume that we consider the linear map $\mathscr{S}_2 : L^{(N)(\text{old})} \longrightarrow L^{(N)(\text{new})}$ only when two pairs of two parameters $\varepsilon_{1,2}^{(\text{old})}$ and $\varepsilon_{1,2}^{(\text{new})}$ are related by

$$\varepsilon_1^{(\text{new})} = \varepsilon_2^{(\text{old})}, \ \varepsilon_2^{(\text{new})} = \varepsilon_1^{(\text{old})}, \tag{3.5.1}$$

which keeps the pairing $\langle\lambda|\lambda\rangle = \langle\lambda^T|\lambda^T\rangle$. The other parameters $a_i (1 \le i \le N)$ are assumed to be kept invariant. The linear map then induces an algebra map $\mathscr{S}_2 : \text{SH}^{(N)(\text{old})} \longrightarrow \text{SH}^{(N)(\text{new})}$ by

$$\mathscr{S}_2(D_{0,l+1}^{(\text{old})}) = \left(\frac{\varepsilon_1^{(\text{new})}}{\varepsilon_2^{(\text{new})}}\right)^l D_{0,l+1}^{(\text{new})}, \mathscr{S}_2(D_{\pm 1,l}^{(\text{old})}) = \left(\frac{\varepsilon_1^{(\text{new})}}{\varepsilon_2^{(\text{new})}}\right)^l D_{\pm 1,l}^{(\text{new})}, \ \mathscr{S}_2(E_l^{(\text{old})}) = \left(\frac{\varepsilon_1^{(\text{new})}}{\varepsilon_2^{(\text{new})}}\right)^l E_l^{(\text{new})}, \ l \ge 0, \tag{3.5.2}$$

The only nontrivial point for $\mathscr{S}_2$ to be an algebra map is the quadratic relation (2.3.41), but we see that it also holds since we have

$$\mathscr{S}_2(D_{\pm 1}^{(\text{old})}(z)) = D_{\pm 1}^{(\text{new})}\left(\left(\frac{\varepsilon_2^{(\text{old})}}{\varepsilon_1^{(\text{old})}}\right)^{-1} z\right), \kappa^{(\text{old})}(u) = \left(\frac{\varepsilon_2^{(\text{old})}}{\varepsilon_1^{(\text{old})}}\right)^3 \kappa^{(\text{new})}\left(\left(\frac{\varepsilon_2^{(\text{old})}}{\varepsilon_1^{(\text{old})}}\right)^{-1} u\right). \tag{3.5.3}$$

3.5.2 The triality

We have the isomorphism even after we specialize the parameters. The $\mathscr{S}_2$-dual of a restricted N-datum $\langle N, N+M, (\tilde{n}_i)_{i=1,\cdots,N}\rangle$ is the N-datum $\langle (N+M, N), (1, n_i)_{1\le i\le N}\rangle$. The corresponding N-Burge condition becomes

$$\lambda^{(i+1)}(l) - \lambda^{(i)}(l) \le \tilde{n}_i, \ (i,l) \in X^{(N)}. \tag{3.5.4}$$

Taking the transposition, we obtain

$$\lambda^{(i+1)T}(l + \tilde{n}_i) \le \lambda^{(i)T}(l), \ (i,l) \in X^{(N)}. \tag{3.5.5}$$

A pair $(i,l) \in X^{(N)}$ now labels the lth column of the ith Young diagram and it is natural to assign it with an integer by

$$\mathscr{X}^T(i,l) \equiv a_i + \varepsilon_1^{(\text{new})}(l-1) = a_i + N(l-1) = \mathscr{X}(i,l). \tag{3.5.6}$$

A similar reasoning shows that the algebra map $\mathscr{S}_2$ intertwines these two representations.

In summary, we obtained two algebra maps $\mathscr{S}_1$ and $\mathscr{S}_2$ among a certain specialized SH^c. They intertwine three minimal model representations of the nonlinear algebra SH^c:

$$\langle (N,N+M),(n_i,1)_{1\le i\le N}\rangle \overset{\mathscr{S}_1}{\longleftrightarrow} \langle (M,N+M),(m_i,1)_{1\le i\le M}\rangle \overset{\mathscr{S}_2}{\longleftrightarrow} \langle (N+M,M),(1,m_i)_{1\le i\le M}\rangle\,. \tag{3.5.7}$$

The maps $\mathscr{S}_1$ and $\mathscr{S}_2$ permute three representations and then form $\mathfrak{S}_3$ in that sense. We have an analog of the triality in $\mathscr{W}_\infty[\mu]$ in terms of the algebra SH^c. In fact, the triality in $\mathscr{W}_\infty[\mu]$ is expressed as the redundancy of the 't Hooft coupling parameter μ:

$$\mathscr{W}_\infty[\mu=N] \longleftrightarrow \mathscr{W}_\infty\left[\mu=-N\left(1+\frac{\varepsilon_1}{\varepsilon_2}\right)\right] \longleftrightarrow \mathscr{W}_\infty\left[\mu=-N\left(1+\frac{\varepsilon_2}{\varepsilon_1}\right)\right]. \tag{3.5.8}$$

In other words, we give another meaning of the triality as the identification of certain special points in the parameter space associated with the algebra SH^c. The corresponding minimal model representations share the same spectrum with respect to the set of commuting operators D_{l+1} after the rescaling by $\mathscr{S}_{1,2}$.

Chapter 4

Conclusion

In the present thesis, we constructed a series of irreducible representations of SH^c which describes minimal model representations. We can express such an action explicitly by using a gauge theoretical basis. Each element of the basis satisfies the N-Burge condition, which is consistent with a related work[42]. Using the explicit representation to analyze the level-rank duality, we find a more appropriate labeling set $X_B^{(N)}$ which parameterizes rows of N-tuple Young diagrams. The set $X_B^{(N)}$ is naturally equipped with a partial order and we proved that the gauge theoretical basis is identical to the set of all P-partitions over the partially ordered set $X_B^{(N)}$. The level-rank duality can be considered as the redundancy how we see a P-partition as a multiple Young diagram. We also have an algebra isomorphism between two minimal model representations which are level-rank dual to each other. The two representations share the same set of eigenvectors with respect to commuting operators $D_{0,l+1} \in \mathrm{SH}^c$, $l \geq 0$. The commutator $E_{l+k} = [D_{-1,l}, D_{1,k}], l, k \geq 0$, is also shared after a constant rescaling. Since a lowest weight representation of SH^c is characterized by the actions of these degree-0 operators on the vacuum state $|\emptyset\rangle$, we may consider the two representations as the same one. We also have another isomorphism associated with the transposition operation of Young diagrams. The isomorphism and the level-rank duality form a triality of SH^c and give an identification of three minimal model representations. This phenomenon is expected for its universal property in a sense that there is an equivalence between $\mathscr{W}_{N+\mathrm{U}(1)}$ and $\mathrm{SH}^{(N)}$ for any N. The universality may be a sign of a triality.

There are some directions following this study.

Back to 4-dimensional gauge theories

We have mentioned in Section 2.2 that the basis $\{|\lambda\rangle\}_{\lambda \in \mathscr{Y}^{(N)}}$ for the space $L^{(N)}$ corresponds to the set of all fixed points in the $\mathrm{U}(N)$-instanton moduli space. We also have seen in Section 3.4 that the Euler characters of some elements of the basis have vanished and then seem to give divergent contributions to the instanton partition function. However, we should notice what leads to the Nekrasov formula (2.2.13). The formula is justified only when each fixed point is isolated, but the vanishing Euler character for a null vector means that the torus $\tilde{D}$ does not localize the null vectors to an isolated point. Instead, we have to integrate a certain function over null subspace. Our results claims there is a certain action of SH^c on such a nontrivial space. It may be interesting to see the geometry of the instanton moduli space from the representation theory of SH^c and to study the role of null vectors in 4-dimensional gauge theories.

Note that when we write down the explicit action of SH^c, the parameters $\varepsilon_{1,2}, a_{1,\cdots,N}$ have appeared as degree-0 rational functions in most cases. It may be interesting to compare Seiberg–Witten curves and our results, if there may exist a limit $\varepsilon_{1,2}, a_{1,\cdots,N} \to 0$ keeping their linear relations.

Applications to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence

We have seen a universal aspect of SH^c in a sense that there is a series of assignments $\mathrm{SH}^c \to \mathrm{SH}^{(N)}$ for any N and then have reached a triality relation in the algebra. In [28], the level-rank redundancy in $\mathcal{W}_\infty[\mu]$ gave an identification of the symmetries in the two sides AdS_3 and CFT_2. We hope SH^c also gives the same identification and its minimal model representation will be realized in the higher spin theory on AdS_3.

On the connection between the Rogers–Ramanujan identities and the Lee–Yang singularity

We revisit the connection between the Rogers–Ramanujan identities and the Lee–Yang singularity from the theory of P-partitions. A generalization of this connection is known for the Virasoro cases ($N=2$) (See [53] and references therein). There are few works beyond them, however[†1]. While the product hand side of the generating function is completely determined[52], it is difficult to have the sum side in general. It is meaningful to study the algebra SH^c itself to reveal new identities.

[†1] Some identities are obtained for the algebra $\mathcal{W}_3$ in [56, 57]

Acknowledgement

I am very grateful to my supervisor Yutaka Matsuo for teaching, collaborations and discussions. I also thank Masayuki Fukuda and Rui Dong Zhu for collaborations and discussions. I greatly thank Yuji Tachikawa for helpful discussions. I would like to thank all the members of High Energy Physics Theory Group in UTokyo.

Appendix A

Proof of Lemma 2.2

Here we give a proof of the lemma 2.2.

Lemma A.1. *For $\lambda \in \mathscr{Y}^{(N)}$ and $l_1, l_2, \cdots, l_n$, we have*

$$
\begin{aligned}
&D_{1,l_n} D_{1,l_{n-1}} \cdots D_{1,l_1} |\lambda\rangle \\
&= (\varepsilon_1 \varepsilon_2)^n \sum_{\substack{\pi \supset \lambda \\ |\pi| = |\lambda| + n}} \left(\sum_{w \in \mathfrak{S}_n} \mathrm{eu}\left(-(1-t_1)(1-t_2) \sum_{i>j}^{n} s_{w(i)} s_{w(j)}^{-1} \right) \prod_{i=1}^{n} \left(\frac{c_1\left(s_{w(i)}\right)}{\varepsilon_1} \right)^{l_i} \right) \\
&\qquad \times \mathrm{eu}\big(((1-t_1)(1-t_2) V_\pi^* - W^*) \otimes (V_\pi - V_\lambda) - n t_1 t_2 \big) |\pi\rangle \,,
\end{aligned}
\tag{A.0.1}
$$

where $V_\pi^* - V_\lambda^* = \sum_{i=1}^{n} s_i$ represents a decomposition of $V_\pi^* - V_\lambda^*$ by the torus action.

Proof. For each pair of N-tuples $\lambda, \pi \in \mathscr{Y}^{(N)}$ such that $\pi \supset \lambda, |\pi| = |\lambda| + n$, we fix a sequence of N-tuples $\pi_0, \pi_1, \cdots, \pi_n \in \mathscr{Y}^{(N)}$ satisfying $\lambda = \pi_0 \subsetneq \pi_1 \subsetneq \cdots \subsetneq \pi_{n-1} \subsetneq \pi_n = \pi$. We denote by $s_i = V_{\pi_i}^* - V_{\pi_{i-1}}^*$ for $1 \le i \le n$. Note that the sequence $(s_i)_{i=1}^n$ characterizes the original sequence of N-tuples. We call a permutation $w \in \mathfrak{S}_n$ admissible if there is such a sequence of N-tuples which gives the permuted sequence $(s_{w(i)})_{i=1}^n$. We see from (2.3.9) that we have

$$
\begin{aligned}
&D_{1,l_n} D_{1,l_{n-1}} \cdots D_{1,l_1} |\lambda\rangle \\
&= (\varepsilon_1 \varepsilon_2)^n \sum_{\substack{\pi \supset \lambda \\ |\pi| = |\lambda| + n}} \left(\sum_{\substack{w \in \mathfrak{S}_n \\ \text{admissible}}} \mathrm{eu}\left(-(1-t_1)(1-t_2) \sum_{j>i}^{n} s_{w(j)} s_{w(i)}^{-1} \right) \prod_{i=1}^{n} \left(\frac{c_1\left(s_{w(i)}\right)}{\varepsilon_1} \right)^{l_i} \right) \\
&\qquad \times \mathrm{eu}\big(((1-t_1)(1-t_2) V_\pi^* - W^*) \otimes (V_\pi - V_\lambda) - n t_1 t_2 \big) |\pi\rangle \,,
\end{aligned}
\tag{A.0.2}
$$

Here we should note that a similar consideration with Figure 2.2 shows that the Euler character $\mathrm{eu}\big(((1-t_1)(1-t_2)V_\pi^* - W^*) \otimes (V_\pi - V_\lambda) - n t_1 t_2\big)$ neither vanishes nor diverges.

The following statement completes the proof.

$$
w \in \mathfrak{S}_n \text{ is not admissible} \Longrightarrow \mathrm{eu}\left(-(1-t_1)(1-t_2) \sum_{j>i}^{n} s_{w(j)} s_{w(i)}^{-1} \right) = 0 \,. \tag{A.0.3}
$$

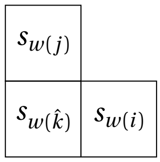


Figure A.1: Three boxes $s_{w(i)}, s_{w(j)}, s_{w(\hat{k})}$ whose positions are given by the torus action.

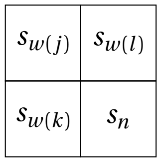


Figure A.2: A pattern how the box s_n is assigned.

We prove this by induction on n. For $n \le 3$, we have $s_i s_j^{-1} \ne 1, t_1 t_2$ for any numbers $i \ne j$, and then we see that (A.0.3) holds. For general n, define the integer $P(w)$ for $w \in \mathfrak{S}_n$ by

$$\begin{aligned} P(w) = \#\{(i,j) | 1 \le i < j \le n,\ s_{w(j)}^{-1} s_{w(i)} = t_1, t_2\} \\ - \#\{(i,j) | 1 \le i < j \le n,\ s_{w(j)}^{-1} s_{w(i)} = t_1 t_2\}. \end{aligned} \tag{A.0.4}$$

The Euler class in (A.0.3) vanishes if $P(w) > 0$ and diverges if $P(w) < 0$. If a pair (i,j) of distinct numbers satisfies $i < j$ and $s_{w(j)}^{-1} s_{w(i)} = t_1 t_2$, there exists a unique number $\hat{k} \ne i, j$ such that $s_{w(\hat{k})}^{-1} s_{w(i)} = t_1$ and $s_{w(j)}^{-1} s_{w(\hat{k})} = t_2$, which can be seen in Figure A.1. Here we again relate each 1-dimensional vector subspace to a box whose position is given by the torus action. Note that the map $(i,j) \mapsto \hat{k}$ is injective. Since $i < \hat{k}$ or $\hat{k} < j$, the subspace $s_{w(\hat{k})}^{-1} s_{w(i)}$ or $s_{w(j)}^{-1} s_{w(\hat{k})}$ contributes to $P(w)$ by $+1$. This means $P(w) \ge 0$. Since π_{n-1} is a N-tuple of Young diagram, we have

$$\begin{aligned} P(w) = \Big[& \#\{(i,j) | 1 \le i < j \le n,\ w(i), w(j) \ne n,\ s_{w(j)}^{-1} s_{w(i)} = t_1, t_2\} \\ & - \#\{(i,j) | 1 \le i < j \le n,\ w(i), w(j) \ne n,\ s_{w(j)}^{-1} s_{w(i)} = t_1 t_2\} \Big] \\ & + \#\{j | w^{-1}(n) < j \le n,\ s_{w(j)}^{-1} s_n = t_1, t_2\} - \#\{j | w^{-1}(n) < j \le n,\ s_{w(j)}^{-1} s_n = t_1 t_2\}. \end{aligned} \tag{A.0.5}$$

The last term can not be negative by a similar discussion. The first term is equal to $P(\tilde{w})$ for $\tilde{w} \in \mathfrak{S}_{n-1}$, which maps $\{1, \cdots, w^{-1}(n) - 1, w^{-1}(n) + 1, \cdots, n\}$ to $\{1, \cdots, n-1\}$ by $\tilde{w}(i) = w(i)$. Assume that $P(w) = 0$, and then we have $P(\tilde{w}) = 0$ and

$$\#\{j | w^{-1}(n) < j \le n,\ s_{w(j)}^{-1} s_n = t_1, t_2\} - \#\{j | w^{-1}(n) < j \le n,\ s_{w(j)}^{-1} s_n = t_1 t_2\} = 0. \tag{A.0.6}$$

The induction hypothesis means that $\tilde{w}$ is admissible. The box s_n is assigned like Figure A.2, where some of the three boxes $s_{w(j)}, s_{w(k)}$ and $s_{w(l)}$ may not exist, but $s_{w(k)}$ and $s_{w(l)}$ should exist if the box $s_{w(j)}$ is given. We have $k, l > j$ since $\tilde{w}$ is admissible. The condition (A.0.6) then holds only when $w^{-1}(n) > j, k, l$, which means that w is admissible. Therefore (A.0.3) holds for general n. □

References

[1] M. Fukuda, S. Nakamura, Y. Matsuo, and R.-D. Zhu, "SH^c realization of minimal model CFT: triality, poset and Burge condition," *JHEP* **11** (2015) 168, arXiv:1509.01000 [hep-th].

[2] O. Schiffmann and E. Vasserot, "Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on $\mathbb{A}_2$," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* **118** no. 1, (2013) 213–342.

[3] L. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-Dimensional Gauge Theories," *Letters in Mathematical Physics* **91** no. 2, (2010) 167–197. http://dx.doi.org/10.1007/s11005-010-0369-5.

[4] P. Bouwknegt and K. Schoutens, "W symmetry in conformal field theory," *Physics Reports* **223** no. 4, (1993) 183–276.

[5] P. Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*. Springer Science & Business Media, 2012.

[6] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," *Nucl. Phys.* **B241** (1984) 333–380.

[7] D. A. Huse, "Exact exponents for infinitely many new multicritical points," *Phys. Rev. B* **30** (Oct, 1984) 3908–3915.

[8] V. A. Fateev and A. B. Zamolodchikov, "Conformal Quantum Field Theory Models in Two-Dimensions Having Z(3) Symmetry," *Nucl. Phys.* **B280** (1987) 644–660.

[9] D. Friedan, Z. Qiu, and S. Shenker, "Conformal Invariance, Unitarity, and Critical Exponents in Two Dimensions," *Phys. Rev. Lett.* **52** (Apr, 1984) 1575–1578.

[10] V. Dotsenko, "Critical behaviour and associated conformal algebra of the $\mathbb{Z}_3$ Potts models," *Nuclear Physics B* **235** no. 1, (1984) 54 – 74. http://www.sciencedirect.com/science/article/pii/0550321384901482.

[11] V. A. Fateev and S. L. Lukyanov, "The Models of Two-Dimensional Conformal Quantum Field Theory with Z(n) Symmetry," *Int. J. Mod. Phys.* **A3** (1988) 507.

[12] M. Jimbo, T. Miwa, and M. Okado, "Local state probabilities of solvable lattice models: An $A^{(1)}_{n-1}$ family," *Nuclear Physics B* **300** (1988) 74 – 108. http://www.sciencedirect.com/science/article/pii/0550321388905871.

[13] T. Nakanishi, "Non-unitary minimal models and RSOS models," *Nuclear Physics B* **334** no. 3, (1990) 745–766.

[14] P. Goddard, A. Kent, and D. I. Olive, "Virasoro Algebras and Coset Space Models," *Phys. Lett.* **B152** (1985) 88.

[15] F. A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, "Extensions of the Virasoro Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir Invariants," *Nucl. Phys.* **B304** (1988) 348–370.

[16] F. A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, "Coset Construction for Extended Virasoro Algebras," *Nucl. Phys.* **B304** (1988) 371–391.

[17] B. Feigin and E. Frenkel, "Quantization of the Drinfeld-Sokolov reduction," *Physics Letters B* **246** no. 1,2, (1990) 75 – 81. http://www.sciencedirect.com/science/article/pii/037026939091310 8.

[18] B. Feigin and D. Fuks, "Invariant skew-symmetric differential operators on the line and Verma modules over the Virasoro algebra," *Functional Analysis and Its Applications* **16** no. 2, (1982) 114–126. http://dx.doi.org/10.1007/BF01081626.

[19] A. Kuniba and T. Nakanishi, "LEVEL RANK DUALITY IN FUSION RSOS MODELS," in *International Colloquium on Modern Quantum Field Theory Bombay, India, January 8-14, 1990.* 1990.

[20] A. Kuniba, T. Nakanishi, and J. Suzuki, "Ferro-and antiferro-magnetizations in RSOS models," *Nuclear Physics B* **356** no. 3, (1991) 750–774.

[21] D. Altschuler, M. Bauer, and H. Saleur, "Level rank duality in nonunitary coset theories," *J. Phys.* **A23** (1990) L789–L794.

[22] H. Saleur and D. Altschuler, "Level-rank duality in quantum groups," *Nuclear Physics B* **354** no. 2, (1991) 579 – 613. http://www.sciencedirect.com/science/article/pii/055032139190367 7.

[23] M. Jimbo and T. Miwa, *Irreducible Decomposition of Fundamental Modules for $A_l^{(1)}$ and $C_l^{(1)}$, and Hecke Modular Forms.* Kyoto University. Research Institute for Mathematical Sciences [RIMS], 1983.

[24] M. Jimbo and T. Miwa, *On a duality of branching rules for affine Lie algebras.* Kyoto University. Research Institute for Mathematical Sciences [RIMS], 1983.

[25] T. Nakanishi and A. Tsuchiya, "Level-rank duality of WZW models in conformal field theory," *Communications in Mathematical Physics* **144** no. 2, (1992) 351–372. http://dx.doi.org/10.1007/BF02101097.

[26] J. Fuchs and C. Schweigert, "Level-Rank Duality of WZW Theories and Isomorphisms of N = 2 Coset Models," *Annals of Physics* **234** no. 1, (1994) 102 – 140. http://www.sciencedirect.com/science/article/pii/S000349168471075X.

[27] M. R. Gaberdiel and T. Hartman, "Symmetries of Holographic Minimal Models," *JHEP* **05** (2011) 031, arXiv:1101.2910 [hep-th].

[28] M. R. Gaberdiel and R. Gopakumar, "Triality in minimal model holography," *Journal of High Energy Physics* **2012** no. 7, (2012) 1–27.

[29] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** (2004) 831–864, arXiv:hep-th/0206161 [hep-th].

[30] N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions," arXiv:hep-th/0306238 [hep-th].

[31] N. Wyllard, "A_{N-1} conformal Toda field theory correlation functions from conformal $\mathcal{N} = 2SU(N)$ quiver gauge theories," *Journal of High Energy Physics* **2009** no. 11, (2009) 002. http://stacks.iop.org/1126-6708/2009/i=11/a=002.

[32] A. Mironov and A. Morozov, "Proving AGT relations in the large-c limit," *Phys. Lett.* **B682** (2009) 118–124, arXiv:0909.3531 [hep-th].

[33] A. Mironov, A. Morozov, and S. Shakirov, "A direct proof of AGT conjecture at beta = 1," *JHEP* **02** (2011) 067, arXiv:1012.3137 [hep-th].

[34] V. A. Fateev and A. V. Litvinov, "On AGT conjecture," *JHEP* **02** (2010) 014, arXiv:0912.0504 [hep-th].

[35] L. Hadasz, Z. Jaskolski, and P. Suchanek, "Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals," *JHEP* **06** (2010) 046, arXiv:1004.1841 [hep-th].

[36] V. A. Alba, V. A. Fateev, A. V. Litvinov, and G. M. Tarnopolskiy, "On combinatorial expansion of the conformal blocks arising from AGT conjecture," *Lett. Math. Phys.* **98** (2011) 33–64, arXiv:1012.1312 [hep-th].

[37] V. A. Fateev and A. V. Litvinov, "Integrable structure, W-symmetry and AGT relation," *JHEP* **01** (2012) 051, arXiv:1109.4042 [hep-th].

[38] S. Kanno, Y. Matsuo, and H. Zhang, "Extended Conformal Symmetry and Recursion Formulae for Nekrasov Partition Function," *JHEP* **08** (2013) 028, arXiv:1306.1523 [hep-th].

[39] A. Negut, "Exts and the AGT Relations," arXiv:1510.05482 [hep-th].

[40] K. B. Alkalaev and V. A. Belavin, "Conformal blocks of W_N minimal models and AGT correspondence," *JHEP* **07** (2014) 024, arXiv:1404.7094 [hep-th].

[41] M. Bershtein and O. Foda, "AGT, Burge pairs and minimal models," *JHEP* **06** (2014) 177, arXiv:1404.7075 [hep-th].

[42] V. Belavin, O. Foda, and R. Santachiara, "AGT, N-Burge partitions and $\mathcal{W}_N$ minimal models," *JHEP* **10** (2015) 073, arXiv:1507.03540 [hep-th].

[43] H. Nakajima and K. Yoshioka, "Instanton counting on blowup. 1.," *Invent.Math.* **162** (2005) 313–355, arXiv:math/0306198 [math-ag].

[44] R. P. Stanley, *Enumerative Combinatorics*. Wadsworth Publ. Co., Belmont, CA, USA, 1986.

[45] S. Ramanujan and L. Rogers, "Proof of certain identities in combinatory analysis," in *Proc. Cambridge Philos. Soc*, vol. 19, pp. 214–216. 1919.

[46] V. G. Kac and M. Wakimoto, "Modular invariant representations of infinite dimensional Lie algebras and superalgebras," *Proc. Nat. Acad. Sci.* **85** (1988) 4956–5960.

[47] N. Arbesfeld and O. Schiffmann, "A presentation of the deformed $W_{1+\infty}$ algebra," arXiv:1209.0429 [math.RT].

[48] Y. Tachikawa, "A review on instanton counting and W-algebras," arXiv:1412.7121 [hep-th].

[49] M. Atiyah, N. J. Hitchin, V. Drinfeld, and Y. Manin, "Construction of Instantons," *Phys.Lett.* **A65** (1978) 185–187.

[50] W. H. Burge, "Restricted partition pairs," *Journal of Combinatorial Theory, Series A* **63** no. 2, (1993) 210 – 222. http://www.sciencedirect.com/science/article/pii/009731659390057F.

[51] I. Gessel and C. Krattenthaler, "Cylindric partitions," *Transactions of the American Mathematical Society* **349** no. 2, (1997) 429–479.

[52] A. Borodin, "Periodic Schur process and cylindric partitions," *Duke Math. J.* **140** no. 3, (12, 2007) 391–468. http://dx.doi.org/10.1215/S0012-7094-07-14031-6.

[53] O. Foda and T. Welsh, "Cylindric partitions, W_r characters and the Andrews-Gordon-Bressoud identities," arXiv:1510.02213 [math-ph].

[54] B. Gordon, "A combinatorial generalization of the Rogers-Ramanujan identities," *American Journal of Mathematics* (1961) 393–399.

[55] G. E. Andrews, *The theory of partitions.* No. v. 2 . Section, Number theory in Encyclopedia of mathematics and its applications / edited by G.-C. Rota. Cambridge University Press, 1984. http://ci.nii.ac.jp/ncid/BA06553742.

[56] G. E. Andrews, A. Schilling, and S. O. Warnaar, "An A_2 Bailey lemma and Rogers–Ramanujan-type identities," math/9807125.

[57] B. Feigin, O. Foda, and T. Welsh, "Andrews-Gordon identities from combinations of Virasoro characters," *Ramanujan J.* **17** (2008) 33–52, arXiv:math-ph/0504014 [math-ph].