Special Lagrangian submanifolds and mean curvature flows

その他のタイトル	特殊ラグランジュ部分多様体と平均曲率流について
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博士論文

論文題目 Special Lagrangian submanifolds and mean curvature flows

(特殊ラグランジュ部分多様体と平均曲率流について)

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Summary

This is a summary of the Ph.D. thesis. In this thesis we study special Lagrangian submanifolds, mean curvature flows (especially, Lagrangian mean curvature flows) and some related topics. This thesis consists of self-contained four parts, Part I, Part II, Part III and Part IV (cf. [19, 20, 21, 22]).

Recently, the study of special Lagrangian submanifolds have acquired an important role in Mirror Symmetry. For example, they are key words in the Strominger-Yau-Zaslow Conjecture [17] which explains Mirror Symmetry of 3-dimensional Calabi-Yau manifolds. Historically, special Lagrangian submanifolds in Calabi-Yau manifolds are defined in the paper of Harvey and Lawson [7] as calibrated submanifolds. As a general property of calibrated submanifolds, a special Lagrangian submanifold is a minimal submanifold. Furthermore the notion of special Lagrangian submanifolds is generalized in almost Calabi-Yau manifolds. Here we give definitions. An almost Calabi-Yau manifold is a triple (M, ω, Ω), where (M, ω) is a Kähler manifold and Ω is a non-vanishing holomorphic volume form on M. On an almost Calabi-Yau manifold (M, ω, Ω), a function $\psi : M \to \mathbb{R}$ is defined by

$$e^{2m\psi}\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega},$$

where m is the complex dimension of M. For a Lagrangian immersion $F : L \to M$, the Lagrangian angle $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ is defined by

$$F^*\Omega = e^{i\theta_F + mF^*\psi} dV_L,$$

where dV_L is the (locally defined up to sign) volume form on L, and $F : L \to M$ is called a special Lagrangian with phase $e^{i\theta}$ if its Lagrangian angle θ_F is identically constant θ. It follows that if $F : L \to M$ is a special Lagrangian immersion then $F : L \to M$ is a minimal submanifold in the conformally changed Riemannian manifold $(M, e^{2\psi}g)$, where g is the original Riemannian manifold on M. Motivated by this fact, we introduce the generalized mean curvature vector field as introduced by Behrndt in [1, §3] and later generalized by Smoczyk and Wang in [16]. The generalized mean curvature vector field H_g of $F : L \to M$ is a normal vector field defined by

$$H_g = H - m(\nabla\psi)^{\perp}.$$

Here H is the ordinary mean curvature vector field of $F : L \to M$, ∇ is the gradient with respect to g, and $\perp$ is the projection from TM to $T^{\perp}L$ it is the g-orthogonal complement of $F_*(TL)$. It is equivalent to that a Lagrangian immersion $F : L \to M$ is a special Lagrangian and $H_g \equiv 0$. In general, constructing explicit examples of special Lagrangian submanifolds is difficult, since these conditions are locally written by nonlinear elliptic PDE. However some examples are constructed in the case that the ambient Calabi-Yau manifold has symmetries, especially in $\mathbb{C}^m$.

In Part I (cf. [19]), we construct some examples of special Lagrangian submanifolds and Lagrangian self-similar solutions in almost Calabi–Yau cones over toric Sasaki manifolds. Let (S, g) be a Sasaki manifold, that is, there exists a complex structure on its cone $C(S) := S \times \mathbb{R}^+$ compatible with the cone metric $\overline{g} := r^2 g + dr^2$ so that it gives a Kähler structure on $C(S)$, where r is the standard coordinate on $\mathbb{R}^+ := \{\, r > 0 \,\}$. We denote the real dimension of S by $2m-1$. For a point $p = (s, r) \in C(S) = S \times \mathbb{R}^+$, we define its position vector following [4] by

$$\overrightarrow{p} = r\frac{\partial}{\partial r},$$

which is a tangent vector on $C(S)$ at p. In this part we further assume that (S, g) is toric, that is, there exists an effective holomorphic action of m-dimensional torus T^m on $C(S)$ preserving $\overline{g}$, and two projections $\pi : C(S) \to S$ and $r : C(S) \to \mathbb{R}^+$ satisfy $\pi(\tau \cdot p) = \tau \cdot \pi(p)$ and $r(\tau \cdot p) = r(p)$ for all $\tau \in T^m$ and$p \in C(S)$. The most typical example of the toric Sasaki manifold is the odd dimensional standard sphere, hence its cone is $\mathbb{C}^m \setminus \{0\}$ and it is toric Kähler. The cone of a toric Sasaki manifold is a kind of generalization of $\mathbb{C}^m$, since it has $\mathbb{R}^+$-action and T^m-action. We call $(C(S), \overline{g})$ the toric Kähler cone. Since $C(S)$ is a toric Kähler manifold, there exists a fan Σ which determines the complex structure on $C(S)$, an anti-holomorphic and anti-symplectic involution $\sigma : C(S) \to C(S)$, a moment map $\mu : C(S) \to \mathfrak{g}^*$, and a Reeb vector $\xi \in \mathfrak{g}$, where $\mathfrak{g} \cong \mathbb{R}^m$ is the Lie algebra of T^m. We denote the set of fixed points of σ by $C(S)^{\sigma}$, called the real form, which is a real m-dimensional submanifold in $C(S)$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_d\} \in \mathbb{Z}^m$ be the primitive generators of the 1-dimensional cones of Σ, then

by the standard algebraic toric geometry theory, it holds that the canonical line bundle $K_{C(S)}$ of $C(S)$ is trivial if and only if there exists a $\gamma \in \mathbb{Z}^m$ such that $\langle \gamma, \lambda_i \rangle = 1$ for all $i = 1, \dots, m$. If $K_{C(S)}$ of $C(S)$ is trivial, that is, there exists a $\gamma = (\gamma_1, \dots, \gamma_m) \in \mathbb{Z}^m$ as above, the non-vanishing holomorphic $(m, 0)$-form Ω_γ is constructed by

$$\Omega_\gamma := \exp(\gamma_1 z^1 + \cdots + \gamma_m z^m) dz^1 \wedge \cdots \wedge dz^m$$

on the open dense $T^m_{\mathbb{C}} (\cong (\mathbb{C}^\times)^m)$-orbit, where $(z^i)_{i=1}^m$ is the logarithmic holomorphic coordinates on this orbit, and actually this form can be extended whole over $C(S)$ as a non-vanishing holomorphic $(m, 0)$-form. In this part, we assume that there exists a $\gamma \in \mathbb{Z}^m$ as above, and call the triple $(C(S), \omega, \Omega_\gamma)$ the toric almost Calabi-Yau cone, where ω is the Kähler form on $C(S)$.

To construct Lagrangian submanifolds, we take $\zeta \in \mathfrak{g}$ and $c \in \mathbb{R}$, and denote the hyperplane $\{\, y \in \mathfrak{g}^* \mid \langle y, \zeta \rangle = c \,\}$ by $H_{\zeta,c}$. We assume that $\mathrm{Int}\Delta \cap H_{\zeta,c} \neq \emptyset$ and $\zeta \notin \mathfrak{z}_y$ for any $y \in \Delta \cap H_{\zeta,c}$, where we define $\mathfrak{z}_y$ for $y \in \Delta$ by $\mathfrak{z}_y = \mathrm{Span}_{\mathbb{R}}\{\, \lambda_i \mid \langle y, \lambda_i \rangle = 0 \}$. We define a real $(m-1)$-dimensional submanifold by

$$C(S)^\sigma_{\zeta,c} := \{\, p \in C(S)^\sigma \mid \langle \mu(p), \zeta \rangle = c \,\}.$$

Take an open interval $I \subset \mathbb{R}$. Let $f : I \to \mathbb{R}$ and $\rho : I \to \mathbb{R}^+$ be two functions on I, and $\tau_0 = (e^{i\nu^1}, \dots, e^{i\nu^m})$ be an element of torus T^m. We assume that $\dot{f}$ is non-vanishing on I. We denote the 1-parameter orbit $\{\exp(f(t)\zeta) \cdot \tau_0\}_{t \in I}$ in torus by $\{\tau(t)\}_{t \in I}$. We define a real m-dimensional manifold by

$$L_{\zeta,c} = C(S)^\sigma_{\zeta,c} \times I$$

and define a map $F : L_{\zeta,c} \to C(S)$ by

$$F(p,t) := \rho(t) \cdot (\tau(t) \cdot p)$$

for $(p,t) \in C(S)^\sigma_{\zeta,c} \times I = L_{\zeta,c}$, where "$\tau(t)\cdot$" denotes the T^m-action and "$\rho(t)\cdot$" denotes the standard $\mathbb{R}^+$-action on the cone $C(S)$. Then the main theorems in Part I are the following.

Theorem 6.1. *Assume that the function $\rho : I \to \mathbb{R}^+$ is identically constant. Take a constant $\theta_0 \in \mathbb{R}$. Then $F : L_{\zeta,c} \to C(S)$ is a special Lagrangian submanifold with phase $e^{i\theta_0}$ if and only if*

$$N := \langle \zeta, \gamma \rangle = 0 \quad \text{and} \quad \theta := \langle \nu, \gamma \rangle = \theta_0 - \frac{\pi}{2}.$$

Theorem 6.2. *We assume that $\zeta = \xi$ (remember that ζ denotes the Reeb vector), and put $\kappa(t) := \log \rho(t)$. Take a constant $\theta_0 \in \mathbb{R}$. Then $F : L_{\zeta,c} \to C(S)$ is a special Lagrangian submanifold with phase $e^{i\theta_0}$ if and only if*

$$\mathrm{Im}(e^{i(\theta-\theta_0)} e^{N(\kappa(t)+if(t))}) = \mathrm{const}.$$

Remark 6.3. If we define the curves $c_j : I \to \mathbb{C}^\times$ by $c_j(t) := \rho^{\xi^j}(t) e^{i(f(t)\xi^j + \nu^j)}$, then the condition in Theorem 6.2 is equivalent to the equality

$$\mathrm{Im}\big(e^{-i\theta_0} c_1^{\gamma_1} \cdots c_m^{\gamma_m}\big) = \mathrm{const}.$$

For example in $\mathbb{C}^m$, the canonical Reeb field is $\xi = (1, \dots, 1)$ and we can take $\gamma = (1, \dots, 1)$. Then if we take $\theta_0 = 0$ and $\nu^1 = \cdots = \nu^m = 0$ for example, then $c_1(t) = \cdots = c_m(t)$, and we put $c(t) := c_1(t)$. Then the condition in Theorem 6.2 becomes $\mathrm{Im}(c^m(t)) = \mathrm{const}$, and the image of $F : L_{\zeta,c} \to \mathbb{C}^m$ coincides with

$$\{\, (c(t)x^1, \dots, c(t)x^m) \in \mathbb{C}^m \mid t \in I,\ x^j \in \mathbb{R},\ (x^1)^2 + \cdots + (x^m)^2 = c \,\}.$$

Hence this is an extension of examples of special Lagrangian submanifolds mentioned in Theorem 3.5 in Section III.3.B. in the paper of Harvey and Lawson [7].

Theorem 7.1. *Let us assume that $\zeta = \xi$, and put $c(t) := \rho(t) e^{if(t)} \in \mathbb{C}^\times$. If there exist a function $\theta : I \to \mathbb{R}/\pi\mathbb{Z}$ and a constant $A \in \mathbb{R}$, and $\theta(t)$ and $c(t)$ satisfy the differential equations*

$$\begin{cases} \dot{c}(t) = e^{i(\theta(t)-\theta)} \overline{c(t)}^{N-1} \\ \dot{\theta}(t) = A\rho(t)^N \sin(f(t)N + \theta - \theta(t)), \end{cases} \tag{1}$$

then $F : L_{\zeta,c} \to C(S)$ is a Lagrangian generalized self-similar solution with

$$2cH^g = A\overrightarrow{F}^{\perp}$$

and Lagrangian angle $\theta_F(p,t) = \theta(t)$.

Remark 7.2. Assume that all $\xi^j \neq 0$. If we define curves $c_j : I \to \mathbb{C}^*$ by $c_j(t) := \rho^{\xi^j}(t)e^{i(f(t)\xi^j+\nu^j)}$, then the differential equations in Theorem 7.1 are equivalent to the following differential equations.

$$\begin{cases} \frac{d}{dt}c_j^{1/\xi^j}(t) = e^{i\theta(t)}\overline{c_1^{\gamma_1}(t)\cdots c_j^{\gamma_j-1/\xi^j}(t)\cdots c_m^{\gamma_m}(t)} \quad (j=1,\dots,m) \\ \frac{d}{dt}\theta(t) = A\,\mathrm{Im}(e^{-i\theta(t)}c_1^{\gamma_1}(t)\cdots c_m^{\gamma_m}(t)). \end{cases}$$

For example in $\mathbb{C}^m$, the canonical Reeb field is $\xi = (1,\dots,1)$ and $\gamma = (1,\dots,1)$. Then if we take $\theta_0 = 0$ and $\nu^1 = \cdots = \nu^m = 0$ for example, then the above differential equality becomes

$$\begin{cases} \frac{d}{dt}c_j(t) = e^{i\theta(t)}\overline{c_1(t)\cdots c_{j-1}(t)\cdot c_{j+1}(t)\cdots c_m(t)} \quad (j=1,\dots,m) \\ \frac{d}{dt}\theta(t) = A\,\mathrm{Im}(e^{-i\theta(t)}c_1(t)\cdots c_m(t)), \end{cases}$$

and the image of $F : L_{\zeta,c} \to \mathbb{C}^m$ coincides with

$$\{\, (c_1(t)x^1,\dots,c_m(t)x^m) \in \mathbb{C}^m \mid t \in I,\ x^j \in \mathbb{R},\ (x^1)^2+\cdots+(x^m)^2 = c\,\}.$$

This differential equations appear in Theorem A in the paper of Joyce, Lee and Tsui [11]. Hence this is one of extension of the paper of Joyce, Lee and Tsui in $\mathbb{C}^m$ to the toric almost Calabi–Yau cone.

As a corollary of Part I, for any integer $g \geq 1$, we can construct a real 6-dimensional toric almost Calabi–Yau cone M_g and a 3-dimensional special Lagrangian submanifold $F_g^1 : L_g^1 \to M_g$ which is diffeomorphic to $\Sigma_g \times \mathbb{R}$, where Σ_g is a closed surface of genus g. This is a generalization of the construction of special Lagrangian submanifold in $\mathbb{C}^m$ by Harvey-Lawson [7] and Joyce [9]. Furthermore, as a corollary of Theorem 7.1, for any integer $g \geq 1$, we can construct a compact Lagrangian (generalized) self-shrinker $F_g^2 : L_g^2 \to M_g$ which is diffeomorphic to $\Sigma_g \times S^1$. The meaning and importance of Lagrangian self-shrinkers are mentioned in other paragraphs below. This construction of Lagrangian self-shrinkers is a generalization of one of Joyce, Lee and Tsui [11] in $\mathbb{C}^m$.

Although the chief aim of Part I is constructing explicit examples of special Lagrangian submanifolds, there is an abstract way to get special Lagrangian submanifolds. It is the Lagrangian mean curvature flow. Let (M,ω,Ω) be an almost Calabi-Yau manifold with complex dimension m, and L be a real m-dimensional manifold. Then, for $T \in (0,\infty]$, a continuous map $F : L\times[0,T) \to M$ which is smooth on $L\times(0,T)$ is called a mean curvature flow, if for each $t \in [0,T)$ a map $F_t := F(\cdot,t) : L \to M$ is an immersion and it satisfies the following parabolic PDE

$$\frac{\partial F_t}{\partial t} = H_g(F),$$

where H_g is the generalized mean curvature vector field for F_t introduced above. There is a well-known magical fact that if an initial immersion $F_0 : L \to M$ is a Lagrangian submanifold then, along the mean curvature flow, each $F_t : L \to M$ is also a Lagrangian immersion automatically, that is, the Lagrangian condition is preserved under the mean curvature flow, and we call F_t the Lagrangian mean curvature flow. (More precisely, this flow should be called the generalized Lagrangian mean curvature flow since we used the generalized mean curvature flow. However we omit such detail of notions in this summary.) A Lagrangian mean curvature flow is one of potential approaches to find a special Lagrangian submanifold in a given (almost) Calabi-Yau manifold as the following meaning. If there exists a long time solution of a Lagrangian mean curvature flow $F : L\times[0,\infty) \to M$ starting from a given Lagrangian immersion $F_0 : L \to M$ and the flow converges to some smooth immersion $F_\infty : L_\infty \to M$, then it is a minimal (in the sense of $H_g \equiv 0$) Lagrangian immersion, that is, a special Lagrangian submanifold.

Indeed, the method of Lagrangian mean curvature flow has more deep background related to Mirror Symmetry proposed by Thomas and Yau [18]. Roughly speaking, they introduce a stability condition on Lagrangian submanifolds and conjecture that the Lagrangian mean curvature flow $F : L\times[0,\infty) \to M$ starting from a stable Lagrangian submanifold exists for all time, that is, $T = \infty$, and converges to a special Lagrangian submanifold in its Hamiltonian deformation class. This conjecture is called Thomas-Yau conjecture. Recently, Joyce [10] has updated the Thomas-Yau conjectures to achieve more plausible statement. In [10], he discussed the possibility that the Lagrangian mean curvature flow develops singularities many times even if an initial Lagrangian submanifold is stable and mentioned the necessity of surgeries of Lagrangian mean curvature flows. Thus it is meaningful to construct examples of Lagrangian mean curvature flows with singularities to understand the motion of Lagrangian mean curvature flows and to develop this program.

In Part II (cf. [20]), we construct explicit examples of special or weighted Hamiltonian stationary Lagrangian submanifolds in toric almost Calabi–Yau manifolds and construct solutions of generalized Lagrangian mean curvature flows with singularities and topological changes starting from these examples. Here, a weighted Hamiltonian stationary Lagrangian submanifold is a Lagrangian immersion $F : L \to M$ to some almost Calabi-Yau manifold with complex dimension m such that $\Delta_f \theta_F := \Delta\theta_F + \langle \nabla\theta_F, \nabla f\rangle = 0$, where $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ is the Lagrangian angle defined in the summary of Part I and $f := -mF^*\psi$ is a smooth function on L constructed by the function ψ on M which is defined in the summary of Part I.

Let $(M, \omega, \Omega_\gamma)$ be a toric almost Calabi-Yau manifold with complex dimension m and a holomorphic volume form Ω_γ which is defined by the way explained in the summary of Part I above. In Part II, we do not assume that M has a cone shape as in Part I. Since M is a toric Kähler manifold, there exist a moment map $\mu : M \to \Delta$ with a moment polytope Δ and an anti-holomorphic and anti-symplectic involution $\sigma : M \to M$. We denote the fixed point set of σ by M^σ and call it the real form of M. This is a real m-dimensional submanifold in M. To construct a Lagrangian immersion, fix an integer n with $0 \le n \le m$, take a set of n vectors $\zeta = \{\zeta_1, \ldots, \zeta_n\} \subset \mathbb{Z}^m$ and a set of n constants $c = \{c_1, \ldots, c_n\} \subset \mathbb{R}$. Then we define the set

$$M^\sigma_{\zeta,c} := \{\, p \in M^\sigma \mid \langle \mu(p), \zeta_i\rangle = c_i,\ i = 1, \ldots, n \,\}.$$

We assume that $M^\sigma_{\zeta,c}$ is a real $(m-n)$-dimensional submanifold in M^σ and $T_\zeta := V_\zeta/(V_\zeta \cap \mathbb{Z}^m)$ is isomorphic to a sub-torus T^n in T^m, where $V_\zeta := \mathrm{Span}_{\mathbb{R}}\{\zeta_1, \ldots, \zeta_n\}$. Then we put a real m-dimensional manifold as

$$L_{\zeta,c} := M^\sigma_{\zeta,c} \times T_\zeta \tag{2}$$

and define a map $F_{\zeta,c} : L_{\zeta,c} \to M$ by

$$F_{\zeta,c}(p, [v]) := \exp v \cdot p.$$

Then the main theorems in Part II are the following.

Theorem 10.1. *$F_{\zeta,c} : L_{\zeta,c} \to M$ is a T^n-invariant weighted Hamiltonian stationary Lagrangian submanifold for all ζ and c, and its Lagrangian angle $\theta_{\zeta,c} : L_{\zeta,c} \to \mathbb{R}/\pi\mathbb{Z}$ is given by $\theta_{\zeta,c}(p, [v]) = 2\pi\langle\gamma, v\rangle + \frac{\pi}{2}n\,(\mathrm{mod.}\,\pi)$. Thus $F_{\zeta,c} : L_{\zeta,c} \to M$ is a special Lagrangian submanifold if and only if $\langle\gamma, \zeta_i\rangle = 0$ for all $i = 1, \ldots, n$.*

Remark 13.4. It is clear that the real form M^σ, that is the case of $n = 0$, is always a special Lagrangian submanifold, and every torus fiber, that is the case of $n = m$, is not a special Lagrangian submanifold with respect to this holomorphic volume form Ω_γ. If $M = \mathbb{C}^m$, we can take $\gamma = (1, \ldots, 1)$. Then the special Lagrangian condition appeared in Joyce [9, Example 9.4] coincides with the condition appeared in Theorem 10.1.

Theorem 10.2. *The family of the images of $\{F_{\zeta,c(t)} : L_{\zeta,c(t)} \to M\}_{0\le t\le T}$ is a solution of generalized Lagrangian mean curvature flow with singularities and topological changes with initial condition $F_{\zeta,c}$, where $c(t) := \{c_1(t), \ldots, c_n(t)\}$ and each $c_j(t)$ is given by $c_j(t) := c_j - 2\pi t\langle\gamma, \zeta_j\rangle$. Here T is the first time that $M^\sigma_{\zeta,c(t)}$ becomes empty set.*

In this summary, we omit the precise definition of "a mean curvature flow with singularities and topological changes". Roughly speaking, it is a mean smooth mean curvature flow except some finite singular times.

Example 15.1. Let $(\mathbb{C}^m, \omega, \Omega)$ be a standard complex plane with a holomorphic volume form $\Omega = dw_1 \wedge \cdots \wedge dw_m$ by the standard coordinates w. If we write $w_i = e^{z_i}$ where $w_i \neq 0$, then Ω is written by $\Omega = e^{z_1+\cdots+z_m} dz_1 \wedge \cdots \wedge dz_m$. Hence we can take γ as $\gamma = (1, \ldots, 1)$. A moment map is given by $\mu(w) = \frac{1}{2}(|w_1|^2, \ldots, |w_m|^2)$ and a moment polytope is given by

$$\Delta = \{\, y \in \mathbb{R}^m \mid \langle y, \lambda_i\rangle \ge 0,\ i = 1, \ldots, m \,\},$$

where $\lambda_i := e_i$, the i-th standard base, and then we have $\langle\gamma, \lambda_i\rangle = 1$ for all i. The real form of $\mathbb{C}^m$ is $\mathbb{R}^m$ and note that $\mathbb{R}^m$ can be constructed by gluing from 2^m-copies of Δ. Take one vector $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{R}^m$ satisfying $\langle\gamma, \zeta\rangle > 0$ and $c = 0$. Since

$$c(t) = c - 2\pi\langle\gamma, \zeta\rangle t = -2\pi t\langle\gamma, \zeta\rangle = -2\pi t\sum_{j=1}^{m}\zeta_j,$$

the image of $F_{\zeta,c(t)} : L_{\zeta,c} \to \mathbb{C}^m$ is given by

$$\left\{ (x_1 e^{2\pi i \zeta_1 s}, \dots, x_m e^{2\pi i \zeta_m s}) \in \mathbb{C}^m \;\middle|\; 0 \le s \le 1,\ \sum_{j=1}^m \zeta_j x_j^2 = -4\pi t \sum_{j=1}^m \zeta_j,\ x = (x_1, \dots, x_m) \in \mathbb{R}^m \right\}.$$

This submanifold coincides with V_t in Theorem 1.1 in Lee and Wang [14], and they proved that V_t is Hamiltonian stationary and $\{V_t\}_{t\in\mathbb{R}}$ forms an eternal solution for Brakke flow. Hence our theorems can be considered as a kind of generalization of example of Lee and Wang to toric almost Calabi–Yau manifolds.

Example 15.2. Let $M = K_{\mathbb{P}^2}$ be the total space of the canonical line bundle of $\mathbb{P}^2$. Then a moment polytope is given by $\Delta = \{ y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \ge \kappa_i,\ i = 1, \dots, 4 \}$, where

$$\lambda_1 = (0,0,1),\ \lambda_2 = (1,0,1),\ \lambda_3 = (0,1,1),\ \lambda_4 = (-1,-1,1)$$

and $\kappa_1 = \kappa_2 = \kappa_3 = 0$, $\kappa_4 = -1$. Of course, M is a toric almost Calabi–Yau manifold since we can take $\gamma = (0,0,1)$ so that $\langle \gamma, \lambda_i \rangle = 1$ for all i. For example, take $n = 1$, and take one vector and one constant as $\zeta = (3,1,5)$ and $c = 5$. Then, by simple calculation, one can easily see that a generalized Lagrangian mean curvature flow constructed by this setting exists for $t \in I = [0, \frac{1}{2\pi})$ and it forms singularities and topological changes when $t = \frac{1}{5\pi}$ and $t = \frac{2}{5\pi}$, and vanishes when $t = \frac{1}{2\pi}$. One can also see the topology of $L_{\zeta,c(t)} = M^{\sigma}_{\zeta,c(t)} \times S^1$ (since now $T_\zeta \cong S^1$) by the same argument as explained in the proof of Proposition A.3 in [19]. In fact, the topology of $M^{\sigma}_{\zeta,c(t)}$ is S^2 when $0 \le t < \frac{1}{5\pi}$, is T^2 when $\frac{1}{5\pi} < t < \frac{2}{5\pi}$, is S^2 when $\frac{2}{5\pi} < t < \frac{1}{2\pi}$.

As mentioned above, it is important to study singularities of Lagrangian mean curvature flows. In the study of mean curvature flows, there is a well-known result of Huisken [8]. He studied asymptotic behavior of a mean curvature flow in $\mathbb{R}^m$ when it develops a singularity of special type I, and proved that its rescaled flow converges to a self-shrinker in $\mathbb{R}^m$. Here a self-shrinker is an immersion $F : L \to \mathbb{R}^m$ from some manifold L which satisfies $H(F)_x = -\frac{1}{2}\overrightarrow{x}^{\perp}$ for all points $x \in F(L)$. Hence a self-shrinker is considered as a local model of a singularity of a mean curvature flow.

In Part III (cf. [21]), we try to generalize the result of Huisken in $\mathbb{R}^m$ to in a more general Riemannian manifold, to study singularities of a Lagrangian mean curvature flow in a Calabi-Yau manifold. As a result of such an attempt, we have generalized the result of Huisken in $\mathbb{R}^m$ for a Ricci-mean curvature flow moving along a Ricci flow constructed from a gradient shrinking Ricci soliton, although it is not a Calabi-Yau manifold. Here we give each definitions. First, in general, a Ricci-mean curvature flow is a coupled parabolic PDE system of a Ricci flow and a mean curvature flow

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t)$$
$$\frac{\partial F_t}{\partial t} = H(F_t),$$

where (N, g_t) is a Ricci flow on some manifold N and $F_t : L \to (N, g_t)$ is a family of immersions from some manifold L to N. Next, a gradient shrinking Ricci soliton is Riemannian manifold $(N, \tilde{g}, \tilde{f})$ with some potential function $\tilde{f} \in C^\infty(N)$ which satisfies

$$\mathrm{Ric}(\tilde{g}) + \mathrm{Hess}\tilde{f} = \frac{1}{2}\tilde{g}.$$

As Hamilton's proof of Theorem 20.1 in [6], one can easily see that $R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f}$ is a constant, where $R(\tilde{g})$ denotes the scalar curvature of $\tilde{g}$. Hence by adding some constant to $\tilde{f}$ if necessary, we may assume that the potential function $\tilde{f}$ also satisfies $R(\tilde{g}) + |\nabla \tilde{f}|^2 - \tilde{f} = 0$. In Part III, we assume that $(N, \tilde{g})$ is complete and has bounded geometry. Given a gradient shrinking Ricci soliton $(N, \tilde{g}, \tilde{f})$ and a positive time $0 < T < \infty$, we can construct the Ricci flow g_t on N (which exists on the time interval $(-\infty, T)$) by

$$g_t := (T - t)\Phi_t^* \tilde{g},$$

where $\{\Phi_t \colon N \to N\}_{t\in(-\infty,T)}$ is the 1-parameter family of diffeomorphisms with $\Phi_0 = \mathrm{id}_N$ generated by the time-dependent vector field $V_t := \frac{1}{T-t}\nabla \tilde{f}$. We define the associated time dependent potential function $f_t \in C^\infty(N)$ by $f_t := \Phi_t^* \tilde{f}$ and denote a smooth function on $N \times [0, T)$ defined by $(p, t) \mapsto f_t(p)$ by f. Then the main theorems in Part III are the following.

Theorem 17.6. *Assume that $(N, \tilde{g}, \tilde{f})$ is compact. Let $F\colon M\times[0,T)\to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t) defined by $g_t := (T-t)\Phi_t^*\tilde{g}$. Assume that M is compact and F satisfies*

$$\limsup_{t\to T}\big(\sqrt{T-t}\max_M |A(F_t)|\big) < \infty,$$

where $A(F_t)$ is the norm of the second fundamental form of F_t. We define the normalized mean curvature flow $\tilde{F}\colon M\times[-\log T,\infty)\to N$ by

$$\tilde{F}_s := \Phi_t \circ F_t \quad \text{with} \quad s = -\log(T-t).$$

Then, for any sequence $s_1 < s_2 < \cdots < s_j < \cdots \to \infty$ and points $\{x_j\}_{j=1}^\infty$ in M, there exist sub-sequence s_{j_k} and x_{j_k} such that the family of immersion maps $\tilde{F}_{s_{j_k}}\colon M\to N$ from pointed manifolds (M, x_{j_k}) converges to an immersion map $\tilde{F}_\infty\colon M_\infty\to N$ from some pointed manifold (M_∞, x_∞). Furthermore, M_∞ is a complete Riemannian manifold with metric $\tilde{F}_\infty^*\tilde{g}$ and $\tilde{F}_\infty$ is a self-shrinker in $(N, \tilde{g}, \tilde{f})$ with $\lambda = -1$, that is, $\tilde{F}_\infty$ satisfies

$$H(\tilde{F}_\infty) = -((\nabla\tilde{f})\circ\tilde{F}_\infty)^\perp.$$

Theorem 17.7. *Assume that $(N, \tilde{g}, \tilde{f})$ is non-compact. Let $F\colon M\times[0,T)\to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t) defined by $g_t := (T-t)\Phi_t^*\tilde{g}$. Assume that M is compact and F satisfies*

$$\limsup_{t\to T}\big(\sqrt{T-t}\max_M |A(F_t)|\big) < \infty.$$

Furthermore, we assume that there exists a point $p_0\in M$ such that when $t\to T$

$$\ell_{F_t(p_0),t}\to f \quad \text{pointwise on } N\times[0,T),$$

where $f\colon N\times[0,T)\to\mathbb{R}$ is a function on $N\times[0,T)$ defined above and $\ell_{*,\bullet}\colon N\times[0,\bullet)\to\mathbb{R}$ is Perelman's reduced distance based at $(*,\bullet)$. Then, for any sequence $s_1 < s_2 < \cdots < s_j < \cdots \to \infty$, there exists a sub-sequence s_{j_k} such that the family of immersion maps $\tilde{F}_{s_{j_k}}\colon M\to N$ from pointed manifolds (M, p_0) converges to an immersion map $\tilde{F}_\infty\colon M_\infty\to N$ from some pointed manifold (M_∞, p_∞). Furthermore, M_∞ is a complete Riemannian manifold with metric $\tilde{F}_\infty^*\tilde{g}$ and $\tilde{F}_\infty$ is a self-shrinker in $(N, \tilde{g}, \tilde{f})$ with $\lambda = -1$, that is, $\tilde{F}_\infty$ satisfies

$$H(\tilde{F}_\infty) = -((\nabla\tilde{f})\circ\tilde{F}_\infty)^\perp.$$

We omit the definition of the reduced distance for Ricci flows in this summary, however we have the following remark which states that Theorem 17.7 completely implies the results of Huisken [8].

Remark 17.10. Consider $\mathbb{R}^n$ as the Gaussian soliton with potential function $\tilde{f}(x) := \frac{1}{4}|x|^2$. Since $\overrightarrow{x} = 2\nabla\tilde{f}(x)$, the definition of self-shrinkers in $\mathbb{R}^n$ coincides with the equations appear in Theorem 17.7. We take $T = 1$ for simplicity. Then we have

$$\Phi_t(x) = \frac{1}{\sqrt{T-t}}x, \quad g_t \equiv g_{\mathrm{st}}, \quad f(x,t) = \frac{|x|^2}{4(T-t)}.$$

Furthermore, one can easily see that in this trivial Ricci flow Perelman's reduced distance bases at $(*,\bullet)$ is given by

$$\ell_{*,\bullet}(x,t) := \frac{|x-*|^2}{4(\bullet - t)}.$$

Hence it is clear that the condition

$$\ell_{F_t(p_0),t}\to f \quad \text{pointwise on } \mathbb{R}^n\times[0,T)$$

is equivalent to the condution $F_t(p_0)\to \mathrm{O}$ as $t\to T$.

Example 17.11. Here we consider compact examples of self-similar solutions embedded in compact gradient shrinking Ricci solitons. Let $(N, \tilde{g}, \tilde{f})$ be a compact gradient shrinking Ricci soliton. Then N itself and a critical point P (0-dimensional submanifold) of $\tilde{f}$ are trivially compact self-similar solutions, since $H = 0$ and $\nabla\tilde{f}^\perp = 0$. The next examples are given in *Kähler-Ricci solitons.* Let $(N, \tilde{g}, \tilde{f})$ be a compact gradient shrinking Kähler Ricci soliton. Let $M\subset N$ be a compact *complex* submanifold such that the gradient $\nabla\tilde{f}$ is tangent to M. Then M is a compact self-similar solution, since $H = 0$ (by a well-known fact that a complex submanifold in a Kähler manifold is minimal) and $\nabla\tilde{f}^\perp = 0$ on M. Actually, Cao [2] and Koiso [12] (for notations and assumptions, see [13]) constructed examples of compact gradient shrinking Kähler Ricci solitons. By their construction, each soliton is the total space of some complex $\mathbb{P}^1$-fibration and the gradient of the potential function is tangent to every $\mathbb{P}^1$-fiber. Hence each $\mathbb{P}^1$-fiber is a compact self-similar solution with real dimension 2.

There are many results about self-shrinkers, more generally, self-similar solutions in $\mathbb{R}^m$, where a self-similar solution is an immersion $F : L \to \mathbb{R}^m$ with $H(F)_x = \lambda((\nabla \frac{|x|^2}{4}) \circ F)^\perp$ for some constant λ. If $\lambda > 0$ we call F a self-expander, and if $\lambda < 0$ we call F a self-shrinker. By a generalization of the notion of a self-similar solution in $\mathbb{R}^m$ to in a gradient shrinking Ricci soliton (N, g, f) as

$$H(F) = \lambda((\nabla f) \circ F)^\perp,$$

we can discuss which results about self-similar solutions in $\mathbb{R}^m$ also hold in a gradient shrinking Ricci soliton (N, g, f). As an example of such results, it is proved that a result due to Smoczyk partially holds in a gradient shrinking Kähler-Ricci soliton. More precisely, in the proof of Theorem 2.3.5 in [15], he proved that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in $\mathbb{C}^n$, and as a generalization of this statement, we can prove that every compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in a gradient shrinking Kähler-Ricci soliton.

In Part IV (cf. [22]), we give further two results which are already established when (N, g, f) is $\mathbb{R}^m$. The first result is an analog of Theorem 4.3 of Futaki, Li and Li [5] under the Lagrangian assumption. The statement is the following.

Theorem 24.2. *Let (N, ω, g, J) be a real $2m$-dimensional gradient shrinking Kähler-Ricci soliton with potential function $f\colon N \to \mathbb{R}$ satisfying $\mathrm{Ric}(g) + \mathrm{Hess} f = g$. Let $F\colon L \to N$ be a compact Lagrangian self-shrinker with*

$$H(F) = -\frac{1}{2}((\nabla f) \circ F)^\perp.$$

Assume that $F(L)$ is not contained in a set $\{\, f = m - \frac{C_0}{2} \,\}$, where $C_0 := R(g) + |\nabla f|^2 - 2f$ is a constant. Then we have

$$\mathrm{diam}(L, F^*g) \geq \frac{\pi}{\sqrt{\frac{3}{4} + \frac{m}{2}(K_0 + A_0^2)}},$$

for constants $K_0, A_0 \geq 0$ satisfying $|K_N| \leq K_0$ and $|A| \leq A_0$, where K_N is the sectional curvature of (N, g) and A is the second fundamental form of F.

The second result is an analog of Proposition 5.3 of Cao and Li [3].

Theorem 24.3. *Let (N, ω, g, J) be a $2m$-dimensional gradient shrinking Kähler-Ricci soliton with potential function $f\colon N \to \mathbb{R}$ satisfying $\mathrm{Ric}(g) + \mathrm{Hess} f = g$. Then there exists no compact Lagrangian self-expander in N if $R(g) < 2m$.*

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